Income Distributions in Multi-Sector Analysis: Miyazawa's Fundamental Equation of Income Formation Revisited

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Abstract: In standard type 1 input-output models, households' activities are part of the exogenous final demand. This means that their scale and composition are exogenously determined. That is, if some other final demand categories change (say public investment or exports) this does not influence the behaviour of the household categories. In type 2 input-output models households' activities are explained endogenously to capture the possibility of mutual interaction between household categories and productive sectors. In this area, Miyazawa (1976) proposed a novel way of modeling the endogenization of households' activities. In modeling terms, Miyazawa's proposition resulted in the so-called 'fundamental equation of income formation', core of which is an extended input coefficients matrix. This extended coefficients matrix produced several new types of multiplier matrices and explains industrial gross output and households' income in terms of non-household final demand in great detail. The model is traditionally solved by inverting the new extended coefficients matrix, which often generates highly complex outcomes in terms of convoluted multiplier matrices. Consequently, the link between final demand impulses, gross outputs and income formation is not straightforward, working sometimes in different directions. Regarding this aspect, as we shall show, there is a second way to solve Miyazawa's fundamental equation, which is much more transparent. This second way shows that Miyazawa type endogenization means that gross output and (remaining) final demand are directly linked via a new type of coefficients matrix. This matrix is the sum of the traditional matrix of intermediate input coefficients and a number of matrices of rank 1, each one corresponding to an endogenized households category. The existence of this matrix makes several new applications possible including the study of shifts over time in the distribution of income and (un)employment between the households categories involved. In an appendix we briefly focus on the link between the use of matrices of rank 1 in, respectively, Leontief, Sraffa and Miyazawa input-output economics.

Keywords: Income distribution; Miyazawa endogenization; Input-output models.

Acknowledgements: We would like to thank the anonymous reviewers for their helpful comments. This work was supported by the Ministerio de Ciencia, Inno- vación y Universidades of Spanish Government under Grants RTI2018-095484-B-100 and RTI2018-093970-B-C33; and the Agencia de Gestió d'Ajudes Universitaris i de Recerca of Catalonia's Government under Grant 2017-SGR-852.

1 Introduction [1]

The standard Leontief input-output (IO) model has been the workhorse of multi-sectoral economic analysis for many years. The model gave us the all-important multipliers, which immediately can be used to calculate output and employment impacts of shifts in final demand. The model is excellent in distinguishing direct and indirect effects and it is known as a "type 1" IO model, indicating that households and other final demand related activities (induced effects) are explained exogenously.

Nonetheless, there are a number of issues that the standard IO model cannot adequately deal with. For example, if household consumption (possibly sub-divided in several categories) is exogenous, this means that size and composition of such consumption are determined via mechanisms that are not explained by the model. Here a problem arises, i.e. the fact that the model cannot handle interactions among households' categories themselves and between them and industrial sectors. As a matter of illustration, let us suppose that two categories of final demand are distinguished, say a consumption bundle for skilled and another one for unskilled workers. Suppose further that the skilled workers receive a higher wage. Via various interactions, this can be expected to influence consumption behavior of the unskilled workers, but without additional knowledge it is difficult to say in which way. Clearly, neglecting the possibilities of this type of interactions –as is done when adopting the standard IO model– probably results in neglecting an important part of economic behavior. Additionally, there can be other issues. Assume, for example, that in such a situation the consumption bundle of the unskilled workers population is specified to stay the same. This would mean that consumption *per head* in that group would decline with increasing employment.

Reality may be different, and we enter here an area where the standard IO model needs to be adapted and/or extended. This leads to "type 2" IO models where household behavior is assumed to be endogenously determined to capture these effects. By now there is a substantial literature on these issues, see Sonis and Hewings (1999, 2000), Miller and Blair (2009) and Kim and Hewings (2019). For comprehensive discussions of extended IO models, see Batey and Rose (1990) or Batey (2018).

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[1] This paper originated out of the special session honouring Miyazawa at the 21th International Input-Output Conference, in Kitakyushu, Japan, 2013. The session was organized as a taking stock of Miyazawa's intellectual inheritance.

This is where Miyazawa made an important contribution in re-structuring the standard IO model. He proposed a well-known variant belonging to the "type 2" class of models. Miyazawa's model (Miyazawa and Masegi, 1963; Miyazawa, 1976) is different from related models in that he proposed a different structure to capture household behavior. He distinguished between several households categories and assumed knowledge of consumption propensities and income characteristics per each category. The proposed new structure was formed around the so-called "fundamental equation of income formation", and brought several new analytical tools, which made possible to study in depth the interaction between consumption and income related activities. Miyazawa's proposition resulted in a separate literature including the studies by Sonis and Hewings (1999), Hewings et al., (2001), or Hewings and Parr (2007), among others, based on several types of 'inward' and 'outward' looking multipliers.

However, Miyazawa's distribution model requires a non-negligible effort regarding data. The kind of information that is crucial to build a Miyazawa system is the one related to the disaggregation of the endogenous consumption categories (normally households by a socioeconomic category) and the link between income groups and consumption groups. In the first case, microdata from households budget surveys and a bridge matrix that links COICOP (classification of individual consumption according to purpose) with CPA products (classification of products by activity) is required. For addressing the second issue, it is needed information on the income sources of each household, normally from living conditions or similar surveys. Of course, availability of such databases, their spatial and industrial representativeness (if we are thinking on a model at a subnational level) and having the specific knowledge for combining these sources make the usage of Miyazawa models quite scarce[2].

Many applications of the Miyazawa model have been focused on what may be called separation issues —e.g. developments in certain metropolitan areas where a separation between the central city and its suburbs can be noticed, sometimes looking more like independent economic zones, see e.g. Hewings and Parr (2007). Nonetheless, there is one field where the Miyazawa model can play a more prominent role, i.e. analysing systematic shifts over time or space in the parameters that measure household consumption or remuneration. The long-term dynamics in the age distribution of the population

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[2] Moreover, most researchers interested in income distribution questions opt for other alternative models such as social accounting matrix (SAM) frameworks or computable general equilibrium (CGE) models (that account also for the secondary distribution of income).

provide an example here. These often cause substantial changes in the source of revenues (from wage to non-wage income and pensions) and in the spending behaviour (from spending on education to spending on health services).

In this paper we aim to show that the Miyazawa approach can provide a substantial part of the insight here. This is possible because, as we shall show, a second way exists to solve Miyazawa's fundamental equation of income distribution. This second method shows another perspective on relations and reveals the existence of direct connections between the main variables. This provides a quite different side which is characterized by directness, and which directly can be compared, in terms of properties, with the properties of "type 1" IO models.

The structure of the paper is as follows. In section 2 we briefly discuss the standard "type 1" and "type 2" IO models. In section 3 we present the standard Miyazawa model and the traditional way of solving the fundamental equation of income determination. In section 4 we present the second way of solving this equation, and in section 5 we discuss the link with income distribution problems and we offer a number of simulations to show how the model responds to shifts in parameters. Summary and conclusions are given in section 6. In the Appendix we briefly discuss the relation between Leontief, Sraffa and Miyazawa types of modeling income distributions.

2 The Standard Leontief model

The probably most well-known Leontief-type model (see Miller & Blair, 2009) is the so-called open, static model with one primary factor –usually identified as homogeneous labor– and one final demand category –often called household consumption–. The model is given as [3]

(1) $$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$$

and the employment equation

(2) $$L = \mathbf{l}'\mathbf{x}$$

[3] Matrices are noted in bold and capital letters; vectors in bold and lower case letters; and scalars in italics and lower case letters. Vectors are columns by definition, so their transposed row vectors are indicated by a prime. Moreover, in this paper we follow the standard IO literature that does not use partitions for partitioned matrices (see Miller & Blair (2009, Ch. 6.4).

Here $\mathbf{x}$ and $\mathbf{f}$ are column vectors that stand for gross output and final demand, respectively; $\mathbf{A}$ is the matrix of input coefficients; and $\mathbf{l}'$ –the row vector of direct labor inputs– represents sectoral value-added. L is a scalar that stands for total employment. In this model, final demand $\mathbf{f}$ is determined exogenously.

Extensions of the above model can distinguish several final demand categories including more than one household category, deliveries to government and other public institutions, and exports or net exports. Similarly, more than one primary input category can be distinguished, which may include more than one type of labor. However, in modern IO tables there are no a-priori one-to-one relationship between final demand and value-added categories (like in the simple case of one final demand and one value-added category); see for example the structure of the WIOD tables (Dietzenbacher et al., 2013).

Equation (2) specifies the total quantity of labor required to produce the net output commodity bundle $\mathbf{f}$. It may be, of course, that there is a constraint on the supply of labour, say $L \leq \tilde{L}$. In that case $\mathbf{f}$ must be such that

$$(3) \qquad L = \mathbf{l}'\mathbf{x} \leq \tilde{L}.$$

If more than one primary factor is distinguished, additional constraints can be in place. From (1) we have straightforwardly the familiar output determining equation

$$(4) \qquad \mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$$

where $(\mathbf{I} - \mathbf{A})^{-1}$ is the multiplier matrix. Much of standard Leontief theory is devoted to finding conditions that guarantee that this matrix has properties such as non-negativity when required by economic theory. [4]

The above model is a so-called "type 1" model, indicating that households activities are part of the predetermined final demand vector $\mathbf{f}$. "Type 2" models include households among the explained variables. A characteristic formulation is (Batey, 1985),

$$(5) \qquad \begin{pmatrix} \mathbf{x}_{\mathrm{I}} \\ \mathrm{x}_{\mathrm{H}} \end{pmatrix} = \begin{pmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{h}_c \\ -\mathbf{h}_w & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{d}_{\mathrm{I}} \\ \mathrm{d}_{\mathrm{H}} \end{pmatrix}$$

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[4] For other aspects, especially regarding the interpretation of vertically integrated labor input coefficients, see Pasinetti (1977, Ch. 5); for a discussion of the relation between the notions of vertical integration and circular interdependency, see Cardinale (2018).

where $\mathbf{h}_w$ and $\mathbf{h}_c$ stand for, respectively, household income from employment and household consumption of commodities; $\mathbf{x}_I$ stands for gross output and x_H for total household income from employment; finally, $\mathbf{d}_I$ and d_H represent exogenous income. We recognize that (5) is an extension of (4) with additional attention for distribution issues.

As we can see, the standard IO model gives a connection between final demand, gross and net output, and employment. However, it has a number of drawbacks. Suppose, for example, that more than one household category is being distinguished and that the additionally incorporated households have their own consumption bundle. According to "type 1" based theory, these final demand categories are completely independent. So, suppose that additionally several types of labor quality are distinguished, such as forms of skilled and non-skilled workers. Assuming that these groups have different consumption preferences, we may ask what happens if one category, say a skilled labor type, receives a higher wage. If this results in a higher consumption demand for this category, other categories may profit indirectly because demand for the products they produce may also increase. Such a situation –where connections exist between the final demand categories– cannot be adequately handled by the standard IO model presented above. Here Miyazawa's views enter.

3 Miyazawa endogenization

Miyazawa models (Miyazawa and Masegi, 1963; Miyazawa, 1976) address the problem of modeling the interactions between household categories among themselves and among the rest of the economy. As an alternative to the standard IO model presented in section 2, Miyazawa proposes to treat household consumption and their factor remuneration endogenously —i.e. not accounting for income transfers between institutions (Pyatt, 2001)—. That is, these activities are not assumed to be exogenous anymore, but are explained as a function of other variables. He thereby assumes that households can be sub-divided in *q* income bracket groups and that full information exists on workers consumption and payments patterns in each income group.

So, following Miyazawa, let there be *q* household groups that we want to endogenize and let us assume full information on these. To formalize, let $\mathbf{C} = [c_{ih}]$ be the *n* x *q* matrix of the amounts of sector *i*'s product consumed per \$ of income of households in income group *h* (h = 1,...,q), and let $\mathbf{V} = [v_{gj}]$ be the *q* x *n* matrix of income paid to a wage earner in income bracket g (g = 1,...,q) per \$ worth of output of sector j. This leads to the expanded IO system

$$(6)\qquad \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{V} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} + \begin{pmatrix}\mathbf{f}^*\\ \mathbf{g}\end{pmatrix}$$

and the new, augmented input coefficients matrix

$$(7)\qquad \mathbf{M} = \begin{pmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{V} & \mathbf{0}\end{pmatrix}$$

Here **x** again stands for the gross output vector, **y** is the vector of total income per income group, **f*** the vector of final demand excluding the q endogenized households categories, and **g** a vector of exogenous income (if any) for the income groups. Solving for $\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$ we have

$$(8)\qquad \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{I}-\mathbf{A} & -\mathbf{C}\\ -\mathbf{V} & \mathbf{I}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{f}^*\\ \mathbf{g}\end{pmatrix}$$

Recalling the procedure for obtaining the inverse of the partitioned matrix on the r.h.s. of equation (7) and with $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$(9)\qquad \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{B}[\mathbf{I}+\mathbf{C}(\mathbf{I}-\mathbf{VBC})^{-1}\mathbf{VB}] & \mathbf{BC}(\mathbf{I}-\mathbf{VBC})^{-1}\\ (\mathbf{I}-\mathbf{VBC})^{-1}\mathbf{VB} & (\mathbf{I}-\mathbf{VBC})^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{f}^*\\ \mathbf{g}\end{pmatrix}$$

Here $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$ is the well-known Leontief multiplier matrix.

Again following Miyazawa, we now simplify the notation and write $\mathbf{L} = \mathbf{VBC}$ and $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1} = (\mathbf{I} - \mathbf{VBC})^{-1}$, where Miyazawa identifies **VBC** as the matrix of inter-income-group coefficients and the inverse $(\mathbf{I} - \mathbf{VBC})^{-1}$ as the interrelational income multiplier matrix. This results in the familiar equation

$$(10)\qquad \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{B}(\mathbf{I}+\mathbf{CKVB}) & \mathbf{BCK}\\ \mathbf{KVB} & \mathbf{K}\end{pmatrix}\begin{pmatrix}\mathbf{f}^*\\ \mathbf{g}\end{pmatrix}$$

To focus on **f*** –thereby following the standard approach–[5] we put $\mathbf{g} = \mathbf{0}$, which results in

$$(11)\qquad \mathbf{x} = [\mathbf{B}(\mathbf{I}+\mathbf{CKVB})]\mathbf{f}^*$$

[5] See e.g. Miller & Blair (2009, Ch. 6.4).

We see that the multiplier matrix of the standard IO model is enlarged by the term **BCKVB**, a consequence of the additional relations that have been accounted for. For the associated remunerations we then have

(12) $$\mathbf{y} = \mathbf{KVBf^*}$$

Equation (11) is Miyazawa's fundamental equation of income formation. This equation is fundamental in the sense that it combines an expression for the direct and indirect effects of a unit change in final demand – i.e. equation (4) – with a multiplier denoting the extent to which the direct and indirect effects are magnified by the induced effect.

Compared to the standard IO model of section 2, the addition of the **BCKVB** component is a significant step towards further endogenization of an economy's main variables. The interpretation of the additional component is straightforward. Exogenous final demand **f*** will generate (through **B**) direct and indirect changes in production. The product **VB** provides the direct and indirect income that will be generated, and the product **CVB** tells us how that income is spent, via connections brought about by **K**, the interrelational income multiplier that indicates how income change in one household group will generate additional income in other groups. The total impact then is given by the term **BCKVB**. In this way, decomposing the Miyazawa approach provides a 'walk through the system', comparable to the $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$ decomposition of SAMs by Pyatt and Round (1985).[6]

Additionally, the **K** matrix shows interesting insights on the relationships between the endogenous institutional sectors considered. In the previous literature, it has been used to present the income generation and distribution between households of different ages (Kim and Hewings, 2019), but also of different regions (Hewings et al., 2001; Hewings and Parr, 2007), revealing important asymmetries that otherwise would remain hidden.

Therefore, the model has been used extensively to trace distribution-related problems including their spatial configuration. Given sufficient data on matrices **B**, **C**, **V**, and **K**, the development over time and space of particular industries and/or categories of workers can be followed to explain shifts in gross output. Equation (12) gives the corresponding composition of earned incomes.

[6] We would like to thank one of the referees for this suggestion.

Conceptually, the model was a great step forward providing researchers with a new set of tools. However, as mentioned, there is a second way to understand what happens when households are endogenized. Below we shall present this second way, and show in which way it helps us in interpreting and applying the model.

4 A second way to solve the fundamental equation of income formation

We start by taking a look at equation (6). Written out we have (assuming again g=0)

(13) $\mathbf{x} = \mathbf{Ax} + \mathbf{Cy} + \mathbf{f}^*$

and

(14) $\mathbf{y} = \mathbf{Vx}$

We can straightforwardly substitute **y** in equation (13), which gives

(15) $\mathbf{x} = \mathbf{Ax} + \mathbf{CVx} + \mathbf{f}^*$

or

(16) $\mathbf{x} = (\mathbf{A} + \mathbf{CV})\mathbf{x} + \mathbf{f}^*$

which provides a different way of looking at the composition of gross output **x**. Matrix **A** + **CV** is a second extended input coefficients matrix of dimension *n* x *n*, and it has an interesting property that we will discuss in section 4.1 below.

4.1. Structure of matrix A + CV

Let us now take a closer look at the matrix product **CV** where, as before, $\mathbf{C} = [c_{ih}]$ is the *n* x *q* matrix of the standardized amounts of sector *i*'s product consumed and $\mathbf{V} = [v_{gj}]$ the *q* x *n* matrix of standardized income paid. We have

Lemma:

Let **C** and **V** be as defined above. With $\mathbf{c}_{.1}$ standing for the first column of matrix **C**, etc., we have $\mathbf{CV} = \mathbf{c}_{.1}\mathbf{v}_{1.} + \mathbf{c}_{.2}\mathbf{v}_{2.}$

Proof:

We only provide the proof for $q = 2$, the extension to $q = n$ being straightforward. We have

(17)
$$\mathbf{CV} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11}v_{11} + c_{12}v_{21} & c_{11}v_{12} + c_{12}v_{22} & c_{11}v_{13} + c_{12}v_{23} \\ c_{21}v_{11} + c_{22}v_{21} & c_{21}v_{12} + c_{22}v_{22} & c_{21}v_{13} + c_{22}v_{23} \\ c_{31}v_{11} + c_{32}v_{21} & c_{31}v_{12} + c_{32}v_{22} & c_{31}v_{13} + c_{32}v_{23} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix} (v_{11} \quad v_{12} \quad v_{13}) + \begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \end{pmatrix} (v_{21} \quad v_{22} \quad v_{23})$$

$$= \mathbf{c}_{.1}\mathbf{v}_{1.} + \mathbf{c}_{.2}\mathbf{v}_{2.}$$

Thus, **CV** is the sum of two matrices which are the outer product of, respectively, the vectors $\mathbf{c}_{.1}$ and $\mathbf{v}_{1.}$ and $\mathbf{c}_{.2}$ and $\mathbf{v}_{2.}$. That is, **CV** is the sum of two matrices of rank 1. Writing $\mathbf{A}_1 = \mathbf{c}_{.1}\mathbf{v}_{1.}$ and $\mathbf{A}_2 = \mathbf{c}_{.2}\mathbf{v}_{2.}$, we obtain

(18)
$$\mathbf{x} = (\mathbf{A} + \mathbf{CV})\mathbf{x} + \mathbf{f}^*$$
$$= (\mathbf{A} + \mathbf{A}_1 + \mathbf{A}_2)\mathbf{x} + \mathbf{f}^*$$

Matrix $\mathbf{A}_1$ is the outer product of vectors $\mathbf{c}_{.1}$ and $\mathbf{v}_{1.}$. This means that the larger the elements of $\mathbf{c}_{.1}$ and/or $\mathbf{v}_{1.}$, the larger the elements of $\mathbf{A}_1$ will be. The same is true, *mutatis mutandis*, regarding matrix $\mathbf{A}_2$. Correspondingly, the elements of matrix $\mathbf{A} + \mathbf{A}_1 + \mathbf{A}_2$ and the inverse $[\mathrm{I} - (\mathbf{A} + \mathbf{A}_1 + \mathbf{A}_2)]^{-1}$ will become larger for larger elements of $\mathbf{A}_1$ and $\mathbf{A}_2$. Vice versa, for elements of $\mathbf{A}_1$ and $\mathbf{A}_2$ becoming smaller, a reverse outcome can be observed.

So, gross output following Miyazawa can alternatively be interpreted as being determined by $\mathbf{f}^*$ via

(19)
$$\mathbf{x} = [\mathrm{I} - (\mathbf{A} + \mathbf{A}_1 + \mathbf{A}_2)]^{-1}\mathbf{f}^*$$

The multiplier matrices in question are equal, as can be verified straightforwardly, i.e.

(20)
$$[\mathbf{I} - (\mathbf{A} + \mathbf{CV})]^{-1} = \mathbf{B}(\mathbf{I} + \mathbf{CKVB})$$

4.2. The Distribution of Incomes

The above description of the origin of factor incomes has additional interesting properties. In fact, the introduction of the two rank 1 matrices results in a new decomposition of gross output. Starting from (18), we have

(21)
$$\mathbf{x} = (\mathbf{A} + \mathbf{CV})\mathbf{x} + \mathbf{f}^*$$

$$= (\mathbf{A} + \mathbf{A}_1 + \mathbf{A}_2)\mathbf{x} + \mathbf{f}^*$$

$$= \mathrm{A}\mathbf{x} + \mathbf{c}_{.1}\mathbf{v}_{1.}\mathbf{x} + \mathbf{c}_{.2}\mathbf{v}_{2.}\mathbf{x} + \mathrm{f}^*$$

Or,

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \gamma_1\mathbf{c}_{.1} + \gamma_2\mathbf{c}_{.2} + \mathbf{f}^*$$

with $\gamma_1 = \mathbf{v}_{1.}\mathbf{x}$ and $\gamma_2 = \mathbf{v}_{2.}\mathbf{x}$.

So, from expressions in equations (21), we can follow the distribution of incomes straightforwardly by following the shift over time in the two gammas (given constant $\mathbf{c}_{.1}$ and $\mathbf{c}_{.2}$). Therefore, they can be taken as indicators of the relative development of the income *shares* over time.

We note that $\mathbf{c}_{.1}$ and $\mathbf{c}_{.2}$ give the standardized and *constant* consumption bundles for household categories 1 and 2, respectively. These bundles are multiplied by the coefficients γ_1 and γ_2 that are positive functions of (the elements of) $\mathbf{x}$. So, if gross output increases, also the gammas will increase in size. The products $\gamma_1\mathbf{c}_{.1}$ and $\gamma_2\mathbf{c}_{.2}$ then record shifts in the distribution of income between the two household categories. (Note that additional households categories can be straightforwardly incorporated in the relations above).

Below we shall present a numerical illustration of the relation over time between the two income shares in equations (21) given certain developments in non-households final demand. So let's take a look at the quotient $\gamma = \gamma_1 / \gamma_2$ over time. We thereby have assumed a development path for $\mathbf{f}^*$ that does not seem unrealistic (see Figure 1).

4.3. <u>A second walk through the system</u>

In section 3 we sketched the potential of the Miyazawa model via a 'walk through the system'. As we shall show below, the newly proposed decomposition offers the possibility of a second 'walk' which is entirely due to the presence of the $\mathbf{A}_1$ and $\mathbf{A}_2$ matrices. While the first one explicitly incorporated the various direct, indirect and induced impacts of a shift in final demand, the proposed second one opens up many possibilities especially in following systematic changes in the basic parameters determining consumption, income, long-term trends, policy changes, etc.

For example, by considering equation (21), we can detect in which way income changes in one group of households will generate additional income in other groups.

We also can detect ways in which changes in consumer preferences in one group influence preferences in another group after an increase in income for that group. Also, our proposed decomposition makes it possible to detect –and follow- the impact of long-term shifts in the basic structure of an economy. We may be thinking here of ageing, automation and the subsequent replacement of labour for capital, etc. An overall characteristic is that we are taking a look at the system where all direct and secondary effects have been accounted for, so we have at our disposal a "total" effect.

We may proceed by taking a closer look at the last equation of expressions (21). Suppose consumer habits in period *t* for group 1 change, and that consumption of one or more goods increases, and can be captured via the coefficients of $\mathbf{c}_{.1}$. This would mean, 'ceteris paribus', a higher $\mathbf{x}_t$ in this period, the size of the effects depending on the relative multiplier effects of the new consumption preferences with respect to the previous consumption preferences. In general, with unchanged $\mathbf{v}_{1.}$ We will observe a higher income for group 1 in its totality. However, it also will mean a higher income for group 2, from a rise in $\mathbf{v}_{2.}\mathbf{x}_t$, given unchanged $\mathbf{v}_{2.}$. From the same equation, we have that both changes also imply a change in employment, in all sectors.

We can say more, however. With $\mathbf{v}_{1.} = (v_{21} \quad v_{22} \quad v_{23})$ we have:

(22) $$\mathbf{v}_{1.}\mathbf{x}_t = \mathbf{v}_{11}\mathbf{x}_{1,t} + \mathbf{v}_{12}\mathbf{x}_{2,t} + \mathbf{v}_{13}\mathbf{x}_{3,t}$$

and

(23) $$\mathbf{v}_{2.}\mathbf{x}_t = \mathbf{v}_{21}\mathbf{x}_{1,t} + \mathbf{v}_{22}\mathbf{x}_{2,t} + \mathbf{v}_{23}\mathbf{x}_{3,t}$$

$\mathbf{x}_{1,t}$ standing for the first element of vector $\mathbf{x}_t$, etc. Again starting from an autonomous rise in one or more elements of $\mathbf{c}_{.1}$, we may wish to see what a rise in the elements of $\mathbf{x}_t$ will mean for incomes in the total group and at the sector level. To this end, we can compare the elements on the r.h.s. of the above two equations pair-wise (with unchanged $\mathbf{v}_{1.}$ and $\mathbf{v}_{2.}$. Suppose the term $\mathbf{v}_{11}$ on the r.h.s. of equation (22) is "large" and that the corresponding coefficient in equation (23), $\mathbf{v}_{21}$, is "small". We then can say that the impact on the second group and in the first sector probably is much smaller than in the first group.

There is another point here. We can also directly note an impact on the income distribution. This is because the three terms on the r.h.s. of equations (22) and (23) add up to total income. So, this effect (i.e. a "large" vs a "small" impact) immediately means that both groups will –as far as this first component is concerned- drift apart. Subsequently, it then depends on the size of the other two terms on each r.h.s. what exactly will happen, and how fast. This can be seen as an illustration how the income distribution will be affected and, also, how difficult it can be to counter certain undesired developments such as increasing divergence in incomes. (In the illustrations we come back to this).

A further insight can be obtained by considering long term trends in either $\mathbf{v}_{1.}$, $\mathbf{v}_{2.}$, $\mathbf{c}_{.1}$, or $\mathbf{c}_{.2}$. Suppose we observe a change over time in income for group 1 where the new income vector $\underline{\mathbf{v}}_{1.}$ is given by the equation $\underline{\mathbf{v}}_{1.} = t\mathbf{v}_{1.}$. With $t < 1$ we then have declining incomes for group 1. By incorporating this trend in our equation for gross output, we immediately obtain the consequences over time, for group 1 as well as for the other two groups. As a next step, we can incorporate policies to correct such unwanted developments. Considering shifts over time (and/or space) in the above mentioned vectors $\mathbf{v}_{1.}$, $\mathbf{v}_{2.}$, $\mathbf{c}_{.1}$, or $\mathbf{c}_{.2}$. may, in terms of the model we have explored here, be the way to capture the most relevant aspects of phenomena like the shifting age distribution of working populations or the arrival of robots in significant amounts. Further exploration will be asked for here.

5 Income distribution exogenously generated and simulations

Above we have seen how the choice of $\mathbf{f}^*$ determines the distribution of income between the two categories of households.[7] However, we can look at a different way at the model; we can see —in particular— what aiming at a certain distribution of income would mean in terms of gross output. We do this by going back to the fourth expression of equation (21), where we recognize the standard way of looking at the model causality, running from non-households final demand $\mathbf{f}^*$ to gross output $\mathbf{x}$. Given $\mathbf{x}$, income shares then are determined as

(24) $$\mathbf{A}_1\mathbf{x} = \mathbf{c}_{.1}\mathbf{v}_{1.}\mathbf{x} = \gamma_1\mathbf{c}_{.1}$$

and

(25) $$\mathbf{A}_2\mathbf{x} = \mathbf{c}_{.2}\mathbf{v}_{2.}\mathbf{x} = \gamma_2\mathbf{c}_{.2}$$

[7] Recall that we know exactly the consumption coefficients of both household categories.

As pointed out, the model has an additional property in that it allows us a certain amount of freedom in selecting which variables we would like to be exogenously determined and which endogenously.

Up to now, (non-households) final demand **f*** has been the variable of the model exogenously determined, while the other ones were endogenous. However, also a vice-versa interpretation is possible. That interpretation leads to a connection with income or employment policies working via gross output as the central variable. From (21), we have that γ_1 is the product of (constant) $\mathbf{v}_{1.}$ and variable $\mathbf{x}$. So, via policies focusing on $\mathbf{x}$, we can influence γ_1 and, therefore, the distribution of incomes.

To see how this might be accomplished, let us return to (21). Exogenously fixing $\mathbf{x}$ means that, via $\mathbf{v}_{1.}\mathbf{x}$ and $\mathbf{v}_{2.}\mathbf{x}$, both gammas are fixed. However, that means that also the income distribution is fixed. And, it means that **f*** becomes an endogenous variable. So, if indeed we can consider the gammas as being determined or 'aimed at' by policy makers, this opens the door to a range of new developments. Policy makers can, according to their insights, attribute specific values to γ_1 and/or γ_2, which then become policy parameters. Each such selection results in a different distribution of incomes.

The above opens up the possibility of using the gammas as 'buttons' to calibrate the model at generating a certain desired distribution of incomes. Further below we shall provide different numerical simulations.[8]

5.1. Policy constraints

In section 2, when discussing the standard Leontief model, we pointed out there may be constraints in the supply of labor, the only primary factor being distinguished. If a certain net output vector **f*** requires more labor than is being supplied, this net output bundle cannot be produced, and output goals have to be changed. The same is true if more than one primary factor is distinguished. If one of these is in short supply, the net output bundle **f*** cannot be produced.

In the model we have presented above in section 4, similar input constraints are present. However, given the structure of the model, they are found via a different way, i.e. via $\gamma_1 = \mathbf{v}_{1.}\mathbf{x}$ and $\gamma_2 = \mathbf{v}_{2.}\mathbf{x}$. If the supply of one of the two categories is below

[8] Different developments are possible. We can, alternatively, start from a given γ_1 and calculate what γ_2 must be in 10 periods to have $\gamma(t) = \gamma_1(t)/\gamma_2(t) = h(t)$, where h(t) is a policy determined variable. However, these other variants are beyond the scope the aim of this paper.

the corresponding gamma value, the ‘program’ is infeasible and, if policy makers wish to continue, the values of the gammas will have to be modified accordingly.

5.2. Shifts in parameters

So, the above approach offers direct insight in the links between the income distribution determining parameters. Clearly, we also can extend our interpretation of model parameters to the elements of matrices **C** and **V** to investigate the impact of systematic shifts in household income and consumption activities on the income distribution. We see that shifts in the elements of these matrices (i.e. **C** and **V**) can be directly related to shifts in **f*** and **x**. That is, we have a direct ‘chain’ from Δ**f*** to Δ**x** and the corresponding change in income distribution.

Simulation 1

This first simulation illustrates the large scope of the Miyazawa model as presented above. Before presenting different simulations, let us start with an initial situation in which three sectors (A) and two households’ categories. Moreover, for all households’ categories, the standardized consumption (C) and income coefficients (V) are available as the next numerical example shows: [9]

$$A = \begin{bmatrix} 0.15 & 0.25 & 0.05 \\ 0.20 & 0.05 & 0.40 \\ 0.30 & 0.25 & 0.05 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.10 & 0.05 \\ 0.20 & 0.10 \\ 0.01 & 0.10 \end{bmatrix}$$

and

$$V = \begin{bmatrix} 0.05 & 0.10 & 0.08 \\ 0.12 & 0.05 & 0.10 \end{bmatrix}$$

This gives the standard Leontief multiplier matrix as

$$B = (I - A)^{-1} = \begin{bmatrix} 1.365 & 0.425 & 0.250 \\ 0.527 & 1.348 & 0.595 \\ 0.569 & 0.489 & 1.288 \end{bmatrix}$$

[9] The numerical values of the coefficients are taken from Miller & Blair (2009, Ch. 6.4).

In addition, there is a final demand category that consists of final demand excluding household consumption (f*); we shall refer to this category as 'remaining final demand', or simply 'final demand' when no confusion is possible:

$$\mathrm{f}^* = \begin{pmatrix} 2000 \\ 4000 \\ 1000 \end{pmatrix}$$

From equation (11), the Miyazawa 'all-in' multiplier matrix is

$$\mathrm{B(I + CKVB)} = \begin{bmatrix} 1.444 & 0.499 & 0.3234 \\ 0.649 & 1.460 & 0.7061 \\ 0.657 & 0.564 & 1.3648 \end{bmatrix}$$

With g = 0, this results in

$$\mathrm{x} = [\mathrm{B(I + CKVB)}]\mathrm{f}^* = \begin{pmatrix} 5210.22 \\ 7848.78 \\ 4937.69 \end{pmatrix}$$

and

$$\mathrm{y} = [\mathrm{KVB})]\mathrm{f}^* = \begin{pmatrix} 1440.40 \\ 1511.43 \end{pmatrix}$$

From this numerical example we can also illustrate the new decomposition we presented above in this paper. We calculate matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, the outer product of vectors $\mathbf{c}_{.1}$ and $\mathbf{v}_{1.}$, and $\mathbf{c}_{.2}$ and $\mathbf{v}_{2.}$, respectively,

$$\mathbf{A}_1 = \mathbf{c}_{.1}\mathbf{v}_{1.} = \begin{pmatrix} 0.0050 & 0.0100 & 0.0080 \\ 0.0100 & 0.0200 & 0.0160 \\ 0.0005 & 0.0010 & 0.0008 \end{pmatrix}$$

and

$$\mathbf{A}_2 = \mathbf{c}_{.2}\mathbf{v}_{2.} = \begin{pmatrix} 0.0060 & 0.0025 & 0.0050 \\ 0.0120 & 0.0050 & 0.0100 \\ 0.0120 & 0.0050 & 0.0100 \end{pmatrix}$$

Numerically, the new coefficients matrix will be

$$\mathbf{A} + \mathbf{A}_1 + \mathbf{A}_2 = \begin{bmatrix} 0.1610 & 0.2625 & 0.0630 \\ 0.2220 & 0.0750 & 0.4260 \\ 0.3125 & 0.2560 & 0.0608 \end{bmatrix}$$

with corresponding multiplier matrix

$$(\mathrm{I} - [\mathbf{A} + \mathbf{A}_1 + \mathbf{A}_2])^{-1} = \begin{bmatrix} 1.4440 & 0.4990 & 0.3234 \\ 0.6490 & 1.4600 & 0.7061 \\ 0.6570 & 0.5640 & 1.3648 \end{bmatrix}$$

which is the same matrix as obtained before, following Miyazawa's approach.

We can also start from calculating the gammas and how the income is distributed between the two household categories:

$$\gamma_1 = \mathbf{v}_{1.}\mathbf{x} = 1440.406$$

$$\gamma_2 = \mathbf{v}_{2.}\mathbf{x} = 1511.437$$

$$\gamma = \gamma_1/\gamma_2 = 0.953$$

Below we shall present a number of numerical illustrations of the decompositions discussed above. These illustrate in particular how the successive incorporation of additional shifts in the coefficients will influence sectoral behaviour and the income distribution.

Figures 1-a to 1-d present the evolution of the household incomes over time given a certain shift in the exogenous variables or parameters. The vertical axis gives the time period. Figure 1-e gives the resulting income distribution.

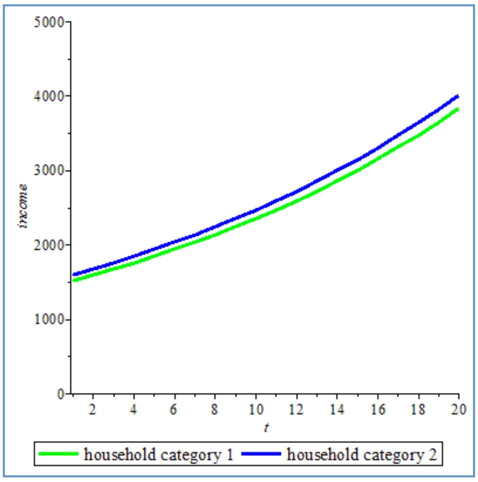
Figure 1-a gives what we have indicated as the basic situation or scenario. In this first exercise we discuss the impact of a 5% change in (remaining) final demand for each period on household incomes and the income distribution. As expected, household incomes will increase proportionally. Clearly, the income distribution remains the same, as given by figure 1-e, the green line.

Figure 1-b gives a similar exercise, but now with an absolute increase of 200 units in final demand for good 1 and an unchanged (final) demand for the other two goods (i.e. 4000 and 1000 units, respectively). Figure 1-c records the impact of a change of 200 units for good 2, the other two being unchanged at 2000 and 1000. Figure 1-d gives the analogous outcomes for good 3. The resulting shift in the income distribution for each exercise is given by the corresponding curves in figure 1-e.

Figure 1: Functional relation over time between γ_1 and γ_2 for given f*

Figure 1-a

Figure 1-b



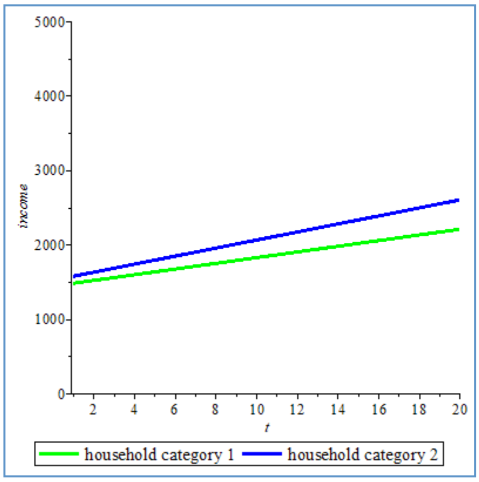


Figure 1-c

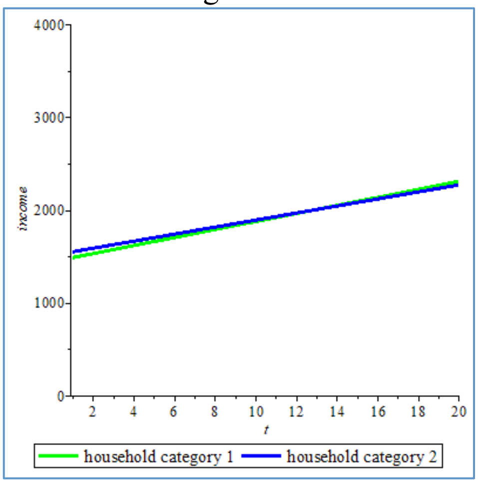


Figure 1-d

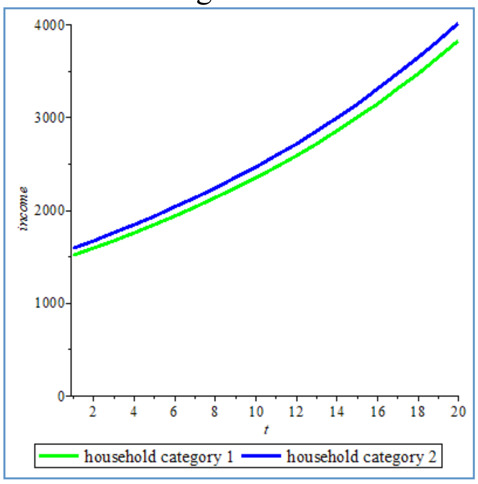
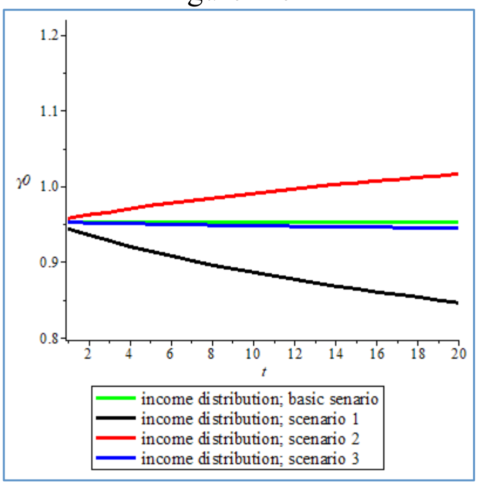


Figure 1-e



Clearly, we can see that different shifts in the remaining final demand cause different outcomes in terms of how the income of this “toy model” economy is distributed between the household categories. For example, we see how a shock in sector 2 would imply a decrease in income differences while an expansion of demand for sector 1's product would aggravate these.

Simulation 2

Simulation 2 explores the effects of a systematic change in the remuneration of workers of both household categories (for the moment we do not go into the causes of this shift, which may reflect technological changes, changes in consumer preferences, competitive positions, or otherwise).

This simulation is superposed on simulation 1. That is, each of the four exercises is repeated, but now combined with a systematic *decrease* of one pro mille in each of the income coefficients of household category 1 and a similar *increase* in the income coefficients of household category 2. That is, indicating the income coefficients matrix $\mathbf{V}$ at t = 0 by the symbol $\mathbf{V}_0$, we have for the initial period 0 (see section 5.2)

$$\mathbf{V}_0 = \begin{bmatrix} 0.050 & 0.100 & 0.080 \\ 0.120 & 0.050 & 0.100 \end{bmatrix}$$

and for period 1

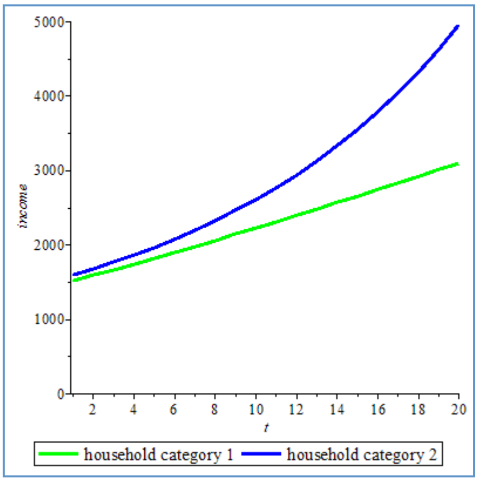
$$\mathbf{V}_1 = \begin{bmatrix} 0.049 & 0.099 & 0.079 \\ 0.121 & 0.051 & 0.101 \end{bmatrix}$$

So, Figure 2-b shows the impact corresponding to a 200 units increase in final demand for good 1 combined with the corresponding shift in the income coefficients, and so on for the rest of Figures. We note a much more rapid change in the income distribution (as given by Figure 2-e) than in the case of no shifts in the income coefficients (note that the green and the blue line almost coincide here). Employing a model of the type discussed above, empirical work will have to decide on the relative speed in which the groups will be drifting apart along the lines signaled in section 4.3.

Figure 2: Effects of systematic changes in worker remuneration for given f*

Figure 2-a

Figure 2-b



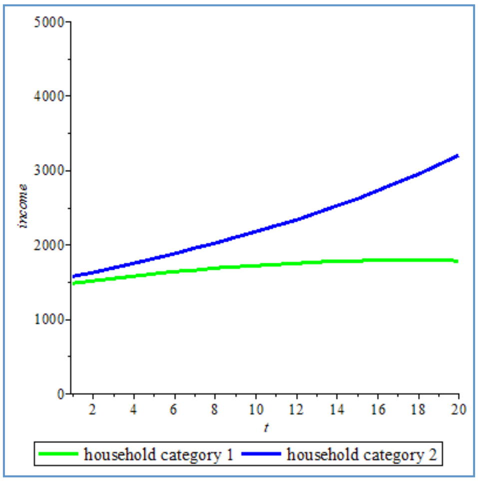


Figure 2-c

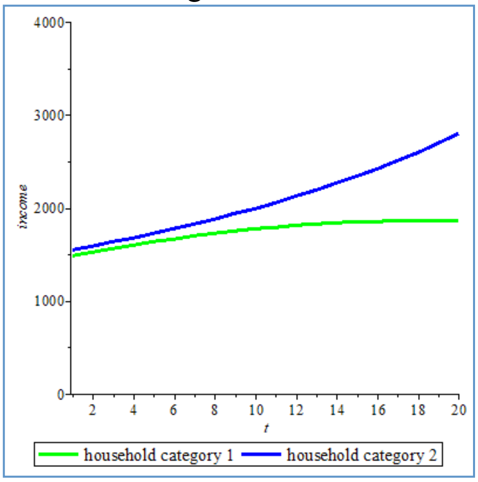


Figure 2-d

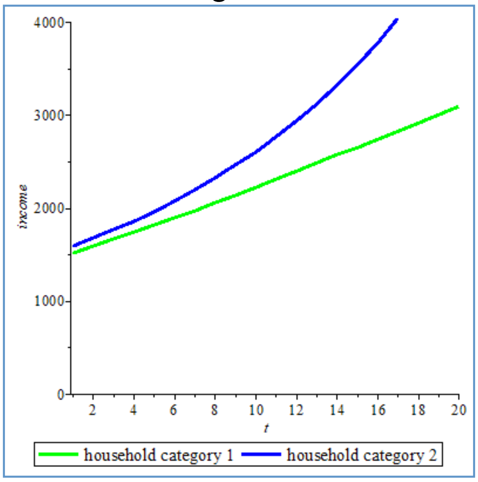
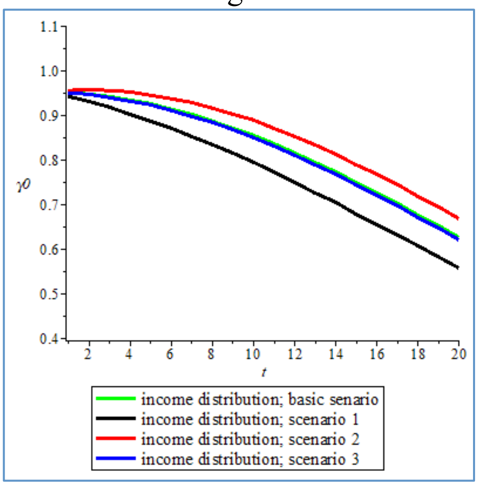


Figure 2-e



5.3. ***Interpretation in terms of socio-political interests***

In a recent article Cardinale (2018) raised the question if socio-political aggregations (labor unions, consumer actions, environmental protection groups, a.s.o.) can be understood from a structural political economics point of view. Here, the author focused on the structure offered by multi-sector models of the Leontief, Sraffa, and Pasinetti type. Central here was the distinction between circular interdependence and vertical integration. Circular interdependence and vertical integration differ in that circular interdependence focuses on (sometimes highly aggregated) sectors which produce a characterizing product, while vertical integration offers an aggregation that focuses on the direct relation between final demand categories and primary factor inputs.

Central concepts in the proposed analysis are viability at various levels of aggregation and the possibility of conflict. This in turn asks for certain limits to be imposed on the system (basically to guarantee continuing viability). Given the model structures, class solidarity can be modeled by focusing, for example, on the value added row(s) in Leontief models, either by looking at individual sectors or at a combination of sectors. Action in the context of, say, climate change programs, can be valued by considering the impact of primary factor variation on commodity prices.

The above offers an interesting context for the Miyazawa-based income distribution presented in this paper. Viability, for example, is guaranteed by the condition that the dominant eigenvalue of matrix $A + A_1 + A_2$ must be smaller than 1. The gamma parameters straightforwardly are derived from the value added side but simultaneously (also) from the consumer side. They allow a direct interpretation in terms of collective versus particular interests (for more details, see also Appendix). As such our Miyazawa proposed structural decomposition and re-arrangement represents a form in-between strictly interpreted circular interdependence and vertical integration options. Additional exploration will be asked for here.

6 Summary and Conclusion

In this paper we have returned to Miyazawa well-known approach at the role of households activities in open Leontief modeling. Traditionally these activities are assumed to be exogenously determined, but –as the literature showed– this can lead to serious misrepresentations in terms of output and income distribution, the reason being that any interactions between households among themselves and between households and the industrial sectors are neglected. Miyazawa's solution was to explain household activities using an endogenization procedure. That is, these activities (sub-divided into a number of categories) were accounted for as if they were industries. In terms of modeling this meant that the corresponding input coefficients became part of an 'extended input coefficients matrix'. In terms of economic tools this meant the introduction of a new type of multipliers to capture the various 'internal' and 'external' effects.

The new approach, however, was at the cost of the relative transparency that characterizes many of the earlier input-output (IO) models. The new model is not particularly transparent in getting a good and quick impression of, say, the impact of changes in (non-households) final demand on the distribution of incomes. The reason is the presence of various types of multiplier effects, some pointing in one direction, others in another one. Central here is the 'fundamental equation of income formation', which explains sectoral gross outputs and the factor incomes in terms of (non-households) final demand. Solving this equation involves inverting a large, partitioned square matrix, the sub-matrices of which themselves are functions of several matrices. It is this inversion procedure that substantially eliminates the IO model's traditional transparency.

Regarding this aspect we show in this paper that there is a *second* way to solve the fundamental equation of income distribution. The existence of this second way is a consequence of the structure of household activities as proposed by Miyazawa. As we show, this is equivalent to introducing a new type of coefficients matrix which is the sum of the traditional full rank matrix of intermediate input coefficients and a number of matrices of rank 1, each one corresponding to an endogenized household category. This new matrix provides a new look at the Miyazawa structure and reveals the existence of direct connections between the main variables of the economy. In particular, the income distribution becomes much more transparent. As we show in the Appendix, the structure of this second approach can directly be compared with the structure of the earlier, 'simpler' IO versions.

Our proposed alternative is, in our view, particularly useful in analyzing the effects of specific long-term trends in an economy's basic variables, such as the shifting age distribution of a country's population and more in particular the shift in the age distribution of the working population over time. We have simulated a number of examples of long-term shifts by first presenting the effects on the income distribution when gross output obeys a steady growth. Hereafter we have superimposed on this certain long-term trends in the distribution of value added over income groups. We have shown that such trends can be a very important factor in determining gross output and the (relative) distribution of incomes.

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Appendix

Matrices of rank 1 in IO analysis

Having put forward in the above our interpretation of the Miyazawa model, it may be useful to take a step back and to also devote a few words to the question how we can understand Miyazawa's approach from a Leontief point of view. We shall thereby focus on the coefficients matrices of rank 1.

We first go back to equations (1) and (2), which give us total output **x** and total employment L. We note that there is only a single primary factor (labour), which obtains the entire net output **f**. We also know **l**′, the (row) vector of factor remuneration per industry. Adding a price equation completes the model

$$\mathbf{p}' = \mathbf{p}'\mathbf{A} + w\mathbf{l}' \tag{26}$$

where w is the wage rate, expressed in euros or dollars, say.

Traditionally equation (26) is solved for **p**′ which gives

$$\mathbf{p}' = w\mathbf{l}'(\mathbf{I} - \mathbf{A})^{-1} \tag{27}$$

Here matrix $(\mathbf{I} - \mathbf{A})^{-1}$ is the well-known multiplier matrix, and prices are in terms of embodied labour. A substantial part of the IO literature is devoted to the properties of this matrix to make the model economically interpretable. For example, in standard applications all elements of the multiplier matrix should be positive. [10]

However, there is a second way to interpret prices. This can be seen as follows. Suppose again that final demand is fully consumed by the single primary factor we have distinguished, labour. We then can consider the commodity bundle **f** as the real wage for the entire work force. If we also adopt the standard assumption of homogeneous labour, consumption per head is **f**/L, a commodity bundle obtained by dividing each element of **f** by L. To link the wage in money terms to the real wage, we impose an standardization, say,

$$w = \mathbf{p}'(\mathbf{f}/L) = \mathbf{1} \tag{28}$$

Substitution in (28) gives

$$\mathbf{p}' = \mathbf{p}'[\mathbf{A} + (\mathbf{f}/L)\mathbf{l}'] \tag{29}$$

[10] For mathematical aspects, see e.g. Takayama (1970, Ch 4), Pasinetti (1977, Mathematical Appendix) or Kemp and Kimura (1978, Ch 2).

We see that $\mathbf{p}'$ is now is modelled as the left-hand Perron-Frobenius eigenvector of the matrix on the r.h.s. of (29). This eigenvector corresponds to the dominant eigenvalue of that matrix, which is equal to one. We can solve (29) starting from $\mathbf{f}$ (which is exogenously given, as in section 2). Let DET stand for the determinant of the matrix $\mathbf{I}$ (the unit matrix) minus the same r.h.s. matrix. We then must have

$$\text{(30)} \qquad \text{DET} = \left| \mathrm{I} - [\mathbf{A} + \left(\frac{\mathbf{f}}{L}\right)\mathbf{l}'] \right| = 0$$

This fixes L which now can be calculated straightforwardly. [11] Hereafter $\mathbf{p}'$ is easily calculated as the corresponding left-hand eigenvector.

So, we observe that the basic equations of the standard Leontief model allows us to interpret the price vector $\mathbf{p}'$ in two ways, once in terms of embodied labour and once in terms of the dominant eigenvector of matrix $\mathbf{A} + \mathbf{H}$, where

$$\text{(31)} \qquad \mathbf{H} \equiv \left(\frac{\mathbf{f}}{L}\right)\mathbf{l}'$$

We see that $\mathbf{H}$ has rank 1, being the outer product of the column vector $\mathbf{f}/L$ and the row vector $\mathbf{l}'$, so we have here a formulation of a Leontief-type price equation in which a matrix of rank 1 plays a central part. We also see that the above representation is possible because there is a single, homogeneous factor the income per head of which (i.e. the vector $\mathbf{f}/L$) is known together with the accompanying factor remuneration vector $\mathbf{l}'$. We note that this one-to-one relation between final demand categories and value added categories is not upheld in the standard Leontief model, which explains the absence of input coefficients matrices of rank 1 in standard Leontief theory. However, as we have seen, retaining this one-to-one relation adds a new tool to our set of instruments.

It may be useful take a brief look at a different but related multi-sector income distribution model, i.e. the Sraffian. Matrices of rank 1 do play a role in Sraffian income distribution analysis as shown in Steenge and Serrano (2012). We again start from the same two equations (1) and (2) and a price equation that contains a rate of profit r on circulating capital. In addition, we again use the homogeneity assumption regarding labour and introduce two standardizations. The Sraffian price equation is

$$\text{(32)} \qquad \mathbf{p}' = \mathbf{p}'(1+r)\mathbf{A} + \omega\mathbf{l}'$$

---

[11] Equation (30) is a single equation in one unknown, L.

where ω is the *share* of net output going to labour. The standardizations are $\mathbf{p}'\mathbf{f} = 1$ and $L = 1$. Equation (32) is a well-known representation of the price vector in Sraffian economics, see e.g. Pasinetti (1977, Ch 5).

However, following Steenge and Serrano (2012), we may rewrite to

$$(33) \qquad \mathbf{p}' = \mathbf{p}'[(1+r)\mathbf{A} + \omega\left(\frac{\mathbf{f}}{L}\right)\mathbf{l}']$$

where we have used $\mathbf{p}'\left(\frac{\mathbf{f}}{L}\right) = 1$. In equation (33), as in equation (29), the price vector is a left-hand Perron-Frobenius eigenvector of the matrix on the r.h.s. This gives an analogous way of solving the system. That is, we have that the determinant of matrix I - $[(1+r)\mathbf{A} + \omega\left(\frac{\mathbf{f}}{L}\right)\mathbf{l}']$ must be equal to 0. Adopting the symbol $\varphi = \varphi(r,w)$ for this determinant, we thus have

$$(34) \qquad \varphi = \left|\mathrm{I} - [(1+r)\mathbf{A} + \omega\left(\frac{\mathbf{f}}{L}\right)\mathbf{l}']\right| = 0$$

which gives us the distribution of income between r and ω. The question to be addressed then is how φ looks like for various values of **f**. As shown in Steenge and Serrano (2012, section 3.2) φ is linear if **f** is the right-hand Perron-Frobenius eigenvalue of matrix **A**. [12] Finally, we observe that equation (29) *also* can be interpreted as an income distribution determining IO-based equation. Here, by construction, the distribution is trivial because labour gets the entire surplus. That is, in terms of equation (34) we have $r = 0$ and $\omega = 1$. Note that in the notation of section 4 above, we would have $\mathbf{H} = \mathbf{A}_1$.

So, also here rank 1 matrices appear in an income distribution context where we have full knowledge of the consumption preferences and the wages and salaries paid out to the various household categories. As we have observed (section 2) the standard Leontief model has no facility (like Miyazawa's model has) to fully benefit from such information, because the model has been constructed without building in the possibility of an equal number of (standardized) consumption preferences bundles and factor remuneration vectors. This may explain why matrices of rank 1 do not play a role in standard IO analysis.[13]

[12] In standardized form, **f** is the well-known Sraffian Standard Commodity.

[13] For further references to rank 1 matrices in multi-sector modelling, see Steenge and Serrano (2012).