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Basics of Malliavin Calculus

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Abstract

This work is an introduction to Malliavin calculus. We start by giving the definition of an integration by parts formula and how they are related to the existence of densities of random variables. The central topic of this work is how using Malliavin calculus we can find integration by parts formulas. In order to accomplish this objective, there are presented tools such as the Wiener chaos decomposition, the multiple Wiener-Itô integral and the fundamental operators which are: the differential operator, the divergence operator and the generator of the Ornstein–Uhlenbeck semigroup. These operators are combined to obtain explicit integration by parts formulas that result in criteria for the existence and regularity of probability densities. Finally, it is provided an example where there are obtained conditions for the Malliavin differentiability of a particular process.

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Introduction

Malliavin calculus is an infinite-dimensional calculus on a Gaussian space, that is, a stochastic calculus of variations. The initial objective of Malliavin for the creation of this calculus, was the study of the existence of densities of Wiener functionals such as solutions of stochastic differential equations. The main ingredient for the study of the properties of the probability law of these functionals, is the integration by parts formula. The integration by parts formula allows to obtain criteria for the existence of densities and its regularity. Malliavin calculus provides the tools for the derivation of integration by parts formulas by means of the fundamental operators.

Malliavin calculus has been proved useful in the study of many areas of probability others than the study of regularity of probability laws, such as the study of stochastic partial differential equations, the extension of Itô 's formulas and even it has found applications in finance for the computation of Greeks.

The aim of this work is to study the regularity of probability laws using Malliavin calculus. In order to achieve this objective we will use integration by parts formulas, which we will see that are related to the existence of densities of random variables. For that reason, the central topic of this work is to find integration by parts formulas using Malliavin calculus.

In Section 1 we introduce the motivation of the work: the integration by parts formula and some results on the existence and smoothness of probability densities related with the integration by parts formulas. The results of this section will be fundamental in Section 5, when we will use Malliavin calculus to derive integration by parts formulas.

Section 2 deals with Malliavin calculus in the special case where the underlying space is finite dimensional. The aim of this section is to "smooth" the introduction to Malliavin calculus. In this section we introduce the finite dimensional version of mainly all concepts that will be used throughout the work. We also derive an integration by parts formula that allows to give sufficient conditions for the existence of a density for random vectors in this space. In this context, the introduced concepts are more intuitive and allows the reader to be better prepared for the next chapters.

In Section 3 we present some tools that will ease the understanding and development of the theory. The main tool is the Wiener chaos decomposition, that is, the decomposition of the space of square integrable random variables into its projections in the spaces generated by the Hermite polynomials. We also introduce the Wiener integral with respect to a white noise and define the multiple Wiener-Itô stochastic integral which result to be related to the Wiener chaos decomposition.

In Section 4 we study the fundamental operators of Malliavin calculus. These operators are: the derivative operator D, the adjoint of D or divergence operator δ and the generator of the Ornstein–Uhlenbeck semigroup L. The derivative operator extends the classical notion of derivative to the derivative of a random variable, which result in an infinite dimensional version of the derivative since our random variables will be functionals of an isonormal Gaussian process. The divergence operator is introduced as the adjoint of the derivative operator, then it is briefly introduced the Skorohod integral which coincides with the divergence operator when the underlying Hilbert space is of the form of an L^2 space. Finally, the generator of the Ornstein–Uhlenbeck semigroup is introduced and related to the other two operators.

Section 5 deals with the existence and regularity of probability densities by means of the Malliavin calculus. We use the studied theory in order to obtain explicit integration by parts formulas in terms of the fundamental operators and derive sufficient conditions for the existence and regularity of densities.

In Section 6 we use the developed theory to introduce some applications. We briefly explain the relation of the Skorohod integral with the Itô stochastic integral and apply the tools of Malliavin calculus to a practical example, inspired in the solution of the heat equation. In this practical example we give sufficient conditions for the existence of the Malliavin derivative.

1 Integration by parts and absolute continuity of probability laws

The aim of this section is to motivate the introduction of Malliavin calculus used for proving the existence and smoothnes of densities and for its computation. This section is based on the first chapter of [3].

Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ be a multiindex and define $|\alpha| = \sum_{i=1}^n \alpha_i$, then, for any function $\varphi : \mathbb{R}^n \to \mathbb{R}$ sufficiently smooth, the differential of φ of order α is denoted by

$$\partial_\alpha \varphi := \partial^{|\alpha|}_{\alpha_1,...,\alpha_n} \varphi = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

By convention, if some α_i is zero for any $i \in \{1, ..., n\}$, the corresponding partial derivative is not considered, and hence if $|\alpha| = 0$ then $\partial^\alpha \varphi = \varphi$.

Definition 1.1. Let F be a $\mathbb{R}^n$-valued random vector, $F = (F_1, ..., F_n)$ and G be an integrable random variable defined on some probability space $(\Omega, \mathcal{F}, P)$. Let α be a multiindex, then, the pair F, G is said to satisfy an *integration by parts formula of degree* α if there exists a random variable $H_\alpha(F, G) \in L^1(\Omega)$ such that

$$E[(\partial_\alpha \varphi)(F)G] = E[\varphi(F)H_\alpha(F, G)], \tag{1}$$

for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$.

The integration by parts formula (1) is recursive in the following sense. If $\alpha = \beta + \gamma$, β and γ multiindices, then $H_\alpha(F, G) = H_\gamma(F, H_\beta(F, G))$ almost surely. Indeed,

$$\begin{aligned} E[\varphi(F)H_\alpha(F,G)] &= E[(\partial_\alpha\varphi)(F)G] \\ &= E[(\partial_\gamma\varphi)(F)H_\beta(F,G)] \\ &= E[\varphi(F)H_\gamma(F, H_\beta(F,G))], \end{aligned} \tag{2}$$

for all $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, whenever there are satisfied the corresponding intermediate integration by parts formulas.

The connection of this definition with the study of probability laws and densities is given by the following proposition.

Proposition 1.1.

1. *Assume that (1) holds for $\alpha = (1, ..., 1)$ and $G = 1$. Then, the probability law of F has a density $p(x)$ with respect to the Lebesgue measure on $\mathbb{R}^n$. Moreover,*

   $$p(x) = E[\mathbb{1}_{(x \le F)} H_{(1,...,1)}(F, 1)],$$

   which in particular means that p is continuous.

2. *Assume that (1) holds for any multiindex α and $G = 1$. Then $p \in \mathcal{C}^{|\alpha|}(\mathbb{R}^n)$ and*

   $$\partial_\alpha p(x) = (-1)^{|\alpha|} E[\mathbb{1}_{(x \le F)} H_{\alpha+1}(F, 1)],$$

   where $\alpha + 1 := (\alpha_1 + 1, ..., \alpha_n + 1)$.

Proof. 1. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varphi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y)dy \in \mathcal{C}_b^\infty$. Then, applying the integration by parts formula and Fubini's theorem we obtain

$$\begin{aligned}
E[f(F)] &= E[(\partial_{(1,\dots,1)}\varphi)(F)] = E[\varphi(F)H_{(1,\dots,1)}(F,1)] \\
&= E\left[\left(\int_{-\infty}^{F_1} \cdots \int_{-\infty}^{F_n} f(y)dy\right) H_{(1,\dots,1)}(F,1)\right] \\
&= E\left[\left(\int_{\mathbb{R}^n} \mathbb{1}_{(y\le F)} f(y)dy\right) H_{(1,\dots,1)}(F,1)\right] \\
&= \int_{\mathbb{R}^n} f(y) E\left[\mathbb{1}_{(y\le F)} H_{(1,\dots,1)}(F,1)\right] dy.
\end{aligned}$$

Now let B be a bounded Borel set of $\mathbb{R}^n$ and consider the sequence of functions $f_n \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ converging pointwise to $\mathbb{1}_B$. Since the set is bounded, we can interchange limit an integral and we will have that

$$\begin{aligned}
E[\mathbb{1}_B(F)] &= E\left[\lim_{n\to\infty} f_n(F)\right] = \lim_{n\to\infty} E[f_n(F)] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^n} f_n(y) E\left[\mathbb{1}_{(y\le F)} H_{(1,\dots,1)}(F,1)\right] dy \\
&= \int_{\mathbb{R}^n} \lim_{n\to\infty} f_n(y) E\left[\mathbb{1}_{(y\le F)} H_{(1,\dots,1)}(F,1)\right] dy \\
&= \int_{\mathbb{R}^n} \mathbb{1}_B(y) E\left[\mathbb{1}_{(y\le F)} H_{(1,\dots,1)}(F,1)\right] dy.
\end{aligned} \tag{3}$$

In particular it is also true for unbounded Borel sets, taking a sequence $\{B_n\}_n$ of Borel sets such that $B_n \subseteq B_{n+1}$ $\forall n \in \mathbb{N}$ we will be able to apply the monotone convergence theorem. It happens that since the law of F is given by $P \circ F^{-1}$, we have that $P \circ F^{-1}(B) = P(F \in B) = \int_{\{\omega: F(\omega)\in B\}} dP = \int_\Omega \mathbb{1}_B(F)dP = E[\mathbb{1}_B(F)]$, for all B Borel sets of $\mathbb{R}^n$. Hence, by (3), the law of F is absolutely continuous with probability density given by $p(x) = E\left[\mathbb{1}_{(x\le F)} H_{(1,\dots,1)}(F,1)\right]$. Let's see that, in particular, $p(x)$ is continuous. We will see that both left and right limits exist and are the same.

- **Left limit:** Let $\{x_n\}_n$ be an increasing sequence converging towards x. Then, we have that the sets $[x_n, \infty)$ decrease towards $[x, \infty)$ in such a way that

$$|\mathbb{1}_{[x_n,\infty)}(F) H_{(1,\dots,1)}(F,1)| \le |H_{(1,\dots,1)}(F,1)| \in L^1(\Omega). \tag{4}$$

  Then, by dominated convergence, we have that $p(x_n) \to p(x)$, when approaching from the left.

- **Right limit:** Let now $\{x_n\}_n$ be a decreasing sequence converging towards x. In this case we have that the sets $[x_n, \infty)$ increase towards (x, ∞) instead of $[x, \infty)$. Nonetheless, as $\mathbb{1}_{[x,\infty)} = \mathbb{1}_{\{x\}} + \mathbb{1}_{(x,\infty)}$ and $E[\mathbb{1}_{\{x\}} H_{(1,\dots,1)}(F,1)] = 0$ since $P(F = x) = 0$, then

$E\left[\mathbb{1}_{(x\leq F)}H_{(1,...,1)}(F,1)\right] = E\left[\mathbb{1}_{(x<F)}H_{(1,...,1)}(F,1)\right]$. Again, by (4) we can apply dominated convergence and hence $p(x_n) \to p(x)$, when approaching from the right.

Since both limits are finite and coincide, we have that p is continuous.

2. Since it is valid for any multiindex α, in particular it is valid for the multiindex $(1,...,1)$ and hence it exist a continuous density function p. We define the weak derivative of p as the $L^1(\mathbb{R}^n)$ function h satisfying that

$$\int_{\mathbb{R}^n} \partial_\alpha \varphi p = (-1)^{|\alpha|}\int_{\mathbb{R}^n} \varphi h,$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Let $\psi_\beta \in \mathcal{C}_b^\infty$ defined for any multiindex β and satisfying $\partial_\beta \psi_\beta = f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, that is, it is the β-integral of f. Then, applying the fact that (1) is satisfied for any α and using Fubini, we obtain

$$\begin{aligned} E[f(F)] &= E[\partial_{\alpha+1}\psi_{\alpha+1}(F)] = E[\psi_{\alpha+1}(F)H_{\alpha+1}(F,1)] \\ &= E\left[\int_{-\infty}^{F_1}\cdots\int_{-\infty}^{F_n}\partial_{(1,...,1)}\psi_{\alpha+1}(x)dxH_{\alpha+1}(F,1)\right] \\ &= E\left[\int_{\mathbb{R}^n}\mathbb{1}_{(x\leq F)}\psi_\alpha(x)dxH_{\alpha+1}(F,1)\right] \\ &= \int_{\mathbb{R}^n}\psi_\alpha(x)E\left[\mathbb{1}_{(x\leq F)}H_{\alpha+1}(F,1)\right]dx. \end{aligned}$$

Since in particular it is also true that

$$E[f(F)] = E[\partial_\alpha\psi_\alpha(F)] = \int_{\mathbb{R}^n}\partial_\alpha\psi_\alpha(x)p(x)dx,$$

we will have that

$$\int_{\mathbb{R}^n}\partial_\alpha\psi_\alpha(x)p(x)dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n}\psi_\alpha(x)(-1)^{|\alpha|}E\left[\mathbb{1}_{(x\leq F)}H_{\alpha+1}(F,1)\right]dx,$$

with $\partial_\alpha\psi_\alpha = f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. We want this equallity holding for all $\psi_\alpha \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, however, for any $\psi_\alpha \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ exists an $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\partial_\alpha\psi_\alpha = f$. Hence, as the equallity holds for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we will have that the weak derivative of p is $h(x) = (-1)^{|\alpha|}E\left[\mathbb{1}_{(x\leq F)}H_{\alpha+1}(F,1)\right]$, but since h is continuous (by the same reason that before), then $p \in \mathcal{C}^{|\alpha|}(\mathbb{R}^n)$ and it will be its derivative in the usual sense.

□

There is another proposition that allows to obtain the existence of densities and its smoothness under weaker assumptions, however, it will be less informative about the specific form of the density or its derivatives. This is Malliavin's criterion on existence of densities.

Proposition 1.2. *1. Assume that for any multiindex α_i such that $|\alpha_i| = 1$ and every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, there exist positive constants C_i, not depending on φ such that*

$$|E[(\partial_{\alpha_i}\varphi)(F)]| \leq C_i||\varphi||_\infty, \tag{5}$$

then the law of F has a density.

2. Assume that for any multiindex α and every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, there exist positive constants C_α, not depending on φ such that

$$|E[(\partial_\alpha\varphi)(F)]| \leq C_\alpha||\varphi||_\infty, \tag{6}$$

then the law of F has a $\mathcal{C}^\infty$ density.

Notice that if an integration by parts formula holds, then (5) also holds. Indeed,

$$|E[(\partial_\alpha\varphi)(F)G]| = |E[\varphi(F)H_\alpha(F,G)]| \leq E[|H_\alpha(F,G)|] \; ||\varphi||_\infty,$$

where $E[|H_\alpha(F,G)|]$ is a positive constant not depending on φ.

2 Finite dimensional Malliavin calculus

In this section we will deal with random vectors defined on a finite dimensional probability space whose measure is the standard Gaussian measure. This will be the equivalent of a finite dimensional Malliavin calculus and will put into context many of the forecoming ideas. This part is mixture of what is in chapter 2 of [3] and the notes of the BGSMath course of Malliavin Calculus presented by David Nualart in the academic year 2018-2019. Most of the exposed conditions can be relaxed as shown in [3], however, we have preferred to focus in explaining a rough idea of Malliavin calculus in finite dimensions rather than a precise one.

Let $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \mu_m)$ be our working probability space, where μ_m is the standard Gaussian measure $\mu_m(dx) = \rho(x)dx$, with

$$\rho(x) = (2\pi)^{-\frac{m}{2}} \exp\left(-\frac{||x||^2}{2}\right).$$

We will denote by E_m the expectation with respect to the measure μ_m. Let $\mathcal{C}_p^k(\mathbb{R}^m)$ be the space of functions $f : \mathbb{R}^m \to \mathbb{R}$, which are k times continuously differentiable and such that the partial derivatives have at most polynomial growth, that is $\exists N \geq 1$ and $\exists C \in \mathbb{R}$ such that $||f^{(k)}(x)|| \leq C(1 + ||x||^N)$. Let D be the *differential operator* defined over $\mathcal{C}^1(\mathbb{R}^m)$ as $Df = \nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_m})$, the gradient, $\forall f \in \mathcal{C}^1(\mathbb{R}^m)$. Now let δ be the adjoint operator of D in $L^2(\mathbb{R}^m, \mu_m)$. The adjoint operator must act on functions $\varphi : \mathbb{R}^m \to \mathbb{R}^m$, such that $\delta\varphi : \mathbb{R}^m \to \mathbb{R}$ and satisfy the duality relationship

$$E_m[\langle Df, \varphi\rangle] = E_m[f\delta\varphi], \quad \forall f \in \mathcal{C}_p^1(\mathbb{R}^m), \quad \forall \varphi_i \in \mathcal{C}_p^1(\mathbb{R}^m), \ i = 1, ..., m,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^m$. Assume $f, \varphi^i \in \mathcal{C}_p^1(\mathbb{R}^m)$ for $i = 1, ..., m$, then by integration by parts we obtain

$$\begin{aligned} E_m[\langle Df, \varphi\rangle] &= E_m\left[\sum_{i=1}^m \partial_i f \varphi^i\right] = \sum_{i=1}^m \int_{\mathbb{R}^m} \partial_i f(x)\varphi^i(x)\mu(dx) \\ &= -\sum_{i=1}^m \int_{\mathbb{R}^m} f(x)\left(\partial_i\rho(x)\varphi^i(x) + \rho(x)\partial_i\varphi^i(x)\right)dx \\ &= \int_{\mathbb{R}^m} f(x)\left(\sum_{i=1}^m \left(x_i\varphi^i(x) - \partial_i\varphi^i(x)\right)\right)\rho(x)dx. \end{aligned}$$

Hence

$$\delta\varphi(x) = \sum_{i=1}^m \left(x_i\varphi^i(x) - \partial_i\varphi^i(x)\right),$$

and δ is the so called *divergence operator*.

Now we will introduce the third operator playing a role in the Malliavin calculus. It is the *Ornstein–Uhlenbeck operator*. It can be defined as a second order differential operator. Let $f \in \mathcal{C}^2(\mathbb{R}^m)$

$$Lf(x) = \sum_{i=1}^m \left(\partial^2_{x_i x_i} f(x) - x_i\partial_{x_i} f(x)\right).$$

This operator satisfies the relationship $Lf = -\delta Df$, connecting the three mentioned operators. That is

$$\delta Df(x) = \sum_{i=1}^{m} (x_i \partial_i f(x) - \partial_i \partial_i f(x)) = -Lf(x).$$

Moreover,

$$\delta(fDg) = -\langle Df, Dg \rangle - fLg. \tag{7}$$

The Ornstein–Uhlenbeck operator is the infinitesimal generator of the Ornstein–Uhlenbeck semigroup[1] $\{P_t, t \geq 0\}$. The *Ornstein–Uhlenbeck semigroup* is defined as

$$P_t f(x) = \int_{\mathbb{R}^m} f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) \mu(dy).$$

Now let's see how we can derive an integration by parts formula using the above-mentioned operators. Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a random vector, $F = (F^1, ..., F^n)$, and assume that each of the components are $\mathcal{C}_p^\infty(\mathbb{R}^m)$, then we define the *Malliavin matrix of F* as

$$\gamma_F(x) = (\langle DF^i(x), DF^j(x) \rangle)_{i,j}, \quad 1 \leq i, j \leq n.$$

Consider now $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$. We will have, using the chain rule, that

$$\partial_i(\varphi(F(x))) = \sum_{k=1}^{n} (\partial_k \varphi)(F(x)) \partial_i F^k(x),$$

hence, we can write

$$\begin{aligned}
\langle D(\varphi(F(x))), DF^j(x) \rangle &= \sum_{l=1}^{m} \sum_{k=1}^{n} (\partial_k \varphi)(F(x)) \partial_l F^k(x) \partial_l F^j \\
&= \sum_{k=1}^{n} (\partial_k \varphi)(F(x)) \langle DF^k(x), DF^j(x) \rangle \\
&= ((D\varphi)(F(x)) \gamma_F(x))_j,
\end{aligned}$$

for any $j \in \{1, ..., n\}$. Now, if we assume that γ_F is invertible μ-almost everywhere, then

$$(D\varphi)(F(x)) = D(\varphi(F(x)))(DF(x))^T \gamma_F^{-1}(x),$$

[1] The definition of infinitesimal generator of a semigroup and other related concepts can be found in the appendix.

where $(DF(x))^T$ is the transposed differential matrix of F. This means that

$$\begin{aligned}E_m[(\partial_i\varphi)(F)] &= \sum_{j=1}^n E_m\left[\langle D(\varphi(F)), DF^j(\gamma_F^{-1})_{j,i}\rangle\right]\\ &= \sum_{j=1}^n E_m\left[\varphi(F)\delta(DF^j(\gamma_F^{-1})_{j,i})\right]\\ &= -\sum_{j=1}^n E_m\left[\varphi(F)\left(\langle DF^j, D((\gamma_F^{-1})_{j,i})\rangle + (\gamma_F^{-1})_{j,i}LF^j\right)\right],\end{aligned}$$

using (7). Which is an integration by parts formula if $H_i(F,1) \in L^1(\mathbb{R}^m,\mu)$, where

$$H_i(F,1) = -\sum_{j=1}^n \delta(DF^j(\gamma_F^{-1})_{i,j}) = -\sum_{j=1}^n \left(\langle DF^j, D((\gamma_F^{-1})_{i,j})\rangle + (\gamma_F^{-1})_{i,j}LF^j\right).$$

Notice that γ_F is symmetric and hence $(\gamma_F^{-1})_{i,j} = (\gamma_F^{-1})_{j,i}$.

Finally, the following proposition provides sufficient conditions for the existence of a density for the random vector F.

Proposition 2.1. *Let $F \in \mathcal{C}_p^\infty(\mathbb{R}^n)$ and assume that*

1. *The matrix $\gamma_F(x)$ is invertible μ-almost every $x \in \mathbb{R}^m$.*
2. $\det\gamma_F^{-1} \in L^p(\mathbb{R}^m,\mu)$, $D(\det\gamma_F^{-1}) \in L^q(\mathbb{R}^m,\mu)$, *for some $p,q \in (1,\infty)$.*

Then the law of F has a density.

Proof. The assumption of invertibility of the matrix was already used for the derivation of the integration by parts formulas $E_m[(\partial_i\varphi)(F)] = E_m\left[\varphi(F)H_i(F,1)\right]$. The second assumption ensures that $H_i(F,1) \in L^1(\mathbb{R}^m,\mu)$. That is, as

$$E_m[|H_i(F,1)|] \leq \sum_{j=1}^n \left(E_m\left[|\langle DF^j, D((\gamma_F^{-1})_{i,j})\rangle|\right] + E_m\left[|(\gamma_F^{-1})_{i,j}LF^j|\right]\right),$$

we just need to see that each of the expected values is finite. Since $F \in \mathcal{C}_p^\infty(\mathbb{R}^n)$ in particular $LF^j \in L^s(\mathbb{R}^m;\mu_m)$ for any $s \geq 1$, then applying Hölder's inequality we obtain that

$$E_m\left[|(\gamma_F^{-1})_{i,j}LF^j|\right] \leq \left(E_m\left[|(\gamma_F^{-1})_{i,j}|^r\right]\right)^{\frac{1}{r}} \left(E_m\left[|LF^j|^{\frac{r}{r-1}}\right]\right)^{\frac{r-1}{r}},$$

for some $1 < r < p$. As

$$(\gamma_F^{-1})_{i,j} = \frac{1}{\det\gamma_F}C_{j,i} = \det\gamma_F^{-1}C_{j,i},$$

where $C_{j,i}$ are the cofactors, that is the determinant of the matrix suppressing the j-th row and i-th column and multiplying by $(-1)^{i+j}$, we can apply again Hölder's inequality and obtain that

$$E_m\left[|(\gamma_F^{-1})_{i,j}|^r\right] \leq \left(E_m\left[|\det \gamma_F^{-1}|^{rs}\right]\right)^{\frac{1}{s}} \left(E_m\left[|C_{j,i}|^{\frac{rs}{s-1}}\right]\right)^{\frac{s-1}{s}} < \infty,$$

if $1 < s < p$ is such that $sr = p$, since by hypothesis $\det \gamma_F^{-1} \in L^p(\mathbb{R}^m; \mu_m)$ and $C_{j,i}$ is just the finite sum of finite products of terms $\langle DF^k, DF^l \rangle$, $1 \leq k, l \leq m$, which applying triangular inequality and Hölder inequality we find is in $L^s(\mathbb{R}^m; \mu_m)$ for any $s \geq 1$. Similarly, for the other term,

$$E_m\left[|\langle DF^j, D((\gamma_F^{-1})_{i,j})\rangle|\right] \leq E_m\left[||DF^j||\ ||D((\gamma_F^{-1})_{i,j})||\right],$$

which again $DF^j \in L^s(\mathbb{R}^m; \mu_m)$ for any $s \geq 1$ and we can apply Hölder inequality. Moreover, since

$$D((\gamma_F^{-1})_{i,j}) = D\left(\det \gamma_F^{-1} C_{j,i}\right) = D\left(\det \gamma_F^{-1}\right) C_{j,i} + \det \gamma_F^{-1} DC_{j,i},$$

we can apply Hölder inequality with the desired exponents since $C_{j,i}, DC_{j,i} \in L^s(\mathbb{R}^m; \mu_m)$ for any $s \geq 1$.

Once $H_i(F, 1)$ has been shown to belong to $L^1(\mathbb{R}^m, \mu)$, then, we have an integration by parts formula, and since $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ we will have

$$E_m[(\partial_i \varphi)(F)] \leq C_i ||\varphi||_\infty,$$

where $C_i := \sum_{j=1}^n E_m\left[|\langle DF^j, D((\gamma_F^{-1})_{i,j})\rangle| + |(\gamma_F^{-1})_{i,j} LF^j|\right]$. Finally, using proposition 1.2 the proof is concluded. $\square$

3 Hermite polynomials, multiple integrals and Chaos expansion

This section is an instrumental one. We will introduce some tools that will ease the understanding and development of the theory. In this section we have mainly followed [2].

First of all, we will introduce the idea of an isonormal Gaussian process, that generalizes the idea of the Brownian motion.

Let H be a real separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$.

Definition 3.1. We say that a stochastic process $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ is an *isonormal Gaussian process* if W is a centered Gaussian family of random variables such that $E[W(h)W(g)] = \langle h, g \rangle_H \ \forall h, g \in H$.

It can be seen that the Wiener-Itô integral, $W(h) = \int h(t)dW_t$, is an isonormal Gaussian process, since it is a Gaussian process and by the isometry property, $E[W(h)W(g)] = E[\int h(t)dW_t \int g(t)dW_t] = E[\int h(t)g(t)dt] = \langle h, g \rangle_H$.

3.1 Hemite Polynomials

We will introduce the Hermite polynomials and some basic properties since they will be used thoughout the work. The Hermite polynomials can be defined in several ways, however, we have opted to follow the definition of [2].

Definition 3.2. The nth *Hermite polynomial* is defined by,

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), \quad n \geq 1,$$

and $H_0(x) = 1$.

An important property of these polynomials is

$$\sum_{n=0}^{\infty} t^n H_n(x) = \exp\left(tx - \frac{t^2}{2}\right). \tag{8}$$

It can be seen by considering $f(y) = \exp\left(-\frac{y^2}{2}\right)$ and expanding $f(x-t)$ around x, that is,

$$\begin{aligned}\exp\left(-\frac{(x-t)^2}{2}\right) &= f(x-t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(-t)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}) = e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} t^n H_n(x),\end{aligned}$$

since by definition of the Hermite polynomials: $\frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}) = (-1)^n n! H_n(x) e^{-\frac{x^2}{2}}$. The result follows multiplying both sides by $e^{\frac{x^2}{2}}$. From this result, we have

$$H_n'(x) = H_{n-1}(x), \tag{9}$$
$$(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x), \tag{10}$$
$$H_n(-x) = (-1)^n H_n(x), \tag{11}$$

where $n \geq 1$. Let $F(x,t) = \exp(tx - \frac{t^2}{2})$, then (9) follows from the fact that $\frac{\partial}{\partial x}F = tF$, which means

$$\sum_{n=0}^{\infty} t^n H_n'(x) = \sum_{n=0}^{\infty} t^{n+1} H_n(x).$$

The identity (10) follows from $\frac{\partial}{\partial t}F = (x-t)F$, that is

$$\sum_{n=0}^{\infty} n t^{n-1} H_n(x) = x \sum_{n=0}^{\infty} t^n H_n(x) - \sum_{n=0}^{\infty} t^{n+1} H_n(x),$$

and (11) from $F(-x,t) = F(x,-t)$, which implies

$$\sum_{n=0}^{\infty} t^n H_n(-x) = \sum_{n=0}^{\infty} (-1)^n t^n H_n(x).$$

The three properties follow from equalling terms of the series.

Additional properties are that $H_{2k+1}(0) = 0$ and $H_{2k}(0) = \frac{(-1)^k}{2^k k!}$, and that the highest order term of $H_n(x)$ is $\frac{x^n}{n!}$.

The Hermite polynomials can be extended to the infinite dimensional case. Let Λ be the set of all sequences $(a_1, a_2, ...)$, $a_i \in \mathbb{N}$, $i \geq 1$ where only a finite number of a_i is different from zero. For any multiindex $a \in \Lambda$, the generalized Hermite polynomial $H_a(x)$ with $x \in \mathbb{R}^{\mathbb{N}}$ is defined by

$$H_a(x) = \prod_{i=1}^{\infty} H_{a_i}(x_i).$$

Now let's see some results that will be useful in the context of Malliavin calculus.

Lemma 3.1. *Let X, Y be two random variables with joint Gaussian distribution, such that $E[X] = E[Y] = 0$ and $E[X^2] = E[Y^2] = 1$. Then, for all $n, m \geq 0$ we have*

$$E[H_n(X)H_m(Y)] = \begin{cases} 0 & \text{if} \quad n \neq m. \\ \frac{1}{n!}(E[XY])^n & \text{if} \quad n = m. \end{cases}$$

Proof. We have for all $s,t \in \mathbb{R}$,

$$E\left[\exp\left(sX - \frac{s^2}{2}\right)\exp\left(tY - \frac{t^2}{2}\right)\right] \\ = E\left[\exp\left(sX+tY\right)\right]\exp\left(-\frac{1}{2}(s^2+t^2)\right) = \exp\left(stE\left[XY\right]\right), \tag{12}$$

where we have used that the expectation of the exponential of a Gaussian random variable $Z \sim N(\mu,\sigma^2)$ is $E[\exp(Z)] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$, and since in our case $Z = sX + tY$, it means that $\mu = 0$ and $\sigma^2 = E[(sX+tY)^2] = E[s^2X^2 + t^2Y^2 + 2stXY] = s^2 + t^2 + 2stE[XY]$.

Taking now the partial derivative $\frac{\partial^{n+m}}{\partial s^n \partial t^m}$ in both sides of (12) and evaluating at $s = t = 0$, we obtain

$$\left.\frac{\partial^{n+m}}{\partial s^n \partial t^m}\right|_{s=t=0} E\left[\exp\left(sX - \frac{s^2}{2}\right)\exp\left(tY - \frac{t^2}{2}\right)\right] = E[n!m!H_n(X)H_m(Y)],$$

by using (8). On the right-hand side of (12), we obtain

$$\begin{aligned} &\left.\frac{\partial^{n+m}}{\partial s^n \partial t^m}\right|_{s=t=0} \exp\left(stE\left[XY\right]\right) \\ &\qquad = \left.\frac{\partial^{n+m}}{\partial s^n \partial t^m}\right|_{s=t=0}\left(\sum_{i=0}^{\infty}\frac{(stE\left[XY\right])^i}{i!}\right) \\ &\qquad = \left[\sum_{i=\max\{n,m\}}^{\infty}\frac{n!s^{i-n}m!t^{i-m}}{i!}(E\left[XY\right])^i\right]_{s=t=0} \\ &\qquad = \begin{cases} 0 & \text{if} \quad n \neq m. \\ n!(E[XY])^n & \text{if} \quad n = m. \end{cases} \end{aligned}$$

□

Let now $\mathcal{G}$ be the σ-field generated by the random variables $\{W(h), h \in H\}$. We present without proof the following technical lemma.

Lemma 3.2. *The random variables $\{e^{W(h)}, h \in H\}$ form a total subset of $L^2(\Omega, \mathcal{G}, P)$.*

For references of the proof see [2].

3.2 Chaos expansion

The Hermite polynomials are instrumental tools for proving the chaos expansion. In this section we will see that we would be able to decompose random variables of $L^2(\Omega)$ into its projections in the spaces generated by the Hermite polynomials. Throughout this section we will consider $\mathcal{G}$ to be the σ-field generated by the random variables $\{W(h), h \in H\}$.

Definition 3.3. For each $n \geq 0$ we denote by $\mathcal{H}_n$ the *Wiener chaos of order* n, defined by the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{H_n(W(h)), h \in H, ||h||_H = 1\}$ whenever $n \geq 1$, and the set of constants when $n = 0$.

We notice that $\mathcal{H}_1$ is the set of zero-mean Gaussian random variables given by $\{W(h), h \in H\}$. Moreover, by Lemma 3.1 the spaces $\mathcal{H}_n$ and $\mathcal{H}_m$ are orthogonal whenever $n \neq m$. We will see in fact, that the Wiener chaos not only are orthogonal, but also generate a whole space of interest.

Theorem 3.1. *The space $L^2(\Omega, \mathcal{G}, P)$ can be decomposed into the infinite sum of orthogonal Wiener chaos $\mathcal{H}_n$:*

$$L^2(\Omega, \mathcal{G}, P) = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

Proof. We already know that the Wiener chaos are orthogonal and, moreover, since $\mathcal{H}_n \subset L^2(\Omega, \mathcal{G}, P)$ for all $n \geq 0$ then $\oplus_{n=0}^{\infty} \mathcal{H}_n \subseteq L^2(\Omega, \mathcal{G}, P)$. We just need to prove that for any random variable of $L^2(\Omega, \mathcal{G}, P)$ exists a non-zero projection over some Wiener chaos. Let's assume that this is not the case, that is, that exists $X \in L^2(\Omega, \mathcal{G}, P)$ such that is orthogonal to $\mathcal{H}_n$ for any n. That means that $E[XH_n(W(h))] = 0$ for all n and for all $h \in H$ with $||h||_H = 1$. Since x^n can be expressed as a linear combination of Hermite polynomials, we have in fact that $E[XW(h)^n] = 0$ for all $n \geq 0$, which implies that in fact it should be orthogonal to the set $\{e^{W(h)}\}$, but by Lemma 3.2 this is a total subset of $L^2(\Omega, \mathcal{G}, P)$ which means that $X = 0$. □

For any infinite multiindex $a \in \Lambda$, let's define

$$\Phi_a = \sqrt{a!} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)), \tag{13}$$

where $\{e_i, i \geq 1\}$ is an orthonormal basis of H.

Proposition 3.1. *The family of random variables $\{\Phi_a, a \in \Lambda\}$ form a complete orthonormal system in $L^2(\Omega, \mathcal{G}, P)$.*

Proof. Let's first see that it is an orthonormal system. Let $a, b \in \Lambda$, then

$$E[\Phi_a \Phi_b] = \sqrt{a!}\sqrt{b!} \prod_{i=1}^{\infty} E\left[H_{a_i}(W(e_i)) H_{b_i}(W(e_i))\right] = \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{if} \quad a \neq b \end{cases},$$

by Lemma 3.1.

Let's see now that it spans $L^2(\Omega, \mathcal{G}, P)$. Let $\mathcal{P}_n^0$ the space of random variables of the form $p(W(h_1), ..., W(h_k))$ where $k \geq 1$, $h_1, ..., h_k \in H$, and p is a real valued polynomial of k variables with degree at most n. Let now $\mathcal{P}_n$ be the L^2-closure of $\mathcal{P}_n^0$.

We have that $\oplus_{i=1}^{n} \mathcal{H}_i \subseteq \mathcal{P}_n$. Let's see that in fact $\oplus_{i=1}^{n} \mathcal{H}_i = \mathcal{P}_n$, that is, we have to see that $\mathcal{P}_n \subseteq \oplus_{i=1}^{n} \mathcal{H}_i$. That means that for any $X \in \mathcal{P}_n$, $X \in \oplus_{i=1}^{n} \mathcal{H}_i$

and hence, since Wiener chaos of different order are orthogonal, it would mean that $E[XH_m(W(h))] = 0$ for all $m > n$ and all $h \in H$, $||h||_H = 1$. In fact it is only necessary to prove that for X of the form $p(W(h_1), ..., W(h_k))$, a polynomial of degree less or equal than n.

As for any $h \in H$ it is possible to rewrite $X = p(W(h_1), ..., W(h_k))$ as $q(W(e'_1), ..., W(e'_j), W(h))$, another polynomial of degree less or equal than n, and $\{e'_1, ..., e'_j, h\}$ an orthonormal family in H (by using Gram-Schmidt orthogonalization method), we will have that $W(e'_1), ..., W(e'_j)$, $W(h)$ are independent Gaussian variables. Then, it only remains to check that the terms $E[W(h)^r H_m(W(h))] = 0$ for all $r \leq n < m$. But since x^r can be expressed as a linear combination of Hermite polynomials of degree less or equal than r it follows immediately that $E[W(h)^r H_m(W(h))] = 0$.

The proof is concluded noticing that the random variables $\{\Phi_a, a \in \Lambda, |a| = n\}$ belong to $\mathcal{P}_n$ and that we can express any of the e'_i and h with the basis $\{e_i, i \geq 1\}$. That is, we can approximate $q(W(e'_1), ..., W(e'_j), W(h))$ with polynomials in $W(e_i)$.

As $L^2(\Omega, \mathcal{G}, P) = \oplus_{n=0}^{\infty} \mathcal{H}_n$, then the family of random variables $\{\Phi_a, a \in \Lambda\}$ also span $L^2(\Omega, \mathcal{G}, P)$. □

3.3 Multiple Wiener-Itô integrals

In this section we will consider an important case, when the Hilbert space H is an L^2 space is of the form $L^2(T, \mathcal{B}, \mu)$, where $(T, \mathcal{B})$ is a measurable space and μ is a σ-finite measure without atoms, that is, it does not exist any $A \in \mathcal{B}$, $\mu(A) < \infty$ such that if $B \in \mathcal{B}$, $B \subseteq A$ then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

Let's first introduce the notion of white noise.

Definition 3.4. A *white noise based on* μ is a random set function W on the sets $A \in \mathcal{B}$, such that $\mu(A) < \infty$, satisfying

(i) $W(A)$ follows a distribution $N(0, \mu(A))$,

(ii) if $A, B \in \mathcal{B}$ are disjoint, then $W(A)$ and $W(B)$ are independent.

As simple functions are dense in L^2, we can characterize the isonormal Gaussian process W by the family of random variables $\{W(A), A \in \mathcal{B}, \mu(A) < \infty\}$, where $W(A) := W(\mathbb{1}_A)$. Let's see that indeed this process defines a white noise based on μ. Each of the random variables $W(A)$ follows a distribution $N(0, \mu(A))$. Moreover, if $A, B \in \mathcal{B}$ are disjoint, then $E[W(A)W(B)] = \langle \mathbb{1}_A, \mathbb{1}_B \rangle = 0$ and since it is a Gaussian process, $W(A)$ and $W(B)$ are independent.

Wiener integral with respect to W

We want to define the wiener integral with respect to the white noise W.

Step 1 Let's consider simple functions, that is, functions of the form

$$h_n = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}, \tag{14}$$

$A_i \in \mathcal{B}$ pairwise disjoint. Then, the Wiener integral is defined as $\int_T h_n dW = \sum_{i=1}^n a_i W(A_i)$, based on the white noise.

Step 2 Let h be any element of $L^2(T)$ and let $(h_n)_n$ be a sequence of simple functions of the form (14) converging to h in $L^2(T)$. Then, we will define the Wiener integral of h, $\int_t hdW$, as the $L^2(\Omega)$-limit of the sequence of Gaussian random variables $\left(\int_T h_n dW\right)_n$.

Hence, the random variable $W(h)$, $h \in L^2(T)$, coincides with the Wiener stochastic integral $\int_T hdW$ that we have just defined.

In the case where $T = [0,1]$ and μ is the Lebesgue measure on $[0,1]$, then we can consider $W(t) = W([0,t))$, $t \in [0,1]$. Considering a continuous version of $B = \{W(t), t \in [0,1]\}$, we will have that B coincides with the standard Brownian motion. Moreover, $W(h)$ coincides with $\int_0^1 h(s)dB_s$, the stochastic integral with respect to the Brownian motion.

Multiple Wiener integrals

Let $m \geq 1$ fixed, we denote by $\mathcal{E}_m \subset L^2(T^m, \mathcal{B}^m, \mu^m)$ the set of elementary functions of the form

$$f(t_1, ..., t_m) = \sum_{i_1,...,i_m=1}^{n} a_{i_1 \dots i_m} \mathbb{1}_{A_{i_1} \times \dots \times A_{i_m}}(t_1, ..., t_m), \tag{15}$$

where $A_1, ..., A_n$ are pairwise-disjoint sets belonging to $\mathcal{B}_0 = \{A \in \mathcal{B} : \mu(A) < \infty\}$ and the coefficients are zero whenever two indices are equal, that is, f vanishes on the diagonal. For a function $f \in \mathcal{E}_m$ of the form (15), the multiple Wiener-Itô integral is defined as

$$I_m(f) = \sum_{i_1,...,i_m=1}^{n} a_{i_1 \dots i_m} W(A_{i_1}) \cdots W(A_{i_m}).$$

The following properties hold:

(i) I_m is linear,

(ii) $I_m(f) = I_m(\tilde{f})$, where $\tilde{f}(t_1, ..., t_m) = \frac{1}{m!} \sum_\sigma f(t_{\sigma(1)}, ..., t_{\sigma(m)})$, with σ running over all permutations of $\{1, ..., m\}$, is the so called symmetrization of f,

(iii) $E[I_m(f)I_q(g)] = \begin{cases} m! \langle \tilde{f}, \tilde{g} \rangle_{L^2(T^m)} & \text{if} \quad m = q, \\ 0 & \text{if} \quad m \neq q. \end{cases}$

We are not giving the proof of these properties, but they can be found in reference [2].

In order to define the stochastic integral in $L^2(T^m)$, it has to be seen that $\mathcal{E}_m$ is dense in $L^2(T^m)$. Since we know that the set of finite linear combination of indicator functions is dense in $L^2(T^m)$, it suffice to show that it can be approximated the indicator function of any set $A = A_1 \times \cdots \times A_m$, $A_i \in \mathcal{B}_0$, $1 \leq i \leq m$, by functions of $\mathcal{E}_m$. Since μ is non-atomic, for any $\epsilon > 0$ we can obtain a system of pairwise-disjoint sets $\{B_1, ..., B_n\} \subset \mathcal{B}_0$, such that $\mu(B_i) < \epsilon$ $\forall i \in \{1, ..., n\}$ and satisfying that each A_k can be expressed as the disjoint union of some of the B_j. Then we have that

$$\mathbb{1}_A = \sum_{i_1,...,i_m=1}^{n} \varepsilon_{i_1...i_m} \mathbb{1}_{B_{i_1}\times\cdots\times B_{i_m}},$$

with $\varepsilon_{i_1...i_m}$ is either 0 or 1. Now we can divide the sum over two index sets, the index set I, which is the set of $m$ples $(i_1, ..., i_m)$ with pairwise different indices i_k, and the index set J, which is composed of the remaining $m$ples. We can define then the elementary function

$$\mathbb{1}_B = \sum_{(i_1,...,i_m)\in I} \varepsilon_{i_1...i_m} \mathbb{1}_{B_{i_1}\times\cdots\times B_{i_m}} \in \mathcal{E}_m.$$

We will have then

$$\begin{aligned} ||\mathbb{1}_A - \mathbb{1}_B||_{L^2(T^m)} &= \sum_{(i_1,...,i_m)\in J} \varepsilon_{i_1...i_m} \mu(B_{i_1}) \cdots \mu(B_{i_m}) \\ &\leq \binom{m}{2} \sum_{i=1}^{n} \mu(B_i)^2 \left(\sum_{i=1}^{n} \mu(B_i) \right)^{m-2} \leq \binom{m}{2} \epsilon \alpha^{m-1}, \end{aligned}$$

where $\alpha = \mu\left(\bigcup_{i=1}^{m} A_i\right)$. In the first inequality, we have made groups of pairwise equal indices, since at least a pair of indices has to be repeated, and extend the remaining set over all indices. In the second inequality we have used that $\mu(B_i) < \epsilon$ $\forall i \in \{1, ..., n\}$ and that $\sum_{i=1}^{n} \mu(B_i) = \alpha$, since the B_i sets are pairwise disjoints and such that each A_j can be expressed as a disjoint union of some B_k. This means that $\mathcal{E}_m$ is dense in $L^2(T^m)$.

From property (iii), we obtain that $E[I_m(f)^2] = m!||\tilde{f}||^2_{L^2(T^m)} \leq m!||f||^2_{L^2(T^m)}$, by triangular inequality. Hence, using a density argument and this previous fact, the operator I_m can be extended to a linear and continuous operator $I_m : L^2(T^m) \to L^2(\Omega, \mathcal{F}, P)$ satisfying the three mentioned properties. We will write $I_m(f) = \int_{T^m} f(t_1, ..., t_m) W(dt_1) \cdots W(dt_m)$.

Now, we will introduce the contraction of indices, that will be used for multiplication of multiple integrals.

Definition 3.5. Let $f \in L^2(T^p)$, $g \in L^2(T^q)$ be symmetric functions, for any $1 \leq r \leq \min(p, q)$, the *contraction of* r *indices* of f and g, denoted by $f \otimes_r g$, is defined by

$$(f \otimes_r g)(t_1, ..., t_{p+q-2r})$$
$$= \int_{T^r} f(t_1, ..., t_{p-r}, s_1, ..., s_r) g(t_{p-r+1}, ..., t_{p+q-2r}, s_1, ..., s_r) \mu(ds_1) \cdots \mu(ds_r).$$

Notice that $f \otimes_r g \in L^2(T^{p+q-2r})$ and that although f and g are symmetric, the contraction is not necessarily symmetric. We will denote by $f \tilde{\otimes}_r g$ the symmetrization of $f \otimes_r g$.

Proposition 3.2. *Let $f \in L^2(T^p)$, $g \in L^2(T^q)$ be symmetric functions, then*

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{16}$$

Corollary 3.1. Let $f \in L^2(T^p)$ be a symmetric functions and $g \in L^2(T)$, then

$$I_p(f) I_1(g) = I_{p+1}(f \otimes g) + p I_{p-1}(f \otimes_1 g). \tag{17}$$

Here we identify the contraction of $r = 0$ indices $\otimes_0$ with the tensor product $\otimes$. These propositions are introduced here without proof, the reader can find the proof at [2] in Proposition 1.1.3 and 1.1.2 respectively.

Let's see, that in fact, there is a relationship between the multiple integral and the Hermite polynomials.

Proposition 3.3. *Let $H_m(x)$ be the m-th Hermite polynomial, and let $h \in H = L^2(T)$ be an element of norm one. Then*

$$m! H_m(W(h)) = \int_{T^m} h(t_1) \cdots h(t_m) W(dt_1) \cdots W(dt_m).$$

Hence, the multiple integral I_m maps $L^2(T^m)$ onto the Wiener chaos $\mathcal{H}_m$.

Proof. Let's proceed by induction. For $m = 1$ it is trivial. Assume that it holds up to $m - 1$ and let $h^{\otimes m}(t_1, ..., t_m) = h(t_1) \cdots h(t_m)$. The identity (17) yields

$$I_m(h^{\otimes m}) = I_{m-1}(h^{\otimes(m-1)}) I_1(h) - (m-1) I_{m-2} \left(h^{\otimes(m-2)} \int_T h(s) \mu(ds) \right)$$
$$= (m-1)! H_{m-1}(W(h)) W(h) - (m-1)(m-2)! H_{m-2}(W(h)) = m! H_m(W(h)),$$

where we have used that h is of norm one and the recursiveness of the Hermite polynomials (10).

We have that the multiple integral satisfies $E[I_m(f)^2] = m! ||f||^2_{L^2(T^m)}$ for all symmetric function f. Since the set of symmetric functions, say $L^2_S(T^m)$, is a closed subspace of $L^2(T^m)$, we have that $I_m(L^2_S(T^m))$ is a closed subspace of $L^2(\Omega, \mathcal{G}, P)$ such that $\mathcal{H}_m \subseteq I_m(L^2_S(T^m))$ as we have just seen. As the multiple integrals of any order are orthogonal, it means that in particular they are orthogonal to any Wiener chaos of order different of m, and since these span $L^2(\Omega, \mathcal{G}, P)$ it means that in fact $\mathcal{H}_m = I_m(L^2_S(T^m))$. □

Proposition 3.4. *Any square integrable random variable $F \in L^2(\Omega, \mathcal{G}, P)$ can be expanded into a series of multiple stochastic integrals*

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{18}$$

where $f_0 = E[F]$ and I_0 is the identity map. Furthermore, we can assume that the functions f_n are symmetric and, in this case, uniquely determined by F.

Proof. From Theorem 3.1 and Proposition 3.3 it is immediate that exist such f_n satisfying (18). The uniqueness in the case of symmetric functions f_n follows from property (iii) of the multiple integral. □

4 The fundamental operators of Malliavin calculus: Ornstein–Uhlenbeck, derivative, divergenge

In this section we will present the main theory of this thesis, we will introduce the fundamental operators of Malliavin calculus. The framework of this section will be $W = \{W(h), h \in H\}$ an isonormal Gaussian process associated with the Hilbert space H and the underlying probability space $(\Omega, \mathcal{F}, P)$, where in this case $\mathcal{F}$ will be the σ-field generated by W.

4.1 The derivative operator

The derivative operator extends the classical definition of derivative to the derivative of a random variable. In fact, it will be the infinite-dimensional version, since our random variables will be functionals of an isonormal Gaussian process.

Let $\mathcal{S}$ denote the class of smooth random variables such that any random variable $F \in \mathcal{S}$ has the form

$$F = f(W(h_1), ..., W(h_n)), \tag{19}$$

where $f \in \mathcal{C}_p^\infty(\mathbb{R}^n)$, $h_1, ..., h_n \in H$ and $n \geq 1$.

Now let's proceed with the definition of derivative in this particular class of random variables.

Definition 4.1. The *derivative* of a smooth random variable $F \in \mathcal{S}$ of the form (19), is the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), ..., W(h_n))h_i. \tag{20}$$

We will interpret then, $\langle DF, h\rangle_H$ as the directional derivative along the direction h.

Lemma 4.1. *Let $F \in \mathcal{S}$ and $h \in H$, then*

$$E[\langle DF, h\rangle_H] = E[FW(h)]. \tag{21}$$

Proof. We can assume that $||h||_H = 1$ and that there exist orthonormal elements of H, $e_1, ..., e_n$ such that $h = e_1$ and $F = f(W(e_1), ..., W(e_n))$, $f \in \mathcal{C}_p^\infty(\mathbb{R}^n)$. Let $\phi(x) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right)$, the density of a multidimensional standard Gaussian distribution. Then

$$\begin{aligned}
E[\langle DF, h\rangle_H] &= E\left[\sum_{i=1}^{n} \partial_i f(W(e_1), ..., W(e_n))\langle e_i, e_1\rangle\right] \\
&= E[\partial_1 f(W(e_1), ..., W(e_n))] = \int_{\mathbb{R}^n} \partial_1 f(x)\phi(x)dx \\
&= \int_{\mathbb{R}^n} f(x)\phi(x)x_1 dx = E[FW(h)],
\end{aligned}$$

where we have performed an integration by parts in the 4-th equality. □

We can extend this lemma to a more general one.

Lemma 4.2. *Let $F, G \in \mathcal{S}$ and let $h \in H$, then*

$$E[G\langle DF, h\rangle_H] = E[-F\langle DG, h\rangle_H + FGW(h)]. \quad (22)$$

Proof. Since FG is also a smooth random variable, we can apply the previous lemma to FG and we will obtain that

$$E[FGW(h)] = E[\langle D(FG), h\rangle_H] = E[G\langle DF, h\rangle_H] + E[F\langle DG, h\rangle_H],$$

because on $\mathcal{S}$, D satisfies the chain rule[2] (it can be easily checked using the definition). □

This, in fact, implies the following important result allowing us to extend the derivative.

Proposition 4.1. *The operator $D : \mathcal{S} \subset L^p(\Omega) \to L^p(\Omega; H)$ is closable for any $p \geq 1$.*

Proof. Let's suppose that $(F_n)_n$ is a sequence of smooth random variables converging to zero in $L^p(\Omega)$ and $(DF_n)_n$ is the sequence of its derivatives that is convergent to some value η in $L^p(\Omega; H)$. In order to prove that the operator D is closable from $L^p(\Omega)$ to $L^p(\Omega; H)$, we have to show that under these assumptions, $\eta = 0$, ω-a.s..

By Lemma 4.2, and since $\eta \in L^p(\Omega; H)$, we will have that for any $G \in \mathcal{S}_b$ such that $GW(h)$ is bounded:

$$\begin{aligned} E[G\langle \eta, h\rangle_H] &= \lim_{N\to\infty} E[G\langle DF_N, h\rangle_H] \\ &= \lim_{N\to\infty} E[-F_N\langle DG, h\rangle_H + F_N GW(h)] = 0, \end{aligned}$$

since the random variables $\langle DG, h\rangle_H$ and $GW(h)$ have been chosen to be bounded, then, we can apply Hölder inequality if necessary and as F_N converges to zero in L^p the result follows. Since the result is true for any G, by density, $\langle \eta, h\rangle_H = 0$ for any $h \in H$, which means that $\eta = 0$.

It is not immediate to see that the class of random variables $G \in \mathcal{S}_b$ satisfying that $GW(h)$ is bounded, is dense in $L^p(\Omega)$. Let's see that indeed it is the case. As $\mathcal{S}_b$ is dense in $L^p(\Omega)$, it suffices to show that our class of random variables is dense in $\mathcal{S}_b$. Let F be any random variable in $\mathcal{S}_b$, then, it exists a sequence of smooth random variables

$$G_N = \begin{cases} F & \text{if} \quad |FW(h)| < N, \\ 0 & \text{if} \quad |FW(h)| > N+1, \end{cases}$$

[2]A more general statement about the chain rule can be found in Proposition 4.4.

taking a smooth decreasing towards zero when $N \leq |FW(h)| \leq N+1$. Clearly each $G_N W(h)$ is bounded and such that

$$E[|F - G_N|^p] \leq E[\mathbb{1}_{\{|FW(h)|>N\}}|F|^p] \leq P(\{|FW(h)| > N\})||F||_\infty^p \xrightarrow[N\to\infty]{} 0,$$

using Hölder's inequality, the fact that F is bounded and by Markov's inequality

$$P(\{|FW(h)| > N\}) \leq \frac{E[|FW(h)|]}{N} \leq \frac{\left(E[|F|^2]\right)^{\frac{1}{2}}\left(E[|W(h)|^2]\right)^{\frac{1}{2}}}{N} \xrightarrow[N\to\infty]{} 0,$$

since both F and $W(h)$ have finite moments of any order, in particular finite second order moments.

□

Let $\mathbb{D}^{1,p}$ be the closure of $\mathcal{S}$ with respect to the seminorm

$$||F||_{1,p} = \left(E[|F|^p] + E[||DF||_H^p]\right)^{\frac{1}{p}}.$$

By Proposition 4.1, $\mathbb{D}^{1,p}$ is the domain of the closure of the operator D, that we will denote again by D. The k-th derivative operator D^k can be defined in a recursive way, such that if $F \in \mathcal{S}$, then $D^k F$ is an $H^{\otimes k}$-valued random variable. Considering now the seminorms

$$||F||_{k,p} = \left(E[|F|^p] + \sum_{i=1}^{k} E[||D^i F||_{H^{\otimes i}}^p]\right)^{\frac{1}{p}},$$

for any $1 \leq p < \infty$, $1 \leq k < \infty$ and $F \in \mathcal{S}$, the operator $D^k : \mathcal{S} \subset L^p(\Omega) \to L^p(\Omega; H^{\otimes k})$ is closable for any $p \geq 1$. We will denote the domain of the closed extension of D^k (that we will just denote by the same symbol), $\mathbb{D}^{k,p}$ which will be the closure of $\mathcal{S}$ with respect to the seminorm $||\cdot||_{k,p}$. It happens that $||F||_{k,p} \leq ||F||_{k',p'}$ for any F whenever $k \leq k'$ and $p \leq p'$. That means in particular that $\mathbb{D}^{k',p'} \subset \mathbb{D}^{k,p}$ if $k \leq k'$ and $p \leq p'$.

Given an element $h \in H$, the operator $D_h : \mathcal{S} \subset L^p(\Omega) \to L^p(\Omega)$ can be defined as $D_h F = \langle DF, h\rangle_H$. This operator is also closable for all $p \geq 1$ and its domain will be denoted by $\mathbb{D}^{h,p}$.

Now, we will characterize the differential operator in $\mathbb{D}^{1,2}$ using the Wiener chaos introduced previously.

Proposition 4.2. *Let $F \in L^2(\Omega)$ with Wiener chaos expansion $F = \sum_{n=0}^{\infty} J_n F$, where $J_n F$ denotes the projection of F into the nth Wiener chaos. Then, $F \in \mathbb{D}^{1,2}$ if and only if*

$$\sum_{n=1}^{\infty} n||J_n F||_2^2 < \infty.$$

In this case,

$$D(J_n F) = J_{n-1}(DF), \quad \forall n \geq 1,$$

and hence

$$E[||DF||_H^2] = \sum_{n=1}^{\infty} n||J_n F||_2^2.$$

Proof. Considering a random variable ϕ_a of the form (13), that is

$$\Phi_a = \sqrt{a!}\prod_{i=1}^{\infty} H_{a_i}(W(e_i)),$$

we can compute its derivative just using the definition, which yields

$$D(\Phi_a) = \sqrt{a!}\sum_{j=1}^{\infty}\prod_{i=1,i\neq j}^{\infty} H_{a_i}(W(e_i))H_{a_j-1}(W(e_j))e_j,$$

since $H'_{a_j} = H_{a_j-1}$ if $a_j \geq 1$ and zero otherwise. If $|a| = n$ then $D(\Phi_a) \in \mathcal{H}_{n-1}$, the Wiener chaos of order $n-1$. In this case

$$\begin{aligned} E[||D(\Phi_a)||_H^2] &= E\left[a!\sum_{j=1}^{\infty}\prod_{i=1,i\neq j}^{\infty} H_{a_i}^2(W(e_i))H_{a_j-1}^2(W(e_j))\right] \\ &= a!\sum_{j=1}^{\infty}\prod_{i=1,i\neq j}^{\infty} E[H_{a_i}^2(W(e_i))]E[H_{a_j-1}^2(W(e_j))] \\ &= \sum_{j=1}^{\infty}\frac{a!}{\prod_{i=1,i\neq j}^{\infty} a_i!(a_j-1)!} = \sum_{j=1}^{\infty} a_j = |a| = n. \end{aligned}$$

As by Proposition 3.1, $\{\Phi_a, a \in \Lambda\}$ form a complete orthonormal system, the result follows. □

By iteration we have the following proposition.

Proposition 4.3. *Let $F \in L^2(\Omega)$ with Wiener chaos expansion $F = \sum_{n=0}^{\infty} J_n F$. Then $F \in \mathbb{D}^{k,2}$ if and only if*

$$\sum_{n=1}^{\infty} n^k ||J_n F||_2^2 < \infty.$$

In this case,

$$D^k(J_n F) = J_{n-k}(D^k F), \quad \forall n \geq k,$$

and hence

$$E[||D^k F||_{H^{\otimes k}}^2] = \sum_{n=k}^{\infty}\frac{n!}{(n-k)!}||J_n F||_2^2.$$

Notice, that although we are denoting the projection into the Wiener chaos of order n as $J_n(\cdot)$ for both the variable and the derivatives, they are projecting into different spaces, since the random variable is of $L^2(\Omega)$ and the k-th derivative of $L^2(\Omega; H^{\otimes k})$.

In the following statement we introduce a chain rule for the derivative operator.

Proposition 4.4 (Chain rule)**.** *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives and let $F = (F^1, ..., F^m)$ be a random vector such that its components belong to $\mathbb{D}^{1,p}$. Then, $\varphi(F) \in \mathbb{D}^{1,p}$, and*

$$D(\varphi(F)) = \sum_{i=1}^{m} \partial_i \varphi(F) DF^i.$$

For the proof of this proposition it is only necessary to consider that $F^i \in \mathcal{S}$, in which case it is easily obtained from the usual chain rule for real-valued functions, then the result follows by density. The chain rule can be extended to Lipschitz functions.

Proposition 4.5 (Extended chain rule)**.** *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a globally Lipschitz and let $F = (F^1, ..., F^m)$ be a random vector such that its components belong to $\mathbb{D}^{1,2}$. Then, $\varphi(F) \in \mathbb{D}^{1,2}$, and there exists a bounded random vector $G = G_1, ..., G_m)$ such that*

$$D(\varphi(F)) = \sum_{i=1}^{m} G_i DF^i.$$

It can be obtained for any p if considering different assumptions.

Proposition 4.6. *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a globally Lipschitz and let $F = (F^1, ..., F^m)$ be a random vector such that its components belong to $\mathbb{D}^{1,p}$, $p > 1$. Then, if the law of F is absolutely continuous with respect to the Lebesgue measure on R^m, $\varphi(F) \in \mathbb{D}^{1,p}$, and*

$$D(\varphi(F)) = \sum_{i=1}^{m} \partial_i \varphi(F) DF^i.$$

The proofs of these results can be found in [2] and [1] respectively. The chain rule can be iterated to obtain the Leibniz's rule. For instance, letting $F \in \mathbb{D}^{k,p}$, $\varphi \in \mathcal{C}_p^{\infty}(\mathbb{R})$

$$D^k(\varphi(F)) = \sum_{l=1}^{k} \sum_{\mathcal{P}_l} c_l \varphi^{(l)}(F) \prod_{i=1}^{l} D^{|p_i|} F, \tag{23}$$

where $\mathcal{P}_l$ is the set of partitions of $\{1, ..., k\}$ consisting of l disjoint sets $p_1, ..., p_l$, $l = 1, ..., k$, $|p_i|$ is the cardinality of p_i and c_l are positive constants.

4.2 The divergence operator

In this section we will introduce the divergence operator, which will be defined as the adjoint operator of the derivative operator. We will consider then, the derivative operator D defined on the dense subset of $L^2(\Omega)$, $\mathbb{D}^{1,2}$.

Definition 4.2. We denote by δ the adjoint operator of the operator D, and we call it the *divergence operator*. That is, δ is an unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$ such that:

1. The domain of delta, Dom δ, is the set of H-valued square integrable random variables $u \in L^2(\Omega; H)$ such that

$$|E[\langle DF, u\rangle_H]| \leq c||F||_2,$$

for all $F \in \mathbb{D}^{1,2}$, and where c is a constant depending on u.

2. If $u \in$ Dom δ, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by

$$E[F\delta(u)] = E[\langle DF, u\rangle_H], \tag{24}$$

for any $F \in \mathbb{D}^{1,2}$.

We will refer to (24) as the duality property. We can observe, that if we consider $F = 1$, as $DF = 0$, by the duality property $E[\delta(u)] = 0$ for all $u \in$ Dom δ. We can also notice that the divergence operator is linear, again, using the duality property.

Let $\mathcal{S}_H \subset L^2(\Omega; H)$ be the class of smooth elementary elements of the form

$$u = \sum_{j=1}^{n} F_j h_j, \quad F_j \in \mathcal{S}, \quad h_j \in H. \tag{25}$$

Given any $u \in \mathcal{S}_H$ of the form (25), using Lemma 4.2, we obtain

$$\begin{aligned} E[\langle DF, u\rangle_H] &= \sum_{j=1}^{n} E[\langle DF, F_j h_j\rangle_H] = \sum_{j=1}^{n} E[F_j \langle DF, h_j\rangle_H] \\ &= \sum_{j=1}^{n} E[-F\langle DF_j, h_j\rangle_H + FF_j W(h_j)] \\ &= E\left[F\left(\sum_{j=1}^{n} F_j W(h_j) - \langle DF_j, h_j\rangle_H\right)\right], \end{aligned}$$

for any $F \in \mathcal{S}$. In particular $|E[\langle DF, u\rangle_H]| \leq c||F||_2$, using Hölder's inequality. Since $\mathcal{S}$ is dense in $L^2(\Omega)$, we will have that $u \in$ Dom δ and

$$\delta(u) = \sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j\rangle_H. \tag{26}$$

Notice that if $n = 1$, then $u = Fh$ and if $F = 1$, then we have the divergence operator of the H-valued element $u = h$ is $\delta(h) = W(h)$.

Proposition 4.7. *Let $u \in \mathcal{S}_H$, then*

$$D_h(\delta(u)) = \langle u, h\rangle_H + \delta(D_h u).$$

Proof. Assuming u of the form (25), by (26) we have

$$D_h(\delta(u)) = \sum_{j=1}^{n}(D_h F_j)W(h_j) + \sum_{j=1}^{n} F_j\langle h_j, h\rangle_H - \sum_{j=1}^{n} D_h(D_{h_j}F_j).$$

We have that $\sum_{j=1}^{n} F_j\langle h_j, h\rangle_H = \langle u, h\rangle_H$, then, if we show that $D_h(D_{h_j}F_j) = D_{h_j}(D_h F_j)$, by (26) we will see that the remaining terms are $\delta(D_h u)$ and the proof will be concluded. Let $F_j = f_j(W(h_{j_1}), ..., W(h_{j_m}))$, then we will have $DF_j = \sum_{i=1}^{m} \partial_i f_j(W(h_1), ..., W(h_m))h_i$ and hence

$$D_h(D_{h_j}F_j) = \sum_{k=1}^{m}\sum_{i=1}^{m} \frac{\partial^2}{\partial_i\partial_k} f_j(W(h_1), ..., W(h_m))\langle h_i, h_j\rangle_H \langle h_k, h\rangle_H =$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{m} \frac{\partial^2}{\partial_k\partial_i} f_j(W(h_1), ..., W(h_m))\langle h_k, h\rangle_H \langle h_i, h_j\rangle_H = D_{h_j}(D_h F_j).$$

□

Let's denote by $\mathbb{D}^{k,p}(H)$ the domain of the closed extension of $D^k : \mathcal{S}_H \subset L^p(\Omega; H) \to L^p(\Omega; H \otimes H^{\otimes k})$ with respect to the semi-norm $||\cdot||_{k,p,H}$ defined by

$$||u||_{k,p,H} = \left(E[||u||_H^p] + \sum_{i=1}^{k} E[||D^i u||^p_{H\otimes H^{\otimes i}}]\right)^{\frac{1}{p}},$$

for any $1 \leq p < \infty$, $1 \leq k < \infty$ and $u \in \mathcal{S}_H$.

Proposition 4.8. *If $u \in \mathbb{D}^{1,2}(H)$, then $u \in$ Dom δ. Moreover, if $u, v \in \mathbb{D}^{1,2}(H)$, then*

$$E[\delta(u)\delta(v)] = E[\langle u, v\rangle_H] + E[Tr(Du \circ Dv)], \tag{27}$$

where $Tr(Du \circ Dv) = \sum_{i,j=1}^{\infty} D_{e_j}\langle u, e_i\rangle_H D_{e_i}\langle v, e_j\rangle_H$, with $(e_j, i \geq 1)$ a complete orthonormal system in H.

Proof. Let's assume that $u, v \in \mathcal{S}_H$, then, using the duality property (24) and

(26)

$$\begin{aligned}
E[\delta(u)\delta(v)] &= E[\langle D(\delta(u)), v\rangle_H] = E\left[\sum_{i=1}^{\infty}\langle v, e_i\rangle_H D_{e_i}(\delta(u))\right] \\
&= E\left[\sum_{i=1}^{\infty}\langle v, e_i\rangle_H(\langle u, e_i\rangle_H + \delta(D_{e_i}u))\right] \\
&= E\left[\langle u, v\rangle_H\right] + \sum_{i=1}^{\infty} E\left[\langle v, e_i\rangle_H \delta(D_{e_i}u)\right] \\
&= E\left[\langle u, v\rangle_H\right] + \sum_{i=1}^{\infty} E\left[\langle D\langle v, e_i\rangle_H, D_{e_i}u\rangle_H\right] \\
&= E\left[\langle u, v\rangle_H\right] + E\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle D_{e_j}v, e_i\rangle_H\langle D_{e_i}u, e_j\rangle_H\right] \\
&= E[\langle u, v\rangle_H] + E[\mathrm{Tr}(Du \circ Dv)].
\end{aligned}$$

Hence,

$$E[\delta(u)^2] \leq E[||u||_H^2] + E[||Du||_{H\otimes H}^2] = ||u||_{1,2,H}^2. \tag{28}$$

Since it is valid for any $u, v \in \mathcal{S}_H$, by density we will obtain that it is also true for any $u, v \in \mathbb{D}^{1,2}$, then (28) implies that $\mathbb{D}^{1,2} \subseteq \mathrm{Dom}\ \delta$ and (27) also holds. □

Proposition 4.9. *Let $F \in \mathbb{D}^{1,2}$ and $u \in$ Dom δ such that $Fu \in L^2(\Omega; H)$. If $F\delta(u) - \langle DF, u\rangle_H \in L^2(\Omega)$, then $Fu \in$ Dom δ and*

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H. \tag{29}$$

Proof. Let $F \in \mathcal{S}$, $u \in \mathcal{S}_H$. Then, for any $G \in \mathcal{S}$

$$\begin{aligned}
E[G\delta(Fu)] &= E[\langle DG, Fu\rangle_H] = E[\langle D(FG) - GDF, u\rangle_H] \\
&= E[G(F\delta(u) - \langle DF, u\rangle_H)],
\end{aligned}$$

where we have used the duality relationship (24). As the sets $\mathcal{S}, \mathcal{S}_H$ are dense, the result holds. □

Although the divergence operator has been defined as an operator from $L^2(\Omega; H)$ into $L^2(\Omega)$, in fact it can be consider acting in elements of $L^p(\Omega; H)$.

Proposition 4.10. *The operator δ is continuous from $\mathbb{D}^{1,p}$ into $L^p(\Omega)$ for all $p > 1$.*

The following proposition give a similar bound as (28) for the L^p norm of the divergence operator.

Proposition 4.11. *Let u be an element of $\mathbb{D}^{1,p}(H)$, $p > 1$. Then we have*

$$||\delta(u)||_p \leq c_p \left(||E[u]||_H + ||Du||_{L^p(\Omega; H\otimes H)}\right).$$

The details of these proofs can be found in the section of *Sobolev spaces and equivalence of norms* of reference [2].

4.2.1 The Skorohod integral

In this section we will work under the same assumptions than in Section 3.3, that is, assuming the special case when H is an L^2 space of the form $L^2(T,\mathcal{B},\mu)$, where $(T,\mathcal{B})$ is a measurable space and μ is a σ-finite measure without atoms.

In this case, the elements of Dom $\delta \subset L^2(T\times\Omega)$ are square integrable processes and $\delta(u)$ coincides with the so called *Skorohod stochastic integral* of u.

The Skorohod integral will be denoted by

$$\delta(u) = \int_T u_t \delta W_t.$$

Consider the decomposition of $u \in L^2(T\times\Omega)$ in its Wiener chaos in terms of the multiple Wiener-Itô integrals

$$u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)), \tag{30}$$

with $f_n \in L^2(T^{n+1})$ symmetric functions in the first n variables, for all $n \geq 1$.

Proposition 4.12. *Let $u \in L^2(T\times\Omega)$ with decomposition in its Wiener chaos given by (30). Then, $u \in$ Dom δ if and only if the series*

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(f_n)$$

converges in $L^2(\Omega)$.

From this previous proposition we can see that Dom δ consists of processes of $L^2(T\times\Omega)$ satisfying

$$E[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)!||\tilde{f}_n||_{L^2(T^{n+1})} < \infty,$$

where

$$\tilde{f}_n(t_1,...,t_n,t) = \frac{1}{n+1}\left(f_n(t_1,...,t_n,t) + \sum_{i=1}^{n} f_n(t_1,...,t_{i-1},t,t_{i+1},...,t_n,t_i)\right),$$

(since in this case f_n is symmetric on the first n variables).

Let $\mathbb{L}^{1,2}$ denote the space $\mathbb{D}^{1,2}(L^2(T))$. This space is the set of processes $u \in L^2(T\times\Omega)$ such that $u_t \in \mathbb{D}^{1,2}$ for almost all t. Let $F \in \mathbb{D}^{1,2}$, we denote $D_tF := DF(t)$ by identifying $L^2(\Omega; L^2(T))$ with $L^2(T\times\Omega)$. Then, if $u \in \mathbb{L}^{1,2}$, $E[\int_T\int_T (D_s u_t)^2\mu(ds)\mu(dt)] < \infty$. In fact $\mathbb{L}^{1,2}$ is a Hilbert space with the norm

$$||u||^2_{1,2,L^2(T)} = ||u||^2_{L^2(T\times\Omega)} + ||Du||^2_{L^2(T^2\times\Omega)}.$$

By Proposition 4.8, $\mathbb{L}^{1,2} \subset$ Dom δ and for $u,v \in \mathbb{L}^{1,2}$, (27) says

$$E[\delta(u)\delta(v)] = \int_T E[u_tv_t]\mu(dt) + \int_T\int_T E[D_su_tD_tv_s]\mu(ds)\mu(dt).$$

The equivalent of Equation (26), that is, the Skorohod integral of an elementary process of the form $u(t) = \sum_{i=1}^{n} F_i h_i(t)$, $F_i \in \mathcal{S}$ and $h_i \in L^2(T)$ is

$$\int_T u_t \delta W_t = \sum_{i=1}^{n} F_i \int_T h_i(t) dW_t - \sum_{i=1}^{n} \int_T D_t F_i h_i(t) \mu(dt).$$

That is, this kind of integral does not factorize the non Hilbert-valued random variables out of the integral, but incorporates an extra term related with the derivative operator. The Malliavin derivative of the Skorohod integral is given by the following proposition.

Proposition 4.13. *Let $u \in \mathbb{L}^{1,2}$. Assume that for almost all $t \in T$, the process $\{D_t u_s, s \in T\}$ belongs to Dom δ and there is a version of the process $\{\int_T D_t u_s dW_s, t \in T\}$ which is in $L^2(T \times \Omega)$. Then $\delta(u) \in \mathbb{D}^{1,2}$ and*

$$D_t(\delta(u)) = u_t + \int_T D_t u_s \delta W_s.$$

4.3 The Ornstein–Uhlenbeck operator

In this section we will introduce the Ornstein–Uhlenbeck operator. We will deal with concepts such as semigroup of operators and infinitesimal generator of the semigroup, which are defined in the appendix.

Definition 4.3. The *Ornstein–Uhlenbeck semigroup* is the one-parameter semigroup $\{P_t, t \geq 0\}$ of contraction operators in $L^2(\Omega)$ defined by

$$P_t(F) = \sum_{n=0}^{\infty} e^{-nt} J_n F, \tag{31}$$

for any $F \in L^2(\Omega)$.

The Ornstein–Uhlenbeck semigroup can be introduced analogously as it was introduced in Section 2, concerning finite dimensional Malliavin calculus. Let $W' = \{W'(h), h \in H\}$ be an independent copy of W, and assume that W, W' are defined in the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$, then, the process $Z = \{Z(h), h \in H\}$ defined by

$$Z(h) = e^{-t} W(h) + \sqrt{1 - e^{-2t}} W'(h), \quad h \in H,$$

is a Gaussian process with the same covariance that W. Indeed,

$$E[Z(h_1) Z(h_2)] = e^{-2t} \langle h_1, h_2 \rangle_H + (1 - e^{-2t}) \langle h_1, h_2 \rangle_H = \langle h_1, h_2 \rangle_H.$$

Now, given a random variable $F \in L^2(\Omega)$, let ψ_F be a measurable mapping from $\mathbb{R}^H$ to $\mathbb{R}$, determined $P \circ W^{-1}$-a.s. by $\psi_F = F \circ W$. Then, $\psi_F(Z(\omega, \omega')) = \psi_F(e^{-t} W(\omega) + \sqrt{1 - e^{-2t}} W'(\omega'))$ is well defined $P \times P'$-a.s.

Proposition 4.14 (Mehler's formula)**.** *Let W' be an independent copy of the isonormal Gaussian process $W = \{W(h), h \in H\}$. Then, for any $t \geq 0$ and any $F \in L^2(\Omega)$, we have*

$$P_t(F) = E'[\psi_F(e^{-t}W + \sqrt{1-e^{-2t}}W')],$$

where E' denotes the expectation with respect to P'.

Proof. Let $\tilde{P}_t$ denote the operator such that $\tilde{P}_t(F) = E'[\psi_F(e^{-t}W+\sqrt{1-e^{-2t}}W')]$. In order to show that P_t defined as (31) is equal to $\tilde{P}_t$, let's show that for each t both P_t and $\tilde{P}_t$ are linear contraction operators on $L^2(\Omega)$ that coincide in a dense set.

It is immediate from the definition, that P_t is a linear contraction operator on $L^2(\Omega)$. Let's see then, that $\tilde{P}_t$ is also a linear contraction operator on $L^2(\Omega)$.

$$\begin{aligned} E[|\tilde{P}_t(F)|^2] &= E[|E'[\psi_F(e^{-t}W + \sqrt{1-e^{-2t}}W')]|^2] \\ &\leq E[E'[|\psi_F(e^{-t}W + \sqrt{1-e^{-2t}}W')|^2]] = E[|F^2|], \end{aligned}$$

where it has been used Jensen inequality and the fact that $e^{-t}W+\sqrt{1-e^{-2t}}W'$ and W have the same law.

As by Lemma 3.2, the random variables of the form $F = \exp(W(h)-\frac{1}{2}||h||_H^2)$, $h \in H$ are dense in $L^2(\Omega)$, it suffices to show that both linear contraction operators coincide for these random variables. We will have

$$F = \exp\left(W(h) - \frac{1}{2}||h||_H^2\right) = \sum_{n=0}^{\infty} ||h||_H^n \mathcal{H}_n\left(\frac{W(h)}{||h||_H}\right)$$

by (8). It is immediate to see that $J_nF = ||h||_H^n \mathcal{H}_n\left(\frac{W(h)}{||h||_H}\right)$ and hence

$$P_t(F) = \sum_{n=0}^{\infty} e^{-nt}||h||_H^n \mathcal{H}_n\left(\frac{W(h)}{||h||_H}\right).$$

On the other hand, since $F = \psi_F \circ W$, it is clear that $\psi_F(\cdot) = \exp(\,\cdot\, - \frac{1}{2}||h||_H^2)$. Therefore

$$\begin{aligned} \tilde{P}_t(F) &= E'\left[\psi_F\left(e^{-t}W(h) + \sqrt{1-e^{-2t}}W'(h)\right)\right] \\ &= E'\left[\exp\left(e^{-t}W(h) + \sqrt{1-e^{-2t}}W'(h) - \frac{1}{2}||h||_H^2\right)\right] \\ &= \exp\left(e^{-t}W(h) - \frac{1}{2}||h||_H^2\right) E'\left[\exp\left(\sqrt{1-e^{-2t}}W'(h)\right)\right] \\ &= \exp\left(e^{-t}W(h) - \frac{1}{2}e^{-2t}||h||_H^2\right) = \sum_{n=0}^{\infty} e^{-nt}||h||_H^n \mathcal{H}_n\left(\frac{W(h)}{||h||_H}\right), \end{aligned}$$

where in the last equality we have used again (8). This proves that in fact both operators are the same on $L^2(\Omega)$. □

The operators P_t are non-negative, that is if $F \geq 0$ then $P_t(F) \geq 0$, which can be seen from the Mehler's formula, and also symmetric:

$$E[GP_t(F)] = E[FP_t(G)] = \sum_{n=0}^{\infty} e^{-nt} E[J_n(F)J_n(G)].$$

Now, let's introduce what will be the infinitesimal generator of the Ornstein–Uhlenbeck semigroup.

Definition 4.4. We define the operator L as

$$LF = \sum_{n=0}^{\infty} -nJ_nF,$$

whenever the series converges in $L^2(\Omega)$. Then, the domain of this operator is

$$\text{Dom } L = \left\{ F \in L^2(\Omega) : \sum_{n=0}^{\infty} n^2 ||J_nF||_2^2 < \infty \right\}.$$

In particular, from the characterization of $\mathbb{D}^{1,2}$ by Proposition 4.2, we can see that Dom $L \subset \mathbb{D}^{1,2}$.

Proposition 4.15. *The operator L is the infinitesimal generator of the Ornstein–Uhlenbeck semigroup $\{P_t, t \geq 0\}$.*

Proof. We have to show that $F \in$ Dom L if and only if $\lim_{t\to 0} \frac{1}{t}(P_tF - F)$ exists in $L^2(\Omega)$ and in this case $LF = \lim_{t\to 0} \frac{1}{t}(P_tF - F)$.

Let's assume that $F \in$ Dom L, then

$$\begin{aligned}\lim_{t\to 0} E\left[\left|\frac{1}{t}(P_tF - F)\right|^2\right] &= \lim_{t\to 0} E\left[\left|\frac{1}{t}\sum_{n=0}^{\infty}(e^{-nt} - 1)J_nF\right|^2\right] \\ &= \lim_{t\to 0}\sum_{n=0}^{\infty}\left|\frac{1}{t}(e^{-nt} - 1)\right|^2 ||J_nF||_2^2 \leq \sum_{n=0}^{\infty} n^2||J_nF||_2^2 < \infty,\end{aligned}$$

since $\left|\frac{1}{t}(e^{-nt} - 1)\right| \leq n$ and $F \in$ Dom L, which means that $\lim_{t\to 0} \frac{1}{t}(P_tF - F)$ exists in $L^2(\Omega)$. Let's see now that it coincides with LF.

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}(P_tF - F) &= \lim_{t\to 0} \frac{1}{t}\sum_{n=0}^{\infty}(e^{-nt} - 1)J_nF \\ &= \sum_{n=0}^{\infty} \lim_{t\to 0} \frac{1}{t}(e^{-nt} - 1)J_nF = \sum_{n=0}^{\infty} -nJ_nF = LF,\end{aligned}$$

where we have used bounded convergence and the fact that $\lim_{t\to 0} \frac{1}{nt}(e^{-nt} - 1) = \frac{d}{dt}(e^{-nt})\big|_{nt=0}$.

Let's show the converse now, let's assume that $\lim_{t\to 0} \frac{1}{t}(P_tF - F) = G$ in $L^2(\Omega)$. Then

$$J_nG = \lim_{t\to 0} \frac{1}{t}(P_tJ_nF - J_nF) = -nJ_nF,$$

using the same argument as before, which implies that indeed $F \in \text{Dom } L$ and $G = LF$. □

Let's introduce a proposition that relates the three operators that were introduced.

Proposition 4.16. *Let $F \in L^2(\Omega)$. $F \in$ Dom L if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in$ Dom δ. In this case $\delta DF = -LF$.*

Proof. Let's assume first that $F \in \text{Dom } L$, using Proposition 4.3 we see that in particular Dom $L \subset \mathbb{D}^{2,2}$, which means that $F \in \mathbb{D}^{2,2} \subset \mathbb{D}^{1,2}$ and $DF \in \mathbb{D}^{1,2}(H) \subset \text{Dom } \delta$. In this case, let $G \in \mathcal{S}$, using the duality property and Proposition 4.2 we obtain that

$$\begin{aligned} E[G\delta(DF)] = E[\langle DG, DF\rangle_H] &= \sum_{n=0}^{\infty} nE[J_nGJ_nF] \\ &= E\left[G\sum_{n=0}^{\infty} nJ_nF\right] = E[G(-LF)], \end{aligned} \tag{32}$$

which by density means $\delta(DF) = -LF$. For the converse, since $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom } \delta$, we can apply directly (32) and hence $F \in \text{Dom } L$. □

The following result shows the behaviour of L as a second order differential operator on sufficiently smooth random variables.

Proposition 4.17. *Let $F \in \mathcal{S}$ of the form (19), then $F \in$ Dom L and*

$$\begin{aligned} LF = \sum_{i,j=1}^{n} &\frac{\partial^2 f}{\partial x_i \partial x_j}(W(h_1), ..., W(h_n))\langle h_i, h_j\rangle_H \\ &- \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))W(h_i). \end{aligned}$$

Proof. By Proposition 4.16, $LF = -\delta DF$, hence

$$\begin{aligned} LF &= -\delta\left(\sum_{i=1}^{n} \partial_i f(W(h_1), ..., W(h_n))h_i\right) \\ &= -\sum_{i=1}^{n} \delta\left(\partial_i f(W(h_1), ..., W(h_n))h_i\right) \\ &= -\sum_{i=1}^{n} \partial_i f(W(h_1), ..., W(h_n))W(h_i) \\ &\quad + \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(W(h_1), ..., W(h_n))\langle h_i, h_j\rangle_H, \end{aligned}$$

by applying the definition of the divergence operator to a smooth random process, i.e. (26). □

Proposition 4.18. *Let $F = (F^1, ..., F^m)$ be a random vector such that $F^i \in \mathbb{D}^{2,4}$, $i \in \{1, ..., m\}$. Let $\varphi \in \mathcal{C}^2(\mathbb{R}^m)$ with bounded first and second partial derivatives. Then $\varphi(F) \in Dom\ L$, and*

$$L(\varphi(F)) = \sum_{i,j=1}^{m} (\partial_i\partial_j\varphi)(F)\langle DF^i, DF^j\rangle_H + \sum_{i=1}^{m} (\partial_i\varphi)(F)LF^i.$$

Proof. Applying the chain rule and Proposition 4.16

$$\begin{aligned} L(\varphi(F)) &= -\delta(D(\varphi(F))) = -\delta(\sum_{i=1}^{m}(\partial_i\varphi)(F)DF^i) = -\sum_{i=1}^{m}\delta((\partial_i\varphi)(F)DF^i) \\ &= -\sum_{i=1}^{m}\big((\partial_i\varphi)(F)\delta(DF^i) - \langle D((\partial_i\varphi)(F)), DF^i\rangle_H\big) \\ &= \sum_{i=1}^{m}\left((\partial_i\varphi)(F)LF^i + \sum_{j=1}^{m}(\partial_i\partial_j\varphi)(F)\langle DF^j, DF^i\rangle_H\right), \end{aligned}$$

where we have used Proposition 4.9 in order to obtain $\delta((\partial_i\varphi)(F)DF^i)$. □

To conclude with the Ornstein–Uhlenbeck operator, we think is worth to mention, although we do not prove it, the hypercontractivity property of P_t.

Theorem 4.1. *Let $p \geq 1$ and $t > 0$, and set $q(t) = e^{2t}(p-1)+1 > p$. Suppose that $F \in L^p(\Omega)$. Then*

$$||P_t(F)||_q(t) \leq ||F||_p.$$

The proof of this theorem can be found in referece [2].

4.4 Local property of the operators

In this section it is commented the local properties of the operators D and δ.

Definition 4.5. An operator $\mathcal{O}$ defined on some space of random variables is said to be *local* on $A \in \mathcal{F}$ if for any random variable F such that $F = 0$ a.s. on A, then $\mathcal{O}(F) = 0$ a.s. on A.

Proposition 4.19. *The derivative operator D is local on $\mathbb{D}^{1,1}$.*

Proposition 4.20. *The divergence operator δ is local on $\mathbb{D}^{1,2}(H)$.*

This local property allows to localize the domains of the operators D and δ. That is, it can be defined the set of random variables $\mathbb{D}^{1,p}_{\text{loc}}$ such that there exists an increasing sequence $\Omega_n \subset \Omega$ converging almost surely to Ω and a sequence $F_n \in \mathbb{D}^{1,p}$, $n \geq 1$ satisfying that $F_n = F$ a.s. on Ω_n. If $F \in \mathbb{D}^{1,p}_{\text{loc}}$, then $DF = DF_n$ on Ω_n. The local property allows to ensure that this is well defined, since the random variable $F - F_n = 0$ a.s. on Ω_n and hence $0 = D(F - F_n) = DF - DF_n$ a.s. on Ω_n , which means that $DF = DF_n$ a.s. on Ω_n.

4.5 Malliavin calculus in Hilbert spaces

Throughout the previous sections we have already worked with Hilbert space valued random variables, without caring much about it. In this section, we are going to justify a little bit what we have done already. This section will ease the understanding of some of the computations of the next section. We have followed references [2] and [1].

Let V be a Hilbert space. For any $p \geq 1$, we denote by $L^p(\Omega; V) = L^p(\Omega, \mathcal{F}, P; V)$ the set of V-valued random variables X, being $\mathcal{F}$-measurable and such that $E[||X||_V^p] < \infty$, where $(\Omega, \mathcal{F}, P)$ is the underlying probability space and that in our case, $\mathcal{F} = \sigma(W)$, with $W = \{W(h), h \in H\}$. Notice that $L^2(\Omega; V)$ will be also a Hilbert space with inner product $\langle X, Y \rangle_{L^2(\Omega;V)} = E[\langle X, Y \rangle_V]$.

Let denote by $\mathcal{S}_V$ the space of V-valued smooth random variables of the form $\sum_{i=1}^n F_i v_i$, with $F_i \in \mathcal{S}$ and $v_i \in V$. Then, the k-th Malliavin derivative of any $F \in \mathcal{S}_V$ is given by the $H^{\otimes k} \otimes V$-valued random variable $D^k F = \sum_{i=1}^n D^k F_i \otimes v_i$. In this case, as with the real-valued random variables it can be seen that the operator $D^k : \mathcal{S}_V \subset L^p(\Omega; V) \to L^p(\Omega; H^{\otimes k} \otimes V)$ is closable. Then we can extend the domain of the operator to the closure of $\mathcal{S}_V$ with respect to the norm $|| \cdot ||_{k,p,V}$ given by

$$||F||_{k,p,V} = \left(E[||F||_V^p] + \sum_{i=1}^k E[||D^i F||^p_{H^{\otimes i} \otimes V}] \right)^{\frac{1}{p}}.$$

We denote by $\mathbb{D}^{k,p}(V)$ to this domain. We will denote

$$\mathbb{D}^\infty(V) := \bigcap_{p\geq 1} \bigcap_{k \geq 1} \mathbb{D}^{k,p}(V).$$

Notice that if $V = \mathbb{R}$ we will simply denote $\mathbb{D}^{k,p}(\mathbb{R}) = \mathbb{D}^{k,p}$ and $\mathbb{D}^{\infty}(\mathbb{R}) = \mathbb{D}^{\infty}$. Observe that D is continuous from $\mathbb{D}^{k,p}(V)$ into $\mathbb{D}^{k-1,p}(H \otimes V)$ and as a consequence D is continuous from $\mathbb{D}^{\infty}(V)$ into $\mathbb{D}^{\infty}(H \otimes V)$. Now we are going to introduce another extension of the chain rule for smooth random vectors.

Proposition 4.21. *Suppose $F = (F^1, ..., F^m)$ is a random vector such that $F^i \in \mathbb{D}^{\infty}$ and $\varphi \in \mathcal{C}_p^{\infty}(\mathbb{R}^m)$. Then, $\varphi(F) \in \mathbb{D}^{\infty}$, and*

$$D(\varphi(F)) = \sum_{i=1}^{m} (\partial_i \varphi)(F) DF^i.$$

We will introduce a proposition in order to compute the derivative of the scalar product of derivatives, that will be useful as a tool for the proofs of next section. If V, W are Hilbert spaces, each of them with inner product $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, then we can define the inner product in $V \otimes W$ as $\langle \cdot, \cdot \rangle_{V \otimes W} = \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_W$.

Proposition 4.22. *Let $F, G \in \mathbb{D}^{2,2}$, then for any $h \in H$*

$$D_h \left(\langle DF, DG \rangle_H \right) = \langle D^2 F, DG \otimes h \rangle_{H \otimes H} + \langle DF \otimes h, D^2 G \rangle_{H \otimes H}.$$

Proof. Let $F, G \in \mathcal{S}$ and let's assume that there exist orthonormal elements of H, $e_1, ..., e_n$ such that $F = f(W(e_1), ..., W(e_n))$, $G = g(W(e_1), ..., W(e_n))$, $f, g \in \mathcal{C}_p^{\infty}(\mathbb{R}^n)$. Then

$$\langle DF, DG \rangle_H = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_*) \frac{\partial g}{\partial x_i}(W_*) \langle e_i, e_i \rangle_H,$$

where we have denoted by $W_* = (W(e_1), ..., W(e_n))$. Hence

$$\begin{aligned}
&D_h \left(\langle DF, DG \rangle_H \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(W_*) \frac{\partial g}{\partial x_i}(W_*) + \frac{\partial f}{\partial x_i}(W_*) \frac{\partial^2 g}{\partial x_i \partial x_j}(W_*) \right) \langle e_i, e_i \rangle_H \langle e_j, h \rangle_H \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle \frac{\partial^2 f}{\partial x_i \partial x_j}(W_*) e_i \otimes e_j, \frac{\partial g}{\partial x_i}(W_*) e_i \otimes h \right\rangle_{H \otimes H} \\
&\qquad + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle \frac{\partial f}{\partial x_i}(W_*) e_i \otimes h, \frac{\partial^2 g}{\partial x_i \partial x_j}(W_*) e_i \otimes e_j \right\rangle_{H \otimes H} \\
&= \langle D^2 F, DG \otimes h \rangle_{H \otimes H} + \langle DF \otimes h, D^2 G \rangle_{H \otimes H}.
\end{aligned}$$

Since $\mathcal{S}$ is dense in $\mathbb{D}^{2,2}$ we can extend the result to any $F, G \in \mathbb{D}^{2,2}$. □

Corollary 4.1. Let $F, G \in \mathbb{D}^{\infty}$, then $\langle DF, DG \rangle_H \in \mathbb{D}^{\infty}$.

Proof. We can prove it recursively applying Proposition 4.22 to the scalar product $\langle \cdot, \cdot \rangle_{H^{\otimes k}}$. □

5 Criteria for existence and regularity of densities

In this section, mainly based on [3] and [2], we will apply Malliavin calculus in order to obtain explicit integration by parts formulas and then by means of the propositions of Section 1, derive criteria for the existence of densities, as well as for the regularity of these densities.

5.1 Existence of densities.

First of all, we will introduce the simpler case, the case of just a random variable.

Proposition 5.1. *Let $F \in \mathbb{D}^{1,2}$ and assume that $\frac{DF}{||DF||_H^2} \in Dom\ \delta$, then, the law of F is absolutely continuous. Moreover, its density is given by*

$$p(x) = E\left[\mathbb{1}_{(F \geq x)} \delta\left(\frac{DF}{||DF||_H^2}\right)\right], \tag{33}$$

hence, it is continuous and bounded.

Proof. This proof relies on Proposition 1.1, we just need to show that

$$E[\varphi'(F)] = E\left[\varphi(F)\delta\left(\frac{DF}{||DF||_H^2}\right)\right],$$

for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R})$. That is, that F and $G = 1$ satisfy an integration by parts formula of degree 1 with $H_1(F, 1) = \delta\left(\frac{DF}{||DF||_H^2}\right)$, since by assumption $\frac{DF}{||DF||_H^2} \in \text{Dom } \delta$.

Applying the chain rule, we obtain that $D(\varphi(F)) = \varphi'(F)DF$, hence

$$\langle D(\varphi(F)), DF\rangle_H = \langle \varphi'(F)DF, DF\rangle_H = \varphi'(F)||DF||_H^2,$$

which means that

$$\varphi'(F) = \left\langle D(\varphi(F)), \frac{DF}{||DF||_H^2}\right\rangle_H.$$

Using the duality property, it leads to the desired integration by parts formula. □

When F is a random vector instead, we have to consider a more involved analysis.

Definition 5.1. Let $F : \Omega \to \mathbb{R}^n$ be a random vector, $F = (F^1, ..., F^n)$, and assume that each of the components are in $\mathbb{D}^{1,2}$, then, we define the *Malliavin matrix of F* as the $n \times n$ matrix

$$\gamma_F(x) = (\langle DF^i(x), DF^j(x)\rangle_H)_{i,j}, \quad 1 \leq i, j \leq n.$$

Whenever there is no possible confusion we will drop the subindex and we will denote the Malliavin matrix of F simply by γ.

Proposition 5.2. *Let $F : \Omega \to \mathbb{R}^n$ be a random vector with components $F^i \in \mathbb{D}^{1,2}$, $i \in \{1, ..., n\}$. Assume that:*

1. *The Malliavin matrix, γ, is invertible a.s.,*
2. *For every $i, j \in \{1, ..., n\}$, the random variables $(\gamma^{-1})_{i,j} DF^j \in$ Dom δ.*

Then, for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$

$$E[\partial_i \varphi(F)] = E[\varphi(F) H_i(F, 1)],$$

with $H_i(F, 1) = \sum_{l=1}^n \delta\left((\gamma^{-1})_{i,l} DF^l\right)$. Thereby, the law of F is absolutely continuous.

Proof. For any given $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, using the chain rule 4.4 we obtain that $D(\varphi(F)) = \sum_{k=1}^n \partial_k \varphi(F) DF^k$. Hence

$$\langle D(\varphi(F)), DF^l \rangle_H = \sum_{k=1}^n \partial_k \varphi(F) \langle DF^k, DF^l \rangle_H = \sum_{k=1}^n \partial_k \varphi(F) \gamma_{k,l},$$

for $l \in \{1, ..., n\}$. As this form a linear system with matrix γ, we can invert this, since it is invertible a.s., and obtain

$$\partial_k \varphi(F) = \sum_{l=1}^n \left\langle D(\varphi(F)), (\gamma^{-1})_{k,l} DF^l \right\rangle_H,$$

for any $k \in \{1, ..., n\}$, a.s.. Notice that γ is symmetric and $(\gamma^{-1})_{k,l} = (\gamma^{-1})_{l,k}$. Now, taking expectations and applying the duality property

$$E[\partial_k \varphi(F)] = \sum_{l=1}^n E\left[\left\langle D(\varphi(F)), (\gamma^{-1})_{k,l} DF^l \right\rangle_H\right] = \sum_{l=1}^n E\left[\varphi(F) \delta\left((\gamma^{-1})_{k,l} DF^l\right)\right],$$

for any $k \in \{1, ..., n\}$, satisfying the first part of the proposition. By assumption 2) $\delta\left((\gamma^{-1})_{k,l} DF^l\right) \in L^2(\Omega) \subset L^1(\Omega)$ for any $k, l \in \{1, ..., n\}$ and hence

$$|E[\partial_k \varphi(F)]| \leq ||\varphi||_\infty \sum_{l=1}^n \left|E\left[\delta\left((\gamma^{-1})_{k,l} DF^l\right)\right]\right|.$$

Now, the existence of the density for the random vector F follows applying Proposition 1.2. ☐

Corollary 5.1. Let $F : \Omega \to \mathbb{R}^n$ be a random vector, $F = (F^1, ..., F^n)$. Assume that its Malliavin matrix γ is invertible a.s. and for any $i, j \in \{1, ..., n\}$, $F^i \in$ Dom L, $(\gamma^{-1})_{i,j} \in \mathbb{D}^{1,2}$, $(\gamma^{-1})_{i,j} DF^j \in L^2(\Omega; H)$, $(\gamma^{-1})_{i,j} \delta(DF^j) \in L^2(\Omega)$ and $\langle D(\gamma^{-1})_{i,j}, DF^j \rangle_H \in L^2(\Omega)$. Then, for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$

$$E[\partial_i \varphi(F)] = E[\varphi(F) H_i(F, 1)],$$

with $H_i(F, 1) = -\sum_{l=1}^n \left(\langle DF^l, D(\gamma^{-1})_{i,l} \rangle_H + (\gamma^{-1})_{i,l} LF^l\right)$. And the law of F is absolutely continuous.

Proof. Following the exact same steps as in the previous proof, we can get to

$$E[\partial_k \varphi(F)] = \sum_{l=1}^{n} E\left[\varphi(F)\delta\left((\gamma^{-1})_{k,l} DF^l\right)\right].$$

As $DF \in \text{Dom}\,\delta$ and $(\gamma^{-1})_{i,j} \in \mathbb{D}^{1,2}$ such that $(\gamma^{-1})_{i,j} DF^j \in L^2(\Omega; H)$ for all $i, j \in \{1, ..., n\}$ by hypothesis, then, we can apply Proposition 4.9 and obtain that $\delta\left((\gamma^{-1})_{k,l} DF^l\right) = -\left(\langle DF^l, D(\gamma^{-1})_{k,l}\rangle_H + (\gamma^{-1})_{k,l} LF^l\right)$, concluding the proof. □

The next theorem is the main result of this section and provides sufficient conditions for the existence of a density which are more natural than the ones provided by Proposition 5.2.

Theorem 5.1. *Let $F : \Omega \to \mathbb{R}^n$ be a random vector satisfying:*

1. *$F^i \in \mathbb{D}^{2,2}$ for any $i \in \{1, ..., n\}$,*
2. *The Malliavin matrix, γ, is invertible almost surely.*

Then, the law of F has a density with respect to the Lebesgue measure on $\mathbb{R}^n$.

Proof. We propose the solution to the linear system of equations as in the proof of Proposition 5.2:

$$\partial_i \varphi(F) = \sum_{l=1}^{n} \left\langle D(\varphi(F)), (\gamma^{-1})_{i,l} DF^l \right\rangle_H, \tag{34}$$

for a given $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and any $i \in \{1, ..., n\}$, a.s.. Since γ^{-1} may not have moments, we need to apply a localizing argument if we want to take expectations on both sides of the previous expression. Consider, for any natural number $N \geq 1$, a non-negative function $\psi_N \in \mathcal{C}_0^\infty(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))$ such that

- $\psi_N(\sigma) = 1$, if $\sigma \in C_N$,
- $\psi_N(\sigma) = 0$, if $\sigma \notin C_{N+1}$,

where $\sigma \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and

$$C_N = \left\{\sigma \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) : |\sigma_{i,j}| \leq N \text{ for all } i, j = 1, ..., n \text{ and } |\det \sigma| \geq \frac{1}{N}\right\}.$$

As

$$(\gamma^{-1})_{i,j} = \frac{1}{\det \gamma} C_{j,i} = \det \gamma^{-1} C_{j,i}, \tag{35}$$

with $C_{j,i}$ being the cofactors, we easily see that

$$|\psi_N(\gamma)(\gamma_F^{-1})_{i,j}| \leq \psi_N(\gamma)(N+1)(n-1)!(N+1)^{n-1},$$

hence, we will have that $E[|\psi_N(\gamma)(\gamma_F^{-1})_{i,j}|] < \infty$. Then, we can multiply the expression (34) on both sides by $\psi_N(\gamma)$ and take expectations to obtain

$$E[\psi_N(\gamma)\partial_i\varphi(F)] = \sum_{l=1}^{n} E\left[\left\langle D(\varphi(F)), \psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l\right\rangle_H\right]. \qquad (36)$$

Let's see that $\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l \in \mathbb{D}^{1,2}(H)$. We have to show that $\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l \in L^2(\Omega; H)$ and $D\left(\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l\right) \in L^2(\Omega; H\otimes H)$. For the first one

$$\begin{aligned} E\left[\psi_N^2(\gamma)(\gamma^{-1})_{i,l}^2||DF^l||_H^2\right] &\leq \left((N+1)(n-1)!(N+1)^{n-1}\right)^2 E\left[\psi_N^2(\gamma)||DF^l||_H^2\right] \\ &\leq \left((N+1)(n-1)!(N+1)^{n-1}\right)^2 (N+1), \end{aligned}$$

since in particular $||DF^l||_H^2 = \gamma_{l,l}$, which means that $\psi_N^2(\gamma)||DF^l||_H^2 < N+1$. For the second one, using the derivative of the product we obtain that

$$\begin{aligned} &D\left(\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l\right) = D\left(\psi_N(\gamma)(\gamma^{-1})_{i,l}\right)DF^l + \psi_N(\gamma)(\gamma^{-1})_{i,l}D^2F^l \\ &= D\left(\psi_N(\gamma)\right)(\gamma^{-1})_{i,l}DF^l + \psi_N(\gamma)D\left((\gamma^{-1})_{i,l}\right)DF^l + \psi_N(\gamma)(\gamma^{-1})_{i,l}D^2F^l. \end{aligned}$$

Let's see that each of this summands belong to $L^2(\Omega; H\otimes H)$. For the last summand

$$\begin{aligned} &E\left[\psi_N(\gamma)^2(\gamma^{-1})_{i,l}^2||D^2F^l||_{H\otimes H}^2\right] \\ &\qquad \leq \left((N+1)(n-1)!(N+1)^{n-1}\right)^2 E\left[||D^2F^l||_{H\otimes H}^2\right] < \infty, \end{aligned}$$

since in particular $F \in \mathbb{D}^{2,2}$. For $\psi_N(\gamma)D\left((\gamma^{-1})_{i,l}\right)DF^l$, let's develop the derivative of the Malliavin matrix:

$$\begin{aligned} D\left((\gamma^{-1})_{i,l}\right) &= D\left(\frac{1}{\det\gamma}C_{l,i}\right) = D\left(\frac{1}{\det\gamma}\right)C_{l,i} + \frac{1}{\det\gamma}D\left(C_{l,i}\right) \\ &= -\frac{1}{(\det\gamma)^2}D(\det\gamma) + \frac{1}{\det\gamma}D\left(C_{l,i}\right). \end{aligned}$$

We have that $\det\gamma$ is a linear combination of elements of the form $\gamma_{i_1,j_1}\cdots\gamma_{i_n,j_n}$ and $C_{l,i}$ is a linear combination of elements of the form $\gamma_{i'_1,j'_1}\cdots\gamma_{i'_{n-1},j'_{n-1}}$, then, we will have that their derivative will be linear combinations of elements of the form $\gamma_{i_1,j_1}\cdots D(\gamma_{i_k,j_k})\cdots\gamma_{i_n,j_n}$ and $\gamma_{i'_1,j'_1}\cdots D(\gamma_{i'_s,j'_s})\cdots\gamma_{i'_{n-1},j'_{n-1}}$, respectively with $k = 1,...,n$, $s = 1,...,n-1$. Once multiplied by $\psi_N(\gamma)$ we will only need to have bounds for elements of the form $\psi_N(\gamma)D(\gamma_{i,j})$, that is $\psi_N(\gamma)D\left((\gamma^{-1})_{i,l}\right)DF^l \in L^2(\Omega; H\otimes H)$ if $\psi_N(\gamma)D(\gamma_{p,q})DF^l \in L^2(\Omega; H\otimes H)$ for any p,q. Since $D(\gamma_{p,q}) = D(\langle DF^p, DF^q\rangle_H)$, from Proposition 4.22, we have that

$$||D(\gamma_{p,q})||_H \leq ||D^2F^p||_{H\otimes H}||DF^q||_H + ||DF^p||_H||D^2F^q||_{H\otimes H},$$

and therefore

$$\begin{aligned} &E[||\psi_N(\gamma)D(\gamma_{p,q})DF^l||_{H\otimes H}^2] \\ &\quad \leq 2E\left[|\psi_N(\gamma)|^2\left(||D^2F^p||_{H\otimes H}^2\gamma_{qq} + \gamma_{pp}||D^2F^q||_{H\otimes H}^2\right)||DF^l||_H^2\right] \\ &\quad \leq CE[||D^2F^p||_{H\otimes H}^2 + ||D^2F^q||_{H\otimes H}^2] < \infty, \end{aligned}$$

where C is a constant, and this is finite since the elements of γ are bounded due to $\psi_N(\gamma)$ and $F^l \in \mathbb{D}^{2,2}$ for all l.

Finally, for $D\left(\psi_N(\gamma)\right)(\gamma^{-1})_{i,l}DF^l$, since $\psi_N \in \mathcal{C}_0^\infty(\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))$ we can apply the chain rule and get

$$D\left(\psi_N(\gamma)\right) = \sum_{i,j=1}^{n} \partial_{i,j}\psi_N(\gamma)D(\gamma_{i,j}),$$

where $\partial_{i,j}$ denotes the partial derivative with respect to the component $\sigma_{i,j}$. See that $D\left(\psi_N(\gamma)\right)(\gamma^{-1})_{i,l}DF^l \in L^2(\Omega; H\otimes H)$ reduces to see that elements of the form $\partial_{p,q}\psi_N(\gamma)D(\gamma_{p,q})(\gamma^{-1})_{i,l}DF^l$, but since $\partial_{p,q}\psi_N(\gamma)$ is also smooth with compact support, we can apply the same argument as with the previous component and see that it is also in $L^2(\Omega; H\otimes H)$.

As $\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l \in \mathbb{D}^{1,2}(H)$ has been shown, then $\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l \in \operatorname{Dom}\delta$, see Proposition 4.8. As a consequence, we can apply the duality property to (36) and obtain

$$\begin{aligned}
|E[\psi_N(\gamma)\partial_i\varphi(F)]| &= \left|\sum_{l=1}^{n} E\left[\varphi(F)\delta\left(\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l\right)\right]\right| \\
&\leq E\left[\left|\sum_{l=1}^{n} \delta\left(\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l\right)\right|\right] ||\varphi||_\infty,
\end{aligned}$$

where $E\left[\left|\sum_{l=1}^n \delta\left(\psi_N(\gamma)(\gamma^{-1})_{i,l}DF^l\right)\right|\right] = C_i$ is a constant not depending on φ. Then, by Proposition 1.2, $[\psi_N(\gamma)P]\circ F^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$. Since by hypothesis $|\gamma_{i,j}| < \infty$ and $|\det\gamma| > 0$ a.s., then $\lim_{N\to\infty}\psi_N(\gamma) = 1$ and hence for any Borel set $B \in \mathcal{B}(\mathbb{R}^m)$ with zero Lebesgue measure we have $P(F^{-1}(B)) = 0$, which means that F has a density. □

In fact, there is a much weaker condition in order to obtain the existence of densities. The following result is a weaker version of the previous theorem.

Theorem 5.2. *Let $F : \Omega \to \mathbb{R}^n$ be a random vector satisfying:*

1. *$F^i \in \mathbb{D}^{1,p}_{loc}$ for any $i \in \{1,...,n\}$, $p > 1$,*
2. *The Malliavin matrix, γ, is invertible almost surely.*

Then, the law of F has a density with respect to the Lebesgue measure on $\mathbb{R}^n$.

The proof of this theorem is not presented in this work since it requires a little bit more involved techniques. However, a proof of this theorem can be found in reference [2].

5.2 Smoothness of densities.

Now we want to use the second part of the propositions of Section 1 in order to obtain sufficient conditions for the smoothness of the densities of random vectors. First, we will introduce a couple of lemmas, whose proofs can be found in [2] and that will help us with the proof of the result.

Lemma 5.1. *Let $(F_n)_n$ be a sequence of random variables converging to F in $L^p(\Omega)$ for some $p > 1$. Suppose that $\sup_n ||F_n||_{s,p} < \infty$ for some s. Then, $F \in \mathbb{D}^{s,p}$.*

Lemma 5.2. *Let $F \in \mathbb{D}^{1,2}$ be a random variable such that $E[|F|^{-2}] < \infty$. Then, $P(F > 0)$ is either 0 or 1.*

Now we will expose sufficient conditions for the smoothness of densities of a random vector.

Proposition 5.3. *Let $F : \Omega \to \mathbb{R}^n$ be a random vector satisfying:*

1. *$F^i \in \mathbb{D}^\infty$, for any $i \in \{1, ..., n\}$,*
2. *The Malliavin matrix of F, γ, is invertible almost surely and*

$$\det \gamma^{-1} \in \bigcap_{p\in[1,\infty)} L^p(\Omega). \tag{37}$$

Then the law of F has an infinitely differentiable density with respect to the Lebesgue measure on $\mathbb{R}^n$.

Proof. First of all, let's see that $(\gamma^{-1}_{i,j}) \in \mathbb{D}^\infty$ for all $i, j \in \{1, ..., n\}$. Let φ_N be defined as $\varphi_N(x) = \left(x + \frac{1}{N}\right)^{-1}$ for $x \geq 0$ and $N \geq 1$. As either $P(\det \gamma > 0)$ is zero or one by Lemma 5.2, we will assume that $\det \gamma > 0$. Let now $Y_N = \left(\det \gamma + \frac{1}{N}\right)^{-1}$, $N \geq 1$. As φ_N can be extended to a function in $\mathcal{C}^\infty_p(\mathbb{R})$ and $\det \gamma \in \mathbb{D}^\infty$ (deduced from Corollary 4.1), then, by an iteration of the chain rule, we have that $Y_N \in \mathbb{D}^\infty$, for any $N \geq 1$. We have that Y_N converges to $\det \gamma^{-1}$ in $L^p(\Omega)$ for any $p \geq 1$, by (37). Then, due to Lemma 5.1, $\det \gamma^{-1}$ belong to $\mathbb{D}^\infty$ if the sequence $(Y_N)_N$ has uniformly bounded derivatives of any order in $L^p(\Omega)$ for any p. Applying Leibniz's rule, (23), we can see that the derivatives are uniformly bounded. Indeed,

$$D^k(\varphi_N(\det \gamma)) = \sum_{l=1}^{k} \sum_{\mathcal{P}_l} c_l \varphi_N^{(l)}(\det \gamma) \prod_{i=1}^{l} D^{|p_i|} \det \gamma,$$

are uniformly bounded since $\det \gamma \in \mathbb{D}^\infty$ and

$$|\varphi_N^{(l)}(\det \gamma)| = l! \left((\det \gamma) + \frac{1}{N}\right)^{-(l+1)} = l! \, (Y_N)^{(l+1)},$$

and the L^p norm of Y_N is bounded by the L^p norm of $\det \gamma^{-1}$ for any $p, N \geq 1$. From the expression of the inverse of γ, (35), it can be seen that all the entries of γ^{-1} belong to $\mathbb{D}^\infty$. This is because both $\det \gamma^{-1}$ and F^i belong to $\mathbb{D}^\infty$, for any $i \in \{1, ..., n\}$.

Now, as in the proof of Proposition 5.2, we can obtain that for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$

$$\partial_i \varphi(F) = \sum_{l=1}^{n} \left\langle D(\varphi(F)), (\gamma^{-1})_{i,l} DF^l \right\rangle_H .$$

Multiplying both sides by a given $G \in \mathbb{D}^\infty$ and taking expectations we obtain

$$E\left[\partial_i \varphi(F) G\right] = \sum_{l=1}^{n} E\left[\left\langle D(\varphi(F)), G(\gamma^{-1})_{i,l} DF^l \right\rangle_H\right] .$$

We have that $G(\gamma^{-1})_{i,l} DF^l \in \mathbb{D}^\infty$, since each of the factors belong to $\mathbb{D}^\infty$, then, we can apply the duality property and obtain

$$E\left[\partial_i \varphi(F) G\right] = E\left[\varphi(F) H_i(F, G)\right], \tag{38}$$

with $H_i(F, G) = \sum_{l=1}^{n} \delta\left(G(\gamma^{-1})_{i,l} DF^l\right)$, for all $i \in \{1, ..., n\}$. It can be shown using an extension of Proposition 4.7 and iterating, that $H_i \in \mathbb{D}^\infty$. Moreover, we can generalize (38) for any multiindex α using induction over $|\alpha|$. The expression was given for $|\alpha| = 1$, now let's assume that is true for $|\alpha| = r - 1$. Given β a multiindex such that $|\beta| = r$, we can express $\beta = \alpha + i$, where α and i are multiindex such that $|\alpha| = r - 1$ and $|i| = 1$. Then, by recursion

$$E[(\partial_\beta \varphi)(F) G] = E[(\partial_i \varphi)(F) H_\alpha(F, G)] = E[\varphi(F) H_i(F, H_\alpha(F, G))],$$

and we can define the $\mathbb{D}^\infty$ random variable $H_\beta(F, G)$ equal to $H_i(F, H_\alpha(F, G))$ almost surely, where $H_i(F, H_\alpha(F, G)) = \sum_{l=1}^{n} \delta\left(H_\alpha(F, G)(\gamma^{-1})_{i,l} DF^l\right)$.

Taking $G = 1$ we obtain that for any multiindex α and every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$

$$E[(\partial_\alpha \varphi)(F)] = E[\varphi(F) H_\alpha(F, 1)].$$

In particular

$$|E[(\partial_\alpha \varphi)(F)]| \leq C_\alpha ||\varphi||_\infty,$$

where $C_\alpha := E[H_\alpha(F, 1)]$ is a constant not depending on φ. Applying Proposition 1.2 we finally show that the law of F has a $\mathcal{C}^\infty$ density. $\square$

6 Applications

In this section we will introduce some applications of the theory developed before, we will briefly explain the relation of the Skorohod integral with the Itô stochastic integral and finally we will present a practical example.

6.1 Density of random variables

We will start considering sufficient conditions for a random variable $F \in L^2(\Omega)$ in order to satisfy conditions of Proposition 5.1, which in addition provide of another expression for the density of the random variable.

Proposition 6.1. *Let F be a random variable belonging to $\mathbb{D}^{2,4}$ such that $E[||DF||_H^{-8}] < \infty$, then $\frac{DF}{||DF||_H^2} \in Dom\ \delta$ and*

$$\delta\left(\frac{DF}{||DF||_H^2}\right) = -\frac{LF}{||DF||_H^2} - 2\frac{\langle DF \otimes DF, D^2F\rangle_{H\otimes H}}{||DF||_H^4}. \tag{39}$$

Proof. Let's see that indeed $\frac{DF}{||DF||_H^2} \in \mathbb{D}^{1,2}(H)$. First of all, applying Cauchy-Schwarz inequality

$$E\left[\left|\left|\frac{DF}{||DF||_H^2}\right|\right|_H^2\right] \leq E\left[||DF||_H^4\right]^{\frac{1}{2}} E\left[||DF||_H^{-8}\right]^{\frac{1}{2}} < \infty,$$

by hypothesis. Now we have to see that $D\left(\frac{DF}{||DF||_H^2}\right) \in L^2(\Omega; H\otimes H)$. Applying the chain rule

$$D\left(\frac{DF}{||DF||_H^2}\right) = \frac{D^2F}{||DF||_H^2} - \frac{D(\langle DF, DF\rangle_H)}{||DF||_H^4}DF,$$

and now applying Proposition 4.22, we obtain that

$$\left|\left|\frac{D(\langle DF, DF\rangle_H)}{||DF||_H^4}DF\right|\right|_{H\otimes H} \leq 2\frac{||D^2F||_{H\otimes H}}{||DF||_H^2}.$$

Hence,

$$\left|\left|D\left(\frac{DF}{||DF||_H^2}\right)\right|\right|_{H\otimes H} \leq 3\frac{||D^2F||_{H\otimes H}}{||DF||_H^2}. \tag{40}$$

Finally, using (40) and applying Cauchy-Schwarz inequality

$$E\left[\left|\left|D\left(\frac{DF}{||DF||_H^2}\right)\right|\right|_{H\otimes H}^2\right] \leq 9E\left[||D^2F||_{H\otimes H}^4\right]^{\frac{1}{2}} E\left[||DF||_H^{-8}\right]^{\frac{1}{2}} < \infty,$$

since both factors on the right-hand side are finite by hypothesis. Therefore $\frac{DF}{||DF||_H^2} \in \mathbb{D}^{1,2}(H) \subset \text{Dom } \delta$. In order to show (39) we will use Proposition 4.9, that says that given $G \in \mathbb{D}^{1,2}$ and $u \in \text{Dom } \delta$, then $\delta(Gu) = G\delta(u) - \langle DG, u\rangle_H$.

Let's consider $G_\epsilon = (||DF||_H + \epsilon)^{-2}$ and $u = DF$. $u \in \text{Dom } \delta$ since in particular $DF \in \mathbb{D}^{1,2}(H) \subset \text{Dom } \delta$ and $G_\epsilon \in \mathbb{D}^{1,2}$ because

$$\begin{aligned}||D((||DF||_H + \epsilon)^{-2})||_H &= ||D(\langle DF, DF\rangle_H)||_H(||DF||_H + \epsilon)^{-4} \\ &\leq 2||D^2F||_{H\otimes H}||DF||_H(||DF||_H + \epsilon)^{-4},\end{aligned}$$

by virtue of Proposition 4.22. Now, since $(||DF||_H + \epsilon)^{-1}$ is bounded by $1/\epsilon$, we can apply Cauchy-Schwarz inequality and see that G_ϵ belongs to $\mathbb{D}^{1,2}$. Hence

$$\begin{aligned}\delta\left(\frac{DF}{(||DF||_H + \epsilon)^2}\right) &= \frac{\delta DF}{(||DF||_H + \epsilon)^2} - \frac{D_u(\langle DF, DF\rangle_H)}{(||DF||_H + \epsilon)^4} \\ &= -\frac{LF}{(||DF||_H + \epsilon)^2} - 2\frac{\langle D^2F, DF \otimes DF\rangle_{H\otimes H}}{(||DF||_H + \epsilon)^4},\end{aligned}$$

where we have used Proposition 4.22 and that $\delta DF = -LF$. Making ϵ tend to zero, we obtain the desired result. □

The result of this proposition, combined with Proposition 5.1, states that the density of such a random variable F is

$$p(x) = E\left[-\mathbb{1}_{(F\geq x)}\left(\frac{LF}{||DF||_H^2} + 2\frac{\langle DF \otimes DF, D^2F\rangle_{H\otimes H}}{||DF||_H^4}\right)\right].$$

From Theorem 5.2, conditions for the existence of density for a random variable F can be considered to be $F \in \mathbb{D}^{1,p}$, $p > 1$, and $||DF||_H > 0$ almost surely. Under stronger conditions, the next proposition bounds the density of the random variable.

Proposition 6.2. *Let q, α, β be three positive real numbers such that $\frac{1}{q} + \frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let F be a random variable belonging to $\mathbb{D}^{2,\alpha}$ satisfying that $E[||DF||_H^{-2\beta}] < \infty$. Then the density of F, $p(x)$, has the following upper bound*

$$p(x) \leq c_{1,\alpha,\beta}(P(|F| > |x|))^{\frac{1}{q}}\left(E[||DF||_H^{-1}] + ||D^2F||_{L^\alpha(\Omega;H\otimes H)}\left|\left|||DF||_H^{-2}\right|\right|_\beta\right). \tag{41}$$

Proof. From Theorem 5.2, we obtain that F has a density. Moreover, we can see that

$$p(x) = E\left[\mathbb{1}_{(F\geq x)}\delta\left(\frac{DF}{||DF||_H^2}\right)\right], \tag{42}$$

still holds when $F \in \mathbb{D}^{1,p_1}$ and $\frac{DF}{||DF||_H^2} \in \mathbb{D}^{1,p_2}(H)$, with $p_1, p_2 > 1$. Let's check that those hypothesis are satisfied. As $F \in \mathbb{D}^{2,\alpha}$, we will have that $F \in \mathbb{D}^{1,p_1}$ for any $p_1 \leq \alpha$. In order to see that $\frac{DF}{||DF||_H^2} \in L^{p_2}(\Omega; H)$ it is required that $E[||DF||_H^{-p_2}] < \infty$, that will hold for any $p_2 \leq 2\beta$. We will have that if $E\left[\left(\frac{||D^2F||_{H\otimes H}}{||DF||_H^2}\right)^{p_2}\right] < \infty$, then using (40), $D\left(\frac{DF}{||DF||_H^2}\right) \in L^{p_2}(\Omega; H \otimes H)$. Applying Hölder's inequality with $\frac{1}{q_1} + \frac{1}{q_2} = 1$

$$E\left[\left(\frac{||D^2F||_{H\otimes H}}{||DF||_H^2}\right)^{p_2}\right] \leq E\left[||D^2F||_{H\otimes H}^{q_1\cdot p_2}\right]^{\frac{1}{q_1}} E\left[||DF||_H^{-2q_2\cdot p_2}\right]^{\frac{1}{q_2}}.$$

Then $E\left[||D^2F||_{H\otimes H}^{q_1\cdot p_2}\right]$ will be finite if $q_1\cdot p_2 \le \alpha$ and $E\left[||DF||_H^{-2q_2\cdot p_2}\right]$ will be finite if $q_2\cdot p_2 \le \beta$. Combining the obtained inequalities we can see that $F \in \mathbb{D}^{1,p_1}$ and $\frac{DF}{||DF||_H^2} \in \mathbb{D}^{1,p_2}(H)$ for $1 < p_1 \le \alpha$ and $1 < p_2 \le \min\left(2\beta, \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^{-1}\right)$ and indeed (42) holds.

Applying Hölder's inequality to (42) with $\frac{1}{p}+\frac{1}{q}=1$ it is obtained that

$$p(x) \le (P(F>x))^{\frac{1}{q}} \left\|\delta\left(\frac{DF}{||DF||_H^2}\right)\right\|_p, \tag{43}$$

since the law of F is absolutely continuous and then $P(F \ge x) = P(F > x)$. Moreover, as $\mathbb{1}_{(F\ge x)} = 1 - \mathbb{1}_{(F<x)}$,

$$\begin{aligned} p(x) &= E\left[\delta\left(\frac{DF}{||DF||_H^2}\right)\right] - E\left[\mathbb{1}_{(F<x)}\delta\left(\frac{DF}{||DF||_H^2}\right)\right] \\ &= -E\left[\mathbb{1}_{(F<x)}\delta\left(\frac{DF}{||DF||_H^2}\right)\right], \end{aligned} \tag{44}$$

since $E\left[\delta\left(\frac{DF}{||DF||_H^2}\right)\right]=0$. Applying again Hölder's inequality to the absolute value of (44), we obtain that

$$p(x) \le (P(F<x))^{\frac{1}{q}} \left\|\delta\left(\frac{DF}{||DF||_H^2}\right)\right\|_p. \tag{45}$$

Putting together (43) and (45) we obtain,

$$p(x) \le (P(|F|>|x|))^{\frac{1}{q}} \left\|\delta\left(\frac{DF}{||DF||_H^2}\right)\right\|_p. \tag{46}$$

Using Proposition 4.11, we obtain that

$$\left\|\delta\left(\frac{DF}{||DF||_H^2}\right)\right\|_p \le c_p\left(\left\|E\left[\frac{DF}{||DF||_H^2}\right]\right\|_H + \left\|D\left(\frac{DF}{||DF||_H^2}\right)\right\|_{L^p(\Omega;H\otimes H)}\right). \tag{47}$$

Finally, combining the equations (46), (47) and (40), the proof is concluded. □

6.2 The Itô integral as a particular case of the Skorohod integral

In the same setting as in Section 3.3 and Section 4.2.1, let's assume $H = L^2(T,\mathcal{B},\mu)$. If $A \in \mathcal{B}$, let $\mathcal{F}_A$ be the σ-field generated by the $\{W(B), B \subset A, B \in \mathcal{B}\}$.

Proposition 6.3. *Assume $F \in \mathbb{D}^{1,2}$ and let $A \in \mathcal{B}$. Then the conditional expectation $E[F|\mathcal{F}_A]$ also belongs to $\mathbb{D}^{1,2}$, and*

$$D_t(E[F|\mathcal{F}_A]) = E[D_tF|\mathcal{F}_A]\mathbb{1}_A(t),$$

a.e. in $T\times\Omega$.

Corollary 6.1. Let $A \in \mathcal{B}$ and suppose that $F \in \mathbb{D}^{1,2}$ is $\mathcal{F}_A$-measurable. Then, D_tF is zero almost everywhere in $A^c \times \Omega$.

Lemma 6.1. *Let $A \in \mathcal{B}$, and let F be a square integrable random variable that is measurable with respect to the σ-field $\mathcal{F}_{A^c}$. Then the process $F\mathbb{1}_A$ is Skorohod integrable and*

$$\delta(F\mathbb{1}_A) = FW(A).$$

Proof. Assuming $F \in \mathbb{D}^{1,2}$ and applying Corollary 6.1 to (29) from Proposition 4.9

$$\delta(F\mathbb{1}_A) = FW(A) - \int_T D_tF\mathbb{1}_A(t)\mu(dt) = FW(A).$$

The general case follows approximating $F \in L^2(\Omega)$ by $F_n \in \mathbb{D}^{1,2}$ random variables and using the fact that δ is closed. □

Now let's see that the Skorohod integral coincides with the Itô integral in a particular class of random variables. Consider now $H = L^2([0,T])$ and μ the Lebesgue measure on $[0,T]$. Let's consider the case of a one dimensional Brownian motion. Let $W = \{W_t, 0 \le t \le T\}$ be a Brownian motion. Let $\mathcal{F}_t$ denote the σ-field generated by $\{W_s, 0 \le s \le t\}$. We will denote by L^2_a the closed subspace of $L^2([0,T] \times \Omega)$ formed by the adapted processes, that is, the processes $u = \{u_t, 0 \le t \le T\}$ such that u_t is $\mathcal{F}_t$-measurable.

Proposition 6.4. *$L^2_a \subset$ Dom δ, and the divergence operator restricted to L^2_a coincides with the Itô integral, that is $\delta(u) = \int_0^T u_t dW_t$.*

Proof. Let u be an elementary adapted process of the form

$$u_t = \sum_{i=1}^{n} F_i \mathbb{1}_{(t_i,t_{i+1}]}(t), \tag{48}$$

where F^i are $\mathcal{F}_{t_i}$-measurable square integrable random variables and $0 \le t_1 < \ldots < t_n \le T$. From Lemma 6.1 u is Skorohod integrable and

$$\delta(u) = \sum_{i=1}^{n} F_i W((t_i, t_{i+1}]) = \sum_{i=1}^{n} F_i(W(t_{i+1}) - W(t_i)),$$

which is the Itô integral of u. For a general process $u \in L^2_a$ the Itô integral of u is defined as the $L^2(\Omega)$ limit of the Itô integral of elementary adapted processes u_n such that converge to u in $L^2(T \times \Omega)$. Since we will have a sequence of $\delta(u_n)$ converging in $L^2(\Omega)$ and δ is closed, then $u \in$ Dom δ and $\delta(u)$ coincides with the Itô integral of u. □

6.3 Practical example: the solution of the Heat equation

In this section we will apply the developed theory in order to find conditions for the differentiability (in the Malliavin sense) of a stochastic process. The stochastic process that we will consider, is motivated by the solution of the stochastic heat equation

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) - \dfrac{\partial^2}{\partial x^2}v(t,x) = f(v(t,x))\dot{W}(t,x), \\ \\ v(0,x) = 0, \end{cases} \tag{49}$$

where $(t,x) \in [0,T] \times \mathbb{R}$. The solution of this stochastic partial differential equation is the stochastic process

$$v(t,x) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(t-s)}} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) f(v(s,y))W(ds,dy), \tag{50}$$

$t \in [0,T]$, $x \in \mathbb{R}$.

Definition of the stochastic process

Let $W = \{W(t,x),(t,x) \in [0,T] \times \mathbb{R}\}$ be a space-time white noise and let $u = \{u(t,x),(t,x) \in [0,T] \times \mathbb{R}\}$ be a stochastic process. Let's define the stochastic process of interest $F = \{F(t,x),(t,x) \in [0,T] \times \mathbb{R}\}$ as

$$F(t,x) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(t-s)}} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) u(s,y)W(ds,dy), \tag{51}$$

$t \in [0,T]$, $x \in \mathbb{R}$.

As it has been mentioned, the motivation for this example comes from the solution of the Heat equation (50). In this case the, integrand is adapted since it does only depend on past time values. The next definition is the formal definition of an adapted process.

Definition 6.1. Let's denote, for each $t \in [0,T]$, $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s,x), s \in [0,t], x \in \mathbb{R}\}$ and the P-null sets. We say that a stochastic process $\{u(t,x),(t,x) \in [0,T] \times \mathbb{R})\}$ is *adapted* if for all (t,x) the random variable $u(t,x)$ is $\mathcal{F}_t$-measurable.

From now on we will consider that u is adapted.

Integrability conditions

By an extension of Itô 's theory on stochastic integration, $F(t,x)$ is a well defined random variable in $L^2(\Omega)$ if and only if

$$||F(t,x)||_{L^2(\Omega)} = E\left[\int_0^t \int_{\mathbb{R}} \left(\frac{1}{\sqrt{4\pi(t-s)}} \exp\left[-\frac{|x-y|^2}{4(t-s)}\right]\right)^2 u^2(s,y)dyds\right] < \infty, \tag{52}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}$.

We want then to find conditions on u ensuring (52). In order to obtain these conditions, let's re-express (52) as

$$\int_0^t \frac{1}{\sqrt{8\pi(t-s)}} \int_{\mathbb{R}} G(t-s, x-y) E\left[u^2(s,y)\right] dyds < \infty, \tag{53}$$

where G is the Gaussian Kernel

$$G(t,x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x|^2}{2t}\right).$$

If we look at the spatial integral of (53), we notice that in fact, it is the convolution of $f(t,x) := E\left[u^2(t,x)\right]$ with the Gaussian Kernel $G(t-s,x)$. Informally, the Gaussian Kernel will smooth the function $f(t,x)$ in the x variable.

In order to fulfill condition (53), the first thing that has to be satisfied is that $E\left[u^2(t,x)\right] < \infty$ almost every $(t,x) \in [0,T] \times \mathbb{R}$. Let us give two different type of conditions ensuring (53).

Condition 1

Let's consider u satisfying

$$\sup_{\substack{t\in[0,T],\\ x\in\mathbb{R}}} E[u^2(t,x)] \leq M,$$

being M a positive constant. In this case we would have

$$\begin{aligned} ||F(t,x)||_{L^2(\Omega)} &\leq M \int_0^t \frac{1}{\sqrt{8\pi(t-s)}} \int_{\mathbb{R}} G(t-s,x-y)dyds \\ &= M \int_0^t \frac{ds}{\sqrt{8\pi(t-s)}} = \frac{M}{\sqrt{2\pi}} t^{\frac{1}{2}} < \infty, \end{aligned}$$

and (52) is satisfied.

Condition 2

Let's consider now u satisfying $E[u^2(t,x)] \leq P(t,x)$, where P is a polynomial function. We will have that the spatial integral in (53) would be bounded by

$$\int_{\mathbb{R}} G(t-s,x-y)P(s,y)dy = E[P(s,Z)],$$

where Z is a random variable such that $Z \sim N(x, t-s)$. Then, $E[P(s,Z)] = Q_t(s,x)$, where Q_t is another polynomial with coefficients depending on t. Hence,

$$||F||_{L^2(\Omega)} \leq \int_0^t \frac{Q_t(s,x)}{\sqrt{8\pi(t-s)}} < \infty,$$

for any polynomial Q_t.

Differentiability conditions

Let's consider $H = L^2([0,T] \times \mathbb{R})$ and W the space-time white noise. Let $v^{t,x} = \{v^{t,x}(s,y), (t,x) \in [0,T] \times \mathbb{R}\}$ be the stochastic process defined by

$$v^{t,x}(s,y) = G(2(t-s), x-y)\mathbb{1}_{[0,t]}(s)u(s,y). \tag{54}$$

Then, we can express our random variable as

$$F(t,x) = \int_0^T \int_{\mathbb{R}} v^{t,x}(s,y)W(ds,dy).$$

By (52) we will have that $v \in L^2([0,T] \times \mathbb{R} \times \Omega)$ under the above mentioned conditions of u. We can consider that this stochastic integral is the Skorohod integral and hence $F(t,x) = \delta(v^{t,x})$. Then, in order to obtain the differentiability of $F(t,x)$ we would like to apply Proposition 4.13. For doing that we need to prove the following conditions:

(a) $v^{t,x} \in \mathbb{L}^{1,2}$,

(b) the process $\{D_{r,z}v^{t,x}(s,y), (s,y) \in [0, T \times \mathbb{R}]\} \in \text{Dom }\delta$ a.e. in $[0,T] \times \mathbb{R}$,

(c) there is a version of the process

$$\left\{\int_0^T \int_{\mathbb{R}} D_{r,z}v^{t,x}(s,y)W(ds,dy), (r,z) \in [0,T] \times \mathbb{R}\right\},$$

which is in $L^2([0,T] \times \mathbb{R} \times \Omega)$.

Let's check (a). We have already seen under what conditions $v^{t,x} \in L^2([0,T] \times \mathbb{R} \times \Omega)$, now we have to see when $Dv^{t,x} \in L^2(([0,T] \times \mathbb{R})^2 \times \Omega)$. We have that

$$D_{r,z}v^{t,x}(s,y) = G(2(t-s), x-y)\mathbb{1}_{[0,t]}(s)D_{r,z}u(s,y),$$

since G and $\mathbb{1}_{[0,t]}$ are functions of the Hilbert space. Then, we would have that $Dv^{t,x} \in L^2(([0,T] \times \mathbb{R})^2 \times \Omega)$ if

$$E\left[\int_0^t \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} (G(2(t-s), x-y))^2 (D_{r,z}u(s,y))^2 dzdrdyds\right] < \infty. \tag{55}$$

In this case we can consider the same bounds as in *condition 1* and *condition 2* but on

$$E[||Du(s,y)||_H^2] = \int_0^T \int_{\mathbb{R}} E[(D_{r,z}u(s,y))^2]dzdr,$$

since we can derive that the expression

$$\int_0^t \frac{1}{\sqrt{8\pi(t-s)}} \int_{\mathbb{R}} G(t-s, x-y)E[||Du(s,y)||_H^2]dyds < \infty, \tag{56}$$

which is equivalent to (53). Notice, that in particular, u must be in $\mathbb{L}^{1,2}$.

For (b) we need to show that $D_{r,z}v^{t,x} \in \text{Dom } \delta$. A sufficient condition is considering $D_{r,z}v^{t,x} \in \mathbb{D}^{1,2}(L^2([0,T] \times R))$. That is, we need that $v^{t,x} \in \mathbb{D}^{2,2}$. The only thing left to see is that $D^2v^{t,x} \in L^2(([0,T]\times\mathbb{R})^3 \times \Omega)$. With the same procedure that before we obtain that we need

$$\int_0^t \frac{1}{\sqrt{8\pi(t-s)}} \int_{\mathbb{R}} G(t-s, x-y) E[||D^2u(s,y)||^2_{H\otimes H}]dyds < \infty, \qquad (57)$$

where

$$E[||D^2u(s,y)||^2_{H\otimes H}] = \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} E[(D_{\tau,\theta}D_{r,z}u(s,y))^2]dzdrd\theta d\tau.$$

Again, *condition 1* and *condition 2* applied to $E[||D^2u(s,y)||^2_{H\otimes H}]$ will work because of the form of (57).

We have that (c) is just a technical condition since $D_{r,z}v^{t,x}$ may not belong to Dom δ for a subset of null measure of $[0,T]\times\mathbb{R}$ and then $\delta(D_{r,z}v^{t,x})$ is not defined in that set. However, it is possible to change the process in that null set in order to fulfill (c) without extra conditions.

Now that we have checked the hypothesis, we can apply Proposition 4.13 and obtain that $F(t,x) \in \mathbb{D}^{1,2}$ and

$$D_{r,z}F(t,x) = D_{r,z}(\delta(v^{t,x})) = v^{t,x}(r,z) + \int_0^T \int_{\mathbb{R}} D_{r,z}v^{t,x}(s,y)W(ds,dy).$$

Notice that that if $r > t$, then $D_{r,z}F(t,x) = 0$. That is, $v^{t,x}(r,z) = 0$ because of the term $\mathbb{1}_{[0,t]}(r)$ and

$$\int_0^T \int_{\mathbb{R}} D_{r,z}v^{t,x}(s,y)W(ds,dy) = \int_0^t \int_{\mathbb{R}} G(2(t-s), x-y)D_{r,z}u(s,y)W(ds,dy) = 0,$$

because $D_{r,z}u(s,y) = 0$ when $r > s$ from the adaptability of u and Corollary 6.1.

Conclusions

In this work I have learned the basics of Malliavin calculus. In the study of this theory I have also became familiar with some useful techniques such as the localizing argument. I have found particularly enriching to redo some of the proofs with a little bit more of detail. For instance, in the proof of Proposition 5.1 I have dealt with the derivative of the inverse of the Malliavin matrix, which at first was hard, but turned out to provide a good insight of how the derivative operator works. I have also been able to see another type of stochastic integral which work with non-adapted processes, that is, the Skorohod integral.

When the project proposal was discussed, the section of applications was thought to be a little bit more extensive. We had in mind to be able to go a little bit further in the practical example. We would like to have obtained sufficient conditions for the existence of densities of the proposed process and also to have obtained some bounds for the probability density. However, I didn't have enough time to do that part properly and we decided to close the topic of applications with the differentiability conditions.

One of the possible continuations of this work is to use Malliavin calculus to study stochastic partial differential equations. That is, continue the what has been started in Section 6 but studying the solutions of stochastic partial differential equations in a rigorous way and apply Malliavin calculus in order to study the existence and regularity of probability densities of these solutions.

A Appendix

There are some concepts of Functional Analysis that will be applied throughout the work which are not directly related with the objectives and that will be presented in this annex for sake of completeness. The definitions and results of this appendix can be found in [4].

Operators

Definition A.1. The *graph* $G(T)$ *of a linear operator* T on Dom $T \subset X$ into Y is the set $\{(x, Tx); x \in \text{Dom } T\}$ in the product space $X \times Y$. Let X, Y be topological vector spaces, then T is called a *a closed linear operator* when the graph $G(T)$ constitutes a closed linear subspace of $X \times Y$.

Definition A.2. A linear operator T on Dom $T \subset X$ into Y is said to be *closable* or *pre-closed* if the closure in $X \times Y$ of the graph $G(T)$ is the graph of a linear operator, say S, on Dom $S \subset X$ into Y.

Proposition A.1. *If* X, Y *are quasi-normed linear spaces, then* T *is closable if and only if,* $(x_n)_n \subset$ *Dom* T *is a sequence such that* $\lim_{n\to\infty} x_n = 0$ *and such that* $\lim_{n\to\infty} Tx_n = y$*, then* $y = 0$*.*

Definition A.3. Let V, W be Hilbert spaces with respective inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, and let T be a linear operator defined on Dom $T \subset V$ into W. The *adjoint operator* of T is T^* defined on Dom $T^* \subset W$ into V satisfying

$$\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V, \; v \in V, w \in W,$$

where

$$\text{Dom } T^* = \{w \in W : \forall v \in \text{Dom } T, |\langle Tv, w \rangle_W| \leq c||v||_V\}.$$

Definition A.4. Let H be a Hilbert space, we say that a sequence $\{\varphi_n\}_{n\geq 1} \subset H$ converges to $\varphi \in H$ in the weak topology, if and only if for all $\psi \in H$,

$$\langle \varphi_n, \psi \rangle_H \xrightarrow[n\to\infty]{} \langle \varphi, \psi \rangle_H.$$

Semigroups

Definition A.5. The one-parameter family of operators $\{T_t, t \geq 0\}$ is said to satisfy the *semi-group property* if

$$T_t T_s = T_{t+s}, \; t, s > 0, \;\; T_0 = I.$$

Definition A.6. The *infinitesimal generator* of the one-parameter family of operators $\{T_t, t \geq 0\}$ is defined as

$$A := \lim_{t\to 0} \frac{1}{t}(T_t - I).$$

Definitions of spaces and notations

Definition of spaces

$\mathcal{C}_p^\infty(\mathbb{R}^n)$	set of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all its partial derivatives are bounded by polynomial growth.
$\mathcal{C}_b^\infty(\mathbb{R}^n)$	set of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all its partial derivatives are bounded (in particular $\mathcal{C}_b^\infty(\mathbb{R}^n) \subset \mathcal{C}_p^\infty(\mathbb{R}^n)$).
$\mathcal{C}_0^\infty(\mathbb{R}^n)$	set of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f has compact support (in particular all partial derivatives have compact support and hence are bounded, that is $\mathcal{C}_0^\infty(\mathbb{R}^n) \subset \mathcal{C}_b^\infty(\mathbb{R}^n)$).
$\mathcal{C}_p^k(\mathbb{R}^n)$	set of k times differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and its partial derivatives are bounded by polynomial growth.
$\mathcal{S}$	set of Gaussian funcionals $f(W(h_1), ..., W(h_n))$, $f \in \mathcal{C}_p^\infty(\mathbb{R}^n)$, $h_1, ..., h_n \in H$.
$\mathcal{S}_b$	set of Gaussian funcionals $f(W(h_1), ..., W(h_n))$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $h_1, ..., h_n \in H$.
$\mathcal{S}_0$	set of Gaussian funcionals $f(W(h_1), ..., W(h_n))$, $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $h_1, ..., h_n \in H$.
$\mathcal{P}$	set of Gaussian funcionals $f(W(h_1), ..., W(h_n))$, f is a polynomial, $h_1, ..., h_n \in H$.
$\mathcal{S}_H$	space of H-valued smooth random variables of the form $\sum_{i=1}^n F_i h_i$, with $F_i \in \mathcal{S}$ and $h_i \in H$.

Notation

Let Λ be the set of all sequences $(a_1, a_2, ...)$, $a_i \in \mathbb{N}$, $i \geq 1$ where only a finite number of a_i is different from zero. Then, if $a \in \Lambda$, we denote $a! = \prod_{i=1}^\infty a_i!$ and $|a| = \sum_{i=1}^\infty a_i$.

Let $F \in L^2(\Omega)$. We denote by $J_n F$ the projection of F into the nth Wiener chaos. Then, we write $F = \sum_{n=0}^n J_n F$.

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