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by

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On the Cohen-Macaulay property of the fiber cone of ideals with reduction number at most one

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Dedicated to the memory of Prof. Dr. Wolfgang Vogel

1 Introduction

Let $(A, \mathfrak{m})$ be a noetherian local ring with maximal ideal $\mathfrak{m}$ and I an ideal of A. The fiber cone of I is defined as $F_{\mathfrak{m}}(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$, which is a noetherian graded algebra over the residue field $A/\mathfrak{m}$. The projective variety associated to $F_{\mathfrak{m}}(I)$ corresponds to the fiber at the maximal ideal $\mathfrak{m}$ of the blowup of A with center I, whose arithmetic properties are those of the ring $F_{\mathfrak{m}}(I)$. This graded ring also

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provides useful information on the ideal I. For instance, its Hilbert function gives the minimal number of generators of the powers of I. Also, when the residue field is infinite, the dimension of $F_{\mathfrak{m}}(I)$ coincides with the minimal number of generators of any minimal reduction of I, that is the analytic spread of I. See J. Sally's book [10] for basic information concerning these facts.

Assume from now on that the residue field $A/\mathfrak{m}$ is infinite and let J be a minimal reduction of I. It is then well known that since J is generated by analytically independent elements $F_{\mathfrak{m}}(J)$ is a polynomial ring over the residue field $A/\mathfrak{m}$, and that $F_{\mathfrak{m}}(I)$ is a finite extension of $F_{\mathfrak{m}}(J)$, i.e. $F_{\mathfrak{m}}(J)$ is a Noether normalization of $F_{\mathfrak{m}}(I)$. As a consequence, $F_{\mathfrak{m}}(I)$ is a free $F_{\mathfrak{m}}(J)$-graded module if and only if $F_{\mathfrak{m}}(I)$ is a Cohen-Macaulay graded ring. In this case, its Hilbert function can be explicitely computed and so the minimal number of generators of the powers of I determined.

In this note we are interested in the Cohen-Macaulay property of the fiber cone. Let J be a minimal reduction of I. Recall that the reduction number of I with respect to J is defined as $r_J(I) = \min\{n \mid JI^n = I^{n+1}\}$, and the reduction number of I as $r(I) = \min\{r_J(I) \ ; \ J \text{ a minimal reduction of } I\}$. If I is generated by a regular sequence, or more in general when I is generated by a family of analytically independent elements, i.e. when $r(I) = 0$, $F_{\mathfrak{m}}(I)$ is trivially Cohen-Macaulay. In [7], C. Huneke and J. Sally proved that if A is Cohen-Macaulay and I is $\mathfrak{m}$-primary with reduction number one then $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay. This result was later extended by K. Shah in [11] to any equimultiple ideal with reduction number one in a Cohen-Macaulay ring.

Recall that the analytic deviation of I is defined as the difference $\mathrm{ad}(I) = l(I) - \mathrm{ht}\,(I)$, being the equimultiple ideals those with analytic deviation zero. In [1] we have shown that if A is Cohen-Macaulay and I is an ideal with $\mathrm{ad}(I) = 1$ and $r(I) \leq 1$ which is generically a complete intersection, then $F_{\mathfrak{m}}(I)$ is always Cohen-Macaulay. Some other isolated results have been also proven in [8] (defining ideals of monomial curves in $\mathbb{P}^3$ lying on a quadric), [3] (defining ideals of monomial varieties of codimension two whose Rees algebra is presented by an ideal generated by elements of degree at most two), and [2] (ideals of minimal mixed multiplicity with reduction number at most one in Cohen-Macaulay rings).

Although the Cohen-Macaulayness of the fiber cone does not require the same property for the associated graded ring $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$, see [1], Example 3.8 and Remark 4.3, in view of the results proved in [1] one has a better chance for $F_{\mathfrak{m}}(I)$ to be Cohen-Macaulay if $G(I)$ is so. On the other hand, unifying and generalizing work by several authors on the Cohen-Macaulay property of the associated graded ring, S. Goto, Y. Nakamura, and K. Nishida have given in [5] a quite general list of conditions which force $G(I)$ to be Cohen-Macaulay. It seems then natural to ask when under the same hypothesis the fiber cone $F_{\mathfrak{m}}(I)$ is also Cohen-Macaulay.

In the main result Theorem 3.2 of this note we give a positive answer to the above question when the reduction number of I is at most one. In order to prove it we shall apply a criteria found in [1] to decide when a given family $a_1, \dots, a_k$ of elements in I provides a regular sequence in $F_{\mathfrak{m}}(I)$ when taking their associated leading forms in $I/\mathfrak{m}I \subset F_{\mathfrak{m}}(I)$, which can be seen as a some kind of "mixed" Valabrega-Valla condition. The precise statement of this result as well as some other technical details we shall need are listed in Section 2, while Section 3 is entirely devoted to prove the main theorem of the paper and give some applications.

2 GNN-ideals

Let $(A, \mathfrak{m})$ be a d-dimensional noetherian local ring with an infinite residue field and $I \subset A$ be an ideal with $\mathrm{ht}\,(I) = h$. For a given element $a \in A$ denote by a^* its image in $I/I^2 \hookrightarrow G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$, and by a^0 its image in $I/\mathfrak{m}I \hookrightarrow F_{\mathfrak{m}}(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^{n+1} = G(I)/\mathfrak{m}G(I)$. Next result tells us when the leading forms in $F_{\mathfrak{m}}(I)$ of a family of elements in I are a regular sequence in $F_{\mathfrak{m}}(I)$.

Proposition 2.1 ([1],Theorem 2.8) *Let $a_1, \dots, a_k$ be a family of elements in I. Assume that*

(i) $a_1, \dots, a_k$ *is a regular sequence in* A,

(ii) $(a_1, \dots, a_k) \cap I^{n+1} = (a_1, \dots, a_k)I^n$ *for all* $n \geq 0$.

Then, $a_1^0, \dots, a_k^0$ *is a regular sequence in* $F_{\mathfrak{m}}(I)$ *if and only if* $(a_1, \dots, a_k) \cap \mathfrak{m}I^{n+1} = (a_1, \dots, a_k)\mathfrak{m}I^n$ *for all* $n \geq 0$.

Note that by the well known criteria of Valabrega-Valla [12], (i) and (ii) in the above proposition is equivalent to $a_1^*, \ldots, a_k^*$ is a regular sequence in $G(I)$. It is then clear that the Cohen-Macaulayness of $G(I)$ will help the Cohen-Macaulayness of the fiber cone to hold. Summarizing and generalizing work by several people, Goto-Nakamura-Nishida have given in [5] a set of quite general conditions on I which imply that $G(I)$ is Cohen-Macaulay. Assume that A is Cohen-Macaulay and let $l = l(I)$ the analytic spread of I, J a minimal reduction of I. Suppose that $a_1, \ldots, a_l$ is a minimal set of generators for J which satisfies the condition

(∗) $(a_1, \ldots, a_i)A_\mathfrak{p}$ is a reduction of $I_\mathfrak{p}$ for any $\mathfrak{p} \in V(I)$ and $i = \mathrm{ht}(\mathfrak{p}) \le l$,

where $V(I)$ denotes the set of prime ideals containing I.

Put $J_i = (a_1, \ldots, a_i)$ for $0 \le i \le l$ and $r_i = \max\{r_{J_{i_\mathfrak{p}}}(I_\mathfrak{p})\,;\ \mathfrak{p} \in V(I) \text{ and } \mathrm{ht}(\mathfrak{p}) = i\}$ for $h \le i \le l$, and assume that $a_1, \ldots, a_l$ also satisfies the condition

(∗∗) $a_i \notin \mathfrak{p}$ if $\mathfrak{p} \in \mathrm{Ass}_A A/J_{i-1} \setminus V(I)$ for any $1 \le i \le l$.

By [5], Lemma 2.1 there always exists a minimal system of generators $a_1, \ldots, a_l$ for J satisfying conditions (∗) and (∗∗). Furthermore, for such a minimal system of generators $a_1, \ldots, a_h$ is a regular sequence. Then, they prove the following nice result:

Proposition 2.2 ([5], Theorem 1.1) *Let A be Cohen-Macaulay and J a minimal reduction of I which satisfies conditions (∗) and (∗∗). Put J_i and r_i as above and let $r \ge 0$ be an integer. If the following four conditions:*

(i) $\mathrm{depth} A/I^n \ge d - l + r - n$ *for all* $1 \le n \le r$,

(ii) $r_i \le \max\{0, i - l + r\}$ *for all* $h \le i < l$,

(iii) $A/(J_i : I)$ *is Cohen-Macaulay for all* $h \le i \le l - r - 1$, *and*

(iv) $r_J(I) \le r$

are satisfied, then $G(I)$ is Cohen-Macaulay.

To simplify notation we shall say that I is a r-GNN ideal if there exists a minimal reduction J of I with a minimal system of generators which satisfies conditons (∗),

$(**)$, and (i), (ii), (iii), (iv) in the above proposition. We refer to [5] for several circumstances under which an ideal is r-GNN. As an example we record the following simple case:

Proposition 2.3 *Assume that A is Gorenstein and let $\mathfrak{p}$ be a prime ideal of positive height with analytic deviation two. Suppose that $A/\mathfrak{p}$ is Cohen-Macaulay and $\mathfrak{p}_\mathfrak{q}$ is a complete intersection for every prime ideal $\mathfrak{q} \subset \mathfrak{p}$ with $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p}) \leq 2$, or $\mathfrak{p}_\mathfrak{q}$ is a complete intersection for every prime ideal $\mathfrak{q} \subset \mathfrak{p}$ with $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p}) \leq 1$ and there exists a minimal reduction J of I with $r_J(I) = 1$. Then, $\mathfrak{p}$ is a 1-GNN ideal.*

Proof. Let J be a minimal reduction of I and $a_1, \ldots, a_l$ a minimal system of generators of J satisfying conditions $(*)$ and $(**)$. Then, condition (i) holds trivially while condition (ii) holds because the only reduction of a complete intersection ideal is the ideal itself. Condition (iii) comes from [9], Lemma 1.3, and (iv) from [6], Corollary 3.3, or by hypothesis.

Example 2.4 Let $R = k[X_{1j}, X_{2j} \mid j = 1,2,3,4]$ be the polynomial ring in eight variables over an infinite field k, and P the prime ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{pmatrix}.$$

Set $M = (X_{1j}, X_{2j} \mid j = 1,2,3,4)$, $A = R_M$, and $\mathfrak{p} = PA$. Then $\mathfrak{p}$ is a Cohen-Macaulay prime ideal of height 3 with $\mathrm{ad}(\mathfrak{p}) = 2$ such that $\mathfrak{p}_\mathfrak{q}$ is a complete intersection for any prime ideal $\mathfrak{p} \subset \mathfrak{q}$ with $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p}) \leq 1$. Furthermore, there exists a minimal reduction J of $\mathfrak{p}$ such that $r_J(\mathfrak{p}) = 1$, see [4], Example 4.7. Thus $\mathfrak{p}$ is a 1-GNN ideal.

In order to prove our main result we shall only need some intersection properties satisfied by GNN-ideals. We summarize them for the case $r = 1$ in the following lemma. We use the same notation as in Proposition 2.2.

Lemma 2.5 *Let A be a Cohen-Macaulay ring and I a 1-GNN ideal. Then*

(i) For all $0 \leq i \leq l$, $J_i \cap I^{n+1} = J_i I^n$ for all $n \geq 0$.

(ii) For all $0 \leq i \leq l-1$, $(J_i : a_{i+1}) \cap I^{n+1} = J_i I^n$ for all $n \geq 0$.

Proof. See [5], Lemmas 3.1 and Corollary 3.3.

Note that if $l = h$, i.e if I is equimultiple, a 1-GNN ideal is always Cohen-Macaulay by condition (i), and if $l = h + 1$, i.e if $\operatorname{ad}(I) = 1$, a 1-GNN ideal satisfies $\operatorname{depth}(A/I) \geq \dim A/I - 1$. It then happens that the intersection properties described in Lemma 2.5 are satisfied by a larger class of ideals than the GNN-ideals. For instance, we have:

Proposition 2.6 *Let A be a Cohen-Macaulay ring with infinite residue field and I an ideal with reduction number at most one. Let J be a minimal reduction of I such that $I^2 = JI$ and assume that I is equimultiple, or I is generically a complete intersection with* $\operatorname{ad}(I) = 1$. *Then, there exists a minimal system of generators $a_1, \ldots, a_l$ of J such that if we put $J_i = (a_1, \ldots, a_i)$ for all i the following holds:*

(i) $J_i \cap I^{n+1} = J_i I^n$ for all $0 \leq i \leq l$ and for all $n \geq 0$,

(ii) $(J_i : a_{i+1}) \cap I^{n+1} = J_i I^n$ for all $0 \leq i \leq l - 1$ and for all $n \geq 0$.

Proof. Assume first that I is equimultiple. Then $l = h$ and $a_1, \ldots, a_l$ is a regular sequence. Trivially, $J \cap I^{n+1} = JI^n$ for all $n \geq 0$ since $I^2 = JI$. Thus by the Valabrega-Valla criteria, $a_1^0, \ldots, a_l^0$ is a regular sequence in $G(I)$ and so $a_1^0, \ldots, a_i^0$ for any $0 \leq i \leq l$. Again by the Valabrega-Valla criteria we get that $J_i \cap I^{n+1} = J_i I^n$ for all $n \geq 0$. Condition (ii) holds simply because $(J_i : a_{i+1}) = J_i$ for all $i < l$.

Now suppose that I is generically a complete intersection with $\operatorname{ad}(I) = 1$. Since $I^2 = JI$ we get $J_l \cap I^{n+1} = J_l I^n$ for all $n \geq 0$. Furthermore, by [1], Lemma 4.1 we may assume that $a_1, \ldots, a_h$ is a regular sequence, $J_h \cap I^{n+1} = J_h I^n$ for all $n \geq 0$, and $(J_h : a_{h+1}) \cap I^{n+1} = J_h$ for all $n \geq 0$. Again by using the Valabrega-Valla criteria we get that $J_i \cap I^{n+1} = J_i I^n$ for all $n \geq 0$ and all $0 \leq i \leq h$. To show (ii), note that if $i = l - 1 = h$ we have for all $n \geq 0$ that $(J_h : a_{h+1}) \cap I^{n+1} = J_h \cap I^{n+1} = J_h I^n$ by (i). Finally, if $i < h$ it suffices to take into account that $(J_i : a_{i+1}) = J_i$.

Consider [6], Example 4.7. This is a generically complete intersection ideal with $\operatorname{depth} A/I < \dim A/I - 1$, analytic deviation one, and reduction number one. Hence it satisfies the intersection properties in Proposition 2.6 but it is not a 1-GNN ideal.

3 Main result

Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field and $I \subseteq A$ be an 1-GNN ideal with $\mathrm{ht}(I) = h$ and $l(I) = l$. Our main purpose is to prove that $F_{\mathfrak{m}}(I)$ is a Cohen-Macaulay graded ring.

Let J be a minimal reduction of I, and $a_1, \ldots, a_l$ a system of generators for J satisfying conditions $(*)$, $(**)$ and (i), (ii), (iii), (iv) in Proposition 2.2. We will show that $a_1^0, \ldots, a_l^0$ is a $F_{\mathfrak{m}}(I)$-regular sequence. First of all we prove the following lemma. We use the same notation as in Section 2.

Lemma 3.1 *Let $h \leq i \leq l$. Then, $J_i \cap \mathfrak{m}I^{n+1} = J_i\mathfrak{m}I^n$ for all $n \geq 0$.*

Proof. By descendent induction on i. Assume first that $i = l$. For $n = 0$ the equality $J_l \cap \mathfrak{m}I = J_l\mathfrak{m}$ is satisfied because $J_l = J$ and $a_1, \ldots, a_l$ is part of a minimal system of generators of I. On the other hand, since $r_J(I) \leq 1$, $I^{n+1} = JI^n$ for all $n \geq 1$ and so $J_l \cap \mathfrak{m}I^{n+1} = J_l\mathfrak{m}I^n$.

Assume now $i = l - 1$. For $n = 0$ the equality $J_{l-1} \cap \mathfrak{m}I = J_{l-1}\mathfrak{m}$ is satisfied as above. Let $n > 0$. Then $J_{l-1} \cap \mathfrak{m}I^{n+1} = J_{l-1} \cap \mathfrak{m}I^nJ = \mathfrak{m}I^nJ_{l-1} + a_l\mathfrak{m}I^n \cap J_{l-1}$. Thus it suffices to see that $a_l\mathfrak{m}I^n \cap J_{l-1} \subseteq \mathfrak{m}I^nJ_{l-1}$. If $xa_l \in J_{l-1}$ with $x \in \mathfrak{m}I^n$ then $x \in (J_{l-1} : a_l) \cap I = J_{l-1}$ by Lemma 2.5 and $x \in J_{l-1} \cap \mathfrak{m}I^n = J_{l-1}\mathfrak{m}I^{n-1}$ by induction on n, so $xa_l \in J_{l-1}\mathfrak{m}I^n$.

Let $h \leq i < l - 1$ and assume $J_k \cap \mathfrak{m}I^{n+1} = J_k\mathfrak{m}I^n$ for all $n \geq 0$ and $i < k \leq l$. If $n = 0$ then again $J_i \cap \mathfrak{m}I = J_i\mathfrak{m}$ because $a_1, \ldots, a_i$ is part of a minimal system of generators of I. Let $n > 0$. Then $J_i \cap \mathfrak{m}I^{n+1} = J_i \cap \mathfrak{m}I^nJ = J_i\mathfrak{m}I^n + (a_{i+1}, \ldots, a_l)\mathfrak{m}I^n \cap J_i$. Let $x_{i+1}a_{i+1} + \cdots + x_la_l \in J_i$ with $x_{i+1}, \ldots, x_l \in \mathfrak{m}I^n$. Then $x_l \in (J_{l-1} : a_l) \cap I = J_{l-1}$ by Lemma 2.5 and $x_l \in J_{l-1} \cap \mathfrak{m}I^n = J_{l-1}\mathfrak{m}I^{n-1}$, so $x_la_l \in J_{l-1}\mathfrak{m}I^n$ and $(a_{i+1}, \ldots, a_l)\mathfrak{m}I^n \cap J_i \subseteq J_{l-1}\mathfrak{m}I^n \cap J_i \subseteq J_i\mathfrak{m}I^n + (a_{i+1}, \ldots, a_{l-1})\mathfrak{m}I^n \cap J_i$. By repeating this argument succesively we get $(a_{i+1}, \ldots, a_l)\mathfrak{m}I^n \cap J_i \subseteq J_i\mathfrak{m}I^n + (a_{i+1})\mathfrak{m}I^n \cap J_i$. But if $x \in \mathfrak{m}I^n$ is such that $a_{i+1}x \in J_i$ then $x \in (J_i : a_{i+1}) \cap I = J_i$ by Lemma 2.5. So $x \in J_i \cap \mathfrak{m}I^n = J_i\mathfrak{m}I^{n-1}$ by induction on n and $a_{i+1}x \in J_i\mathfrak{m}I^n$.

Now we are ready to prove our main result.

Theorem 3.2 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field and I a 1-GNN ideal. Then $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay.*

Proof. We divide the proof in two steps. First we will see that we may assume $\mathrm{ht}\,(I) = 0$. Then we will prove the statement under this assumption.

First step. $a_1, \ldots, a_h$ is a regular sequence and $J_h \cap I^{n+1} = J_h I^n$ for $n \geq 0$ by Lemma 2.5. Then by Proposition 2.1 the sequence $a_1^0, \ldots, a_h^0$ is regular in $F_{\mathfrak{m}}(I)$ if and only if $J_h \cap \mathfrak{m} I^{n+1} = J_h \mathfrak{m} I^n$ for all $n \geq 0$, and this follows directly from the Lemma 3.1. So $a_1^0, \ldots, a_h^0$ is a regular sequence in $F_{\mathfrak{m}}(I)$. Now consider the Cohen-Macaulay ring $A/(a_1, \ldots, a_h)$. Then, the ideal $I/(a_1, \ldots, a_h)$ is 1-GNN and $J/(a_1, \ldots, a_h)$ is a minimal reduction of $I/(a_1, \ldots, a_h)$ satisfying the conditions $(*)$, $(**)$ and (i), (ii), (iii), (iv) in the Proposition 2.1 (see [5, Lemma 3.4]). On the other hand one can easily see that $F_{\mathfrak{m}}(I)/(a_1^0, \ldots, a_h^0) \simeq F_{\mathfrak{m}}(I/(a_1, \ldots, a_h))$, and so we may assume $\mathrm{ht}\,(I) = 0$.

Second Step. Assume $\mathrm{ht}\,(I) = 0$. We want to see that $a_1^0, \ldots, a_k^0$ is a $F_{\mathfrak{m}}(I)$-regular sequence. This is equivalent to show the equalities

$$(\mathfrak{m} I^{n+1} + J_i I^n : a_{i+1}) \cap I^n = \mathfrak{m} I^n + J_i I^{n-1} \text{ for } 0 \leq i \leq l-1 \text{ and } n \geq 0.$$

If $n = 0$ we have to prove $(\mathfrak{m} I + J_i : a_{i+1}) = \mathfrak{m} + J_i$. Let x be an element of A such that $x a_{i+1} \in \mathfrak{m} I + J_i$. Then there exists $z \in \mathfrak{m} I$ and $y_1, \ldots, y_i \in A$ verifying $x a_{i+1} = z + y_1 a_1 + \ldots y_i a_i$, so $z \in \mathfrak{m} I \cap J_{i+1} = \mathfrak{m} J_{i+1} \subseteq \mathfrak{m} \subseteq \mathfrak{m} + J_i$.

Assume $n > 0$. Then $(\mathfrak{m} I^{n+1} + J_i I^n : a_{i+1}) \cap I^n = (\mathfrak{m} I^n J + J_i I^n : a_{i+1}) \cap I^n$. If $x \in I^n$ and $x a_{i+1} \in \mathfrak{m} I^n J + J_i I^n = J_i I^n + (a_{i+1}, \ldots, a_l) \mathfrak{m} I^n$ then there exist $\alpha_1, \ldots, \alpha_i \in I^n$ and $\beta_{i+1}, \ldots, \beta_l \in \mathfrak{m} I^n$ such that $x a_{i+1} = \alpha_1 a_1 + \ldots + \alpha_i a_i + \beta_{i+1} a_{i+1} + \ldots + \beta_l a_l$. So $\beta_l \in (J_{l-1} : a_l) \cap \mathfrak{m} I^n \subseteq (J_{l-1} : a_l) \cap I^n = J_{l-1} I^{n-1}$ by Lemma 2.5 and $\beta_l \in \mathfrak{m} I^n \cap J_{l-1} = J_{l-1} \mathfrak{m} I^{n-1}$ by Lemma 3.1. Then, there exist $y_1, \ldots, y_{l-1} \in \mathfrak{m} I^{n-1}$ such that $x a_{i+1} = (\alpha_1 + y_1 a_l) a_1 + \ldots + (\alpha_i + y_i a_l) a_i + (\beta_{i+1} + y_{i+1} a_l) a_{l+1} + \ldots + (\beta_{l-1} + y_{l-1} a_l) a_{l-1}$ with $\alpha_j + y_j a_l \in I^n$, $j = 1, \ldots, i$ and $\beta_k + y_k a_l \in \mathfrak{m} I^n$, $k = i+1, \ldots, l-1$. If we repeat this argument we obtain that $x a_{i+1} = \alpha_1 a_1 + \cdots + \alpha_i a_i + \beta_{i+1} a_{i+1}$ with $\alpha_j \in I^n$ and $\beta_{i+1} \in \mathfrak{m} I^n$. Then $x - \beta_{i+1} \in (J_i : a_{i+1}) \cap I^n = J_i I^{n-1}$ by Lemma 2.5, and so $x \in \mathfrak{m} I^n + J_i I^{n-1}$.

As cited in the introduction, once we know that the fiber cone of I is Cohen-Macaulay we may explicitely compute its Hilbert function and, consequently, the minimal number of generators of the powers of I. One can also deduce that the reduction number of I with respect to a minimal reduction is independent of the reduction. Thus, as a direct consequence of [1], Remark 5.4 and Corollary 5.7, we may formulate the following corollary:

Corollary 3.3 *Let A be a Cohen-Macaulay local ring with an infinite residue field and I a 1-GNN ideal which is not generated by a familly of analitically independent elements. Then,*

(i) $\mu(I^n) = \binom{n+l(I)-1}{l(I)-1} + (\mu(I) - l(I))\binom{n+l(I)-2}{l(I)-1}$ *for all* $n \geq 0$.

(ii) For any minimal reduction J of I, $r_J(I) = 1$.

Finally, taking into account the considerations made at the end of Section 2 we may also obtain:

Corollary 3.4 *Let A be a Cohen-Macaulay ring with infinite residue field and I an ideal with reduction number at most one. Assume that I is equimultiple or I is generically a complete intersection with* $\mathrm{ad}(I) = 1$. *Then, the fiber cone of I is Cohen-Macaulay.*

In particular, the above result can be applied to [6], Example 4.7. See also [2] for another approach to show the Cohen-Macaulayness of that fiber cone.

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