A mechanism for package allocation problems with gross substitutes

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Abstract

We consider a package allocation problem in which a seller owns many indivisible objects and the rest of the agents, buyers, are interested in packages of these objects. Buyers' valuations satisfy monotonicity and the gross substitutes condition (Kelso and Crawford, 1982). The aim of this paper is to analyse the following mechanism: simultaneously, each buyer requests to the seller a package by announcing how much he would pay for it; once buyers have played, the seller decides the final assignment of packages and the prices, as long as this assignment makes no buyer worse off than with his initial request. The subgame perfect equilibrium outcomes of the mechanism correspond to the Vickrey outcome (Vickrey, 1961) of the market.

Keywords: Mechanism, gross substitutes, Vickrey outcome, subgame perfect equilibria

1. Introduction

Package allocation problems are a subclass of resource allocation problems and commonly deal with situations where a set of buyers wish to acquire several indivisible objects from one seller. See for instance Bikhchandani and Ostroy (2002), Milgrom (2007) and Day and Milgrom (2008). In this paper, we approach the package allocation problem assuming that all parties take an active role in the allocation problem. This could be the case in the dissolution of a private company, where the main shareholder sells part of her/his stock to other shareholders.

We consider a situation where the seller owns many indivisible objects on sale and each buyer wants to buy a package of objects and has a non-negative valuation for each package. As usual in package allocation problems, preferences are assumed to be quasi-linear with respect to money and buyers' valuations satisfy monotonicity and the gross substitutes condition.[2] An outcome for this allocation problem specifies an

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[1]The authors thank René van den Brink, Javier Martínez de Albéniz, Jordi Massó and Roberto Serrano for their helpful comments. The support by research grant ECO2017-86481-P (AEI/FEDER,UE) and 2017SGR778 (Government of Catalonia) are acknowledged. The authors are responsible for any remaining error.

[2]Condition introduced by Kelso and Crawford (1982).

Preprint submitted to Journal of Mathematical Economics *November 6, 2019*

assignment of the objects to the buyers and the payment each buyer makes for his assigned package of objects.

We study the strategic interaction of all agents, by means of a simple mechanism we introduce. The mechanism works as follows: first, each buyer requests (for instance, bidding in a sealed envelope) a package he would like to buy and how much he would pay for it; second, the seller decides the final allocation of packages and their prices.

A requirement for allocating objects is efficiency. When buyers request packages of objects simultaneously, and the seller is restricted to choose only among requested packages, an overlapping problem may arise. As a consequence, the outcome of this equilibrium may not be efficient and do not belong to the core. In order to avoid this problem, in our mechanism, the seller is allowed to allocate non-requested packages as long as this does not make any buyer worse off than with his initial request. Our main result is that in any subgame perfect equilibrium (SPE), the final allocation of the objects is efficient and every SPE outcome coincides with the Vickrey outcome of the market.

Our work is related to Bernheim and Whinston (1986), where a set of completely informed buyers want to buy packages of heterogeneous objects. In the mechanism they propose, each buyer reports how much he would pay for each package and the auctioneer chooses an allocation of the packages. If a buyer receives a package, then he pays his bid. This game has multiple equilibria, some of them non-efficient. To overcome that, the authors restrict the strategies of the buyers, the so-called truthful strategies, to obtain Nash equilibria with good properties. In our mechanism all SPE lead to an efficient and core outcome, which is the Vickrey outcome, at the cost of assuming the seller plays a more active role.

The mechanism we propose is also inspired in the two-phase buying and selling procedure for assignment games introduced in Pérez-Castrillo and Sotomayor (2002), in the setting of the Shapley and Shubik (1972) assignment game, to implement the most favorable core allocation for the sellers. Both mechanisms have in common that there are two sides of the market, one side acts first simultaneously and the second side acts later sequentially. In their paper sellers act first setting prices for their objects (one object each), and buyers act later sequentially determining the matching (that assigns at most one object to each buyer). In our case the main differences are that buyers are willing to buy packages of objects and act in first place (each buyer demands a package at a price) and there is only one seller that owns all objects and acts secondly to determine who gets what. As in Perez-Castrillo and Sotomayor (2002), the sector that moves first (in our case the buyers) has an advantage since the other sector has only the freedom to determine the matching. We also obtain that the sector that acts first obtains the maximum possible core payoff.

To sum up, the paper is organized as follows. The next section is devoted to an introduction of the market and the coalitional game associated with it. In Section 3, the mechanism is defined: Theorem 4 proves that any SPE produces an efficient allocation and finally Theorem 6 shows that the payoff of any SPE is the Vickrey payoff vector. The Appendix contains some technical lemmas needed to establish the main results.

2. The market and some preliminaries

Consider a market with m buyers and one seller. The finite set of buyers is denoted by $M = \{1, 2, ..., m\}$ and the seller is denoted by 0. She owns a finite set of indivisible objects on sale, denoted by Q. The set of objects Q includes copies of a dummy object j_0, as many as the number of buyers. Each buyer $i \in M$ has a *valuation* for each package of objects,[3] $w_i : 2^Q \to \mathbb{R}$ such that $w_i(\emptyset) = 0$ and we assume that for each buyer i and for each dummy object j_0, $w_i(R \cup \{j_0\}) = w_i(R)$ for all $R \subseteq Q \setminus \{j_0\}$. Moreover, each agent has a preference relation on the set of bundles formed by a package and an amount of money, $2^Q \times \mathbb{R}$, that is a quasi-linear preference with respect to money.

We will assume that the buyers' valuations w_i satisfy *monotonicity* and the *gross substitutes* condition. Monotonicity says that for any buyer, the more objects in a package, the better. In particular, we have that for each buyer $i \in M$, $w_i(S) \geq 0$ for all $S \subseteq Q$. The gross substitutes condition was introduced by Kelso and Crawford (1982) and has been widely used in Gul and Stacchetti (1999). In fact, we will only use it to guarantee the submodularity property of our coalitional game.

To sum up, our market is described by $(M, \{0\}, Q, w)$ where w stands for buyers' valuations, $w = (w_i)_{i \in M}$, which satisfy monotonicity and the gross substitute condition, and all agents have complete information.

Given a subset of buyers $\emptyset \neq S \subseteq M$, an *allocation* of Q to S consists of a partition of the set of all objects among agents in S, that is, $(A_i)_{i \in S}$ such that $\emptyset \neq A_i \subseteq Q$ is the set of objects allocated to $i \in S$, $\bigcup_{i \in S} A_i = Q$ and $A_i \cap A_{i'} = \emptyset$ if $i \neq i'$. We denote by $\mathcal{A}(S)$ the set of all allocations of Q to S. We say that an allocation $A \in \mathcal{A}(S)$ is efficient for S if

$$\sum_{i \in S} w_i(A_i) \geq \sum_{i \in S} w_i(A'_i) \quad \text{for all } A' \in \mathcal{A}(S). \tag{1}$$

We denote by $\mathcal{A}^*(S)$ the set of efficient allocations for S. Notice that $\mathcal{A}^*(S)$ is never empty for any non-empty coalition of buyers $S \subseteq M$.

Given a market $(M, \{0\}, Q, w)$, let us consider the coalitional game[4] associated with $(M, \{0\}, Q, w)$ as in Ausubel and Milgrom (2002). This game is denoted by $(M \cup \{0\}, v)$ where the set of players is the set of

[3] For each set S, we will denote by $|S|$ the cardinality of S and by 2^S the power set of S. Also, given two sets S and T, we denote $S \setminus T = \{k \in S \mid k \notin T\}$.

[4] A *game in coalitional form with transferable utility* is a pair (N, v) formed by a finite set of players N and a characteristic function v that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. The core of a game (N, v) is $C(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$. We say that a game (N, v) satisfies monotonicity if $v(T) \leq v(S)$ for all $T \subseteq S \subseteq N$.

agents of the market $M \cup \{0\}$ and the worth of each coalition is given as follows. The worth of the empty coalition and the worth of any coalition formed by only one type of agents is zero because in these cases there is no trade. When a coalition is formed by a group of buyers $\emptyset \neq S \subseteq M$ and the seller, the worth is given by

$$v(S \cup \{0\}) = \max_{A \in \mathcal{A}(S)} \left\{ \sum_{i \in S} w_i(A_i) \right\}. \tag{2}$$

Notice that by its definition the game $(M \cup \{0\}, v)$ is monotonic.

A payoff vector $u \in \mathbb{R}^{M \cup \{0\}}$ consists of a payoff for each agent of the market. That is, u_i is the payoff associated to buyer $i \in M$ and u_0 is the seller's payoff. Following Ausubel and Milgrom (2002), a payoff vector $u^* \in \mathbb{R}^{M \cup \{0\}}$ is the *Vickrey payoff vector*[5] of the market $(M, \{0\}, Q, w)$ if for each buyer $i \in M$, we have that

$$u_i^* = v(M \cup \{0\}) - v((M \setminus \{i\}) \cup \{0\}), \tag{3}$$

and for the seller,

$$u_0^* = v(M \cup \{0\}) - \sum_{i \in M} u_i^*.$$

A drawback of the Vickrey payoff vector is that it may lie outside the core and then it could generate a low payoff for the seller (Milgrom, 2004). Ausubel and Milgrom (2002) shows that if monotonicity and the gross substitutes condition are satisfied by each buyer's valuation function, then the coalitional game is *buyers-submodular*.[6] Buyers-submodularity means that the marginal contribution of any buyer to any coalition containing the seller decreases as the coalition grows larger. More precisely, the game $(M \cup \{0\}, v)$ is buyers-submodular if for all $i \in M$ and all $T \subseteq S \subseteq M \setminus \{i\}$, it holds that

$$v(T \cup \{0\} \cup \{i\}) - v(T \cup \{0\}) \geq v(S \cup \{0\} \cup \{i\}) - v(S \cup \{0\}). \tag{4}$$

An equivalent expression to (4) is the following one:

$$v(S \cup \{0\}) - v(T \cup \{0\}) \geq \sum_{i \in S \setminus T} \Big(v(S \cup \{0\}) - v((S \setminus \{i\}) \cup \{0\}) \Big), \tag{5}$$

for all $T \subseteq S \subseteq M$.

It is well known that when the game $(M \cup \{0\}, v)$ is buyers-submodular, then the Vickrey payoff vector belongs to the core (Bikhchandani and Ostroy, 2002; Ausubel and Milgrom, 2002).

The aim of the next section is to provide a mechanism for our market such that the payoff vector in any *subgame perfect equilibrium* is the Vickrey payoff vector of the market.

[5] Notice that the Vickrey payoff vector is unique. The Vickrey payoff vector is the payoff vector associated to the *Vickrey auction* or VCG mechanisms (Vickrey, 1961; Clarke, 1971; Groves, 1973). See *e.g.* Ausubel and Milgrom (2002) and Milgrom (2004) for details.

[6] The reader can also find the proof of this implication in Section 5.5 of Vohra (2011)

3. A mechanism to implement the Vickrey outcome

In this section, we introduce a two-phase mechanism Γ in a complete information setting to implement the Vickrey payoff vector of our market with m buyers and one seller. Let us first describe the mechanism in an informal way. First, each buyer announces a package of objects he wants to acquire and the price he would pay for it. All these requests are made simultaneously. In the second phase, the final allocation and the prices are determined: with the information of buyers' requests, the seller chooses a coalition of buyers and assigns to each of these buyers a package at a price. The seller is allowed to allocate the requested package to a buyer at his proposed price or a different package at a price that makes this buyer not worse off than with his initial request.

In more detail, let $(M, \{0\}, Q, w)$ be a market such that all valuations satisfy monotonicity and the gross substitutes condition and all agents have complete information.

The two phases of the **mechanism** Γ are:

1. Buyers play simultaneously. Each buyer $i \in M$ announces a tentative package $\emptyset \neq B_i \subseteq Q$ and how much he would pay for it, $(B_i, x_i) \in 2^Q \times \mathbb{R}_+$.

   We denote by (B, x) the requests of all buyers to the seller, where $B = (B_i)_{i\in M}$ and $x = (x_i)_{i\in M}$.
2. Once the seller receives the requests (B, x) of all buyers, the seller chooses a triple (S, A, p) where: a) $S \subseteq M$ is a non-empty coalition of buyers; b) $A \in \mathcal{A}(S)$ is an allocation of Q to S; and c) $p = (p_i)_{i\in S} \in \mathbb{R}^S_+$ denotes the payment each buyer $i \in S$ makes for package A_i, under the constraint[7]

$$w_i(A_i) - p_i \geq w_i(B_i) - x_i \quad \text{for each} \quad i \in S. \tag{6}$$

Once the seller has played, the mechanism Γ ends. The payoff of each agent is the following. If a buyer $i \in M$ belongs to S, he receives the package A_i, he pays p_i and his payoff is $w_i(A_i) - p_i$. If a buyer $i \in M$ does not receive a package, that is $i \in M \setminus S$, he pays nothing and his payoff is zero. The seller's payoff is $\sum_{i\in S} p_i$.

Once the mechanism Γ ends, its outcome is $(A, p) \in \mathcal{A}(S) \times \mathbb{R}^S_+$, that is, the coalition $S \subseteq M$ of buyers, the allocation chosen by the seller and the payment p_i each buyer $i \in S$ has to make for the package allocated to him. We say that an outcome $(A, p) \in \mathcal{A}(S) \times \mathbb{R}^S_+$ of the mechanism Γ is a Vickrey outcome[8] if the payoff vector associated to (A, p) is the Vickrey payoff vector of the market.

Given a buyers' strategy profile (B, x), we say that (S, A, p) is a best reply of the seller to (B, x) if it maximizes the seller's payoff over all admissible triples (see 6). Since the set of the seller's feasible replies is a non-empty compact set, there always exists a best reply.

[7]Notice that, given (B, x), the seller can at least choose (S, A, p) where $S = \{i\}$ for some $i \in M$, the allocation is $A = (A_i)$ with $A_i = Q$ and $p_i = x_i$.

[8]It is known that different allocations may produce the Vickrey payoff vector, see *e.g.* Gul and Stacchetti (1999).

The following lemma remarks that when the seller chooses the outcome that maximizes her payoff given any buyers' strategy profile, she will price packages as high as possible given constraint (6). As a consequence, in any subgame perfect equilibrium (SPE), inequality in (6) is satisfied as an equality.

Lemma 1. *Consider any market $(M,\{0\},Q,w)$ and let (B,x) be an arbitrary buyers' strategy profile in the mechanism Γ. Then, in any best reply to (B,x), the seller chooses (S,A,p) such that $\emptyset \neq S \subseteq M$, $A \in \mathcal{A}(S)$ and $p \in \mathbb{R}^S_+$ satisfying*

$$w_i(A_i) - p_i = w_i(B_i) - x_i \quad \text{for all } i \in S. \tag{7}$$

Proof. Given (B,x), let (S,A,p) be a best reply of the seller *i.e.*, $\emptyset \neq S \subseteq M$, $A \in \mathcal{A}(S)$ and $w_i(A_i)-p_i \geq w_i(B_i) - x_i$ for each $i \in S$. By way of contradiction, suppose that $w_{i^*}(A_{i^*}) - p_{i^*} > w_{i^*}(B_{i^*}) - x_{i^*}$ for some $i^* \in S$. Consider the triple (S,A,p') where $\emptyset \neq S \subseteq M$, $A \in \mathcal{A}(S)$ and $p' \in \mathbb{R}^S_+$, that satisfies $p'_i = p_i$ for all $i \in S \setminus \{i^*\}$ and $p'_{i^*} = w_{i^*}(A_{i^*}) - (w_{i^*}(B_{i^*}) - x_{i^*}) \geq 0$. Notice that p'_{i^*} satisfies constraint (6), $p'_{i^*} > p_{i^*}$ and $\sum_{i\in S} p'_i > \sum_{i \in S} p_i$, which contradicts that the seller was maximizing her payoff at (S,A,p). □

The next lemma says that, given any market, if the objects are efficiently allocated (1) to a coalition S of buyers, then each buyer $i \in S$ values the package he receives above his marginal contribution to $S \cup \{0\}$ in the game $(M \cup \{0\}, v)$, see expression (2).

Lemma 2. *Consider any market $(M,\{0\},Q,w)$ and the related game $(M\cup\{0\},v)$, see expression (2). For any set of buyers $\emptyset \neq S \subseteq M$ and any $A = (A_i)_{i\in S} \in \mathcal{A}^*(S)$, we have that*

$$w_i(A_i) \geq v(S \cup \{0\}) - v((S\setminus\{i\}) \cup \{0\}) \quad \text{for all } i \in S. \tag{8}$$

Proof. Take any set of buyers $\emptyset \neq S \subseteq M$, any $A = (A_i)_{i \in S} \in \mathcal{A}^*(S)$ and any $i_1 \in S$. If $S = \{i_1\}$, then $A_{i_1} = Q$ and the result follows immediately. Otherwise, if $|S| > 1$, choose $i_2 \in S \setminus \{i_1\}$ and define the following allocation $A' \in \mathcal{A}(S \setminus \{i_1\})$ where $A'_{i_2} = A_{i_2} \cup A_{i_1}$ and $A'_i = A_i$ for each $i \in S \setminus \{i_1, i_2\}$. Notice that, $w_{i_2}(A_{i_2} \cup A_{i_1}) \geq w_{i_2}(A_{i_2})$ because of the monotonicity assumption on buyers' valuations. Then, we have

$$\begin{aligned} w_{i_1}(A_{i_1}) &= \sum_{i\in S} w_i(A_i) - \sum_{i \in S\setminus\{i_1\}} w_i(A_i) \geq \sum_{i \in S} w_i(A_i) - \sum_{i\in S\setminus\{i_1\}} w_i(A'_i) \\ &\geq v(S\cup\{0\}) - v((S\setminus\{i_1\})\cup\{0\}). \end{aligned}$$

□

Now, we start the analysis of the mechanism Γ. Notice first that this mechanism, as it is also the case in Bernheim and Whinston (1986), may have multiple Nash equilibria, some of them non-efficient.

Example 1. Consider a market with two buyers 1 and 2, and a seller that owns two objects A and B. The buyers' valuations are:

	A	B	AB
1	4	5	7
2	3	5	8

Assume the buyers' strategy (the same for both buyers) is to demand $\{A, B\}$ at the price of 6: $(B_1, x_1) = (B_2, x_2) = (\{A, B\}, 6)$. Assume also that the seller's strategy is to select $S = \{1\}$, $A_1 = \{A, B\}$ and $p_1 = x_1 = 6$, as long as $(B_1, x_1) = (\{A, B\}, 6)$, and otherwise the seller chooses $S = \{2\}$, with $A_2 = \{A, B\}$ and $p_2 = w_2(\{A, B\}) - w_2(B_2) + x_2$.

Notice that buyer 1 gets 0 payoff in any deviation from $(B_1, x_1) = (\{A, B\}, 6)$. Also, buyer 2 cannot obtain any improvement as long as buyer 1 and the seller follow the above strategies. Finally, let us analyze if the seller has any incentives for deviation when the two buyers play $(B_1, x_1) = (B_2, x_2) = (\{A, B\}, 6)$. If the seller selects $S = \{1\}$ and allocates $\{A, B\}$ to buyer 1, then the price must be $p_1 \leq 6$. Similarly, if the seller selects $S = \{2\}$ and allocates $\{A, B\}$ to buyer 2, then the price must be $p_2 \leq 6$. Finally, if the seller selects $S = \{1, 2\}$ there are two possible allocations. Either $\{A\}$ is allocated to 1 at $p_1 \leq 3$ and $\{B\}$ is allocated to 2 at price $p_2 \leq 3$, or $\{B\}$ is allocated to 1 at price $p_1 \leq 4$ and $\{A\}$ is allocated to 2 at price $p_2 \leq 1$. In any case, the seller's payoff is at most 6.

Clearly this Nash equilibrium is inefficient and it is not subgame perfect.

As a consequence, we will focus on the SPE of this mechanism in pure strategies. Notice first that, if the market contains only one buyer, the payoff vector in any SPE is precisely the Vickrey payoff vector of the market. Indeed, if $M = \{i\}$, the Vickrey payoff of buyer i is $u_i^* = v(\{i\} \cup \{0\}) - v(\{0\}) = w_i(Q)$ while the seller's payoff is $u_0^* = 0$. By the rules of the mechanism Γ, the seller has to allocate Q to buyer i at some non-negative price p_i such that $w_i(Q) - p_i \geq w_i(B_i) - x_i$, where $(B_i, x_i) \in 2^Q \times \mathbb{R}_+$ is the strategy played by buyer i. Taking this into account, buyer i will play, in any SPE, (B_i, x_i) such that $w_i(B_i) = w_i(Q)$ and $x_i = 0$.

We will assume from now on that the number of buyers in the market is at least two, *i.e.*, $|M| \geq 2$. First we will show that any SPE of Γ is efficient, that is to say, its final outcome (S, A, p) attains the worth of the grand coalition, $\sum_{i \in S} w_i(A_i) = v(M \cup \{0\})$. This fact will be used later on in the proof of the main theorem.

The following technical lemma, which is proved in the Appendix, will be needed.

Lemma 3. *Consider any market $(M, \{0\}, Q, w)$ and let (B, x) be the buyers' strategy profile in any SPE of the mechanism Γ. For any non-empty coalition of buyers $S \subseteq M$ and any $J \subseteq M \setminus S$ we have either:*

1. *there exists an efficient allocation $A = (A_i)_{i \in S \cup J} \in \mathcal{A}^*(S \cup J)$ such that $w_i(A_i) \geq w_i(B_i) - x_i$ for all $i \in S$, or*
2. *there exist a subcoalition of buyers $T \subsetneq S$, with $T \neq \emptyset$ whenever $J = \emptyset$, and an efficient allocation $A = (A_i)_{i \in T \cup J} \in \mathcal{A}^*(T \cup J)$ such that*
   (a) *$w_i(A_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and*
   (b) *$\sum_{i \in S \setminus T} \left(w_i(B_i) - x_i \right) > v(S \cup J \cup \{0\}) - v(T \cup J \cup \{0\})$.*

Theorem 4. *For any market $(M, \{0\}, Q, w)$, any subgame perfect equilibrium of mechanism Γ is efficient.*

Proof. We first need to prove the following claim.
Claim:

If (B, x) is the buyers' strategy profile in an arbitrary subgame perfect equilibrium of Γ, then in any best reply to (B, x) the seller chooses (S, A, p), $\emptyset \neq S \subseteq M$, $A \in \mathcal{A}(S)$ and $p \in \mathbb{R}^Q_+$, such that

$$\sum_{i \in S} w_i(A_i) = v(S \cup \{0\}) \tag{9}$$

To prove the claim, consider any SPE of Γ with buyers' strategies (B, x). Let (S, A, p), where $\emptyset \neq S \subseteq M$, $A = (A_i)_{i \in S} \in \mathcal{A}(S)$ and $p \in \mathbb{R}^S_+$ satisfies (6), be a best reply of the seller to the buyers' strategy profile (B, x). Notice that by the definition of the game $(M \cup \{0\}, v)$, see (2), we have $\sum_{i \in S} w_i(A_i) \leq v(S \cup \{0\})$. Assume on the contrary that

$$\sum_{i \in S} w_i(A_i) < v(S \cup \{0\}). \tag{10}$$

Case 1. There is an allocation $A' = (A'_i)_{i \in S} \in \mathcal{A}^*(S)$ such that $w_i(A'_i) \geq w_i(B_i) - x_i$, for all $i \in S$.

We then define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S$. We have

$$\begin{aligned}\sum_{i \in S} p'_i &= \sum_{i \in S} \Big(w_i(A'_i) - (w_i(B_i) - x_i) \Big) = v(S \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) \\ &> \sum_{i \in S} \Big(w_i(A_i) - (w_i(B_i) - x_i) \Big) = \sum_{i \in S} p_i,\end{aligned}$$

where the last equality follows from Lemma 1. This contradicts the fact that (S, A, p) maximizes the seller's payoff given (B, x).

Case 2. For every allocation $A' = (A'_i)_{i \in S} \in \mathcal{A}^*(S)$, there is some buyer $i \in S$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma 3, taking $J = \emptyset$, there exist $\emptyset \neq T \subseteq S$ and an allocation $\bar{A} = (\bar{A}_i)_{i \in T} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$. Moreover, by the assumption of Case 2, we have $T \neq S$. Then, Lemma 3 also guarantees

$$\sum_{i \in S \setminus T} \Big(w_i(B_i) - x_i \Big) > v(S \cup \{0\}) - v(T \cup \{0\}). \tag{11}$$

Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for all $i \in T$. Notice that $(T, \overline{A}, \overline{p})$ satisfies the requirement in expression (6) given (B, x). Thus, since (S, A, p) maximizes the seller's payoff given (B, x), we obtain

$$\begin{aligned}\sum_{i \in S} \Big(w_i(A_i) - (w_i(B_i) - x_i) \Big) &= \sum_{i \in S} p_i \geq \sum_{i \in T} \bar{p}_i = \sum_{i \in T} \Big(w_i(\bar{A}_i) - (w_i(B_i) - x_i) \Big) \\ &= v(T \cup \{0\}) - \sum_{i \in T} \Big(w_i(B_i) - x_i \Big),\end{aligned} \tag{12}$$

where the first equality follows from Lemma 1. Since $T \subsetneq S$, then we have[9]

$$v(S \cup \{0\}) - v(T \cup \{0\}) > \sum_{i \in S} w_i(A_i) - v(T \cup \{0\}) \geq \sum_{i \in S \setminus T} \Big(w_i(B_i) - x_i \Big),$$

[9]For any two sets T and S, we say that $T \subsetneq S$ if $[T \subseteq S$ and $T \neq S]$.

where the strict inequality comes from (10) and the second inequality from (12). This fact contradicts (11). Hence $\sum_{i\in S} w_i(A_i) = v(S \cup \{0\})$ and the proof of the claim is completed.

Now we complete the proof of the theorem. Consider any SPE of Γ with buyers' strategies (B, x). Let (S, A, p), where $\emptyset \neq S \subseteq M$, $A = (A_i)_{i\in S} \in \mathcal{A}(S)$ and $p \in \mathbb{R}^S_+$ satisfies (6), be a best reply of the seller to the buyers' strategy profile (B, x). We will show that $v(S \cup \{0\}) = v(M \cup \{0\})$.

If $S = M$, we are done. Otherwise, by monotonicity of the game v, we have that $v(S\cup\{0\}) \leq v(M\cup\{0\})$. Assume on the contrary that $v(S \cup \{0\}) < v(M \cup \{0\})$.

Let $\mathcal{I} = \{I \subseteq M \setminus S \mid v(S \cup \{0\}) < v(S \cup I \cup \{0\})\}$. Notice that the set $\mathcal{I}$ is non-empty since $M \setminus S \in \mathcal{I}$. Let I_1 be a minimal coalition in $\mathcal{I}$ with respect to the inclusion relation, notice that $I_1 \neq \emptyset$. Fix any buyer $i_1 \in I_1$. We have

$$v(S \cup \{i_1\} \cup \{0\}) - v(S \cup \{0\}) \geq v(S \cup I_1 \cup \{0\}) - v(S \cup (I_1 \setminus \{i_1\}) \cup \{0\}) > 0, \tag{13}$$

where the first inequality comes from buyers-submodularity (4) and the strict inequality from the minimality of I_1 and the monotonicity of the game $(M \cup \{0\}, v)$. Now, we consider two cases.

Case 1: There exists $A' \in \mathcal{A}^*(S \cup \{i_1\})$ such that $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in S$.

Define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S$. Since by assumption, (S, A, p) is a best reply to (B, x), by Lemma 1 we have that

$$\sum_{i\in S} p_i = \sum_{i\in S} w_i(A_i) - \sum_{i\in S} \Big(w_i(B_i) - x_i\Big).$$

Now, since we have already proved $\sum_{i\in S} w_i(A_i) = v(S \cup \{0\})$, we have

$$\begin{aligned}\sum_{i\in S} p_i &= v(S \cup \{0\}) - \sum_{i\in S} \Big(w_i(B_i) - x_i\Big) < v(S \cup \{i_1\} \cup \{0\}) - \sum_{i\in S} \Big(w_i(B_i) - x_i\Big) \\ &= \sum_{i\in S\cup\{i_1\}} w_i(A'_i) - \sum_{i\in S} \Big(w_i(B_i) - x_i\Big) = \sum_{i\in S} p'_i + w_{i_1}(A'_{i_1}),\end{aligned} \tag{14}$$

where the strict inequality follows from expression (13).

Moreover, notice that $w_{i_1}(A'_{i_1}) \geq v(S \cup \{i_1\} \cup \{0\}) - v(S \cup \{0\}) > 0$ where the first inequality follows from Lemma 2 and the second one from (13).

Let $\varepsilon > 0$ be such that

$$0 < \varepsilon < w_{i_1}(A'_{i_1}) \text{ and } \sum_{i\in S} p_i < \sum_{i\in S} p'_i + w_{i_1}(A'_{i_1}) - \varepsilon. \tag{15}$$

We will prove that buyer $i_1 \in M \setminus S$ has incentives to unilaterally deviate from (B_{i_1}, x_{i_1}) to (A'_{i_1}, x'_{i_1}) with $x'_{i_1} = w_{i_1}(A'_{i_1}) - \varepsilon$. This will contradict that (B, x) forms part of a SPE of Γ and will complete the proof of $v(S \cup \{0\}) = v(M \cup \{0\})$ for Case 1.

To this end, let $(\tilde{S}, \tilde{A}, \tilde{p})$ be the best reply of the seller when buyer i_1 unilaterally deviates to (A'_{i_1}, x'_{i_1}). We show that $i_1 \in \tilde{S}$. By way of contradiction, assume that $i_1 \notin \tilde{S}$. By Lemma 1, we know that $\tilde{p}_i = w_i(\tilde{A}_i) - (w_i(B_i) - x_i)$ for all $i \in \tilde{S}$. Recall that (S, A, p) is a best reply of the seller to the original buyers' strategies (B, x). Then, since we are assuming that $i_1 \notin \tilde{S}$, we have

$$\sum_{i\in S} p_i \geq \sum_{i\in \tilde{S}} \tilde{p}_i. \tag{16}$$

Consider now $(S \cup \{i_1\}, A', p')$ where, under the assumption of Case 1, $A' = (A'_i)_{i\in S\cup\{i_1\}} \in \mathcal{A}^*(S \cup \{i_1\})$ satisfies $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in S$, and $p' = (p'_i)_{i\in S\cup\{i_1\}} \in \mathbb{R}^{S\cup\{i_1\}}$ is defined by $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S$ and $p'_{i_1} = x'_{i_1} = w_{i_1}(A'_{i_1}) - \varepsilon$. Making use of expressions (15) and (16), we have

$$\sum_{i\in S\cup\{i_1\}} p'_i = \sum_{i\in S} p'_i + x'_{i_1} = \sum_{i\in S} p'_i + \Big(w_{i_1}(A'_{i_1}) - \varepsilon\Big) > \sum_{i\in S} p_i \geq \sum_{i\in \tilde{S}} \tilde{p}_i,$$

which contradicts that the triple $(\tilde{S}, \tilde{A}, \tilde{p})$ is a best reply of the seller when only buyer i_1 deviates to (A'_{i_1}, x'_{i_1}). Hence, this implies that buyer $i_1 \in \tilde{S}$.

Recall that $(\tilde{S}, \tilde{A}, \tilde{p})$ is the reply of the seller when only buyer i_1 deviates from (B_{i_1}, x_{i_1}) to (A'_{i_1}, x'_{i_1}). We know that $i_1 \in \tilde{S}$. Hence the payoff of buyer i_1 is $w_{i_1}(\tilde{A}_{i_1}) - \tilde{p}_{i_1}$. By Lemma 1, we know that $w_{i_1}(\tilde{A}_{i_1}) - \tilde{p}_{i_1} = w_{i_1}(A'_{i_1}) - x'_{i_1} = \varepsilon > 0$. Since $i_1 \in M \setminus S$, the last strict inequality shows that buyer i_1 has incentives to unilaterally deviate as it was claimed, which contradicts that (B, x) forms part of a SPE. Therefore, $v(S \cup \{0\}) = v(M \cup \{0\})$ under the assumption of Case 1.

Case 2: For all $A' \in \mathcal{A}^*(S \cup \{i_1\})$, there is some $i \in S$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma 3, taking (B, x), (S, A, p) and $J = \{i_1\}$, with $i_1 \in M \setminus S$, there exist a coalition $T \subsetneq S$ and $\bar{A} \in \mathcal{A}^*(T \cup \{i_1\})$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\sum_{i \in S \setminus T} \Big(w_i(B_i) - x_i \Big) > v(S \cup \{i_1\} \cup \{0\}) - v(T \cup \{i_1\} \cup \{0\})$$
$$\geq v(S \cup \{0\}) - v(T \cup \{i_1\} \cup \{0\}), \tag{17}$$

where the last inequality comes from the monotonicity of v. Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for each $i \in T$. Taking (17) into account, we get

$$\sum_{i \in S} p_i = \sum_{i \in S} w_i(A_i) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) = v(S \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big)$$
$$< v(T \cup \{i_1\} \cup \{0\}) - \sum_{i \in T} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in T} \bar{p}_i + w_{i_1}(\bar{A}_{i_1}),$$

where the first equality follows from Lemma 1. Therefore, buyer i_1 has incentives to deviate (the argument is analogous to the one in Case 1 in this proof), which contradicts that (B, x) forms part of a SPE. This completes the proof and hence $v(S \cup \{0\}) = v(M \cup \{0\})$. □

The next theorem is the main result of this paper: the payoff in any subgame perfect equilibrium of Γ is the Vickrey payoff vector. As a consequence of the assumption on the buyers' valuations, it turns out that the Vickrey payoff vector belongs to the core of the associated coalitional game. Hence, once the agents have played any SPE of the game Γ, no coalition of agents can improve their current payoff by trading only among themselves.

Another lemma will be used. Its proof is consigned to the Appendix.

Lemma 5. *Consider any SPE of Γ. Let (B, x) be the buyers' strategy profile in this SPE and let (S, A, p) be the reply of the seller to (B, x) in this SPE. For each buyer $t \in S$, there is a triple (S^t, A^t, p^t) with $\emptyset \neq S^t \subseteq M \setminus \{t\}$, $A^t \in \mathcal{A}(S^t)$, $p^t = (p^t_i)_{i \in S^t} \in \mathbb{R}^{S^t}_+$ such that:*

1. *$w_i(A^t_i) - p^t_i = w_i(B_i) - x_i$ for all $i \in S^t$.*
2. *$\sum_{i \in S^t} p^t_i = \sum_{i \in S} p_i$.*
3. *$\sum_{i \in S^t} w_i(A^t_i) = v(S^t \cup \{0\}) = v(M \cup \{0\})$.*
4. *$v(((S \cup S^t) \setminus \{t\}) \cup \{0\}) = v((M \setminus \{t\}) \cup \{0\})$.*
5. *$w_i(B_i) - x_i = 0$ for all $i \in S^t \setminus S$.*

Theorem 6. *The payoff of any SPE of Γ is the Vickrey payoff vector of the market $(M, \{0\}, Q, w)$.*

Proof. Fix any SPE of Γ. Let (B, x) be the buyers' strategy profile in this SPE and denote by (S, A, p) the seller's reply to (B, x). Fix any $i_1 \in S$. Let $(S^{i_1}, A^{i_1}, p^{i_1})$ be as in the statement of Lemma 5 taking $t = i_1$. Define a coalition of buyers $D \subseteq M$ by $D = S \cup S^{i_1}$.

Firstly, we claim that

$$\text{for any } \tilde{A} \in \mathcal{A}^*(D \setminus \{i_1\}), \text{ we have } w_i(\tilde{A}_i) \geq w_i(B_i) - x_i \text{ for all } i \in D \setminus \{i_1\}. \tag{18}$$

To prove that, assume on the contrary there exists $\widehat{A} \in \mathcal{A}^*(D \setminus \{i_1\})$ and some $i_2 \in D \setminus \{i_1\}$ such that $w_{i_2}(B_{i_2}) - x_{i_2} > w_{i_2}(\widehat{A}_{i_2})$. Notice that

$$\begin{aligned} w_{i_2}(B_{i_2}) - x_{i_2} > w_{i_2}(\widehat{A}_{i_2}) &\geq v((D \setminus \{i_1\}) \cup \{0\}) - v((D \setminus \{i_1, i_2\}) \cup \{0\}) \\ &\geq v(D \cup \{0\}) - v((D \setminus \{i_2\}) \cup \{0\}) \geq 0, \end{aligned} \tag{19}$$

where the second inequality comes from Lemma 2 and the third one follows from buyers-submodularity of v.

Now we consider two cases and show that each of them leads to a contradiction.

Case 1: For some $A' \in \mathcal{A}^*(D \setminus \{i_2\})$ it holds $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in D \setminus \{i_2\}$.

Define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for all $i \in D \setminus \{i_2\}$. Therefore

$$\begin{aligned} \sum_{i \in S} p_i = \sum_{i \in S} w_i(A_i) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) &= v(S \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) \\ &= v(D \cup \{0\}) - \sum_{i \in D} \Big(w_i(B_i) - x_i \Big) \\ &< v((D \setminus \{i_2\}) \cup \{0\}) - \sum_{i \in D \setminus \{i_2\}} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in D \setminus \{i_2\}} p'_i, \end{aligned} \tag{20}$$

where the first equality comes from Lemma 1, the second equality from the claim (9) in Theorem 4, the third equality follows from Theorem 4, the monotonicity of v and $w_i(B_i) - x_i = 0$ for all $i \in S^{i_1} \setminus S$ (see part 5 of Lemma 5) and the inequality from (19). This fact, (20), contradicts that (S, A, p) maximizes the seller's payoff at (B, x).

Case 2: For all $A' \in \mathcal{A}^*(D \setminus \{i_2\})$, there is a buyer $i \in D \setminus \{i_2\}$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma 3, taking $S = D \setminus \{i_2\}$ and $J = \emptyset$, there exist $\emptyset \neq T \subsetneq D \setminus \{i_2\}$ and $\bar{A} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\sum_{i \in (D \setminus \{i_2\}) \setminus T} \Big(w_i(B_i) - x_i \Big) > v((D \setminus \{i_2\}) \cup \{0\}) - v(T \cup \{0\}).$$

Making use of (19) notice that

$$\begin{aligned} \sum_{i \in D \setminus T} \Big(w_i(B_i) - x_i \Big) &> v((D \setminus \{i_2\}) \cup \{0\}) - v(T \cup \{0\}) \\ &\quad + v(D \cup \{0\}) - v((D \setminus \{i_2\}) \cup \{0\}) = v(D \cup \{0\}) - v(T \cup \{0\}). \end{aligned} \tag{21}$$

Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for each $i \in T$. We have

$$\begin{aligned} v(D \cup \{0\}) - \sum_{i \in D} \Big(w_i(B_i) - x_i \Big) &= v(S \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) \\ &= \sum_{i \in S} w_i(A_i) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in S} p_i \\ &\geq \sum_{i \in T} \bar{p}_i = v(T \cup \{0\}) - \sum_{i \in T} \Big(w_i(B_i) - x_i \Big), \end{aligned}$$

where the first equality follows from Theorem 4, monotonicity of v and $w_i(B_i)-x_i=0$ for all $i\in S^{i_1}\setminus S$ (see part 5 in Lemma 5), the second equality comes from the claim (9) in Theorem 4, the third equality comes from Lemma 1 and the first inequality follows from the fact that (S,A,p) maximizes the seller's payoff given (B,x). Then,

$$v(D\cup\{0\})-v(T\cup\{0\})\geq\sum_{i\in D\setminus T}\Big(w_i(B_i)-x_i\Big).$$

This fact contradicts (21).

Hence, we have proved the claim (18): for every $i_1\in S$ and any allocation $\tilde{A}\in\mathcal{A}^*(D\setminus\{i_1\})$, we have $w_i(\tilde{A}_i)\geq w_i(B_i)-x_i$ for all $i\in D\setminus\{i_1\}$.

Now, we prove that the payoff vector of any SPE is the Vickrey payoff vector of the market. Fix any $i_1\in S$ and take $D=S\cup S^{i_1}$, where S^{i_1} is as in the statement of Lemma 5. Fix any $\tilde{A}\in\mathcal{A}^*(D\setminus\{i_1\})$. Now, define a price vector $\tilde{p}=(\tilde{p}_i)_{i\in D\setminus\{i_1\}}\in\mathbb{R}_+^{D\setminus\{i_1\}}$ such that $\tilde{p}_i=w_i(\tilde{A}_i)-(w_i(B_i)-x_i)$ for all $i\in D\setminus\{i_1\}$. We have

$$\begin{aligned}v(M\cup\{0\})-\sum_{i\in S}\Big(w_i(B_i)-x_i\Big)&=\sum_{i\in S}w_i(A_i)-\sum_{i\in S}\Big(w_i(B_i)-x_i\Big)=\sum_{i\in S}p_i\\&\geq\sum_{i\in D\setminus\{i_1\}}\tilde{p}_i=v((D\setminus\{i_1\})\cup\{0\})-\sum_{i\in D\setminus\{i_1\}}\Big(w_i(B_i)-x_i\Big)\\&=v((M\setminus\{i_1\})\cup\{0\})-\sum_{i\in D\setminus\{i_1\}}\Big(w_i(B_i)-x_i\Big),\end{aligned}$$

where the first equality follows from Theorem 4, the second equality from Lemma 1, the inequality since (S,A,p) maximizes the seller's payoff given (B,x) and the last equality from part 4 in Lemma 5. Then,

$$v(M\cup\{0\})-v((M\setminus\{i_1\})\cup\{0\})\geq\sum_{i\in S}\Big(w_i(B_i)-x_i\Big)-\sum_{i\in D\setminus\{i_1\}}\Big(w_i(B_i)-x_i\Big).$$

Since $D=S\cup S^{i_1}$ and $w_i(B_i)-x_i=0$ for all $i\in S^{i_1}\setminus S$ (see part 5 of Lemma 5), we obtain

$$v(M\cup\{0\})-v((M\setminus\{i_1\})\cup\{0\})\geq w_{i_1}(B_{i_1})-x_{i_1}. \tag{22}$$

As a consequence, since i_1 was an arbitrary buyer in S, we have that for every buyer $i\in S$,

$$M_i^v\geq w_i(B_i)-x_i. \tag{23}$$

We see now that, in fact, $w_i(B_i)-x_i=M_i^v$ for all $i\in S$.

Fix any buyer $i_1\in S$. Let $(S^{i_1},A^{i_1},p^{i_1})$ be as in the statement of Lemma 5 taking $t=i_1$. Then

$$\begin{aligned}v(M\cup\{0\})-\sum_{i\in S}\Big(w_i(B_i)-x_i\Big)&=\sum_{i\in S}w_i(A_i)-\sum_{i\in S}\Big(w_i(B_i)-x_i\Big)=\sum_{i\in S}p_i\\&=\sum_{i\in S^{i_1}}p_i^{i_1}=\sum_{i\in S^{i_1}}w_i(A_i^{i_1})-\sum_{i\in S^{i_1}}\Big(w_i(B_i)-x_i\Big)=v(S^{i_1}\cup\{0\})-\sum_{i\in S^{i_1}}\Big(w_i(B_i)-x_i\Big),\end{aligned}$$

where the first equality follows from Theorem 4, the second equality follows from Lemma 1, the third equality follows from part 2 of Lemma 5 and the two last equalities follow from parts 1 and 3 of Lemma 5. Since, by part 5 of the same lemma, we know that $w_i(B_i)-x_i=0$ for all $i\in S^{i_1}\setminus S$, we have obtained

$$v(M\cup\{0\})-v(S^{i_1}\cup\{0\})=\sum_{i\in S\setminus S^{i_1}}\Big(w_i(B_i)-x_i\Big). \tag{24}$$

On the other hand, by buyers-submodularity (5), we know that

$$v(M \cup \{0\}) - v(S^{i_1} \cup \{0\}) \geq \sum_{i \in M \setminus S^{i_1}} M_i^v \geq \sum_{i \in S \setminus S^{i_1}} M_i^v, \tag{25}$$

where the last inequality follows since $M_i^v \geq 0$ for each $i \in M$. Making use of expressions (24) and (25), we obtain

$$\sum_{i \in S \setminus S^{i_1}} \left(w_i(B_i) - x_i \right) \geq \sum_{i \in S \setminus S^{i_1}} M_i^v. \tag{26}$$

If inequality (22) were strict, $M_{i_1}^v > w_{i_1}(B_{i_1}) - x_{i_1}$, since $i_1 \in S \setminus S^{i_1}$, and taking into account (23), we would contradict (26). Therefore, we conclude that $w_{i_1}(B_{i_1}) - x_{i_1} = M_{i_1}^v$. As i_1 was an arbitrary buyer belonging to S, we obtain that $w_i(B_i) - x_i = M_i^v$ for all $i \in S$. This shows that in any SPE of the mechanism Γ, if a buyer i obtains a package of objects, *i.e.*, $i \in S$, he requests (B_i, x_i) such that $w_i(B_i) - x_i = M_i^v$. By Lemma 1, we obtain that the payoff for each buyer $i \in S$ under any SPE is his marginal contribution M_i^v. Moreover, the payoff for each buyer $i \in M \setminus S$ is zero which is exactly his marginal contribution M_i^v (see Theorem 4). Since the reply of the seller in any SPE produces an efficient allocation for the market, we conclude that the payoff vector given in any SPE is the Vickrey payoff vector of the market. This completes the proof. □

We have just proved that the payoff associated with any SPE of the mechanism Γ is the Vickrey payoff vector. The reader may ask whether indeed SPE exist for that game. It is not difficult to see that for any efficient allocation, $A \in \mathcal{A}^*(S)$ with $\sum_{i \in S} w_i(A_i) = w_A(M \cup \{0\})$, there is a SPE where each buyer $i \in S$ plays (A_i, x_i) such that $w_i(A_i) - x_i = M_i^v$ and the seller selects a best reply to the buyers' demands.

Even when there is only one efficient allocation of the packages, there may be multiple SPE in our game. On one side this multiplicity comes from the fact that different strategies may give a buyer the same outcome. If (B_1, x_1) is a SPE strategy of buyer 1 and there is another package $\emptyset \neq R_1 \subseteq Q$ and $y_1 \geq 0$ such that $w_1(B_1) - x_1 = w_1(R_1) - y_1$, then if buyer 1 replaces strategy (B_1, x_1) with (R_1, y_1), all the other agents keeping their strategies, we have another SPE of the game.

4. Concluding remarks

Buyers-submodularity is an essential assumption to obtain our results. When there are complementarities, the SPE of our mechanism may change. Take for instance the following market taken from Milgrom (2007).

Example 2. Consider a market with three buyers, 1, 2 and 3, and one seller with two objects A and B.

	A	B	AB
1	0	0	12
2	10	10	10
3	10	10	10

If we apply the same mechanism Γ to this market situation, notice first that the strategies we have mentioned to be a SPE in the buyers-submodular case, are not a Nash equilibrium in this case. Since an optimal matching is to allocate A to buyer 2 and B to buyer 3, and the marginal contributions of the three buyers are 0, 8 and 8 respectively, let us assume that buyers take strategies $(B_1, x_1) = (\{A\}, 0)$, $(B_2, x_2) = (\{A\}, 2)$ and $B_3 = (\{B\}, 2)$. And assume the strategy of the seller is to select a coalition of buyers and maximize the seller's reward allocating the objects to buyers in that coalition while preserving the net profit they demand.

Now, buyer 1 has incentives to deviate to $(B_1', x_1') = (\{A, B\}, 12 - \varepsilon)$, with $0 < \varepsilon < 8$, since she knows that she will be selected by the seller and will obtain a positive payoff.

The above argument also shows that the Vickrey outcome will not be reached by any SPE of the game. Notice that the Vickrey outcome (0,8,8; 4) does not belong to the core of the coalitional game, since the coalition formed by buyer 1 and the seller should get at least 12.

But our game applied to this market still has some SPE. Assume $(B_1, x_1) = (\{A, B\}, 12)$, $(B_2, x_2) = (\{A\}, 6)$ and $(B_3, x_3) = (\{B\}, 6)$. Assume that when the buyers play like that the seller selects $S = \{2, 3\}$, and in any other contingency the seller chooses any best reply (any coalition that allows him to maximize his reward while preserving the buyers' demanded net profit). This constitutes a SPE and the payoff vector is $(0, 4, 4; 12)$. This vector belongs to the core but is not buyers-optimal (in fact this coalitional game has no buyers-optimal core allocation).

Future research could analyze the mechanism Γ in the presence of complementarities and also when there is more than one seller in the market.

5. Appendix

The following lemmas are used in the main results of this paper.

Lemma 3. *Consider any market $(M, \{0\}, Q, w)$ and let (B, x) be the buyers' strategy profile in any SPE of the mechanism Γ. For any non-empty coalition of buyers $S \subseteq M$ and any $J \subseteq M \setminus S$ we have either:*

1. *there exists an efficient allocation $A = (A_i)_{i \in S \cup J} \in \mathcal{A}^*(S \cup J)$ such that $w_i(A_i) \geq w_i(B_i) - x_i$ for all $i \in S$, or*
2. *there exist a subcoalition of buyers $T \subsetneq S$, with $T \neq \emptyset$ whenever $J = \emptyset$, and an efficient allocation $A = (A_i)_{i \in T \cup J} \in \mathcal{A}^*(T \cup J)$ such that*
   (a) *$w_i(A_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and*
   (b) *$\sum_{i \in S \setminus T} \left(w_i(B_i) - x_i \right) > v(S \cup J \cup \{0\}) - v(T \cup J \cup \{0\})$.*

Proof. We first prove the result whenever coalition S is a singleton, $S = \{i_1\}$ for some $i_1 \in M$. In this case, if $J = \emptyset$ we can take $A = (A_{i_1}) \in \mathcal{A}^*(\{i_1\})$ with $A_{i_1} = Q$ and then, by the monotonicity of buyers' valuations, we have $w_{i_1}(A_{i_1}) = w_{i_1}(Q) \geq w_{i_1}(B_{i_1}) - x_{i_1}$, which means that item 1 is satisfied. If $S = \{i_1\}$, $J \neq \emptyset$ and there exists $A \in \mathcal{A}^*(\{i_1\} \cup J)$ such that $w_{i_1}(A_{i_1}) \geq w_{i_1}(B_{i_1}) - x_{i_1}$ we also are done. Otherwise,

$S = \{i_1\}$, $J \neq \emptyset$ and for any $A \in \mathcal{A}^*(\{i_1\} \cup J)$ it holds $w_{i_1}(B_{i_1}) - x_{i_1} > w_{i_1}(A_{i_1})$. In that case, take $T = \emptyset$ and notice that statement 2 (a) holds trivially. Moreover, for all $A \in \mathcal{A}^*(\{i_1\} \cup J)$ we have

$$w_{i_1}(B_{i_1}) - x_{i_1} > w_{i_1}(A_{i_1}) \geq v(\{i_1\} \cup J \cup \{0\}) - v(J \cup \{0\}) = v(S \cup J \cup \{0\}) - v(T \cup J \cup \{0\}), \tag{27}$$

where the second inequality follows from Lemma 2.

Now, we proceed to prove the lemma for any coalition $S \subseteq M$ with $|S| \geq 2$. If there exists an efficient allocation $A^1 = (A_i^1)_{i \in S \cup J} \in \mathcal{A}^*(S \cup J)$ such that $w_i(A_i^1) \geq w_i(B_i) - x_i$ for all $i \in S$, we are done. Otherwise, define $T_1 = S$ and fix $A^1 = (A_i^1)_{i \in T_1 \cup J} \in \mathcal{A}^*(T_1 \cup J)$. We know that there is some buyer $i_1 \in T_1$ such that

$$w_{i_1}(A_{i_1}^1) < w_{i_1}(B_{i_1}) - x_{i_1}. \tag{28}$$

Denote now $T_2 = T_1 \setminus \{i_1\}$. We may assume that $T_2 \neq \emptyset$, since otherwise $T_1 = S = \{i_1\}$ and we are done. Moreover, by inequality (28) and Lemma 2, we have

$$\begin{aligned} w_{i_1}(B_{i_1}) - x_{i_1} > w_{i_1}(A_{i_1}^1) &\geq v(S \cup J \cup \{0\}) - v((S \setminus \{i_1\}) \cup J \cup \{0\}) \\ &= v(S \cup J \cup \{0\}) - v(T_2 \cup J \cup \{0\}). \end{aligned} \tag{29}$$

Now, if there is an allocation $A^2 = (A_i^2)_{i \in T_2 \cup J} \in \mathcal{A}^*(T_2 \cup J)$ such that $w_i(A_i^2) \geq w_i(B_i) - x_i$ for all $i \in T_2$, we are done taking $T = T_2$. Otherwise, fix $A^2 = (A_i^2)_{i \in T_2 \cup J} \in \mathcal{A}^*(T_2 \cup J)$. We know that there is some $i_2 \in T_2$ such that

$$w_{i_2}(A_{i_2}^2) < w_{i_2}(B_{i_2}) - x_{i_2}. \tag{30}$$

Denote now $T_3 = T_2 \setminus \{i_2\}$ and notice that if $T_3 \neq \emptyset$, by inequality (30) and Lemma 2 , we have

$$w_{i_2}(B_{i_2}) - x_{i_2} > w_{i_2}(A_{i_2}^2) \geq v(T_2 \cup J \cup \{0\}) - v((T_2 \setminus \{i_2\}) \cup J \cup \{0\}) \tag{31}$$
$$= v(T_2 \cup J \cup \{0\}) - v(T_3 \cup J \cup \{0\}). \tag{32}$$

By adding (29) and (31), we get

$$\sum_{i \in S \setminus T_3} \Big(w_i(B_i) - x_i \Big) > v(S \cup J \cup \{0\}) - v(T_3 \cup J \cup \{0\}).$$

By proceeding recursively, we construct a sequence of sets $\{T_1, ..., T_{k+1}\}$ such that $T_1 = S$, $T_l \setminus T_{l+1} = \{i_l\}$ for $l = 1, ..., k$, $A^l \in \mathcal{A}^*(T_l \cup J)$ for $l = 1, ..., k+1$, $w_{i_l}(A_{i_l}^l) < w_{i_l}(B_{i_l}) - x_{i_l}$ for $l = 1, ..., k$ and

$$\sum_{i \in S \setminus T_{l+1}} \Big(w_i(B_i) - x_i \Big) > v(S \cup J \cup \{0\}) - v(T_{l+1} \cup J \cup \{0\}) \quad \text{for } l = 1, ..., k. \tag{33}$$

Now if $T_{k+1} \neq \emptyset$ and there is an efficient allocation $A^{k+1} \in \mathcal{A}^*(T_{k+1} \cup J)$ such that $w_i(A_i^{k+1}) \geq w_i(B_i) - x_i$ for all $i \in T_{k+1}$, we are done taking $T = T_{k+1}$. Otherwise, we continue the procedure one more step. Notice that, since S is a finite set, we will eventually reach T_r with $|T_r| = 1$. In that case, as shown at the beginning of the proof, we either conclude taking $T = T_r = \{i_r\}$ when $J = \emptyset$ and also when $J \neq \emptyset$ but there exists $A \in \mathcal{A}^*(\{i_r\} \cup J)$ such that $w_{i_r}(A_{i_r}) \geq w_{i_r}(B_{i_r}) - x_{i_r}$, or taking $T = \emptyset$ otherwise. □

The next lemma shows that in any SPE, and for any buyer who receives a package, there is an alternative action that gives the seller the same payoff.

Lemma 5. *Consider any SPE of* Γ*. Let* (B, x) *be the buyers' strategy profile in this SPE and let* (S, A, p) *be the reply of the seller to* (B, x) *in this SPE. For each buyer* $t \in S$*, there is a triple* (S^t, A^t, p^t) *with* $\emptyset \neq S^t \subseteq M \setminus \{t\}$*,* $A^t \in \mathcal{A}(S^t)$*,* $p^t = (p_i^t)_{i \in S^t} \in \mathbb{R}_+^{S^t}$ *such that:*

1. $w_i(A_i^t) - p_i^t = w_i(B_i) - x_i$ *for all* $i \in S^t$.
2. $\sum_{i \in S^t} p_i^t = \sum_{i \in S} p_i$.
3. $\sum_{i \in S^t} w_i(A_i^t) = v(S^t \cup \{0\}) = v(M \cup \{0\})$.
4. $v(((S \cup S^t) \setminus \{t\}) \cup \{0\}) = v((M \setminus \{t\}) \cup \{0\})$.
5. $w_i(B_i) - x_i = 0$ *for all* $i \in S^t \setminus S$.

Proof. First, we prove statements 1 and 2 together: for each buyer $t \in S$, there is a triple (S^t, A^t, p^t), with $\emptyset \neq S^t \subseteq M \setminus \{t\}$, $A^t \in \mathcal{A}(S^t)$, $p^t = (p_i^t)_{i \in S^t} \in \mathbb{R}_+^{S^t}$, that satisfies $w_i(A_i^t) - p_i^t = w_i(B_i) - x_i$ for all $i \in S^t$ and $\sum_{i \in S^t} p_i^t = \sum_{i \in S} p_i$. To this end, we consider three cases: 1) $p_t = 0$ and $S \setminus \{t\} \neq \emptyset$; 2) $p_t = 0$ and $S \setminus \{t\} = \emptyset$ and 3) $p_t > 0$.

Case 1: $p_t = 0$ *and* $S \setminus \{t\} \neq \emptyset$. Take $k \in S \setminus \{t\}$, $A' = (A'_i)_{i \in S \setminus \{t\}} \in \mathcal{A}(S \setminus \{t\})$ with $A'_i = A_i$ for each $i \in S \setminus \{t, k\}$ and $A'_k = A_k \cup A_t$ and $p' = (p'_i)_{i \in S \setminus \{t\}} \in \mathbb{R}_+^{S \setminus \{t\}}$ with $p'_i = p_i$ for each $i \in S \setminus \{t\}$. Notice that $(S \setminus \{t\}, A', p')$ satisfies $w_i(A'_i) - p'_i \geq w_i(B_i) - x_i$ for all $i \in S \setminus \{t\}$ and $\sum_{i \in S \setminus \{t\}} p'_i = \sum_{i \in S} p_i$. Hence, $(S \setminus \{t\}, A', p')$ is also a best reply of the seller to (B, x) and from Lemma 1 we get $w_i(A'_i) - p'_i = w_i(B_i) - x_i$ for all $i \in S \setminus \{t\}$, which proves statements 1 and 2 for this case.

Case 2: $p_t = 0$ *and* $S \setminus \{t\} = \emptyset$. Take any $i' \in M \setminus \{t\}$ (recall that $|M| \geq 2$) and consider $S^t = \{i'\}$, $A^t = (A_{i'}^t) \in \mathcal{A}(\{i'\})$ with $A_{i'}^t = Q$ and $p^t = (p_{i'}^t)$ with $p_{i'}^t = 0$. Notice that (S^t, A^t, p^t) also satisfies $w_{i'}(A_{i'}^t) - p_{i'}^t \geq w_{i'}(B_{i'}) - x_{i'}$ and $\sum_{i \in S^t} p_i^t = p_{i'}^t = 0 = p_t = \sum_{i \in S} p_i$. Again, this implies that (S^t, A^t, p^t) is a best reply of the seller to (B, x) and hence by Lemma 1 we get that $w_{i'}(A_{i'}^t) - p_{i'} = w_{i'}(B_{i'}) - x_{i'}$, which proves statments 1 and 2 for this case.

Case 3: $p_t > 0$. We define the two following sets:

$$\begin{aligned} \mathcal{C} &= \{(S^t, A^t, p^t) | \emptyset \neq S^t \subseteq M \setminus \{t\}, A^t \in \mathcal{A}(S^t), p^t \in \mathbb{R}_+^{S^t} \text{such that } w_i(A_i^t) - p_i^t \geq w_i(B_i) - x_i \text{ for all } i \in S^t\} \\ \overline{\mathcal{C}} &= \{(S^t, A^t, p^t) \in \mathcal{C} \mid w_i(A_i^t) - p_i^t = w_i(B_i) - x_i \text{ for all } i \in S^t\} \end{aligned}$$

Notice that $\overline{\mathcal{C}} \subseteq \mathcal{C}$, and the set $\overline{\mathcal{C}}$ is non-empty, *e.g.* consider $S^t = \{i\}$ for some $i \in M \setminus \{t\}$, $A_i^t = Q$ and $p_i^t = w_i(Q) - (w_i(B_i) - x_i)$. Moreover, the set $\overline{\mathcal{C}}$ is a finite set.

Assume that, on the contrary to the statement we want to prove, for all $(S^t, A^t, p^t) \in \overline{\mathcal{C}}$, it holds

$$\sum_{i \in S^t} p_i^t \neq \sum_{i \in S} p_i. \tag{34}$$

Since (S, A, p) is the reply of the seller to (B, x), we have that $\sum_{i \in S^t} p_i^t \leq \sum_{i \in S} p_i$ for all $(S^t, A^t, p^t) \in \mathcal{C}$. Hence, for all $(S^t, A^t, p^t) \in \overline{\mathcal{C}}$, (34) turns out to be

$$\sum_{i \in S^t} p_i^t < \sum_{i \in S} p_i. \tag{35}$$

Since the set $\overline{\mathcal{C}}$ is finite, there is $\varepsilon > 0$ such that

$$\sum_{i \in S^t} p_i^t < \sum_{i \in S} p_i - \varepsilon \quad \text{for all } (S^t, A^t, p^t) \in \overline{\mathcal{C}}. \tag{36}$$

We see now that buyer $t \in S$ has incentives to unilaterally deviate. Assume buyer t deviates from (B_t, x_t) to $(A_t, p_t - \alpha)$ with $0 < \alpha < \varepsilon$ such that $p_t - \alpha \geq 0$ (recall that $p_t > 0$ by assumption of this Case 3).. Now, let $(\tilde{S}, \tilde{A}, \tilde{p})$ be the reply of the seller to this deviation. We show now that $t \in \tilde{S}$. By way of contradiction, assume that $t \notin \tilde{S}$. Indeed, since $t \notin \tilde{S}$ and by Lemma 1, we know that the reply $(\tilde{S}, \tilde{A}, \tilde{p})$ to this unilateral deviation belongs to $\overline{\mathcal{C}}$. Consider $(S, A, \underline{p})$ where $\underline{p}_i = p_i$ for all $i \in S \setminus \{t\}$ and $\underline{p}_t = p_t - \alpha$. Hence

$$\sum_{i \in \tilde{S}} \tilde{p}_i < \sum_{i \in S} p_i - \varepsilon < \sum_{i \in S} p_i - \alpha = \sum_{i \in S} \underline{p}_i, \tag{37}$$

where the first inequality comes from (36) applied to $(\tilde{S}, \tilde{A}, \tilde{p}) \in \overline{\mathcal{C}}$ and the second one since $\alpha < \varepsilon$. But then, (37) contradicts that $(\tilde{S}, \tilde{A}, \tilde{p})$ is a best reply of the seller and then, this shows that $t \in \tilde{S}$.

Fix now any best reply $(\tilde{S}, \tilde{A}, \tilde{p})$ of the seller when only buyer $t \in S$ deviates from (B_t, x_t) to $(A_t, p_t - \alpha)$. We know that buyer t belongs to $\tilde{S}$. Hence, the payoff of buyer t is $w_t(\tilde{A}_t) - \tilde{p}_t$. By Lemma 1, we know that $w_t(\tilde{A}_t) - \tilde{p}_t = w_t(A_t) - (p_t - \alpha) > w_t(A_t) - p_t$. This shows that buyer t has incentives to deviate from (B_t, x_t) to $(A_t, p_t - \alpha)$ as claimed, which is a contradiction with the fact that the agents were following a SPE. As a consequence, for each buyer $t \in S$, there is a triple $(S^t, A^t, p^t) \in \mathcal{C}$ with $\sum_{i \in S^t} p_i^t = \sum_{i \in S} p_i$.

We know that (S, A, p) is the reply of the seller to (B, x). Since (S^t, A^t, p^t) is a possible reply of the seller to (B, x) and $\sum_{i \in S^t} p_i^t = \sum_{i \in S} p_i$, then (S^t, A^t, p^t) is also a best reply of the seller to (B, x). Hence, as a consequence of Lemma 1, we have that $w_i(A_i^t) - p_i^t = w_i(B_i) - x_i$ for all $i \in S^t$. This finishes the proof of the existence, for each $t \in S$, of a triple (S^t, A^t, p^t), with $\emptyset \neq S^t \subseteq M \setminus \{t\}$, $A^t \in \mathcal{A}(S^t)$ and $p^t = (p_i^t)_{i \in S^t} \in \mathbb{R}_+^{S^t}$, that satisfies statements 1 and 2.

To prove statement 3, simply apply Theorem 4 to (S^t, A^t, p^t), which we have just seen is also a best reply of the seller to (B, x).

Moreover, to prove statement 4, monotonicity of v gives

$$v(S^t \cup \{0\}) \leq v(((S \cup S^t) \setminus \{t\}) \cup \{0\}) \leq v((M \setminus \{t\}) \cup \{0\}) \leq v(M \cup \{0\})$$

and, together with $v(S^t \cup \{0\}) = v(M \cup \{0\})$, this implies $v(((S \cup S^t) \setminus \{t\}) \cup \{0\}) = v((M \setminus \{t\}) \cup \{0\})$ which proves statement 4.

We finally prove part 5 of this lemma. Notice first that if for some $i' \in S^t \setminus S$, $w_{i'}(B_{i'}) - x_{i'} < 0$, then the seller can give a dummy object to buyer i' at a specific positive price, that is, the triple $(S \cup \{i'\}, A', p')$ where $A'_i = A_i$ for all $i \in S$, $A'_{i'}$ consists of a dummy object j_0, $p'_i = p_i$ for all $i \in S$ and $p'_{i'} = x_{i'} - w_{i'}(B_{i'}) > 0$ is a feasible reply of the seller to (B, x), and since $\sum_{i \in S \cup \{i'\}} p'_i > \sum_{i \in S} p_i$, this contradicts that (S, A, p) is a best reply to (B, x).

On the other hand, if for some $i' \in S^t \setminus S$, $w_{i'}(B_{i'}) - x_{i'} > 0$, buyer i' could deviate to $(B_{i'}, x_{i'} + \varepsilon)$, with $w_{i'}(B_{i'}) - (x_{i'} + \varepsilon) > 0$. Then, $(S^t, A^t, \overline{p}^t)$ where $\overline{p}_i^t = p_i^t$ for all $i \in S^t \setminus \{i'\}$ and $\overline{p}_{i'}^t = p_{i'}^t + \varepsilon$ is a feasible reply of the seller to this deviation and

$$\sum_{i \in S} p_i = \sum_{i \in S^t} p_i^t < \sum_{i \in S^t} \overline{p}_i^t, \tag{38}$$

where the equality follows from statement 2 in this lemma. Take $(\tilde{S}, \tilde{A}, \tilde{p})$ a best reply of the seller to the above buyer i' deviation to $(B_{i'}, x_{i'} + \varepsilon)$. If $i' \in \tilde{S}$, then $w_{i'}(\tilde{A}_{i'}) - \tilde{p}_{i'} = w_{i'}(B_{i'}) - (x_{i'} + \varepsilon) > 0$ makes i' better off (since $i' \notin S$) and this fact contradicts that (B, x) forms part of a SPE of Γ. Otherwise, if $i' \notin \tilde{S}$, then $(\tilde{S}, \tilde{A}, \tilde{p})$ is also a feasible reply of the seller to (B, x) and moreover

$$\sum_{i \in S} p_i = \sum_{i \in S^t} p_i^t < \sum_{i \in S^t} \overline{p}_i^t \leq \sum_{i \in \tilde{S}} \tilde{p}_i, \tag{39}$$

where the first equality follows from statement 2, the strict inequality from (38) and the last inequality since $(\tilde{S}, \tilde{A}, \tilde{p})$ is a best reply when buyer i' unilaterally deviates to $(B_{i'}, x_{i'} + \varepsilon)$. Finally, (39) contradicts that (S, A, p) is a best reply to (B, x) and this completes the proof of statement 5. □

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