Assortative multisided assignment games: the extreme core points

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Abstract

We analyze assortative multisided assignment games, following Sherstyuk (1999) and Martínez-de-Albéniz et al. (2019). In them players' abilities are complementary across types (i.e. supermodular), and also the output of the essential coalitions is increasing depending on types.

We study the extreme core points and show a simple mechanism to compute all of them. In this way we describe the whole core. This mechanism works from the original data array and the maximum number of extreme core points is obtained.

Keywords: Assortative market, assignment game, multisided assignment game, core, extreme core allocations

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1. Introduction

Economic markets with indivisible goods have been considered using worthy matching models. In this setting there are different but related models. In a two-sided matching game or assignment game there are essential coalitions formed from two different types of agents containing one agent of each type. The bilateral assignment game comes initially from Shapley (1955), but Shapley and Shubik (1971) is the paper most cited. In it the authors introduce and analyze a housing market as a bilateral assignment market. We refer now to another seminal paper, Becker (1973). In it, pursuing a general theory of marriage, Becker introduces a special class of assignment games, the two-sided assortative ones. In some assignment problems Becker displays the well-known effect of *mating of the likes.* Finally, Crawford and Knoer (1981) develops a model of labor market by using matching and assignment tools. This last model easily allows to motivate the relevance to study m-sided matching games, with $m \geq 3$. It is easy to think of situations where m types of different skills' workers are needed to achieve valuable essential coalitions. Precisely the main purpose of this paper is to analyze m-sided assortative games.

In all these previous models the most relevant set solution is the core. Roughly speaking the core is formed by all those allocations in which no coalition of agents can improve its reward on its own.

Multisided assignment games were analyzed for the first time in Quint (1991). After showing a three-sided example with an empty core, Quint presents a class of games with the property that the core is non-empty, i.e. balanced. Stuart (1997) proposes another balanced class of multisided assignment games, not related to Quint's class (none of them includes the other). A proof of the non-emptiness is provided, but no description or characterization of the core is given in any of the two models.

Sherstyuk (1999) introduces another important class of m-sided matching games. She analyzes for the first time the assortative multisided assignment games. The definition of this class relies on two conditions imposed on the assignment array: supermodularity and monotonicity. Both conditions assume that agents in each sector can be ranked by some trait or ability. Supermodularity is a complementary property of agents' ability across types. Monotonicity means that ability is aligned with the worth generated by the essential coalitions.

Assortative multisided assignment games form a large class of m-sided assignment games: a full-dimensional cone. In Sherstyuk's paper it is proved the non-emptiness of the core and she describes some extreme core allocations, $m!$ of them, by using the associated characteristic function.

Finally, to describe papers which study the core of multisided assignment markets, Tejada and Núñez (2012) and Tejada (2013) generalize to the m-sided case the Böhm-Bawerk horse markets introduced in Shapley and Shubik (1971), and recently Atay and Núñez (2019) describes a model with a graph relating the different sectors to obtain the characteristic function.

In this paper we analyze a simple mechanism to describe the whole core of any assortative m-sided assignment game. Our method characterizes for the first time all the extreme core allocations of any assortative m-sided matching game. The procedure can be applied for the two-sided case as well as the generic m-sided case. The mechanism depends only on the assignment array data, with no need to compute the characteristic function of the game. We also determine the maximum number of extreme core allocations, $m \cdot$

$(m!)^{n-1}$, where m is the number of sectors and n is the number of agents in each sector. As a by-product we obtain the number of extreme core allocations when we deal only with two sectors, 2^n. Finally our mechanism is a generalization of the one established in Martínez-de-Albéniz et al. (2019), developed only for two-sided assortative games. We want to point out that proofs in the above paper are completely different. In the present paper proofs are simplified and the arguments are distinct. Essentially here we use the natural order structure of the essential coalitions. Moreover our proofs include the two-sided case as a particular case.

In Eriksson et al. (2000) the two-sided assortative case is also analyzed and they show that the core is ordered in payoffs inside each sector. This property remains true for the general m-sided case, as we show.

Although the core of two-sided assignment games has been extensively studied, the core of m-sided assignment games, $m \geq 3$, has not got the same attention. It is not that the subject was not found interesting, but NP-hardness of computational aspects (see Garey and Johnson, 1979) and negative results are the reason behind the scarcity of literature on multisided assignment games.

2. Preliminaries on the multisided assignment markets

A multisided assignment market $(N^1, N^2, \ldots, N^m; A)$ is formed by m non-empty pairwise disjoint finite sets of agents,[2] $N^k = \left\{1^k, 2^k, \ldots, n_k^k\right\}$ for $k \in M = \{1, \ldots, m\}$ and a non-negative m-dimensional array $A = (a_E)_{E \in \Pi_{k=1}^m N^k}$. Each entry a_E represents some measure of the joint produc-

[2]To simplify notation, when no confusion arises, we will drop the superscript to describe the agents in N^k, i.e. $N^k = \{1, 2, \ldots, n_k\}$. Its cardinality is $|N^k| = n_k$.

tivity of agents in $E = (i_1, i_2, \ldots, i_m) \in \Pi_{k=1}^m N^k$, one of each set when they are matched together. We assume that we need exactly one agent of each type to realize the value of a transaction. Clearly we can think of array A as a function f defined on the set $\Pi_{k=1}^m N^k$, that is $a_E = f(E)$. Each set N^k is called a sector and corresponds to a different type of agents, having different skills. Any m-tuple of agents $E = (i_1, \ldots, i_m) \in \Pi_{k=1}^m N^k$ is called an *essential coalition* and we use E either as the m-tuple or as the set of elements formed by its components. In the case of two sectors, $m = 2$, matrix A is known as the assignment matrix (Shapley and Shubik, 1971). When the number of agents is the same in each sector $|N^1| = |N^2| = \ldots = |N^m|$ the assignment market is said to be square.

A *matching* μ among $N^1, \ldots, N^m$ is a set of essential coalitions such that any agent belongs at most to one coalition in μ, and $|\mu| = min_{k \in M} |N^k|$. An agent who does not belong to any of the essential coalitions of μ is unmatched by μ. The set of all matchings is denoted by $\mathcal{M}\left(N^1, \ldots, N^m\right)$. A matching μ is *optimal* if it maximizes $\sum_{E \in \mu} a_E$ over the set $\mathcal{M}\left(N^1, \ldots, N^m\right)$. The set of all optimal matchings is denoted by $\mathcal{M}_A^*\left(N^1, \ldots, N^m\right)$.

Shapley and Shubik (1971) associates any bilateral assignment market with a cooperative game[3], the *assignment game*. In the multisided assignment game of Quint (1991), the set of players is $N = \bigcup_{k=1}^m N^k$ and the characteristic function w_A is defined for any $S \subseteq N$ such that $S \cap N^k \neq \varnothing$

[3]In a cooperative game (N, v), the set of players is given by $N = \{1, \ldots, n\}$ and v is a function that assigns a real number $v(S)$ for any coalition $S \subseteq N$ with $v(\emptyset) = 0$. Its core is defined as $C(v) := \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N) \text{ and for all } S \subseteq N, \ \sum_{i \in S} x_i \geq v(S)\}$. A game is named balanced if its core is non-empty.

for all $k \in M$, by

$$w_A(S) = \max_{\mu \in \mathcal{M}(S\cap N^1,\ldots,S\cap N^m)} \sum_{E\in\mu} a_E, \quad \text{and 0 otherwise.}$$

Notice that any essential coalition evaluates its worth by exactly the corresponding entry, and any other coalition determines its worth by essential coalition combinations its members can form.

The agents of a multisided assignment market may divide among themselves their worth, $w_A(N)$, in any way they like. Thus an *allocation* is a non-negative vector $x = (x^1, x^2, \ldots, x^m) \in \Pi_{k=1}^m \mathbb{R}_+^{n_k}$. Vector $x^k \in \mathbb{R}_+^{n_k}$ is interpreted as the payoffs to agents in N^k, i.e. x_i^k is the payoff associated to player i of sector k. For any essential coalition $E = (i_1, \ldots, i_m) \in \Pi_{k=1}^m N^k$ we write $x(E) = \sum_{k=1}^m x_{i_k}^k$.

The *core of the multisided assignment game* $C(w_A)$ is described for any fixed optimal matching $\mu \in \mathcal{M}_A^*\left(N^1, \ldots, N^m\right)$ as those allocations $x \in \Pi_{k=1}^m \mathbb{R}_+^{n_k}$ satisfying

$$x(E) = a_E \quad \text{for all } E \in \mu,$$
$$x(E) \geq a_E \quad \text{for all } E \notin \mu,$$

and unassigned agents by μ receive a zero payoff in any core allocation.

In the two-sided case, Shapley and Shubik (1971) proves that the core of any assignment game is always non-empty, but in the multisided case, $m \geq 3$, it is known (Kaneko and Wooders, 1982, or Quint, 1991) that the core may be empty.

Becker (1973) introduces two-sided assortative assignment markets. For multisided assignment markets, we assume that the elements of each sector are ordered by some trait and then N^k for $k \in M$ is an ordered set with the natural order. Therefore $\Pi_{k=1}^m N^k$ is a lattice and for any pair of essential coalitions $E, E' \in \Pi_{k=1}^m N^k$ we can define $E \vee E'$ as the maximum component-wise and $E \wedge E'$ as the minimum component-wise.

A multisided assignment market $(N^1, N^2, \ldots, N^m; A)$ is an *assortative market* if it satisfies:

a) supermodularity:[4]

$$a_E + a_{E'} \leq a_{E \vee E'} + a_{E \wedge E'} \quad \text{for all} \quad E, E' \in \Pi_{k=1}^m N^k. \tag{1}$$

b) monotonicity (non-decreasing rows, columns, etc.):

$$a_E \leq a_{E'} \quad \text{for all} \quad E \leq E', \quad E, E' \in \Pi_{k=1}^m N^k. \tag{2}$$

Whenever these two conditions are met, array A is called assortative.

From the supermodularity condition, in a multisided assortative assignment market at least one optimal matching $\mu \in \mathcal{M}_A^*\left(N^1, \ldots, N^m\right)$ is *monotone*,[5] i.e.

$$\text{for any } E, E' \in \mu, \quad \text{either } E \leq E' \text{ or } E' \leq E.$$

When the assortative assignment market is square, $|N^1| = |N^2| = \ldots = |N^m| = n$ there is only one monotone matching which is placed in the main diagonal. If we denote the following essential coalitions: $E_{\boldsymbol{i}} = (i, i, \ldots, i)$, for $i = 1, 2, \ldots, n$, this monotone matching is $\mu = \{E_{\boldsymbol{1}}, E_{\boldsymbol{2}}, \ldots, E_{\boldsymbol{n}}\}$. This is, by the previous observation, optimal in the square supermodular case, maybe not unique.

[4]Notice that this condition implies that array entries form a supermodular function in the lattice $N^1 \times N^2 \times \ldots \times N^m$ with the usual order (see Topkis, 1998).

[5]Notice that if there are two essential coalitions E, E' of μ that are not comparable , we can use supermodularity to obtain a new optimal matching with $E \vee E'$ and $E \wedge E'$. Sherstyuk (1999) calls such a matching consecutive.

From now on, we concentrate in the square case, since any non-square assortative array could be analyzed by adding null rows of entries at the beginning of the array, to make it square. In this way we preserve supermodularity and the monotonicity conditions.

We give some new features of any square multisided assortative assignment market. To this end, the *central strip* in a square multisided assignment market are those essential coalitions

$$E = (i_1, i_2, \ldots, i_m) \quad \text{such that} \quad \max_{k \in M} i_k - \min_{k \in M} i_k \leq 1.$$

or equivalently those essential coalitions such that

$$E_{\boldsymbol{i-1}} \leq E \leq E_{\boldsymbol{i}} \quad \text{for } i = 2, \ldots, n. \tag{3}$$

Theorem 2.1. *For any square multisided assortative assignment market* $(N^1, N^2, \ldots, N^m; A)$ *we have:*

(a) The main diagonal of the assignment array A is an optimal matching (maybe not unique).

(b) An allocation $x \in \Pi_{k=1}^m \mathbb{R}_+^{n_k}$ belongs to the core $C(w_A)$ if and only if [6]

$$x(E) = a_E \qquad \textit{for all } E = E_{\boldsymbol{1}}, E_{\boldsymbol{2}}, \ldots, E_{\boldsymbol{n}}, \tag{4}$$

$$x(E) \geq a_E \qquad \textit{for all } E \in \Pi_{k=1}^m N^k \textit{ such that}$$

$$E_{\boldsymbol{i-1}} < E < E_{\boldsymbol{i}} \quad \textit{for } i = 2, \ldots, n. \tag{5}$$

(c) At any core allocation $x \in C(w_A)$ we have for all $k \in M$

$$0 \leq x_1^k \leq x_2^k \leq \ldots \leq x_n^k.$$

[6] We denote $E < E'$ for $E \leq E'$ and $E \neq E'$.

Proof. Item (a) follows by our previous comments. To prove (b) notice that the only if part is obvious from the definition of the core. Now assume that $x \in \Pi_{k=1}^{m} \mathbb{R}_{+}^{n_k}$ satisfies (4) and (5). We prove that $x(E) \geq a_E$ for all essential coalitions $E = (i_1, i_2, \ldots, i_m)$ by induction on $r = \max_{k \in M} i_k - \min_{k \in M} i_k$. Assume the induction hypothesis: If E is such that $\max_{k \in M} i_k - \min_{k \in M} i_k \leq r$ then $x(E) \geq a_E$. Notice that for $r = 1$ the inequalities are just (4) and (5). Let $E = (i_1, i_2, \ldots, i_m)$ such that $\max_{k \in M} i_k - \min_{k \in M} i_k = r \geq 2$. Denote $j = 1 + \min_{k \in M} i_k$. Then, by supermodularity, $a_E + a_{E_{\boldsymbol{j}}} \leq a_{E \wedge E_{\boldsymbol{j}}} + a_{E \vee E_{\boldsymbol{j}}}$. Clearly $E \wedge E_{\boldsymbol{j}}$ belongs to the central strip, $E \vee E_{\boldsymbol{j}}$ satisfies the induction hypothesis, and $x(E_{\boldsymbol{j}}) = a_{E_{\boldsymbol{j}}}$. Therefore, $a_E \leq x(E \wedge E_{\boldsymbol{j}}) + x(E \vee E_{\boldsymbol{j}}) - x(E_{\boldsymbol{j}}) = x(E)$. To see (c), assume for instance $x \in C(w_A)$. Then for $i = 1, \ldots, n-1$ we have $x(E_{\boldsymbol{i}}) = a_{E_{\boldsymbol{i}}}$, and take the essential coalition E' given by $(i+1, i, \ldots, i)$. Then we have $\sum_{k=1}^{m} x_i^k = a_{E_{\boldsymbol{i}}}$, and $x_{i+1}^1 + \sum_{k=2}^{m} x_i^k \geq a_{E'}$. Thus, $0 \leq a_{E'} - a_{E_{\boldsymbol{i}}} \leq x_{i+1}^1 - x_i^1$. $\square$

Notice that item (b) means that only the central strip of array A is necessary to determine the core conditions. Item (c) means that in any square assortative market, payoffs in the core are such that for any sector, agents are ranked in the same way.

Remark 2.1. *Looking at the proof of Theorem 2.1, notice that the proof of items (a) and (b) only uses the supermodularity condition (1) of the assignment array.*

Item (c) is implied by the monotonicity condition (2) and the fact that we have an optimal matching in the main diagonal. It could be interesting to know which conditions on the array A characterize the results of the above theorem.

A different proof of item (b) in the supermodular two-sided case can be found in Martínez-de-Albéniz and Rafels (2014). The fact that payoffs to agents in the core are ordered is known for two-sided assortative matrices (see Eriksson et al., 2000).

3. Extreme core allocations

Now we give a simple procedure to obtain all the extreme core points. To this end, for notational convenience we introduce, for any square assortative multisided assignment market, an auxiliary agent 0 for any sector. We denote $E_{\mathbf{0}} = (0, 0, \ldots, 0)$ with $a_{E_{\mathbf{0}}} = 0$ and also for any E such that $E_{\mathbf{0}} < E < E_{\mathbf{1}}$ we denote $a_E = 0$.

A *path* p is a sequence of essential coalitions connecting the initial one $E_{\mathbf{0}}$ with the last one $E_{\boldsymbol{n}}$ passing through all essential coalitions $E_{\mathbf{0}}, E_{\mathbf{1}}, \ldots E_{\boldsymbol{n}}$ where $E_{\boldsymbol{i}} = (i, i, \ldots, i)$ for $i = 0, 1, \ldots, n$. Moreover, between $E_{\boldsymbol{i-1}}$ and $E_{\boldsymbol{i}}$, $i = 1, \ldots, n$, the essential coalitions are such that from one essential coalition to the next one we change the agent of only one sector, moving from agent $i-1$ to agent i. Then path p is

$$p = \left(E_{\mathbf{0}}, \ldots, E_{\mathbf{1}}, \ldots, E_{\boldsymbol{i-1}}, E_i^1, E_i^2, \ldots, E_i^{m-1}, E_{\boldsymbol{i}}, \ldots, E_{\boldsymbol{n}}\right),$$

where $E_{\boldsymbol{i-1}} < E_i^1 < E_i^2 < \ldots < E_i^{m-1} < E_{\boldsymbol{i}}$, for $i = 1, 2, \ldots, n$. As a consequence, these paths are included in the central strip, see (3). Given a path p, notice that each block $E_{\boldsymbol{i-1}} < E_i^1 < E_i^2 < \ldots < E_i^{m-1} < E_{\boldsymbol{i}}$, for $i = 1, 2, \ldots, n$ can also be described by a particular permutation $\theta^i \in \Theta(M)$ indicating the order of the sectors that are sequentially increased. The set of all paths is denoted by $\mathcal{P}_n^m$.

For each path $p \in \mathcal{P}_n^m$ we associate an allocation vector, which we name the p-vector, $x^p \in \Pi_{k=1}^m \mathbb{R}_+^{n_k}$ by solving the linear equations given by all the places of the selected path

$$x^p(E) = a_E \quad \text{for } E \text{ belonging to } p, \tag{6}$$

where we use $(x^p)_0^k = 0$, for $k = 1, \ldots, m$, that is any auxiliary agent 0 gets a null payoff.

For each path p the above linear system has a unique non-negative solution. We prove uniqueness and non-negativeness by induction over n. Firstly notice that if $n = 1$ there are $m!$ different paths between $E_{\mathbf{0}}$ and $E_{\mathbf{1}}$, but vector x^p is $a_{E_{\mathbf{1}}} e_k$ for some $k \in M$ where e_k is the canonical vector. Assume that the solution is unique and non-negative up to $E_{\boldsymbol{i-1}}$, and without loss of generality assume that the next essential coalition E_i^1 of path p is $(i, i-1, \ldots, i-1)$. Then by (6) we have

$$\begin{aligned} \sum_{k=1}^{m} x_{i-1}^k &= a_{E_{\boldsymbol{i-1}}}, \qquad \text{and} \\ x_i^1 + \sum_{k=2}^{m} x_{i-1}^k &= a_{E_i^1}, \end{aligned}$$

where we have dropped the superscript p for the path. Then, using the monotonicity (2) and the induction hypothesis we obtain

$$x_i^1 = x_{i-1}^1 + (a_{E_i^1} - a_{E_{\boldsymbol{i-1}}}) \geq x_{i-1}^1 \geq 0.$$

Therefore for each path $p \in \mathcal{P}_n^m$ we have a unique and non-negative p-vector.

Now let us write $\mathcal{E}xt(C(w_A))$ the set of all extreme core points.[7] We prove next that any extreme core point is linked to a path, that is, there is a correspondence between paths and extreme core points. This is our

[7] If $X \subseteq \mathbb{R}^n$ is a convex set, an element of this convex set $x \in X$ is an extreme point if $x = \frac{1}{2}y + \frac{1}{2}z$ for some $y, z \in X$, then $x = y = z$.

following theorem. In order to show our main result, let us first introduce some notation and prove two technical lemmas.

Lemma 3.1. *Let* $(N^1, N^2, \ldots, N^m; A)$ *be a square multisided assortative assignment market. For any extreme core point* $x \in C(w_A)$ *we have* $x_1^{k^*} = a_{E_1}$ *for some* $k^* \in M$ *and* $x_1^k = 0$ *for all* $k \in M \setminus \{k^*\}$.

Proof. Suppose, on the contrary, that there are two sectors, $k', k'' \in M$ such that $x_1^{k'} > 0$ and $x_1^{k''} > 0$ and define $\varepsilon = \min\{x_1^{k'}, x_1^{k''}\} > 0$. Now define $y, z \in \Pi_{k=1}^m \mathbb{R}_+^{n_k}$ as follows, for $t = 1, \ldots, n$,

$$y_t^k = \begin{cases} x_t^k, & \text{for } k \in M \setminus \{k', k''\}, \\ x_t^{k'} + \varepsilon, & \text{for } k = k', \\ x_t^{k''} - \varepsilon, & \text{for } k = k'', \end{cases} \quad z_t^k = \begin{cases} x_t^k, & \text{for } k \in M \setminus \{k', k''\}, \\ x_t^{k'} - \varepsilon, & \text{for } k = k', \\ x_t^{k''} + \varepsilon, & \text{for } k = k''. \end{cases}$$

Clearly by Theorem 2.1(c) and the definition of ε these are non-negative vectors, and since $y(E) = x(E)$ and $z(E) = x(E)$ for all essential coalitions E, we have $y, z \in C(w_A)$. As a consequence $x = \frac{1}{2}y + \frac{1}{2}z$ with $y \neq x$ and $z \neq x$, getting a contradiction with the fact that x is an extreme core point. □

Now we introduce for any $i \in \{1, 2, \ldots, n\}$ the submarket given by all the first i agents from any sector, and the corresponding restricted array. Formally, that is $(N_i^1, N_i^2, \ldots, N_i^m; A^i)$ where $N_i^k = \{1, \ldots, i\}$ for all $k \in M$ and A^i is given by $A^i = (a_E)_{E \in \Pi_{k=1}^m N_i^k}$. Each of these markets is assortative and an optimal matching is given by the main diagonal when the original market is assortative and square.

Next we relate the extreme core points of these markets with our original square multisided assortative assignment market. To this end, for each $x \in$

$C(w_A)$ we denote by $\bar{x}^i$ the restriction of vector x to the coordinates of $\Pi_{k=1}^m N_i^k$, i.e.

$$\bar{x}^i = (x_1^1, \ldots, x_i^1, x_1^2, \ldots, x_i^2, \ldots, x_1^m, \ldots, x_i^m) \in \Pi_{k=1}^m \mathbb{R}_+^{N_i^k}. \tag{7}$$

Clearly $\bar{x}^i \in C(w_{A^i})$ for all $i \in \{1, 2, \ldots, n\}$ if $x \in C(w_A)$.

In our next lemma we prove that whenever we take an extreme core point of an assortative multisided game we also obtain an extreme core point for all submarkets previously defined.

Lemma 3.2. *Let $(N^1, N^2, \ldots, N^m; A)$ be a square multisided assortative assignment market, and $x \in C(w_A)$ be an extreme core point. Then, $\bar{x}^i \in C(w_{A^i})$ is an extreme core point for $i = 1, \ldots, n-1$.*

Proof. Suppose on the contrary that $i^* \in \{1, \ldots, n-1\}$ is the first index such that $\bar{x}^{i^*}$ is not an extreme point of $C(w_{A^{i^*}})$. By Lemma 3.1, $i^* > 1$.

Since we are assuming $\bar{x}^{i^*} \in C(w_{A^{i^*}})$ but not an extreme core point, there are two points $y^*, z^* \in C(w_{A^{i^*}})$ such that

$$\bar{x}^{i^*} = \frac{1}{2}y^* + \frac{1}{2}z^* \quad \text{with} \quad y^* \neq \bar{x}^{i^*} \quad \text{and} \quad z^* \neq \bar{x}^{i^*}. \tag{8}$$

Notice that for all $i < i^*$ we have $y_i^k = z_i^k = x_i^k$ for all $k \in M$, because the corresponding restriction $\bar{x}^{i^*-1}$ gives an extreme core point.

Now define the following vectors $y, z \in \Pi_{k=1}^m \mathbb{R}^{n_k}$ as follows: for all $k \in M$,

$$y_i^k = \begin{cases} x_i^k, & \text{for } i = 1, \ldots, i^*-1, \\ x_i^k + \varepsilon^k, & \text{for } i = i^*, \ldots, n, \end{cases} \qquad z_i^k = \begin{cases} x_i^k, & \text{for } i = 1, \ldots, i^*-1, \\ x_i^k - \varepsilon^k, & \text{for } i = i^*, \ldots, n. \end{cases}$$

where $\varepsilon^k = (y^*)_{i^*}^k - x_{i^*}^k$ for all $k \in M$. Notice that because of (8), at least one ε^k must be different from zero, and we have $(z^*)_{i^*}^k - x_{i^*}^k = -\varepsilon^k$ for all $k \in M$. Moreover

$$\sum_{k \in M} \varepsilon^k = \sum_{k \in M} (y^*)_{i^*}^k - x_{i^*}^k = y^*(E_{\boldsymbol{i^*}}) - x(E_{\boldsymbol{i^*}}) = a_{E_{\boldsymbol{i^*}}} - a_{E_{\boldsymbol{i^*}}} = 0. \tag{9}$$

We claim $y, z \in C(w_A)$ and $x = \frac{1}{2}y + \frac{1}{2}z$ with $y \neq x$ and $z \neq x$.

Firstly we show $y \geq 0$ and $z \geq 0$. Clearly $y_i^k \geq 0$ and $z_i^k \geq 0$ for $i = 1, \ldots, i^* - 1$, and all $k \in M$. Moreover, for all $k \in M$ we have $y_n^k \geq y_{n-1}^k \geq \ldots \geq y_{i^*}^k$ and $z_n^k \geq z_{n-1}^k \geq \ldots \geq z_{i^*}^k$, and to conclude notice that $y_{i^*}^k = x_{i^*}^k + \varepsilon^k = (y^*)_{i^*}^k \geq 0$ and also $z_{i^*}^k \geq 0$.

Secondly, $y(E_{\boldsymbol{i}}) = a_{E_{\boldsymbol{i}}},\ z(E_{\boldsymbol{i}}) = a_{E_{\boldsymbol{i}}}$ for $i = 1, \ldots, n$, by their definitions.

Finally, we show that $y(E) \geq a_E$ and $z(E) \geq a_E$ for all essential coalitions E in the central strip. For all essential coalitions in the central strip such that $E_{\boldsymbol{i^*}} \leq E$, by (9) $y(E) = x(E) + \sum_{k \in M} \varepsilon^k = x(E) \geq a_E$ and analogously $z(E) \geq a_E$. By its definition $y(E) = z(E) = x(E) \geq a_E$ for all essential coalitions E, in the central strip such that $E \leq E_{\boldsymbol{i^*-1}}$. For the case $E_{\boldsymbol{i^*-1}} < E < E_{\boldsymbol{i^*}}$, we claim that $y(E) = y^*(E)$ and $z(E) = z^*(E)$, since we have that $y_{i^*}^k = (y^*)_{i^*}^k$ and $z_{i^*}^k = (z^*)_{i^*}^k$ for all $k \in M$.

By Theorem 2.1(b) we have $y, z \in C(w_A)$ and $x = \frac{1}{2}y + \frac{1}{2}z$ with $y \neq x$ and $z \neq x$, contradicting x is an extreme core point. $\square$

These two lemmas allow to establish our main theorem.

Theorem 3.1. *Let $(N^1, N^2, \ldots, N^m; A)$ be a square multisided assortative assignment market. In it, p-vectors coincide with extreme core points, i.e.*

$$\mathcal{E}xt(C(w_A)) = \{x^p\}_{p \in \mathcal{P}_n^m}.$$

Proof. We prove first that for all path $p \in \mathcal{P}_n^m$ we have $x^p \in C(w_A)$. To this end we prove $x^p(E) \geq a_E$ for all $E_{\boldsymbol{i-1}} < E < E_{\boldsymbol{i}}$ for all $i = 1, \ldots, n$. By Theorem 2.1(b) this is enough to justify $x^p \in C(w_A)$.

Without loss of generality we assume that the essential coalitions of path p between $E_{\boldsymbol{i-1}}$ and $E_{\boldsymbol{i}}, i = 1, \ldots n$, are given by

$$E_{\boldsymbol{i-1}}, (i, i-1, \ldots, i-1), (i, i, i-1, \ldots, i-1), \ldots, E_{\boldsymbol{i}}$$

that is, they follow the natural order of sectors, first moves the first sector, second the second sector and so forth. We denote by $E_i^t = (i, \ldots, i, \underbrace{i}^{t}, i - 1, i-1, \ldots, i-1), 1 \le t \le m-1$, the essential coalition in the previous path such that t is the position of the last i agent, $i = 1, \ldots, n$. As a matter of notation, $E_i^0 = E_{\boldsymbol{i-1}}$ and $E_i^m = E_{\boldsymbol{i}}$.

Given any essential coalition $E = (i_1, i_2, \ldots, i_m)$ with $E_{\boldsymbol{i-1}} < E < E_{\boldsymbol{i}},\ i = 1, \ldots, n$, we define $r(E) = \#\{k \mid i_k = i\}$, the number of i agents in the essential coalition E. Now, we prove $x^p(E) \ge a_E$ with $E_{\boldsymbol{i-1}} < E < E_{\boldsymbol{i}}$ by induction on the number $r(E)$. Clearly $1 \le r(E) \le m-1$. If $r(E) = 1$ let l be the position of the only i. If $l = 1$ there is nothing to prove, and if $l > 1$ notice that $E \wedge E_i^{l-1} = E_{\boldsymbol{i-1}}$ and $E \vee E_i^{l-1} = E_i^l$. Therefore, by supermodularity and the way essential coalitions of path p have been chosen, $a_E + a_{E_i^{l-1}} \le a_{E_{\boldsymbol{i-1}}} + a_{E_i^l}$, and then $a_E + x^p(E_i^{l-1}) \le x^p(E_{\boldsymbol{i-1}}) + x^p(E_i^l)$. Now clearly $x^p(E) \ge a_E$. Assume our induction hypothesis is true up to $r-1$ and let E be such that $r(E) = r$. There are then r positions with agent i and let l be the last of these positions. Then $a_E + a_{E_i^{l-1}} \le a_{E \wedge E_i^{l-1}} + a_{E_i^l}$, by supermodularity. We can apply the induction hypothesis to $E \wedge E_i^{l-1}$ since it has $r(E \wedge E_i^{l-1}) = r - 1$ positions with an i. Now $a_E + x(E_i^{l-1}) \le a_{E \wedge E_i^{l-1}} + a_{E_i^l} \le x(E \wedge E_i^{l-1}) + x(E_i^l)$, and therefore $a_E \le x(E)$ to finish with this part of the proof.

Moreover, vector x^p for $p \in \mathcal{P}_n^m$ is an extreme core point. To see it, just notice that if it were the midpoint of two other core points, these core points must satisfy with equality all the entries of path p. By uniqueness of the solution, they coincide with x^p. We have established that each path gives an extreme core point.

Now we prove that any extreme core point is associated to some path.

Let $x \in C(w_A)$ be an extreme core point. Then by Lemma 3.2, $\bar{x}^i$ is also an extreme core point of $C(w_{A^i})$ for all $i \in \{1, \ldots, n\}$, see (7) for notations.

Suppose on the contrary that x is not a p-vector for any path $p \in \mathcal{P}_n^m$, and let $i^* \in \{1, \ldots, n\}$ be the first index such that $\bar{x}^{i^*}$ is not a p-vector for any $p \in \mathcal{P}_{i^*}^m$. Notice that $|N_{i^*}^1| = |N_{i^*}^2| = \ldots = |N_{i^*}^m| = i^*$.

Clearly, by Lemma 3.1, $i^* > 1$ since any path between $E_{\mathbf{0}}$ and $E_{\mathbf{1}}$ gives $a_{E_{\mathbf{1}}}$ to some agent and zero to the others. Vector $\bar{x}^{i^*-1}$ is a p-vector for some path $p_{i^*-1} \in \mathcal{P}_{i^*-1}^m$ and consider the set of paths in $\mathcal{P}_{i^*}^m$ that coincide with p_{i^*-1} for all essential coalitions in the central strip $E \leq E_{\boldsymbol{i^*-1}}$. Denote this set by $\mathcal{B}_{i^*}$.

Consider now the set given by convex hull of the p-vectors corresponding to paths in $\mathcal{B}_{i^*}$, that is $Conv\{x^p\}_{p \in \mathcal{B}_{i^*}}$. This is a non-empty, compact and convex set and clearly vector $\bar{x}^{i^*}$ cannot be a convex combination of these core points $\{x^p\}_{p \in \mathcal{B}_{i^*}}$. Then we can apply the separating hyperplane theorem (see Boyd and Vandenberghe, 2004) to this point and set. Therefore there exists vector

$$r = (r_1^1, r_2^1, \ldots, r_{i^*}^1, r_1^2, r_2^2, \ldots, r_{i^*}^2, \ldots, r_1^m, r_2^m, \ldots, r_{i^*}^m) \in \Pi_{k=1}^m \mathbb{R}^{n_{i^*}^k}$$

such that

$$r \cdot \bar{x}^{i^*} < r \cdot x^p \quad \text{for all} \quad p \in \mathcal{B}_{i^*}. \tag{10}$$

Let $\theta \in \Theta(M)$ be an ordering of sectors M such that $r_{i^*}^{\theta(1)} \geq r_{i^*}^{\theta(2)} \geq \ldots \geq r_{i^*}^{\theta(m)}$, and define the following sequence of sets: $S_0 = \emptyset, S_1 = \{\theta(1)\}, S_2 = \{\theta(1), \theta(2)\}, \ldots, S_m = M$.

For each $S \subseteq M$ we associate the corresponding essential coalition

$$E^S = (i_1, i_2, \ldots, i_m) \quad \text{with} \quad i_k = i^* \text{ if } \; k \in S \quad \text{and} \quad i_k = i^* - 1 \text{ if } \; k \notin S.$$

Notice that $E^{S_0} = E_{\boldsymbol{i^*-1}} = (i^*-1, i^*-1, \ldots, i^*-1)$ and $E^{S_m} = E_{\boldsymbol{i^*}} = (i^*, i^*, \ldots, i^*)$ and take a path $\bar{p} \in \mathcal{B}_{i^*}$ such that $E^{S_1}, E^{S_2}, \ldots, E^{S_{m-1}}$ are the essential coalitions of the path $\bar{p}$ between $E_{\boldsymbol{i^*-1}}$ and $E_{\boldsymbol{i^*}}$. Then the p-vector associated to the above path $\bar{p} \in \mathcal{B}_{i^*}$ satisfies

$$x^{\bar{p}}(E^{S_k}) = a_{E^{S_k}} \quad \text{for} \quad k = 0, 1, \ldots, m. \tag{11}$$

The previous system (11) gives

$$(x^{\bar{p}})_{i^*}^{\theta(k)} = (x^{\bar{p}})_{i^*-1}^{\theta(k)} + a_{E^{S_k}} - a_{E^{S_{k-1}}} \text{ for } \quad k = 1, 2, \ldots, m.$$

By construction of path $\bar{p}$ we have that

$$(x^{\bar{p}})_i^k = (\bar{x}^{i^*})_i^k = x_i^k \quad \text{for} \quad 1 \leq i \leq i^* - 1 \quad \text{and all} \quad k \in M. \tag{12}$$

Now,

$$
\begin{aligned}
r\cdot x^{\bar p} &= \sum_{k=1}^{m}\sum_{i=1}^{i^*-1} r_i^k\cdot (x^{\bar p})_i^k + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot (x^{\bar p})_{i^*}^{\theta(k)}\\
&= \sum_{k=1}^{m}\sum_{i=1}^{i^*-1} r_i^k\cdot x_i^k + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot \left((x^{\bar p})_{i^*-1}^{\theta(k)} + a_{E^{S_k}} - a_{E^{S_{k-1}}}\right)\\
&= \sum_{k=1}^{m}\sum_{i=1}^{i^*-1} r_i^k\cdot x_i^k + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot (x^{\bar p})_{i^*-1}^{\theta(k)}\\
&\quad + \sum_{k=1}^{m-1}\left(r_{i^*}^{\theta(k)} - r_{i^*}^{\theta(k+1)}\right)\cdot a_{E^{S_k}} - r_{i^*}^{\theta(1)}\cdot a_{E^{S_0}} + r_{i^*}^{\theta(m)}\cdot a_{E^{S_m}}\\
&\le \sum_{k=1}^{m}\sum_{i=1}^{i^*-1} r_i^k\cdot x_i^k + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot (x^{\bar p})_{i^*-1}^{\theta(k)}\\
&\quad + \sum_{k=1}^{m-1}\left(r_{i^*}^{\theta(k)} - r_{i^*}^{\theta(k+1)}\right)\cdot x(E^{S_k}) - r_{i^*}^{\theta(1)}\cdot x(E^{S_0}) + r_{i^*}^{\theta(m)}\cdot x(E^{S_m})\\
&= \sum_{k=1}^{m}\sum_{i=1}^{i^*-1} r_i^k\cdot x_i^k + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot (x^{\bar p})_{i^*-1}^{\theta(k)} + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot \left(x(E^{S_k}) - x(E^{S_{k-1}})\right)\\
&= \sum_{k=1}^{m}\sum_{i=1}^{i^*-1} r_i^k\cdot x_i^k + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot (x^{\bar p})_{i^*-1}^{\theta(k)} + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot x_{i^*}^{\theta(k)} - \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot x_{i^*-1}^{\theta(k)}\\
&= r\cdot \bar x^{i^*} + \sum_{k=1}^{m} r_{i^*}^{\theta(k)}\cdot\left((x^{\bar p})_{i^*-1}^{\theta(k)} - x_{i^*-1}^{\theta(k)}\right) = r\cdot \bar x^{i^*},
\end{aligned}
$$

where the inequality comes from $x\in C(w_A)$ and the fact that $r_{i^*}^{\theta(k)} - r_{i^*}^{\theta(k+1)} \ge 0$ for $k=1,\dots,m-1$, and the last equality by (12).

We have reached a contradiction with (10). Consequently any extreme core point is a p-vector. □

Once we have established the main result of the paper, we move to some related questions. We have just proved that paths from $E_{\mathbf{0}}$ to $E_{\boldsymbol{n}}$ characterize the extreme core allocations of any square assortative multisided assignment market. We discuss now which is the maximum number of extreme core allocations.

Remark 3.1. *For an arbitrary square assortative multisided game with m sectors and n agents in each sector, the maximum number of extreme core allocations is*

$$m \cdot (m!)^{n-1}. \tag{13}$$

To justify the above Remark, notice that, as any path is composed of n subpaths, one for each subpart from $E_{\boldsymbol{i-1}}$ to $E_{\boldsymbol{i}}$, for $i = 1, \ldots, n$, we easily obtain that the total number of paths from $E_{\mathbf{0}}$ to $E_{\boldsymbol{n}}$ is given by $(m!)^n$. Since we are interested in counting how many extreme core allocations, we have to take into account that at the beginning of any path, that is, from $E_{\mathbf{0}}$ to $E_{\mathbf{1}}$, only m different allocations are possible. At this part $m!$ paths collapse at most into m different vectors, precisely those vectors where the worth $a_{E_{\mathbf{1}}}$ is allocated to a particular agent and give a zero payoff to the rest of agents, see Lemma 3.1. By all these arguments, formula (13) is justified.

For the special case in which array A satisfies[8]

$$a_E + a_{E'} = a_{E \vee E'} + a_{E \wedge E'} \qquad \text{for any essential coalitions } E, E', \tag{14}$$

the formula (13) reduces to m if $a_{E_{\mathbf{1}}} > 0$ or to 1 if $a_{E_{\mathbf{1}}} = 0$.

As a numerical illustration, take the following $2 \times 2 \times 2$ array A, with $N^k = \left\{1^k, 2^k\right\}$ for $k = 1, 2, 3$, which is a valuation array,

$$A = \begin{pmatrix} 10 & 11 \\ 12 & 13 \end{pmatrix} \quad \begin{pmatrix} 14 & 15 \\ 16 & 17 \end{pmatrix}.$$

[8]These are supermodular and submodular arrays, and they are called *valuation* arrays.

In it the rows correspond to agents in the first sector, columns to agents in the second sector and matrices to agents in the third sector. Then, for example, $a_{(1,2,2)} = 15$. Its extreme core allocations are

$$x_1 = (10, 12; 0, 1; 0, 4),$$
$$x_2 = (0, 2; 10, 11; 0, 4),$$
$$x_3 = (0, 2; 0, 1; 10, 14).$$

They can be computed by applying the p-vectors mechanism. For instance, vector x_1 is obtained by path

$$p = (E_{\mathbf{0}}, (0, 0, 1), (0, 1, 1), E_{\mathbf{1}}, (1, 2, 1), (1, 2, 2), E_{\mathbf{2}}).$$

Notice that to apply the mechanism we have to check the monotonicity condition (2), not implied by the fact that the array is a valuation.

Moreover any square valuation array A, monotonic or not, is *fully-optimal* in the sense that all its matchings are optimal, i.e. $\mathcal{M}_A^*\left(N^1, \ldots, N^m\right) = \mathcal{M}\left(N^1, \ldots, N^m\right)$. Any pair of non-comparable essential coalitions E, E' in any matching can be changed by $E \vee E'$ and $E \wedge E'$ without losing efficiency. The converse is not true,[9] as the next example shows. The $2 \times 2 \times 2$ array A,

$$A = \begin{pmatrix} 3 & 6 \\ 6 & 6 \end{pmatrix} \quad \begin{pmatrix} 6 & 6 \\ 6 & 9 \end{pmatrix}$$

is a fully-optimal multisided assignment matrix, but not a valuation, since $12 = a_{(1,1,2)} + a_{(2,1,1)} > a_{(1,1,1)} + a_{(2,1,2)} = 3 + 6 = 9$.

Moreover, it has an empty core, since being a fully-optimal matrix, any core allocation must satisfy with equality all the array' entries, but, as the reader can check, they form a non-compatible linear system of equations.

[9] For two-sided square assignment matrices, valuation and fully-optimal are equivalent.

Another important feature of a valuation array is that its entries can always be arranged monotonically by a suitable permutation of the agents. Therefore they can be seen as assortative markets. A way to see which permutation is suitable is the following. Take any core element and from it derive a permutation of agents in each sector such that arranges the components in a non-decreasing way. Notice that this core element satisfies with equality all entries in the array. In this way we obtain an assortative array, that is, where the monotonicity property also holds. As a consequence we can apply our results to any square valuation array. This fact simplifies the assertions made in Sherstyuk (1999) since there is no need to distinguish valuation markets from assortative ones.

It is easy to generate examples in which the maximum number of extreme core points given in (13) is attained.

Example 3.1. *Consider the following* $2 \times 2 \times 2$ *array* A,

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 7 \\ 5 & 10 \end{pmatrix}.$$

Notice that all inequalities of the supermodular property concerning non-comparable essential coalitions are strict. There are $3 \cdot (3!)^1 = 18$ *different* *extreme core allocations, that correspond to different paths.*

$$\begin{array}{rclrcl}
x_1 & = & (1,2;0,2;0,6), & x_2 & = & (0,1;1,3;0,6), \\
\mathbf{x_3} & = & (\mathbf{0},\mathbf{1};\mathbf{0},\mathbf{2};\mathbf{1},\mathbf{7}), & x_4 & = & (1,2;0,5;0,3), \\
\mathbf{x_5} & = & (\mathbf{0},\mathbf{1};\mathbf{1},\mathbf{6};\mathbf{0},\mathbf{3}), & x_6 & = & (0,1;0,5;1,4), \\
x_7 & = & (1,3;0,1;0,6), & x_8 & = & (0,2;1,2;0,6), \\
\mathbf{x_9} & = & (\mathbf{0},\mathbf{2};\mathbf{0},\mathbf{1};\mathbf{1},\mathbf{7}), & \mathbf{x_{10}} & = & (\mathbf{1},\mathbf{4};\mathbf{0},\mathbf{1};\mathbf{0},\mathbf{5}), \\
x_{11} & = & (0,3;1,2;0,5), & x_{12} & = & (0,3;0,1;1,6), \\
x_{13} & = & (1,3;0,5;0,2), & \mathbf{x_{14}} & = & (\mathbf{0},\mathbf{2};\mathbf{1},\mathbf{6};\mathbf{0},\mathbf{2}), \\
x_{15} & = & (0,2;0,5;1,3), & \mathbf{x_{16}} & = & (\mathbf{1},\mathbf{4};\mathbf{0},\mathbf{4};\mathbf{0},\mathbf{2}), \\
x_{17} & = & (0,3;1,5;0,2), & x_{18} & = & (0,3;0,4;1,3).
\end{array}$$

The six vectors in boldface correspond to the $m! = 3! = 6$ *vectors given in Sherstyuk (1999). They are those in which all agents from a sector get their maximum payoff simultaneously, as was pointed out by Sherstyuk.*

We already know that properties valid for bilateral assignment games do not transfer to multisided assignment games. Now, and since we have a description of all the extreme core allocations for assortative multisided assignment games, it is relevant to discuss the extension of some properties.

Firstly, Núñez and Solymosi (2017) characterize for any bilateral assignment game the extreme core allocations as the σ-lemaral vectors.[10] This property does not hold for general multisided assignment games as the reader may check in the following example.

[10] Lemaral vectors for a game (N, v) are defined as follows: for any permutation σ of the agents, the components of the σ-lemaral vector $\bar{r}^{\sigma,v} \in \mathbb{R}^N$ are given recursively by $\bar{r}^{\sigma,v}_{\sigma_i} = \min\left\{v^*(Q \cup \{\sigma_i\}) - \bar{r}^{\sigma,v}(Q) \;:\; Q \subseteq P^{\sigma}_{\sigma_i}\right\}$, with $v^*(S) = v(N) - v(N \setminus S)$ and $P^{\sigma}_{\sigma_i}$ the set of predecessors of player σ_i in the permutation σ.

Example 3.2. *Consider the following* $2 \times 2 \times 2$ *array* A,

$$A = \begin{pmatrix} \mathbf{4} & 6 \\ 3 & 5 \end{pmatrix} \quad \begin{pmatrix} 5 & 3 \\ 2 & \mathbf{6} \end{pmatrix}.$$

Now consider the permutation $\sigma = (2^1, 1^1, 1^3, 1^2, 2^2, 2^3)$ *of all agents, where agent* 2 *of the first sector enters first and agent 1 of the first sector enters second, and so forth. We have indicated an optimal matching in boldface. The lemaral associated to permutation* σ *pays to agent 2 of the first sector its marginal contribution to the grand coalition* $w_A(N) - w_A(N \setminus \{2^1\}) = 4$, *and so forth. The* σ*-lemaral is* $\bar{r}^{\sigma, w_A} = (4, 4; 0, 1; 0, 0)$, *which is not efficient, and then it does not belong to the core.*

Notice that the array in Example 3.2 is not assortative. It is an open question if lemaral vectors and extreme core allocations will coincide for square assortative arrays.

Lastly, Hamers et al. (2002) introduces the CoMa-Property and proves that it is satisfied by bilateral assignment games. Essentially it means that any extreme core allocation is a marginal worth vector.[11] This property does not hold for multisided assignment games. Indeed, as the reader may check, vector $x_2 = (0, 1; 1, 3; 0, 6)$ of our previous Example 3.1 is not a marginal worth vector.

[1] Atay, A., Núñez, M., 2019. Multi-sided assignment games on m-partite graphs. Annals of Operations Research 279, 271–290.

[11] A marginal worth vector associated to a permutation of agents is obtained by giving to an agent what he/she can obtain by joining his/her predecessors minus what they have got previously (see Shapley, 1971).

[2] Becker, G.S., 1973. A Theory of Marriage: Part I. The Journal of Political Economy 81(4), 813–846.

[3] Boyd, S., Vandenberghe, L., 2004. Convex optimization. Cambridge University Press, Cambridge, U.K.

[4] Crawford, V.P., Knoer, E.M., 1981. Job matching with heterogeneous firms and workers. Econometrica, 49(2), 437–450.

[5] Eriksson, K., Karlander, J., Öller, L.-E., 2000. Becker's assortative assignments: stability and fairness. Mathematical Social Sciences 39, 109–118.

[6] Garey, M.R., Johnson, D.S., 1979. Computers and intractability: a guide to the theory of NP-completeness. W.H. Freeman. New York.

[7] Hamers, H.J.M., Klijn, F., Solymosi, T., Tijs, S.H., Villar, J.P., 2002. Assignment games satisfy the CoMa property. Games and Economic Behavior 38(2), 231–239.

[8] Kaneko, M., Wooders, M.H., 1982. Cores of partitioning games. Mathematical Social Sciences 3(4), 313–327.

[9] Martínez-de-Albéniz, F.J., Rafels, C., 2014. Cooperative assignment games with the inverse Monge property. Discrete Applied Mathematics 162, 42–50.

[10] Martínez-de-Albéniz, F.J., Rafels, C., Ybern, N., 2019. Solving Becker's assortative assignments and extensions, Games and Economic Behavior 113, 248–261.

[11] Núñez, M., Solymosi, T., 2017. Lexicographic allocations and extreme core payoffs: the case of assignment games. Annals of Operations Research 254, 211–234.

[12] Quint, T., 1991. The core of an m-sided assignment game. Games and Economic Behavior 3, 487–503.

[13] Shapley, L.S., 1955. Markets as cooperative games. Rand Research Memorandum P–629. Rand Corp., Santa Monica, pp. 1–7.

[14] Shapley, L.S., Shubik, M., 1971. The assignment game I: the core. International Journal of Game Theory 1, 111–130.

[15] Shapley, L.S., 1971. Cores of convex games. International Journal of Game Theory 1, 11–26.

[16] Sherstyuk, K., 1999. Multisided matching games with complementarities. International Journal of Game Theory 28, 489–509.

[17] Stuart, H.W., 1997. The supplier–firm–buyer game and its m-sided generalization. Mathematical Social Sciences 34(1), 21–27.

[18] Tejada, O., 2013. Analysis of the core of multisided Böhm-Bawerk assignment markets. TOP 21(1), 189–205.

[19] Tejada, O., Núñez, M., 2012. The nucleolus and the core-center of multi-sided Böhm-Bawerk assignment markets. Mathematical Methods of Operations Research 75(2), 199–220.

[20] Topkis, D. M. (1998). Supermodularity and complementarity. Princeton University Press.