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MÀSTER DE MATEMÀTICA AVANÇADA

Facultat de Matemàtiques i Informàtica
Universitat de Barcelona

FLUCTUATION IN THE ZERO SET OF THE PARABOLIC GAUSSIAN ANALYTIC FUNCTION

Author: Alexis Arraz Almirall

Supervisor: Dr. Xavier Massaneda Clares
Department: Departament de Matemàtiques i Informàtica

Barcelona, June 27, 2019

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Abstract

In this project we study the fluctuation of the zero set process of the *parabolic Gaussian analytic function*, denoted $\mathbb{S}^2$-GAF and where $\mathbb{S}^2$ is the Riemann sphere. There exist several ways to measure such fluctuations. One of them is to compute the *variance* of certain variables counting the number of points of the process inside a given region. Some asymptotics of such variables will lead us to conclude that the $\mathbb{S}^2$-GAF process is more rigid than the Poisson process on $\mathbb{S}^2$ having, in mean, the same number of points as the $\mathbb{S}^2$-GAF process. Also, we will see that the $\mathbb{S}^2$-GAF process tends, as the intensity goes to infinity, to the planar GAF. Another point of view to study the fluctuations of the $\mathbb{S}^2$-GAF is the so-called *large deviations*, i.e., to measure how certain linear statistics deviate from its average by a fraction of its same average. The latter allows us to estimate the *hole probability*, i.e., the probability that the point process does not meet a given disk.

2010 Mathematics Subject Classification. 30B20, 30C15, 60F10, 60G55.

Keywords: fluctuation, Gaussian analytic function, hole probability, large deviations, linear statistic, zero point process.

Acknowledgments

To my advisor, Dr. Xavier Massaneda Clares, for his weekly dedication and patience, and for offering me the chance to deepen in the topic of Gaussian analytic functions.

Chapter 1

Introduction

The main object of study in this project is a specific type of *point process* in the Riemann sphere $\mathbb{S}^2 := \mathbb{C} \cup \{\infty\}$, that is, a random sequence of points in $\mathbb{S}^2$. Probably the best known point process is the *Poisson process*, denoted $\mathcal{X}$, which is characterized by two properties:

- For all $A \subseteq \mathbb{S}^2$, the random variable $n_A := \#(A \cap \mathcal{X})$ follows a Poisson distribution of parameter $\lambda(A)$, which is the area of A in $\mathbb{S}^2$.
- If for all $A, B \subset \mathbb{S}^2$ we have $A \cap B = \emptyset$, then n_A and n_B are independent random variables.

Independence is natural sometimes, but sometimes is not. For example, some physical phenomena in quantum mechanics cannot be explained by using such a process, due to the clumping points. In the 90's some physicists realized that a good model for point processes with local repulsion are the zero sets of some *random analytic functions.*

1.1 Gaussian analytic functions

Several random analytic functions can be considered, but we are going to focus on *Gaussian analytic functions*, GAFs for short, and see some properties of their zero sets.

We say that the random variable Z follows a *standard complex Gaussian* distribution, denoted $Z \sim N_{\mathbb{C}}(0, 1)$, if its density function, with respect to the Lebesgue measure, is

$$f_Z(z) = \frac{1}{\pi} e^{-|z|^2}, \quad z \in \mathbb{C}.$$

Now, assume that $(e_n)_{n=0}^{+\infty}$ is a sequence of analytic functions in a region $\Omega \subseteq \mathbb{C}$ and that $(\xi_n)_{n=0}^{+\infty}$ is a sequence of i.i.d. $N_{\mathbb{C}}(0, 1)$ random variables. Under some assumptions about convergence, we say that f is a GAF in Ω if:

$$f(z) = \sum_{n=0}^{+\infty} \xi_n e_n(z), \quad z \in \Omega.$$

From this definition and fixing $z \in \Omega$, f follows a complex Gaussian distribution with null mean due to the linear combination of $(e_n)_{n=0}^{+\infty}$ and $(\xi_n)_{n=0}^{+\infty} \sim N_{\mathbb{C}}(0,1)$ and since the linear combination of Gaussian random variables is a Gaussian random variable. So, if we denote the *normalized GAF* as

$$\hat{f}(z) = \frac{f(z)}{\mathrm{Var}[f(z)]}, \quad z \in \Omega,$$

it follows a $N_{\mathbb{C}}(0,1)$ distribution. We have that the *covariance kernel* of a GAF f is

$$\mathcal{K}_f(z,w) := \mathrm{Cov}\left[f(z), \overline{f(w)}\right] = \mathbb{E}\left[f(z)\overline{f(w)}\right] = \sum_{n=0}^{+\infty} e_n(z)\overline{e_n(w)}, \quad z,w \in \Omega.$$

In particular we have:

$$\mathrm{Var}[f(z)] = \sqrt{\mathcal{K}_f(z,z)}.$$

Therefore, all the probabilistic properties of a GAF are encoded in its covariance kernel.
Some remarkable properties about GAFs are:

- A GAF is an analytic function in Ω a.s. Fixed $z \in \Omega$, $f(z)$ converges up to a possible set, that depends on z, with probability zero. The problem is that the uncountable union of these sets can be the whole space or a significant part of it. This can be solved by using a version of Kolmogorov's inequality for Hilbert spaces.

- A standard way to construct a GAF on a given space Ω as in the definition is to consider the orthonormal basis $(e_n)_{n=0}^{+\infty}$ of a Hilbert space $\mathcal{H}$ of analytic functions in Ω. In that case the covariance kernel coincides with the *Bergman kernel* of $\mathcal{H}$, defined as

$$B(z,w) = \sum_{n=0}^{+\infty} e_n(z)\overline{e_n(w)}, \quad z,w \in \Omega.$$

  Notice that $B_w(z) = B(z,w)$ is the *reproducing kernel* of $\mathcal{H}$ at $w \in \Omega$. Such kernel is independent of the choice of the orthonormal basis.

- Let $(e_n)_{n=0}^{+\infty}$ be an orthonormal basis in $\mathcal{H}$. Then, the GAF does not belong to $\mathcal{H}$ a.s., because on the contrary we would have that $\sum_{n=0}^{+\infty} |\xi_n|^2$ converges, something that happens with probability zero for $(\xi_n)_{n=0}^{+\infty}$ i.i.d. with $N_{\mathbb{C}}(0,1)$ distribution.

As canonical examples of this construction, we introduce three of the most studied families of Hilbert spaces of analytic functions.

- The *planar* space or the *Bargmann-Fock* space with real parameter $L > 0$ in $\mathbb{C}$ is defined as

$$\mathcal{F}_L := \left\{ f \in \mathcal{A}(\mathbb{C}) \ : \ \|f\|^2_{\mathcal{F}_L} = \frac{L}{\pi} \int_{\mathbb{C}} |f(z)|^2 \, e^{-L|z|^2} dm(z) < +\infty \right\},$$

where $z \in \mathbb{C}$ and $\mathcal{A}(\mathbb{C})$ is the space of analytic functions in $\mathbb{C}$. The orthonormal basis obtained by normalizing the monomials z^n, $n \in \mathbb{N}$, is:

$$e_n(z) = \sqrt{\frac{L^n}{n!}} z^n.$$

The GAF, also called $\mathbb{C}$-GAF or *planar GAF*, is therefore

$$f_L(z) = \sum_{n=0}^{+\infty} \xi_n \sqrt{\frac{L^n}{n!}} z^n,$$

where $(\xi_n)_{n=0}^{+\infty}$ is a sequence of i.i.d. $N_{\mathbb{C}}(0,1)$ random variables. The covariance kernel is

$$\mathcal{K}_{f_L}(z,w) = \sum_{n=0}^{+\infty} \frac{L^n}{n!} (z\overline{w})^n = e^{Lz\overline{w}}.$$

- The *hyperbolic space* or the *weighted Bergman space* with real parameter $L > 1$ in $\mathbb{D}$ is defined as

$$\mathcal{B}_L := \left\{ f \in \mathcal{A}(\mathbb{D}) \ : \ \|f\|^2_{\mathcal{B}_L} = \frac{L}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, (1 - |z|^2)^{L-2} dm(z) < +\infty \right\},$$

where $z \in \mathbb{D}$ and $\mathcal{A}(\mathbb{D})$ is the space of analytic functions in $\mathbb{D}$. An orthonormal basis is:

$$e_n(z) = \binom{L+n-1}{n}^{1/2} z^n, \quad n \geq 0.$$

The GAF, also called $\mathbb{D}$-GAF or *hyperbolic GAF*, is then

$$f_L(z) = \sum_{n=0}^{+\infty} \xi_n \binom{L+n-1}{n}^{1/2} z^n,$$

which has sense for $L > 0$. The covariance kernel is

$$\mathcal{K}_{f_L}(z,w) = \sum_{n=0}^{+\infty} \binom{L+n-1}{n} (z\overline{w})^n = (1 - z\overline{w})^{-L}.$$

- The *parabolic space* or the *space of polynomials of degree at most* $L \in \mathbb{N}$ in $\mathbb{C}$ is described as:

$$\mathcal{P}_L := \left\{ f \in P_L[\mathbb{C}] \ : \ \|f\|^2_{\mathcal{P}_L} = \frac{L+1}{\pi} \int_{\mathbb{C}} \frac{|f(z)|^2}{(1+|z|^2)^{L+2}} dm(z) < +\infty \right\},$$

where $z \in \mathbb{C}$ and $P_L[\mathbb{C}]$ is the vector space of polynomials of degree at most L with complex coefficients. The space $P_L[\mathbb{C}]$ can be seen as the projection to $\mathbb{C}$ of the space of sections of the L-th power of the canonical bundle of $\mathbb{S}^2$. The norm defined in $\mathcal{P}_L$ makes sense. Indeed, the term

$$\frac{|f(z)|^2}{(1+|z|^2)^L}$$

is the normalization by the degree of f, and

$$\frac{dm(z)}{\pi(1+|z|^2)^2}$$

is the area measure of $\mathbb{S}^2$ projected to $\mathbb{C}$.
An orthonormal basis is:

$$e_n(z) = \binom{L}{n}^{1/2} z^n, \quad 0 \le n \le L.$$

The GAF, also called $\mathbb{S}^2$-GAF or *parabolic GAF*, is then

$$f_L(z) = \sum_{n=0}^{L} \xi_n \binom{L}{n}^{1/2} z^n,$$

and the covariance kernel is

$$\mathcal{K}_{f_L}(z,w) = \sum_{n=0}^{L} \binom{L}{n} (z\overline{w})^n = (1+z\overline{w})^L.$$

1.2 First intensity and the Edelman-Kostlan formula

We will focus our attention on this last space and on the zero set of an $\mathbb{S}^2$-GAF f_L of parameter $L \in \mathbb{N}$ in a region $\Omega \subseteq \mathbb{C}$. The zero set of f_L, denoted $\mathcal{Z}_{f_L}$, will be studied through its *empirical measure*

$$\nu_{f_L} = \sum_{a \in \mathcal{Z}_{f_L}} \delta_a = \frac{1}{2\pi} \Delta \log |f_L|,$$

where δ_a is the Dirac delta measure at a. Notice that ν_{f_L} is a measure supported precisely on the zeros of f_L. The *first intensity* of the GAF f_L is the measure $\mathbb{E}[\nu_{f_L}]$ defined by the action

$$\int_\Omega \varphi d\mathbb{E}[\nu_{f_L}] = \mathbb{E}\left[\int_\Omega \varphi d\nu_{f_L}\right], \quad \varphi \in \mathcal{C}_c^\infty(\Omega).$$

The first intensity measures the average number of points of the point process. According to the well-known *Edelman-Kostlan formula*:

$$\mathbb{E}\left[\nu_{f_L}\right] = \frac{1}{2\pi}\Delta \log \mathbb{E}[|f_L|] = \frac{1}{2\pi}\Delta \log \sqrt{\mathcal{K}_{f_L}(z,z)}, \quad z \in \Omega.$$

For the spaces before introduced, we have that:

- For a $\mathbb{C}$-GAF f_L of real parameter $L > 0$, the first intensity is

$$\mathbb{E}[\nu_{f_L}] = \frac{1}{2\pi}\Delta \log e^{L|z|^2/2} = \frac{L}{\pi} dm(z),$$

  where dm stands for the Lebesgue measure on the plane.

- For a $\mathbb{D}$-GAF f_L of real parameter $L > 1$, the first intensity is

$$\mathbb{E}[\nu_{f_L}] = \frac{1}{4\pi}\Delta \log(1 - zw)^{-L} = \frac{L}{\pi}\frac{dm(z)}{(1 - |z|^2)^2}.$$

- For an $\mathbb{S}^2$-GAF f_L of parameter $L \in \mathbb{N}$, the first intensity is

$$\mathbb{E}[\nu_{f_L}] = \frac{1}{4\pi}\Delta \log\left[(1 + |z|^2)^L\right] = \frac{L}{\pi}\frac{dm(z)}{(1 + |z|^2)^2}.$$

A remarkable feature of these processes is the invariance by the natural translations on each space.

- For a $\mathbb{C}$-GAF the zero point process is invariant by translations

$$\phi_a(z) = z - a, \quad z, a \in \mathbb{C}.$$

- For a $\mathbb{D}$-GAF the zero point process is invariant by automorphisms in $\mathbb{D}$

$$\phi_a(z) = \frac{z - a}{1 - \overline{a}z}e^{i\theta}, \quad z, a \in \mathbb{D}, \quad \theta \in [0, 2\pi).$$

- For an $\mathbb{S}^2$-GAF the zero point process is invariant by rotations in $\mathbb{S}^2$, which in the $\mathbb{C}$-chart are seen as the Möbius transformations

$$\phi_a(z) = \frac{z - a}{1 + \overline{a}z}, \quad z, a \in \mathbb{C}.$$

Since the first intensity determines the distribution in mean of $\mathcal{Z}_{f_L}$, it is also invariant by the suitable transformations just introduced.

1.3 Fluctuations of the parabolic GAF

Having this basic background in GAF theory, we can face the main problem of this project. We know how the zeros of a GAF are distributed in average according to the Edelman-Kostlan formula, but how do they interact? Or what is equivalent, how do they *fluctuate*?

We quantify this fact from different points of view:

- The variance of the random variable $\nu_{f_L}(U) = \#\left(\mathcal{Z}_{f_L} \cap U\right)$, where $U \subseteq \mathbb{C}$. We are going to make an exhaustive study of the variance, denoted $\mathbb{V}$, in a disk of radius 2ρ, $\rho > 0$, endowed with the *chordal metric* (see (2.4.4)). Denoting the chordal disk as $D_{ch} := D_{ch}(z_0, 2\rho)$, for $z_0 \in \mathbb{C}$, using the definition of the variance and the properties of the GAF we prove that

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = \frac{L^2}{2\pi}\rho\sqrt{1-\rho^2}\int_0^{4\rho^2(1-\rho^2)} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\frac{1}{\sqrt{1-\frac{x}{4\rho^2(1-\rho^2)}}}dx.$$

  For the proof see Theorem 3.2.2.
  This is the integral of a positive function in a bounded interval, from which we can extract some information. For example:

  1. *Asymptotics as* $L \to +\infty$ (see Subsection 3.2.3). We will show that:

     Proposition 1.3.1. *Let f_L be an $\mathbb{S}^2$-GAF of intensity $L \in \mathbb{N}$. Consider a chordal disk $D_{ch} := D_{ch}(z_0, 2\rho)$, for $z_0 \in \mathbb{C}$. Then*

     $$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = \left(\frac{\sqrt{L}}{4\sqrt{\pi}}\zeta(3/2)\rho\sqrt{1-\rho^2}\right)(1+o(1)), \quad \text{as } L \to +\infty.$$

     Here ζ stands for the Riemann's zeta function and $o(1)$ is a term tending to 0 as $L \to +\infty$.
     For the Poisson process $\mathcal{X}$ with underlying measure

     $$\frac{L}{\pi}dm,$$

     the random variable

     $$n_L(D(0,r)) := \#\left(\mathcal{X} \cap D(0,r)\right) \tag{1.3.1}$$

     has the same average number of points as our GAF:

     $$\mathbb{E}\left[n_L(D(0,r))\right] = Lr^2. \tag{1.3.2}$$

     But the variance is much larger:

     $$\mathbb{V}\left[n_L(D(0,r))\right] = Lr^2. \tag{1.3.3}$$

     It is in this sense that the GAF process is more *rigid*.

2. *Asymptotics as $\rho \to 0$* (see Subsection 3.2.4). It is intuitive that the variance will tend to zero as $\rho \to 0$. We will quantify the speed of convergence. More precisely:

   Proposition 1.3.2. *Let f_L be an $\mathbb{S}^2$-GAF of intensity $L \in \mathbb{N}$. Consider a chordal disk $D_{ch} := D_{ch}(z_0, 2\rho)$, for $z_0 \in \mathbb{C}$. Then*

   $$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = L\rho^2(1 + o(1)) \quad \text{as } \rho \to 0.$$

   In here, we see that the speed of convergence is equivalent to the Poisson process as the radius tends to zero (see 1.3.3).

3. *$\mathcal{F}_1$ as the limit of $\mathcal{P}_L$ as $L \to +\infty$* (see Subsection 3.2.5). There is a result stating that the functions in $\mathcal{F}_1$ can be seen as limits of rescaled polynomials of $\mathcal{P}_L$ as $L \to +\infty$. More precisely:

   Lemma 1.3.3. *Given a GAF $f_1^{\mathbb{C}} \in \mathcal{F}_1$ and a constant $M > 0$, there is $L_0 \in \mathbb{N}$ such that for all $L \geq L_0$, there exist GAFs $f_L^{\mathbb{S}^2} \in \mathcal{P}_L$ such that*

   $$\int_{\{|z| \leq M/\sqrt{L}\}} \left| f_1^{\mathbb{C}}(\sqrt{L}z) - f_L^{\mathbb{S}^2}(z) \right|^2 e^{-L|z|^2} dm(z) \lesssim \frac{1}{L} \left\| f_1^{\mathbb{C}} \right\|_{\mathcal{F}_1}^2$$

   and

   $$\int_{\{|z| > M/\sqrt{L}\}} \frac{\left| f_L^{\mathbb{S}^2}(z) \right|^2}{\pi(1 + |z|^2)^{L+2}} dm(z) \lesssim \frac{1}{L} \left\| f_1^{\mathbb{C}} \right\|_{\mathcal{F}_1}^2.$$

   In accordance with this lemma, we prove that:

   Proposition 1.3.4. *The limit of the variance of an $\mathbb{S}^2$-GAF as $L \to +\infty$ coincides with the variance of a $\mathbb{C}$-GAF of parameter $L = 1$, that is,*

   $$\lim_{L \to +\infty} \mathbb{V}\left[\nu_{f_L^{\mathbb{S}^2}}\left(D_{ch}(z_0, 2r/\sqrt{L})\right)\right] = \mathbb{V}\left[\nu_{f_1^{\mathbb{C}}}\left(D(z_0, r)\right)\right], \quad z_0 \in \mathbb{C}.$$

- *Large deviations* (see Section 4.1). Consider test-functions $\varphi \in \mathcal{C}_c^2(\mathbb{C})$. We define the *linear statistic* associated to φ as

  $$I_L(\varphi) := \int_{\mathbb{C}} \varphi d\nu_{f_L}.$$

  We have, by the Edelman-Kostlan formula,

  $$\mathbb{E}\left[I_L(\varphi)\right] = \mathbb{E}\left[\int_{\mathbb{C}} \varphi d\nu_{f_L}\right] = \int_{\mathbb{C}} \varphi d\mathbb{E}[\nu_{f_L}].$$

  Here we will study how much $I_L(\varphi)$ deviates from its mean $\mathbb{E}\left[I_L(\varphi)\right]$ by a fraction of the same mean. More concretely:

  Theorem 1.3.5. *For all $\varphi \in \mathcal{C}_c^2(\mathbb{C})$ and for all $\delta > 0$, there exist constants $c = c(\varphi, \delta)$ and $L_0 = L_0(\varphi, \delta) \in \mathbb{N}$ such that, for all $L \geq L_0$,*

  $$\mathbb{P}\left[\left|\frac{I_L(\varphi)}{\mathbb{E}\left[I_L(\varphi)\right]} - 1\right| > \delta\right] \leq e^{-cL^2}.$$

- *Hole probability* (see Section 4.2). As a consequence of the large deviations we will estimate the probability that there is a hole in the zero point process, that is, a disk without zeros of f_L. More precisely:

  Theorem 1.3.6. *For a given $\rho > 0$, there exist $C_1 = C_1(\rho) > 0$, $C_2 = C_2(\rho) > 0$ and $L_0 \in \mathbb{N}$ such that, for all $L \geq L_0$ and a disk $D_{ch} := D_{ch}(z_0, \rho) \subset \mathbb{C}$ with $z_0 \in \mathbb{C}$,*
  $$e^{-C_1 L^2} \leq \mathbb{P}\left[\mathcal{Z}_{f_L} \cap D_{ch} = \emptyset\right] \leq e^{-C_2 L^2}.$$

  For the Poisson process in the planar case with intensity $L > 0$, we have that
  $$\mathbb{P}\left[n_{D(z_0,r)} = 0\right] = e^{-Lr^2}, \quad z_0 \in \mathbb{C},\ r > 0.$$

  But for the $\mathbb{S}^2$-GAF we have lower and upper bounds such that the power of the exponential depends on L^2. Then, it is more unlikely to have a hole in the zero point process of an $\mathbb{S}^2$-GAF than in the Poisson process.

This manuscript is divided in three parts. Chapter 2 is devoted to introduce the basis of GAF theory. I followed [1], [2] and [9].

Chapter 3 is devoted to compute explicitly the variance of the random variables of the zero point process of a $\mathbb{C}$-GAF and an $\mathbb{S}^2$-GAF in suitable disks. Then we give the asymptotic results described above for an $\mathbb{S}^2$-GAF. Here I used [2], [5], [6] and [9].

In Chapter 4 we give a full description of the large deviations and the hole probability of an $\mathbb{S}^2$-GAF. The statements, results and proofs are analogous to [6]. Also [4] was helpful to write these pages.

Chapter 2

Preliminaries

In this chapter we will introduce the basic elements and results about the theory of *Gaussian analytic functions*. Since the project is focused on the zero set of Gaussian analytic functions on the Riemann sphere $\mathbb{S}^2 := \mathbb{C} \cup \{\infty\}$, we are going to explicit these elements on this space and the suitable Hilbert space we are going to use. The following pages are strongly based on [1] and [2], so all the statements and proofs come from those references. The source [9] was also helpful at some points.

2.1 Complex Gaussian distribution

Recall that a real-valued random variable X follows a *real Gaussian distribution*, denoted $X \sim N_{\mathbb{R}}(\mu, \sigma^2)$, if its density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ is the *mean* and $\sigma^2 \in (0, +\infty)$ is the *variance*. We can define also the complex version of the standard Gaussian distribution.

Definition 2.1.1. A complex-valued random variable Z follows a *standard complex Gaussian distribution*, denoted $Z \sim N_{\mathbb{C}}(0, 1)$, if its density function, with respect to the Lebesgue measure, is

$$f_Z(z) = \frac{1}{\pi} e^{-|z|^2}, \quad z \in \mathbb{C}.$$

We outline the following result:

Proposition 2.1.2. *i) If $Z \sim N_{\mathbb{C}}(0, 1)$, then $|Z|^2$ is an exponential random variable of parameter 1, i.e., $\mathbb{P}\left[|Z|^2 > t\right] = e^{-t}$ for all $t > 0$.*

ii) If $(\xi_n)_{n=0}^{+\infty}$ is a sequence of i.i.d. $N_{\mathbb{C}}(0, 1)$ random variables, then

$$\limsup_{n \to +\infty} |\xi_n|^{1/n} = 1, \quad \text{a.s.}$$

Proof. i) By applying polar coordinates and the change of variable $s = r^2$, we have for all $t > 0$,

$$\mathbb{P}\left[|Z|^2 > t\right] = 1 - \mathbb{P}\left[|Z|^2 \le t\right] = 1 - \int_{\{0\le|z|^2\le t\}} \frac{1}{\pi} e^{-|z|^2} dm(z) = 1 - \int_0^{\sqrt{t}} 2re^{-r^2} dr$$
$$= 1 - \int_0^t e^{-s} ds = e^{-t}.$$

ii) It is a consequence of the Borel-Cantelli lemma. See for instance [2], p. 15. □

2.2 Gaussian analytic functions

Here we introduce the main object of study of the project, the so-called *Gaussian analytic function*. For this we endow the space of analytic functions over a region Ω with the topology of uniform convergence on compact sets of Ω. Denote by $\mathcal{A}(\Omega)$ the space just described. A very natural question is: how can we generate GAFs? The next definition addresses this.

Definition 2.2.1. Let $(e_n)_{n=0}^{+\infty}$ be a sequence in $\mathcal{A}(\Omega)$ and let $(\xi_n)_{n=0}^{+\infty}$ be a sequence of i.i.d. random variables with $N_{\mathbb{C}}(0,1)$ distribution. Assume that $\sum_{n=0}^{\infty} |e_n(z)|^2$ converges locally uniformly on Ω. A *Gaussian analytic function* (GAF from now on) is the linear combination

$$f(z) = \sum_{n=0}^{+\infty} \xi_n e_n(z), \quad z \in \Omega. \tag{2.2.1}$$

Fixing $z \in \Omega$, the random variable f described in Definition 2.2.1 follows a complex Gaussian distribution with null mean due to the linear combination of $(e_n)_{n=0}^{+\infty}$ and $(\xi_n)_{n=0}^{+\infty} \sim N_{\mathbb{C}}(0,1)$ and since the linear combination of Gaussian random variables is a Gaussian random variable. So, if we denote the *normalized GAF* as

$$\hat{f}(z) = \frac{f(z)}{\mathrm{Var}[f(z)]}, \quad z \in \Omega,$$

it follows a $N_{\mathbb{C}}(0,1)$ distribution. We have that the *covariance kernel* of a GAF f is

$$\mathcal{K}_f(z,w) := \mathrm{Cov}\left[f(z), \overline{f(w)}\right] = \mathbb{E}\left[f(z)\overline{f(w)}\right] = \sum_{n=0}^{+\infty} e_n(z)\overline{e_n(w)}, \quad z, w \in \Omega. \tag{2.2.2}$$

In particular, we have

$$\mathrm{Var}[f(z)] = \sqrt{\mathcal{K}_f(z,z)}.$$

Therefore, all the probabilistic properties of a GAF are encoded in its covariance kernel.

Remark 2.2.2. To justify the last equality of (2.2.2) notice that

$$\mathcal{K}_f(z,w)=\mathbb{E}\left[f(z)\overline{f(w)}\right]=\sum_{n,m=0}^{+\infty}e_n(z)\overline{e_n(w)}\mathbb{E}\left[\xi_n\overline{\xi_m}\right].$$

Since $(\xi_n)_{n=0}^{+\infty}$ is a sequence of i.i.d. random variables with $N_{\mathbb{C}}(0,1)$ distribution, we have that

$$\mathbb{E}\left[\xi_n\overline{\xi_m}\right]=\delta_{n,m},$$

where $\delta_{n,m}$ denotes the Kronecker delta function. Thus the equality follows.

Let us state a few remarks and properties about GAFs:

- A GAF is an analytic function in Ω a.s. Fixed $z\in\Omega$, $f(z)$ converges up to a possible set, that depends on z, with probability zero. The problem is that the uncountable union of these sets can be the whole space or a significant part of it. This can be solved by using a version of Kolmogorov's inequality for Hilbert spaces (see [2], Lemma 2.2.3).

- The radius of convergence of a GAF is computed with the conditions of growth of the sequence $(e_n)_{n=0}^{+\infty}$ and with ii) of Proposition 2.1.2.

- A standard way to construct a GAF on a given space Ω is to consider the orthonormal basis $(e_n)_{n=0}^{+\infty}$ of a Hilbert space $\mathcal{H}$ of analytic functions in Ω and consider (2.2.1). In that case the covariance kernel (2.2.2) coincides with the *Bergman kernel* of $\mathcal{H}$, defined as

$$B(z,w)=\sum_{n=0}^{+\infty}e_n(z)\overline{e_n(w)},\quad z,w\in\Omega.$$

  Notice that $B_w(z)=B(z,w)$ is the *reproducing kernel* of $\mathcal{H}$ at $w\in\Omega$. Such kernel is independent of the choice of the orthonormal basis.

- Let $(e_n)_{n=0}^{+\infty}$ be an orthonormal basis in $\mathcal{H}$. Then, (2.2.1) does not belong to $\mathcal{H}$ a.s., because on the contrary we would have that $\sum_{n=0}^{+\infty}|\xi_n|^2$ converges, something that happens with probability zero for $(\xi_n)_{n=0}^{+\infty}$ i.i.d. with $N_{\mathbb{C}}(0,1)$ distribution.

Lemma 2.2.3. *The normalized kernel*

$$K(z,w)=\frac{|\mathcal{K}_f(z,w)|^2}{\mathcal{K}_f(z,z)\mathcal{K}_f(w,w)}$$

satisfies $|K(z,w)|\le 1$.

Proof. To check this inequality just notice that, since $B(z,w)=\mathcal{K}_f(z,w)$,

$$|\mathcal{K}_f(z,w)|=|(B_z,B_w)_{\mathcal{H}}|\le\|B_z\|_{\mathcal{H}}\|B_w\|_{\mathcal{H}},$$

where $\mathcal{H}$ is the Hilbert space we are working in and B_z (respectively B_w) is the reproducing kernel of $\mathcal{H}$ at the point $z\in\Omega$ (respectively at $w\in\Omega$). By squaring at both sides and passing the RHS to the left, we see that $|K(z,w)|\le 1$. □

2.3 The Gamma function

The *Gamma function*, denoted Γ, is defined for all $y > 0$ as

$$\Gamma(y) = \int_0^{+\infty} x^{y-1} e^{-x} dx.$$

We state a few well-known properties:

Lemma 2.3.1. *i)* $\Gamma(1) = 1$.

ii) For all $y > 0$, $\Gamma(y+1) = y\Gamma(y)$.

iii) $\Gamma(1/2) = \sqrt{\pi}$.

iv) For all $n \in \mathbb{N}$, $\Gamma(n+1) = n!$

v) For $L \in \mathbb{N}$ *and* $n \leq L$:

$$\binom{L}{n} = \frac{\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)} = \frac{L!}{n!(L-n)!}. \tag{2.3.1}$$

vi) For all $a \in \mathbb{R}$,

$$\lim_{\ell \to +\infty} \frac{\Gamma(\ell)}{\Gamma(\ell + a)} = \ell^{-a}. \tag{2.3.2}$$

We shall also use the *Beta function*, denoted B, which is defined, for all $x, y > 0$, as

$$\mathrm{B}(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds = \int_0^{+\infty} \frac{s^{x-1}}{(1+s)^{x+y}} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.3.3}$$

2.4 The $\mathbb{C}$-GAF and the $\mathbb{S}^2$-GAF

We are going to introduce two families of Hilbert spaces of analytic functions that depend on a positive parameter L. These will be used to generate GAFs and to study the properties of their zero sets.

2.4.1 The spaces $\mathcal{F}_L$

Given a real value $L > 0$, we define the *planar space*, also known as the *Bargmann-Fock space*, as

$$\mathcal{F}_L := \left\{ f \in \mathcal{A}(\mathbb{C}) \ : \ \|f\|^2_{\mathcal{F}_L} = \frac{L}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-L|z|^2} dm(z) < +\infty \right\},$$

where dm stands for the Lebesgue measure. The factor L/π is chosen so that

$$\frac{L}{\pi} e^{-L|z|^2} dm(z), \quad z \in \mathbb{C},$$

is a probability measure, i.e., $\|1\|_{\mathcal{F}_L} = 1$.
In this space,

$$e_n(z) = \sqrt{\frac{L^n}{n!}} z^n, \quad n \geq 0,$$

is an orthonormal basis. Let us show this property.

Lemma 2.4.1. *$(e_n)_{n=0}^{+\infty}$ is an orthonormal basis in $\mathcal{F}_L$.*

Proof. We have to show that $(e_n, e_m)_{\mathcal{F}_L} = \delta_{n,m}$ and that $(e_n)_{n=0}^{+\infty}$ is complete. Let us focus on the first part. For $n, m \in \mathbb{N}$ we have, using polar coordinates:

$$\begin{aligned}(e_n, e_m)_{\mathcal{F}_L} &= \frac{L}{\pi} \int_{\mathbb{C}} \sqrt{\frac{L^{n+m}}{n!m!}} e^{-L|z|^2} dm(z) \\ &= \frac{L}{\pi} \sqrt{\frac{L^{n+m}}{n!m!}} \int_0^{2\pi} \int_0^{+\infty} r^{n+m+1} e^{i\theta(n-m)} e^{-Lr^2} dr d\theta.\end{aligned}$$

However, if $n \neq m$,

$$\int_0^{2\pi} e^{i\theta(n-m)} d\theta = 0. \tag{2.4.1}$$

Hence, if $n \neq m$, we conclude that $(e_n, e_m)_{\mathcal{F}_L} = 0$. Otherwise, if $n = m$, we have, by using the definition and properties of the Gamma function and the change of variable $t = Lr^2$, that

$$(e_n, e_n)_{\mathcal{F}_L} = \frac{L^{n+1}}{n!} \int_0^{+\infty} 2r^{2n+1} e^{-Lr^2} dr = \frac{1}{n!} \int_0^{+\infty} t^n e^{-t} = \frac{\Gamma(n+1)}{n!} = 1.$$

Thus $(e_n)_{n=0}^{+\infty}$ is an orthonormal system in $\mathcal{F}_L$.
For completeness just notice that $(e_n)_{n=0}^{+\infty}$ is a normalization of the monomial basis. □

Using Definition 2.2.1, for every $L > 0$ we can generate the GAF

$$f_L(z) = \sum_{n=0}^{+\infty} \xi_n \sqrt{\frac{L^n}{n!}} z^n, \tag{2.4.2}$$

where $(\xi_n)_{n=0}^{+\infty}$ is a sequence of i.i.d. random variables with $N_{\mathbb{C}}(0,1)$ distribution. This is known as the *planar Gaussian analytic function* ($\mathbb{C}$-GAF, for short). The covariance kernel of this GAF is

$$\mathcal{K}_{f_L}(z, w) = \sum_{n=0}^{+\infty} \frac{L^n}{n!} z^n \overline{w}^n = e^{Lz\overline{w}}. \tag{2.4.3}$$

2.4.2 The Riemann sphere $\mathbb{S}^2$

We deal mainly with GAFs on the Riemann sphere. Let us reveal some properties of it. The topology is defined in the following way:

- If $w \in \mathbb{C}$ the neighbourhood system is generated by the family of disks $\{D(w,r)\}_{r>0}$.
- Otherwise, if $w = \infty$ the neighbourhood system is generated by the family $\mathbb{S}^2 \setminus \{\overline{D(0,r)}\}_{r>0}$.

Let us set $S^2 := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \ : \ x_1^2+x_2^2+x_3^2 = 1\}$. Denote $N := (0,0,1)$.

Lemma 2.4.2. *The stereographic projection*

$$\begin{aligned} p: \quad S^2 \setminus \{N\} &\longrightarrow \qquad \mathbb{C} \\ x = (x_1,x_2,x_3) &\longmapsto \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3} \end{aligned}$$

establishes a homeomorphism between $S^2 \setminus \{N\}$ and $\mathbb{C}$.

Proof. The mapping p is well-defined, continuous and bijective. For the inverse mapping, take $p(x) = a+ib$, for all $a,b \in \mathbb{R}$. So, imposing $a = x_1/(1-x_3)$ and $b = x_2/(1-x_3)$, we get

$$|z|^2 = a^2+b^2 = \frac{x_1^2+x_2^2}{(1-x_3)^2} = \frac{1-x_3^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3}.$$

Thus, since $a = (z+\overline{z})/2$ and $b = (z-\overline{z})/2i$, from

$$(1-x_3)|z|^2 = 1+x_3$$

we deduce that

$$x_3 = \frac{|z|^2-1}{1+|z|^2}.$$

Also it follows that

$$\begin{aligned} x_1 &= a(1-x_3) = a\left(1-\frac{|z|^2-1}{1+|z|^2}\right) = \frac{z+\overline{z}}{1+|z|^2}, \\ x_2 &= b(1-x_3) = b\left(1-\frac{|z|^2-1}{1+|z|^2}\right) = \frac{z-\overline{z}}{i(1+|z|^2)}. \end{aligned}$$

Hence the inverse mapping is defined as

$$\begin{aligned} p^{-1}: \mathbb{C} &\longrightarrow \qquad S^2 \setminus \{N\} \\ z &\longmapsto \left(\frac{z+\overline{z}}{1+|z|^2}, \frac{z-\overline{z}}{i(1+|z|^2)}, \frac{|z|^2-1}{1+|z|^2}\right), \end{aligned}$$

and it is continuous. Therefore p is a homeomorphism between $S^2 \setminus \{N\}$ and $\mathbb{C}$. We extend p to the mapping

$$\begin{aligned} p_\infty : \quad S^2 &\longrightarrow \quad \mathbb{S}^2 \\ x &\longmapsto \begin{cases} p(x), & \text{if } x \neq N, \\ \infty, & \text{if } x = N. \end{cases} \end{aligned}$$

Then we have that p_∞ is bijective and that $\lim_{x\to N} |p_\infty(x)| = \infty$. This implies that p_∞ is a homeomorphism between S^2 and $\mathbb{S}^2$ and, since S^2 is compact, $\mathbb{S}^2$ is also compact. Therefore $\mathbb{S}^2$ can be understood as the Alexandroff compactification of $\mathbb{C}$ with the point $\{\infty\}$. □

The chordal distance

The metric we are going to consider is the *chordal distance*, which is the Euclidian distance in $\mathbb{R}^3$ projected to $\mathbb{C}$ by the stereographic projection. It has the expression:

$$d_{ch}(z,w) := \frac{2|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}}, \quad z,w \in \mathbb{C}. \tag{2.4.4}$$

Remark 2.4.3. Let us state a few properties about the chordal distance:

i) We define the *chordal disk* as

$$D_{ch}(z_0,\rho) := \{z \in \mathbb{C} \ : \ d_{ch}(z_0,z) < \rho\}, \quad z_0 \in \mathbb{C}, \ \rho > 0.$$

ii) The relation between the radii of $D(0,r)$ and $D_{ch}(0,\rho)$, for $r,\rho > 0$, is:

$$D_{ch}(0,\rho) = D\left(0, \frac{\rho}{\sqrt{4-\rho^2}}\right)$$

and

$$D(0,r) = D_{ch}\left(0, \frac{2r}{\sqrt{1+r^2}}\right).$$

iii) The expression

$$dm^*(z) := \frac{dm(z)}{\pi(1+|z|^2)^2}, \quad z \in \mathbb{C},$$

called the *parabolic measure*, is the push-forward in $\mathbb{C}$ of the surface area measure in $\mathbb{S}^2$, by the stereographic projection. We will see in Corollary 2.4.5 that the parabolic measure is invariant by the rotations of $\mathbb{S}^2$ projected to $\mathbb{C}$.

iv) Let us compute the parabolic measure of a chordal disk $D_{ch} := D_{ch}(z_0,\rho)$ with $z_0 \in \mathbb{C}$ and $\rho > 0$. Denote

$$|z| \leq \tilde{\rho} := \frac{\rho}{\sqrt{4-\rho^2}}.$$

Using polar coordinates and the change of variable $t = 1 + r^2$:

$$m^*(D_{ch}) = \int_{D_{ch}} dm^*(z) = \int_{D_{ch}} \frac{dm(z)}{\pi(1+|z|^2)^2} = \int_0^{\tilde{\rho}} \frac{2r}{(1+r^2)^2} dr$$
$$= \int_1^{1+\tilde{\rho}^2} \frac{dt}{t^2} = \frac{\tilde{\rho}^2}{1+\tilde{\rho}^2} = \frac{\rho^2}{4}.$$

Transformations

The transformations we consider are the rotations of $\mathbb{S}^2$ projected to $\mathbb{C}$, which are the Möbius transformations of the form

$$\phi_a^\theta(z) = \frac{z-a}{1+\bar{a}z} e^{i\theta}, \quad z, a \in \mathbb{C} \text{ and } \theta \in [0, 2\pi).$$

As notation, $\phi_a \equiv \phi_a^0$, for all $a \in \mathbb{C}$.

Remark 2.4.4. From the expression of ϕ_a it follows

$$1 + |\phi_a(z)|^2 = \frac{(1+|a|^2)(1+|z|^2)}{|1+\bar{a}z|^2}, \quad a, z \in \mathbb{C}. \tag{2.4.5}$$

Due to (2.4.4) and (2.4.5), it can be easily checked that

$$(1 + |\phi_a(z)|^2)^{-1} = 1 - \left(\frac{d_{ch}(z,a)}{2}\right)^2, \quad a, z \in \mathbb{C}. \tag{2.4.6}$$

Also notice that

$$\phi_a'(z) = \frac{1+|a|^2}{(1+\bar{a}z)^2}. \tag{2.4.7}$$

Corollary 2.4.5. *The parabolic measure is invariant by ϕ_a, for all $a \in \mathbb{C}$.*

Proof. Recalling that dm^* is the surface area measure of $\mathbb{S}^2$ projected to $\mathbb{C}$, we have, by using (2.4.5) and (2.4.7), that for all $z \in \mathbb{C}$:

$$dm^*(\phi_a(z)) = \frac{|\phi_a'(z)|^2}{\pi(1+|\phi_a(z)|^2)^2} dm(z) = \frac{dm(z)}{\pi(1+|z|^2)^2} = dm^*(z).$$

□

2.4.3 The spaces $\mathcal{P}_L$

We represent the Hilbert spaces of holomorphic functions used in the definition of GAF in the ℂ-chart. Given $L \in \mathbb{N}$ we define the *parabolic space* or the *space of polynomials of degree at most* L as:

$$\mathcal{P}_L := \left\{ f \in P_L[\mathbb{C}] \ : \ \|f\|^2_{\mathcal{P}_L} = (L+1) \int_{\mathbb{C}} \frac{|f(z)|^2}{(1+|z|^2)^L} dm^*(z) < +\infty \right\},$$

where $P_L[\mathbb{C}]$ is the vector space of polynomials of degree at most L with complex coefficients and dm stands for the Lebesgue measure on $\mathbb{C}$. The space $P_L[\mathbb{C}]$ can be seen as the projection to $\mathbb{C}$ of the space of sections of the L-th power of the canonical bundle of $\mathbb{S}^2$.
It is not strange to consider such a norm. We normalize $|f(z)|^2$ by $(1+|z|^2)^L$, which is a rescaled by the degree of f, to avoid that the first terms tends to infinite. The factor $L+1$ is chosen so that

$$\frac{L+1}{(1+|z|^2)^L}dm^*(z), \quad z \in \mathbb{C},$$

is a probability measure, i.e., $\|1\|_{\mathcal{P}_L} = 1$.

Lemma 2.4.6. *The family*

$$e_n(z) = \binom{L}{n}^{1/2} z^n, \quad n = 0, ..., L,$$

forms an orthonormal basis of $\mathcal{P}_L$*.*

Proof. We have to show that $(e_n, e_m)_{\mathcal{P}_L} = \delta_{n,m}$ and that $(e_n)_{n=0}^{+\infty}$ is complete. For $n, m \in \mathbb{N}$ we have, by applying a change of variables in polar coordinates:

$$\begin{aligned}(e_n, e_m)_{\mathcal{P}_L} &= \frac{L+1}{\pi}\int_{\mathbb{C}}\binom{L}{n}^{1/2}\binom{L}{m}^{1/2}\frac{z^n\overline{z}^m}{(1+|z|^2)^{L+2}}dm(z) \\ &= \frac{L+1}{\pi}\binom{L}{n}^{1/2}\binom{L}{m}^{1/2}\int_0^{2\pi}\int_0^{+\infty}\frac{r^{n+m+1}e^{i\theta(n-m)}}{(1-r^2)^{L+2}}drd\theta.\end{aligned}$$

Thus, if $n \neq m$, $(e_n, e_m)_{\mathcal{P}_L} = 0$ by (2.4.1). Otherwise, if $n = m$, we get, by using the property (2.3.1), the definition of the Beta function (2.3.3) and the change of variable $t = r^2$, that

$$\begin{aligned}(e_n, e_n)_{\mathcal{P}_L} &= \frac{(L+1)\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}\int_0^{+\infty}\frac{2r^{2n+1}}{(1+r^2)^{L+2}}dr \\ &= \frac{(L+1)\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}\int_0^{+\infty}\frac{t^n}{t^{L+2}}dt \\ &= \frac{(L+1)\Gamma(L+1)}{\Gamma(n+1)\Gamma(L-n+1)}\frac{\Gamma(n+1)\Gamma(L-n+1)}{\Gamma(L+2)} \\ &= \frac{(L+1)\Gamma(L+1)}{\Gamma(L+2)} = 1.\end{aligned}$$

Hence $(e_n)_{n=0}^{L}$ is an orthonormal system in $\mathcal{P}_L$.
Completeness follows since $(z^n)_{n=0}^{L}$ is an orthogonal basis and $(e_n)_{n=0}^{L}$ is its orthonormalization. □

By using Definition 2.2.1, for all $L \in \mathbb{N}$ we can define the GAF

$$f_L(z) = \sum_{n=0}^{L} \xi_n \binom{L}{n}^{1/2} z^n, \tag{2.4.8}$$

where $(\xi_n)_{n=0}^L$ is a sequence of i.i.d. random variables $N_{\mathbb{C}}(0,1)$. This is known as the *parabolic Gaussian analytic function* ($\mathbb{S}^2$-GAF, for short). The covariance kernel of this GAF is

$$\mathcal{K}_{f_L}(z,w) = \sum_{n=0}^{L} \binom{L}{n} z^n \overline{w}^n = (1 + z\overline{w})^L. \tag{2.4.9}$$

2.5 Distribution, intensity and invariance of the zero set of a GAF

In this section we would like to see how these zero point sets are distributed. Assume that f is a GAF in a region $\Omega \subseteq \mathbb{C}$ and denote $\mathcal{Z}_f$ its zero set.

Definition 2.5.1. The *empirical measure* of f is

$$\nu_f = \sum_{a \in \mathcal{Z}_f} \delta_a = \frac{1}{2\pi} \Delta \log |f|,$$

where δ_a is the Dirac delta measure at a and the Laplacian Δ must be understood in the distributional sense.

The empirical measure encodes all the information of $\mathcal{Z}_f$, and allows the use of the tools of the theory of distributions to extract information from it.

Definition 2.5.2. Let μ be measure that is finite over compact sets of Ω. We say that $u \in L^1_{\text{loc}}(\Omega)$ is a *solution* of

$$\Delta u = \mu$$

on Ω in the sense of distributions if for all $\varphi \in C_c^2(\Omega)$ it is satisfied

$$\int_\Omega u(z) \Delta \varphi(z) dm(z) = \int_\Omega \varphi(z) d\mu(z).$$

Now we are ready to state a fundamental result describing the average distribution of $\mathcal{Z}_f$.

Theorem 2.5.3. (The Edelman-Kostlan formula) (see [2], p. 24, 25) *Assume that f is a GAF in Ω with zero mean and covariance kernel $\mathcal{K}_f(z,w)$, for all $z, w \in \Omega$. Then*

$$\mathbb{E}[\nu_f] = \frac{1}{2\pi} \log \mathbb{E}[|f(z)|] = \frac{1}{4\pi} \Delta \log \mathcal{K}_f(z,z),$$

where Δ must be understood in the sense of distributions and $\mathbb{E}[\nu_f]$ is a deterministic measure called the first intensity.

Proof. Consider a function $\varphi \in \mathcal{C}_c^2(\Omega)$. By Definition 2.5.1 we get

$$\int_\Omega \varphi(z)d\nu_f(z) = \int_\Omega \frac{1}{2\pi}\log|f(z)|\Delta\varphi(z)dm(z),$$

which implies

$$\mathbb{E}\left[\int_\Omega \varphi(z)d\nu_f(z)\right] = \mathbb{E}\left[\int_\Omega \frac{1}{2\pi}\log|f(z)|\Delta\varphi(z)dm(z)\right].$$

In order to apply Fubini's theorem on the RHS of the equation we must verify that:

$$\mathbb{E}\left[\int_\Omega \left|\frac{1}{2\pi}\log|f(z)|\Delta\varphi(z)dm(z)\right|\right] < \infty.$$

Using the linearity of the expectation and taking into account that $\Delta\varphi$ is deterministic, we have

$$\mathbb{E}\left[\int_\Omega \left|\frac{1}{2\pi}\log|f(z)|\Delta\varphi(z)dm(z)\right|\right] = \int_\Omega \frac{1}{2\pi}\mathbb{E}\left[|\log|f(z)||\right]|\Delta\varphi(z)|dm(z).$$

By denoting $\hat{f}(z) = f(z)/\sqrt{\mathcal{K}_f(z,z)}$, which is a $N_\mathbb{C}(0,1)$ random variable, we have:

$$\begin{aligned}
\mathbb{E}\left[|\log|f(z)||\right] &= \mathbb{E}\left[\left|\log\left|\frac{f(z)}{\sqrt{\mathcal{K}_f(z,z)}}\sqrt{\mathcal{K}_f(z,z)}\right|\right|\right] \\
&= \mathbb{E}\left[|\log|\hat{f}(z)||\right] + \log\left|\sqrt{\mathcal{K}_f(z,z)}\right| \\
&\overset{(*)}{=} \int_\mathbb{C} |\log|\xi||\frac{e^{-|\xi|^2}}{\pi}dm(\xi) + \frac{1}{2}\log|\mathcal{K}_f(z,z)| \\
&\overset{(**)}{=} \int_0^{+\infty} 2r|\log(r)|e^{-r^2}dr + \frac{1}{2}\log|\mathcal{K}_f(z,z)| \\
&\overset{(***)}{=} \int_0^{+\infty} |\log(s)|e^{-s}ds + \frac{1}{2}\log|\mathcal{K}_f(z,z)| = K_1 + \frac{1}{2}\log|\mathcal{K}_f(z,z)|,
\end{aligned}$$

where in $(*)$ we applied the definition of expectation, taking into account that the density function of a $N_\mathbb{C}(0,1)$ random variable is $e^{-|z|^2}/\pi$, for all $z \in \mathbb{C}$. The step $(**)$ follows by a change into polar coordinates and $(***)$ by the change of variable $s = r^2$. Recall that K_1 is a constant value.

Notice that $\log|\mathcal{K}_f(z,z)|$ is locally integrable for all $z \in \Omega$. However we must study the case when $\mathcal{K}_f(a,a) = 0$, for $a \in \Omega$. For this we will show that:

$$\mathcal{K}_f(z,z) = |z-a|^{2m}G(z,z),$$

where m is a natural number and G is a function such that $G(a,a) \neq 0$, for those values $a \in \Omega$ such that $\mathcal{K}_f(a,a) = 0$.

Indeed, by Definition 2.2.1, we can write

$$\mathcal{K}_f(z,z) = \sum_{n=0}^{+\infty} e_n(z)\overline{e_n(z)}.$$

Each element of $(e_n)_{n=0}^{+\infty}$ is of the form

$$e_n(z) = (z-a)^{m_n} g_n(z),$$

where m_n is the multiplicity of a and g_n is a function such that $g_n(a) \neq 0$. Thus, and by denoting $m = \min_{n\in\mathbb{N}} m_n$:

$$\begin{aligned}\mathcal{K}_f(z,z) &= \sum_{n=0}^{+\infty} |z-a|^{2m_n} |g_n(z)|^2 = |z-a|^{2m} \sum_{n=0}^{+\infty} |z-a|^{2m_n-2m} |g_n(z)|^2 \\ &= |z-a|^{2m} G(z,z),\end{aligned}$$

where $G(a,a) \neq 0$ since $g_n(a) \neq 0$. Hence $\log |\mathcal{K}_f(z,z)|$ is locally integrable in a neighbourhood of a. Therefore

$$\mathbb{E}\left[\int_\Omega \left|\frac{1}{2\pi} \log|f(z)|\Delta\varphi(z) dm(z)\right|\right] < \infty,$$

and we are under conditions of Fubini's theorem. We get:

$$\mathbb{E}\left[\int_\Omega \frac{1}{2\pi} \log|f(z)|\Delta\varphi(z) dm(z)\right] = \int_\Omega \frac{1}{2\pi} \Delta\mathbb{E}\left[\log|f(z)|\right] \varphi(z) dm(z).$$

Proceeding in a similar way as before we have

$$\mathbb{E}\left[\log|f(z)|\right] = K_2 + \frac{1}{2}\log\mathcal{K}_f(z,z),$$

where K_2 is a constant value. Finally, we reach the equality

$$\mathbb{E}\left[\int_\Omega \varphi(z) d\nu_f(z)\right] = \int_\Omega \frac{1}{4\pi} \Delta\log\mathcal{K}_f(z,z)\varphi(z) dm(z).$$

By Fubini's theorem and recalling that φ is deterministic, we get that

$$\mathbb{E}\left[\int_\Omega \varphi(z) d\nu_f(z)\right] = \int_\Omega \varphi(z) d\mathbb{E}\left[\nu_f\right].$$

By Definition 2.5.2 we conclude that the first intensity is, with respect to the Lebesgue measure:

$$\mathbb{E}\left[\nu_f\right] = \frac{1}{4\pi}\Delta\log\mathcal{K}_f(z,z).$$

□

2.5.1 First intensity of a $\mathbb{C}$-GAF

We see here that the average number of points of the $\mathbb{C}$-GAF of intensity $L > 0$ is L times the Lebesgue measure in $\mathbb{C}$.

Let f_L be a $\mathbb{C}$-GAF with real parameter $L > 0$. From its covariance kernel expression, (2.4.3), we have

$$\mathbb{E}[\nu_{f_L}] = \frac{1}{4\pi}\Delta \log \mathcal{K}_{f_L}(z,z) = \frac{1}{4\pi}\Delta \log \left(e^{Lz\overline{w}}\right) = \frac{L}{\pi}\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}(z\overline{z}) = \frac{L}{\pi}dm(z).$$

The zero set process of a $\mathbb{C}$-GAF has another property: it is invariant by translations in $\mathbb{C}$.

Proposition 2.5.4. *Let f_L be a $\mathbb{C}$-GAF of real parameter $L > 0$. Its zero point process is invariant under the transformations*

$$\phi_a(z) = z - a, \quad z, a \in \mathbb{C}.$$

Proof. Denote

$$f_a(z) = f_L(\phi_a(z)).$$

This function has covariance kernel

$$\mathcal{K}_{f_a}(z,w) = \mathcal{K}_{f_L}(\phi_a(z), \phi_a(w)) = e^{Lz\overline{w} - Lz\overline{a} - La\overline{w} + L|a|^2}. \tag{2.5.1}$$

Consider the function

$$T_a f_L(z) = f_a(z) e^{Lz\overline{a} - \frac{L}{2}|a|^2}.$$

We want to show that

$$f_L(z) = T_a f_L(z)$$

in distribution. To check this we must ensure that their covariance kernels are equal. Indeed:

$$\mathcal{K}_{T_a f_L}(z,w) = \mathcal{K}_{f_a}(z,w) e^{Lz\overline{a} - L|a|^2 + La\overline{w}} \overset{(*)}{=} e^{Lz\overline{w}} = \mathcal{K}_{f_L}(z,w),$$

where in $(*)$ we used (2.5.1). Since the covariance kernels coincide, the proof is finished. □

Proposition 2.5.5. *Let f_L be a $\mathbb{C}$-GAF of real parameter $L > 0$. Then f_L and $T_a f_L$ are isometric.*

Proof. We must show that

$$\|f_L\|^2_{\mathcal{F}_L} = \|T_a f_L\|^2_{\mathcal{F}_L}.$$

Noticing that

$$\|T_a f_L\|^2_{\mathcal{F}_L} = \frac{L}{\pi}\int_{\mathbb{C}} |f_L(z-a)|^2 \left| e^{L\overline{a}z - \frac{L}{2}|a|^2}\right|^2 e^{-L|z|^2} dm(z),$$

doing the change of variable $w = z - a$ and recalling that the Lebesgue measure is invariant under translations, the equality is simple to verify. □

Proposition 2.5.4 implies that the zero set of f_a is the same one in distribution than f_L. Indeed, since $e^{Lz\overline{a} - \frac{L}{2}|a|^2}$ does not vanish anywhere, the property is trivially accomplished.

Since the first intensity of a $\mathbb{C}$-GAF determines in mean the distribution of its zero set process, it is also invariant by translations in $\mathbb{C}$.

2.5.2 First intensity of an $\mathbb{S}^2$-GAF

We see here that the average number of points of the $\mathbb{S}^2$-GAF in a region is proportional to the area of the region at $\mathbb{S}^2$.
Let f_L be an $\mathbb{S}^2$-GAF of parameter $L \in \mathbb{N}$. From its covariance kernel, (2.4.9), we have

$$\mathbb{E}[\nu_{f_L}] = \frac{1}{4\pi}\Delta \log \mathcal{K}_{f_L}(z,z) = \frac{1}{4\pi}\Delta \log\left[(1+|z|^2)^L\right] = \frac{L}{\pi}\frac{dm(z)}{(1+|z|^2)^2} = Ldm^*(z).$$

Notice that the Poisson process and the zero set process of an $\mathbb{S}^2$-GAF have the same average number of points (see (1.3.2)).
The zero set process of an $\mathbb{S}^2$-GAF is invariant by rotations in $\mathbb{S}^2$, which in the $\mathbb{C}$-chart are seen as certain Möbius transformations.

Proposition 2.5.6. *Let f_L be an $\mathbb{S}^2$-GAF. Its zero point process is invariant under the Möbius transformations*

$$\phi_a(z) = \frac{z-a}{1+\overline{a}z}, \qquad a, z \in \mathbb{C}. \tag{2.5.2}$$

Proof. As in the proof of Proposition 2.5.4, consider

$$f_a(z) = f_L(\phi_a(z)).$$

Its covariance kernel is

$$\mathcal{K}_{f_a}(z,w) = \mathcal{K}_{f_L}(\phi_a(z),\phi_a(w)) = \left(\frac{(1+|a|^2)(1+z\overline{w})}{(1+\overline{a}z)(1+a\overline{w})}\right)^L. \tag{2.5.3}$$

Consider

$$T_a f_L(z) = \left(\frac{1+|a|^2}{(1+\overline{a}z)^2}\right)^{-L/2} f_a(z).$$

What we want to prove is

$$f_L(z) = T_a f_L(z)$$

in distribution. To see this we must check that their covariance kernels coincide. We have:

$$\begin{aligned}\mathcal{K}_{T_a f_L}(z,w) &= \mathcal{K}_{f_a}(z,w)\left(\frac{1+|a|^2}{(1+\overline{a}z)^2}\right)^{-L/2}\overline{\left(\frac{1+|a|^2}{(1+\overline{a}w)^2}\right)^{-L/2}} \\ &\stackrel{(*)}{=} (1+z\overline{w})^L = \mathcal{K}_{f_L}(z,w),\end{aligned}$$

where in $(*)$ we applied (2.5.3). Since the covariance kernels are equal, we finally have the result we desired. $\square$

Proposition 2.5.7. *Let f_L be an $\mathbb{S}^2$-GAF of parameter $L \in \mathbb{N}$. We have that T_a is an isometry from $\mathcal{P}_L$ to $\mathcal{P}_L$.*

Proof. The statement is equivalent to proving that

$$\|f_L\|^2_{\mathcal{P}_L} = \|T_a f_L\|^2_{\mathcal{P}_L}.$$

Using (2.4.7) we have that

$$(1+|z|^2)\,|\phi_a'(z)| = \frac{(1+|a|^2)(1+|z|^2)}{|1+\overline{a}z|^2} = (1+|\phi_a(z)|)^{-1}.$$

Hence, by doing the change of variable $w = \phi_a(z)$ (see (2.5.2)), using (2.4.7) and applying Corollary 2.4.5, we get:

$$\begin{aligned}\|T_a f_L\|^2_{\mathcal{P}_L} &= (L+1)\int_{\mathbb{C}} \frac{|f_L(\phi_a(z))|^2}{(1+|z|^2)^L}\frac{|1+\overline{a}z|^{2L}}{(1+|a|^2)^L} dm^*(z) \\ &= (L+1)\int_{\mathbb{C}} \frac{|f_L(\phi_a(z))|^2}{(1+|\phi_a(z)|^2)^L} dm^*(z) \\ &= (L+1)\int_{\mathbb{C}} \frac{|f_L(w)|^2}{(1+|w|^2)^L} dm^*(w) = \|f_L\|^2_{\mathcal{P}_L}.\end{aligned}$$

□

By Proposition 2.5.6 we get that the zero set point of an $\mathbb{S}^2$-GAF f_L is the same than $T_a f_L$ in distribution. Furthermore, the zero set of f_L is equal to f_a in distribution. Indeed, if for all $a, z \in \mathbb{C}$ we had

$$\left(\frac{1+|a|^2}{(1+\overline{a}z)^2}\right)^{-L/2} = 0,$$

we would conclude that $|a|^2 = -1$, which is a contradiction by the definition of modulus.
Since the first intensity of an $\mathbb{S}^2$-GAF determines in mean the distribution of its zero set process, it is also invariant by the Möbius rotational transformation. Indeed, by the change of variable $w = \phi_a(z)$ (see (2.5.2)), using (2.4.5) and taking f_a as in the proof of Proposition 2.5.6, we get

$$\begin{aligned}\mathbb{E}[\nu_{f_L}] &= \frac{L}{\pi}\frac{dm(z)}{(1+|\phi_a(z)|^2)^2} = \frac{L}{\pi}\frac{(1-a\overline{w})^2(1-\overline{a}w)^2(1+|a|^2)^2}{(1-a\overline{w})^2(1-\overline{a}w)^2(1+|a|^2)^2(1+|w|^2)^2} dm(w) \\ &= \frac{L}{\pi}\frac{dm(w)}{(1+|w|^2)^2} = \mathbb{E}[\nu_{f_a}].\end{aligned}$$

Chapter 3

Fluctuation of the zero set of an $\mathbb{S}^2$-GAF

Here we address the main goal of this project: the study of the fluctuation of an $\mathbb{S}^2$-GAF in Ω. This can be measured in different ways. One possibility is to measure how the number of zeros in a given region $D \subset\subset \Omega$, with ∂D regular, deviates from its mean (given by the Edelman-Kostlan formula). This deviation can be quantified as the variance of the random variables counting the number of points on D, i.e., $\nu_f(D)$, where f is a GAF in Ω.
We will first give a general formula for the variance of $\nu_f(D)$. Later we shall use this formula to write the particular case of the $\mathbb{S}^2$-GAF f_L of parameter $L \in \mathbb{N}$ and D a disk in $\mathbb{C}$. From this explicit formula we will deduce the asymptotic behaviour of the variance of $\nu_{f_L}(D)$ in different ways.
To write this chapter I followed a similar scheme as in [5]. The source [6] was essential for this chapter. The references [2] and [9] were also helpful for the understanding of some proofs.

3.1 Variance of a GAF

We are going to compute the variance of a GAF f defined in Ω in a region $D \subset\subset \Omega$ with ∂D regular. For simplicity, we are going to denote

$$\nu_f(D) = I(\mathbb{1}_D) = \int_D d\nu_f = \#(\mathcal{Z}_f \cap D).$$

In addition, we will use $\mathbb{V}$ for the variance. We state the following theorem.

Theorem 3.1.1. *Let f be a GAF in Ω and let $D \subset\subset \Omega$ with ∂D regular. Then*

$$\mathbb{V}\left[\nu_f(D)\right] = -\frac{1}{4\pi^2}\int_{\partial D}\int_{\partial D}\frac{1}{1-K(z,w)}\frac{\partial}{\partial \overline{z}}\left(\frac{\mathcal{K}_f(w,z)}{\mathcal{K}_f(z,z)}\right)\frac{\partial}{\partial \overline{w}}\left(\frac{\mathcal{K}_f(z,w)}{\mathcal{K}_f(w,w)}\right)d\overline{z}d\overline{w}, \tag{3.1.1}$$

where

$$K(z,w) = \frac{|\mathcal{K}_f(z,w)|^2}{\mathcal{K}_f(z,z)\mathcal{K}_f(w,w)}. \tag{3.1.2}$$

Proof. By definition

$$\mathbb{V}\left[\nu_f(D)\right] = \mathbb{E}\left[\left(\nu_f(D) - \mathbb{E}\left[\nu_f(D)\right]\right)^2\right].$$

By Definition 2.5.1 and Theorem 2.5.3 we can state:

$$\begin{aligned}\nu_f(D) - \mathbb{E}\left[\nu_f(D)\right] &= \int_D \left(\frac{1}{2\pi}\Delta \log|f(z)| - \frac{1}{2\pi}\Delta\sqrt{\mathcal{K}_f(z,z)}\right)\\ &= \int_D \frac{1}{2\pi}\Delta \log\left(\frac{|f(z)|}{\sqrt{\mathcal{K}_f(z,z)}}\right)\\ &\overset{(*)}{=} \int_D \frac{1}{2\pi}\Delta \log|\hat{f}(z)|,\end{aligned}$$

where in $(*)$ we denote $\hat{f} = f/\sqrt{\mathcal{K}_f(z,z)} \sim N_{\mathbb{C}}(0,1)$. To go further in the calculations consider the 1-form

$$\omega = -\frac{i}{\pi}\overline{\partial}_z \log|\hat{f}(z)|d\overline{z}.$$

Applying the exterior derivative to ω, remembering that $d^2 = 0$ and recalling the fact that $dz \wedge d\overline{z} = -2i dx \wedge dy$, we get:

$$\begin{aligned}dw &= -\frac{i}{\pi}\frac{\partial^2}{\partial z \partial \overline{z}} \log|\hat{f}(z)|d\overline{z}\wedge dz = \frac{i}{4\pi}\Delta \log|\hat{f}(z)|dz\wedge d\overline{z} = \frac{1}{2\pi}\Delta\log|\hat{f}(z)|dx\wedge dy\\ &= \frac{1}{2\pi}\Delta \log|\hat{f}(z)|dm(z).\end{aligned}$$

By Stokes' theorem we conclude that

$$\int_D \frac{1}{2\pi}\Delta \log|\hat{f}(z)| = -\int_{\partial D}\frac{i}{\pi}\overline{\partial}_z \log|\hat{f}(z)|d\overline{z}.$$

So we have

$$\begin{aligned}\mathbb{V}\left[\nu_f(D)\right] &= \mathbb{E}\left[\left(\nu_f(D) - \mathbb{E}\left[\nu_f(D)\right]\right)^2\right]\\ &= \mathbb{E}\left[\int_{\partial D}\frac{i}{\pi}\overline{\partial}_z \log|\hat{f}(z)|d\overline{z}\int_{\partial D}\frac{i}{\pi}\overline{\partial}_w \log|\hat{f}(w)|d\overline{w}\right]\\ &= -\frac{1}{\pi^2}\int_{\partial D}\int_{\partial D}\overline{\partial}_z\overline{\partial}_w \mathbb{E}\left[\log|\hat{f}(z)|\log|\hat{f}(w)|\right]d\overline{z}d\overline{w},\end{aligned}$$

where in the last equality we used Fubini's theorem and the differentiation under the integral sign. Recall that, if X and Y are two random variables, their *covariance* is defined as

$$\operatorname{Cov}\left[X,Y\right] = \mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right].$$

Here

$$\operatorname{Cov}\left[\log|\hat{f}(z)|, \log|\hat{f}(w)|\right] = \mathbb{E}\left[\log|\hat{f}(z)|\log|\hat{f}(w)|\right] - \mathbb{E}\left[\log|\hat{f}(z)|\right]\mathbb{E}\left[\log|\hat{f}(w)|\right].$$

Since $\hat{f} \sim N_{\mathbb{C}}(0,1)$, $\mathbb{E}\left[\log|\hat{f}(z)|\right]$ is constant (independent of z). Hence we have

$$\overline{\partial}_z\overline{\partial}_w \mathrm{Cov}\left[\log|\hat{f}(z)|, \log|\hat{f}(w)|\right] = \overline{\partial}_z\overline{\partial}_w \mathbb{E}\left[\log|\hat{f}(z)|\log|\hat{f}(w)|\right]$$

and then we can write

$$\mathbb{V}\left[\nu_f(D)\right] = -\frac{1}{\pi^2}\int_{\partial D}\int_{\partial D} \overline{\partial}_z\overline{\partial}_w \mathrm{Cov}\left[\log|\hat{f}(z)|, \log|\hat{f}(w)|\right] d\overline{z}d\overline{w}.$$

Notice that

$$\mathbb{E}\left[\hat{f}(z)\overline{\hat{f}(z)}\right] = \mathbb{E}\left[\left|\hat{f}(z)\right|^2\right] = \mathbb{E}\left[\frac{|f(z)|^2}{\mathcal{K}_f(z,z)}\right] = \frac{\mathcal{K}_f(z,z)}{\mathcal{K}_f(z,z)} = 1.$$

Also we get

$$\Theta(z,w) := \mathbb{E}\left[\hat{f}(z)\overline{\hat{f}(w)}\right] = \frac{\mathbb{E}\left[\hat{f}(z)\overline{\hat{f}(w)}\right]}{\sqrt{\mathcal{K}_f(z,z)}\sqrt{\mathcal{K}_f(w,w)}} = \frac{\mathcal{K}_f(z,w)}{\sqrt{\mathcal{K}_f(z,z)}\sqrt{\mathcal{K}_f(w,w)}}.$$

Lemma 3.1.2 ([2], p. 44-46)**.** *Let Z_1 and Z_2 be complex normal random variables such that $\mathbb{E}\left[Z_1\overline{Z_1}\right] = \mathbb{E}\left[Z_2\overline{Z_2}\right] = 1$ and $\mathbb{E}\left[Z_1\overline{Z_2}\right] = \theta$, then*

$$\mathrm{Cov}\left[\log|Z_1|, \log|Z_2|\right] = \sum_{j=1}^{+\infty}\frac{|\theta|^{2j}}{4j^2} = \frac{1}{4}\mathrm{Li}_2\left(|\theta|^2\right).$$

The function

$$\mathrm{Li}_2(x) := \sum_{j=1}^{+\infty}\frac{x^j}{j^2}, \quad x\in[0,1),$$

is called the dilogarithm.

Applying this lemma in our case:

$$\mathbb{V}\left[\nu_f(D)\right] = -\frac{1}{4\pi^2}\int_{\partial D}\int_{\partial D} \overline{\partial}_z\overline{\partial}_w \mathrm{Li}_2\left(|\Theta(z,w)|^2\right) d\overline{z}d\overline{w}.$$

We shall compute $\overline{\partial}_z\overline{\partial}_w\mathrm{Li}_2\left(K(z,w)\right)$. For simplicity, denote (3.1.2) by K. We have

$$\overline{\partial}_w\mathrm{Li}_2\left(K\right) = \overline{\partial}_w\sum_{j=1}^{+\infty}\frac{K^j}{j^2} = \overline{\partial}_w(K)\sum_{j=1}^{+\infty}\frac{K^{j-1}}{j},$$

and

$$\overline{\partial}_z\overline{\partial}_w\mathrm{Li}_2\left(K\right) = \overline{\partial}_z\left(\overline{\partial}_w(K)\sum_{j=1}^{+\infty}\frac{K^{j-1}}{j}\right) \tag{3.1.3}$$

$$= \frac{\partial^2 K}{\partial\overline{z}\partial\overline{w}}\sum_{j=1}^{+\infty}\frac{K^{j-1}}{j} + \overline{\partial}_w(K)\overline{\partial}_z(K)\sum_{j=1}^{+\infty}\frac{j-1}{j}K^{j-2}. \tag{3.1.4}$$

We claim that:

$$\frac{\partial^2 K}{\partial \overline{z} \partial \overline{w}} = \frac{1}{K} \frac{\partial K}{\partial \overline{z}} \frac{\partial K}{\partial \overline{w}}.$$

On the one hand

$$\frac{\partial K}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}} \left(\frac{|\mathcal{K}_f(z,w)|^2}{\mathcal{K}_f(z,z)\mathcal{K}_f(w,w)} \right) = \frac{1}{\mathcal{K}_f(w,w)} \frac{\partial}{\partial \overline{z}} \left(\frac{\mathcal{K}_f(z,w)\overline{\mathcal{K}_f(w,z)}}{\mathcal{K}_f(z,z)} \right) \tag{3.1.5}$$

$$= \frac{\mathcal{K}_f(z,w)}{\mathcal{K}_f(w,w)} \frac{\partial}{\partial \overline{z}} \left(\frac{\mathcal{K}_f(w,z)}{\mathcal{K}_f(z,z)} \right). \tag{3.1.6}$$

On the other hand

$$\frac{\partial K}{\partial \overline{w}} = \frac{\partial}{\partial \overline{w}} \left(\frac{|\mathcal{K}_f(z,w)|^2}{\mathcal{K}_f(z,z)\mathcal{K}_f(w,w)} \right) = \frac{1}{\mathcal{K}_f(z,z)} \frac{\partial}{\partial \overline{w}} \left(\frac{\mathcal{K}_f(z,w)\overline{\mathcal{K}_f(w,z)}}{\mathcal{K}_f(w,w)} \right) \tag{3.1.7}$$

$$= \frac{\mathcal{K}_f(w,z)}{\mathcal{K}_f(z,z)} \frac{\partial}{\partial \overline{w}} \left(\frac{\mathcal{K}_f(z,w)}{\mathcal{K}_f(w,w)} \right). \tag{3.1.8}$$

All combined

$$\begin{aligned} \frac{1}{K} \frac{\partial K}{\partial \overline{z}} \frac{\partial K}{\partial \overline{w}} &= \frac{1}{K} \frac{\mathcal{K}_f(z,w)\mathcal{K}_f(w,z)}{\mathcal{K}_f(w,w)\mathcal{K}_f(z,z)} \frac{\partial}{\partial \overline{z}} \left(\frac{\mathcal{K}_f(w,z)}{\mathcal{K}_f(z,z)} \right) \frac{\partial}{\partial \overline{w}} \left(\frac{\mathcal{K}_f(z,w)}{\mathcal{K}_f(w,w)} \right) \\ &= \frac{\partial^2 K}{\partial \overline{z} \partial \overline{w}}. \end{aligned}$$

Now we can use this identity to finish the computations of $\overline{\partial}_z \overline{\partial}_w \mathrm{Li}_2(K)$. Going back to (3.1.3), we have

$$\begin{aligned} \overline{\partial}_z \overline{\partial}_w \mathrm{Li}_2(K) &= \frac{\partial^2 K}{\partial \overline{z} \partial \overline{w}} \sum_{j=1}^{+\infty} \frac{K^{j-1}}{j} + \overline{\partial}_w(K)\overline{\partial}_z(K) \sum_{j=1}^{+\infty} \frac{j-1}{j} K^{j-2} \\ &= \frac{1}{K} \frac{\partial K}{\partial \overline{z}} \frac{\partial K}{\partial \overline{w}} \sum_{j=1}^{+\infty} \frac{K^{j-1}}{j} + \frac{\partial K}{\partial \overline{z}} \frac{\partial K}{\partial \overline{w}} \sum_{j=1}^{+\infty} \frac{j-1}{j} K^{j-2} \\ &= \frac{1}{K} \frac{\partial K}{\partial \overline{z}} \frac{\partial K}{\partial \overline{w}} \left(\sum_{j=1}^{+\infty} K^{j-1} \right) = \frac{1}{K(1-K)} \frac{\partial K}{\partial \overline{z}} \frac{\partial K}{\partial \overline{w}}, \end{aligned}$$

where the last equality follows because $|K| \leq 1$ (see Lemma 2.2.3). Hence

$$\mathbb{V}[\nu_f(D)] = -\frac{1}{4\pi^2} \int_{\partial D} \int_{\partial D} \frac{1}{K(1-K)} \frac{\partial K}{\partial \overline{z}} \frac{\partial K}{\partial \overline{w}} d\overline{z} d\overline{w}.$$

Using (3.1.5) and (3.1.7), we finish with the expression

$$\mathbb{V}[\nu_f(D)] = -\frac{1}{4\pi^2} \int_{\partial D} \int_{\partial D} \frac{1}{1-K} \frac{\partial}{\partial \overline{z}} \left(\frac{\mathcal{K}_f(w,z)}{\mathcal{K}_f(z,z)} \right) \frac{\partial}{\partial \overline{w}} \left(\frac{\mathcal{K}_f(z,w)}{\mathcal{K}_f(w,w)} \right) d\overline{z} d\overline{w},$$

as we wanted. □

3.2 Fluctuation of the zero set of an $\mathbb{S}^2$-GAF

In this section we will use Theorem 3.1.1 to get expressions for disks of the variance for a $\mathbb{C}$-GAF with real parameter $L > 0$ and for a $\mathbb{S}^2$-GAF with parameter $L \in \mathbb{N}$. We will compare these values with the analogous ones for a Poisson process with the same average of points. Also we will show that the Bargmann-Fock space of parameter $L = 1$ can be seen as the limit of parabolic spaces as $L \to +\infty$.

3.2.1 Variance of a $\mathbb{C}$-GAF

The starting point of our analysis is the explicit formula of the variance for a $\mathbb{C}$-GAF.

Theorem 3.2.1. *Let f_L be a $\mathbb{C}$-GAF of real parameter $L > 0$. For a disk $D(z_0, r) \subset \mathbb{C}$ for $z_0 \in \mathbb{C}$ and $r > 0$ we have:*

$$\mathbb{V}\left[\nu_{f_L}\left(D(z_0,r)\right)\right] = \frac{\sqrt{Lr^2}}{2\pi}\int_0^{4Lr^2}\frac{1}{e^x-1}\frac{\sqrt{x}}{\sqrt{1-\frac{x}{4Lr^2}}}dx. \tag{3.2.1}$$

Proof. The value does not depend on z_0 by the invariance of the translations. Then we can assume that $z_0 = 0$. Denote $D := D(0, r)$.
We apply Theorem 3.1.1 to this case. We know that $\mathcal{K}_{f_L}(z, w) = e^{Lz\overline{w}}$. Then, by (3.1.2), $K(z, w) = e^{-L|z-w|^2}$. It is trivial to compute the following:

$$\frac{\mathcal{K}_{f_L}(z,w)}{\mathcal{K}_{f_L}(w,w)} = e^{Lz\overline{w}-L|w|^2}, \qquad \frac{\partial}{\partial \overline{w}}\left(\frac{\mathcal{K}_{f_L}(z,w)}{\mathcal{K}_{f_L}(w,w)}\right) = L(z-w)e^{Lz\overline{w}-L|w|^2},$$

$$\frac{\mathcal{K}_{f_L}(w,z)}{\mathcal{K}_{f_L}(z,z)} = e^{Lw\overline{z}-L|z|^2}, \qquad \frac{\partial}{\partial \overline{z}}\left(\frac{\mathcal{K}_{f_L}(w,z)}{\mathcal{K}_{f_L}(z,z)}\right) = L(w-z)e^{Lw\overline{z}-L|z|^2},$$

$$\frac{1}{1-K(z,w)} = \frac{1}{1-e^{-L|z-w|^2}}.$$

Hence

$$\mathbb{V}\left[\nu_f\left(D\right)\right] = \frac{L^2}{4\pi^2}\int_{\partial D}\int_{\partial D}\frac{(w-z)^2}{e^{L|z-w|^2}-1}d\overline{z}d\overline{w}.$$

Denoting $z = re^{i\theta}$, $w = re^{i\phi}$, for all $\theta, \phi \in (0, 2\pi)$, we have

$$\begin{aligned}(w-z)^2\left|\mathcal{J}(\theta,\phi)\right| &= r^2(e^{2i\theta}-2e^{i(\theta+\phi)}+e^{2i\phi})\begin{vmatrix}-ire^{-i\theta} & 0\\ 0 & -ire^{-i\phi}\end{vmatrix}\\ &= -r^4(e^{i(\theta-\phi)}+e^{i(\phi-\theta)}-2) = r^4\left|1-e^{i(\theta-\phi)}\right|^2,\end{aligned}$$

where $\mathcal{J}$ is the Jacobian matrix. Applying the change of variables with respect to (θ, ϕ), we get

$$\mathbb{V}\left[\nu_{f_L}\left(D\right)\right] = \frac{Lr^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{Lr^2\left|1-e^{i(\theta-\phi)}\right|^2}{e^{Lr^2\left|1-e^{i(\theta-\phi)}\right|^2}-1}d\theta d\phi \overset{(*)}{=} \frac{Lr^2}{2\pi}\int_0^{2\pi}\frac{Lr^2\left|1-e^{it}\right|^2}{e^{Lr^2|1-e^{it}|^2}-1}dt,$$

where in $(*)$ we applied the change $t=\theta-\phi$. Now, notice that $\left|1-e^{it}\right|^2=2-2\cos t$. Using the change of variable

$$x=2Lr^2(1-\cos t),\qquad dx=2Lr^2\sin t dt,$$

where

$$\sin t=\sqrt{1-\cos^2 t}=\sqrt{\frac{x}{Lr^2}}\sqrt{1-\frac{x}{4Lr^2}},$$

we have, by using the fact that the integrand is even,

$$\mathbb{V}\left[\nu_{f_L}(D)\right]=\frac{\sqrt{Lr^2}}{2\pi}\int_0^{4Lr^2}\frac{1}{e^x-1}\frac{\sqrt{x}}{\sqrt{1-\frac{x}{4Lr^2}}}dx.$$

□

3.2.2 Variance of an $\mathbb{S}^2$-GAF

Let us give the variance expression for an $\mathbb{S}^2$-GAF.

Theorem 3.2.2. *Let f_L be an $\mathbb{S}^2$-GAF of parameter $L\in\mathbb{N}$. For a chordal disk $D_{ch}(z_0,\tilde\rho)\subset\mathbb{C}$ for $z_0\in\mathbb{C}$ and $\tilde\rho>0$ we have:*

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}(z_0,\tilde\rho)\right)\right]=\frac{L^2}{2\pi}\frac{\tilde\rho\sqrt{4-\tilde\rho^2}}{4}\int_0^{\frac{\tilde\rho^2(4-\tilde\rho^2)}{4}}\frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\frac{1}{\sqrt{1-\frac{4}{\tilde\rho^2(4-\tilde\rho^2)}x}}dx,\tag{3.2.2}$$

Proof. The value does not depend on z_0 by the invariance of the rotations. Then we can assume $z_0=0$. By i) of Remark 2.4.3, we have that, if $z\in D_{ch}(0,\tilde\rho)$:

$$|z|\le\frac{\tilde\rho}{\sqrt{4-\tilde\rho^2}}=\hat\rho.$$

We shall apply Theorem 3.1.1 to the case in $D:=D(0,\hat\rho)$. Since f_L is an $\mathbb{S}^2$-GAF, it is easy to verify that

$$K(z,w)=\frac{|1+z\overline{w}|^{2L}}{(1+|z|^2)^L(1+|w|^2)^L}.$$

By straightforward computations we get

$$\frac{\mathcal{K}_{f_L}(z,w)}{\mathcal{K}_{f_L}(w,w)}=\frac{(1+z\overline{w})^L}{(1+|w|^2)^L},\qquad\frac{\partial}{\partial\overline{w}}\left(\frac{\mathcal{K}_{f_L}(z,w)}{\mathcal{K}_{f_L}(w,w)}\right)=L(z-w)\frac{(1+z\overline{w})^{L-1}}{(1+|w|^2)^{L+1}},$$

$$\frac{\mathcal{K}_{f_L}(w,z)}{\mathcal{K}_{f_L}(z,z)}=\frac{(1+w\overline{z})^L}{(1+|z|^2)^L},\qquad\frac{\partial}{\partial\overline{z}}\left(\frac{\mathcal{K}_{f_L}(w,z)}{\mathcal{K}_{f_L}(z,z)}\right)=-L(z-w)\frac{(1+w\overline{z})^{L-1}}{(1+|z|^2)^{L+1}},$$

$$\frac{1}{1-K(z,w)}=\frac{(1+|z|^2)^L(1+|w|^2)^L}{(1+|z|^2)^L(1+|w|^2)^L-|1+z\overline{w}|^{2L}}.$$

Thus

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}(0,\tilde\rho)\right)\right] =$$
$$= \frac{L^2}{4\pi^2}\int_{\partial D}\int_{\partial D}\frac{|1+z\overline{w}|^{2L-2}(z-w)^2 d\overline{z}d\overline{w}}{(1+|z|^2)^{L+1}(1+|w|^2)^{L+1}-|1+z\overline{w}|^{2L}(1+|z|^2)(1+|w|^2)}.$$

Fixing $\hat\rho>0$ of the disk D and denoting $z=\hat\rho e^{i\theta}$, $w=\hat\rho e^{i\phi}$, for all $\theta,\phi\in(0,2\pi)$, we have

$$(z-w)^2\left|\mathcal{J}(\theta,\phi)\right| = \hat\rho^2(e^{2i\theta}-2e^{i(\theta+\phi)}+e^{2i\phi})\begin{vmatrix}-i\hat\rho e^{-i\theta} & 0\\ 0 & -i\hat\rho e^{-i\phi}\end{vmatrix}$$
$$= -\hat\rho^4(e^{i(\theta-\phi)}+e^{i(\phi-\theta)}-2) = \hat\rho^4\left|1-e^{i(\theta-\phi)}\right|^2,$$

where $\mathcal{J}$ stands for the Jacobian matrix. By trivial calculations and applying the change of variable $t=\theta-\phi$, we get now

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}(0,\tilde\rho)\right)\right] = \frac{L^2\hat\rho^4}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\left|1+\hat\rho^2e^{i(\theta-\phi)}\right|^{2L-2}\left|1-e^{i(\theta-\phi)}\right|^2}{(1+\hat\rho^2)^{2L+2}-\left|1+\hat\rho^2e^{i(\theta-\phi)}\right|^{2L}(1+\hat\rho^2)^2}d\theta d\phi$$
$$= \frac{L^2}{2\pi}\frac{\hat\rho^4}{(1+\hat\rho^2)^2}\int_0^{2\pi}\frac{|1+\hat\rho^2e^{it}|^{2L}}{(1+\hat\rho^2)^{2L}-|1+\hat\rho^2e^{it}|^{2L}}\frac{|1-e^{it}|^2}{|1+\hat\rho^2e^{it}|^2}dt.$$

Notice that

$$\left|1+\hat\rho^2e^{it}\right|^2 = (1+\hat\rho^2)^2\left[1-\frac{2\hat\rho^2}{(1+\hat\rho^2)^2}(1-\cos t)\right],\quad \left|1-e^{it}\right|^2 = 2-2\cos t.$$

By using these expressions and that the integrand is even we have

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}(0,\tilde\rho)\right)\right] =$$
$$= \frac{2L^2}{\pi}\frac{\hat\rho^4}{(1+\hat\rho^2)^4}\int_0^{\pi}\frac{\left[1-\frac{2\hat\rho^2}{(1+\hat\rho^2)^2}(1-\cos t)\right]^L}{1-\left[1-\frac{2\hat\rho^2}{(1+\hat\rho^2)^2}(1-\cos t)\right]^L}\frac{1-\cos t}{1-\frac{2\hat\rho^2}{(1+\hat\rho^2)^2}(1-\cos t)}dt.$$

If we use the change of variable

$$x=\frac{2\hat\rho^2}{(1+\hat\rho^2)^2}(1-\cos t),\qquad dx=\frac{2\hat\rho^2}{(1+\hat\rho^2)^2}\sin t dt,$$

where

$$\sin t=\sqrt{1-\cos^2 t}=\frac{1+\hat\rho^2}{\hat\rho}\sqrt{x}\sqrt{1-\frac{(1+\hat\rho^2)^2}{4\hat\rho^2}x},$$

we have

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}(0,\tilde\rho)\right)\right] = \frac{L^2}{2\pi}\frac{\hat\rho}{1+\hat\rho^2}\int_0^{\frac{4\hat\rho^2}{(1+\hat\rho^2)^2}}\frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\frac{1}{\sqrt{1-\frac{(1+\hat\rho^2)^2}{4\hat\rho^2}x}}dx$$
$$= \frac{L^2}{2\pi}\frac{\tilde\rho\sqrt{4-\tilde\rho^2}}{4}\int_0^{\frac{\tilde\rho^2(4-\tilde\rho^2)}{4}}\frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\frac{1}{\sqrt{1-\frac{4}{\tilde\rho^2(4-\tilde\rho^2)}x}}dx.$$

□

We are going to use a simplified version of (3.2.2), which is just the variance by applying the change of variable $\tilde{\rho} = 2\rho$:

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}(0,2\rho)\right)\right] = \frac{L^2}{2\pi}\rho\sqrt{1-\rho^2}\int_0^{4\rho^2(1-\rho^2)} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\frac{1}{\sqrt{1-\frac{x}{4\rho^2(1-\rho^2)}}}dx. \tag{3.2.3}$$

We are going to use this formula to study the asymptotic behaviour of the zero set of an $\mathbb{S}^2$-GAF.

3.2.3 Asymptotics as $L \to +\infty$

Our first case of asymtotic behaviour is when the degree of the polynomial of an $\mathbb{S}^2$-GAF, $L \in \mathbb{N}$, tends to infinite. This can be translated as increasing the average number of points in a chordal disk $D_{ch}(z_0, 2\rho)$, $z_0 \in \mathbb{C}$, which is L times the surface of the chordal disk. In physical models where the points represent gas particles this is called the *transition to the liquid phase*.
We want to show the next:

Proposition 3.2.3. *Let f_L be an $\mathbb{S}^2$-GAF of intensity $L \in \mathbb{N}$. Consider a chordal disk $D_{ch}(z_0, 2\rho)$, with $z_0 \in \mathbb{C}$. Then*

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}(z_0,2\rho)\right)\right] = \left(\frac{\sqrt{L}}{4\sqrt{\pi}}\zeta(3/2)\rho\sqrt{1-\rho^2}\right)(1+o(1)), \quad \text{as } L \to +\infty,$$

where ζ stands for the Riemann's zeta function and $o(1)$ is a term tending to 0 as $L \to +\infty$.

Remark 3.2.4. Recall that the variance of the Poisson process was of order L (see (1.3.3)). However, the variance of the zero set points of an $\mathbb{S}^2$-GAF is of order $\sqrt{L}$. In this sense, the process $\mathcal{Z}_{f_L}$ is more *rigid* than the Poisson process as $L \to +\infty$.

Proof. By invariance, we can choose $z_0 = 0$. By denoting $D_{ch} := D_{ch}(0, 2\rho)$ and $\tau = 4\rho^2(1-\rho^2) \in (0,1)$, we get from (3.2.3):

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = \frac{L^2}{4\pi}\sqrt{\tau}\int_0^{\tau} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\frac{1}{\sqrt{1-x/\tau}}dx. \tag{3.2.4}$$

We want to isolate the leading term as $L \to +\infty$. Specifically, we want to show that this expression can be rewritten as

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = \left(\frac{L^2}{4\pi}\sqrt{\tau}\int_0^{\tau} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx\right)(1+o(1)), \quad L \to +\infty.$$

Denote

$$A_L = \int_0^{\tau} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\frac{1}{\sqrt{1-x/\tau}}dx, \quad \tilde{A}_L = \int_0^{\tau} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx.$$

By (3.2.4), the variance can be expressed as

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = \frac{L^2}{4\pi}\sqrt{\tau}\tilde{A}_L\left(1+\frac{A_L-\tilde{A}_L}{\tilde{A}_L}\right).$$

In order to get the result, we have to consider

$$\lim_{L\to+\infty}\frac{A_L-\tilde{A}_L}{\tilde{A}_L} = \lim_{L\to+\infty}\frac{\displaystyle\int_0^\tau \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\left(\frac{1}{\sqrt{1-x/\tau}}-1\right)dx}{\displaystyle\int_0^\tau \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx} = 0, \quad (3.2.5)$$

and ensure that the order of $A_L - \tilde{A}_L$ is bigger than the order of $\tilde{A}_L$. The following lemma addresses this:

Lemma 3.2.5. *It is satisfied that:*

1. $\tilde{A}_L = \mathcal{O}(L^{-3/2})$.
2. $A_L - \tilde{A}_L = \mathcal{O}(L^{-5/2})$.

Using Lemma 3.2.5, we verify (3.2.5) and we can write

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = \left(\frac{L^2}{\pi}\sqrt{\tau}\int_0^\tau \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx\right)(1+o(1)).$$

Noticing that the mass of the integral is concentrated around zero, the integral from 0 to 1 is of the same order. Now, using the geometric series for $(1-x)^L$ since $|1-x|<1$, the fact that a power series converges uniformly over compact sets in the interval of convergence and using the definition of Beta function (2.3.3), we have that the variance is

$$\begin{aligned}
\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] &= \left(\frac{L^2}{4\pi}\sqrt{\tau}\int_0^1\sum_{k=1}^{+\infty}(1-x)^{kL-1}\sqrt{x}dx\right)(1+o(1)) \\
&= \left(\frac{L^2}{4\pi}\sqrt{\tau}\sum_{k=1}^{+\infty}\int_0^1(1-x)^{kL-1}\sqrt{x}dx\right)(1+o(1)) \\
&= \left(\frac{L^2}{4\pi}\sqrt{\tau}\sum_{k=1}^{+\infty}\mathrm{B}(3/2,kL)\right)(1+o(1)) \\
&= \left(\frac{L^2}{8\sqrt{\pi}}\sqrt{\tau}\sum_{k=1}^{+\infty}\frac{\Gamma(kL)}{\Gamma(kL+3/2)}\right)(1+o(1)).
\end{aligned}$$

By the asymtotics of Γ, (2.3.2), we conclude

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = \left(\frac{L^2}{8\sqrt{\pi}}\sqrt{\tau}\sum_{k=1}^{+\infty}(kL)^{-3/2}\right)(1+o(1)) = \left(\frac{\sqrt{L}}{8\sqrt{\pi}}\zeta(3/2)\sqrt{\tau}\right)(1+o(1)).$$

□

Proof of Lemma 3.2.5. 1. Notice that $\tilde{A}_L$ is equivalent to

$$\begin{aligned}\tilde{A}_L &= \int_0^\tau \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx = \int_0^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx - \int_\tau^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx \\ &= \left(\int_0^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx\right)\left(1-\frac{\int_\tau^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx}{\int_0^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx}\right) \\ &= \left(\int_0^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx\right)\left(1-\frac{J_1}{J_2}\right),\end{aligned}$$

where

$$J_1 := \int_\tau^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx$$

and

$$J_2 := \int_0^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx.$$

Let us check that the order of the numerator J_1 is bigger than the order of the denominator J_2. On the one hand, since $|1-x|<1$ and using that a power series converges uniformly over compact sets in the interval of convergence, we have, by the Beta function (2.3.3),

$$\begin{aligned}J_2 &= \sum_{k=1}^{+\infty}\int_0^1 (1-x)^{Lk-1}\sqrt{x}dx = \sum_{k=1}^{+\infty} \mathrm{B}(3/2, Lk) = \sum_{k=1}^{+\infty}\frac{\Gamma(3/2)\Gamma(Lk)}{\Gamma(Lk+3/2)} \\ &= \frac{\sqrt{\pi}}{2}\sum_{k=1}^{+\infty}\frac{\Gamma(Lk)}{\Gamma(Lk+3/2)}.\end{aligned}$$

By the asymptotics of Γ given in (2.3.2), we get that

$$J_2 = \frac{\sqrt{\pi}}{2}\sum_{k=1}^{+\infty}\frac{\Gamma(Lk)}{\Gamma(Lk+3/2)} \simeq \frac{\sqrt{\pi}}{2}\sum_{k=1}^{+\infty}(Lk)^{-3/2} = \frac{\sqrt{\pi}}{2}\zeta(3/2)L^{-3/2}.$$

Then $J_2 = \mathcal{O}(L^{-3/2})$. On the other hand

$$J_1 = \int_\tau^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx \le \int_\tau^1 \frac{(1-\tau)^{L-1}}{1-(1-\tau)^L}dx = \frac{(1-\tau)^L}{1-(1-\tau)^L}.$$

Hence $J_1 = \mathcal{O}\left((1-\tau)^L\right)$. Thus $\lim_{L\to+\infty} J_1/J_2 = 0$ and

$$\tilde{A}_L = \left(\int_0^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx\right)(1+o(1)).$$

As we did for J_2, we obtain, as $L \to +\infty$,

$$\int_0^1 \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}dx = \sum_{k=1}^{+\infty}\int_0^1 (1-x)^{Lk-1}\sqrt{x}dx = \frac{\sqrt{\pi}}{2}\sum_{k=1}^{+\infty}\frac{\Gamma(Lk)}{\Gamma(Lk+3/2)}$$
$$= \left(\frac{\sqrt{\pi}}{2}\sum_{k=1}^{+\infty}(Lk)^{-3/2}\right)(1+o(1))$$
$$= \left(\frac{\sqrt{\pi}}{2}\zeta(3/2)L^{-3/2}\right)(1+o(1)) \neq 0.$$

2. We have that

$$A_L - \tilde{A}_L = \int_0^\tau \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\left(\frac{1}{\sqrt{1-x/\tau}}-1\right)dx.$$

Notice that the integrand takes high values as x approaches to 0. For this reason we are going to split the integral into two:

$$A_L - \tilde{A}_L = \int_0^{\tau/2} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\left(\frac{1}{\sqrt{1-x/\tau}}-1\right)dx$$
$$+ \int_{\tau/2}^{\tau} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\left(\frac{1}{\sqrt{1-x/\tau}}-1\right)dx =: J_3 + J_4.$$

Let us focus on J_3. Developing $(1-x/\tau)^{-1/2}$ by Taylor at zero we have that, for $x \leq \tau/2$:

$$(1-x/\tau)^{-1/2} = 1 + \frac{x}{2\tau} + \mathcal{O}(x^2) \leq 1 + \frac{x}{\tau}.$$

Thus $(1-x/\tau)^{-1/2} - 1 \leq x/\tau$. Therefore

$$J_3 \leq \int_0^{\tau/2} \frac{(1-x)^L}{1-(1-x)^L}\frac{\sqrt{x}}{1-x}\frac{x}{\tau}dx = \frac{1}{\tau}\int_0^{\tau/2}\frac{(1-x)^{L-1}}{1-(1-x)^L}x^{3/2}dx.$$

Recall that the mass of the integral is concentrated around zero, so the integral from 0 to 1 is of the same order. Using again the uniform convergence of a power series over compact sets of the interval of convergence:

$$J_3 \leq \frac{1}{\tau}\sum_{k=1}^{+\infty}\int_0^1 (1-x)^{kL-1}x^{3/2}dx = \frac{1}{\tau}\sum_{k=1}^{+\infty}\mathrm{B}\left(5/2, kL\right) = \frac{1}{\tau}\sum_{k=1}^{+\infty}\frac{\Gamma(5/2)\Gamma(kL)}{\Gamma(kL+5/2)}$$
$$= \frac{3\sqrt{\pi}}{4\tau}\sum_{k=1}^{+\infty}\frac{\Gamma(kL)}{\Gamma(kL+5/2)}.$$

By using again the asymotitcs of Γ given in (2.3.2) we have that

$$J_3 \leq \frac{3\sqrt{\pi}}{4\tau}\sum_{k=1}^{+\infty}\frac{\Gamma(kL)}{\Gamma(kL+5/2)} \simeq \frac{3\sqrt{\pi}}{4\tau}\sum_{k=1}^{+\infty}(kL)^{-5/2} = \frac{3\sqrt{\pi}}{4\tau}\zeta(5/2)L^{-5/2}.$$

Thus $J_3 = \mathcal{O}(L^{-5/2})$. With this we conclude that $\lim_{L\to+\infty} J_3/\tilde{A}_L = 0$. For J_4 we have that

$$J_4 = \int_{\tau/2}^{\tau} \frac{(1-x)^L}{1-(1-x)^L} \frac{\sqrt{x}}{1-x} \left(\frac{1}{\sqrt{1-x/\tau}} - 1 \right) dx$$
$$\leq \int_{\tau/2}^{\tau} \frac{(1-\tau/2)^L}{1-(1-\tau/2)^L} \frac{\sqrt{\tau}}{1-\tau/2} \frac{dx}{\sqrt{1-x/\tau}}.$$

By the change of variable $y = x/\tau$:

$$J_4 \leq \frac{(1-\tau/2)^L}{1-(1-\tau/2)^L} \frac{\tau^{3/2}}{1-\tau/2} \int_{1/2}^{1} \frac{dy}{\sqrt{1-y}} = \frac{(1-\tau/2)^L}{1-(1-\tau/2)^L} \frac{\sqrt{2}\tau^{3/2}}{1-\tau/2}.$$

Therefore $J_4 = \mathcal{O}\left((1-\tau/2)^L\right)$. This tends to zero much faster than any power of L as $L \to +\infty$. Then $\lim_{L\to+\infty} J_4/\tilde{A}_L = 0$.

□

3.2.4 Asymptotics as $\rho \to 0$

Here we are going to study the case when $\rho \to 0$ and L is fixed. In such a case it is intuitive that the variance of an $\mathbb{S}^2$-GAF tends to zero as ρ does, but here we also see how fast it goes to zero.
We want to show the following:

Proposition 3.2.6. *Let f_L be an $\mathbb{S}^2$-GAF of intensity $L \in \mathbb{N}$. Consider a chordal disk $D_{ch}(z_0, 2\rho)$, with $z_0 \in \mathbb{C}$. Then*

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}(z_0, 2\rho)\right)\right] = L\rho^2(1+o(1)) \quad \text{as } \rho \to 0.$$

Remark 3.2.7. The Poisson process and the zero set process of an $\mathbb{S}^2$-GAF have the same speed of convergence.

Proof. By invariance, we can choose $z_0 = 0$. Denote $D_{ch} := D_{ch}(0, 2\rho)$. We start from the expression (3.2.4), which is

$$\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] = \frac{L^2}{4\pi}\sqrt{\tau} \int_0^{\tau} \frac{(1-x)^L}{1-(1-x)^L} \frac{\sqrt{x}}{1-x} \frac{1}{\sqrt{1-x/\tau}} dx,$$

where $\tau = 4\rho^2(1-\rho^2)$. Notice that, since $\tau \to 0$, the Taylor series at a neighbourhood of zero guarantees us that

$$(1-x)^L = e^{L\log(1-x)} = e^{-Lx}(1+o(1))$$

and

$$1-(1-x)^L = (1-e^{-Lx})(1+o(1)) = Lx(1+o(1)).$$

Thus, noticing that the integral takes high values near of zero and making the change of variable $t = x/\tau$:

$$\begin{aligned}\mathbb{V}\left[\nu_{f_L}\left(D_{ch}\right)\right] &= \left(\frac{L^2}{4\pi}\sqrt{\tau}\int_0^\tau \frac{e^{-Lx}}{Lx}\frac{\sqrt{x}}{1-x}\frac{1}{\sqrt{1-x/\tau}}dx\right)(1+o(1)) \\ &= \left(\frac{L}{4\pi}\sqrt{\tau}\int_0^\tau e^{-Lx}\frac{dx}{\sqrt{x}\sqrt{1-x/\tau}}\right)(1+o(1)) \\ &= \left(\frac{L}{4\pi}\sqrt{\tau}\int_0^1 e^{-L\tau t}\frac{\tau}{\sqrt{t\tau}\sqrt{1-t}}dt\right)(1+o(1)) \\ &= \left(\frac{L}{4\pi}\tau\int_0^1 e^{-L\tau t}\frac{dt}{\sqrt{t}\sqrt{1-t}}\right)(1+o(1)).\end{aligned}$$

Denote

$$F(\tau) := \int_0^1 e^{-L\tau t}\frac{dt}{\sqrt{t}\sqrt{1-t}}.$$

We have that

$$\begin{aligned}F(\tau) &= \sum_{k=0}^{+\infty}\frac{(-1)^k L^k}{k!}\left(\int_0^1 t^{k-1/2}\frac{dt}{\sqrt{1-t}}\right)\tau^k = \sum_{k=0}^{+\infty}\frac{(-1)^k L^k}{k!}\frac{\Gamma(k+1/2)\Gamma(1/2)}{\Gamma(k+1)}\tau^k \\ &= \sqrt{\pi}\sum_{k=0}^{+\infty}\frac{(-1)^k L^k}{(k!)^2}\Gamma(k+1/2)\tau^k = \sqrt{\pi}\Gamma(1/2) - L\sqrt{\pi}\Gamma(3/2)\tau + ... \\ &= \pi - \frac{L\pi}{2}\tau + ...\end{aligned}$$

Hence, as $\rho \to 0$:

$$\begin{aligned}\mathbb{V}\left[\nu_{f_L}(D_{ch})\right] &= \left[\frac{L}{4\pi}\tau(\pi - \frac{L\pi}{2}\tau + ...)\right](1+o(1)) \\ &= \left(L\rho^2(1-\rho^2)\right)(1+o(1)) = L\rho^2(1+o(1)).\end{aligned}$$

□

3.2.5 $\mathcal{F}_1$ as the limit of $\mathcal{P}_L$ as $L \to +\infty$

There is a result stating that $\mathcal{F}_1$ can be understood as a limit space of $\mathcal{P}_L$ as $L \to +\infty$. More concretely:

Lemma 3.2.8. *Given a GAF $f_1^{\mathbb{C}} \in \mathcal{F}_1$ and a constant $M > 0$, there is $L_0 \in \mathbb{N}$ such that for all $L \geq L_0$, there exist GAFs $f_L^{\mathbb{S}^2} \in \mathcal{P}_L$ such that*

$$\int_{\{|z|\leq M/\sqrt{L}\}}\left|f_1^{\mathbb{C}}(\sqrt{L}z) - f_L^{\mathbb{S}^2}(z)\right|^2 e^{-L|z|^2}dm(z) \lesssim \frac{1}{L}\left\|f_1^{\mathbb{C}}\right\|_{\mathcal{F}_1}^2$$

and

$$\int_{\{|z|>M/\sqrt{L}\}}\frac{\left|f_L^{\mathbb{S}^2}(z)\right|^2}{\pi(1+|z|^2)^{L+2}}dm(z) \lesssim \frac{1}{L}\left\|f_1^{\mathbb{C}}\right\|_{\mathcal{F}_1}^2.$$

A deeper explanation can be found in [7], p. 32.
In accordance with Lemma 3.2.8, we are going to see whether the limit as $L \to +\infty$ of the variance of the random variables of the zero point process of an $\mathbb{S}^2$-GAF $f_L^{\mathbb{S}^2}$ of parameter $L \in \mathbb{N}$ in $D_{ch}(z_0, 2r/\sqrt{L})$, with $z_0 \in \mathbb{C}$, is the variance of a $\mathbb{C}$-GAF $f_1^{\mathbb{C}}$ of parameter $L = 1$.

Proposition 3.2.9. *The limit of the variance of an $\mathbb{S}^2$-GAF as $L \to +\infty$ coincides with the variance of a $\mathbb{C}$-GAF of parameter $L = 1$, that is,*

$$\lim_{L\to+\infty} \mathbb{V}\left[\nu_{f_L^{\mathbb{S}^2}}\left(D_{ch}(z_0, 2r/\sqrt{L})\right)\right] = \mathbb{V}\left[\nu_{f_1^{\mathbb{C}}}\left(D(z_0, r)\right)\right], \quad z_0 \in \mathbb{C}.$$

Proof. By invariance, we can choose $z_0 = 0$. Denote $D_{ch} := D_{ch}(0, 2r/\sqrt{L})$ and $D := D(0, r)$. Choosing $\rho = r/\sqrt{L}$, using the new expression of τ in (3.2.4) and applying the change of variable $y = (L^2 x)/(4r^2(L - r^2))$, we have that

$$\begin{aligned}
\mathbb{V}\left[\nu_{f_L^{\mathbb{S}^2}}(D_{ch})\right] &= \frac{L^2}{4\pi}\sqrt{\frac{4r^2(L-r^2)}{L^2}} \int_0^{\frac{4r^2(L-r^2)}{L^2}} \frac{(1-x)^L}{1-(1-x)^L} \frac{\sqrt{x}}{1-x} \frac{dx}{\sqrt{1-\frac{L^2 x}{4r^2(L-r^2)}}} \\
&= \frac{4}{\pi}\frac{r^4(L-r^2)^2}{L^2} \int_0^1 \frac{\left(1-\frac{4r^2(L-r^2)}{L^2}y\right)^{L-1}}{1-\left(1-\frac{4r^2(L-r^2)}{L^2}y\right)^L} y^{1/2}(1-y)^{-1/2} dy.
\end{aligned}$$

Denote

$$g_{L,r}(y) = \frac{\left(1-\frac{4r^2(L-r^2)}{L^2}y\right)^{L-1}}{1-\left(1-\frac{4r^2(L-r^2)}{L^2}y\right)^L},$$

which is a decreasing function with respect to L, bounded below by 0 and

$$\lim_{L\to+\infty} g_{L,r}(y) = \frac{1}{e^{4r^2 y} - 1}.$$

By Monotone Convergence Theorem we get that

$$\lim_{L\to+\infty} \int_0^1 g_{L,r}(y)dy = \int_0^1 \lim_{L\to+\infty} g_{L,r}(y)dy = \int_0^1 \frac{dy}{e^{4r^2 y} - 1}.$$

Notice that

$$\lim_{L\to+\infty} \frac{r^4(L-r^2)^2}{L^2} = r^4.$$

Since both limits exist:

$$\begin{aligned}
\lim_{L\to+\infty} \mathbb{V}\left[\nu_{f_L^{\mathbb{S}^2}}(D_{ch})\right] &= \frac{4}{\pi} \lim_{L\to+\infty} \left(\frac{r^4(L-r^2)^2}{L^2} \int_0^1 g_{L,r}(y) y^{1/2}(1-y)^{-1/2} dy\right) \\
&= \frac{4}{\pi}\left(\lim_{L\to+\infty} \frac{r^4(L-r^2)^2}{L^2}\right)\left(\lim_{L\to+\infty} \int_0^1 g_{L,r}(y) y^{1/2}(1-y)^{-1/2} dy\right) \\
&= \frac{4}{\pi} r^4 \int_0^1 \frac{y^{1/2}(1-y)^{-1/2}}{e^{4r^2 y} - 1} dy.
\end{aligned}$$

Now, let us compute the variance of a $\mathbb{C}$-GAF for $L = 1$. Due to (3.2.1), we can write:

$$\mathbb{V}\left[\nu_{f_1^{\mathbb{C}}}(D)\right] = \frac{r}{2\pi}\int_0^{4r^2} \frac{1}{e^x - 1}\frac{\sqrt{x}}{\sqrt{1-\frac{x}{4r^2}}}dx.$$

Applying the change of variable $y = x/(4r^2)$:

$$\mathbb{V}\left[\nu_{f_1^{\mathbb{C}}}(D)\right] = \frac{4}{\pi}r^4\int_0^1 \frac{y^{1/2}(1-y)^{-1/2}}{e^{4r^2y}-1}dy,$$

and the result follows. □

3.3 Variance of an $\mathbb{S}^2$-GAF via linear statistics

In this section we study the fluctuations of an $\mathbb{S}^2$-GAF f_L of parameter $L \in \mathbb{N}$ through *linear statistics*, that is, for all test-functions $\varphi \in \mathcal{C}_c^2(\Omega)$, where $\Omega \subseteq \mathbb{C}$,

$$I_L(\varphi) = \int_\Omega \varphi d\nu_{f_L}.$$

We have the following theorem:

Theorem 3.3.1 ([2], p. 42-44)**.** *Consider a linear statistic $I_L(\varphi)$, where $\varphi \in \mathcal{C}_c^2(\Omega)$ and $\Omega \subseteq \mathbb{C}$. Then*

$$\mathbb{V}\left[I_L(\varphi)\right] = \frac{\zeta(3)}{32L}\left\|\Delta^*\varphi\right\|^2_{L^2(\Omega,m^*)} + \mathcal{O}\left(\frac{\log L}{L^2}\right),$$

as $L \to +\infty$, where $\Delta^* := (1+|z|^2)^2\Delta$.

Remark 3.3.2. Δ^* is called the *invariant Laplacian*, and it is invariant in the sense

$$\Delta^*(u \circ \phi_a) = \Delta^* u \circ \phi_a, \quad u \in \mathcal{C}_c^2(\mathbb{C}),$$

where ϕ_a is the transformation defined in (2.5.2).

Proof. By definition

$$\mathbb{V}\left[I_L(\varphi)\right] = \mathbb{E}\left[\left(I_L(\varphi) - \mathbb{E}\left[I_L(\varphi)\right]\right)^2\right],$$

where

$$I_L(\varphi) = \frac{1}{2\pi}\int_\Omega \varphi(z)\Delta\log|f_L(z)|$$

and by the Edelman-Kostlan formula (see Theorem 2.5.3)

$$\mathbb{E}\left[I_L(\varphi)\right] = \frac{1}{2\pi}\int_\Omega \varphi(z)\Delta\log\sqrt{\mathcal{K}_{f_L}(z,z)}.$$

Letting $\hat{f}_L \sim N_{\mathbb{C}}(0,1)$ we get

$$I_L(\varphi) - \mathbb{E}\left[I_L(\varphi)\right] = \frac{1}{2\pi}\int_\Omega \varphi(z)\Delta\log\frac{|f_L(z)|}{\sqrt{\mathcal{K}_{f_L}(z,z)}} = \frac{1}{2\pi}\int_\Omega \varphi(z)\Delta\log|\hat{f}_L(z)|$$
$$= \frac{1}{2\pi}\int_\Omega \Delta\varphi(z)\log|\hat{f}_L(z)|.$$

Thus

$$\mathbb{V}\left[I_L(\varphi)\right] = \frac{1}{4\pi^2}\mathbb{E}\left[\int_\Omega \Delta\varphi(z)\log|\hat{f}_L(z)|dm(z)\int_\Omega \Delta\varphi(w)\log|\hat{f}_L(w)|dm(w)\right]$$
$$= \frac{1}{4\pi^2}\int_\Omega\int_\Omega \Delta\varphi(z)\Delta\varphi(w)\mathbb{E}\left[\log|\hat{f}_L(z)|\log|\hat{f}_L(w)|\right]dm(z)dm(w).$$

Since $\hat{f}_L \sim N_{\mathbb{C}}(0,1)$, we have that $\mathbb{E}\left[\log|\hat{f}_L(z)|\right]$ is constant (independent of z). Therefore:

$$\overline{\partial}_z\overline{\partial}_w\mathrm{Cov}\left[\log|\hat{f}_L(z)|, \log|\hat{f}_L(w)|\right] = \overline{\partial}_z\overline{\partial}_w\mathbb{E}\left[\log|\hat{f}_L(z)|\log|\hat{f}_L(w)|\right],$$

and hence

$$\mathbb{V}\left[I_L(\varphi)\right] = \frac{1}{4\pi^2}\int_\Omega\int_\Omega \Delta\varphi(z)\Delta\varphi(w)\mathrm{Cov}\left[\log|\hat{f}_L(z)|, \log|\hat{f}_L(w)|\right]dm(z)dm(w)$$
$$= \frac{1}{4}\int_\Omega\int_\Omega \Delta^*\varphi(z)\Delta^*\varphi(w)\mathrm{Cov}\left[\log|\hat{f}_L(z)|, \log|\hat{f}_L(w)|\right]dm^*(z)dm^*(w).$$

For simplicity, denote $\kappa_L(z,w) = \mathrm{Cov}\left[\log|\hat{f}_L(z)|, \log|\hat{f}_L(w)|\right]$. To estimate the integrals we will separate the points that are close and far from the diagonal due to:

Lemma 3.3.3. *For all $x \in [0,1]$, the next estimate holds:*

$$\frac{x}{4} \le \sum_{j=1}^{+\infty}\frac{x^j}{4j^2} \le \frac{x}{2}.$$

By using Lemma 3.3.3, we get the estimates

$$\frac{(1+|\phi_w(z)|^2)^{-L}}{4} \le \kappa_L(z,w) \le \frac{(1+|\phi_w(z)|^2)^{-L}}{2}. \tag{3.3.1}$$

Hence, for a value C_L, that we are going to specify later, we can write:

$$\Omega^2 = \Omega\times\Omega = \{(z,w)\ :\ \kappa_L(z,w)\le C_L\}\cup\{(z,w)\ :\ \kappa_L(z,w) > C_L\}.$$

So, the points $(z,w)\in\Omega^2$ such that $\kappa_L(z,w) > C_L$ are those near the diagonal. Otherwise, the points $(z,w)\in\Omega^2$ such that $\kappa_L(z,w)\le C_L$ are those far from the diagonal. We will see that the variance above is only relevant near the diagonal,

and its principal term will be $\|\Delta^*\varphi\|^2_{L^2(\Omega,m^*)}$.
We have

$$\begin{aligned}
\mathbb{V}\left[I_L(\varphi)\right] &= \frac{1}{4}\int_\Omega\int_\Omega \Delta^*\varphi(z)\Delta^*\varphi(w)\kappa_L(z,w)dm^*(z)dm^*(w)\\
&= \frac{1}{4}\left[\int_{\{(z,w)\ :\ \kappa_L(z,w)\le C_L\}} \Delta^*\varphi(z)\Delta^*\varphi(w)\kappa_L(z,w)dm^*(z)dm^*(w)\right.\\
&\left. + \int_{\{(z,w)\ :\ \kappa_L(z,w)> C_L\}} \Delta^*\varphi(z)\Delta^*\varphi(w)\kappa_L(z,w)dm^*(z)dm^*(w)\right]\\
&= \frac{1}{4}\left[\int_{\{(z,w)\ :\ \kappa_L(z,w)\le C_L\}} \Delta^*\varphi(z)\Delta^*\varphi(w)\kappa_L(z,w)dm^*(z)dm^*(w)\right.\\
& + \int_{\{(z,w)\ :\ \kappa_L(z,w)> C_L\}} \left(\Delta^*\varphi(w)-\Delta^*\varphi(z)\right)\Delta^*\varphi(z)\kappa_L(z,w)dm^*(z)dm^*(w)\\
&\left. + \int_{\{(z,w)\ :\ \kappa_L(z,w)> C_L\}} \left(\Delta^*\varphi(z)\right)^2\kappa_L(z,w)dm^*(z)dm^*(w)\right].
\end{aligned}$$

Let us name the integrals as:

$$I_1 := \int_{\{(z,w)\ :\ \kappa_L(z,w)\le C_L\}} \Delta^*\varphi(z)\Delta^*\varphi(w)\kappa_L(z,w)dm^*(z)dm^*(w),$$

$$I_2 := \int_{\{(z,w)\ :\ \kappa_L(z,w)> C_L\}} \left(\Delta^*\varphi(w)-\Delta^*\varphi(z)\right)\Delta^*\varphi(z)\kappa_L(z,w)dm^*(z)dm^*(w)$$

and

$$I_3 := \int_{\{(z,w)\ :\ \kappa_L(z,w)> C_L\}} \left(\Delta^*\varphi(z)\right)^2\kappa_L(z,w)dm^*(z)dm^*(w).$$

The bound for I_1 is straightforward. Indeed:

$$\begin{aligned}
|I_1| &\le \int_{\{(z,w)\ :\ \kappa_L(z,w)\le C_L\}} |\Delta^*\varphi(z)|\,|\Delta^*\varphi(w)|\,\kappa_L(z,w)dm^*(z)dm^*(w)\\
&\le C_L\int_{\{(z,w):\ \kappa_L(z,w)\le C_L\}} |\Delta^*\varphi(z)|\,|\Delta^*\varphi(w)|\,dm^*(z)dm^*(w)\\
&\le C_L\int_\Omega |\Delta^*\varphi(z)|\,dm^*(z)\int_\Omega |\Delta^*\varphi(w)|\,dm^*(w) = C_L\,\|\Delta^*\varphi\|^2_{L^1(\Omega,m^*)}.
\end{aligned}$$

For I_2, since $\varphi\in\mathcal{C}^2_c(\Omega)$, $\Delta^*\varphi$ is uniformly continuous in supp φ. Hence it exists $\varepsilon(t)$ with $\lim_{t\to 1}\varepsilon(t)=0$ such that, for all $z,w\in\Omega$ and taking ϕ_w as in (2.5.2) we have

$$|\Delta^*\varphi(w)-\Delta^*\varphi(z)| \le \varepsilon\left((1+|\phi_w(z)|^2)^{-1}\right).$$

Observe that, from (2.4.6),

$$(1+|\phi_w(z)|^2)^{-L} = \left[1-\left(\frac{d_{ch}(z,w)}{2}\right)^2\right]^L = |\Theta_L(z,w)|^2,$$

recalling that

$$\Theta_L(z,w) := \frac{\mathcal{K}_{f_L}(z,w)}{\sqrt{\mathcal{K}_{f_L}(z,z)}\sqrt{\mathcal{K}_{f_L}(w,w)}}.$$

For $(z,w) \in \Omega^2$ such that $\kappa_L(z,w) > C_L$, we get that

$$(2C_L)^{1/L} < (1+|\phi_w(z)|^2)^{-1},$$

and therefore, since the function $\varepsilon(t)$ is decreasing,

$$\varepsilon\left((2C_L)^{1/L}\right) > \varepsilon\left((1+|\phi_w(z)|^2)^{-1}\right).$$

Thus, by denoting $\Omega_{\kappa,\varphi} = \{(z,w) \ : \ \kappa_L(z,w) > C_L\} \cap (\text{supp } \varphi \times \text{supp } \varphi)$:

$$\begin{aligned}
|I_2| &\leq \int_{\Omega_{\kappa,\varphi}} |\Delta^*\varphi(w) - \Delta^*\varphi(z)|\, |\Delta^*\varphi(z)|\kappa_L(z,w)dm^*(z)dm^*(w) \\
&\leq \frac{1}{2}\int_{\Omega_{\kappa,\varphi}} \varepsilon\left((1+|\phi_w(z)|^2)^{-1}\right) |\Delta^*\varphi(z)|(1+|\phi_w(z)|^2)^{-L}dm^*(z)dm^*(w) \\
&\leq \frac{\varepsilon\left((2C_L)^{1/L}\right)}{2}\int_{\Omega_{\kappa,\varphi}} |\Delta^*\varphi(z)|(1+|\phi_w(z)|^2)^{-L}dm^*(z)dm^*(w) \\
&\overset{(*)}{=} \frac{\varepsilon\left((2C_L)^{1/L}\right)}{2}\int_{\text{supp } \varphi} |\Delta^*\varphi(z)|dm^*(z)\int_{\{w\in\Omega \ : \ \kappa_L(0,w)>C_L\}} (1+|w|^2)^{-L}dm^*(w) \\
&= \frac{\varepsilon\left((2C_L)^{1/L}\right)}{2}\|\Delta^*\varphi\|_{L^1(\Omega,m^*)}\int_{\{w\in\Omega \ : \ \kappa_L(0,w)>C_L\}} (1+|w|^2)^{-L}dm^*(w),
\end{aligned}$$

where in $(*)$ we just applied the invariance of the measure dm^*. Since $\varepsilon(t) \lesssim |1-t|$ for all t in a neighbourhood of 1, we obtain by choosing $C_L = 1/(2L^2)$ and applying Taylor in a neighbourhood of $1/L \simeq 0$ (as $L \to +\infty$):

$$\varepsilon\left((2C_L)^{1/L}\right) \lesssim 1 - (2C_L)^{1/L} = 1 - L^{-2/L} \simeq \frac{\log L}{L}.$$

Hence

$$|I_2| \lesssim \frac{\log L}{L}\frac{\|\Delta^*\varphi\|_{L^1(\Omega,m^*)}}{2}\int_{\{w\in\Omega \ : \ \kappa_L(0,w)>C_L\}} (1+|w|^2)^{-L}dm^*(w).$$

For I_3 notice that, by using again the invariance of dm^*:

$$I_3 = \|\Delta^*\varphi\|^2_{L^2(\Omega,m^*)}\int_{\{w\in\Omega \ : \ \kappa_L(0,w)>C_L\}} (1+|w|^2)^{-L}dm^*(w).$$

By the fact that $\lim_{L\to+\infty} C_L^{1/L} = 1$ we can write that $I_2 = o(I_3)$ and

$$\mathbb{V}\left[I_L(\varphi)\right] = I_3\left[1 + \mathcal{O}\left(\frac{\log L}{L}\right)\right] + \mathcal{O}(C_L).$$

We must compute

$$J_w := \int_{\{w\in\Omega\,:\,\kappa_L(0,w)>C_L\}} (1+|w|^2)^{-L} dm^*(w) = \int_{\{w\in\Omega\,:\,\kappa_L(0,w)>C_L\}} |\Theta_L(0,w)|^2 dm^*(w).$$

By the definition of Θ_L and using Lemma 3.1.2 we have

$$\begin{aligned} J_w &= \int_{\{w\in\Omega\,:\,\kappa_L(0,w)>C_L\}} \sum_{j=1}^{+\infty} \frac{(1+|w|^2)^{-jL}}{4j^2} dm^*(w) \\ &= \sum_{j=1}^{+\infty} \frac{1}{4j^2} \int_{\{w\in\Omega\,:\,\kappa_L(0,w)>C_L\}} (1+|w|^2)^{-jL} dm^*(w). \end{aligned}$$

From (3.3.1),

$$\{|w|^2 < (4C_L)^{-1/L} - 1\} \subset \{\kappa_L(0,w) > C_L\} \subset \{|w|^2 < (2C_L)^{-1/L} - 1\}.$$

From here we see that

$$\begin{aligned} &\{\kappa_L(0,w) > C_L\} = \\ &= \left\{|w|^2 < (2C_L)^{-1/L} - 1\right\} \setminus \left(\left\{|w|^2 < (2C_L)^{-1/L} - 1\right\} \cap \{\kappa_L(0,w) \le C_L\}\right). \end{aligned}$$

Therefore

$$\begin{aligned} J_w = \sum_{j=1}^{+\infty} \frac{1}{4j^2} \Bigg[&\int_{\{|w|^2<(2C_L)^{-1/L}-1\}} (1+|w|^2)^{-jL} dm^*(w) \\ &- \int_{\{|w|^2<(2C_L)^{-1/L}-1\}\cap\{\kappa_L(0,w)\le C_L\}} (1+|w|^2)^{-jL} dm^*(w) \Bigg]. \end{aligned}$$

The negative sum is negligible. More precisely:

Lemma 3.3.4. *It is satisfied that*

$$\sum_{j=1}^{+\infty} \frac{1}{4j^2} \int_{\{|w|^2<(2C_L)^{-1/L}-1\}\cap\{\kappa_L(0,w)\le C_L\}} (1+|w|^2)^{-jL} dm^*(w) = \mathcal{O}\left(\frac{\log L}{L^3}\right).$$

By Lemma 3.3.4 we get

$$J_w = \sum_{j=1}^{+\infty} \frac{1}{4j^2} \int_{\{|w|^2<(2C_L)^{-1/L}-1\}} (1+|w|^2)^{-jL} dm^*(w) + \mathcal{O}\left(\frac{\log L}{L^3}\right).$$

By denoting

$$I_j = \int_{\{|w|^2<(2C_L)^{-1/L}-1\}} (1+|w|^2)^{-jL} dm^*(w),$$

calling $r_L = (2C_L)^{-1/L} - 1$ and changing into polar coordinates

$$I_j = \int_{\{|w|^2<r_L\}} (1+|w|^2)^{-jL} dm^*(w) = \int_0^{\sqrt{r_L}} (1+r^2)^{-jL-2} r dr$$
$$= \frac{1}{2}\int_0^{r_L} (1+t)^{-jL-2} dt = \frac{1}{2(1+jL)} \left[1-(1+r_L)^{-jL-1}\right].$$

We reach the expression:

$$J_w = \frac{1}{8}\sum_{j=1}^{+\infty} \frac{1}{j^2(1+jL)} \left[1-(1+r_L)^{-jL-1}\right] + \mathcal{O}\left(\frac{\log L}{L^3}\right).$$

Notice that

$$\frac{1}{j^2(1+jL)} \leq \frac{1}{j^2} \quad \text{and} \quad \frac{1}{j^2(1+jL)(1+r_L)^{jL+1}} \leq \frac{1}{j^2}.$$

So we can separate J_w into two sums because both sums are convergent:

$$J_w = \frac{1}{8}\left[\sum_{j=1}^{+\infty} \frac{1}{j^2(1+jL)} - \sum_{j=1}^{+\infty} \frac{(1+r_L)^{-jL-1}}{j^2(1+jL)}\right] + \mathcal{O}\left(\frac{\log L}{L^3}\right).$$

Again, the negative sum is negligible. That is:

Lemma 3.3.5. *It is satisfied that*

$$\sum_{j=1}^{+\infty} \frac{(1+r_L)^{-jL-1}}{j^2(1+jL)} = \mathcal{O}\left(\frac{\log L}{L^2}\right).$$

By Lemma 3.3.5, we can state that

$$J_w = \frac{1}{8}\sum_{j=1}^{+\infty} \frac{1}{j^2(1+jL)} + \mathcal{O}\left(\frac{\log L}{L^2}\right).$$

We have that $\lim_{L\to+\infty}(jL)/(1+jL) = 1$, uniformly in $j \geq 1$. Thus

$$J_w = \frac{1}{8}\sum_{j=1}^{+\infty} \frac{1}{j^3 L} + \mathcal{O}\left(\frac{\log L}{L^2}\right) = \frac{\zeta(3)}{8L} + \mathcal{O}\left(\frac{\log L}{L^2}\right).$$

Hence, as $L \to +\infty$, and recalling that $I_1 = \mathcal{O}(L^{-2})$ for such a chosen C_L, we finally have

$$\mathbb{V}\left[I_L(\varphi)\right] = \frac{\zeta(3)}{32L} \|\Delta^*\varphi\|^2_{L^2(\Omega,m^*)} + \mathcal{O}\left(\frac{\log L}{L^2}\right).$$

□

Let us show the three lemmas stated along the proof of Theorem 3.3.1.

Proof of Lemma 3.3.3. For all $x \in [0,1]$ we must see that

$$\frac{x}{4} \leq \sum_{j=1}^{+\infty} \frac{x^j}{4j^2} \leq \frac{x}{2}.$$

The left bound is clear, since it is the $j = 1$ term of the sum. For the right bound denote

$$F(x) = x - \frac{1}{2}\sum_{j=1}^{+\infty} \frac{x^j}{j^2}.$$

We want to see that $F(x) \geq 0$ for all $x \in [0,1]$. By computing the derivative we have

$$F'(x) = 1 - \frac{1}{2x}\sum_{j=1}^{+\infty} \frac{x^j}{j} = 1 - \frac{1}{2x}\log(1+x),$$

and notice that $F'(x) \geq 0$ since $\log(1+x) \leq 2x$ by Taylor for $x \geq 0$. Then the upper bound holds. ☐

Proof of Lemma 3.3.4. We want to show that

$$S = \sum_{j=1}^{+\infty} \frac{1}{4j^2} \int_{\{|w|^2<(2C_L)^{-1/L}-1\}\cap\{\kappa_L(0,w)\leq C_L\}} (1+|w|^2)^{-jL} dm^*(w) = \mathcal{O}\left(\frac{\log L}{L^3}\right).$$

We have

$$\begin{aligned}
S &\leq \sum_{j=1}^{+\infty} \frac{1}{4j^2} \int_{\{(2C_L)^{1/L}\leq(1+|w|^2)^{-1}\leq(4C_L)^{1/L}\}} (1+|w|^2)^{-jL} dm^*(w) \\
&= \frac{1}{8\pi}\sum_{j=1}^{+\infty} \frac{1}{j^2} \int_{\{(2C_L)^{1/L}\leq(1+|w|^2)^{-1}\leq(4C_L)^{1/L}\}} (1+|w|^2)^{-jL-2} dm(w) \\
&\leq \frac{(4C_L)^{1+2/L}}{8\pi} m\left(\left\{(4C_L)^{-1/L} - 1 \leq |w|^2 \leq (2C_L)^{-1/L} - 1\right\}\right) \sum_{j=1}^{+\infty}\frac{1}{j^2} \\
&\leq \frac{\pi}{48}(4C_L)^{1+2/L}\left[(2C_L)^{-1/L} - 1\right] = \frac{\pi}{48}\left(\frac{2}{L^2}\right)^{1+2/L}\left(L^{2/L}-1\right) \\
&= \frac{\pi}{24}\frac{1}{L^2} 2^{2/L} L^{-4/L}\left(L^{2/L}-1\right).
\end{aligned}$$

By Taylor series of $\exp(x)$ around 0, we have:

$$\begin{aligned}
2^{2/L} &= e^{\frac{2}{L}\log 2} = 1 - \frac{2}{L}\log 2 + \mathcal{O}\left(\frac{1}{L^2}\right) = 1 + \mathcal{O}\left(\frac{1}{L}\right), \\
L^{-4/L} &= e^{-\frac{4}{L}\log L} = 1 - \frac{4}{L}\log L + \mathcal{O}\left(\frac{\log^2 L}{L^2}\right) = 1 + \mathcal{O}\left(\frac{\log L}{L}\right), \\
L^{2/L} - 1 &= e^{\frac{2}{L}\log L} - 1 = \frac{2}{L}\log L + \mathcal{O}\left(\frac{\log^2 L}{L^2}\right) = \mathcal{O}\left(\frac{\log L}{L}\right).
\end{aligned}$$

Therefore,

$$S \leq \frac{\pi}{24}\frac{1}{L^2}\left(1+\mathcal{O}\left(\frac{1}{L}\right)\right)\left(1+\mathcal{O}\left(\frac{\log L}{L}\right)\right)\left(\mathcal{O}\left(\frac{\log L}{L}\right)\right)$$
$$= \mathcal{O}\left(\frac{\log L}{L^3}\right).$$

□

Proof of Lemma 3.3.5. We have to see that

$$S = \sum_{j=1}^{+\infty} \frac{(1+r_L)^{-jL-1}}{j^2(1+jL)} = \mathcal{O}\left(\frac{\log L}{L^2}\right).$$

Recall that $r_L = (2C_L)^{-1/L} - 1$. By direct computation we get that $(1+r_L)^{-jL-1} = L^{-2j-\frac{2}{L}}$. Since $L \to +\infty$, we obtain $1 + jL \simeq jL$ and $L^{-2/L} = -2\log L/L + \mathcal{O}(\log^2 L/L^2) \leq \mathcal{O}(\log L/L)$. Thus

$$S = \sum_{j=1}^{+\infty} \frac{L^{-2j-\frac{2}{L}}}{j^2(1+jL)} \leq \left(\sum_{j=1}^{+\infty} \frac{1}{j^3 L}\right)\mathcal{O}\left(\frac{\log L}{L}\right) = \mathcal{O}\left(\frac{\log L}{L^2}\right).$$

□

Chapter 4

Large deviations and the Hole probability of an $\mathbb{S}^2$-GAF

In this chapter we are going to develop another point of view to understand the fluctuation of a process generated by the zero set of an $\mathbb{S}^2$-GAF. For the first section of this chapter, consider a test-function $\varphi \in \mathcal{C}_c^2(\mathbb{C})$, an $\mathbb{S}^2$-GAF f_L of parameter $L \in \mathbb{N}$ and the linear statistic

$$I_L(\varphi) = \int_{\mathbb{C}} \varphi d\nu_{f_L}.$$

As seen before, its mean is

$$\mathbb{E}[I_L(\varphi)] = \mathbb{E}\left[\int_{\mathbb{C}} \varphi d\nu_{f_L}\right] = \int_{\mathbb{C}} \varphi d\mathbb{E}[\nu_{f_L}] = L \int_{\mathbb{C}} \varphi dm^*.$$

We study how much I_L deviates from its mean $\mathbb{E}\left[I_L(\varphi)\right]$ by a fraction of this same mean. In particular, we will see that such deviation happens with very small probability.

The second part of this chapter is a consequence of this study. We estimate the probability that there is a hole, that is, a disk without any zeros of f_L.

We are going to prove all the results of this chapter in terms of the chordal distance. This chapter is strongly based on [6]. Also [4] was helpful to write these pages.

4.1 Large deviations

We begin stating the main theorem.

Theorem 4.1.1 (Large deviations)**.** *For all $\varphi \in \mathcal{C}_c^2(\mathbb{C})$ and for all $\delta > 0$, there exist constants $c = c(\varphi, \delta) > 0$ and $L_0 = L_0(\varphi, \delta) \in \mathbb{N}$ such that for all $L \geq L_0$*

$$\mathbb{P}\left[\left|I_L(\varphi) - \mathbb{E}\left[I_L(\varphi)\right]\right| > \delta \mathbb{E}\left[I_L(\varphi)\right]\right] \leq e^{-cL^2}.$$

Notice that this can be rewritten as

$$\mathbb{P}\left[\left|\frac{I_L(\varphi)}{\mathbb{E}\left[I_L(\varphi)\right]} - 1\right| > \delta\right] \leq e^{-cL^2}.$$

To prove Theorem 4.1.1 we need a sequence of results.

Proof of Theorem 4.1.1. Consider $\varphi \in \mathcal{C}_c^2(\mathbb{C})$. By the definition of distributional derivative we have

$$I_L(\varphi) - \mathbb{E}[I_L(\varphi)] = \frac{1}{2\pi}\int_{\mathbb{C}} \varphi \Delta \log|\hat{f}_L| = \frac{1}{4\pi}\int_{\text{supp }\varphi} (\Delta\varphi) \log|\hat{f}_L|^2,$$

where $\hat{f}_L = f_L/\sqrt{\mathcal{K}_{f_L}(z,z)} \sim N_{\mathbb{C}}(0,1)$. Hence

$$\begin{aligned} |I_L(\varphi) - \mathbb{E}[I_L(\varphi)]| &\leq \|\Delta\varphi\|_\infty \int_{\text{supp }\varphi} \left|\log|\hat{f}_L(z)|^2\right| (1+|z|^2)^2 dm^*(z) \\ &\leq \sup_{z\in\text{supp }\varphi} \left[(1+|z|^2)^2\right] \|\Delta\varphi\|_\infty \int_{\text{supp }\varphi} \left|\log|\hat{f}_L(z)|^2\right| dm^*(z) \\ &= C_\varphi \int_{\text{supp }\varphi} \left|\log|\hat{f}_L(z)|^2\right| dm^*(z), \end{aligned}$$

where C_φ is a constant that depends on φ. Now we have to see that this integral is small. This is not strange because $\log|\hat{f}_L|^2$ has values near to zero since $|\hat{f}_L|^2$ has mean one.

Lemma 4.1.2. *For all regular compact set $K \subset \mathbb{C}$ and for all $\delta > 0$ there is a constant $c = c(K,\delta)$ such that*

$$\mathbb{P}\left[\int_K \left|\log|\hat{f}_L(z)|^2\right| dm^*(z) > \delta L\right] \leq e^{-cL^2}.$$

To show this we need another lemma that controls the average of $\left|\log|\hat{f}_L(z)|^2\right|$ over disks.

Lemma 4.1.3. *There is a constant $c > 0$ such that for all $D_{ch} := D_{ch}(z_0,\rho) \subset \mathbb{C}$, with $z_0 \in \mathbb{C}$,*

$$\mathbb{P}\left[\frac{1}{m^*(D_{ch})}\int_{D_{ch}} \left|\log|\hat{f}_L(w)|^2\right| dm^*(w) > 5L\right] \leq e^{-cL^2}.$$

Let us see how Lemma 4.1.2 is implied by Lemma 4.1.3.

Proof of Lemma 4.1.2. Since K is a compact set, we can use a minimal covering consisting on a finite number of open disks $D_j := D_{ch}(z_j,\varepsilon)$, for all $j = 1,...,N$, such that all the disks have the same invariant volume, say $m^*(D_j) = v$. We will choose v later on.

By Lemma 4.1.3, outside an exceptional set of probability $N\exp(-cL^2) \le \exp(-c'L^2)$:

$$\int_K \left|\log|\hat{f}_L(w)|^2\right| dm^*(w) \le \int_{\bigcup_{j=1}^N D_j} \left|\log|\hat{f}_L(w)|^2\right| dm^*(w)$$
$$\le \sum_{j=1}^N \int_{D_j} \left|\log|\hat{f}_L(w)|^2\right| dm^*(w)$$
$$\le \sum_{j=1}^N 5Lm^*(D_j) = 5NLv.$$

By choosing $v = \delta/(5N)$, we conclude that

$$\int_K \left|\log|\hat{f}_L(w)|^2\right| dm^*(w) \le \delta L.$$

□

Now we shall show Lemma 4.1.3. For this we need more estimates.

Lemma 4.1.4. *For all $\rho < 1$ and $\delta > 0$ there exist $c = c(\rho,\delta) > 0$ and $L_0 = L_0(\rho,\delta) \in \mathbb{N}$ such that for all $L \ge L_0$ and $z_0 \in \mathbb{C}$:*

(a) $\mathbb{P}\left[\max_{z\in D_{ch}(z_0,\rho)} \log|\hat{f}_L(z)|^2 < -\delta L\right] \le e^{-cL^2}$,

(b) $\mathbb{P}\left[\max_{z\in D_{ch}(z_0,\rho)} \log|\hat{f}_L(z)|^2 > \delta L\right] \le e^{-ce^{L\delta/2}}$.

Combining both estimates, we get

$$\mathbb{P}\left[\max_{z\in D_{ch}(z_0,\rho)} \left|\log|\hat{f}_L(z)|^2\right| > \delta L\right] \le e^{-cL^2}.$$

Proof. We know that the distribution of $\mathcal{Z}_{f_L}$ is invariant by translations, hence we can choose $z_0 = 0$. By i) of Remark 2.4.3, the condition $z \in D_{ch}(0,\rho)$ is equivalent to

$$|z| \le \hat{\rho} := \frac{\rho}{\sqrt{4-\rho^2}}.$$

(a) Consider the event

$$\mathcal{E}_1 := \left\{\max_{|z|\le\hat{\rho}} \log|\hat{f}_L(z)|^2 < -\delta L\right\}.$$

Under this event $\mathcal{E}_1$:

$$\log|\hat{f}_L(z)|^2 = \log|f_L(z)|^2 - L\log(1+|z|^2) < -\delta L$$

and therefore, for $|z| \le \hat{\rho}$:

$$\log|f_L(z)|^2 < L\left(\log(1+|z|^2) - \delta\right) \le L\left(\log(1+\hat{\rho}^2) - \delta\right).$$

Since $\log(1+|z|^2)$ is subharmonic in $\mathbb{C}$, we can use the Maximum principle to state:

$$\mathcal{E}_1 \subset \left\{ \max_{|z|=\hat{\rho}} \log|f_L(z)|^2 \le L\left(\log(1+\hat{\rho}^2) - \delta\right) \right\}.$$

Thus

$$\begin{aligned}\mathbb{P}\left[\mathcal{E}_1\right] &\le \mathbb{P}\left[\max_{|z|=\hat{\rho}} \frac{\log|f_L(z)|}{L} \le \frac{1}{2}\left(\log(1+\hat{\rho}^2) - \delta\right)\right] \\ &= \mathbb{P}\left[\max_{|z|=\hat{\rho}} \frac{\log|f_L(z)|}{L} \le \left(\frac{1}{2} - \tilde{\delta}\right)\log(1+\hat{\rho}^2)\right],\end{aligned}$$

where $\tilde{\delta} = \delta / \left[2\log(1+\hat{\rho}^2)\right]$. To continue we need another lemma.

Lemma 4.1.5. *For all $\delta \in (0, 1/2)$ and $r > 0$ there exist $c = c(r,\delta)$ and $L_0 = L_0(r,\delta) \in \mathbb{N}$ such that for all $L \ge L_0$:*

$$\mathbb{P}\left[\max_{|z|=r} \frac{\log|f_L(z)|}{L} \le \left(\frac{1}{2} - \delta\right)\log(1+r^2)\right] \le e^{-cL^2}.$$

Proof. Under such event we have

$$\max_{|z|=r} |f_L(z)| \le (1+r^2)^{L\left(\frac{1}{2}-\delta\right)}.$$

We shall see that this forces some coefficients of the series of f_L to be small, something that can happen only with probability $\exp(-cL^2)$.
Since

$$f_L(z) = \sum_{n=0}^{L} \frac{f_L^{(n)}(0)}{n!} z^n = \sum_{n=0}^{L} \xi_n \binom{L}{n}^{1/2} z^n,$$

where $(\xi_n)_{n=0}^{L}$ is a sequence of i.i.d. $N_{\mathbb{C}}(0,1)$ random variables, we have

$$\xi_n = \binom{L}{n}^{-1/2} \frac{f_L^{(n)}(0)}{n!},$$

for all $n \in \mathbb{N}$. Cauchy estimates yield

$$|\xi_n| \le \binom{L}{n}^{-1/2} \frac{1}{r^n} \max_{|z|=r} |f_L(z)| \le \binom{L}{n}^{-1/2} \frac{1}{r^n} (1+r^2)^{L\left(\frac{1}{2}-\delta\right)}.$$

Taking squares:

$$|\xi_n|^2 \le \frac{(1+r^2)^{L(1-\delta)}}{\binom{L}{n} r^{2n}},$$

where δ stands for the old δ^2.
This event happens with low probability. Assume that $n \in \mathbb{N}$ takes high values, that is, $n/L \geq \varepsilon$ for some $\varepsilon > 0$ with $n \leq L$. By Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

as $n \to +\infty$. Hence

$$\binom{L}{n} \sim \frac{1}{\sqrt{L-n}}\left(\frac{L}{L-n}\right)^{L-n}\left(\frac{L}{n}\right)^n.$$

Therefore, and using $L - n \leq L$, we get

$$|\xi_n|^2 \leq \frac{(1+r^2)^{L(1-\delta)}}{\binom{L}{n} r^{2n}} \lesssim \sqrt{L}\frac{(1+r^2)^{L(1-\delta)}}{r^{2n}}\left(\frac{L-n}{L}\right)^{L-n}\left(\frac{n}{L}\right)^n$$
$$= \exp\left[\frac{1}{2}\log L + L(1-\delta)\log(1+r^2) - n\log r^2 + (L-n)\log\left(1-\frac{n}{L}\right) + n\log\left(\frac{n}{L}\right)\right].$$

Denote

$$\Phi(t) = \frac{1}{2}\log L + L(1-\delta)\log(1+r^2) - t\log r^2 + (L-t)\log\left(1-\frac{t}{L}\right) + t\log\left(\frac{t}{L}\right).$$

We have that $|\xi_n|^2 \leq \exp\Phi(n)$. Let us study the behaviour of $\Phi(t)$, for all $t \in [1, L]$. We have that

$$\Phi(L) = \frac{1}{2}\log L + L(1-\delta)\log(1+r^2) - L\log r^2$$

and

$$\Phi'(t) = \log\left[\frac{t}{L}\frac{1}{r^2\left(1-\frac{t}{L}\right)}\right] > 0.$$

Define

$$\tilde{\Phi}(s) = \Phi(sL) = \frac{1}{2}\log L + L(1-\delta)\log(1+r^2) - sL\log r^2 + L(1-s)\log(1-s) + sL\log(s).$$

Since

$$\tilde{\Phi}'(s) = -L\log r^2 - L\log(1-s) + L\log s,$$

we have that $\tilde{\Phi}'(\alpha) = 0$ at $\alpha = r^2/(1+r^2)$. In this minimum,

$$\tilde{\Phi}(\alpha) = \frac{1}{2}\log L - \delta L\log(1+r^2),$$

where the negative term is the dominant one as $L \to +\infty$. So, for $L \in \mathbb{N}$ large enough and taking $s \in [(1-\delta/2)\alpha, (1+\delta/2)\alpha]$, we have

$$\tilde{\Phi}(s) \leq -\frac{\delta}{2}L\log(1+r^2).$$

With this, for $t \in I_{r,L,\delta} := [(1-\delta/2)L\alpha, (1+\delta/2)L\alpha]$, we have

$$\Phi(t) \leq -\frac{\delta}{2} L \log(1+r^2).$$

In particular, for all $n \in I_{r,L,\delta}$,

$$|\xi_n|^2 \leq e^{\Phi(n)} \leq e^{-\frac{\delta}{2} L \log(1+r^2)}.$$

Denote $N = \#\{n \in \mathbb{N} \ : \ n \in I_{r,L,\delta}\}$. Note that $N \simeq \delta L \alpha$. Thus

$$\begin{aligned}
\mathbb{P}\left[\max_{|z|=r} \frac{\log|f_L(z)|}{L} \leq \left(\frac{1}{2}-\delta\right)\log(1+r^2)\right] &\leq \prod_{n \in I_{r,L,\delta}} \mathbb{P}\left[|\xi_n|^2 \leq e^{-\frac{\delta}{2} L \log(1+r^2)}\right] \\
&= \left(\mathbb{P}\left[|\xi_n|^2 \leq e^{-\frac{\delta}{2} L \log(1+r^2)}\right]\right)^N.
\end{aligned}$$

Using the Taylor development of $1-\exp(-x)$ around zero we get

$$\left(1 - e^{-e^{-\frac{\delta}{2} L \log(1+r^2)}}\right)^N \leq e^{-\frac{\delta}{2} N L \log(1+r^2)} = e^{-cL^2},$$

for some c that depends on r and δ, specifically

$$c = \frac{\delta^2}{2} \frac{r^2}{1+r^2} \log(1+r^2).$$

□

Applying Lemma 4.1.5, we finish the proof of section (a).

(b) Consider the event

$$\mathcal{E}_2 := \left\{\max_{|z| \leq \hat{\rho}} \log|\hat{f}_L(z)|^2 > \delta L\right\} = \left\{\max_{|z| \leq \hat{\rho}} \left[\log|f_L(z)|^2 - L\log(1+|z|^2)\right] > \delta L\right\}.$$

To estimate the probability of $\mathcal{E}_2$ we will control the coefficients of the series of f_L. Let $C > 0$ be a constant such that $C\delta < 1$. From the definition of $\mathbb{S}^2$-GAF (2.4.8) we have that

$$|f_L(z)| \leq \sum_{n=0}^{[C\delta L]} |\xi_n| \binom{L}{n}^{1/2} |z|^n + \sum_{n=[C\delta L]+1}^{L} |\xi_n| \binom{L}{n}^{1/2} |z|^n =: S_1 + S_2.$$

Let us estimate both parts separately.
For S_1 we use Cauchy-Schwarz inequality:

$$\begin{aligned}
S_1 &\leq \left(\sum_{n=0}^{[C\delta L]} |\xi_n|^2\right)^{1/2} \left(\sum_{n=0}^{[C\delta L]} \binom{L}{n} |z|^{2n}\right)^{1/2} \leq \left(\sum_{n=0}^{[C\delta L]} |\xi_n|^2\right)^{1/2} \left(\sum_{n=0}^{L} \binom{L}{n} |z|^{2n}\right)^{1/2} \\
&= \left(\sum_{n=0}^{[C\delta L]} |\xi_n|^2\right)^{1/2} (1+|z|^2)^{L/2}.
\end{aligned}$$

For S_2 we have, using that $|z| \leq \hat{\rho}$:

$$S_2 \leq \sum_{n=[C\delta L]+1}^{L} |\xi_n| \binom{L}{n}^{1/2} \hat{\rho}^n.$$

In the proof of Lemma 4.1.2 we chose disks with very small radii. So, for convenience of the actual proof, we can select $\rho^2 < 2$ so that $\hat{\rho} < 1$. Hence we can consider $\beta > 0$ such that $\hat{\rho} = \exp(-\beta)$. Also take $\gamma \in (0, \beta)$ and $\varepsilon > 0$ such that $0 < \gamma < \gamma + \varepsilon < \beta$. Define the event

$$\mathcal{A} := \{|\xi_n| \leq e^{\gamma n} \ : \ n > [C\delta L]\}.$$

Applying again the Stirling's formula we get

$$\binom{L}{n}^{1/2} \lesssim \frac{L^{L/2}}{n^{n/2}(L-n)^{(L-n)/2}}.$$

Therefore

$$S_2 \leq \sum_{[C\delta L]+1}^{L} e^{(\gamma-\beta)n} \binom{L}{n}^{1/2} \leq \sum_{[C\delta L]+1}^{L} e^{(\gamma-\beta)n} \frac{L^{L/2}}{n^{n/2}(L-n)^{(L-n)/2}}.$$

Lemma 4.1.6. *For a given $\varepsilon > 0$ there is $D > 0$ large enough such that for all $n > [C\delta L]$:*

$$\frac{L^L}{n^n (L-n)^{(L-n)}} \leq D e^{\varepsilon n}.$$

Proof. Notice that

$$\frac{L^L}{n^n (L-n)^{(L-n)}} = e^{L \log L - n \log n - (L-n)\log(L-n)}.$$

Hence, we have to check whether for some $\varepsilon > 0$ there is $D > 0$ large enough such that for all $n > [C\delta L]$:

$$L \log L - n \log n - (L-n)\log(L-n) \leq \varepsilon n + \log D.$$

Define

$$\Phi_\varepsilon(t) := L \log L - t \log t - (L-t)\log(L-t) - \varepsilon t$$

for all $t \in [[C\delta L]+1, L]$. We want to see that

$$\Phi_\varepsilon(t) \leq \log D.$$

By taking $t = Ls$ we define the rescaling of Φ_ε:

$$\tilde{\Phi}_\varepsilon(s) := \Phi_\varepsilon(Ls) = L \log L - Ls \log(Ls) - (L-Ls)\log(L-Ls) - \varepsilon Ls$$

for all $s \in (\alpha, 1]$. Since

$$\tilde{\Phi}'_\varepsilon(s) = L\log(1-s) - L\log s - L\varepsilon = L\left[\log\left(\frac{1-s}{s}\right) - \varepsilon\right],$$

we see that $\tilde{\Phi}'_\varepsilon(s) < 0$ if $1/(1+\exp(\varepsilon)) \leq s$. So, choosing $\alpha = 1/(1+\exp(\varepsilon))$, $\tilde{\Phi}_\varepsilon$ is a decreasing function. Clearly, the maximum is at $s_\varepsilon = 1/(1+\exp(\varepsilon))$. Therefore, for all $s \in (s_\varepsilon, 1]$:

$$\tilde{\Phi}_\varepsilon(s) \leq C_{\varepsilon,L} := \tilde{\Phi}_\varepsilon(s_\varepsilon).$$

Thus,

$$\Phi_\varepsilon(t) \leq C_{\varepsilon,L}$$

if $t/L \geq s_\varepsilon$, i.e., if $t \geq s_\varepsilon L$. □

Taking D appropriately as in Lemma 4.1.6 and since $0 < \gamma < \gamma + \varepsilon < \beta$, we get

$$S_2 \lesssim \sum_{n=[C\delta L]+1}^{L} e^{-[\beta-(\gamma+\varepsilon)]n} \leq \sum_{n=0}^{+\infty} e^{-[\beta-(\gamma+\varepsilon)]n} = \frac{1}{1-e^{-[\beta-(\gamma+\varepsilon)]}}.$$

Let us show that $\mathcal{A}$ happens with high probability. Indeed,

$$\mathbb{P}[\mathcal{A}] = \prod_{n=[C\delta L]+1}^{L} \mathbb{P}\left[|\xi_n|^2 \leq e^{2\gamma n}\right] = \prod_{n=[C\delta L]+1}^{L} \left(1 - e^{-e^{2\gamma n}}\right).$$

Using the fact that $\log(1-x) \simeq -x$ for $x \simeq 0$, we write

$$\log \mathbb{P}[\mathcal{A}] = \sum_{n=[C\delta L]+1}^{L} \log\left(1 - e^{-e^{2\gamma n}}\right) \simeq -\sum_{n=[C\delta L]+1}^{L} e^{-e^{2\gamma n}}.$$

There is a constant $L_0 = L_0(\rho, \delta) \in \mathbb{N}$ such that for all $L \geq L_0$ and $n > [C\delta L]$:

$$-e^{-e^{2\gamma n}} \geq -e^{-e^{\gamma C\delta L}}.$$

Thus

$$\log \mathbb{P}[\mathcal{A}] \geq -\sum_{n=[C\delta L]+1}^{L} e^{-e^{\gamma C\delta L}} \simeq -e^{-e^{\gamma C\delta L}},$$

and this implies that

$$\mathbb{P}[\mathcal{A}] \geq e^{-e^{-e^{\gamma C\delta L}}}.$$

What we get after all is, under $\mathcal{A}$,

$$|f(z)| \leq \left(\sum_{n=0}^{[C\delta L]} |\xi_n|^2\right)^{1/2} (1+|z|^2)^{L/2} + \frac{1}{1-e^{-[\beta-(\gamma+\varepsilon)]}}.$$

The condition in $\mathcal{E}_2$, that is $|f_L(z)|^2/(1+|z|^2)^L > e^{\delta L}$, implies

$$\begin{aligned}\sum_{n=0}^{[C\delta L]} |\xi_n|^2 &\geq \left[(1+|z|^2)^{-L/2}\left(|f_L(z)| - \left(1-e^{-[\beta-(\gamma+\varepsilon)]}\right)^{-1}\right)\right]^2 \\ &= \left[\frac{|f_L(z)|}{(1+|z|^2)^{L/2}} - (1+|z|^2)^{-L/2}\left(1-e^{-[\beta-(\gamma+\varepsilon)]}\right)^{-1}\right]^2 \\ &> \left[e^{\delta L/2} - (1+|z|^2)^{-L/2}\left(1-e^{-[\beta-(\gamma+\varepsilon)]}\right)^{-1}\right]^2 > e^{\delta L}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{P}\left[\mathcal{E}_2 \cap \mathcal{A}\right] &\leq \mathbb{P}\left[\sum_{n=0}^{[C\delta L]} |\xi_n|^2 \geq e^{\delta L}\right] \leq \sum_{n=0}^{[C\delta L]} \mathbb{P}\left[|\xi_n|^2 \geq e^{\delta L}\right] = \sum_{n=0}^{[C\delta L]} e^{-e^{\delta L}} \\ &\lesssim e^{-e^{\delta L/2}}.\end{aligned}$$

Finally

$$\mathbb{P}\left[\mathcal{E}_2\right] = \frac{\mathbb{P}\left[\mathcal{E}_2 \cap \mathcal{A}\right]}{\mathbb{P}\left[\mathcal{A}\right]} \leq e^{-e^{\delta L/2}} e^{e^{-e^{\gamma C\delta L}}} \leq e^{-ce^{L\delta/2}},$$

for a suitable $c > 0$.

□

We have to show Lemma 4.1.3. However, we need a result that estimates the average of $\log|\hat{f}_L(z_0)|^2$, for $z_0 \in \mathbb{C}$.

Lemma 4.1.7. *Let $D_{ch}^{z_0} := D_{ch}(z_0, \rho)$ be a disk with $z_0 \in \mathbb{C}$ and $\rho > 0$. Then*

$$\log|\hat{f}_L(z_0)|^2 \leq \frac{1}{m^*(D_{ch}^{z_0})} \int_{D_{ch}^{z_0}} \log|\hat{f}_L(w)|^2 dm^*(w) + L\varepsilon(\hat{\rho}, z_0),$$

where, by i) of Remark 2.4.3,

$$\hat{\rho} := \frac{\rho}{\sqrt{4-\rho^2}}$$

and

$$\varepsilon(\hat{\rho}, z_0) := \begin{cases} 1 - \dfrac{\log(1+\hat{\rho}^2)}{\hat{\rho}^2}, & \text{if } |z_0\hat{\rho}| < 1, \\ \left(1 - \dfrac{\log(1+\hat{\rho}^2)}{\hat{\rho}^2}\right) - \dfrac{1}{m^*(D_{ch}^0)} \displaystyle\int_{1/|z_0|}^{\hat{\rho}} \log|\overline{z_0}r|^2 \frac{2r}{(1+r^2)^2} dr, & \text{if } |z_0\hat{\rho}| > 1. \end{cases}$$

For any case, $\varepsilon(\hat{\rho}, z_0) \leq 1$.

Remark 4.1.8. The function ε can be written in terms of ρ and the integral is computable, but we do not need that. We will need only the inequality of ε.

Proof. By the subharmonicity of $\log|\hat{f}_L|^2$, we have

$$\begin{aligned}\log|\hat{f}_L(z_0)|^2 &= \log|f_L(z_0)|^2 - L\log\mathcal{K}_{f_L}(z_0,z_0) = \log|f_L(z_0)|^2 - L\log(1+|z_0|^2)\\ &\le \frac{1}{m^*(D_{ch}^{z_0})}\int_{D_{ch}^{z_0}}\log|f_L(w)|^2dm^*(w) - L\log(1+|z_0|^2)\\ &= \frac{1}{m^*(D_{ch}^{z_0})}\int_{D_{ch}^{z_0}}\log|\hat{f}_L(w)|^2dm^*(w)\\ &\quad + L\left[\frac{1}{m^*(D_{ch}^{z_0})}\int_{D_{ch}^{z_0}}\log(1+|w|^2)dm^*(w) - \log(1+|z_0|^2)\right].\end{aligned}$$

Name

$$I := \int_{D_{ch}^{z_0}}\log(1+|w|^2)dm^*(w).$$

By (2.4.5), we get

$$\begin{aligned}I &= \int_{D_{ch}^{0}}\log(1+|\phi_{z_0}(w)|^2)dm^*(w) = \int_{D_{ch}^{0}}\log\left[\frac{(1+|z_0|^2)(1+|w|^2)}{|1+\overline{z_0}w|^2}\right]dm^*(w)\\ &= \int_{D_{ch}^{0}}\log(1+|z_0|^2)dm^*(w) + \int_{D_{ch}^{0}}\log(1+|w|^2)dm^*(w) - \int_{D_{ch}^{0}}\log|1+\overline{z_0}w|^2dm^*(w).\end{aligned}$$

Since $m^*(D_{ch}^{z_0}) = m^*(D_{ch}^{0})$ and using the new expression of I, we have

$$\begin{aligned}\log|\hat{f}_L(z_0)|^2 &= \frac{1}{m^*(D_{ch}^{z_0})}\int_{D_{ch}^{z_0}}\log|\hat{f}_L(w)|^2dm^*(w)\\ &\quad + L\left[\frac{1}{m^*(D_{ch}^{0})}\int_{D_{ch}^{0}}\log(1+|w|^2)dm^*(w) - \frac{1}{m^*(D_{ch}^{0})}\int_{D_{ch}^{0}}\log|1+\overline{z_0}w|^2dm^*(w)\right].\end{aligned}$$

Using a polar coordinate change of variables,

$$\int_{D_{ch}^{0}}\log|1+\overline{z_0}w|^2dm^*(w) = \frac{1}{\pi}\int_0^{\hat{\rho}}\left(\int_0^{2\pi}\log|1+\overline{z_0}re^{i\theta}|^2d\theta\right)\frac{r}{(1+r^2)^2}dr.$$

Let us study the integral in θ by distinguishing cases:

- If $|z_0\hat{\rho}|<1$, the harmonicity of the integrand implies

$$\int_0^{2\pi}\log|1+\overline{z_0}re^{i\theta}|^2d\theta = 2\pi\log(1) = 0.$$

- If $|z_0\hat{\rho}|>1$, by using the previous case and the change of variable $\psi=-\theta$ we can conclude that

$$\begin{aligned}\int_0^{2\pi}\log|1+\overline{z_0}re^{i\theta}|^2d\theta &= \int_0^{2\pi}\log|\overline{z_0}r|^2d\theta + \int_0^{2\pi}\log|1+(\overline{z_0}r)^{-1}e^{-i\theta}|^2d\theta\\ &= 2\pi\log|\overline{z_0}r|^2 + \int_0^{2\pi}\log|1+(\overline{z_0}r)^{-1}e^{-i\psi}|^2d\psi\\ &= 2\pi\log|\overline{z_0}r|^2.\end{aligned}$$

Thus, if $|z_0\hat{\rho}| > 1$:

$$\int_{D_{ch}^0} \log|1+\overline{z_0}w|^2 dm^*(w) = \frac{1}{\pi}\int_{1/|z_0|}^{\hat{\rho}} \left(\int_0^{2\pi} \log|1+\overline{z_0}re^{i\theta}|^2 d\theta\right) \frac{r}{(1+r^2)^2}dr$$
$$= \int_{1/|z_0|}^{\hat{\rho}} \log|\overline{z_0}r|^2 \frac{2r}{(1+r^2)^2}dr.$$

By using another change into polar coordinates, the change of variable $t = 1+r^2$ and integration by parts with $u = \log t$, $dv = dt/t^2$, we have

$$\int_{D_{ch}^0} \log(1+|w|^2)dm^*(w) = \int_0^{\hat{\rho}} \log(1+r^2)\frac{2r}{(1+r^2)^2}dr = \int_1^{1+\hat{\rho}^2} \frac{\log t}{t^2}dt$$
$$= -\frac{\log t}{t}\Big|_{t=1}^{t=1+\hat{\rho}^2} + \int_1^{1+\hat{\rho}^2}\frac{dt}{t^2} = \frac{\hat{\rho}^2 - \log(1+\hat{\rho}^2)}{1+\hat{\rho}^2}$$
$$= m^*(D_{ch}^0)\left(1 - \frac{\log(1+\hat{\rho}^2)}{\hat{\rho}^2}\right).$$

Hence the expression of $\varepsilon(\hat{\rho}, z_0)$ follows and it is trivial that $\varepsilon(\hat{\rho}, z_0) \le 1$ for all $\hat{\rho} > 0$ and $z_0 \in \mathbb{C}$.

□

Proof of Lemma 4.1.3. Using section (a) of Lemma 4.1.4, outside of an exceptional set of probability $\exp(-cL^2)$, there exists $\lambda \in D_{ch} := D_{ch}(z_0, \rho) \subset \mathbb{C}$ such that

$$\log|\hat{f}_L(\lambda)|^2 > -Lm^*(D_{ch}).$$

By Lemma 4.1.7,

$$-Lm^*(D_{ch}) < \frac{1}{m^*(D_{ch})}\int_{D_{ch}} \log|\hat{f}_L(w)|^2 dm^*(w) + L,$$

which implies

$$0 < \frac{1}{m^*(D_{ch})}\int_{D_{ch}} \log|\hat{f}_L(w)|^2 dm^*(w) + 2L.$$

Separating the logarithm into the positive and negative part:

$$\frac{1}{m^*(D_{ch})}\int_{D_{ch}} \log^-|\hat{f}_L(w)|^2 dm^*(w) \le \frac{1}{m^*(D_{ch})}\int_{D_{ch}} \log^+|\hat{f}_L(w)|^2 dm^*(w) + 2L.$$

Adding the positive part of the logarithm,

$$\frac{1}{m^*(D_{ch})}\int_{D_{ch}} \left|\log|\hat{f}_L(w)|^2\right| dm^*(w) \le \frac{2}{m^*(D_{ch})}\int_{D_{ch}} \log^+|\hat{f}_L(w)|^2 dm^*(w) + 2L.$$

By Lemma 4.1.4, outside an exceptional set of probability $\exp(-cL^2)$ we conclude

$$\frac{1}{m^*(D_{ch})}\int_{D_{ch}} \left|\log|\hat{f}_L(w)|^2\right| dm^*(w) \le 2\max_{w\in D_{ch}} \log^+|\hat{f}_L(w)|^2 + 2L$$
$$\le 3Lm^*(D_{ch}) + 2L \le 5L.$$

□

By applying Lemma 4.1.3, the proof of Theorem 4.1.1 is finished. □

A direct consequence of Theorem 4.1.1, useful to show the Hole probability theorem, is the following.

Corollary 4.1.9. *Let U be a bounded open set in $\mathbb{C}$. For all $\delta \in (0,1)$ there are constants $c = c(U,\delta) > 0$ and $L_0 = L_0(U,\delta) \in \mathbb{N}$ such that for all $L \geq L_0$*

$$\mathbb{P}\left[\left|\frac{I_L(\mathbb{1}_U)}{\mathbb{E}\left[I_L(\mathbb{1}_U)\right]} - 1\right| > \delta\right] = \mathbb{P}\left[\left|\frac{\nu_{f_L}(U)}{Lm^*(U)} - 1\right| > \delta\right] \leq e^{-cL^2}.$$

Proof. Consider test-functions $\varphi_1, \varphi_2 \in \mathcal{C}_c^2(\mathbb{C})$ such that $0 \leq \varphi_1 \leq \mathbb{1}_U \leq \varphi_2 \leq 1$,

$$\int_{\mathbb{C}} \varphi_1 dm^* \geq m^*(U)(1-\delta)$$

and

$$\int_{\mathbb{C}} \varphi_2 dm^* \leq m^*(U)(1+\delta).$$

By Theorem 4.1.1 we have, outside an exceptional set of probability $\exp(-cL^2)$:

$$\begin{aligned}\nu_{f_L}(U) &= \int_U d\nu_{f_L} \leq \int_{\mathbb{C}} \varphi_2 d\nu_{f_L} \leq (1+\delta)\mathbb{E}\left[\int_{\mathbb{C}} \varphi_2 d\nu_{f_L}\right] = (1+\delta)\int_{\mathbb{C}} \varphi_2 d\mathbb{E}[\nu_{f_L}] \\ &= L(1+\delta)\int_{\mathbb{C}} \varphi_2 dm^* \leq L(1+\delta)^2 m^*(U).\end{aligned}$$

Analogously, outside an exceptional set of probability $\exp(-cL^2)$, we get:

$$\nu_{f_L}(U) \geq L(1-\delta)^2 m^*(U).$$

Thus:

$$\frac{\nu_{f_L}(U)}{Lm^*(U)} - 1 \leq 2\delta + \delta^2 \leq 3\delta.$$

Also

$$\frac{\nu_{f_L}(U)}{Lm^*(U)} - 1 \geq -2\delta + \delta \geq -3\delta.$$

□

4.2 Hole probability

In the last section of the project we focus on the *Hole probability*.

Theorem 4.2.1. *For a given $\rho > 0$, there exist $C_1 = C_1(\rho) > 0$, $C_2 = C_2(\rho) > 0$ and $L_0 \in \mathbb{N}$ such that, for all $L \geq L_0$ and for all $z_0 \in \mathbb{C}$,*

$$e^{-C_1 L^2} \leq \mathbb{P}\left[\mathcal{Z}_{f_L} \cap D_{ch}(z_0,\rho) = \emptyset\right] \leq e^{-C_2 L^2},$$

where $\mathcal{Z}_{f_L}$ is the zero set of the $\mathbb{S}^2$-GAF f_L.

Proof. Upper bound. Since

$$\{\nu_{f_L}(D_{ch}(z_0,\rho))=0\} \subset \left\{\left|\frac{\nu_{f_L}(D_{ch}(z_0,\rho))}{Lm^*(D_{ch}(z_0,\rho))}-1\right|>\delta\right\},$$

for all $\delta < 1$, Corollary 4.1.9 gives us the upper bound.
Lower bound. By the invariance we can assume that $z_0 = 0$. Denote $D_{ch} := D_{ch}(0,\rho)$. We are going to choose two events that force f_L to have $\mathcal{Z}_{f_L} \cap D_{ch} = \emptyset$. Clearly

$$|f_L(z)| \geq |\xi_0| - \left|\sum_{n=1}^{L} \xi_n \binom{L}{n}^{1/2} z^n\right|.$$

The first event is

$$\mathcal{E}_1 := \{|\xi_0| \geq 1\}.$$

By i) of Proposition 2.1.2, $\mathcal{E}_1$ has probability

$$\mathbb{P}[\mathcal{E}_1] = \mathbb{P}\left[|\xi_0|^2 \geq 1\right] = e^{-1}.$$

By i) of Remark 2.4.3, take

$$\hat{\rho} := \frac{\rho}{\sqrt{4-\rho^2}}.$$

For the second term we use the Cauchy-Schwarz inequality for $|z| \leq \hat{\rho}$:

$$\begin{aligned}\left|\sum_{n=1}^{L} \xi_n \binom{L}{n}^{1/2} z^n\right| &\leq \sum_{n=1}^{L} |\xi_n| \binom{L}{n}^{1/2} \hat{\rho}^n \leq \left(\sum_{n=1}^{L} |\xi_n|^2\right)^{1/2} \left(\sum_{n=1}^{L} \binom{L}{n} \hat{\rho}^{2n}\right)^{1/2} \\ &\leq \left(\sum_{n=1}^{L} |\xi_n|^2\right)^{1/2} (1+\hat{\rho}^2)^{L/2}.\end{aligned}$$

Choosing the second event as

$$\mathcal{E}_2 := \left\{|\xi_n|^2 \leq \frac{1}{16L(1+\hat{\rho}^2)^L},\ n=1,...,L\right\},$$

we see that, under $\mathcal{E}_2$:

$$\left|\sum_{n=1}^{L} \xi_n \binom{L}{n}^{1/2} z^n\right| \leq \frac{1}{4}.$$

Knowing that $1-\exp(-x) \geq x/2$ for $x \leq 1$, we get

$$\begin{aligned}\mathbb{P}[\mathcal{E}_2] &= \prod_{n=1}^{L} \mathbb{P}\left[|\xi_n|^2 \leq \frac{1}{16L(1+\hat{\rho}^2)^L}\right] = \left[1-\exp\left(-\frac{1}{16L(1+\hat{\rho}^2)^L}\right)\right]^L \\ &\geq \left(\frac{1}{32L(1+\hat{\rho}^2)^L}\right)^L = e^{-L^2\log[32L(1+\hat{\rho}^2)]} = e^{-CL^2},\end{aligned}$$

where $C = C(\hat{\rho}, L) := \log[32L(1+\hat{\rho}^2)]$.
All combined, under the event $\mathcal{E}_1 \cap \mathcal{E}_2$ we have that $f_L(z) \geq 3/4 > 0$ and

$$\mathbb{P}[\mathcal{E}_1 \cap \mathcal{E}_2] = \mathbb{P}[\mathcal{E}_1]\,\mathbb{P}[\mathcal{E}_2] \geq e^{-1}e^{-CL^2} \geq e^{-C_1L^2},$$

for a suitable $C_1 = C_1(\rho) > 0$. □

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