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Steenrod squares in signed simplicial sets

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Abstract

The cup-i products in semi-simplicial sets are developed following the work in [1]. The category of signed simplicial sets is presented, and using the cup-i product formulas, we find an operation in the cochain complex of a signed semi-simplicial set that gives rise to the Steenrod squares. Explicit formulas are given to compute them.

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1 Introduction

In elementary algebraic topology, one comes across the notion of a simplicial complex, which is a geometric construction produced by gluing together geometrical simplices of different dimension with rather restrictive rules. Those can be used to model topological spaces, with the advantage that all the information is encoded in a purely combinatorial way and that homology and cohomology are easy to compute.

Semi-simplicial sets are a generalization of the notion of a simplicial complex, and simplicial sets are yet a further generalization. In simplicial sets, the geometric point of view is not as apparent as in simplicial complexes (for instance, they have *degenerate simplices*, which are invisible in the geometric realization), but they have properties that makes them advantageous. For example, there is a really naturally defined product in simplicial sets, but there is not one in semi-simplicial sets or simplicial complexes (at least not as organic as the one in simplicial sets).

In the cohomology of a space (say a simplicial set, or a topological space, or a semi-simplicial set) one has the structure of a graded ring, with the product given by the cup product. This is a product in the level of cochains that induces a product in the cohomology classes. This product on the cochains can be generalized to find cup-i products, where the cup-0 coincides with the cup product. These cup-i products are used to define the Steenrod squares

$$\mathrm{Sq}^i : H^n(X; \mathbb{F}_2) \to H^{n+i}(X; \mathbb{F}_2).$$

Here $\mathbb{F}_2$ denotes the ring with two elements. Steenrod squares are *stable* cohomology operations, which means that they commute with the suspension operation. As said by Mosher and Tangora in [5], "In the past two decades, cohomology operations have been the center of a major area of activity in algebraic topology. This technique for supplementing and enriching the algebraic structure of the cohomology ring led to important progress, both in general homotopy theory and in specific geometric applications. [...] We focus attention on the single most important sort of operations, the Steenrod squares."

In this work, the goal is to find Steenrod squares in the category of *signed simplicial sets*. The category of signed sets has sets as objects, but each morphism carry more information than a usual set map; the morphism assigns a sign to some elements, either a $+$ or a $-$.

The method will consist on finding a simplicial set model for our signed simplicial set and a morphism that connects their chain complexes, and use the cup-i product formulas given in [1].

Why signed simplicial sets?

In 2000, Khovanov [9] defined a knot invariant that assigns to each knot diagram a sequence of chain complexes of R-modules indexed by the integers (this index is called the *quantum grading*, compared to the usual *homologic grading* intern to each chain complex). When the diagram is modified with one of the three

Reidemeister moves, the family of chain complexes is modified but the homology remains invariant. Therefore the homology of these chain complexes is a knot invariant, called the *Khovanov homology*. Later [11], another similar invariant was discovered, called the *odd Khovanov homology*.

In 2014, Lipschitz and Sarkar ([10], [12]) improved the Khovanov invariant by discovering another knot invariant that assigns to each knot diagram a sequence of spectra indexed by the integers such that the cohomology of the j-th spectrum is the homology of the j-th Khovanov chain complex. Again, the homotopy type of these spectra does not change if the diagram is modified by a Reidemeister move, therefore this family of spectra is a knot invariant. This improvement was extended to the odd Khovanov homology in 2018 ([13]).

The immediate consequence of this development is that the Khovanov homology with coefficients in $\mathbb{F}_2$ is provided with Steenrod squares (since it is the homology of a spectrum).

The definition of the Lipshitz and Sarkar spectra uses the notion of an *augmented semi-simplicial object in the Burnside 2-category*, which is a generalization of the notion a simplicial set. And the definition of the Sarkar, Scaduto and Sotffregen uses the notion of an *augmented semi-simplicial object in the signed Burnside 2-category*, which is a generalization of the notion of a signed semi-simplicial set.

Lipshitz and Sarkar ([14]) determined a formula for the second Steenrod square Sq^2 and found knots with the same Khovanov homology but with different Sq^2, thus proving that their invariant was stronger than Khovanov's. Six months ago, Cantero ([15]) found a formula for computing Steenrod squares in augmented semi-simplicial objects in the Burnside 2-category. As of today, there is not a known formula for computing Steenrod squares in the odd Khovanov homology.

Given D a knot diagram, let $j_{\min}$ the minimum quantum degree for which the Khovanov chain complex is nonzero (altough it may be contractible).

Two years ago, Manchón, González-Meneses and Silvero ([16]) observed that the Khovanov spectrum in quantum degree $j_{\min}$ can be constructed as the realisation of a simplicial set, and therefore as the realization of a semi-simplicial set as well. This means that in this case, the classic formulas can be used to compute the Steenrod squares.

Manchón, González-Meneses and Silvero's reasoning can be adapted to prove that the odd Khovanov spectrum of a knot diagram is, in quantum degree $j_{\min}$, a signed semi-simplicial set. Therefore the Steenrod squares can be computed with the formulas found in this work.

Combining the techniques found here with Cantero's work, we hope to find formulas for the Steenrod squares in the odd Khovanov homology.

2 The simplex category

2.1 Simplicial sets

Definition 2.1. A *simplicial set* X is a sequence of sets $\{X_n\}_{n=0}^{\infty}$ together with maps for each $n \geq 0$ and for each $0 \leq i \leq n$

$$d_i^n : X_n \longrightarrow X_{n-1}$$

$$s_i^n : X_n \longrightarrow X_{n+1}$$

satisfying the following relations:

$$d_i^{n-1} d_j^n = d_{j-1}^{n-1} d_i^n \qquad \text{for } i < j \tag{2.1}$$

$$d_i^{n+1} s_j^n = s_{j-1}^{n-1} d_i^n \qquad \text{for } i < j \tag{2.2}$$
$$d_i^{n+1} s_j^n = \mathrm{id}^n \qquad \text{for } i = j \text{ or } i = j+1 \tag{2.3}$$
$$d_i^{n+1} s_j^n = s_j^{n-1} d_{i-1}^n \qquad \text{for } i > j+1 \tag{2.4}$$

$$s_i^{n+1} s_j^n = s_{j+1}^{n+1} s_i^n \qquad \text{for } i \leq j. \tag{2.5}$$

The d's are called *face maps*, the s's are the *degeneracy maps*, and the elements of each X_n are called *simplices*. The simplices in the image of some s_i^n will be called *degenerate*, and those who are not will be called *nondegenerate*.

A morphism of simplicial sets $f\colon X \to Y$, also called *simplicial map*, is a collection of functions $f_n\colon X_n \to Y_n$ for each $n \geq 0$ that commute with face and degeneration maps:

$$f_n d_i^{n+1} = d_i^{n+1} f_{n+1}$$
$$f_{n+1} s_i^n = s_i^n f_n.$$

Remark 2.2. Simplicial sets and simplicial maps constitute a category with the levelwise composition (if $f\colon X \to Y$ and $g\colon Y \to Z$, then $g \circ f\colon X \to Z$ is defined by $(g \circ f)_n = g_n \circ f_n$) and levelwise identity.

Remark 2.3. The superindex of the face or degeneracy maps will be omitted when it is clear from the context.

Remark 2.4. Let $[k] = \{0 \leq 1 \leq ... \leq k\}$ be the kth ordinal. Denote by Δ the *simplicial category*, made of finite ordinals as objects and order-preserving maps as morphisms. Then a simplicial set can be seen as a functor

$$X : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

where $X_n \coloneqq X([n])$. Notice that there are exactly $n+1$ order-preserving injective morphisms $[n-1] \to [n]$, one for each i that is skipped in the codomain. Denoting this maps by $\tilde{d_i}$, define $d_i = X(\tilde{d_i})$. Take also, for each $0 \leq i \leq n$,

$\tilde{s}_i\colon [n+1] \to [n]$ to be the only order-preserving surjective map that sends two elements to $i \in [n]$. Then conditions (2.1)-(2.5) are equivalent to the functoriality of X. Moreover, what was called a simplicial map is now a natural transformation between functors.

This allows us to identify the functor category $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ with the category of simplicial sets.

More generally, a *simplicial object* in a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$, and they form the category $\mathcal{C}^{\Delta^{\mathrm{op}}}$. For example, a simplicial object in the category of groups is called a *simplicial group*. It turns out to be a set of groups $\{G_n\}_{n=0}^{\infty}$ with the usual degeneracy and face maps, in this case being group homomorphisms. One can similarly work on simplicial abelian groups, or simplicial R-modules for a commutative ring R.

2.2 Simplicial sets and topological spaces

Definition 2.5. Define the *standard topological n-simplex* Δ^n by

$$\Delta^n = \{(x_1, ..., x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i = 1 \text{ and } x_i \geq 0 \text{ for any } 0 \leq i \leq n\}.$$

The ith *vertex* of Δ^n is $(x_1, ..., x_n)$ with $x_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$

Example 2.6. Let Y be a topological space. The *singular simplicial set* $\mathcal{S}(Y)$ of Y is defined as follows:

$$\mathcal{S}(Y)_n = \{f\colon \Delta^n \longrightarrow Y \mid f \text{ continuous}\}.$$

Denote by $\partial_i\colon \Delta^n \to \Delta^{n+1}$ the inclusion of Δ^n into Δ^{n+1} as each i-face, and by $\sigma_i\colon \Delta^n \to \Delta^{n-1}$ the unique affine map that brings together the vertices i and $i+1$. More precisely:

$$\begin{aligned} \partial_i(x_0, ..., x_n) &= (x_0, ..., x_{i-1}, 0, x_i, ..., x_{n-1}) \\ \sigma_i(x_0, ..., x_n) &= (x_0, ..., x_i + x_{i+1}, ..., x_n). \end{aligned}$$

Now for any $f \in \mathcal{S}(Y)_n$ define

$$d_i(f) = f \circ \partial_i \in \mathcal{S}(Y)_{n-1}$$

$$s_i(f) = f \circ \sigma_i \in \mathcal{S}(Y)_{n+1}.$$

With these operators $\mathcal{S}(Y)$ turns out to be a simplicial set; even better, this construction gives a functor $\mathcal{S}\colon \mathbf{Top} \longrightarrow \mathbf{Set}^{\Delta^{\mathrm{op}}}$. It is called the *singular chains functor*.

Proposition 2.7. *There is a geometric realization functor for simplicial sets,* $|\;|\colon \mathbf{Set}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Top}$*, defined in the following way:*

$$|X| = \bigsqcup_{n \geq 0} \Delta^n \times X_n \Big/ \sim$$

where the relation is given by

$$(\partial_i(p), d_i x) \sim (p, x)$$
$$(\sigma_i(p), s_i x) \sim (p, x).$$

Proof. To see functoriality one have to say how $|\ |$ acts on morphisms. Let X, Y be two simplicial sets and $f\colon X \to Y$ be a simplicial map. Then define $|f|\colon |X| \to |Y|$ as $|f|(p,x) = (p, f(x))$. This map is well defined since f is a simplicial map, thus it commutes with the face and degeneracy maps. Clearly $|Id| = Id_{\mathbf{Top}}$, and if $g\colon Y \to Z_R$ is another simplicial map, $|g \circ f| = |g| \circ |f|$. □

Proposition 2.8. *This geometric realization gives a CW-complex.*

A previous lemma is needed in order to prove this, which will be used later as well.

Lemma 2.9. *Each element in $|X|$ has a unique representative (p, x) such that x is nondegenerate and p is an interior point of the geometrical simplex corresponding to x.*

Proof. Let us prove two previous statements.

First claim: *any simplex $x \in X_n$ can be expressed in a unique way as $x = s_{j_r}...s_{j_1}y$, where y is nondegenerate and $0 \leq j_1 < ... < j_r < n$.* Notice first that we can always write $x = s_{k_r}...s_{k_1}y$ for a nondegenerate simplex y (simplices of order 0 are always nondegenerate, so the process of applying degeneration maps will end). And then use that $s_i s_j = s_{j+1} s_i$ when $i \leq j$ to permute the degeneracy maps so that the indices are ordered as we want. For uniqueness, suppose that $x = sy = s'y'$, where y, y' are nondegenerate simplices and s, s' are sequences of degeneration maps. Then, since $d_k s_k = \text{id}$, you can take d, d' with $ds = d's' = \text{id}$. Now $sy = s'y'$ implies that $y = dsy = ds'y'$, and similarly $d'sy = y'$. If $\dim(y) = m$, $\dim(y') = m'$, recall that the compositions ds' and $d's$ correspond to order-preserving maps $[m] \to [m']$ and $[m'] \to [m]$ respectively. But since y and y' are nondegenerate, then those maps have to be surjective, so $m = m'$ and $d's = ds' = \text{id}$, thus $y = y'$ and $s = s'$, as we wanted.

Second claim: *any $p \in \Delta^n$ can be written in a unique way as $p = \sigma_{i_s}...\sigma_{i_1}q$ where $q \in \Delta^{n-s}$ is an interior point and $0 \leq i_1 \leq ... \leq i_s \leq n$.* Notice that we can think about the maps ∂ as morphisms from Δ to itself, and similarly the σ can be inclusions. Now if p is not in the interior of Δ^n, then it will be in the image of some ∂. Then again, if it's not in the interior of this image, it's in the image of another ∂ (now $\partial : \Delta^{n-1} \to \Delta^{n-2}$), so eventually (since, in the worst case, in dimension 0 we finish) we obtain that p is an interior point of $\partial_{i_1}...\partial_{i_s}(\Delta^n)$. Notice that $\partial_j\partial_i = \partial_i\partial_{j+1}$ if $j > i$ (from the definition of ∂), thus we can rearrange the maps in a way so that $0 \leq i_1 \leq ... \leq i_s \leq n$. So we have $\partial_{i_1}...\partial_{i_s}p = p$, and then $p = \sigma_{i_s}...\sigma_{i_1}p$ (since $\sigma_k\partial_k q = q$ if $\partial_k q = q$). So we have written p in the way that was asked. The prove of the uniqueness is analogous to the one in the previous claim, since the σ and ∂ satisfy the same relations as the d and s before.

We are now able to prove the lemma. Let $\overline{X} = \bigsqcup_{n\geq 0} \Delta^n \times X_n$, that is, the step of the geometric realization without applying the relation $\sim$. Consider the map

$$\lambda\colon \overline{X} \longrightarrow \overline{X}$$
$$\lambda(p,x) = (\partial_{j_1}...\partial_{j_r}p, y)$$

for $x \in X_n$, and with $y \in X_{n-r}$, $0 \leq j_1 \leq ... \leq j_r < n$ chosen as in the first claim. And on the other hand define

$$\rho\colon \overline{X} \longrightarrow \overline{X}$$
$$\rho(p,x) = (q, d_{i_1}...d_{i_s}x)$$

for $x \in X_n$, and choosing $q \in \Delta^{n-s}$ and $0 \leq i_1 \leq ... \leq i_s \leq n$ as in the second claim. Now notice that the composition $\lambda\rho$ sends each point to an equivalent point through $\sim$ (since both λ and ρ send points to equivalent ones), and it will be of the form (q,z) where z is nondegenerate (because we applied λ) and q is interior (since we applied ρ, and λ maintains this property). And the uniqueness of this representative comes from the uniqueness proved in both previous claims, since they tell that this construction is the only way to find a suitable representative. □

We wanted to see that for any simplicial set X, $|X|$ will always produce a CW-complex. Indeed, take the images of the nondegenerate simplices of X as the cells of $|X|$. By lemma 2.9, the interior of those cells give a partition of $|X|$, so this space will indeed be a CW-complex.

2.3 Homology and cohomology of a simplicial set

Given a commutative ring R, denote by R-**Mod** the category of R-modules.

Definition 2.10. Given a set X and a commutative ring R, let $R\langle X\rangle$ denote the *free R-module on* X, that is, the free R-module generated by the elements of X. This construction defines a functor:

$$A_R\colon \mathbf{Set} \longrightarrow R\text{-}\mathbf{Mod}$$

where $A_R(X) = R\langle X\rangle$ and if $f\colon X \to Y$, then $A_R(f)\colon R\langle X\rangle \to R\langle Y\rangle$ is given by $A_R(f)(\sum_{x\in X}\lambda_x x) = \sum_{x\in X}\lambda_x f(x)$.

Now let X be a simplicial set. Remember that it can be seen as a functor $X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$, and similarly a simplicial R-module is a functor $\Delta^{\mathrm{op}} \to R$-**Mod**. So using A_R one can define a functor Z_R from simplicial sets to simplicial R-modules by:

$$Z_R\colon \mathbf{Set}^{\Delta^{\mathrm{op}}} \longrightarrow R\text{-}\mathbf{Mod}^{\Delta^{\mathrm{op}}}$$

where $Z_R(X) = A_R \circ X$.

Definition 2.11. Let $\mathbf{Ch}_*(R)$ denote the category of chain complexes of R-modules.

Definition 2.12. Define a functor $C_*\colon R\text{-}\mathbf{Mod}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Ch}_*(R)$ that takes a simplicial R-module A to its *Moore chain complex* $C_*(A)$, which is defined as follows. For each n, denote by $(DA)_n$ the submodule of A_n generated by all degenerate simplices. Then define $C_n(A) = A_n/(DA)_n$, with the differential induced by the homomorphism:

$$\partial_n = \sum_{i=0}^{n}(-1)^i d_i.$$

If $s_j a \in A_n$, then $\partial_n(s_j a) = \sum_{i=0}^{n}(-1)^i d_i s_j a$ which is degenerate since we can swap the face and degeneracy maps for each summand except for $d_j s_j a$ and $d_{j+1} d_j a$, which are both equal to a with different sign, so they cancel each other. So $\partial_n((DA)_n) \subseteq (DA)_{n-1}$, thus ∂ indeed induces a morphism between the quotients.

The differential needs to satisfy $\partial\partial = 0$ in order to truly have a chain complex. Certainly:

$$\begin{aligned}
\partial_n\partial_{n+1} &= \sum_{i,j}(-1)^{i+j}d_i d_j \\
&= \sum_{i\geq j}(-1)^{i+j}d_i d_j + \sum_{i<j}(-1)^{i+j}d_i d_j \\
&= \sum_{i\geq j}(-1)^{i+j}d_i d_j + \sum_{i<j}(-1)^{i+j}d_{j-1} d_i \\
&= \sum_{i\geq j}(-1)^{i+j}d_i d_j - \sum_{i\leq k}(-1)^{k+l}d_k d_i = 0.
\end{aligned}$$

Given a simplicial set X and a commutative ring R, we will denote:

$$C_*(X;R) = C_*(Z_R(X)).$$

This defines a functor $C_*(-;R)\colon \mathbf{Set}^{\Delta^{\mathrm{op}}} \to \mathbf{Ch}_*(R)$.

Now one can define the homology groups of this chain complex as usual, and for a simplicial set X we will write:

$$H_n(X;R) = H_n(C_*(X;R)).$$

Similarly one can work towards the cohomology groups. Define $Z_R^*(X)_n = \mathrm{Hom}(\mathbb{Z}\langle X_n\rangle, R)$, and the new face maps d_i^* turn into *ascending* maps defining them as $d_i^*(\varphi) = \varphi \circ d_i$, for $\varphi \in Z_R^*(X)_n$. Define

$$C^n(X;R) = \mathrm{Hom}(\mathbb{Z}\langle X_n\rangle, R) \big/ \mathrm{Hom}(\mathbb{Z}\langle (DX)_n\rangle, R)$$

and take the differential to be the analogous to the one used in the chain complexes:

$$\delta_n = \sum_{i=0}^{n}(-1)^i d_i^*.$$

The natural question now is how this cohomology is related to the singular cohomology through the singular chain functor and the geometrical realization functor. This is studied by Milnor in [4], and the result is pretty satisfactory.

Proposition 2.13. *For any simplicial set X,*

$$H^*(X) \cong H^*(S(|X|))$$

and for any topological space Y,

$$H^*(Y) \cong H^*(|S(Y)|).$$

Proof. See [4]. $\square$

2.4 Semi-simplicial sets

Sometimes it will be more convenient to work with semi-simplicial sets rather than with simplicial sets. Those are just like simplicial sets, but without the degeneracy maps.

Definition 2.14. A *semi-simplicial set* X is a sequence of sets $\{X_n\}_{n=0}^{\infty}$ together with face maps for each $n \geq 0$ and for each $0 \leq i \leq n$

$$d_i : X_n \longrightarrow X_{n-1}$$

satisfying the following relations:

$$d_i d_j = d_{j-1} d_i \qquad\qquad \text{for } i < j.$$

A morphism of semi-simplicial sets $f\colon X \to Y$, also called *semi-simplicial map*, is a collection of functions $f_n\colon X_n \to Y_n$ for each $n \geq 0$ that commute with face maps:

$$f_n d_i = d_i f_{n+1}$$

Remark 2.15. Like in simplicial sets, semi-simplicial sets and semi-simplicial maps constitute a category with the levelwise composition (if $f\colon X \to Y$ and $g\colon Y \to Z$, then $g \circ f\colon X \to Z$ is defined by $(g \circ f)_n = g_n \circ f_n$) and levelwise identity.

Remark 2.16. Recall that Δ denotes the simplicial category. Denote by Δ_{inj} the subcategory of Δ containing only the injective functions (and the same objects). A semi-simplicial set can be seen as a functor

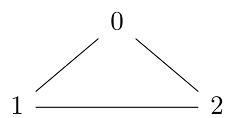
$$X : \Delta_{\text{inj}}^{\text{op}} \longrightarrow \mathbf{Set}$$

where $X_n \coloneqq X([n])$, and what was called a semi-simplicial map is now a natural transformation between functors.

Therefore, just like we identified the simplicial category with the category of functors $\mathbf{Set}^{\Delta^{\text{op}}}$, the semi-simplicial category can be identified with $\mathbf{Set}^{\Delta_{\text{inj}}^{\text{op}}}$.

One might ask now for the relations between semi-simplicial sets and simplicial sets; more specifically, for good behaving functors between the categories. We will be interested in two functors, one in each direction, which are inverse one from another.

Let's see an example to motivate the definitions that will follow. Take the standard simplicial (as in ordered simplicial complex) model of S^1:



Seen as a semi-simplicial set, we have

$$X_0 = \{[0], [1], [2]\} \qquad X_1 = \{[01], [02], [12]\}$$

where face maps d_i act by removing the vertex in position i. Notice that we can make this construction with any ordered simplicial complex, so we actually have a functor from the category of ordered simplicial complexes to the category of semi-simplicial sets. Back to the example, now to build a simplicial set define the morphisms s_i to be the maps that add a repetition of the vertex in position i, for example $s_1([01]) = [011]$, and add all the images of this maps to the corresponding X_i. Then our simplicial set is now

$$X_0 = \{[0], [1], [2]\} \qquad X_1 = \{[01], [02], [12], [00], [11], [22]\}$$
$$X_2 = \{[000], [001], [002], [011], [111], [112], [022], [122], [222]\} \qquad X_i = \{...\}$$

Note that what we did is consider the simplices of the initial semi-simplicial set as the nondegenerate simplices of a simplicial set, and then we added all the degenerate simplices. Conversely, now one could eliminate all degenerate simplices and forget about the degeneracy maps and would have the initial semi-simplicial set X.

Proposition 2.17.

1. *There is a functor $N \colon \mathbf{Set}^{\Delta^{\mathrm{op}}_{\mathrm{inj}}} \longrightarrow \mathbf{Set}^{\Delta^{\mathrm{op}}}$ that consist on adding a new simplex for each combination of s_i's applied to each simplex, and then identifying the new simplices between them using the relations involving each s_i as stated in the definition of a simplicial set. That is, given X a Δ-set,*
$$(NX)_n = X_n \cup \{s_{i_1} s_{i_2} ... s_{i_k}(x) \mid \ x \in X_{n-k}\}.$$

2. *There is another functor $M : \mathbf{Set}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Set}^{\Delta^{\mathrm{op}}_{\mathrm{inj}}}$, consisting on removing all degenerate simplices.*

Remark 2.18. These functors are inverse one from another.

Remark 2.19. A realization functor $|\ |$ has been defined for simplicial sets. Notice that this gives a realization functor for semi-simplicial sets composing with N.

But by looking at the definition of $|\ |$, notice that all the simplices added with N are not used; in other words, $|\ |$ only cares about nondegenerate simplices. So simply by removing the degenerate simplices in the definition, one can define

$$|\ |\colon \mathbf{Set}^{\Delta_{\mathrm{inj}}^{\mathrm{op}}} \longrightarrow \mathbf{Top}$$

which is more simple than the previous one. So there is a factorization

$$\begin{tikzcd}
\mathbf{Set}^{\Delta^{\mathrm{op}}} \arrow[d, "M"'] & \\
\mathbf{Set}^{\Delta_{\mathrm{inj}}^{\mathrm{op}}} \arrow[r, "|\ |"] & \mathbf{Top} \arrow[ul, "S"']
\end{tikzcd}$$

of the geometrical realization of a simplicial set through a semi-simplicial set.

With semi-simplicial sets and given a ring R, one can apply the free functor the same way we did with simplicial sets, and then pass to a chain complex with the alternate face differential the same way. So it's immediate that this diagram

$$\begin{tikzcd}
\mathbf{Set}^{\Delta^{\mathrm{op}}} \arrow[d, "C_*(-;R)"'] \arrow[r, shift right, "M"'] & \mathbf{Set}^{\Delta_{\mathrm{inj}}^{\mathrm{op}}} \arrow[l, shift right, "N"'] \arrow[dl, "C_*(-;R)"] \\
\mathbf{Ch_*(R)} &
\end{tikzcd}$$

commutes. In particular, we can compute the homologies working with a simplicial set or with its corresponding semi-simplicial set, whichever suits us. In what follows, jumping from simplicial to semi-simplicial and back will be common, even without referring to the functors N and M.

The following diagram, that commutes up to quasi-isomorphism, sums up what we have seen.

$$\begin{tikzcd}
\mathbf{Set}^{\Delta^{\mathrm{op}}} \arrow[d, "C_*(-;R)"'] \arrow[r, shift right, "M"'] & \mathbf{Set}^{\Delta_{\mathrm{inj}}^{\mathrm{op}}} \arrow[l, shift right, "N"'] \arrow[dl, "C_*(-;R)"] \arrow[r, "|\ |"] & \mathbf{CWcplex} \arrow[r, "i"] & \mathbf{Top} \arrow[lll, bend right, "S"'] \arrow[dlll, bend left, "S_*"] \\
\mathbf{Ch_*(R)} & & &
\end{tikzcd}$$

Here S_* denotes the singular chains functor.

3 Products

Definition 3.1. Given simplicial sets X and Y, the *Cartesian product* $X \times Y$ is the simplicial set defined by

$$(X \times Y)_n = X_n \times Y_n$$

with face and degeneracy maps

$$\begin{aligned} d_i(x, y) &= (d_i x, d_i y) \\ s_i(x, y) &= (s_i x, s_i y). \end{aligned}$$

It's immediate that this gives a well-defined simplicial set.

Proposition 3.2. *Let X and Y be two countable simplicial sets. Then there is a natural homeomorphism*

$$|X \times Y| \cong |X| \times |Y|.$$

Proof. See [4]. □

In order to study how the homology behaves under products, we introduce now the tensor product operation in $R\text{-}\mathbf{Mod}^{\Delta^{\mathrm{op}}}$ and $\mathbf{Ch}_*(R)$.

Definition 3.3. Let A and B be two simplicial R-modules. Define the simplicial R-module $A \otimes B$ as:

$$\begin{aligned} (A \otimes B)_n &= A_n \otimes B_n \\ d_i(a \otimes b) &= (d_i a \otimes d_i b) \\ s_i(a \otimes b) &= (s_i a \otimes s_i b). \end{aligned}$$

On the other hand, if C and D are two chain complexes, then

$$\begin{aligned} (C \otimes D) &= \bigoplus_{p+q=n} C_p \otimes D_q \\ \partial(a \otimes b) &= \partial a \otimes b + (-1)^{\dim a} a \otimes \partial b. \end{aligned}$$

Again, it is immediate that $\partial\partial = 0$ under this definition, so the tensor product is well defined.

Remark 3.4. Notice that the definition of tensor products for simplicial R-modules is analogous to the definition of the cartesian product of simplicial sets. It doesn't come as a surprise then that given X, Y simplicial sets, there is a natural isomorphism

$$Z_R(X \times Y) \cong Z_R(X) \otimes Z_R(Y)$$

since $Z_R(X \times Y)_n = A_R((X \times Y)_n)$ is the free simplicial group generated by the elements in $(X \times Y)_n = X_n \times Y_n$. On the other hand $Z_R(X)_n \otimes Z_R(Y)_n = A_R(X_n) \otimes A_R(Y_n)$, and since $A_R(X_n) \otimes A_R(Y_n) = A_R(X_n \times Y_n)$, again this is the free R-module generated by $X_n \times Y_n$.

Definition 3.5. Let A and B be two simplicial R-modules. Then there is a map

$$\nabla\colon C_*(A) \otimes C_*(B) \longrightarrow C_*(A \otimes B)$$

called the *shuffle* or the Eilenberg-Zilber map, defined if $a \in A_p$ and $b \in B_q$ by

$$\nabla(a \otimes b) = \sum_{(\mu,\nu)} \operatorname{sign}(\mu,\nu)(s_\nu a) \otimes (s_\mu b) \in C_{p+q}(A \otimes B)$$

where the sum ranges over all (p, q) shuffles, that is, over all permutations of $p + q$ elements

$$(\mu, \nu) = (\mu_1, ..., \mu_p, \nu_1, ..., \nu_q)$$

such that $\mu_1 < ... < \mu_p$ and $\nu_1 < ... < \nu_q$ and $\operatorname{sign}(\mu, \nu)$ is the signature of the permutation. The morphisms s_μ and s_ν are defined by $s_\mu = s_{\mu_p} ... s_{\mu_p}$ and $s_\nu = s_{\nu_p} ... s_{\nu_q}$.

Definition 3.6. There is another map, the Alexander-Whitney map

$$f\colon C_*(A \otimes B) \longrightarrow C_*(A) \otimes C_*(B)$$

defined for $a \otimes b \in A_n \otimes B_n = (A \otimes B)_n$ as

$$f(a \otimes b) = \sum_{p+q=n} d_{n-p+1} ... d_{n-1} d_n\, a \otimes \underbrace{d_0 ... d_0}_{q}\, b.$$

Proposition 3.7. *The shuffle and the Alexander-Whitney map are well defined.*

Proof. Let $a \in A_n$ and $b \in B_n$. We want to see that

$$\partial f(a \otimes b) = f(\partial(a \otimes b)).$$

Let's develop each expression separately. On the one hand

$$f(\partial(a \otimes b)) = f(\partial a \otimes \partial b) = f\left(\sum_{i=0}^{n} (-1)^i d_i a \otimes d_i b\right) = \sum_{i=0}^{n} (-1)^i f\,(d_i a \otimes d_i b)$$

$$= \sum_{i=0}^{n} (-1)^i \sum_{p+q=n-1} (d_{n-p} d_{n-p+1} ... d_{n-1} d_i) a \otimes (\underbrace{d_0 ... d_0}_{q}\, d_i) b = \star$$

and, on the other hand

$$\partial f(a \otimes b) = \partial \sum_{p+q=n} (d_{n-p+1} d_{n-p+2} ... d_n) a \otimes (\underbrace{d_0 ... d_0}_{q}) b =$$

$$= \sum_{p+q=n} \sum_{i=0}^{q} (-1)^i (d_i d_{n-p+1} ... d_n) a \otimes (\underbrace{d_0 ... d_0}_{q}) b$$

$$+ \sum_{p+q=n} \sum_{i=0}^{p} (-1)^{q+i} (d_{n-p+1} ... d_n) a \otimes (d_i\, \underbrace{d_0 ... d_0}_{q}) b = \star\star\,.$$

Both expressions live in $(C_*(A)\otimes C_*(B))_{n-1}=\bigoplus_{s=0}^{n-1}C_s(A)\otimes C_{n-1-s}(B)$. So the equality has to be checked for each $0\le s\le n-1$. Fix some s, and then restricting to $C_s(A)\otimes C_{n-1-s}(B)$ one has

$$\begin{aligned}
\star\star &= \sum_{i=0}^{s+1}(-1)^i(d_id_{s+2}...d_n)a\otimes(\underbrace{d_0...d_0}_{s+1})b+\sum_{i=0}^{n-s}(-1)^{i+s}(d_{s+1}...d_n)a\otimes(d_i\underbrace{d_0...d_0}_{s})b\\
&= \sum_{i=0}^{s}(-1)^i(d_id_{s+2}...d_n)a\otimes(\underbrace{d_0...d_0}_{s+1})b+\sum_{i=1}^{n-s}(-1)^{i+s}(d_{s+1}...d_n)a\otimes(d_i\underbrace{d_0...d_0}_{s})b\\
&= \sum_{i=0}^{s}(-1)^i(d_{s+1}...d_{n-1}d_i)a\otimes(\underbrace{d_0...d_0}_{s+1})b+\sum_{i=1}^{n-s}(-1)^{i+s}(d_{s+1}...d_n)a\otimes(\underbrace{d_0...d_0}_{s}d_{i+s})b\\
&= \sum_{i=0}^{s}(-1)^i(d_{s+1}...d_{n-1}d_i)a\otimes(\underbrace{d_0...d_0}_{s}d_i)b\\
&+ \sum_{i=1}^{n-s}(-1)^{i+s}(d_{s+1}...d_{n-1}d_{i+s})a\otimes(\underbrace{d_0...d_0}_{s}d_{i+s})b\\
&= \sum_{i=0}^{s}(-1)^i(d_{s+1}...d_{n-1}d_i)a\otimes(\underbrace{d_0...d_0}_{s}d_i)b\\
&+ \sum_{i=s+1}^{n}(-1)^i(d_{s+1}...d_{n-1}d_i)a\otimes(\underbrace{d_0...d_0}_{s}d_i)b\\
&= \sum_{i=0}^{n}(-1)^i(d_{s+1}...d_{n-1}d_i)a\otimes(\underbrace{d_0...d_0}_{s}d_i)b=\star.
\end{aligned}$$

□

Theorem 3.8 (Eilenberg-Zilber)**.** *The shuffle and Alexander-Whitney maps give a chain homotopy equivalence between $C_*(A)\otimes C_*(B)$ and $C_*(A\otimes B)$. In particular, their homology modules are isomorphic.*

For the proof of this theorem we will use the acyclic models idea, which we proceed to develop.

Fix a ring R, and for $n\ge 0$, define a simplicial R-module Δ^n, the n-dimensional *model*, where $(\Delta^n)_p$ is the free module generated by the order-preserving maps $\lambda\colon[p]\to[n]$ (which also can be seen as all p-dimensional faces with n vertices, allowing vertex repetition, or in other words, degenerate faces). The face and degeneracy maps are given by $d_i\lambda:=\lambda\tilde{d_i}$ ($\tilde{d_i}$ defined as in remark 2.4) and similarly $s_i\lambda:=\lambda\tilde{s}_i$. Denote by $\kappa\in\Delta^n$ the n-simplex given by the identity map $[n]\to[n]$, which is called the *basic cell* on this model. Notice also that there is only one element in $(\Delta^n)_0$, the identity $\mathrm{id}_0\colon[0]\to[0]$. Define a map $\varepsilon:(\Delta^n)_0\to R$ by $\varepsilon([p])=1$ for any $0\le p\le n$. Notice that $\varepsilon\partial\colon(\Delta^n)_1\to R$ in the zero map; we say then that Δ^n is *augmented* by ε. For an augmented simplicial module A, the corresponding chain complex $C_*(A)$ is *acyclic* if $H_k(A)=0$ for $k>0$ and $\varepsilon\colon H_0(A)\cong R$.

Lemma 3.9. *The chain complex $C_*(\Delta^n)$ (with Δ^n augmented by ε) is acyclic for any $n \geq 0$.*

Proof. We show that the chain is contractible. Looking at generators of $(\Delta^n)_p$ as p-faces, define $h_p\colon (\Delta^n)_p \longrightarrow (\Delta^n)_{p+1}$ by

$$h_p([i_1 ... i_p]) = [0\, i_1 ... i_p].$$

Notice that these maps send degenerate faces to degenerate faces, so they induce a map $h_p\colon C_p(\Delta^n) \longrightarrow C_{p+1}(\Delta^n)$ which is a chain homotopy between the zero map and the identity chain map, since $d_i h = h d_{i-1}$ for $i > 0$ and $d_0 h = \mathrm{id}$, so

$$h\partial + \partial h = \mathrm{id}\,.$$

So the chain complex $C_*(\Delta^n)$ is contractible, therefore $H_p(\Delta^n) = 0$ for $p > 0$ and $H_0(\Delta^n) \cong R$, with ε inducing this isomorphism. Thus $C_*(\Delta^n)$ is acyclic. □

Lemma 3.10. *If A, B are simplicial R-modules and $C_*(A)$, $C_*(B)$ are acyclic, then $C_*(A \otimes B)$ and $C_*(A) \otimes C_*(B)$ are acyclic as well.*

Proof. See [3], chapter 8. □

Lemma 3.11. *Let A be a simplicial R-module, and $a \in A_n$ an n-simplex. Then there is a unique simplicial map $f\colon \Delta^n \longrightarrow A$ satisfying $f(\kappa) = a$.*

Proof. Take any generator $\lambda \in (\Delta^n)_p$, so $\lambda\colon [p] \longrightarrow [n]$. Thinking about λ as a composition of face and degeneracy maps, we have $\lambda\kappa = \kappa\tilde{\lambda} = \tilde{\lambda}$, so $f(\tilde{\lambda}) = f(\kappa\tilde{\lambda}) = \lambda f(\kappa) = \lambda a$, which defines a simplicial map, which is unique since it is defined only by $f(\kappa)$. □

Now two lemmas more in the context of the Eilenberg-Zilber theorem, and the proof will follow.

Lemma 3.12. *Let A, B be two simplicial R-modules. If two chain homomorphisms $f, g\colon C_*(A \otimes B) \longrightarrow C_*(A \otimes B)$ are the identity in dimension zero, then they are naturally homotopic.*

Proof. We want to define an homotopy h between f and g, that is with $f_n - g_n = \partial_{n+1} h_n + h_{n-1}\partial_n$. Define $h_0 = 0$, which works since $f_0 = g_0$. Now by induction, assume we already defined h_p for $p < n$. We assume first that $A = \Delta^n = B$, and define h_n on $\kappa \otimes \kappa$. We need

$$\partial h_n(\kappa \otimes \kappa) = f_n(\kappa \otimes \kappa) - g_n(\kappa \otimes \kappa) - h_{n-1}\partial(\kappa \otimes \kappa).$$

By induction hypothesis, $\partial(f_n - g_n - h_{n-1}\partial) = f_{n-1}\partial - g_{n-1}\partial - \partial h_{n-1}\partial - h_{n-2}\partial\partial = 0$. So the right side of the equality is a cycle in an acyclic complex, so it is a boundary of some chain c. Define then $h_n(\kappa \otimes \kappa) = c$, and the equality will be satisfied, and then for $a \otimes b \in (A \otimes B)_n$ define $h_n(a \otimes b) = (\alpha \otimes \beta)d$, where α and β are compositions of face maps with $\alpha\kappa = a$, $\beta\kappa = b$ (the existence and uniqueness of this maps is proven in the second previous lemma). Then h_n working on the model implies that it also satisfies $\partial h_n + h_{n-1}\partial = f_n - g_n$ in $A \otimes B$, so the lemma is proven. □

Remark 3.13. Given the acyclicity of $C_*(\Delta^n) \otimes C_*(\Delta^n)$, one can show by an analogous argument that the previous lemma also works for an automorphism on this chain.

Proof of the theorem 3.8. Notice now that both the shuffle and the Eilenberg-Zilber map are the identity in dimension zero, so the composition (in any order) is also the identity in dimension zero. Therefore both compositions are homotopic to the identity, thus they define an homotopy equivalence, and the Eilenberg-Zilber theorem is proven. $\square$

It's established then that for simplicial R-modules A, B, there is an isomorphism:

$$H_*(C_*(A) \otimes C_*(B)) \cong H_*(C_*(A \otimes B)).$$

Which, using remark 3.4, implies that for X, Y simplicial sets one has:

$$H_*(C_*(X; R) \otimes C_*(X; R)) \cong H_*(C_*(X \times Y; R)) = H_*(X \times Y; R).$$

The last step is to work on the left hand side of the equality to translate the expression into a tensor product of the homologies. We will assume that R is a field, since it's the case we are interested in, to get a simple version of Künneth theorem.

Theorem 3.14 (Künneth, algebraic version)**.** *Let R be a field, and let C, C' be two chain complexes of R-modules. Then there is an isomorphism*

$$H_*(C) \otimes H_*(C') \cong H_*(C \otimes C')$$

that is, for each n, one has

$$\bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \cong H_n(C \otimes C').$$

Proof. See, for example, chapter 3B of [7]. $\square$

Combining this with theorem 3.8, one has the following.

Corollary 3.15 (Künneth)**.** *Let R be a field, and let X, Y be two simplicial sets. Then there is an isomorphism*

$$H_*(X; R) \otimes H_*(Y; R) \cong H_*(X \times Y; R).$$

3.1 Cohomology and cup product

Remark 3.16. By the universal coefficient theorem for cohomology, if R is a field, for any n:

$$H^n(X; R) \cong \mathrm{Hom}(H_n(X; \mathbb{Z}), R).$$

For a simplicial set X, consider the diagonal map $D : X \longrightarrow X \times X$ given by $D(x) = (x, x)$. It induces a map in the correspondent chain complexes $D_*\colon C_*(X;R) \longrightarrow C_*(X \times X;R)$. Composing with the Alexander-Witney map, one obtains a morphism of chains:

$$fD_*\colon C_*(X;R) \longrightarrow C_*(X \times X;R) \longrightarrow C_*(X;R) \otimes C_*(X;R)$$

which induces in cohomology

$$\smile\colon H^p(X;R) \otimes H^q(X;R) \longrightarrow H^{p+q}(X;R).$$

This map is called the *cup* product.

Corollary 3.17. *Under the cup product, the cohomology of a simplicial set constitute a graded R-algebra.*

Remark 3.18. Since the Alexander-Whitney map has been given explicitly, we can deduce an explicit form for the cup product as well. Let $\varphi \in C^p(X;R)$ and $\psi \in C^q(X;R)$. Then the cochain $\varphi \smile \psi \in C^{p+q}(X;R)$ can be computed by the formula

$$(\varphi \smile \psi)(\sigma) = \varphi \cdot (d_{p+1}...d_{n-1}d_n\sigma)\, \psi(\underbrace{d_0...d_0}_{n-q}\,\sigma)$$

for any $\sigma \in X_n$ (a generator of $C_n(X;R)$).

4 Cup-i products and Steenrod squares

Let $\mathbb{F}_2$ denote the ring with two elements. In [6], Steenrod introduced the cup-i products in the $\mathbb{F}_2$-cochains of simplicial complexes by means of formulae, and showed that those give rise to what now are called Steenrod squares. Those are cohomology operations that are stable, that is, commute with suspension, and are widely used in homotopy theory.

The formulae that Steenrod gave extend the Alexander-Whitney product on cochains. Moreover, the cup-0 coincides with the cup product we have already seen. To present the cup-i products we will follow the presentation given by Medina-Mardones in [1], in which the cup-i product formulae is seen as a symmetric multiplication in the cochains.

Notice that we will be working with semi-simplicial sets in this section for simplicity, since neither degeneration maps or degenerated simplices are needed. But one can define the cup-i products in simplicial sets simply switching to semi-simplicial and then back to simplicial, if needed.

A bit of notation will be needed before presenting the explicit formulas.

Definition 4.1. For any positive integer q, define the set

$$P_q = \{(u_1, ..., u_q) \in \mathbb{N}^q \mid 0 \leq u_1 < ... < u_q\}$$

and for any $n \geq q$, define the subset

$$P_q(n) = \{(u_1, ..., u_q) \in P_q \mid u_q \leq n\}.$$

Given P_q and $U \in P_q$, for each $u_i \in U = (u_1, ..., u_q) \in P_q$, define the *index of* u_i *in* U as

$$\mathrm{ind}_U(u_i) = u_i + i,$$

and define as well

$$U^+ = \{u_i \in U \mid \mathrm{ind}_U(u_i) \text{ even}\}$$
$$U^- = \{u_i \in U \mid \mathrm{ind}_U(u_i) \text{ odd}\}.$$

Lastly, set $P_0 = \{\emptyset\}$ and $d_\emptyset = \mathrm{id}$.

Definition 4.2. Let C be a chain complex of $\mathbb{F}_2$-modules. For any $n \geq 0$, define the *doubling map*

$$\begin{aligned} D_n \colon C_n &\longrightarrow (C \otimes C)_{2n} \\ x &\longmapsto x \otimes x. \end{aligned}$$

for any x generator of C_n.

Definition 4.3. Let X be a semi-simplicial set and let $C = C_*(X; \mathbb{F}_2)$. Then if $U = (u_1, ..., u_q) \in P_q$, define $d_U \colon C \to C$ in the following way. For any $\sigma \in C_n$,

$$d_U(x) = \begin{cases} d_{u_1} ... d_{u_q}(x) & \text{if } q, u_q \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.4. For integers k and n define the map $\nabla^{\binom{n}{n-k}} \colon C_n \to (C \otimes C)_{n+k}$ by

$$\nabla^{\binom{n}{n-k}} = \sum_{U \in P_{n-k}(n)} (d_{U^-} \otimes d_{U^+}) D_n$$

when $0 \leq k \leq n$, and by $\nabla^{\binom{n}{n-k}} = 0$ otherwise. And for any integer k define the map $\nabla_k \colon C \to C \otimes C$ by

$$\nabla_k = \bigoplus_n \nabla^{\binom{n}{n-k}}.$$

Definition 4.5. Let C be a chain complex of $\mathbb{F}_2$-modules. The *twisting map* T is defined by

$$\begin{aligned} T \colon C \otimes C &\longrightarrow C \otimes C \\ a \otimes b &\longmapsto b \otimes a. \end{aligned}$$

The following theorem is the main result proven in [1], and makes ∇_k a *symmetric multiplication.*

Theorem 4.6. *For integers q and n*

$$\partial_{2n-q} \nabla^{\binom{n}{q}} + \nabla^{\binom{n-1}{q-1}} \partial_n = (1+T) \nabla^{\binom{n}{q+1}}.$$

All these maps are depicted in the following diagram.

$$\begin{tikzcd}
C_{n-1} \arrow[dr, "\nabla^{\binom{n-1}{q-1}}"] & & C_n \arrow[ll, "\partial_n"'] \arrow[dl, "\nabla^{\binom{n}{q+1}}"'] \arrow[dr, "\nabla^{\binom{n}{q}}"] & \\
& (C \otimes C)_{2n-q-1} \arrow[loop below, "T"] & & (C \otimes C)_{2n-q} \arrow[ll, "\partial_{2n-q}"]
\end{tikzcd}$$

Proof. See [1]. $\square$

Definition 4.7. Let X be a semi-simplicial set and let i be a positive integer. There is a map called the *cup-i product*:

$$\smile_i \colon C^*(X; \mathbb{F}_2) \otimes C^*(X; \mathbb{F}_2) \longrightarrow C^*(X; \mathbb{F}_2)$$

that, given positive integers p and q, restricts to

$$\smile_i \colon C^p(X; \mathbb{F}_2) \otimes C^q(X; \mathbb{F}_2) \longrightarrow C^{p+q-i}(X; \mathbb{F}_2)$$

and is defined in the following way: if $\varphi \in C^p(X; \mathbb{F}_2)$, $\psi \in C^q(X; \mathbb{F}_2)$ and $\sigma \in C_{p+q-i}(X; \mathbb{F}_2)$, then

$$(\varphi \smile_i \psi)(\sigma) \coloneqq (\varphi \otimes \psi)(\nabla_i \sigma).$$

Remark 4.8. The map ∇_0 coincides with the Alexander-Whitney map described in 3.6. Thus the cup-0 product coincides with the cup product.

Proof. Let σ be a n-chain. Then

$$\begin{aligned}\nabla_0\,\sigma = \nabla^{\binom{n}{n}}\sigma &= \sum_{U\in P_n(n)} d_{U^-}\,\sigma\otimes d_{U^+}\,\sigma = \sum_{j=0}^{n+1} d_{j+1}...d_n\,\sigma\otimes d_0...d_{j-1}\,\sigma \\ &= \sum_{j=0}^{n+1} d_{j+1}...d_n\,\sigma\otimes \underbrace{d_0...d_0}_{j}\,\sigma = \sum_{p+q=n} d_{n-p+1}...d_n\,\sigma\otimes\underbrace{d_0...d_0}_{q}\,\sigma\end{aligned}$$

which is exactly the Alexander-Whitney map. □

Proposition 4.9. *The cup-i products satisfy the* coboundary formulas*, that is:*

$$\delta(\varphi\smile_i\psi) = \delta\varphi\smile_i\psi + \varphi\smile_i\delta\psi + \varphi\smile_{i-1}\psi + \psi\smile_{i-1}\varphi.$$

Proof. This is a consequence of theorem 4.6; moreover, it is actually equivalent to that fact.

Let φ and ψ be cochains of dimension p and q respectively, write $n = p+q$ and let σ be a $(n-i+1)$-chain. Then, using theorem 4.6:

$$\begin{aligned}\big(\delta(\varphi\smile_i\psi)\big)(\sigma) =& (\varphi\smile_i\psi)(\delta\sigma) = (\varphi\otimes\psi)(\nabla_i\partial\sigma) = (\varphi\otimes\psi)(\nabla^{\binom{n-i}{n-2i}}\partial\sigma) \\ =&(\varphi\otimes\psi)\Big(\partial\nabla^{\binom{n-i+1}{n-2i+1}}\sigma + (1+T)\nabla^{\binom{n-i+1}{n-2i+2}}\sigma\Big) \\ =&\delta(\varphi\otimes\psi)(\nabla^{\binom{n-i+1}{n-2i}}\sigma) + (\varphi\otimes\psi)\Big((1+T)\nabla^{\binom{n-i+1}{n-2i+2}}\sigma\Big) \\ =&(\delta\varphi\otimes\psi)(\nabla^{\binom{n-i+1}{n-2i}}\sigma) + (\varphi\otimes\delta\psi)(\nabla^{\binom{n-i+1}{n-2i}}\sigma) + \\ &(\varphi\otimes\psi)(\nabla^{\binom{n-i+1}{n-2i+2}}\sigma) + (\psi\otimes\varphi)(\nabla^{\binom{n-i+1}{n-2i+2}}\sigma) \\ =&(\delta\varphi\smile_i\psi)(\sigma) + (\varphi\smile_i\delta\psi)(\sigma) + \\ &(\varphi\smile_{i-1}\psi)(\sigma) + (\psi\smile_{i-1}\varphi)(\sigma)\end{aligned}$$

as wanted. □

This formula is crucial since it is used to prove that the operation of sending a cochain to the cup-i product against itself sends cocycles to cocycles. This together with some other properties allow the cup-i squarings to give rise to to operations in cohomology.

Proposition 4.10. *Let X be a semi-simplicial set and let $\varphi, \psi \in C^n(X; \mathbb{F}_2)$ be two cocycles (that is, $\delta\varphi = \delta\psi = 0$). Denote by $[\cdot]$ the cohomology class of a cycle. Then:*

$$\delta(\varphi \smile_i \varphi) = 0 \tag{4.1}$$

$$[\varphi] = [\psi] \implies [\varphi \smile_i \varphi] = [\psi \smile_i \psi] \tag{4.2}$$

$$[(\varphi + \psi) \smile_i (\varphi + \psi)] = [\varphi \smile_i \varphi + \psi \smile_i \psi] \tag{4.3}$$

Proof. For 4.1, using proposition 4.9:

$$\delta(\varphi \smile_i \varphi) = \delta\varphi \smile_i \varphi + \varphi \smile_i \delta\varphi + \varphi \smile_{i+1} \varphi + \varphi \smile_{i+1} \varphi = 0.$$

For 4.2, notice that

$$\varphi \smile_{i+1} \psi = \varphi \smile_i \psi + \psi \smile_i \varphi$$

thus $\delta(\varphi \smile_i \psi + \psi \smile_i \varphi) = 0$. Now

$$\begin{aligned} &[\varphi] = [\psi] \implies \delta(\varphi + \psi) = 0 \underset{4.1}{\implies} \delta((\varphi + \psi) \smile_i (\varphi + \psi)) = 0 \implies \\ &\delta(\varphi \smile_i \varphi + \varphi \smile_i \psi + \psi \smile_i \varphi + \psi \smile_i \psi) \implies \\ &\delta(\varphi \smile_i \varphi + \psi \smile_i \psi) \implies [\varphi \smile_i \varphi] = [\psi \smile_i \psi]. \end{aligned}$$

Finally, 4.3 is immediate:

$$\begin{aligned} &\delta\big((\varphi + \psi) \smile_i (\varphi + \psi) + \varphi \smile_i \varphi + \psi \smile_i \psi\big) = \\ &\delta\big(\varphi \smile_i \varphi + \varphi \smile_i \psi + \psi \smile_i \varphi + \psi \smile_i \psi + \varphi \smile_i \varphi + \psi \smile_i \psi\big) = 0 \end{aligned}$$

Therefore $[(\varphi + \psi) \smile_i (\varphi + \psi)] = [\varphi \smile_i \varphi + \psi \smile_i \psi]$. □

Notice in this proof that working on $\mathbb{F}_2$ is needed, so repeated items get canceled. There are analogous operations for $\mathbb{F}_p$, but they will not be treated here.

Definition 4.11. Proposition 4.10 tells us that the operation $\varphi \to \varphi \smile_i \varphi$ sends cocycles to cocycles and cohomologous cycles into cohomologous cycles. Therefore it induces an homomorphism:

$$\mathrm{Sq}_i \colon H^n(X; \mathbb{F}_2) \longrightarrow H^{2n-i}(X; \mathbb{F}_2).$$

A change of index is applied so it is consistent with the degree of the homomorphism. Specifically let $\mathrm{Sq}^i = \mathrm{Sq}_{n-i}$, and then:

$$\mathrm{Sq}^i \colon H^n(X; \mathbb{F}_2) \longrightarrow H^{n+i}(X; \mathbb{F}_2).$$

This is called the *i-th Steenrod square*.

In [8], Steenrod and Epstein proved the following properties of the Steenrod squares:

1. $\mathrm{Sq}^i\colon H^n(X;\mathbb{F}_2) \to H^{n+i}(X;\mathbb{F}_2)$ is a natural homomorphism.
2. Sq^0 is the identity homomorphism.
3. If $\varphi \in H^n(X;\mathbb{F}_2)$, then $\mathrm{Sq}^n x = x \smile x$.
4. If $i > n$, then $\mathrm{Sq}^i(\varphi) = 0$ for any $\varphi \in H^n(X;\mathbb{F}_2)$.
5. *Cartan formula*: $\mathrm{Sq}^k(\varphi \smile \psi) = \sum_{i=0}^k (\mathrm{Sq}^i \varphi) \smile (\mathrm{Sq}^{k-i} \psi)$.
6. Sq^1 is the Bockstein homomorphism.
7. *Adem relations*:

   $$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{k=0}^{\lceil i/2 \rceil} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k,$$

   where the binomial coefficient is taken mod 2.
8. Sq^i commutes with respect to suspension; Steenrod squares are *stable* cohomology operations.

Moreover, they proved that the first five imply the others, and that this properties completely characterize the Steenrod squares, thus can be taken as axioms. Notice that properties 2, 3 and 4 are immediate from our construction. For a detailed treatment of Steenrod squares and this properties, see [8] or [5].

5 Signed simplicial sets

In this section we introduce signed simplicial sets, together with the functors needed to compute homology and cohomology in this objects like we did with simplicial sets.

Definition 5.1. A *pointed set* S is an ordered pair $(X, *)$ where X is a set and $*$ is an element of X called the *base point*. The category of pointed sets $\mathbf{Set}_*$ has pointed sets as objects and maps which send base point to base point as morphisms.

Definition 5.2. Given pointed sets X and Y, the *wedge sum* $X \vee Y$ is the pointed set given by:

$$X \vee Y = {}^{X \cup Y}\!\diagup_{*_X \sim *_Y}.$$

Definition 5.3. For pointed sets X and Y, the *smash product* $X \wedge Y$ is the pointed set given by:

$$X \wedge Y = {}^{X \times Y}\!\diagup_{(X \times \{*\} \cup \{*\} \times Y)}.$$

Definition 5.4. Let X, Y be two pointed semi-simplicial sets. The *join* $X * Y$ of X and Y is a semi-simplicial set that has the following set of n-simplices:

$$(X * Y)_n = \Big(\bigvee_{i+j=n+1} X_i \wedge Y_j \Big) \vee X_n \vee Y_n.$$

The face maps are defined in the following way.

- If $x \in X_n \subset (X * Y)_n$ or $y \in Y_n \subset (X * Y)_n$, the face maps are the ones in X_n and Y_n respectively, and the images fall into $X_{n-1} \subset (X * Y)_{n-1}$ or $Y_{n-1} \subset (X * Y)_{n-1}$ respectively.
- If $(x, y) \in X_i \times Y_j$

  $$d_k(x, y) = \begin{cases} (d_k x, y) & \text{if } k \leq i, i \neq 0 \\ (x, d_{k-i-1} y) & \text{if } k > i, j \neq 0. \end{cases}$$

  If $i = 0$

  $$d_0(x, y) = y \in Y_{n-1} \subset (X * Y)_{n-1}.$$

  If $j = 0$

  $$d_n(x, y) = x \in X_{n-1} \subset (X * Y)_{n-1}.$$

Definition 5.5. There is a functor, analogous to the functor defined in 2.10:

$$\begin{aligned} A_R^* \colon \mathbf{Set}_* &\longrightarrow R\text{-}\mathbf{Mod} \\ X &\longmapsto {}^{R\langle X \rangle}\!\diagup_{R\langle * \rangle} \end{aligned}$$

and if $f \colon X \to Y$ and x is a generator of ${}^{R\langle X \rangle}\!\diagup_{R\langle * \rangle}$, then $A_R^*(f)(x) = f(x) \in {}^{R\langle Y \rangle}\!\diagup_{R\langle * \rangle}$.

Definition 5.6. Let us introduce the *category of signed sets*, denoted by $\mathbf{Set}_{\pm}$.

- The objects are pointed sets.
- Given $X, Y \in \mathbf{Set}_{\pm}$, a morphism between X and Y in $\mathbf{Set}_{\pm}$ is an ordered pair of morphisms of pointed sets $f^+, f^- \colon X \to Y$ that satisfy

$$(f^+)^{-1}(Y \smallsetminus \{*\}) \cap (f^-)^{-1}(Y \smallsetminus \{*\}) = \varnothing.$$

- Let $f \colon X \to Y$, $g \colon Y \to Z$ be two morphisms of signed sets, with $f = (f^+, f^-)$ and $g = (g^+, g^-)$. Then $(h^+, h^-) = h = g \circ f$ is defined by:

$$h^+(x) = \begin{cases} (g^+ \circ f^+)(x) \text{ if } f^+(x) \neq * \\ (g^- \circ f^-)(x) \text{ otherwise} \end{cases}$$

$$h^+(x) = \begin{cases} (g^- \circ f^+)(x) \text{ if } f^+(x) \neq * \\ (g^+ \circ f^-)(x) \text{ otherwise} \end{cases}$$

  for any $x \in X$.

- The identity in $\mathbf{Set}_{\pm}$ is $(\mathrm{id}, *)$, where $*$ denotes the constant map to the base point.

Remark 5.7. Alternatively, one can think of morphisms in $\mathbf{Set}_{\pm}$ as an ordered pair of maps (f, σ), where $f \colon X \to Y$ is a morphism of pointed sets and

$$\sigma \colon f^{-1}(Y \smallsetminus \{*\}) \longrightarrow \{+1, -1\}.$$

Then the identity is $(id, +1)$, and the composition is defined by

$$(f_2, \sigma_2) \circ (f_1, \sigma_1) = (f_2 \circ f_1,\, (\sigma_2 \circ f_1) \cdot \sigma_1).$$

The first definition is the one that will be used, but this one may be a little more intuitive. In the signed set category, for any morphism, each element of X that does not go to the base point have an assigned sign; either positive or negative.

Proof. If $(f^+, f^-) \colon X \to Y$, take (f, σ) in the following way:

$$f(x) = \begin{cases} f^+(x) \text{ if } f^+(x) \neq * \\ f^-(x) \text{ otherwise} \end{cases}$$

and for $x \in f^{-1}(Y \smallsetminus \{*\})$,

$$\sigma(x) = \begin{cases} +1 \text{ if } (f^+)(x) \neq * \\ -1 \text{ otherwise.} \end{cases}$$

On the other hand, given $(f, \sigma)\colon X \to Y$, take (f^+, f^-) in the following way:

$$f^+(x) = \begin{cases} f(x) \text{ if } \sigma(x) = +1 \\ * \text{ otherwise} \end{cases}$$

$$f^-(x) = \begin{cases} f(x) \text{ if } \sigma(x) = -1 \\ * \text{ otherwise.} \end{cases}$$

□

As in the definition 2.10, one wants to have the "free" functor associated to a commutative ring R. The morphisms in signed sets encode more information than in ordinary sets; this makes possible that the induced homomorphisms send generators to generators (as usual) but also to generators multiplied by -1.

Definition 5.8. There is a functor

$$\begin{aligned} A_R^{\pm}\colon \mathbf{Set}_{\pm} &\longrightarrow R\text{-}\mathbf{Mod} \\ X &\longmapsto R\langle X\rangle / R\langle * \rangle \end{aligned}$$

which sends $f = (f^+, f^-)\colon X \to Y$ to the homomorphism

$$\begin{aligned} A_R^{\pm}(f)\colon R\langle X\rangle / R\langle * \rangle &\longrightarrow R\langle Y\rangle / R\langle * \rangle \\ x &\longmapsto f^+(x) - f^-(x) \end{aligned}$$

for any x generator of $R\langle X\rangle / R\langle * \rangle$.

The following category will be useful because it contains the category $\mathbf{Set}_{\pm}$ (in a sense that will be specified later) and it can be easier to work with.

Definition 5.9. Denote by Σ_2-$\mathbf{Set}_*$ the category of pointed sets with a free action of Σ_2 (the base point is invariant by the action). Σ_2 denotes the group with two elements. A morphism in Σ_2-$\mathbf{Set}_*$ is a map which is invariant with respect to the action.

Definition 5.10. There is a functor G:

$$\begin{aligned} G\colon \mathbf{Set}_{\pm} &\longrightarrow \Sigma_2\text{-}\mathbf{Set}_* \\ X &\longmapsto X \vee \overline{X} \end{aligned}$$

where $\overline{X}$ is a disjoint copy of X through the assignment $x \to \overline{x}$. The group Σ_2 acts by sending $x \to \overline{x}$ and $\overline{x} \to x$ for every $x \in X$. If $f\colon X \to Y$ is a morphism of signed sets, then $G(f)\colon G(X) \to G(Y)$ is defined by

$$G(f)(x) = \begin{cases} f^+(x) \text{ if } f^+(x) \neq * \\ \overline{f^-(x)} \text{ otherwise} \end{cases}$$

$$G(f)(\overline{x}) = \begin{cases} \overline{f^+(x)} \text{ if } f^+(x) \neq * \\ f^-(x) \text{ otherwise.} \end{cases}$$

Remark 5.11. The functor G is injective in objects and faithful (the function between hom sets is injective). Thus $\mathbf{Set}_{\pm}$ can be seen as a subcategory of Σ_2-$\mathbf{Set}_*$, with G being the inclusion.

Having introduced the $\mathbf{Set}_*$ and $\mathbf{Set}_{\pm}$ categories, one can look now at the simplicial (and semi-simplicial) objects in these categories. For example:

Definition 5.12. A *pointed simplicial set* is a functor

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}_*.$$

Remark 5.13. Follows from the definition that a pointed simplicial set is just like a simplicial set but with a distinguished n-simplex for each n, corresponding to a the base point of the set X_n. Notice also that all face and degeneration maps from one of those simplices goes to another of them, in the appropriate dimension.

By convention, we will only write the "base simplex" of dimension 0, denoting it by $*$, and we will omit the others when listing n-simplices of higher dimensions. The symbol $*$ can also be used to refer to a base simplex of a dimension greater than 0 when clear from the context.

This convention is also used for pointed semi-simplicial sets and signed simplicial or signed semi-simplicial sets.

Using the functors A^*_R and $A^{\pm}_R$ one can compute the homology and cohomology of a pointed simplicial set or a signed simplicial set respectively proceeding like in section 2.3. The chain and cochain complexes of a signed simplicial or signed semi-simplicial set X will be denoted by $C^{\pm}_*(X;R)$ and $C^*_{\pm}(X;R)$ respectively.

6 Steenrod squares for signed simplicial sets

6.1 Construction of the Steenrod squares

Now that we have the tools to compute the cohomology of a signed simplicial set, our aim is to find Steenrod squares in that cohomology. First, some justification of what we are about to do.

First, the functor G will be used to turn our signed semi-simplicial set into a pointed semi-simplicial set with a Σ_2 action. Now given a pointed semi-simplicial set Y with a Σ_2 action, one defines its *signed realization* as the homotopic quotient of $S^1 * |Y|$ by the diagonal action of Σ_2, making Σ_2 act on S^1 by reflection. Here $*$ denotes the join operation in topological spaces. Let B be a pointed semi-simplicial set with a Σ_2 action which is a model of S^1 with the reflection action. Then

$$S^1 * |Y| \simeq |B| * |Y| \underset{\star}{=} |B * Y|.$$

For $\star$, see [17].

Since the Σ_2 action is free on $G(X)$, then it is also free in $B*G(X)$, therefore the homotopic quotient is the same as the topological quotient ${|B * G(X)|}\big/{\Sigma_2} = \left|{B * G(X)}\big/{\Sigma_2}\right|$.

We are going to compute the chain complex $C_*\left({B * G(X)}\big/{\Sigma_2}; \mathbb{F}_2\right)$ and prove that it has the same homology as the complex $C_*^{\pm}(X;\mathbb{F}_2)$, as expected. Then the formulas we have in the first complex will provide us of Steenrod squares in the second one.

Definition 6.1. Define the semi-simplicial pointed set $B \in \mathbf{Set}_*^{\Delta_{\text{inj}}^{\text{op}}}$ by:

$$\begin{aligned} B_1 &= \{a_0, a_1\} \\ B_0 &= \{*, v\} \end{aligned}$$

and

$$\begin{aligned} d_0 a_0 &= * & \qquad d_1 a_0 &= v \\ d_0 a_1 &= * & \qquad d_1 a_1 &= v. \end{aligned}$$

And let Σ_2 act on B by switching the 1-simplices a_0 and a_1 and leaving the 0-simplices invariant.

Let's compute ${B * G(X)}\big/{\Sigma_2}$. First without the action:

$$\begin{aligned} (B * G(X))_{n+2} = &\big(\{a_0, a_1, *\} \wedge (X_n \vee \overline{X}_n)\big) \vee \big(\{v, *\} \wedge (X_{n+1} \vee \overline{X}_{n+1})\big) \\ &\vee B_{n+2} \vee (X_{n+2} \vee \overline{X}_{n+2})\big) \end{aligned}$$

Taking the quotient:

$$\left(B * G(X) \big/ \Sigma_2\right)_{n+2} = \bigcup_{x \in X_n} \{\underset{\substack{\| \\ [(a_1,\overline{x})]}}{[(a_0,x)]}, \underset{\substack{\| \\ [(a_1,x)]}}{[(a_0,\overline{x})]}\} \cup \bigcup_{x \in X_{n+1}} \{\underset{\substack{\| \\ [(v,\overline{x})]}}{[(v,x)]}\}$$

To simplify the notation, we will write:

$$\begin{aligned} [(a_0,x)] &= x \\ [(a_0,\overline{x})] &= \overline{x} \\ [(v,x)] &= (v,x). \end{aligned}$$

The face maps are given by:

$$\begin{aligned} &d_0(v,x) = * \\ &d_i(v,x) = \begin{cases} (v, d^+_{i-1}x) \text{ if } d^+_{i-1} \neq * \\ (v, d^-_{i-1}x) \text{ otherwise} \end{cases} \quad \text{if } i \geq 1 \end{aligned}$$

$$\begin{aligned} &d_0 x = d_0 \overline{x} = * \\ &d_1 x = d_1 \overline{x} = (v,x) \\ &d_i x = \begin{cases} d^+_{i-2}x \text{ if } d^+_{i-2}x \neq * \\ \overline{d^-_{i-2}x} \text{ otherwise} \end{cases} \quad \text{if } i \geq 2 \\ &d_i \overline{x} = \begin{cases} \overline{d^+_{i-2}x} \text{ if } d^+_{i-2}x \neq * \\ d^-_{i-2}x \text{ otherwise} \end{cases} \quad \text{if } i \geq 2. \end{aligned}$$

One can proceed now to compute the Moore chain complex of this semi-simplicial set.

$$C\left(B * G(X) \big/ \Sigma_2; \mathbb{F}_2\right)_{n+2} = \mathbb{F}_2 \left\langle \bigcup_{x \in X_n} \{x, \overline{x}\} \cup \bigcup_{x \in X_{n+1}} \{(v,x)\} \right\rangle.$$

The face maps can be written in a cleaner way now.

$$\begin{aligned} &d_0(v,x) = 0 \\ &d_i(v,x) = (v, d^+_{i-1}x) + (v, d^-_{i-1}x) \text{ if } i \geq 1 \end{aligned}$$

$$\begin{aligned} &d_0 x = d_0 \overline{x} = 0 \\ &d_1 x = d_1 \overline{x} = (v,x) \\ &d_i x = d^+_{i-2}x + \overline{d^-_{i-2}x} \text{ if } i \geq 2 \\ &d_i \overline{x} = \overline{d^+_{i-2}}x + d^-_{i-2}x \text{ if } i \geq 2. \end{aligned}$$

Before introducing the crucial map that lets us compute Steenrod squares in signed simplicial sets through the complex $C_*\left({}^{B*G(X)}\!\diagup_{\Sigma_2};\mathbb{F}_2\right)$, a quick definition.

Definition 6.2. Let C be a chain complex. The *suspension* of C, denoted by ΣC, is the chain complex obtained by shifting the degrees up by one, that is:

$$\Sigma C_n = C_{n-1}$$

for each n.

Consider the following map.

$$\begin{aligned} \Phi\colon \Sigma^2 C_*^{\pm}(X;\mathbb{F}_2) &\longrightarrow C_*\left({}^{B*G(X)}\!\diagup_{\Sigma_2};\mathbb{F}_2\right) \\ x &\longmapsto x+\overline{x} \end{aligned}$$

Remark 6.3. The map Φ is a chain map.

Proof. Let $x\in X$ be a generator in $\Sigma^2 C_*^{\pm}(X;\mathbb{F}_2)$. Then

$$\begin{aligned} \partial\Phi(x) &= \partial x+\partial\overline{x} \\ &= (x,v)+\sum_{i=0}^{n} d_i^+ x+\sum_{i=0}^{n}\overline{d_i^- x}+(x,v)+\sum_{i=0}^{n}\overline{d_i^+ x}+\sum_{i=0}^{n} d_i^- x \\ &= \Phi(\sum_{i=0}^{n} d_i^+ x+\sum_{i=0}^{n} d_i^- x)=\Phi(\partial x). \end{aligned}$$

□

Theorem 6.4. *The chain map Φ induces an isomorphism in cohomology.*

Proof. For simplicity, let us write

$$\Sigma^2 C_*^{\pm}(X;\mathbb{F}_2)=C_*^{\pm}$$

$$C_*\left({}^{B*G(X)}\!\diagup_{\Sigma_2};\mathbb{F}_2\right)=C_*.$$

Notice that Φ is injective. Thus there is a short exact sequence of chain complexes

$$0 \longrightarrow \Sigma^2 C_*^{\pm} \xrightarrow{\Phi} C_* \xrightarrow{\pi} \operatorname{coker}\Phi \longrightarrow 0.$$

Recall that $\operatorname{coker}\Phi_n = {}^{C_n}\!\diagup_{\operatorname{Im}\Phi_n}$, and the cokernel chain complex is given by

$$\ldots \xleftarrow{\partial_{n-1}} \operatorname{coker}\Phi_{n-1} \xleftarrow{\partial_n} \operatorname{coker}\Phi_n \xleftarrow{\partial_{n+1}} \operatorname{coker}\Phi_{n+1} \xleftarrow{\partial_{n+2}} \ldots$$

We claim that this chain complex has trivial homology, moreover, it is actually null-homotopic. Let us see that. Notice that for any $x\in C_n$ generator, $[x]=[\overline{x}]$

in coker Φ_n since $x - \overline{x} \in \operatorname{Im}\Phi_n$. Omitting the square brackets for simplicity define, for each n, the homotopy map:

$$h_n(x) = 0$$
$$h_n((v,x)) = x.$$

Now $\mathrm{id}_n = h_{n-1}\partial_n + \partial_{n+1}h_n$. Indeed:

$$h_{n-1}\partial_n(x) + \partial_{n+1}h_n(x) = h_{n-1}\Big((v,x) + \sum_{i=0}^{n-2}(d_i^+x + d_i^-x)\Big) + 0 = x = \mathrm{id}(x)$$

and

$$h_{n-1}\partial_n((v,x)) + \partial_{n+1}h_n((v,x)) = h_{n-1}\Big(\sum_{i=0}^{n-1}((v,d_i^+x) + (v,d_i^+x))\Big) + \partial_{n+1}(x) =$$

$$\sum_{i=0}^{n-1}(d_i^+x + d_i^+x) + (v,x) + \sum_{i=0}^{n-1}(d_i^+x + d_i^+x) = (v,x) = \mathrm{id}((v,x)).$$

Therefore, by Künneth, the cokernel chain complex has trivial cohomology as well. Now going back to the short exact sequence of chain complexes

$$0 \longrightarrow \Sigma^2 C_*^{\pm} \xrightarrow{\Phi} C_* \xrightarrow{\pi} \operatorname{coker}\Phi \longrightarrow 0$$

one can take the induced long sequence in cohomology, where since the cohomology of coker Φ is zero for any n, then the cohomology modules for $C_*^{\pm}$ and C_* are isomorphic, with the isomorphism induced by Φ. □

Using this morphism, we can define a product in the cochains of a signed semi-simplicial set. Let $\varphi, \psi \in \Sigma^2 C^*_{\pm}(X;\mathbb{F}_2)$. Define cochains

$$\varphi', \psi' \in C_*\left({B * G(X)}/{\Sigma_2}; \mathbb{F}_2\right)$$

by

$$\varphi'(x) = \varphi(x) \qquad \psi'(x) = \psi(x)$$
$$\varphi'(\overline{x}) = 0 \qquad \psi'(\overline{x}) = 0$$
$$\varphi'((v,x)) = 0 \qquad \psi'((v,x)) = 0.$$

Notice that these satisfy:

$$\varphi = \Phi^*(\varphi') \qquad \psi = \Phi^*(\psi')$$

in other words, $\varphi = \varphi' \circ \Phi$ and $\psi = \psi' \circ \Phi$. Now define a non-linear product on the cochains of a signed simplicial sets using the cup-i product in simplicial sets:

$$\varphi \vee_i \psi \coloneqq \Phi^*(\varphi' \smile_{i+2} \psi')$$

that will give rise to the Steenrod squares, since Φ induces isomorphism in cohomology.

Remark 6.5. The adjustment of the degree is due to the double suspension applied to the chain complex $C^*_\pm(X;\mathbb{F}_2)$. For example, assume $\varphi \in C^p_\pm(X;\mathbb{F}_2)$, $\phi \in C^p_\pm(X;\mathbb{F}_2)$ and $x \in C^\pm_n(X;\mathbb{F}_2)$. If we want $\varphi \vee_i \psi$ to have degree n, we need to set $i = p+q-n$ (just as in section 4). Notice that, in $C_*(B * G(X)/_{\Sigma_2};\mathbb{F}_2)$:

$$\begin{aligned}\dim \varphi' &= p+2\\ \dim \psi' &= q+2\\ \dim \Phi(x) &= n+2\end{aligned}$$

so the right cup-i product is $\smile_{i+2}$, since $p+2+q+2-(n+2) = i+2$.

Corollary 6.6. *Steenrod squares in signed simplicial sets can be computed by*

$$\mathrm{Sq}^i(\varphi) \coloneqq [\varphi \vee_{n-i} \varphi]$$

for any $\varphi \in C^n_\pm(X;\mathbb{F}_2)$.

6.2 Formulae

In chapter 4, explicit formulas for the cup-i were given. Now we can develop the Steenrod squares we constructed using the cup-i product to find more explicit formulae in terms of the simplicial set X.

Let $x \in C^\pm_n(X;\mathbb{F}_2)$ be a generator and $\varphi, \psi \in C^*_\pm(X;\mathbb{F}_2)$ with $\dim\varphi + \dim\psi = n+i$.

$$(\varphi \vee_i \psi)(x) = (\varphi' \smile_{i+2} \psi')(\Phi(x)) = (\varphi' \smile_{i+2} \psi')(x+\overline{x}) =$$
$$(\varphi' \otimes \psi')(\nabla_{i+2}(x+\overline{x})) = (\varphi' \otimes \psi')(\nabla^{\binom{n+2}{n-i}}(x+\overline{x}))$$

Developing inside the parenthesis:

$$\nabla^{\binom{n+2}{k}}(x+\overline{x}) = \sum_{U\in P_k(n+2)} (d_{U^-} \otimes d_{U^+})(x\otimes x + \overline{x}\otimes\overline{x}) =$$
$$\sum_{U\in P_k(n+2)} ((d^+_{U^-} + d^-_{U^-}) \otimes (d^+_{U^+} + d^-_{U^+}))(x\otimes x + \overline{x}\otimes\overline{x}) =$$

expanding:

$$\begin{aligned}
&\sum_{U\in P_k(n+2)} ((d^+_{U^-}) \otimes (d^+_{U^+})(x\otimes x) &&+ \sum_{U\in P_k(n+2)} ((d^+_{U^-}) \otimes (d^-_{U^+})(x\otimes x)+\\
&\sum_{U\in P_k(n+2)} ((d^-_{U^-}) \otimes (d^+_{U^+})(x\otimes x) &&+ \sum_{U\in P_k(n+2)} ((d^-_{U^-}) \otimes (d^-_{U^+})(x\otimes x)+\\
&\sum_{U\in P_k(n+2)} ((d^+_{U^-}) \otimes (d^+_{U^+})(\overline{x}\otimes \overline{x}) &&+ \sum_{U\in P_k(n+2)} ((d^+_{U^-}) \otimes (d^-_{U^+})(\overline{x}\otimes \overline{x})+\\
&\sum_{U\in P_k(n+2)} ((d^-_{U^-}) \otimes (d^+_{U^+})(\overline{x}\otimes \overline{x}) &&+ \sum_{U\in P_k(n+2)} ((d^-_{U^-}) \otimes (d^-_{U^+})(\overline{x}\otimes \overline{x})
\end{aligned}$$

Recall that φ' and ψ' send all $\overline{x}$ elements to zero, so those can be taken out from the expression since they do not matter for computing the $\vee_i$ product. So each tensor product that has at least one element of the $\overline{x}$ kind (after applying de d maps) is irrelevant. Removing these, we are left with:

$$\sum_{U \in P_k(n+2)} (d^+_{U^-} \otimes d^+_{U^+})(x \otimes x) + \sum_{U \in P_k(n+2)} (d^-_{U^-} \otimes d^-_{U^+})(\overline{x} \otimes \overline{x}).$$

In section 6.1, the face maps in $C_*(B * G(X)/_{\Sigma_2}; \mathbb{F}_2)$ were explicitly given. Recall that $d_0 x = d_0 \overline{x} = 0$ and that $d_1 x = d_1 \overline{x} = (v, x)$. And since $\varphi'((v,x)) = \psi'((v,x)) = 0$, tensor products with at least one of those elements are irrelevant for computing $\varphi \vee_i \psi$. Removing them we are left with:

$$\sum_{\substack{U \in P_k(n+2) \\ 0 \notin U \\ 1 \notin U}} (d^+_{U^-} \otimes d^+_{U^+})(x \otimes x) + \sum_{\substack{U \in P_k(n+2) \\ 0 \notin U \\ 1 \notin U}} (d^-_{U^-} \otimes d^-_{U^+})(\overline{x} \otimes \overline{x}) =$$
$$\sum_{U \in P_k(n)} (d^+_{U^-} \otimes d^+_{U^+})(x \otimes x) + \sum_{U \in P_k(n)} (d^-_{U^-} \otimes d^-_{U^+})(\overline{x} \otimes \overline{x}).$$

This last expression is close to the one we had for simplicial sets. Let's then redefine the $\vee_i$ product with this formulae.

Definition 6.7. Let X be a signed semi-simplicial set. For integers k and n define the map $\dot{\nabla}^{\binom{n}{n-k}} \colon C^{\pm}_n(X; \mathbb{F}_2) \to (C^{\pm}_*(X; \mathbb{F}_2) \otimes C^{\pm}_*(X; \mathbb{F}_2))_{n+k}$ by

$$\dot{\nabla}^{\binom{n}{n-k}} = \sum_{U \in P_{n-k}(n)} (d^+_{U^-} \otimes d^+_{U^+} + d^-_{U^-} \otimes d^-_{U^+}) D_n$$

when $0 \leq k \leq n$, and by $\dot{\nabla}^{\binom{n}{n-k}} = 0$ otherwise. And for any integer k define the map $\dot{\nabla}_k \colon C \to C \otimes C$ by

$$\dot{\nabla}_k = \bigoplus_n \dot{\nabla}^{\binom{n}{n-k}}.$$

Definition 6.8. Given $\varphi, \psi \in C^*_{\pm}(X; \mathbb{F}_2)$, define the cochain $\varphi \vee_i \psi$ by:

$$(\varphi \vee_i \psi)(\sigma) = (\varphi \otimes \psi)(\dot{\nabla}_i \sigma)$$

for any $\sigma \in C^{\pm}_*(X; \mathbb{F}_2)$.

6.3 Final considerations

A product in the cochains of a signed simplicial set has been presented, which induces Steenrod squares in the cohomology. The natural question now is if the $\dot{\nabla}$ product defines a symmetric multiplication, that is, satisfies the formula in theorem 4.6. In other words, is the $\vee_i$ product *a* cup-i product in signed simplicial sets?

The answer is no, since it already fails to satisfy

$$\partial_{2n-1}\dot{\nabla}^{\binom{n}{1}} + \dot{\nabla}^{\binom{n-1}{0}}\partial_n = (1+T)\dot{\nabla}^{\binom{n}{2}}.$$

Could this product exist, and what could be the techniques to find it? A way to solve it would be making the map in theorem 6.4 an isomorphism. This is not possible, at least with the model B of S^1 we have used here. But attempts with handful other more complicate models show that this may be an ineffective path.

A second approach is to find a way to define the symmetric multiplication $\nabla^{\binom{n}{k}}$ inductively over k using the formula in theorem 4.6. One should proof here that you can always find the next formula, which is not necessarily true;

$$\partial_{2n-k}\dot{\nabla}^{\binom{n}{k}} + \dot{\nabla}^{\binom{n-1}{k-1}}\partial_n = (1+T)\dot{\nabla}^{\binom{n}{k+1}}$$

notice that you need the left hand side to be symmetric for each k.

Having worked "by hand" up to $\dot{\nabla}^{\binom{n}{3}}$, seems like:

- There are lots of choices in each iteration, so a lot cup-i products could exist.
- The formulas get complicated really fast, and it is difficult to organize them in a compact way.
- Proving that you can always find the next formula (that is, showing that this method works) is rather difficult.

Still, we think that an adequate treatment of this approach might produce the desired outcome; if not explicit formulas, at least proving that the symmetric multiplication exists.

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