Expected Shortfall computation with multiple control variates

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Abstract

In this work we derive an exact formula to calculate the Expected Shortfall (ES) value for the one-factor delta-gamma approach which, to the best of our knowledge, was still missing in the literature. We then use the one-factor delta-gamma as a control variate to estimate the ES of the multi-factor delta-gamma approach. A one-factor delta-gamma approximation is used for each risk factor appearing in the problem. Since the expected values of control variates are computed by means of an exact formula, the additional effort of computation with respect to the naive estimator of the multi-factor delta-gamma can be neglected. With this method, we achieve a considerable reduction of the variance. We have established a theorem to prove that the variance is further reduced when we use all the risk factors instead of just some of them. We show that one of the main potential applications takes place in the insurance industry regulation within the Swiss solvency test framework. We perform a model risk analysis and illustrate these results with numerical experiments.

Key words. Swiss solvency test, market risk, nonlinear portfolio, delta-gamma approximation, Expected Shortfall, exact formula, control variates

AMS subject classifications. 62P05, 65C05, 65C50

1 Introduction

A classical but still important problem in market risk management is the estimation of a profit and loss distribution of a portfolio over a specified time horizon and the associated risk measures. Value-at-Risk (VaR) has become an important measure for estimating and managing portfolio market risk. VaR is defined as a certain quantile of the change in value of a portfolio during a specified holding period. While the basic concept of VaR is simple, many complications can arise in practical use. An important complication is caused by nonlinearity in the portfolio payoff structure. This problem arises for all portfolios that include assets with nonlinear payoffs, such as option positions. For such nonlinear portfolios, VaR cannot be computed directly from a risk factor distribution. Instead, the risk factor distribution first needs to be converted into a profit and loss distribution for the portfolio. VaR is then computed from this profit and loss distribution. The Basel Committee of Banking Supervision initiated a fundamental review of the trading book regime (see [2, 3]), beginning with an assessment of those things that went wrong during the financial turmoil. The revised standards for minimum capital requirements for market risk were established in [4]. The Committee has focused, among other things, on moving from VaR to the ES for measuring the portfolio risk. ES is a coherent measure (in the sense of [1]) that takes into account the tail of the portfolio loss distribution beyond the VaR value.

Monte Carlo (MC) simulation is typically used to calculate these risk measures, first simulating changes in the risk factors of a portfolio, then the portfolio is evaluated at each new price and the change in value of the portfolio is estimated. This method is known as *full revaluation*. However, accurate VaR and ES estimates with a full revaluation method, are obtained at the cost of a considerable computational effort, since there might be a large number of instruments in the portfolio and when the confidence level is high, a large number of simulations may be required to obtain accurate estimates of the tail probability.

In the present work, we adopt the *delta-gamma* approximation [6, 11, 18] as an alternative approach to the full revaluation method. This approach is based on the assumption that the change in portfolio value is a quadratic function of the changes in the risk factors. This method is sometimes called partial MC, since the only part of the procedure that involves simulation is the one related to the price change. This fact makes the delta-gamma method much more competitive from the computational standpoint. The boost in terms of CPU time is at the cost of less accurate values when measuring the risk, since it is a second order Taylor approximation to the true change in portfolio value (the suitability of the delta-gamma approach is studied in [21]). Typically, the changes in the risk factors are assumed to be normally distributed. The assumption of normality is found, for instance, in the insurance industry. As pointed out by [15, 16, 19], the standard market model of the Swiss solvency test (SST), in force since 2011, defines the capital required by a Swiss insurance company to absorb negative financial scenarios. The capital requirement imposed by the Swiss financial market supervisory authority (FINMA) corresponds to the ES calculated at 99% confidence level by means of a delta-gamma approach where the risk factors follow a multivariate normal distribution, and this computation is generally done by means of Monte Carlo simulation. Thus, the possibility of rare events such as financial crises are ignored under the multivariate normal framework. As stated in [19], the FINMA has amended the model to take into account the possibility of exceptionally high losses in financial markets, so we consider this model as well in the present work. Other potential applications of the delta-gamma approach appear in counterparty risk, and more precisely, to speed up the calculation of initial margin payments as suggested by [20]. It has also been employed to compute the VaR value of straddles, strangles and spread options in [7].

The delta-gamma method has also been studied in [8, 19] and [21] within the context of Fourier inversion methods, avoiding this way MC simulation. The probability density function (PDF) and the cumulative distribution function (CDF) are recovered through Fourier inversion from the characteristic function of the random variable representing the change in value of the portfolio. However, as pointed out in [8], for certain holding periods the risk measures are difficult to estimate by means of Fourier inversion, since numerical errors may hamper the accurate estimation of the PDF. Further, when a numerical method is employed, then some parameters have to be fixed, and that is generally a trial and error problem. The implementation of a MC method is usually a more simple task, and this is why simulation is often preferred by financial companies.

The main contributions of this work within the delta-gamma framework are the following.

- We derive an exact formula to calculate the ES value for the one-factor delta-gamma approach under the assumption of normal changes in the risk factor. While the density function of the one-factor delta-gamma approach was already known in closed-form (see for instance [13] or [21]), an exact formula for estimating the ES value was still missing. We therefore overcome the numerical problems stated before in the presence of a single risk factor when using Fourier inversion.

- We extend the one-factor exact formula to the one-factor SST framework. We provide an exact formula for the ES where extreme scenarios are taken into account, and we give the mathematical expression that relates the VaR value with and without extreme scenarios.

- We propose a conservative estimation of the ES for separable and multivariate portfolios. This is achieved by means of the exact formula and the subadditivity property of the ES. For those portfolios involving only one asset, like for instance the trading strategies described in [17], the exact formula can be applied straightforwardly.

- We use the one-factor delta-gamma as a control variate to estimate the ES of the multi-factor delta-gamma approach in the normal case as well as when extreme scenarios are considered. A one-factor delta-gamma approximation is used for each risk factor appearing in the financial context. Since the expected values of control variates are computed by means of an exact formula, the additional effort of computation with respect to the naive estimator of the multi-factor delta-gamma can be neglected. We achieve a considerable variance reduction factor (VRF). This fact will be shown in the numerical experiments. It is worth remarking that the closed-form formula given for the univariate case is a key aspect for the successful application of the variance reduction method put forward in this work.

- When we use a control variates technique in the context of multi-factor delta-gamma, we can potentially include as many control variates as risk factors appearing in the problem. So we have to make a decision on which factor(s) should be used. We prove in Theorem 1 that the variance is reduced (or at least equal) when we use all the risk factors as control variates. To the best of our knowledge, this is a new contribution to the literature. We illustrate this result in the numerical experiments part.

Other authors have used control variates with a rather different approach in the context of portfolio losses for computing the VaR value (see for instance [11]) or valuation of derivatives (like [9] for basket and Asian options).

The paper is organized as follows. We introduce the delta-gamma approach in Section 1.1. We derive an exact formula for a non-central chi-squared distribution with one degree of freedom in Section 2. That formula will be used in Section 3 to obtain an exact formula for the ES under the one-factor delta-gamma approach. We extend the computation of the ES with an exact formula to the SST model in Section 4. The problem of multiple control variates is put forward in Section 5, and the numerical experiments, along with an analysis of model risk, are presented in Section 6. Finally, Section 7 concludes.

1.1 The delta-gamma approach

Suppose the current value of a portfolio is $V(t)$, the holding period is Δt, and the value of the portfolio at time $t+\Delta t$ is $V(t+\Delta t)$. The change in the portfolio value during the holding period is ΔV, where $\Delta V = V(t+\Delta t) - V(t)$. The VaR value of ΔV at a confidence level α, is given by q_α, where,

$$\mathbb{P}(\Delta V < q_\alpha) = \alpha.$$

In practice, Δt ranges from one day to two weeks and $\alpha \in (0,1)$ is close to 1. By definition, the ES risk measure at confidence level α is given by,

$$\mathrm{ES}_\alpha(\Delta V) = \mathbb{E}\left(\Delta V | \Delta V > q_\alpha\right),$$

or, alternatively,

$$\mathrm{ES}_\alpha(\Delta V) = \frac{1}{1-\alpha}\int_{q_\alpha}^{+\infty} x f_{\Delta V}(x)dx, \tag{1}$$

where $f_{\Delta V}$ is the PDF of ΔV.

We assume that there are p risk factors and that $S(t) = (S_1(t), \ldots, S_p(t))^T$ denotes the value of these factors at time t. Define $\Delta S = S(t+\Delta t) - S(t)$ to be the change in the risk factors during the interval $[t, t+\Delta t]$. Then, the *delta-gamma approximation* is given by,

$$\Delta V \simeq \Delta V_\gamma = \Theta \Delta t + \delta^T \Delta S + \frac{1}{2}\Delta S^T \Gamma \Delta S, \tag{2}$$

where $\Theta = \frac{\partial V}{\partial t}, \delta_i = \frac{\partial V}{\partial S_i}$, $\Gamma_{i,j} = \frac{\partial^2 V}{\partial S_i \partial S_j}$, are called the Greeks, and all partial derivatives are being evaluated at $S(t)$. Further, ΔS is governed by a normal distribution with mean zero and covariance matrix Σ. Observe that in the univariate case ($p=1$), we have,

$$\Delta V_\gamma = \sum_{i=1}^{n} x_i \frac{\partial v_i}{\partial t}\Delta t + \sum_{i=1}^{n} x_i \frac{\partial v_i}{\partial S}\Delta S + \frac{1}{2}\sum_{i=1}^{n} x_i \frac{\partial^2 v_i}{\partial S^2}(\Delta S)^2, \tag{3}$$

where n represents the number of assets in the portfolio, x_i is the amount of asset i and v_i the value of asset i. In this particular case,

$$\Theta = \sum_{i=1}^{n} x_i \frac{\partial v_i}{\partial t}, \ \delta = \sum_{i=1}^{n} x_i \frac{\partial v_i}{\partial S}, \ \Gamma = \sum_{i=1}^{n} x_i \frac{\partial^2 v_i}{\partial S^2}.$$

2 Non-central chi-squared distribution with one degree of freedom

We devote this section to the study of the non-central chi-squared distributions with one degree of freedom, since as we will see in the next section, they are the building blocks of the delta-gamma approach. More precisely, if X is a random variable that follows a non-central chi-squared distribution with one degree of freedom and non-centrality parameter ζ, we will compute its CDF, its VaR value and give a closed-form expression for the ES. The starting point is the PDF of X, given by (see [14] for details),

$$f_\zeta(x) = \frac{e^{-\frac{1}{2}(x+\zeta)}}{\sqrt{2\pi x}} \cosh\left(\sqrt{\zeta x}\right), \tag{4}$$

where,

$$\cosh\left(\sqrt{\zeta x}\right) = \frac{e^{\sqrt{\zeta x}} + e^{-\sqrt{\zeta x}}}{2}, \quad x \in (0, +\infty), \zeta > 0.$$

We can derive the CDF of X,

$$F_\zeta(x) = \int_0^x f_\zeta(y)dy, \quad x \in (0, +\infty), \zeta > 0,$$

by integrating,

$$F_\zeta(x) = \int_0^x \frac{e^{-\frac{1}{2}(y+\zeta)}}{\sqrt{2\pi y}} \cdot \frac{e^{\sqrt{\zeta y}} + e^{-\sqrt{\zeta y}}}{2} dy, \quad x \in (0, +\infty), \zeta > 0.$$

If we make the change of variables $y = t^2$ we obtain,

$$F_\zeta(x) = \Phi\left(\sqrt{x} - \sqrt{\zeta}\right) + \Phi\left(\sqrt{x} + \sqrt{\zeta}\right) - 1, \quad x \in (0, +\infty), \zeta > 0,$$

or, alternatively,

$$F_\zeta(x) = \Phi\left(\sqrt{x} - \sqrt{\zeta}\right) - \Phi\left(-\sqrt{x} - \sqrt{\zeta}\right), \quad x \in (0, +\infty), \zeta > 0,$$

where Φ is the CDF of the standard normal distribution.

Let q_α^ζ be the VaR value of X calculated at the confidence level $\alpha \in (0, 1)$, this is $F_\zeta(q_\alpha^\zeta) = \alpha$. We derive a closed-form formula for the ES of X by means of its density (4) and expression (1),

$$\text{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_{q_\alpha^\zeta}^{+\infty} x f_\zeta(x) dx.$$

Proposition 1. *The ES at confidence level α of a non-central chi-squared random variable X with one degree of freedom and non-centrality parameter ζ is,*

$$ES_\alpha(X) = \frac{1}{1-\alpha}\left[(\zeta+1)(1-\alpha) + \phi\left(\sqrt{q_\alpha^\zeta} + \sqrt{\zeta}\right)\left(\left(\sqrt{q_\alpha^\zeta} + \sqrt{\zeta}\right)e^{2\sqrt{q_\alpha^\zeta \zeta}} + \sqrt{q_\alpha^\zeta} - \sqrt{\zeta}\right)\right],$$

where ϕ denotes the PDF of the standard normal distribution.

Proof. If we make the change of variables $x = t^2$ then,

$$\text{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_{\sqrt{q_\alpha^\zeta}}^{+\infty} t^2 \left(\phi\left(t - \sqrt{\zeta}\right) + \phi\left(-t - \sqrt{\zeta}\right)\right) dt.$$

If we define,

$$g(t) = (\zeta+1)\left(\Phi\left(t - \sqrt{\zeta}\right) + \Phi\left(t + \sqrt{\zeta}\right) - 1\right) - \phi\left(t + \sqrt{\zeta}\right) \cdot \left(\left(t + \sqrt{\zeta}\right)e^{2\sqrt{\zeta}t} + t - \sqrt{\zeta}\right),$$

then,

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\left(g\left(+\infty\right) - g\left(\sqrt{q_\alpha^\zeta}\right)\right).$$

Finally, we conclude by using that,

$$F_\zeta(q_\alpha^\zeta) = \Phi\left(\sqrt{q_\alpha^\zeta} - \sqrt{\zeta}\right) + \Phi\left(\sqrt{q_\alpha^\zeta} + \sqrt{\zeta}\right) - 1 = \alpha.$$

□

3 Closed-form Expected Shortfall in the delta-gamma framework

In this section we derive a closed formula to calculate the ES of ΔV_γ for a single risk factor. We will use that formula for each risk factor to reduce the variance of the multi-factor delta-gamma approach estimator. The details will be given in Section 5 and Section 6. We follow the results provided in [21], where it is established the link between the density function of ΔV_γ and the density function of a non-central chi-squared distribution with one degree of freedom, called Q, with non-centrality parameter ζ and density function f_Q. It is worth mentioning that the density function of the one-factor delta-gamma approach was also given in [13]. More precisely,

$$f_{\Delta V_\gamma}(x) = \frac{2}{|\lambda|} f_Q\left(\frac{2}{\lambda}\left(x - \Theta\Delta t - \bar{\mu}_Q\right)\right), \tag{5}$$

where $\lambda = \Gamma C^2, d = \delta C, C = \sigma\sqrt{\Delta t}S(t), \bar{\mu}_Q = -\frac{1}{2}\frac{d^2}{\lambda}$ and $\zeta = \left(\frac{d}{\lambda}\right)^2$. For completeness, we give at the beginning of Section 6 a brief explanation on the selection of the variance $\Sigma = C^2$ considered to simulate the price change of individual risk factors $\Delta S_j, j = 1, \ldots, p$. Further, the VaR value of ΔV_γ is given by,

$$q_\alpha = \begin{cases} \frac{\lambda}{2} q_\alpha^\zeta + \bar{\mu}_Q + \Theta\Delta t, & \text{if } \lambda > 0, \\ \frac{\lambda}{2} q_{1-\alpha}^\zeta + \bar{\mu}_Q + \Theta\Delta t, & \text{if } \lambda < 0, \end{cases} \tag{6}$$

where the quantiles q_α^ζ and $q_{1-\alpha}^\zeta$ represent the VaR value of Q at confidence level α and $1-\alpha$ respectively.

In order to derive the expression for the ES, we differentiate between positive and negative λ. We start by assuming that $\lambda > 0$. In this case, it is shown in [21] that $f_{\Delta V_\gamma}$ is either unimodal or bimodal in its domain of definition $(\bar{\mu}_Q + \Theta\Delta t, +\infty)$. If we take into account (5) the ES is given by,

$$\mathrm{ES}_\alpha(\Delta V_\gamma) = \frac{1}{1-\alpha}\int_{q_\alpha}^{+\infty} x f_{\Delta V_\gamma}(x)dx = \frac{1}{1-\alpha}\cdot\frac{2}{\lambda}\int_{q_\alpha}^{+\infty} x f_Q\left(\frac{2}{\lambda}\left(x - \Theta\Delta t - \bar{\mu}_Q\right)\right)dx.$$

If we make the change of variables $y = \frac{2}{\lambda}\left(x - \Theta\Delta t - \bar{\mu}_Q\right)$, then by (6) we get,

$$\mathrm{ES}_\alpha(\Delta V_\gamma) = \frac{1}{1-\alpha}\int_{q_\alpha^\zeta}^{+\infty}\left(\frac{\lambda}{2}y + \Theta\Delta t + \bar{\mu}_Q\right) f_Q(y)dy = \frac{\lambda}{2}\mathrm{ES}_\alpha(Q) + \frac{1}{1-\alpha}\cdot\left(\Theta\Delta t + \bar{\mu}_Q\right)\int_{q_\alpha^\zeta}^{+\infty} f_Q(y)dy,$$

where $\mathrm{ES}_\alpha(Q)$ denotes the Expected Shortfall of the random variable Q. Finally, taking into account that,

$$\int_{q_\alpha^\zeta}^{+\infty} f_Q(y)dy = 1-\alpha,$$

we conclude that,

$$\mathrm{ES}_\alpha(\Delta V_\gamma) = \frac{\lambda}{2}\mathrm{ES}_\alpha(Q) + \Theta\Delta t + \bar{\mu}_Q.$$

We now consider $\lambda < 0$. In this case, it is shown in [21] that $f_{\Delta V_\gamma}$ is either unimodal or bimodal in its domain of definition $(-\infty, \bar{\mu}_Q + \Theta\Delta t)$, where $x = \bar{\mu}_Q + \Theta\Delta t$ is a vertical asymptote for $f_{\Delta V_\gamma}$. If we take that observation into account then by (5) the ES is given by,

$$\mathrm{ES}_\alpha(\Delta V_\gamma) = \frac{1}{1-\alpha}\int_{q_\alpha}^{\bar{\mu}_Q+\Theta\Delta t} x f_{\Delta V_\gamma}(x)dx = \frac{-1}{1-\alpha}\cdot\frac{2}{\lambda}\int_{q_\alpha}^{\bar{\mu}_Q+\Theta\Delta t} x f_Q\left(\frac{2}{\lambda}\left(x-\Theta\Delta t-\bar{\mu}_Q\right)\right)dx.$$

If we make the change of variables $y = \frac{2}{\lambda}\left(x-\Theta\Delta t-\bar{\mu}_Q\right)$, then by (6) we get,

$$\begin{aligned}\mathrm{ES}_\alpha(\Delta V_\gamma) &= \frac{-1}{1-\alpha}\int_{q^\zeta_{1-\alpha}}^{0}\left(\frac{\lambda}{2}y+\Theta\Delta t+\bar{\mu}_Q\right)f_Q(y)dy \\ &= \frac{1}{1-\alpha}\left[\frac{\lambda}{2}\int_0^{q^\zeta_{1-\alpha}} y f_Q(y)dy + \left(\Theta\Delta t+\bar{\mu}_Q\right)\int_0^{q^\zeta_{1-\alpha}} f_Q(y)dy\right].\end{aligned}$$

Taking into account that,

$$\int_0^{q^\zeta_{1-\alpha}} f_Q(y)dy = 1-\alpha,$$

then,

$$\mathrm{ES}_\alpha(\Delta V_\gamma) = \frac{1}{1-\alpha}\cdot\frac{\lambda}{2}\int_0^{q^\zeta_{1-\alpha}} y f_Q(y)dy + \left(\Theta\Delta t+\bar{\mu}_Q\right). \tag{7}$$

Since Q is a non-central chi-squared distribution with one degree of freedom and non-centrality parameter ζ, we follow the same steps as in Section 2 to obtain,

$$\begin{aligned}&\int_0^{q^\zeta_{1-\alpha}} y f_Q(y)dy = g\left(\sqrt{q^\zeta_{1-\alpha}}\right) - g(0) \\ &= (\zeta+1)\left(\Phi\left(\sqrt{q^\zeta_{1-\alpha}}-\sqrt{\zeta}\right)+\Phi\left(\sqrt{q^\zeta_{1-\alpha}}+\sqrt{\zeta}\right)-1\right) - \phi\left(\sqrt{q^\zeta_{1-\alpha}}+\sqrt{\zeta}\right)\cdot \\ &\cdot\left(\left(\sqrt{q^\zeta_{1-\alpha}}+\sqrt{\zeta}\right)e^{2\sqrt{q^\zeta_{1-\alpha}\zeta}}+\sqrt{q^\zeta_{1-\alpha}}-\sqrt{\zeta}\right) \\ &= (\zeta+1)(1-\alpha) - \phi\left(\sqrt{q^\zeta_{1-\alpha}}+\sqrt{\zeta}\right)\cdot\left(\left(\sqrt{q^\zeta_{1-\alpha}}+\sqrt{\zeta}\right)e^{2\sqrt{q^\zeta_{1-\alpha}\zeta}}+\sqrt{q^\zeta_{1-\alpha}}-\sqrt{\zeta}\right).\end{aligned} \tag{8}$$

Then, by (7) and (8),

$$\begin{aligned}\mathrm{ES}_\alpha(\Delta V_\gamma) &= \frac{\lambda}{2}\cdot\frac{1}{1-\alpha}\left[(\zeta+1)(1-\alpha) - \phi\left(\sqrt{q^\zeta_{1-\alpha}}+\sqrt{\zeta}\right)\cdot\left(\left(\sqrt{q^\zeta_{1-\alpha}}+\sqrt{\zeta}\right)e^{2\sqrt{q^\zeta_{1-\alpha}\zeta}}+\sqrt{q^\zeta_{1-\alpha}}-\sqrt{\zeta}\right)\right] \\ &+ \Theta\Delta t+\bar{\mu}_Q.\end{aligned}$$

We can summarize in the following proposition the value of the ES for all values of λ.

Proposition 2. *The ES at confidence level α of the delta-gamma approximation ΔV_γ reads,*

$$\begin{aligned}ES_\alpha(\Delta V_\gamma) &= \frac{\lambda}{2}\cdot\frac{1}{1-\alpha}\left[(\zeta+1)(1-\alpha) + sign(\lambda)\phi\left(\sqrt{q^\zeta}+\sqrt{\zeta}\right)\cdot\left(\left(\sqrt{q^\zeta}+\sqrt{\zeta}\right)e^{2\sqrt{q^\zeta\zeta}}+\sqrt{q^\zeta}-\sqrt{\zeta}\right)\right]+ \\ &+ \Theta\Delta t+\bar{\mu}_Q,\end{aligned} \tag{9}$$

where $sign(\lambda)$ is the sign function (takes the value 1 for positive λ and -1 for negative λ) and $q^\zeta = q^\zeta_\alpha$ for positive λ and $q^\zeta = q^\zeta_{1-\alpha}$ for negative λ.

Proof. The result follows from the expressions above. □

3.1 Quantile computation

Looking at formula (9), the only required computation for obtaining the ES value is the quantile q^ζ. The quantile q^ζ satisfies $F_\zeta\left(q^\zeta\right) = \eta$, where $\eta = \alpha$ for positive λ, $\eta = 1 - \alpha$ for negative λ, and F_ζ is the distribution function of a non-central chi-squared random variable with one degree of freedom and non-centrality parameter ζ. We define,

$$G_\zeta(x) = F_\zeta(x) - \eta,$$

where,

$$F_\zeta(x) = \Phi\left(\sqrt{x} - \sqrt{\zeta}\right) - \Phi\left(-\sqrt{x} - \sqrt{\zeta}\right), \quad x \in (0, +\infty), \zeta > 0,$$

as seen in Section 2. We observe that,

$$\begin{aligned}
G_\zeta(0) &= F_\zeta(0) - \eta = -\eta < 0, \quad \text{and,} \\
G_\zeta(\zeta) &= F_\zeta(\zeta) - \eta = \Phi(0) - \Phi\left(-2\sqrt{\zeta}\right) - \eta = \Phi\left(2\sqrt{\zeta}\right) - \eta - \frac{1}{2}.
\end{aligned}$$

Since $G'_\zeta(x) > 0$ for all $x \in (0, +\infty)$, there is a unique solution of $G_\zeta(x) = 0$ in the interval $[0, \zeta]$ provided that $\Phi\left(2\sqrt{\zeta}\right) - \eta - \frac{1}{2} > 0$. In that case, we can safely apply a bisection method to the function $G_\zeta(x)$ with initial interval $[0, \zeta]$. When $\Phi\left(2\sqrt{\zeta}\right) - \eta - \frac{1}{2} < 0$, then the unique root is located at some point beyond ζ and we apply a Newton-Raphson method with initial seed ζ (we prefer not to use the Newton-Raphson method in the first case to avoid negative values in subsequent iterations).

4 Shocks in the risk factors: the SST model

In this section we derive a closed formula to calculate the ES of the change in value of the portfolio for a single risk factor under the SST model. To do this, we consider multiple scenarios, which occur with small probabilities and are mutually exclusive. To be more precise, the new model considers $l + 1$ scenarios with associated probabilities of occurrence $p_0, p_1, \ldots, p_l$, where p_0 stands for the normal scenario and,

$$\sum_{i=0}^{l} p_i = 1.$$

For scenarios $i \geq 1$ the change in value of the portfolio V is modified by the additive term,

$$s_i := \overline{\Delta S}_i^T \Gamma \overline{\Delta S}_i + \delta^T \overline{\Delta S}_i,$$

where $\overline{\Delta S}_i$ represents the change in value of the risk factors corresponding to the scenario $i \geq 1$. In summary, the scenario-adjusted value of ΔV is given by,

$$\Delta V_\gamma^s := \Delta V_\gamma + \mathcal{S}, \tag{10}$$

where ΔV_γ is given in (2), $\mathcal{S} = \sum_{i=0}^{l} I_i s_i$ and the indicator random variables I_i select which scenario occurs, i.e., with probability p_i, $I_i = 1$ and $I_k = 0$ for $k \neq i$ (this is, the random variables are mutually exclusive) and the indicator variables I_i are independent of the risk factors. Let $f_\mathcal{S}$ be the density function of $\mathcal{S}$, then,

$$f_\mathcal{S}(x) = p_0 \delta(x) + \sum_{i=1}^{l} p_i \delta(x - s_i),$$

where δ stands for the Dirac delta function. Since ΔV_γ and $\mathcal{S}$ are assumed to be independent, then the density of ΔV_γ^s is the convolution product between both densities,

$$f_{\Delta V_\gamma^s}(x) = \left(f_{\Delta V_\gamma} * f_S\right)(x) = \int_{\mathbb{R}} f_{\Delta V_\gamma}(y) f_S(x-y)dy = \int_{\mathbb{R}} f_{\Delta V_\gamma}(y)\left[p_0\delta(x-y) + \sum_{i=1}^{l} p_i \delta(x-y-s_i)\right]dy,$$

where $f_{\Delta V_\gamma}$ is considered to be zero outside its domain. Finally, if we take into account that $\delta(x) = \delta(-x)$ and, if we define $s_0 = 0$, then,

$$f_{\Delta V_\gamma^s}(x) = \sum_{i=0}^{l} p_i f_{\Delta V_\gamma}(x - s_i) = \frac{2}{|\lambda|}\sum_{i=0}^{l} p_i f_Q\left(\frac{2}{\lambda}\left(x - s_i - \Theta\Delta t - \bar{\mu}_Q\right)\right), \tag{11}$$

where, for the second equality in (11), we have used the expression (5). It is worth remarking that the first equality in (11) is also given in [19]. If we restrict ourselves to the single risk factor case, then we will show that an exact formula for the ES can be developed also in this case.

Let us assume that $\lambda > 0$ in (11). In this case, since the domain of definition of $f_{\Delta V_\gamma}(x - s_i)$ is $(s_i + \bar{\mu}_Q + \Theta\Delta t, +\infty)$, then the distribution function $F_{\Delta V_\gamma^s}$ of ΔV_γ^s reads,

$$F_{\Delta V_\gamma^s}(x) = \sum_{i=0}^{l} p_i \int_{s_i+\Theta\Delta t+\bar{\mu}_Q}^{x} f_{\Delta V_\gamma}(y - s_i)dy = \frac{2}{\lambda}\sum_{i=0}^{l} p_i \int_{s_i+\Theta\Delta t+\bar{\mu}_Q}^{x} f_Q\left(\frac{2}{\lambda}(y - s_i - \Theta\Delta t - \bar{\mu}_Q)\right)dy.$$

If we make the change of variables $z = \frac{2}{\lambda}(y - s_i - \Theta\Delta t - \bar{\mu}_Q)$, then,

$$F_{\Delta V_\gamma^s}(x) = \sum_{i=0}^{l} p_i \int_{0}^{\frac{2}{\lambda}(x-s_i-\Theta\Delta t-\bar{\mu}_Q)} f_Q(z)dz = \sum_{i=0}^{l} p_i F_\zeta\left(\frac{2}{\lambda}(x - s_i - \Theta\Delta t - \bar{\mu}_Q)\right).$$

If q_α^s represents the VaR of ΔV_γ^s at confidence level α then it can be obtained by solving the equation $F_{\Delta V_\gamma^s}(q_\alpha^s) = \alpha$, this is,

$$\sum_{i=0}^{l} p_i F_\zeta\left(\frac{2}{\lambda}(q_\alpha^s - s_i - \Theta\Delta t - \bar{\mu}_Q)\right) = \alpha.$$

We now use the VaR value to compute the ES,

$$\begin{aligned}
\mathrm{ES}_\alpha(\Delta V_\gamma^s) &= \frac{1}{1-\alpha}\sum_{i=0}^{l} p_i \int_{q_\alpha^s}^{+\infty} x f_{\Delta V_\gamma}(x - s_i)dx \\
&= \frac{2}{\lambda}\cdot\frac{1}{1-\alpha}\sum_{i=0}^{l} p_i \int_{q_\alpha^s}^{+\infty} x f_Q(\frac{2}{\lambda}(x - s_i - \Theta\Delta t - \bar{\mu}_Q))dx.
\end{aligned}$$

If we make the change of variables $z = \frac{2}{\lambda}(x - s_i - \Theta\Delta t - \bar{\mu}_Q)$, then,

$$\mathrm{ES}_\alpha(\Delta V_\gamma^s) = \frac{1}{1-\alpha}\sum_{i=0}^{l} p_i \left[\frac{\lambda}{2}\int_{l_e}^{u_e} z f_Q(z)dz + (s_i + \Theta\Delta t + \bar{\mu}_Q)\int_{l_e}^{u_e} f_Q(z)dz\right],$$

where $l_e = \frac{2}{\lambda}(q_\alpha^s - s_i - \Theta\Delta t - \bar{\mu}_Q)$ and $u_e = +\infty$. Finally, if we use the function g defined in Section 2 we end up with,

$$\mathrm{ES}_\alpha(\Delta V_\gamma^s) = \frac{1}{1-\alpha}\sum_{i=0}^{l} p_i \left[\frac{\lambda}{2}\left(\zeta + 1 - g\left(\sqrt{l_e}\right)\right) + (s_i + \Theta\Delta t + \bar{\mu}_Q)\left(1 - F_\zeta(l_e)\right)\right].$$

Let us assume that $\lambda < 0$ in (11). In this case, since the domain of definition of $f_{\Delta V_\gamma}(x - s_i)$ is $(-\infty, s_i + \bar{\mu}_Q + \Theta\Delta t)$, then the distribution function $F_{\Delta V_\gamma^s}$ of ΔV_γ^s reads,

$$F_{\Delta V^s_\gamma}(x) = \int_{-\infty}^x f_{\Delta V^s_\gamma}(y)dy = \sum_{i=0}^l p_i \int_{-\infty}^x f_{\Delta V_\gamma}(y - s_i)dy = -\frac{2}{\lambda}\sum_{i=0}^l p_i \int_{-\infty}^x f_Q\left(\frac{2}{\lambda}(y - s_i - \Theta\Delta t - \bar{\mu}_Q)\right) dy.$$

If we make the change of variables $z = \frac{2}{\lambda}(y - s_i - \Theta\Delta t - \bar{\mu}_Q)$, then,

$$\begin{aligned} F_{\Delta V^s_\gamma}(x) &= \sum_{i=0}^l p_i \int_{\frac{2}{\lambda}(x - s_i - \Theta\Delta t - \bar{\mu}_Q)}^{+\infty} f_Q(z)dz = \sum_{i=0}^l p_i \left[1 - F_\zeta\left(\frac{2}{\lambda}(x - s_i - \Theta\Delta t - \bar{\mu}_Q)\right)\right] \\ &= 1 - \sum_{i=0}^l p_i F_\zeta\left(\frac{2}{\lambda}(x - s_i - \Theta\Delta t - \bar{\mu}_Q)\right). \end{aligned}$$

If q^s_α represents the VaR of ΔV^s_γ at confidence level α then it can be obtained by solving the equation $F_{\Delta V^s_\gamma}(q^s_\alpha) = \alpha$, this is,

$$\sum_{i=0}^l p_i F_\zeta\left(\frac{2}{\lambda}(q^s_\alpha - s_i - \Theta\Delta t - \bar{\mu}_Q)\right) = 1 - \alpha. \tag{12}$$

We now use the VaR value to compute the ES,

$$\begin{aligned} \mathrm{ES}_\alpha(\Delta V^s_\gamma) &= \frac{1}{1-\alpha}\sum_{i=0}^l p_i \int_{q^s_\alpha}^{s_i + \Theta\Delta t + \bar{\mu}_Q} x f_{\Delta V_\gamma}(x - s_i)dx \\ &= -\frac{2}{\lambda}\cdot\frac{1}{1-\alpha}\sum_{i=0}^l p_i \int_{q^s_\alpha}^{s_i + \Theta\Delta t + \bar{\mu}_Q} x f_Q(\frac{2}{\lambda}(x - s_i - \Theta\Delta t - \bar{\mu}_Q))dx. \end{aligned}$$

If we make the change of variables $z = \frac{2}{\lambda}(x - s_i - \Theta\Delta t - \bar{\mu}_Q)$, then,

$$\mathrm{ES}_\alpha(\Delta V^s_\gamma) = \frac{1}{1-\alpha}\sum_{i=0}^l p_i \left[\frac{\lambda}{2}\int_{\bar{l}_e}^{\bar{u}_e} z f_Q(z)dz + (s_i + \Theta\Delta t + \bar{\mu}_Q)\int_{\bar{l}_e}^{\bar{u}_e} f_Q(z)dz\right],$$

where $\bar{l}_e = 0$ and $\bar{u}_e = \frac{2}{\lambda}(q^s_\alpha - s_i - \Theta\Delta t - \bar{\mu}_Q)$. Finally, if we use the function g defined in Section 2 we end up with,

$$\mathrm{ES}_\alpha(\Delta V^s_\gamma) = \frac{1}{1-\alpha}\sum_{i=0}^l p_i \left[\frac{\lambda}{2} g\left(\sqrt{\bar{u}_e}\right) + (s_i + \Theta\Delta t + \bar{\mu}_Q)F_\zeta(\bar{u}_e)\right].$$

5 Multiple control variates

As pointed out by [10], the method of control variates is one of the most effective methods for improving the efficiency of MC simulation. The method takes advantage of the information about the errors in estimates of known quantities to reduce the error in an estimate of an unknown quantity. For the time being, we consider a unique control variate and the new estimator,

$$E^* = E - c_1(E_1 - \tau_1),$$

where E stands for the naive MC estimator, E_1 is the control variate with known expected value τ_1, and c_1 is the optimal coefficient minimizing the variance of E^*, this is,

$$c_1 = \frac{\mathrm{Cov}(E, E_1)}{\mathrm{Var}(E_1)}, \tag{13}$$

where Cov and Var are the covariance and variance, respectively. If we use the optimal c_1 in (13) then,

$$\mathrm{Var}(E^*) = \big(1 - \mathrm{Corr}(E, E_1)^2\big)\,\mathrm{Var}(E), \tag{14}$$

where $\mathrm{Corr}(E, E_1)$ denotes the correlation between E and E_1. Thus, the variance of the new estimator is dramatically reduced with respect to the variance of the naive estimator when the correlation between E and E_1 is close to one (in absolute value). As pointed out in [9], the coefficient c_1 can be estimated by using a pilot run with a smaller sample size or by using the full sample of the simulation. The former approach leads to an unbiased estimate while the second one has a bias which is negligible when the sample size is large (the bias is of order $\mathcal{O}(1/n)$). In this work, we will use the full sample of the simulation for estimating c_1, since the sample size will be large.

Let us now consider multiple control variates. The general formulae for an arbitrary number d of control variates are taken from [12]. The estimator in this case reads,

$$E^* = E - c^T(\mathcal{E} - \tau),$$

where $c^T = (c_1, \dots, c_d)$ is the vector of coefficients minimizing the variance of E^* and $\mathcal{E} = (E_1, \dots, E_d)$ is the vector of control variates with known expected value $\tau = (\tau_1, \dots, \tau_d)$. Then, c is selected as the optimal value of the problem,

$$\min \mathrm{Var}(E^*) = \mathrm{Var}(E) - 2c^T\mathcal{C} + c^T\mathcal{D}c, \tag{15}$$

where $\mathcal{C}$ is the d-dimensional vector of covariances of E with each of the components of $\mathcal{E}$, and $\mathcal{D}$ is the covariance matrix of $\mathcal{E}$. If we assume that $\mathcal{D}$ is a non-singular matrix, then the first and second order optimality conditions of the minimization problem (15) imply that there is an optimal and unique solution given by $c = \mathcal{D}^{-1}\mathcal{C}$ with optimal value,

$$\mathrm{Var}(E^*) = (1 - R^2)\mathrm{Var}(E),$$

where,

$$R^2 = \frac{\mathcal{C}^T\mathcal{D}^{-1}\mathcal{C}}{\mathrm{Var}(E)}. \tag{16}$$

In particular, we can easily calculate the variance reduction factor achieved when using two control variates (this is, for $d = 2$) and compare it with the variance reduction factor obtained if we use a unique control variate. Thus,

$$\mathcal{C} = \begin{pmatrix} \mathrm{Cov}(E, E_1) \\ \mathrm{Cov}(E, E_2) \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} \mathrm{Var}(E_1) & \mathrm{Cov}(E_1, E_2) \\ \mathrm{Cov}(E_1, E_2) & \mathrm{Var}(E_2) \end{pmatrix},$$

and,

$$c = \frac{1}{|\mathcal{D}|} \cdot \begin{pmatrix} \mathrm{Var}(E_2) & -\mathrm{Cov}(E_1, E_2) \\ -\mathrm{Cov}(E_1, E_2) & \mathrm{Var}(E_1) \end{pmatrix} \mathcal{C}. \tag{17}$$

With the optimal solution $c = (c_1, c_2)^T$ in (17), we have,

$$\begin{aligned} R^2 &= \frac{1}{|\mathcal{D}|\mathrm{VaR}(E)} \cdot \mathcal{C}^T \begin{pmatrix} \mathrm{Var}(E_2) & -\mathrm{Cov}(E_1, E_2) \\ -\mathrm{Cov}(E_1, E_2) & \mathrm{Var}(E_1) \end{pmatrix} \mathcal{C} \\ &= \frac{\mathrm{Var}(E)\mathrm{Var}(E_1)\mathrm{Var}(E_2)}{|\mathcal{D}|\mathrm{VaR}(E)} \cdot \big(\mathrm{Corr}(E, E_1)^2 + \mathrm{Corr}(E, E_2)^2 - 2\mathrm{Corr}(E, E_1)\mathrm{Corr}(E, E_2)\mathrm{Corr}(E_1, E_2)\big) \\ &= \frac{\mathrm{Corr}(E, E_1)^2 + \mathrm{Corr}(E, E_2)^2 - 2\mathrm{Corr}(E, E_1)\mathrm{Corr}(E, E_2)\mathrm{Corr}(E_1, E_2)}{1 - \mathrm{Corr}(E_1, E_2)^2}, \end{aligned} \tag{18}$$

since $|\mathcal{D}| = 1 - \mathrm{Corr}(E_1, E_2)^2\mathrm{Var}(E_1)\mathrm{Var}(E_2)$.

Within the delta-gamma framework, we can choose as many control variates as risk factors, so the natural and first question arising at this stage is whether it would be better to use one or two control variates for measuring the risk with a greater reduction of variance. We answer this question in the following lemma.

Lemma 1. *If we define $R_1^2 = Corr(E,E_1)^2$ as in expression (14) and,*

$$R_2^2 = \frac{Corr(E,E_1)^2 + Corr(E,E_2)^2 - 2Corr(E,E_1)Corr(E,E_2)Corr(E_1,E_2)}{1 - Corr(E_1,E_2)^2},$$

as in (18) then $R_2^2 \geq R_1^2$, and the equality holds if and only if $Corr(E,E_1)Corr(E_1,E_2) - Corr(E,E_2) = 0$.

Proof. Indeed, $R_2^2 \geq R_1^2$ if and only if,

$$\mathrm{Corr}(E,E_1)^2 + \mathrm{Corr}(E,E_2)^2 - 2\mathrm{Corr}(E,E_1)\mathrm{Corr}(E,E_2)\mathrm{Corr}(E_1,E_2) \geq (1 - \mathrm{Corr}(E_1,E_2)^2)\mathrm{Corr}(E,E_1)^2,$$

this is,

$$(\mathrm{Corr}(E,E_1)\mathrm{Corr}(E_1,E_2) - \mathrm{Corr}(E,E_2))^2 \geq 0,$$

and this completes the proof. □

Note that the same result applies if we select E_2 instead of E_1. We generalize the result of Lemma 1 to an arbitrary dimension.

Theorem 1. *Let $\mathcal{D}_l$ and $\mathcal{C}_l$ be the matrix $\mathcal{D}$ and vector $\mathcal{C}$, respectively, of expression (16) corresponding to the first l control variates. Let $b_{l-1}^T = (Cov(E_1,E_l), Cov(E_2,E_l), \ldots, Cov(E_{l-1},E_l))$ and $\mathcal{C}(l) = Cov(E,E_l)$ the last component of vector $\mathcal{C}_l$. If we define,*

$$R_l^2 = \frac{\mathcal{C}_l^T \mathcal{D}_l^{-1} \mathcal{C}_l}{Var(E)},$$

then, $R_k^2 \geq R_{k-1}^2$ for all $k \geq 2$, and the equality holds if and only if $b_{k-1}^T \mathcal{D}_{k-1}^{-1} \mathcal{C}_{k-1} - \mathcal{C}(k) = 0$.

Proof. We can write matrix $\mathcal{D}_k$ in block form,

$$\mathcal{D}_k = \begin{pmatrix} \mathcal{D}_{k-1} & b_{k-1} \\ b_{k-1}^T & \mathrm{Var}(E_k) \end{pmatrix},$$

with inverse matrix (see for instance [22] for details),

$$\mathcal{D}_k^{-1} = \begin{pmatrix} \mathcal{D}_{k-1}^{-1} + \frac{1}{\mathcal{F}} \mathcal{D}_{k-1}^{-1} b_{k-1} b_{k-1}^T \mathcal{D}_{k-1}^{-1} & -\frac{1}{\mathcal{F}} \mathcal{D}_{k-1}^{-1} b_{k-1} \\ -\frac{1}{\mathcal{F}} b_{k-1}^T \mathcal{D}_{k-1}^{-1} & \frac{1}{\mathcal{F}} \end{pmatrix},$$

where $\mathcal{F} = \mathrm{Var}(E_k) - b_{k-1}^T \mathcal{D}_{k-1}^{-1} b_{k-1}$.

We have that $R_k^2 \geq R_{k-1}^2$ if and only if,

$$\mathcal{C}_k^T \mathcal{D}_k^{-1} \mathcal{C}_k \geq \mathcal{C}_{k-1}^T \mathcal{D}_{k-1}^{-1} \mathcal{C}_{k-1}. \tag{19}$$

After some basic algebraic manipulation we get,

$$\begin{aligned} \mathcal{C}_k^T \mathcal{D}_k^{-1} \mathcal{C}_k = {} & \mathcal{C}_{k-1}^T \mathcal{D}_{k-1}^{-1} \mathcal{C}_{k-1} + \frac{1}{\mathcal{F}} \mathcal{C}_{k-1}^T \mathcal{D}_{k-1}^{-1} b_{k-1} b_{k-1}^T \mathcal{D}_{k-1}^{-1} \mathcal{C}_{k-1} - \frac{1}{\mathcal{F}} \mathcal{C}_{k-1}^T \mathcal{D}_{k-1}^{-1} b_{k-1} \mathcal{C}(k) \\ & - \frac{1}{\mathcal{F}} b_{k-1}^T \mathcal{D}_{k-1}^{-1} \mathcal{C}_{k-1} \mathcal{C}(k) + \frac{1}{\mathcal{F}} \mathcal{C}(k)^2. \end{aligned}$$

Expression (19) is satisfied if and only if,

$$\frac{1}{\mathcal{F}} \mathcal{C}_{k-1}^T \mathcal{D}_{k-1}^{-1} b_{k-1} b_{k-1}^T \mathcal{D}_{k-1}^{-1} \mathcal{C}_{k-1} - \frac{1}{\mathcal{F}} \mathcal{C}_{k-1}^T \mathcal{D}_{k-1}^{-1} b_{k-1} \mathcal{C}(k) - \frac{1}{\mathcal{F}} b_{k-1}^T \mathcal{D}_{k-1}^{-1} \mathcal{C}_{k-1} \mathcal{C}(k) + \frac{1}{\mathcal{F}} \mathcal{C}(k)^2 \geq 0.$$

If we use that $(\mathcal{D}_{k-1}^{-1})^T = \mathcal{D}_{k-1}^{-1}$ then we derive an equivalent expression,

$$\frac{1}{\mathcal{F}}\left(b_{k-1}^T\mathcal{D}_{k-1}^{-1}\mathcal{C}_{k-1}\right)^2 - \frac{2}{\mathcal{F}}b_{k-1}^T\mathcal{D}_{k-1}^{-1}\mathcal{C}_{k-1}\mathcal{C}(k) + \frac{1}{\mathcal{F}}\mathcal{C}(k)^2 \geq 0,$$

which holds if and only if,

$$\left(b_{k-1}^T\mathcal{D}_{k-1}^{-1}\mathcal{C}_{k-1} - \mathcal{C}(k)\right)^2 \geq 0,$$

provided that $\mathcal{F} > 0$. Indeed, $\mathcal{F}$, called the Schur complement of $\mathcal{D}_{k-1}$ in $\mathcal{D}_k$, is positive semi-definite, since every covariance matrix is positive semi-definite and by [23] $\mathcal{D}_k$ is positive semi-definite if and only if $\mathcal{D}_{k-1}$ and $\mathcal{F}$ are positive semi-definite. □

Note that Lemma 1 is a particular case of Theorem 1 for $k = 2$.

6 Expected Shortfall and control variates

In this section we focus our attention in the efficient computation of the ES value under the multi-factor delta-gamma approach given in (2). We aim at calculating the aforementioned risk measure by means of an improved version of crude MC simulation. For this purpose, we consider each risk factor of the delta-gamma approximation as a control variate to reduce the variance. We will perform a consistency check in Section 6.1 for the one-factor delta-gamma approach. In Section 6.2 we will consider a European call option where the asset and the interest rate are the risk factors and only one control variate will be used. Finally, we calculate the ES in Section 6.3 for a basket put option, which has a payoff depending on two assets. In this last case, we illustrate the results obtained in Theorem 1 by using a different number of control variates in separated examples.

We will assume the geometric Brownian motion (GBM), also called Black-Scholes model, for option valuation. The GBM model assumes that $\log\left(\frac{S_j(t+\Delta t)}{S_j(t)}\right)$ is normally distributed with mean $\mu_j\Delta t$ and standard deviation $\sigma_j\sqrt{\Delta t}$, for $j = 1\ldots,p$ and $S(t) = (S_1(t),\ldots,S_p(t))$. Thus, there seems to be an inconsistency between the valuation model (this is GBM) and the model used for path simulation (this is the normal distribution introduced in Section 1.1). However, for small Δt (as the holding period is),

$$\frac{S_j(t+\Delta t)}{S_j(t)} = 1 + \frac{\Delta S_j}{S_j(t)} \simeq \exp\left(\frac{\Delta S_j}{S_j(t)}\right),$$

which is log-normally distributed if ΔS_j is normally distributed. In that case, ΔS_j follows a normal distribution with mean $\mu_j\Delta t$ and standard deviation $S_j(t)\cdot\sigma_j\sqrt{\Delta t}$. A common assumption within the delta-gamma framework consists of approximating the mean by zero and we therefore have that ΔS_j follows a normal distribution with mean zero and variance $\Sigma = \left(\sigma_j\sqrt{\Delta t}S_j(t)\right)^2$.

6.1 Consistency check of the one-factor delta-gamma approach

We consider in this section the one-factor delta-gamma approach given in (3). Under this assumption, we can compute the ES value by means of the exact formula of Proposition 2, once the quantile value q^ζ has been obtained following the steps given in Section 3.1. In all of our experiments we fix a tolerance error of 10^{-6} for obtaining the quantile. As pointed out in Section 3.1, when the solution q^ζ is located within the interval $[0,\zeta]$, we can apply a bisection method. We use a faster version of the classical bisection method called Brent's method (see [5]) which combines the bisection procedure with linear or quadratic inverse interpolation[1].

Our first test portfolio is taken from [8]. It consists of one short European call and half a short European put with maturity 60 days ($T = 60/365$). The underlying asset at time t is $S(t) = 100$ with volatility level $\sigma = 0.3$, interest rate $r = 0.1$ and strike price $K = 101$ for each option. The pricing formula and the Greeks are detailed in Appendix A. We consider different holding periods Δt, ranging from one to thirty days. We present in Table 1 the VaR and ES values obtained by means of

[1] Computations were carried out in R code, and Brent's method is implemented in the function `uniroot`.

MC simulation at confidence level $\alpha = 0.99$, as well as the exact values calculated with the formula of Proposition 2. We show in the last column of the table, the value reported in [8]. This value was obtained by means of a numerical method based on the inversion of the characteristic function of ΔV_γ with wavelets. As it was explained in [8], for the two last cases $\Delta t = 10/365, 30/365$ there are some numerical difficulties that hamper the computation of the VaR value and it is replaced by the (known) loss upper bound. The density plots in Figure 1 illustrate this fact (for concrete details see [8]).

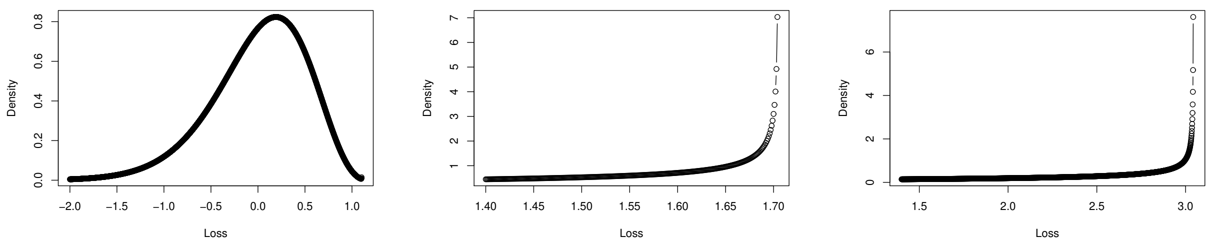
Δt	VaR (MC)	VaR (Exact)	ES (MC)	ES (Exact)	VaR (reported in [8])
1/365	0.90311421	0.90307268	0.96464937	0.96460523	0.9038
10/365	1.70443274	1.70443156	1.70478356	1.70478331	1.7050
30/365	3.04335253	3.04335308	3.04368835	3.04306448	3.0439

Table 1: VaR and ES values corresponding to different holding periods Δt and $\alpha = 0.99$. MC values are calculated with 10^8 simulations of price change ΔS.

We report in Table 2 the MC and exact values for VaR and ES at very high confidence levels. We know from [8] and [21] that the loss upper bound in this case is 1.102455.

α	VaR (MC)	VaR (Exact)	ES (MC)	ES (Exact)
0.999	1.03507382	1.03536925	1.06162291	1.06187675
0.9999	1.09139676	1.09120022	1.09765512	1.09757439

Table 2: VaR and ES values corresponding to holding period $\Delta t = 1/365$ and different confidence levels $\alpha = 0.999, 0.9999$. MC values are calculated with 10^8 simulations of price change ΔS.



(a) $\Delta t = 1/365$.

(b) $\Delta t = 10/365$.

(c) $\Delta t = 30/365$.

Figure 1: Density plots of ΔV_γ.

In the following sections we use this exact formula within the control variates technique to provide a solution for the multi-factor delta-gamma approach.

6.2 Delta-gamma approach with two risk factors and one control variate

In this section we address the efficient computation of ES within a two-factor delta-gamma framework by using the one-factor delta-gamma as the unique control variate. For the one-factor delta-gamma, the risk factor considered is the underlying asset. We therefore apply the formula of Proposition 2 to the one-factor delta-gamma approach with a unique risk factor (the underlying asset). We consider a short European call option with strike $K = 101$, maturity $T = 60/365$, $S(t) = 100$, $r(t) = 0.1$, and the volatility of the underlying asset is $\sigma = 0.3$, being the underlying asset and the interest rate the risk factors. In this case, the two-factor version of (2) reads,

$$\Delta V_\gamma = \Theta \Delta t + \delta_S \Delta S + \delta_r \Delta r + \frac{1}{2}\left(\Gamma_{SS}(\Delta S)^2 + \Gamma_{rr}(\Delta r)^2\right) + \Gamma_{Sr}\Delta S \Delta r,$$

where $\delta_S, \delta_r, \Gamma_{SS}, \Gamma_{rr}, \Gamma_{Sr}$ are the corresponding delta and gamma Greeks of the option (observe that we have replaced the notation $\delta_i, \Gamma_{i,j}$ of Section 1.1 by $\delta_S, \delta_r, \Gamma_{SS}, \Gamma_{rr}, \Gamma_{Sr}$ to emphasize the risk factors). The pricing formula and the Greeks are detailed in Appendix A. If we assume that,

$$\log\left(\frac{r(t+\Delta t)}{r(t)}\right) \sim \mathcal{N}(0, \bar{\sigma}^2),$$

where $\mathcal{N}(0, \bar{\sigma}^2)$ denotes a normal distribution with mean zero and variance $\bar{\sigma}^2$, then following the argument given at the beginning of Section 6, we can assume that $\Delta r \sim \mathcal{N}(0, (\bar{\sigma} r(t))^2)$ and $\Delta S \sim \mathcal{N}(0, (\sigma\sqrt{\Delta t} S(t))^2)$. Further, we assume correlated normals with correlation parameter $\rho = 0.5$. Then,

$$\Sigma = \begin{pmatrix} \sigma_S^2 & \sigma_S \sigma_r \rho \\ \sigma_S \sigma_r \rho & \sigma_r^2 \end{pmatrix},$$

is the covariance matrix, with $\sigma_S = \sigma\sqrt{\Delta t} S(t), \sigma_r = \bar{\sigma} r(t)$. We set $\bar{\sigma} = 0.1$. We calculate the ES with the naive estimator and the control variates estimator and present the results in the second and third column of Table 3, respectively. Two different confidence levels are considered and specified in the first column of the table.The holding period is set to $\Delta t = 1/365$. The VRF achieved at confidence level $\alpha = 0.99$ is 7.7, where it has been computed as the ratio between the variance of the naive estimator and the variance of the control variate estimator. The VRF is almost the double for $\alpha = 0.9$.

α	ES (naive)	ES (CV)	VRF
0.9	1.487314	1.487373	14.1
0.99	2.129052	2.129427	7.7

Table 3: ES values for $\Delta t = 1/365$. MC values are calculated with 10^8 simulations of price change ΔS.

It is worth mentioning that the additional computational cost of carrying out the estimation with the control variate can be neglected, since the expected value of the control variate is obtained with an exact formula (the only numerical part involved is the quantile computation, which is done efficiently).

6.3 Delta-gamma approach with two risk factors and two control variates

In this section we calculate the ES within a two-factor delta-gamma framework by using the one-factor delta-gamma as the control variate for each risk factor. For this purpose, we consider a geometric basket put option with pricing formula and Greeks detailed in Appendix B. In this case, the risk factors are the underlying assets S_1 and S_2 and the parameters employed are as follows. Strike $K = 100$, maturity $T = 1$, $S_1(t) = 90, S_2(t) = 110$, $r = 0.04$, and the volatilities of the underlying assets are $\sigma_1 = 0.2$ and $\sigma_2 = 0.3$. We assume a correlation $\rho = 0.75$ between the assets with normal distributions for the price change,

$$\Delta S_1 \sim \mathcal{N}(0, (\sigma_1\sqrt{\Delta t} S_1(t))^2), \quad \Delta S_2 \sim \mathcal{N}(0, (\sigma_2\sqrt{\Delta t} S_2(t))^2).$$

The two-factor delta-gamma approach (2) reads,

$$\Delta V_\gamma = \Theta \Delta t + \delta_{S_1}\Delta S_1 + \delta_{S_2}\Delta S_2 + \frac{1}{2}\left(\Gamma_{S_1S_1}(\Delta S_1)^2 + \Gamma_{S_2S_2}(\Delta S_2)^2\right) + \Gamma_{S_1S_2}\Delta S_1 \Delta S_2.$$

We consider a time horizon of $\Delta t = 10/365$ and two different confidence levels $\alpha = 0.9$ and $\alpha = 0.99$. We present in Table 4 the VRF obtained in three different situations. We consider first a single control variate corresponding to the risk factor S_1, in second place we perform a similar experiment with S_2 and finally we take the two risk factors, this is, we use two control variates. The outcome is in line with the statement of Theorem 1, and a greater reduction variance is achieved when we employ the one-factor delta-gamma approach for each risk factor as a control variate.

α	VRF(S_1)	VRF (S_2)	VRF (S_1, S_2)
0.9	2.2	3.3	5.6
0.99	1.5	2.0	2.9

Table 4: VRF for different factors and confidence levels.

6.4 The SST model

We devote this section to show how the variance reduction technique developed in this work can be used within the SST framework. Extreme scenarios definitions and probabilities of occurrence currently used in practice can be found in [15]. Further, the standard estimation of risk capital by means of the delta-gamma model takes into account 96 risk factors (82 market risk factors plus 14 life risk factors). For sake of clarity and brevity, we will consider an arbitrary set of probabilities associated to the extreme scenarios as well as the two-risk factor portfolio of Section 6.3 with two control variates.

In order to appreciate the difference in risk between the normal case of Section 6.3 and the SST model, we use the same set of parameters as before. The extreme scenarios considered in this section are defined in Table 5,

Scenario	p_i	$\overline{\Delta S_1}$	$\overline{\Delta S_2}$
0	0.4	0	0
1	0.3	3	0
2	0.2	0	4
3	0.1	5	5

Table 5: Extreme scenarios.

The ES values calculated at confidence levels $\alpha = 0.9, 0.99$ are shown in Table 6 with the corresponding VRF presented in Table 7.

	$\alpha = 0.9$		$\alpha = 0.99$	
	ES without shocks	ES with shocks	ES without shocks	ES with shocks
Naive	2.340301	3.062148	3.236419	4.216899
CV	2.340422	3.062348	3.236460	4.217128

Table 6: ES values for $\Delta t = 10/365$. MC values are calculated with 10^8 simulations of price change ΔS.

α	VRF (S_1, S_2)
0.9	4.0
0.99	2.2

Table 7: VRF for different confidence levels with shocks.

We observe a considerable reduction of the variance under the SST model, although the reduction factor is lower with shocks than without shocks.

6.5 Model risk

We start by analyzing the model risk in the univariate case by comparing the VaR value q_α with the VaR value q_α^s corresponding to ΔV_γ and ΔV_γ^s, respectively. We consider the case $\lambda < 0$ as it is the situation in Section 6.2 and Section 6.4. The case $\lambda > 0$ can be treated analogously. From expression (6) we know that,

$$q_\alpha = \frac{\lambda}{2} q^\zeta_{1-\alpha} + \bar{\mu}_Q + \Theta \Delta t,$$

this is,

$$F_\zeta\left(\frac{2}{\lambda}(q_\alpha - \Theta\Delta t - \bar{\mu}_Q)\right) = 1 - \alpha. \tag{20}$$

Then, from expression (20) and (12) we have,

$$\sum_{i=0}^{l} p_i F_\zeta\left(\frac{2}{\lambda}(q^s_\alpha - s_i - \Theta\Delta t - \bar{\mu}_Q)\right) = F_\zeta\left(\frac{2}{\lambda}(q_\alpha - \Theta\Delta t - \bar{\mu}_Q)\right). \tag{21}$$

Finally, we can isolate q_α from (21) and we end up with,

$$q_\alpha = \frac{\lambda}{2} F_\zeta^{-1}\left(\sum_{i=0}^{l} p_i F_\zeta\left(\frac{2}{\lambda}(q^s_\alpha - s_i - \Theta\Delta t - \bar{\mu}_Q)\right)\right) + \Theta\Delta t + \bar{\mu}_Q. \tag{22}$$

The expression (22) shows the relation between the risk measured by the VaR value within the SST model and the delta-gamma without shocks in the underlying factor. We consider the example in Section 6.4. The VaR value at confidence level $\alpha = 0.9$ of the univariate version with risk factor S_1 is $q_\alpha = 0.859797$ without shocks and $q^s_\alpha = 1.294235$ with shocks. We illustrate in Figure 2 the relation given in formula (22) for this particular example. The upper extreme of the horizontal axis corresponds to the value $\Theta\Delta t + \bar{\mu}_Q$, which is the maximum level of losses for the model without shocks. We can observe that under the SST model, the univariate risk grows (almost) linearly with respect to the model without shocks.

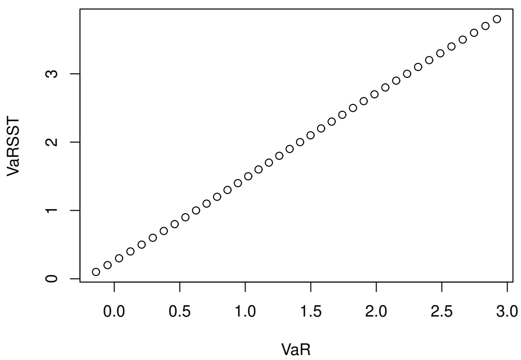


Figure 2: Relation given by formula (22) for a range of quantiles.

6.6 Separable portfolios

If we assume that portfolio V is separable, this is, it can be decomposed into a sum of one-dimensional subportfolios,

$$V(S_1(t), \ldots, S_p(t)) = V_1(S_1) + \cdots + V_p(S_p), \tag{23}$$

where each $V_j(S_j)$ depends only on the risk factor S_j, then we can then decompose the p-dimensional delta-gamma approach (2) with normal ΔS as,

$$\Delta V_\gamma = \sum_{j=1}^{p} \Delta V^j_\gamma,$$

where $\Delta V_\gamma^j := \Theta^j \Delta t + \delta^j \Delta S_j + \frac{1}{2}\Gamma^j \Delta S_j^2$, and $\Theta^j = \frac{\partial V_j}{\partial t}, \delta^j = \frac{\partial V_j}{\partial S_j}, \Gamma^j = \frac{\partial^2 V_j}{\partial S_j^2}$. Since the ES enjoys the subadditivity property of a coherent risk measure then,

$$\mathrm{ES}_\alpha(\Delta V_\gamma) \leq \sum_{j=1}^{p} \mathrm{ES}_\alpha(\Delta V_\gamma^j), \tag{24}$$

where $\mathrm{ES}_\alpha(\Delta V_\gamma^j)$ can be readily computed with the exact formula of Section 3. The same argument applies if we consider the scenario-adjusted ΔV_γ^s in (10). Thus, the right-hand-side of (24) gives us a conservative but fast alternative of computing the ES value avoiding MC simulation. When V is not separable, we can combine MC simulation for those financial instruments which depend on more than one risk factor, with the exact ES for the instruments written on a unique risk factor.

7 Conclusions

In this work we have further investigated the well-known delta-gamma approach for computing the ES of the change in portfolio value. We have derived an exact formula to calculate the ES value for the one-factor delta-gamma approach which was still missing in the literature. We then use the one-factor delta-gamma as a control variate to estimate the ES of the multi-factor delta-gamma approach. A one-factor delta-gamma approximation is used for each risk factor appearing in the problem. Since the expected values of control variates are computed by means of an exact formula, the additional effort of computation with respect to the naive estimator of the multi-factor delta-gamma can be neglected. With this method, we achieve a considerable VRF. We have established a theorem to prove that the variance is further reduced when we use all the risk factors and we have illustrated these results with numerical experiments. Two models have been presented for driving the dynamics of the risk factors, the normal model and the SST model, and we have included an analysis of model risk. Finally, we consider the case of separable portfolios, and we provide an upper bound of the ES by using the exact formula of the univariate case. The possibility of either combining control variates with other variance reduction techniques or using nonlinear controls has not been explored in this work, and we leave it for future research.

Appendix A. Greeks for European calls and puts

The Black-Scholes formula for pricing a European call reads,

$$v(S(t), \sigma, T, r, K) = S(t)\Phi(d_1) - e^{-r(T-t)} K \Phi(d_2),$$

where,

$$d_1 = \frac{\ln(S(t)/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

The price of the corresponding put option is,

$$v(S(t), \sigma, T, r, K) = e^{-r(T-t)} K \Phi(-d_2) - S(t)\Phi(-d_1).$$

The Greeks used in Section 6.1 are,

- theta (call),
$$\frac{\partial v}{\partial t} = -\frac{S(t)\phi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2),$$
- theta (put),
$$\frac{\partial v}{\partial t} = -\frac{S(t)\phi(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}\Phi(-d_2),$$

- delta (call),

$$\frac{\partial v}{\partial S} = \Phi(d_1),$$

- delta (put),

$$\frac{\partial v}{\partial S} = -\Phi(-d_1),$$

- gamma (call and put),

$$\frac{\partial^2 v}{\partial S^2} = \frac{\phi(d_1)}{S(t)\sigma\sqrt{T-t}}.$$

The Greeks used in Section 6.2 (evaluated at $t = 0$) are, theta (call), delta (call), gamma (call) and,

- rho (call),

$$\frac{\partial v}{\partial r} = KTe^{-rT}\Phi(d_2),$$

- Γ_{rr} (call),

$$\frac{\partial^2 v}{\partial r^2} = KTe^{-rT}\left(-T\Phi(d_2) + \frac{\sqrt{T}}{\sigma}\phi(d_2)\right),$$

- Γ_{Sr} (call),

$$\frac{\partial v^2}{\partial S \partial r} = \frac{\sqrt{T}}{\sigma}\phi(d_1).$$

Appendix B. Greeks for the geometric basket put option

The formula for pricing a geometric basket put option under the Black-Scholes dynamics for assets S_1 and S_2 with maturity T, strike K and payoff,

$$\max\left(K - \sqrt{S_1(T)S_2(T)}, 0\right),$$

reads,

$$v(\hat{S}(t), \hat{\sigma}, T, r, K) = e^{-r(T-t)}K\Phi(-\hat{d}_2) - \hat{S}(t)\Phi(-\hat{d}_1),$$

where,

$$\hat{d}_1 = \frac{\ln(\hat{S}(t)/K) + (r + \frac{1}{2}\hat{\sigma}^2)(T-t)}{\hat{\sigma}\sqrt{T-t}}, \quad \hat{d}_2 = \hat{d}_1 - \hat{\sigma}\sqrt{T-t},$$

and $\hat{S}(t) = \sqrt{S_1(t)S_2(t)}$, $\hat{\sigma} = \frac{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_1\sigma_2\rho}}{2}$, being σ_1 and σ_2 the volatility of asset S_1 and S_2, respectively, ρ their correlation and r the risk-free rate.

The Greeks used in Section 6.3 (evaluated at $t = 0$) are,

- theta (put),

$$\frac{\partial v}{\partial t} = -\frac{\hat{S}(0)\phi(\hat{d}_1)\hat{\sigma}}{2\sqrt{T}} + rKe^{-rT}\Phi(-\hat{d}_2),$$

- δ_{S_1} (put),

$$\frac{\partial v}{\partial S_1} = -\frac{S_2(0)}{2\hat{S}(0)}\Phi(-\hat{d}_1),$$

- δ_{S_2} (put),

$$\frac{\partial v}{\partial S_2} = -\frac{S_1(0)}{2\hat{S}(0)}\Phi(-\hat{d}_1),$$

- $\Gamma_{S_1 S_1}$ (put),

$$\frac{\partial^2 v}{\partial S_1^2} = \frac{S_2(0)}{4S_1(0)\hat{S}(0)} \left(\Phi(-\hat{d}_1) + \frac{\phi(\hat{d}_1)}{\hat{\sigma}\sqrt{T}} \right),$$

- $\Gamma_{S_2 S_2}$ (put),

$$\frac{\partial^2 v}{\partial S_2^2} = \frac{S_1(0)}{4S_2(0)\hat{S}(0)} \left(\Phi(-\hat{d}_1) + \frac{\phi(\hat{d}_1)}{\hat{\sigma}\sqrt{T}} \right),$$

- $\Gamma_{S_1 S_2} = \Gamma_{S_2 S_1}$ (put),

$$\frac{\partial^2 v}{\partial S_1 \partial S_2} = \frac{\partial^2 v}{\partial S_2 \partial S_1} = \frac{1}{4\hat{S}(0)} \left(-\Phi(-\hat{d}_1) + \frac{\phi(\hat{d}_1)}{\hat{\sigma}\sqrt{T}} \right).$$

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