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Abstract

It was proved by Dow and Simon that there are 2^{ω_1} (as many as possible) pairwise non-homeomorphic compact, T_2, scattered spaces of height ω_1 and width ω. In this paper, we prove that if α is an ordinal with $\omega_1 \leq \alpha < \omega_2$ and $\theta = \langle \kappa_\xi : \xi < \alpha \rangle$ is a sequence of cardinals such that either $\kappa_\xi = \omega$ or $\kappa_\xi = \omega_1$ for every $\xi < \alpha$, then there are 2^{ω_1} pairwise non-homeomorphic compact, T_2, scattered spaces whose cardinal sequence is θ.

Keywords: Cantor-Bendixson derivatives; scattered spaces; cardinal sequences.

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A topological space X is called *scattered*, if every closed subspace of X has an isolated point. A useful tool in the study of scattered spaces is the Cantor-Bendixson process for topological spaces. If X is a topological space and α is an ordinal, we define the *α-derivative of* X by induction on α as follows: $X^0 = X$; if $\alpha = \beta + 1$, $X^\alpha = \{x \in X : x \text{ is an accumulation point of } X^\beta\}$; and if α is limit, $X^\alpha = \bigcap\{X^\beta : \beta < \alpha\}$. For every ordinal β, we define the *β-level of* X by $I_\beta(X) = X^\beta \setminus X^{\beta+1}$. It is well-known that a space X is scattered if and only if there is an ordinal α such that $X^\alpha = \emptyset$.

Suppose that X is a scattered space. Then we define the *height of* X by $ht(X)$ = the least ordinal β such that X^β is finite, and we define the

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cardinal sequence of X by $CS(X) = \langle | I_\beta(X) | : \beta < ht(X) \rangle$. All the spaces we consider are Hausdorff. By an LCS-*space* we mean a locally compact, Hausdorff, scattered space. Note that if X is an LCS-space with cardinal sequence θ and X is not compact, then the one-point compactification of X has also cardinal sequence θ. If $\alpha > 0$ is an ordinal and X is an LCS-space, we say that X is an (ω, α)-*space* if $CS(X) = \theta$ where θ is the sequence $\langle \kappa_\beta : \beta < \alpha \rangle$ with $\kappa_\beta = \omega$ for every $\beta < \alpha$. An LCS-space X is called *thin-tall*, if X is an (ω, ω_1)-space. It was proved by Rajagopalan and, independently, by Juhász and Weiss that there exists a thin-tall space. In [3], it was even proved by Juhász and Weiss that for every ordinal α such that $0 < \alpha < \omega_2$, there exists an (ω, α)-space. However, it is known that the existence of an (ω, ω_2)-space is independent of the axioms of Set Theory (see [1]). On the other hand, it was proved by Dow and Simon in [2] that there are 2^{ω_1} (as many as possible) pairwise non-homeomorphic thin-tall spaces. From the proof of this result we can infer by using a standard argument that for every ordinal α such that $\omega_1 \leq \alpha < \omega_2$, there are also 2^{ω_1} pairwise non-homeomorphic (ω, α)-spaces. The aim of this paper is then to prove that if α is an ordinal with $\omega_1 \leq \alpha < \omega_2$ and $\theta = \langle \kappa_\xi : \xi < \alpha \rangle$ is a sequence of cardinals such that either $\kappa_\xi = \omega$ or $\kappa_\xi = \omega_1$ for every $\xi < \alpha$, then there are 2^{ω_1} pairwise non-homeomorphic LCS-spaces whose cardinal sequence is θ.

This paper is divided in two sections. In the first one, we consider the case of cardinal sequences of length ω_1. In the second section, we first prove that for every ordinal $\alpha < \omega_2$ and every cardinal sequence $\theta = \langle \kappa_\xi : \xi < \alpha \rangle$ where $\kappa_\xi \in \{\omega, \omega_1\}$ for each $\xi < \alpha$, there is an LCS-space with cardinal sequence θ, and then we prove that the construction given in Section 1 can be generalized to any uncountable ordinal $< \omega_2$.

We want to remark that results on cardinal sequences for LCS-spaces have a direct translation to the context of superatomic Boolean algebras (i.e. Boolean algebras in which every subalgebra is atomic), since it is known that the notion of a compact, Hausdorff, scattered space is the dual notion of a superatomic Boolean algebra.

1 Cardinal sequences of length ω_1

We fix a cardinal sequence $\theta = \langle \kappa_\xi : \xi < \omega_1 \rangle$ where $\kappa_\xi \in \{\omega, \omega_1\}$ for every $\xi < \omega_1$. Then, by using a refinement of the argument carried out in [2, Section 2], we shall construct 2^{ω_1} pairwise non-homeomorphic LCS-spaces with cardinal sequence θ. The underlying set of the 2^{ω_1} spaces we want to construct will be the set $D = \bigcup\{\{\xi\} \times \kappa_\xi : \xi < \omega_1\}$. For every $n < \omega$, we define the *column* C_n by $\omega_1 \times \{n\}$. Now suppose that X is an LCS-space of underlying set D such that $I_\xi(X) = \{\xi\} \times \kappa_\xi$ for any $\xi < \omega_1$. Let S be a stationary subset of ω_1. Then, for $n < \omega$, we say that S is *associated* to C_n in X, if for every $x = (\xi, n) \in C_n$ where ξ is a limit ordinal, the following holds:

(1) If $\xi \in S$, then for every neighbourhood U of x there is a $\zeta < \xi$ such that $\{(\mu, n) : \zeta < \mu \leq \xi\} \subseteq U$.

(2) If $\xi \notin S$, there is a neighbourhood U of x such that $U \cap C_n = \{x\}$.

Then we say that X is an *admissible θ-space*, if the following conditions hold:

(*) (1) For each $n < \omega$, C_n is a closed subset of X.

(2) For each $n < \omega$, there is a stationary subset of ω_1 associated to C_n in X.

(3) For every $x \in X$ there is a neighbourhood U of x such that $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$.

Lemma 1 *If X and Y are admissible θ-spaces and $f : X \longrightarrow Y$ is a homeomorphism, then for every $k < \omega$ there are an $n < \omega$ and a $\xi < \omega_1$ such that $f''(C_k \cap X^\xi) = C_n \cap Y^\xi$.*

Proof. It is clear that for every $x \in X$, if $x \in I_\beta(X)$ then $f(x) \in I_\beta(Y)$. We consider ω_1 with the order topology. Then, if $N \subseteq \omega_1$ we write $N' = \{\xi < \omega_1 : \xi \text{ is an accumulation point of } N\}$. Let S be the stationary subset associated to C_k in X. We have that $f''(C_k) \setminus \bigcup\{C_n : n < \omega\}$ is countable. To check this point, note that otherwise if we put $N = \{\zeta < \omega_1 : (\zeta, \mu) \in f''(C_k) \setminus \bigcup\{C_n : n < \omega\}$ for some $\mu < \omega_1\}$, then there is a $\rho \in S \cap N'$. Now, by using (*)(3), we infer that no point of Y can be the image under f of the point (ρ, k). On the other hand, if for $k < \omega$ there are $m, n < \omega$ with $m \neq n$ such that $C_m \cap f''(C_k)$ and $C_n \cap f''(C_k)$ are uncountable, then if we put $M = \{\zeta < \omega_1 : (\zeta, m) \in f''(C_k)\}$ and $N = \{\zeta < \omega_1 : (\zeta, n) \in f''(C_k)\}$, we have that there is a $\rho \in S \cap M' \cap N'$. Now, we would infer from (*)(1) that no point of Y can be the image under f of (ρ, k). $\dashv$

In what follows, if x is a point of an LCS-space X, when we consider a neighbourhood U of x, we shall tacitly assume that if β is the ordinal such that $x \in I_\beta(X)$, then $U \cap X^\beta = \{x\}$.

By a *decomposition* of an infinite set a, we mean a partition of a in infinite subsets.

Theorem 1 *Let S be a stationary subset of ω_1. Then, there is an admissible θ-space X such that for each $n < \omega$, S is the stationary subset associated to C_n in X.*

Proof. We construct by transfinite induction on $\xi < \omega_1$ a space X_ξ satisfying the following conditions:

(1) The underlying set of X_ξ is $\bigcup\{X_\xi^{(\mu)} : \mu \leq \xi\}$ where $X_\xi^{(\mu)} = \{\mu\} \times \omega$ if $\kappa_\mu = \omega$ or $\xi \leq \omega$, $X_\xi^{(\mu)} = \{\mu\} \times \xi$ if $\kappa_\mu = \omega_1$ and $\xi > \omega$.

(2) X_ξ is an LCS-space such that $I_\mu(X_\xi) = X_\xi^{(\mu)}$ for every $\mu \leq \xi$.

(3) For every $n < \omega$, $C_n \cap X_\xi$ is a closed subset of X_ξ.

(4) If ξ is limit and $\xi \in S$, then for every $n < \omega$ and every neighbourhood U of (ξ, n) there is a $\zeta < \xi$ such that $\{(\mu, n) : \zeta < \mu \leq \xi\} \subseteq U$.

(5) If ξ is limit and $\xi \notin S$, then for each $n < \omega$ there is a neighbourhood U of (ξ, n) such that $U \cap C_n = \{(\xi, n)\}$.

(6) For every $x \in X_\xi$ there is a neighbourhood U of x such that $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$.

(7) If $\xi < \eta$ and $x \in X_\xi$, then a neighbourhood basis of x in X_ξ is also a neighbourhood basis of x in X_η.

(8) If $\xi < \eta$, then every compact subset of X_ξ is a compact subset of X_η.

We define X_0 as the ordinal ω with the order topology. Then, assume $\xi > 0$. Without loss of generality, we may assume that $\xi \geq \omega$ and $\kappa_\xi = \omega_1$. First, we suppose $\xi = \mu + 1$. To construct X_ξ we previously define for each $\alpha \leq \mu$ an LCS-space Y_α such that $ht(Y_\alpha) = \xi$, $I_\beta(Y_\alpha) = \{\beta\} \times \xi$ if $\beta \leq \alpha$ and $\kappa_\beta = \omega_1$, and $I_\beta(Y_\alpha) = I_\beta(X_\mu)$ otherwise. In addition, we shall have that if $\beta < \alpha \leq \mu$ and $x \in Y_\beta$, then a neighbourhood basis of x in Y_β is also a neighbourhood basis of x in Y_α. The construction of Y_0 is immediate. Then, assume that α is limit. Let Y be the direct union of $\{Y_\beta : \beta < \alpha\}$. If $\kappa_\alpha = \omega$, we put $Y_\alpha = Y$. Then, suppose $\kappa_\alpha = \omega_1$. We have to define a neighbourhood basis of the point (α, μ). Let $\{x_n : n < \omega\}$ be an enumeration of Y. For each $n < \omega$, we construct an open compact neighbourhood U_n of some y_n in Y as follows. We take U_0 as an open compact neighbourhood of x_0 such that $U_0 \setminus \{x_0\} \subseteq \bigcup\{C_n : n < \omega\}$. If $n > 0$, let y_n be the first element in the enumeration $\{x_n : n < \omega\}$ such that $y_n \notin U_0 \cup \ldots \cup U_{n-1}$. Then we choose U_n as an open compact neighbourhood of y_n such that:

(+) (1) $U_n \setminus \{y_n\} \subseteq \bigcup\{C_k : k < \omega\}$.

(2) For all $m \leq n$, if $y_n \notin C_m$ then $U_n \cap C_m = \emptyset$.

(3) $U_n \cap (U_0 \cup \ldots \cup U_{n-1}) = \emptyset$.

Let $\{z_n : n < \omega\}$ be an enumeration of $X_\mu^{(\mu)}$. Note that for every $n < \omega$ there is a $k_n < \omega$ such that $z_n = y_{k_n}$. Then, we define $W_n = U_{k_n}$. Let $\langle \beta_n : n < \omega \rangle$ be a sequence of ordinals converging to α in a strictly increasing way. Now, for each $n < \omega$ we choose an element $v_n \in I_{\beta_n}(X_\mu) \cap W_n$ and an open compact neighbourhood V_n of v_n with $V_n \subseteq W_n$. Put $v = (\alpha, \mu)$. Then we define a basic neighbourhood of v as a set of the form $\{v\} \cup \bigcup\{V_n : n > k\}$ where $k < \omega$. If α is a successor ordinal, we would proceed in a similar way. Now, put $Z = Y_\mu$. The underlying set of X_ξ is $Z \cup \{\xi\} \times \xi$. If $x \in Z$, a basic neighbourhood of x in X_ξ is a basic neighbourhood of x in Z. Proceeding as above, we construct for each $n < \omega$ an open compact neighbourhood U_n of some y_n in Z satisfying $(+)(1) - (3)$ in such a way that $\{U_n : n < \omega\}$ is a partition of Z. For each $n < \omega$, put $v_n = (\mu, n)$ and then consider the neighbourhood V_n chosen for v_n. Let $\{t_n : n < \omega\}$ be an enumeration of $\{\xi\} \times \xi$. Let $\{a_n : n < \omega\}$ be a decomposition of ω. For $n < \omega$, we define a basic neighbourhood of t_n in X_ξ as a set of the form $\{t_n\} \cup \bigcup\{V_k : k \in a_n \setminus m\}$ where $m < \omega$.

Now suppose that ξ is a limit ordinal. If $\xi \notin S$, we can construct X_ξ by means of an argument similar to the one given in the successor case. So, we assume that $\xi \in S$. Let Z be the direct union of $\{X_\mu : \mu < \xi\}$. The underlying set of X_ξ is $Z \cup (\{\xi\} \times \xi)$. If $x \in Z$, a basic neighbourhood of x in X_ξ is a basic neighbourhood of x in Z. As above, for every $n < \omega$ we choose a neighbourhood U_n of some y_n in Z verifying $(+)(1) - (3)$ in such a way that $\{U_n : n < \omega\}$ is a partition of Z. Put $Y = \{y_n : n < \omega\}$. For every $n < \omega$, put $t_n = (\xi, n)$. Let $\{t'_n : n < \omega\}$ be an enumeration of the set $\{(\xi, \zeta) : \omega \leq \zeta < \xi\}$. Fix $n < \omega$. Our purpose is to define a neighbourhood basis of t_n. By using $(+)(2)$, it is easy to check that for every $\zeta < \xi$, $Y \cap \{(\mu, n) : \zeta < \mu < \xi\}$ is infinite. Set $Y \cap C_n = \{v_m : m < \omega\}$. For each $m < \omega$, let V_m be the neighbourhood chosen for v_m. We put $W_n = \bigcup\{V_m : m < \omega\}$. Note that there is a $\zeta < \xi$ such that $\{(\mu, n) : \zeta < \mu < \xi\} \subseteq W_n$. Then, we define a basic neighbourhood of t_n as a set of the form $\{t_n\} \cup \bigcup\{V_m : m > k\}$ where $k < \omega$. Note that $\{W_n : n < \omega\}$ is pairwise disjoint. To define a neighbourhood basis of a point t'_n, we consider a sequence of ordinals $\langle \xi_n : n < \omega \rangle$ converging to ξ in a strictly increasing way and then, for each $k < \omega$, we choose $u_k \in Y \cap C_k \cap Z^{\xi_k}$. Now, for $k < \omega$, consider the neighbourhood V_k' chosen for u_k (as an element of Y). Note that $V_k' \subseteq W_k$ for each $k < \omega$. Let $\{a_n : n < \omega\}$ be a decomposition of ω. Fix $n < \omega$. Then, we define a basic neighbourhood of t_n' as a set of the form $\{t_n'\} \cup \bigcup\{V_m' : m \in a_n \setminus k\}$ where $k < \omega$.

Now we define the desired space X as the direct union of the spaces X_ξ for $\xi < \omega_1$. ⊣

Theorem 2 *Let $\theta = \langle \kappa_\alpha : \alpha < \omega_1 \rangle$ where $\kappa_\alpha \in \{\omega, \omega_1\}$ for each $\alpha < \omega_1$. Then, there are 2^{ω_1} pairwise non-homeomorphic LCS-spaces with cardinal sequence θ.*

Proof. Let $\langle S_\xi : \xi < 2^{\omega_1} \rangle$ be a sequence of stationary subsets of ω_1 such that if $\mu < \xi < 2^{\omega_1}$, $S_\xi \setminus S_\mu$ is stationary. By using Theorem 1, for every $\xi < 2^{\omega_1}$ there is an admissible θ-space X_ξ such that S_ξ is associated to each column in X_ξ. Now, we infer from Lemma 1 that if $\mu < \xi < 2^{\omega_1}$, then X_μ and X_ξ are not homeomorphic. ⊣

2 Cardinal sequences of length greater than ω_1

Our aim here is to extend the construction given in Section 1 to any uncountable ordinal $< \omega_2$. First, we need to prove the following result:

Theorem 3 *Let α be an ordinal such that $0 < \alpha < \omega_2$. Let $\theta = \langle \kappa_\xi : \xi < \alpha \rangle$ be a sequence of cardinals such that either $\kappa_\xi = \omega$ or $\kappa_\xi = \omega_1$ for every $\xi < \alpha$. Then, there is an LCS-space X such that $CS(X) = \theta$.*

In the proof of Theorem 3 we will extend the argument given by Juhász and Weiss in [3]. If β is an ordinal and $\tau = \langle \lambda_\xi : \xi < \beta \rangle$ is a sequence of cardinals with $\lambda_\xi \in \{\omega, \omega_1\}$ for every $\xi < \beta$, we denote by K_τ the class of all the LCS-spaces X such that $CS(X) = \tau$.

Suppose that $\tau_1 = \langle \lambda_\xi : \xi \leq \alpha_1 \rangle$, $\tau_2 = \langle \lambda_\xi' : \xi \leq \alpha_2 \rangle$ are sequences of cardinals such that $\lambda_\xi \in \{\omega, \omega_1\}$ for every $\xi < \alpha_1$, $\lambda_{\alpha_1} = \omega$, $\lambda_0{}' = \omega$ and $\lambda_\xi' \in \{\omega, \omega_1\}$ for every ξ such that $0 < \xi \leq \alpha_2$. Assume that $X \in K_{\tau_1}$ is a σ-compact space such that $I_{\alpha_1+1}(X) = \emptyset$ and $Y \in K_{\tau_2}$ is a space such that $X \cap Y = \emptyset$. Then we define the LCS-space $X \otimes Y$ as follows. The underlying set of $X \otimes Y$ is $X \cup (Y \setminus I_0(Y))$. Let us consider an enumeration $\{u_n : n < \omega\}$ of $I_{\alpha_1}(X)$ and an enumeration $\{v_n : n < \omega\}$ of $I_0(Y)$. Since X is a paracompact space, for every $n < \omega$ we can choose a compact open neighbourhood U_n of u_n in such a way that $\{U_n : n < \omega\}$ is a discrete family. Then, if $x \in X$ we define a basic neighbourhood of x as a neighbourhood of x in X, and if $x \in Y \setminus I_0(Y)$ we define a basic neighbourhood of x as a set of the form $(V \setminus I_0(Y)) \cup \bigcup\{U_n : v_n \in V\}$, where V is a basic neighbourhood of x in Y. Consider $\tau = \langle \kappa_\xi : \xi \leq \alpha_1 + \alpha_2 \rangle$ where $\kappa_\xi = \lambda_\xi$ for $\xi \leq \alpha_1$ and $\kappa_\xi = \lambda'_\mu$ if $\xi = \alpha_1 + \mu$ where $0 < \mu \leq \alpha_2$. Then, it can be proved that $X \otimes Y \in K_\tau$. Note that if in addition Y is σ-compact, then $X \otimes Y$ is also σ-compact.

Let β be an ordinal such that $cf(\beta) \leq \omega$. Let $\tau = \langle \lambda_\xi : \xi < \beta \rangle$ be a sequence of cardinals such that $\lambda_\xi \in \{\omega, \omega_1\}$ for every $\xi < \beta$. Suppose that $X \in K_\tau$ is a σ-compact space with $I_\beta(X) = \emptyset$ and $T = \{t_\xi : \xi < \omega_1\}$ is a set of different elements which do not occur in X. Then we define a space $H(X,T)$ of underlying set $X \cup T$ such that $H(X,T)$ is an LCS-space with $ht(H(X,T)) = \beta + 1$, $I_\xi(H(X,T)) = I_\xi(X)$ for $\xi < \beta$, $I_\beta(H(X,T)) = T$ and $I_{\beta+1}(H(X,T)) = \emptyset$. First we assume that $\beta = \gamma + 1$ is a successor ordinal. Then, if $x \in X$ we define a basic neighbourhood of x as a neighbourhood of x in X. Since X is σ-compact, we infer that $I_\gamma(X)$ is a countable set. Let $\{y_n : n < \omega\}$ be an enumeration of $I_\gamma(X)$.. For every $n < \omega$ we consider a compact open neighbourhood U_n of y_n in such a way that $\{U_n : n < \omega\}$ is a discrete family. Let $\{a_\xi : \xi < \omega_1\}$ be an almost disjoint family of ω. Then, for every $\xi < \omega_1$, a basic neighbourhood of t_ξ is a set of the form $\{t_\xi\} \cup \bigcup\{U_m : m \in a_\xi, m > k\}$ where $k < \omega$. Analogously, if $cf(\beta) = \omega$ we consider a sequence of ordinals $\langle \beta_n : n < \omega \rangle$ converging to β in a strictly increasing way, and then for each $n < \omega$ we choose a point $z_n \in I_{\beta_n}(X)$ and a compact open neighbourhood U_n of z_n in such a way that $\{U_n : n < \omega\}$ is a discrete family. As above we consider an almost disjoint family $\{a_\xi : \xi < \omega_1\}$ of ω, and then we define as a basic neighbourhood of t_ξ a set of the form $\{t_\xi\} \cup \bigcup\{U_m : m \in a_\xi, m > k\}$ where $k < \omega$. Proceeding in a similar way, we can define a space $H(X,T)$ if T is an infinite countable set of elements not occurring in X. Note that in this case $H(X,T)$ is σ-compact.

Proof of Theorem 3. We show that for every ordinal $\alpha < \omega_2$ and every sequence of cardinals $\theta = \langle \kappa_\xi : \xi \leq \alpha \rangle$ where $\kappa_\xi \in \{\omega, \omega_1\}$ for each $\xi \leq \alpha$, we can construct a space $X \in K_\theta$ with $I_\xi(X) = \{\xi\} \times \kappa_\xi$ for every $\xi \leq \alpha$

and $I_{\alpha+1}(X) = \emptyset$. We construct the space X by transfinite induction on α. Without loss of generality we may assume that $\kappa_\alpha = \omega_1$. The case $\alpha = 0$ is immediate. Then suppose $\alpha = \beta+1$. Let $\theta_\beta = \langle \kappa_\xi : \xi \leq \beta \rangle$. By the induction hypothesis, $K_{\theta_\beta} \neq \emptyset$. Let $\theta'_\beta = \langle \kappa_\xi : \xi < \beta \rangle$. Since $K_{\theta_\beta} \neq \emptyset$, it follows that there is a compact space $Z_0 \in K_{\theta'_\beta}$. Let Z_1 be the topological sum of a family of ω disjoint copies of Z_0. Then we define $Z = H(Z_1, \{\alpha\} \times \omega_1)$. Now let us consider a $Y \in K_{\theta_\beta}$ such that $Y \cap Z = \emptyset$. Let X be the topological sum of Y and Z. Then, it follows that $X \in K_\theta$.

Next assume that α is a limit ordinal such that $cf(\alpha) = \omega$. Let $\langle \alpha_n : n < \omega \rangle$ be a sequence of ordinals converging to α in a strictly increasing way. For each $n < \omega$, we put $\theta_n = \langle \kappa_\xi : \xi \leq \alpha_n \rangle$. By the induction hypothesis, for each $n < \omega$ there is a compact space $Y_n \in K_{\theta_n}$. We may assume that the Y_n are pairwise disjoint. Let Y be the topological sum of the Y_n for $n < \omega$. Then we define $X = H(Y, \{\alpha\} \times \omega_1)$. We have $X \in K_\theta$.

Now assume that α is a limit ordinal such that $cf(\alpha) = \omega_1$. Let $\langle \gamma_\mu : \mu < \omega_1 \rangle$ be a closed sequence of ordinals converging to α in a strictly increasing way such that $cf(\gamma_\mu) \leq \omega$ for each $\mu < \omega_1$. Let $\langle \alpha_\xi : \xi < \nu \rangle$ be the order-preserving enumeration of the γ_μ such that $\kappa_{\gamma_\mu} = \omega_1$. Without loss of generality we may suppose that $\nu = \omega_1$. In order to find a space $X \in K_\theta$, we construct by transfinite induction on $\xi \in [\omega, \omega_1)$ an "approximation" X_ξ such that the following conditions hold:

(1) The underlying set of X_ξ is $\bigcup\{X_\xi^{(\beta)} : \beta \leq \alpha_\xi\} \cup X_\xi^{(\alpha)}$ where $X_\xi^{(\beta)} = \{\beta\} \times \kappa_\beta$ if $\beta \notin \{\alpha_\mu : \mu \leq \xi\} \cup \{\alpha\}$ and $X_\xi^{(\beta)} = \{\beta\} \times \xi$ if $\beta \in \{\alpha_\mu : \mu \leq \xi\} \cup \{\alpha\}$.

(2) X_ξ is a σ-compact LCS-space such that $X_\xi^{(\beta)} = I_\beta(X_\xi)$ for each $\beta \leq \alpha_\xi$ and $X_\xi^{(\alpha)} = I_{\alpha_\xi+1}(X_\xi)$.

(3) $X_\xi \setminus X_\xi^{(\alpha)}$ with the relative topology of X_ξ is a σ-compact LCS-space.

(4) If $\omega \leq \mu < \xi$ and $x \in X_\mu^{(\beta)}$ for some $\beta \leq \alpha_\mu$, then a neighbourhood basis of x in X_μ is also a neighbourhood basis of x in X_ξ.

(5) If $\omega \leq \mu < \xi$ and $C \subseteq X_\mu \setminus X_\mu^{(\alpha)}$ is a compact subset of X_μ, then C is a compact subset of X_ξ.

Moreover if $\omega \leq \xi < \omega_1$, we will define for each $x \in X_\xi^{(\alpha)}$ a canonical neighbourhood $W_x^{(\xi)}$ of x in X_ξ in such a way that the following two conditions hold:

(1) If $\omega \leq \mu < \xi < \omega_1$ and $x \in X_\mu^{(\alpha)}$, then $W_x^{(\mu)} \subseteq W_x^{(\xi)}$.

(2) If $\omega \leq \mu < \xi < \omega_1$ and $x, y \in X_\mu^{(\alpha)}$ with $x \neq y$, then $W_x^{(\mu)} \cap W_y^{(\mu)} = W_x^{(\xi)} \cap W_y^{(\xi)}$.

For each $x \in X_\xi^{(\alpha)}$, we will define a clopen neighbourhood basis of x in X_ξ from the canonical neighbourhood $W_x^{(\xi)}$. Furthermore, we shall have that $W_x^{(\xi)}$ is a compact neighbourhood of x.

In order to construct X_ω, we define by induction on $n < \omega$ a σ-compact LCS-space Y_n with $ht(Y_n) = \alpha_n + 1$, $I_{\alpha_n+1}(Y_n) = \emptyset$ and such that if $m < n < \omega$, Y_m is an open subspace of Y_n and for any $\zeta \leq \alpha_m$, $I_\zeta(Y_m) = I_\zeta(Y_n)$. We assume $\alpha_0 > 0$. Let $\tau_0 = \langle \kappa_\beta : \beta < \alpha_0 \rangle$. By the induction hypothesis, there is a compact space $Z_0 \in K_{\tau_0}$. Then we define Y_0 as the topological sum of ω disjoint copies of Z_0. Next assume $n = m + 1$. Let $\delta = o.t.(\alpha_n \setminus \alpha_m)$. Let $\tau = \langle \lambda_\zeta : \zeta < \delta \rangle$ where $\lambda_0 = \omega$ and $\lambda_\zeta = \kappa_{\alpha_m + \zeta}$ if $0 < \zeta < \delta$. Again by the induction hypothesis, there is a compact space $Z_0 \in K_\tau$. Let Z_1 be the topological sum of ω disjoint copies of Z_0. Then we define $Y_n = Y_m \otimes Z_1$. Let Y' be the direct union of the spaces Y_n for $n < \omega$. Without loss of generality we may suppose that α_ω is the limit of $\{\alpha_n : n < \omega\}$. Then we put $Y = H(Y', \{\alpha_\omega\} \times \omega)$. We define the underlying set of X_ω as $Y \cup (\{\alpha\} \times \omega)$. If $x \in Y$, a basic neighbourhood of x in X_ω is a neighbourhood of x in Y. For each $n < \omega$, we put $y_n = (\alpha_\omega, n)$ and $x_n = (\alpha, n)$. For each $n < \omega$ we can choose a compact open neighbourhood U_n of y_n in Y in such a way that $\{U_n : n < \omega\}$ is a discrete family. Let $\{a_n : n < \omega\}$ be a decomposition of ω. Then we define for each $n < \omega$, the canonical neighbourhood of x_n in X_ω by $W_{x_n}^{(\omega)} = \{x_n\} \cup \bigcup\{U_k : k \in a_n\}$. Now, for every $n < \omega$, we define a basic neighbourhood of x_n in X_ω as a set of the form $W_{x_n}^{(\omega)} \setminus C$ where $C \subseteq W_{x_n}^{(\omega)} \setminus \{x_n\}$ is a compact open subset of Y.

Now we assume $\xi = \mu + 1$ with $\omega \leq \mu < \omega_1$. In order to construct X_ξ we define for each $\zeta \leq \mu$ a σ-compact LCS-space Y_ζ such that $ht(Y_\zeta) = \alpha_\mu + 2, I_\beta(Y_\zeta) = \{\beta\} \times \xi$ if $\beta \in \{\alpha_\rho : \rho \leq \zeta\}, I_\beta(Y_\zeta) = I_\beta(X_\mu)$ otherwise. First we fix an enumeration $\{x_n : n < \omega\}$ of $\{\alpha\} \times \mu$. In order to define Y_0, we assume that α_0 is a successor ordinal, say $\alpha_0 = \beta_0 + 1$. If α_0 is a limit ordinal, we would use a similar argument by using the fact that $cf(\alpha_0) = \omega$. For every $x \in X_\mu$, we define a basic neighbourhood of x in Y_0 as a neighbourhood of x in X_μ. Now we consider a discrete family $\{V_n : n < \omega\}$ of compact open neighbourhoods of the points x_n in X_μ. For each $n < \omega$ we consider a $z_n \in V_n \cap I_{\beta_0}(X_\mu)$ and a compact open neighbourhood U_n of z_n with $U_n \subseteq V_n$. We put $y = (\alpha_0, \mu)$. Then we define a basic neighbourhood of y as a set of the form $\{y\} \cup \bigcup\{U_k : k > m\}$ where $m < \omega$. Proceeding in a similar way, we can construct $Y_{\zeta+1}$ from Y_ζ, and Y_ζ from the union of the Y_η for $\eta < \zeta$ if ζ is limit. Now we put $Y = Y_\mu$. Again since Y is a paracompact space, we can choose a discrete collection $\{V_n : n < \omega\}$ of compact open neighbourhoods of the points x_n in Y. For each $n < \omega$, we consider V_n with the relative topology of Y. Then, for every $n < \omega$ we define a σ-compact LCS-space Z_n such that $ht(Z_n) = \alpha_\xi + 1, I_\beta(Z_n) = I_\beta(V_n)$ for each $\beta \leq \alpha_\mu$ and in such a way that the Z_n are pairwise disjoint. Let $\delta = o.t.(\alpha_\xi \setminus \alpha_\mu)$. Let $\tau = \langle \lambda_\rho : \rho < \delta \rangle$ where $\lambda_0 = \omega$ and $\lambda_\rho = \kappa_{\alpha_\mu + \rho}$ if $0 < \rho < \delta$. Let $\{a_n : n < \omega\}$ be a decomposition of $\{\alpha_\xi\} \times \xi$. Let us fix a natural number n. We put $a_n = \{y_m : m < \omega\}$. For each $m < \omega$, we consider a compact space $Z_{y_m} \in K_\tau$ such that $I_\delta(Z_{y_m}) = \{y_m\}$. We suppose that the Z_{y_m} are pairwise disjoint. Then we define Z'_n as the topological sum of the family $\{Z_{y_m} : m < \omega\}$, and we put $Z_n = (V_n \setminus \{x_n\}) \otimes Z'_n$. Now we define Z as the topological sum of the family $\{Z_n : n < \omega\}$. We then define X_ξ as follows. The underlying set of X_ξ is $Y \cup Z \cup \{(\alpha, \mu)\}$. If $x \in Y \setminus \{\alpha\} \times \xi$, a basic neighbourhood of x in X_ξ is a basic neighbourhood of x in Y. Analogously, if $x \in Z_n$ for some $n < \omega$, a basic neighbourhood of x in X_ξ is a basic neighbourhood of x in Z_n. For every $n < \omega$, we define the canonical neighbourhood of x_n in X_ξ as the set $W^{(\xi)}_{x_n} = W^{(\mu)}_{x_n} \cup Z_n$. Then we define a basic neighbourhood of x_n in X_ξ as a set of the form $W^{(\xi)}_{x_n} \setminus C$, where C is a compact open subset of $W^{(\xi)}_{x_n} \setminus \{x_n\}$. We put $y = (\alpha, \mu)$. For each $n < \omega$, we consider a point $z_n \in I_{\alpha_\xi}(Z_n)$ and a compact open neighbourhood U_n of z_n in the space Z_n. Then we define the canonical neighbourhood of y in X_ξ as the set $W^{(\xi)}_y = \{y\} \cup \bigcup\{U_m : m < \omega\}$. So, we define a basic neighbourhood of y in X_ξ as a set of the form $W^{(\xi)}_y \setminus C$, where $C \subseteq W^{(\xi)}_y$ is a compact open subset of Z.

Now suppose that ξ is a limit ordinal. Without loss of generality we may assume that α_ξ is the limit of $\{\alpha_\mu : \mu < \xi\}$. First we define the σ-compact LCS-space Y of underlying set $\bigcup\{X_\mu : \omega \leq \mu < \xi\}$ as follows. If $x \in X_\mu \setminus (\{\alpha\} \times \mu)$ for some $\mu < \xi$, a basic neighbourhood of x is a basic neighbourhood of x in X_μ. If $x \in \{\alpha\} \times \xi$, we define the canonical neighbourhood of x in Y by $W^*_x = \bigcup\{W^{(\mu)}_x : \omega \leq \mu < \xi\}$, and then we define a basic neighbourhood of x in Y as a set of the form $W^*_x \setminus C$ where $C \subseteq W^*_x \setminus \{x\}$ is a compact open subset of X_μ for some $\mu < \xi$. Now we define the space X_ξ as follows. The underlying set of X_ξ is $Y \cup (\{\alpha_\xi\} \times \xi)$. As above, if $x \in X_\mu \setminus (\{\alpha\} \times \mu)$ for some $\mu < \xi$, a basic neighbourhood of x in

X_ξ is a basic neighbourhood of x in X_μ. Let $\{x_n : n < \omega\}$ be an enumeration of $\{\alpha\} \times \xi$. We choose a discrete collection $\{V_n : n < \omega\}$ of compact open neighbourhoods of the points x_n in Y. Let us consider a decomposition $\{a_n : n < \omega\}$ of $\{\alpha_\xi\} \times \xi$. Let $\langle \beta_m : m < \omega \rangle$ be a sequence of ordinals converging to α_ξ in a strictly increasing way. We fix a natural number n. We consider V_n with the relative topology of Y. For each $m < \omega$, we consider a $z_m \in I_{\beta_m}(V_n)$ and a compact open neighbourhood U_m of z_m in V_n such that $\{U_m : m < \omega\}$ is a discrete family in $V_n \setminus \{x_n\}$. We set $a_n = \{y_k : k < \omega\}$. We fix a decomposition $\{b_k : k < \omega\}$ of ω. Then we define a basic neighbourhood of a point y_k in X_ξ as a set of the form $\{y_k\} \cup \bigcup\{U_m : m \in b_k, m > l\}$ where $l < \omega$. Now we define the canonical neighbourhood of a point x_n in X_ξ by $W^{(\xi)}_{x_n} = W^*_{x_n} \cup a_n$. Then, a basic neighbourhood of x_n in X_ξ is a set of the form $W^{(\xi)}_{x_n} \setminus C$ where C is a compact open subset of $W^{(\xi)}_{x_n} \setminus \{x_n\}$.

Finally we define the space X as follows. The underlying set of X is $\bigcup\{X_\xi : \omega \leq \xi < \omega_1\}$. If $x \in X_\xi \setminus \{\alpha\} \times \omega_1$ for some $\xi < \omega_1$, a basic neighbourhood of x in X is a basic neighbourhood of x in X_ξ. If $x \in \{\alpha\} \times \omega_1$, we put $W_x = \bigcup\{W^{(\xi)}_x : \omega \leq \xi < \omega_1\}$. Then we define a basic neighbourhood of x in X as a set of the form $W_x \setminus C$ where $C \subseteq W_x \setminus \{x\}$ is a compact open subset of X_ξ for some $\xi < \omega_1$. It can be verified that $X \in K_\theta$. $\dashv$

Theorem 3 is in a sense best possible, since under CH we have that if $\theta = \langle \kappa_\xi : \xi < \eta \rangle$ is such that $\kappa_\alpha = \omega$ and $\kappa_\beta = \omega_2$ for some $\alpha < \beta < \eta$, then there is no LCS-space X such that $CS(X) = \theta$. To check this point, assume on the contrary that there is an LCS-space X with $CS(X) = \theta$. For every $x \in X^\alpha$ consider a clopen neighbourhood U_x of x. Now, we put $a_x = U_x \cap I_\alpha(X)$. Since we are assuming that if γ is the ordinal such that $x \in I_\gamma(X)$ then $U_x \cap X^\gamma = \{x\}$, we have that $x \neq y$ implies $a_x \neq a_y$. Hence, we can identify every point of X^α with a subset of $I_\alpha(X)$. Also, it was proved by Baumgartner in [1] that if it is consistent that there exists an inaccessible cardinal, then it is consistent with $2^\omega = \omega_2$ that there is no LCS-space with cardinal sequence $\theta = \langle \kappa_\xi : \xi \leq \omega_1 \rangle$ where $\kappa_\xi = \omega_1$ for each $\xi < \omega_1$ and $\kappa_{\omega_1} = \omega_2$. On the other hand, Juhász has pointed out that in a collaboration with Weiss, they have proved that if $\theta = \langle \kappa_\xi : \xi < \omega_1 \rangle$ is a sequence of cardinals such that $\kappa_\xi \leq 2^\omega$ for each $\xi < \omega_1$, then there is an LCS-space X such that $CS(X) = \theta$.

Next, combining the arguments given in the proofs of Theorem 1 and Theorem 3 we can show the following result, whose proof is left to the reader.

As above, we write $C_n = \omega_1 \times \{n\}$ for $n < \omega$.

Lemma 2 *Suppose that $\theta = \langle \kappa_\xi : \xi \leq \omega_1 \rangle$ is a sequence of cardinals such that $\kappa_\xi \in \{\omega, \omega_1\}$ for every $\xi < \omega_1$ and $\kappa_{\omega_1} = \omega_1$. Then, there is an LCS-space X with $I_\xi(X) = \{\xi\} \times \kappa_\xi$ for $\xi \leq \omega_1$ and $I_{\omega_1+1}(X) = \emptyset$ such that the following two conditions are satisfied:*

(1) For every $x \in X \setminus I_{\omega_1}(X)$ and every $n < \omega$ there is a neighbourhood U of x such that $(U \setminus \{x\}) \cap C_n = \emptyset$.

(2) For every $x \in X$ there is a neighbourhood U of x such that $U \setminus \{x\} \subseteq \bigcup\{C_n : n < \omega\}$.

Now, we can prove the main result.

Theorem 4 *Let α be an ordinal such that $\omega_1 \leq \alpha < \omega_2$. Let $\theta = \langle \kappa_\xi : \xi < \alpha \rangle$ be a sequence of cardinals such that either $\kappa_\xi = \omega$ or $\kappa_\xi = \omega_1$ for every $\xi < \alpha$. Then, there are 2^{ω_1} pairwise non-homeomorphic LCS-spaces with cardinal sequence θ.*

Proof. Let $\tau = \langle \kappa_\xi : \xi < \omega_1 \rangle$. Consider $\langle X_\xi : \xi < 2^{\omega_1} \rangle$ a sequence of pairwise non-homeomorphic admissible τ-spaces constructed as in Theorem 2. Let X'_ξ be the one-point compactification of X_ξ. Then, let Y_ξ be the topological sum of ω disjoint copies of X'_ξ. Let $\beta = o.t.(\alpha \setminus \omega_1)$. Now let $\tau' = \langle \kappa'_\xi : \xi < \beta \rangle$ where $\kappa'_0 = \omega$, $\kappa'_\xi = \kappa_{\omega_1+\xi}$ if $0 < \xi < \beta$. By Theorem 3, there is an LCS-space Y such that $CS(Y) = \tau'$. For $\xi < 2^{\omega_1}$, we may assume that the underlying sets of Y and Y_ξ are disjoint. Then, we define $Z_\xi = Y_\xi \otimes Y$ for every $\xi < 2^{\omega_1}$. Note that if $\kappa_{\omega_1} = \omega$, we infer from the proof of Lemma 1 that the spaces Z_ξ are pairwise non-homeomorphic LCS-spaces with cardinal sequence θ. So, assume that $\kappa_{\omega_1} = \omega_1$. Let $\tau^* = \langle \kappa_\xi : \xi \leq \omega_1 \rangle$. Let Z be an LCS-space of cardinal sequence τ^* which verifies the conditions of Lemma 2. We may assume that for every $\xi < 2^{\omega_1}$, the underlying sets of Z and Z_ξ are disjoint. Then, we define Z'_ξ as the topological sum of Z and Z_ξ. By using the argument given in Lemma 1, it is now easy to check that the spaces Z'_ξ are pairwise non-homeomorphic LCS-spaces with cardinal sequence θ. $\dashv$

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