STRICT DENSITY INEQUALITIES FOR SAMPLING AND INTERPOLATION IN WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. Answering a question of Lindholm, we prove strict density inequalities for sampling and interpolation in Fock spaces of entire functions in several complex variables defined by a plurisubharmonic weight. In particular, these spaces do not admit a set that is simultaneously sampling and interpolating. To prove optimality of the density conditions, we construct sampling sets with a density arbitrarily close to the critical density.

The techniques combine methods from several complex variables (estimates for $\bar\partial$) and the theory of localized frames in general reproducing kernel Hilbert spaces (with no analyticity assumed). The abstract results on Fekete points and deformation of frames may be of independent interest.

1. INTRODUCTION AND RESULTS

Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function, and assume that there are constants $m, M > 0$ such that

$$(1) \qquad im\partial\bar\partial|z|^2 \leq i\partial\bar\partial\phi \leq Mi\partial\bar\partial|z|^2$$

in the sense of positive currents [15]. For $1 \leq p < \infty$, we let A_ϕ^p be the space of entire functions on $\mathbb{C}^n$ equipped with the norm

$$\|f\|_{\phi,p}^p := \int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(z)} dm(z),$$

where dm denotes the Lebesgue measure. For $p = \infty$, we use the norm

$$\|f\|_{\phi,\infty} = \sup_{z \in \mathbb{C}^n} |f(z)| e^{-\phi(z)}.$$

Point evaluations are bounded linear functionals, and therefore A_ϕ^2 is a reproducing kernel Hilbert space. We denote its reproducing kernel by $K_\phi(z, w)$, or just $K(z, w)$ when it is not ambiguous. We also write $K_{\phi,w}(z) := K_\phi(z, w)$.

A set $\Lambda \subseteq \mathbb{C}^n$ is called a *sampling set* for A_ϕ^p if there are constants $A, B > 0$ such that

$$A\|f\|_{\phi,p}^p \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^p \, e^{-p\phi(\lambda)} \leq B\|f\|_{\phi,p}^p, \qquad \text{for all } f \in A_\phi^p,$$

with the usual modification for $p = \infty$. The constants A, B are called the stability constants. A set Λ is called an *interpolating set* for A_ϕ^p, if for every $a \in \ell_\phi^p(\Lambda)$ there exists a function $f \in A_\phi^p$ such that

$$f(\lambda) = a_\lambda, \qquad \lambda \in \Lambda.$$

In this case, there is always $C_p > 0$ and a choice of f such that $\|f\|_{\phi,p} \leq C_p\|a\|_{p,\phi}$.

This article is concerned with the density of sampling and interpolating sets. The upper and lower *weighted Beurling upper densities* of Λ are defined by

(2) $$D_\phi^+(\Lambda) := \limsup_{r \to \infty} \sup_{z \in \mathbb{C}^n} \frac{\#(\Lambda \cap B_r(z))}{\int_{B_r(z)} K(w, w)e^{-2\phi(w)}dm(w)},$$

(3) $$D_\phi^-(\Lambda) := \liminf_{r \to \infty} \inf_{z \in \mathbb{C}^n} \frac{\#(\Lambda \cap B_r(z))}{\int_{B_r(z)} K(w, w)e^{-2\phi(w)}dm(w)}.$$

For the standard weight $\phi(z) = \pi |z|^2 /2$, one recovers Beurling's classical densities, since the reproducing kernel is $K(w, z) = e^{\pi \bar{w}z}$. In dimension $n = 1$, sampling and interpolating sets are characterized completely by density conditions [18, 19, 22]. In higher dimensions, only the necessity of the conditions can be expected to hold. We investigate this matter in several directions.

For 2-homogeneous weights, Lindholm [17] showed the following necessary density conditions:

(4)   If Λ is a sampling set for A_ϕ^p, then $D_\phi^-(\Lambda) \geq 1$.

(5)   If Λ is an interpolating set for A_ϕ^p, then $D_\phi^+(\Lambda) \leq 1$.

We will show that the homogeneity of ϕ may be removed and that the necessary conditions (4) and (5) are valid for all weights ϕ satisfying (1). This more general result follows from the abstract density theory [4, 12], which is applicable due to the off-diagonal decay of the reproducing kernel. See Section 2.3 for the details.

Strictly speaking, the densities in (2) and (3) differ from those used in $[17, 19]$, where the following densities are used instead:

$$\widetilde{D}_\phi^+(\Lambda) := \limsup_{r \to \infty} \sup_{z \in \mathbb{C}^n} \frac{\#(\Lambda \cap B_r(z))}{\int_{B_r(z)} (i\partial\bar{\partial}\phi)^n},$$

$$\widetilde{D}_\phi^-(\Lambda) := \liminf_{r \to \infty} \inf_{z \in \mathbb{C}^n} \frac{\#(\Lambda \cap B_r(z))}{\int_{B_r(z)} (i\partial\bar{\partial}\phi)^n}.$$

It can be shown by combining results of $[19]$, $[6]$ and $[12]$ that in 1 dimension we have

$$(6) \qquad\qquad\qquad \widetilde{D}_\phi^+(\Lambda) = \frac{1}{\pi^n n!} D_\phi^+(\Lambda),$$

and similarly for the lower densities. The relation (6) holds also in several variables if we assume in addition that ϕ is 2-homogeneous $[17]$ - see also $[12, \text{Section } 5.4]$. In general, it remains an open problem to decide when both densities coincide.

As our first contribution we show that these results are sharp.

Theorem 1.1. *Let* $\phi : \mathbb{C}^n \to \mathbb{R}$ *be a plurisubharmonic function satisfying* (1). *Then given* $\varepsilon > 0$, *there exists a set* $\Lambda \subseteq \mathbb{C}^n$ *that is interpolating for* $A_{(1+\varepsilon)\phi}^2$ *and sampling for* $A_{(1-\varepsilon)\phi}^2$. *As a consequence,*

$$(7) \qquad\qquad\qquad \inf_{\Lambda \in SS} D_\phi^+(\Lambda) = \sup_{\Lambda \in SI} D_\phi^-(\Lambda) = 1,$$

where the infimum runs over all sampling sets for A_ϕ^2 *and the supremum over all interpolation sets for* A_ϕ^2.

The sampling part of Theorem 1.1 is closely related to the main result in $[3]$, which establishes the existence of frames in an abstract setting whose density is arbitrarily close to the critical density. Although it may be possible to apply the results of $[3]$ to our setting, it is far from clear how to overcome certain technical challenges caused by the subtleties of general plurisubharmonic weights, such as the construction of an adequate "reference frame" or the identification of the corresponding abstract densities with $D_\phi^\pm$. Instead, in this paper we resort to a new technique based on Fekete points introduced in $[16]$. This approach also yields the existence of interpolating sets with density arbitrary close to the critical density.

Our second contribution is to show that the supremum and infimum in (7) are not attained. We will prove that the inequalities in (4) and (5) are in fact strict. This question was mentioned as an open problem by Lindholm $[17]$, but remained unanswered even for the special case of 2-homogeneous weights.

Theorem 1.2. *Let* $\phi : \mathbb{C}^n \to \mathbb{R}$ *be a plurisubharmonic function satisfying* (1) *and let* $p \in [1, \infty]$.

(a) *If Λ is a sampling set for A_ϕ^p, then $D_\phi^-(\Lambda) > 1$.*

(b) *If Λ is an interpolating set for A_ϕ^p, then $D_\phi^+(\Lambda) < 1$.*

Corollary 1.3. *There does not exist a set $\Lambda \subseteq \mathbb{C}^n$ that is simultaneously sampling and interpolating for A_ϕ^2. Equivalently, there is no Riesz basis for A_ϕ^2 that consists of reproducing kernels.*

The strict density conditions of Theorem 1.2 are substantially different from the necessary, non-strict conditions in (4) and (5). So far, strict density conditions have been proved only in few situations, namely (i) for weighted Fock spaces A_ϕ^p *in dimension* 1 [19], and (ii) for Gabor frames, where the result is known as the Balian-Low theorem. Although there is an extensive literature on the theorem for Gabor frames over a lattice, strict density conditions for non-uniform Gabor frames were shown only recently in [2] (with pseudodifferential operators) and [14].

We note that the proofs in [19] for weighted Fock spaces in one variable rely on the sufficiency of density conditions for sampling and interpolation [6], and these are not available in higher dimension. We will adopt a different approach that combines the strategies of [19] and of [14]. To circumvent arguments that are specific to one-dimensional complex analysis, we resort instead to techniques from [14] that were introduced originally to study the stability of Gabor frames under quite general deformations. More precisely, we consider the notion of *Lipschitz convergence of sets*. Roughly, a sequence of sets $\Lambda_j \subseteq \mathbb{C}^n$ converges Lipschitz-wise to $\Lambda \subseteq \mathbb{C}^n$, if there exist maps $\tau_j : \Lambda \to \mathbb{C}^n$, such that $\Lambda_j = \tau_j(\Lambda)$, $\tau_j(\lambda) \to \lambda$ for all $\lambda \in \Lambda$ and distances are preserved locally. See Section 3.5 for the precise, technical definition. Our third contribution is the following deformation result for sampling sets (interpolating sets) in weighted Fock spaces.

Theorem 1.4. *Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function satisfying* (1), *and $p \in [1, \infty]$. Assume that Λ_j is a sequence of sets that converge to Λ in Lipschitz-wise, $\Lambda_j \overset{Lip}{\longrightarrow} \Lambda$.*

(a) If Λ is a sampling set for A_ϕ^p, then Λ_j is also a sampling set for A_ϕ^p for sufficiently large j.

(b) If Λ is an interpolating set for A_ϕ^p, then Λ_j is also an interpolating set for A_ϕ^p for sufficiently large j.

Theorem 1.2 then follows from the deformation stability by choosing a sequence of dilated sets $\Lambda_j = (1 + \frac{1}{j})\Lambda$. If Λ is a sampling set for A_ϕ^p, then so is Λ_j for large j. Then $D_\phi^-(\Lambda) > D_\phi^-(\Lambda_j)$ and by the necessary density condition (4) we obtain $D_\phi^-(\Lambda) > D_\phi^-(\Lambda_j) \geq 1$. See Section 6.1 for details.

For the proof of the main results we will enrich the outline of [19] and [14] by several new aspects:

(i) *Universality.* Λ is a sampling set (interpolating set) for A_ϕ^p for *some* $p \in [1, \infty]$, if and only if Λ is a sampling set (interpolating set) for A_ϕ^p for *all* $p \in [1, \infty]$. The technical novelty is a Wiener-type lemma for infinite matrices with off-diagonal decay that are left-invertible on a subspace (Lemma 7.1). This clarifies some subtleties in [1] and [14], and enhances their applicability (in [14] we used so-called Wilson bases to reach similar conclusions).

(ii) *Weak limits* play an important role in sampling theory and in deformation results. The main obstacle in weighted Fock spaces is their lack of translation invariance. This was circumvented in [19] by noting that, while a single space A_ϕ^p may not be translation invariant, the union of all A_ϕ^p is translation invariant. This insight was leveraged by developing abstract translation operators that map a weighted Fock space into another weighted Fock space. The extension of these ideas to several complex variables requires considerable technicalities (Section 4). In particular, we will show that the map $\phi \mapsto K_\phi$ (for the reproducing kernel of A_ϕ^2) obeys some continuity property (Proposition 4.1).

(iii) The *theory of localized frames in reproducing kernel Hilbert spaces* enters several times in the proof of the universality of sampling sets and of Theorem 1.4. These arguments do not rely on analyticity and may be of independent interest for further applicability. In particular, Theorem 7.6 contains an abstract version of the construction of Fekete points in reproducing kernel Hilbert spaces.

Finally we comment on a question raised in [19] on the difference between Paley-Wiener and Fock spaces. Whereas the necessary density condition in weighted Fock spaces is strict, it is not so in the Paley-Wiener space, and consequently, Paley-Wiener space admits sequences that are both sampling and interpolating. We believe that the difference lies in the off-diagonal decay of reproducing kernels. Whereas the reproducing kernel of Paley-Wiener space is not even in L^1, the adjusted reproducing kernel of weighted Fock space decays exponentially. In the end, this difference may contribute to the different behavior of the two spaces.

The article is organized as follows. In Section 2 we collect some facts about Fock spaces and $\bar{\partial}$-equations, while in Section 3 we introduce the key definitions and tools. The abstract translation operators are introduced in Section 4. These tools are used to characterize sampling and interpolating sets in Section 5. Theorem 1.2 is derived in Section 6. For clarity, the more general arguments that are applicable to abstract reproducing kernel Hilbert spaces with a certain off-diagonal decay are postponed to Section 7.

2. Preliminaries

2.1. **Notation.** We are mainly interested in functions of n complex variables, but we develop some auxiliary results on the Euclidean spaces $\mathbb{R}^d$. Of course, when we apply these to $\mathbb{C}^n$, we let $d = 2n$.

A set $\Lambda \subseteq \mathbb{R}^d$ is called *relatively separated* if

$$(8) \qquad \text{rel}(\Lambda) := \sup\{\#(\Lambda \cap B_1(x)) : x \in \mathbb{R}^d\} < \infty,$$

and it is called *separated* if

$$(9) \qquad \text{sep}(\Lambda) := \inf\{|\lambda - \lambda'| : \lambda \neq \lambda' \in \Lambda\} > 0.$$

Separated sets are relatively separated, and relatively separated sets are finite unions of separated sets. A set Λ is called *relatively dense* if there exists $R > 0$ such that $\mathbb{R}^d = \bigcup_{\lambda \in \Lambda} B_R(\lambda)$.

2.2. **The reproducing kernel.** Recall that we denote the reproducing kernel of A_ϕ^2 by K_ϕ and write $K_{\phi,z}(w) = K_\phi(w, z)$. The diagonal of the reproducing kernel satisfies

$$(10) \qquad 0 < c \leq K_\phi(z, z)e^{-2\phi(z)} \leq C < \infty,$$

for some constants c, C that only depend on the constants in (1); see e.g. [21, Proposition 2.5]. In addition, the reproducing kernel K_ϕ satisfies the following off-diagonal decay estimate [9]:

$$(11) \qquad |K_\phi(z, w)|e^{-\phi(z)-\phi(w)} \leq Ce^{-c|z-w|},$$

for all $z, w \in \mathbb{C}^n$ and some constants $c, C > 0$ which only depend on the bounds in (1). See [7] for more general conditions for off-diagonal decay.

2.3. **Non-strict density conditions.** The following statement offers a small extension of Lindholm's density theorem [17].

Theorem 2.1. *Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function satisfying* (1).
- (a) *If Λ is a sampling set for A_ϕ^2, then $D_\phi^-(\Lambda) \geq 1$.*
- (b) *If Λ is an interpolating set for A_ϕ^2, then $D_\phi^+(\Lambda) \leq 1$.*

Proof. We apply the abstract density result in [12, Corollary 4.1] to the metric space $\mathbb{C}^n = \mathbb{R}^{2d}$, $d\mu(z) = dm(z)$, and the reproducing kernel Hilbert space

$$V^2 := \left\{ e^{-\phi} f : f \in A_\phi^2 \right\}.$$

The density theorem [12, Corollary 4.1] requires certain assumptions on the metric and measure, which are indeed satisfied by the Euclidean space and Lebesgue measure, and assumptions on the reproducing kernel of V^2 (behavior of the diagonal and off-diagonal

decay), which is $K_\phi(z, w)e^{-\phi(z)-\phi(w)}$. The required conditions on the reproducing kernel are easily seen to hold due to (10) and (11). □

Remark 2.2. For 2-homogeneous weights ϕ, Theorem 2.1 is essentially due to Lindholm [17]. To be precise, the density condition in (2) and (3) and those of Lindholm are formally different, but they were shown to coincide for 2-homogeneous weights by [12, Section 5.4]. The generalization in Theorem 2.1 may be taken as a hint that the new notion of density is perfectly appropriate for weighted Fock spaces.

By (10), the unweighted Beurling density $D^\pm(\Lambda)$ is comparable to the weighted density $D_\phi^\pm(\Lambda)$, namely $cD^\pm(\Lambda) \leq D_\phi^\pm(\Lambda) \leq CD^\pm(\Lambda)$. Since a set of positive lower Beurling density is relatively dense, we obtain the following corollary.

Corollary 2.3. *Assume that ϕ satisfies* (1). *Then every sampling set for A_ϕ^2 is relatively dense.*

2.4. The $\partial\bar{\partial}$ equation.

Lemma 2.4. *Let $\theta = \sum_{1 \leq j,k \leq n} \theta_{jk} dz_j \wedge d\bar{z}_k$ be a positive, d-closed $(1,1)$-current satisfying $\theta \leq Mi\partial\bar{\partial}|z|^2$. Then there exists $u : \mathbb{C}^n \to \mathbb{C}$ solving the equation $i\partial\bar{\partial}u = \theta$, and such that*

$$(12) \qquad |u(z)| \leq CM(1 + |z|)^2 \log(1 + |z|),$$

where the constant C depends only on the dimension n.

Proof. The solution is found in two stages. In the first step, we let $d = \partial + \bar{\partial}$ and solve $dv = \theta$. The solution v is as in Poincaré's lemma: $v = v_{0,1} + v_{1,0}$ where

$$v_{0,1}(z) = \sum_{1 \leq j,k \leq n} \left(\int_{t=0}^1 \theta_{j,k}(tz)tz_j dt \right) d\bar{z}_k, \quad v_{1,0} = -\overline{v_{0,1}}.$$

We have $\bar{\partial}v_{0,1} = \partial v_{1,0} = 0$ and check easily that $dv = \theta$. Furthermore, v satisfies the estimate $v(z) \leq CM(1 + |z|)$, where C depends only on the dimension of the space.

In the second step, we solve the equation $i\bar{\partial}w = v_{0,1}$. By [5, Theorem 9'], there exists a solution to this equation, given by an explicit integral formula, satisfying

$$|w(z)| \leq CM(1 + |z|)^2 \log(1 + |z|).$$

Now, it is readily checked $u := 2\mathrm{Re}\, w$ solves the equation $i\partial\bar{\partial}u = \theta$ and also satisfies the desired growth estimate. □

Remark 2.5. It follows from standard regularity theory for the Poisson equation that the solution u in Lemma 2.4 has derivatives of order 1 that are locally α-Hölder continuous, i.e., $u \in C^{1,\alpha}$ for every $\alpha \in (0, 1)$.

2.5. **Size control.** In some of the results we introduce the following extra size assumption on ϕ:

(13) $$|\phi(z)| \leq C(1+|z|)^2 \log(1+|z|), \qquad z \in \mathbb{C}^n.$$

As explained below, this extra condition can always be achieved without changing the class of sampling or interpolating sets.

Proposition 2.6. *Let $\phi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function satisfying* (1). *Then there exists a plurisubharmonic function $\tilde{\phi} : \mathbb{C}^n \to \mathbb{R}$ satisfying* (1) *and* (13), *and with the following property: a set $\Lambda \subseteq \mathbb{C}^n$ is a sampling (resp. interpolating) set for A_ϕ^p and some $p \in [1, \infty]$ if and only if Λ is a sampling (resp. interpolating) set for $A_{\tilde{\phi}}^p$. Furthermore, the density remains invariant under this change, $D_\phi^\pm(\Lambda) = D_{\tilde{\phi}}^\pm(\Lambda)$.*

Proof. Lemma 2.4 provides a function $\tilde{\phi}$ satisfying (13), and such that $\partial\bar{\partial}\tilde{\phi} = \partial\bar{\partial}\phi$. This implies that there exists an entire function G such that $\operatorname{Re} G = \tilde{\phi} - \phi$, and therefore,

$$e^{-\phi(z)} = |e^{G(z)}|e^{-\tilde{\phi}(z)}.$$

Hence, multiplication by e^G gives an isometry from A_ϕ^p to $A_{\tilde{\phi}}^p$. The sampling and interpolating sets are therefore the same in A_ϕ^p and $A_{\tilde{\phi}}^p$. The kernels of A_ϕ^2 and $A_{\tilde{\phi}}^2$ are related by

$$K_{\tilde{\phi}}(z,w) = e^{G(z)-\overline{G(w)}} K_\phi(z,w).$$

Consequently the Bergman measure of the ball $B_r(z)$ in the definition of the density is

$$\int_{B_r(z)} K_{\tilde{\phi}}(w,w)e^{-2\tilde{\phi}(w)}\, dm(w) = \int_{B_r(z)} K_\phi(w,w)\,|e^{G(w)}|^2 e^{-2\tilde{\phi}(w)}\, dm(w)$$
$$= \int_{B_r(z)} K_\phi(w,w)e^{-2\phi(w)}\, dm(w),$$

and thus $D_{\tilde{\phi}}^\pm(\Lambda) = D_\phi^\pm(\Lambda)$. $\qquad\square$

3. Some tools

3.1. **Bessel bounds for weighted analytic functions.** The following lemma follows from [17, Lemmas 7 and 17].

Lemma 3.1. *Let f be a holomorphic function on $B_1(z) \subseteq \mathbb{C}^n$. Let $\psi : B_1(z) \to \mathbb{R}$ be a plurisubharmonic function such that $i\partial\bar{\partial}\psi(w) \leq Mi\partial\bar{\partial}|w|^2$. Then, for all $p \in [1, \infty)$,*

(14) $$|f(z)|^p e^{-p\psi(z)} \leq C_1 \int_{B_1(z)} |f(w)|^p e^{-p\psi(w)} dm(w).$$

In addition, if $f(z) \neq 0$, then for $r > 0$

$$(15) \qquad |\nabla \left(|f(z)|^r e^{-r\psi(z)} \right)| \leq C_2 \left[\int_{B_1(z)} |f(w)|^p e^{-p\psi(w)} dm(w) \right]^{r/p}.$$

The constants depend only on p, r, M, and the dimension n.

As a consequence of (14), we obtain the following local Bessel bound.

Corollary 3.2. *Let $\Lambda \subseteq \mathbb{C}^n$ be relatively separated and let f be a holomorphic function on $\Lambda + B_1(0)$, and suppose that ψ is a plurisubharmonic function satisfying $i\partial\bar{\partial}\psi(w) \leq Mi\partial\bar{\partial}|w|^2$ on $\Lambda + B_1(0)$. Then, for all $p \in [1, \infty)$,*

$$\left(\sum_{\lambda \in \Lambda} |f(\lambda)|^p \, e^{-p\psi(\lambda)} \right)^{1/p} \leq C \operatorname{rel}(\Lambda) \left[\int_{\Lambda + B_1(0)} |f(w)|^p e^{-p\psi(w)} dm(w) \right]^{1/p}.$$

We also derive the following fact.

Corollary 3.3. *Let ψ be a plurisubharmonic function such that $i\partial\bar{\partial}\psi(w) \leq Mi\partial\bar{\partial}|w|^2$, and let $1 \leq p \leq +\infty$. Then every interpolating set for A_ψ^p is separated.*

Proof. As in [19, Proposition 9], if Λ is interpolating and $\lambda, \lambda' \in \Lambda$ are two different points, we may find $f \in A_\psi^p$ such that $f(\lambda) = e^{\psi(\lambda)}$, $f(\lambda') = 0$, and $\|f\|_{p,\psi} \leq C_\Lambda$. Lemma 3.1 now implies that $1 = \left| \left| f(\lambda)e^{-\psi(\lambda)} \right| - \left| f(\lambda')e^{-\psi(\lambda')} \right| \right| \lesssim C_\Lambda |\lambda - \lambda'|$. $\qquad \square$

3.2. Amalgam spaces.

The *amalgam space* $W(L^\infty, L^1)(\mathbb{R}^d)$ consists of all functions $f \in L^\infty(\mathbb{R}^d)$ such that

$$\|f\|_{W(L^\infty, L^1)} := \int_{\mathbb{R}^d} \|f\|_{L^\infty(B_1(x))} dx \asymp \sum_{k \in \mathbb{Z}^d} \|f\|_{L^\infty([0,1]^d + k)} < \infty.$$

The (closed) subspace of $W(L^\infty, L^1)(\mathbb{R}^d)$ of continuous functions is denoted $W(C_0, L^1)(\mathbb{R}^d)$, and is a convenient space of test functions. Its dual space will be denoted $W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ and consists of all complex-valued Borel measures μ such that

$$\|\mu\|_{W(\mathcal{M}, L^\infty)} := \sup_{x \in \mathbb{R}^d} \|\mu\|_{B_1(x)} = \sup_{x \in \mathbb{R}^d} |\mu| \, (B_1(x)) < \infty.$$

Such measures are often called *translation-bounded*. We refer the reader to [10] for a general theory of Wiener amalgam spaces.

3.3. Universality of sampling and interpolating sets.

The following universality results are a central technical tool.

Theorem 3.4. *Assume that ϕ satisfies* (1).

(a) If Λ is a sampling set for A_ϕ^p for some $p \in [1, \infty]$, then it is a sampling set for all A_ϕ^p with $p \in [1, \infty]$.

(b) If Λ is an interpolating set A_ϕ^p for some $p \in [1, \infty]$, then it is an interpolating set for all A_ϕ^p with $p \in [1, \infty]$.

The proof, which is postponed to Section 7, follows from the decay of the reproducing kernel and a non-commutative Wiener's Lemma.

3.4. **Weak convergence of sets.** Let $\Lambda \subseteq \mathbb{R}^d$ be a set. A sequence $\{\Lambda_j : j \geq 1\}$ of subsets of $\mathbb{R}^d$ *converges weakly* to Λ, in short $\Lambda_j \xrightarrow{w} \Lambda$, if for every $R > 0$ and $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\Lambda \cap B_R(0) \subseteq \Lambda_j + B_\varepsilon(0) \quad \text{and} \quad \Lambda_j \cap B_R(0) \subseteq \Lambda + B_\varepsilon(0).$$

For a relatively separated set $\Lambda \subseteq \mathbb{R}^d$, we let $W(\Lambda)$ denote the set of weak limits of the translated sets $\Lambda + x, x \in \mathbb{R}^d$, i.e., $\Gamma \in W(\Lambda)$ if there exists a sequence $\{x_j : j \geq 1\} \subseteq \mathbb{R}^d$ such that $\Lambda + x_j \xrightarrow{w} \Gamma$. It is easy to see that then Γ is always relatively separated.

3.5. **Lipschitz convergence of sets.** Given a set $\Lambda \subseteq \mathbb{R}^d$, we say that a sequence of sets $\{\Lambda_j : j \geq 1\}$ *converges to Λ in a Lipschitz fashion*, denoted by $\Lambda_j \xrightarrow{Lip} \Lambda$, if there is a sequence of maps $\tau_j : \Lambda \to \mathbb{R}^d$ with the following properties:

(a) $\Lambda_j = \tau_j(\Lambda) = \{\tau_j(\lambda) : \lambda \in \Lambda\}$.
(b) $\tau_j(\lambda) \longrightarrow \lambda$, as $j \longrightarrow \infty$, for all $\lambda \in \Lambda$.
(c) Given $R > 0$,

$$\sup_{\substack{\lambda, \lambda' \in \Lambda \\ |\lambda - \lambda'| \leq R}} |(\tau_j(\lambda) - \tau_j(\lambda')) - (\lambda - \lambda')| \to 0, \quad \text{as } j \longrightarrow \infty.$$

(d) Given $R > 0$, there exist $R' > 0$ and $j_0 \in \mathbb{N}$ such that if $|\tau_j(\lambda) - \tau_j(\lambda')| \leq R$ for some $j \geq j_0$ and some $\lambda, \lambda' \in \Lambda$, then $|\lambda - \lambda'| \leq R'$.

We also say that $\{\Lambda_j : j \geq 1\}$ is a Lipschitz deformation of Λ, with the understanding that a sequence of underlying maps $\{\tau_j : j \geq 1\}$ is also given. We think of each sequence of points $\{\tau_j(\lambda) : j \geq 1\}$ as a (discrete) path moving towards the endpoint λ.

The main example of Lipschitz convergence is $\Lambda_j = \tau_j \Lambda$, where $\tau_j : \mathbb{R}^d \to \mathbb{R}^d$, $\tau_j(0) = 0$ and their differential maps $D\tau_j$ satisfy $D\tau_j \longrightarrow I$ in $L^p(\mathbb{R}^d, \mathbb{R}^{d \times d})$ for $p > d$ [14, Lemma 6.4]. In particular if $\{A_j : j \geq 1\}$ is a sequence of matrices such that $A_j \longrightarrow I$, then $A_j \Lambda \xrightarrow{Lip} \Lambda$. The notion of Lipschitz deformation is a suitable concept of a global deformation of sets [8, 14]. In many situations, as in this article, Lipschitz deformations preserve sampling sets and interpolating sets.

The following lemma from [14] connects Lipschitz convergence and weak convergence of translates.

Lemma 3.5 (Lemma 6.8 in [14]). *Let Λ be relatively separated, $\{\Lambda_j : j \geq 1\}$ a Lipschitz deformation of Λ, and $\Gamma \subseteq \mathbb{R}^d$.*

   (i) *For $j \geq 1$ let $\lambda_j \in \Lambda_j$. If $\Lambda_j - \lambda_j \xrightarrow{w} \Gamma$, then $\Gamma \in W(\Lambda)$.*
   (ii) *Let $\{x_j : j \geq 1\} \subseteq \mathbb{R}^d$ and assume that Λ is relatively dense. If $\Lambda_j - x_j \xrightarrow{w} \Gamma$, then $\Gamma \in W(\Lambda)$.*

4. Translation type operators

We assume that ϕ satisfies (1) and (13), and extend the construction of the translation type operators from [19] to several complex variables.

4.1. **Translated weights.** Given $\zeta \in \mathbb{C}^n$, we let ϕ_ζ be a solution of the equation

$$\partial \bar{\partial} \phi_\zeta(z) = \partial \bar{\partial} \phi(z - \zeta), \tag{16}$$

given by Lemma 2.4 with $\theta = \partial \bar{\partial} \phi(\cdot - \zeta)$. Thus, the functions ϕ_ζ satisfy the estimate

$$\phi_\zeta(z) \leq CM(1 + |z|)^2 \log(1 + |z|). \tag{17}$$

We emphasize that the constant C is independent of ζ and depends only on the dimension n. For each $\zeta \in \mathbb{C}^n$ we thus fix a choice of ϕ_ζ and call it *translated weight*. For $\zeta = 0$, we simply let

$$\phi_0 = \phi.$$

This choice is possible because we assumed (13).

4.2. **Translation operators.** Let $q(z, \zeta)$ be a function that is entire in z and satisfies

$$\mathrm{Re}(q(z, \zeta)) := \phi_\zeta(z) - \phi(z - \zeta).$$

We now define the translation type operators T_ζ as

$$T_\zeta f(z) := e^{q(z,\zeta)} f(z - \zeta).$$

They satisfy

$$T_\zeta f(z) e^{-\phi_\zeta(z)} = e^{q(z,\zeta)} f(z - \zeta) e^{-\phi(z-\zeta) - \mathrm{Re}(q(z,\zeta))} = e^{i \mathrm{Im}(q(z,\zeta))} f(z - \zeta) e^{-\phi(z-\zeta)}.$$

Consequently,

$$|T_\zeta f(z)| \, e^{-\phi_\zeta(z)} = |f(z - \zeta)| \, e^{-\phi(z-\zeta)}. \tag{18}$$

Therefore $T_\zeta : A_\phi^p \to A_{\phi_\zeta}^p$ is an isometric isomorphism for all $1 \leq p \leq \infty$. Furthermore, if Λ is a sampling or interpolating set for A_ϕ^p, then $\Lambda + \zeta$ is a sampling or interpolating set for $A_{\phi_\zeta}^p$ with the same stability constants. In addition,

$$
(19) \quad
\begin{aligned}
K_{\phi_\zeta}(z,w)e^{-\phi_\zeta(z)-\phi_\zeta(w)} &= (T_\zeta \otimes \overline{T_\zeta}) K_\phi(z,w) e^{-\phi_\zeta(z)-\phi_\zeta(w)} \\
&= e^{i\mathrm{Im}(q(z,\zeta)-q(w,\zeta))} K_\phi(z-\zeta, w-\zeta) e^{-\phi(z-\zeta)-\phi(w-\zeta)}.
\end{aligned}
$$

As a consequence, we have the following covariance formula:

$$
(20) \quad T_\zeta \left(e^{-\phi(\lambda)} K_\phi(\cdot, \lambda) \right) = e^{i\mathrm{Im}(q(\lambda+\zeta,\zeta))} e^{-\phi_\zeta(\lambda+\zeta)} K_{\phi_\zeta}(\cdot, \lambda+\zeta).
$$

4.3. **Compactness.** Given a sequence of numbers $\{\zeta_j : j \geq 1\} \subseteq \mathbb{C}^n$, the family $\{\phi_{\zeta_j} : j \geq 1\}$ satisfies the condition (1) with the same constants as ϕ. As a consequence we prove the following compactness result that asserts a continuous dependence of the reproducing kernel K_ψ on ψ.

Proposition 4.1. *Assume that ϕ satisfies (1) and (13). Then for every sequence $\{\zeta_j : j \geq 1\} \subseteq \mathbb{C}^n$ there exists a subsequence $\{\zeta_{j_k} : k \geq 1\}$ such that $\phi_{\zeta_{j_k}}$ converges to a plurisubharmonic function ψ uniformly on compact sets. The function ψ satisfies*

$$
(21) \quad m i \partial \bar\partial |z|^2 \leq i \partial \bar\partial \psi \leq M i \partial \bar\partial |z|^2 \,,
$$

in the sense of positive currents, and the growth bound (13). In addition, convergence

$$
K_{\phi_{\zeta_{j_k}}} \to K_\psi
$$

holds uniformly on compact subsets of $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. Step 1. (Existence of the convergent subsequence). By assumption and (17), both $\{\phi_{\zeta_j} : j \geq 1\}$ and $\{\Delta \phi_{\zeta_j} : j \geq 1\}$ are locally bounded sequences. By the regularity of Poisson's equation, the functions ϕ_{ζ_j} belong locally to $C^{1,\alpha}$ for every $\alpha \in (0,1)$, i.e.,

$$
\sup_{j \geq 1} \sup_{z \in \mathbb{C}} \|\phi_{\zeta_j}\|_{C^{1,\alpha}(B_1(z))} < \infty.
$$

By the Arzela-Ascoli Theorem and the diagonal argument, it follows that $\{\phi_{\zeta_j} : j \geq 1\}$ has a subsequence that converges locally in the C^1-norm to a certain function ψ. In particular, $\phi_{\zeta_j} \longrightarrow \psi$, $\partial_z \phi_{\zeta_j} \longrightarrow \partial_z \psi$, and $\bar\partial_z \phi_{\zeta_j} \longrightarrow \bar\partial_z \psi$, uniformly on compact sets. With this information, we can deduce (21) by integrating against a test function.

Step 2. (Convergence of reproducing kernels). By (17) and (11), we know that $K_{\phi_{\zeta_j}}$ is locally uniformly bounded in $\mathbb{C}^n \times \mathbb{C}^n$ in the sense that $\sup_{j \in \mathbb{N}} \sup_{(z,w) \in C} |K_{\phi_{\zeta_j}}(z,w)| < \infty$ for every compact set $C \subseteq \mathbb{C}^n \times \mathbb{C}^n$. By Montel's theorem, we can pass to a subsequence and assume that

$$
K_{\phi_{\zeta_j}} \to K
$$

with uniform convergence on compact sets and a kernel $K(z, w), z, w \in \mathbb{C}^n$ that is analytic in z and $\bar{w}$. We have to show that $K = K_\psi$, and for this it is enough to show that $K(z, z) = K_\psi(z, z)$, because an entire function in $(z, \bar{w})$ is determined by its values on the diagonal.

We first prove $K(z, z) \geq K_\psi(z, z)$. We fix $z \in \mathbb{C}^n$ and define $f(w) := K_{\psi, z}(w)$. So

$$\|f\|^2_{\psi, 2} = f(z) = K_\psi(z, z).$$

For $\varepsilon > 0$ we choose $R > 0$ such that

$$\int_{|w-z|>R-1} |f(w)|^2 e^{-2\psi(w)} dm(w) \leq \varepsilon.$$

We also choose $j_0 = j_0(z)$ such that $e^{-2\phi_{\zeta_j}(w)} \leq 2e^{-2\psi(w)}$ for all $j \geq j_0$ and $w \in B_r(z)$, and

$$(22) \qquad \int_{|w-z|<R} |f(w)|^2 e^{-2\phi_{\zeta_j}(w)} dm(w) \geq f(z) - 2\varepsilon = K_\psi(z, z) - 2\varepsilon.$$

Let χ be a cut-off function which equals 1 on $B_{R-1}(z)$, 0 on $\mathbb{C}^n \setminus B_r(z)$ and $|\bar{\partial}\chi| \lesssim 1$ everywhere, and set $h = f\chi$. Note that, by (22),

$$(23) \qquad \|h\|^2_{\phi_{\zeta_j}, 2} \leq K_\psi(z, z) + C\varepsilon,$$

for some constant C. We will modify h to a holomorphic function using Hörmander's estimate for $\bar{\partial}$. This guarantees a solution u_j in $L^2(\mathbb{C}^n, e^{-2\phi_{\zeta_j}})$ of the equation $\bar{\partial}u = \bar{\partial}h = f\bar{\partial}\chi$ such that

$$\int_{\mathbb{C}^n} |u_j(z)|^2 e^{-2\phi_{\zeta_j}(z)} dm(z) \leq C \int_{\mathbb{C}^n} |f(w)|^2 |\bar{\partial}\chi(w)|^2 e^{-2\phi_{\zeta_j}(w)} dm(w) \leq C'\varepsilon,$$

where the constants C, C' depend on m in (1), but not on j. By the choice of χ, u is holomorphic on $B_{R-1}(z)$, therefore, by Lemma 3.1, $|u_j(z)|^2 e^{-2\phi_{\zeta_j}(z)} \leq C''\varepsilon$. Combining this with (17) we conclude that

$$|u_j(z)|^2 \leq C_z^2 \varepsilon,$$

where C_z depends on z. Since $h(z) = f(z)$, the difference $h_j^* := h - u_j$ satisfies

$$|h_j^*(z) - f(z)| = |u_j(z)| \leq C_z \varepsilon^{1/2}.$$

Furthermore, by (23),

$$\|h_j^*\|_{\phi_{\zeta_j}, 2} \leq \sqrt{K_\psi(z, z)} + C_1\varepsilon, \quad |h_j^*(z)| = |f(z) - u_j(z)| \geq K_\psi(z, z) - C_z \varepsilon^{1/2},$$

for some constant C_1 which depends only on the constants m and M in (1), and the growth bound (17). By the extremal characterization of the diagonal values of reproducing

kernels, we obtain

$$K_{\phi_{\zeta_j}}(z,z) = \sup_{g \in A^2_{\phi_{\zeta_j}}} \frac{|g(z)|^2}{\|g\|^2_{\phi_{\zeta_j},2}} \geq \frac{|h_j^*(z)|^2}{\|h_j^*\|^2_{\phi_{\zeta_j},2}} \geq K_\psi(z,z) - C_z' \varepsilon^{1/2},$$

where C_z' may depend on z. Since this inequality holds for arbitrarily small ε and large enough j, we deduce that $K(z,z) \geq K_\psi(z,z)$.

The opposite inequality is obtained similarly by reversing the roles of ϕ_{ζ_j} and ψ.   □

In Proposition 4.1, the convergence of the reproducing kernels holds uniformly on compact sets. In the next proposition, we show that in certain situations, the convergence of the diagonal entries is in fact uniform.

Proposition 4.2. *Assume that ϕ satisfies* (1) *and* (13)*. Then, as $\delta \longrightarrow 0$,*

$$(24) \qquad K_{(1+\delta)\phi}(z,z)e^{-2(1+\delta)\phi(z)} \longrightarrow K_\phi(z,z)e^{-2\phi(z)}, \quad and$$

$$(25) \qquad \frac{K_{(1+\delta)\phi}(z,z)e^{-2(1+\delta)\phi(z)}}{K_\phi(z,z)e^{-2\phi(z)}} \longrightarrow 1,$$

uniformly on $\mathbb{C}^n$. (Here, δ may be positive or negative.)

Proof. Arguing as in the proof of Proposition 4.1 with $(1+\delta_j)\phi$ replacing ϕ_{ζ_j}, we can show that for any $\varepsilon > 0$, there exists $\delta_0 \in (0, 1/4)$ such that for all $\delta \in (-\delta_0, \delta_0)$,

$$(26) \qquad |K_{(1+\delta)\phi}(0,0)e^{-2(1+\delta)\phi(0)} - K_\phi(0,0)e^{-2\phi(0)}| < \varepsilon.$$

The constant δ_0 depends on ϕ only through the bounds m and M in (1) and the growth bound (17). Therefore, (26) holds also for all weights $(a\phi)_\xi$, $\xi \in \mathbb{C}^n$, and $a \in (1/2, 3/2)$, with the same constant δ_0. By (19),

$$K_{\phi_{-\xi}}(z,z)e^{-2\phi_{-\xi}(z)} = K_\phi(z+\xi, z+\xi)e^{-2\phi(z+\xi)}.$$

As a consequence,

$$|K_{(1+\delta)\phi}(\xi,\xi)e^{-2(1+\delta)\phi(\xi)} - K_\phi(\xi,\xi)e^{-2\phi(\xi)}| < \varepsilon,$$

for all $\xi \in \mathbb{C}^n$, and all $\delta \in (-\delta_0, \delta_0)$. Hence, (24) holds uniformly over $\mathbb{C}^n$. This, together with the lower bound in (10), implies that (25) also holds uniformly for $z \in \mathbb{C}^n$.   □

Remark 4.3. The uniform convergence in Proposition 4.2 relies on the fact that $\partial\bar{\partial}(1+\delta)\phi$ is uniformly bounded for small δ.

5. SAMPLING AND INTERPOLATION

Whereas Beurling's theory of sampling and interpolation in Paley-Wiener spaces requires only weak limits of sets, the theory of weighted Fock spaces requires weak limits of sets and weight functions. This is the price for the lack of translation invariance of the A_ϕ^2's. Precisely, given a set Λ and a weight $\phi : \mathbb{C}^n \to \mathbb{R}$, we say that $(\Gamma, \psi) \in W(\Lambda, \phi)$ if there exists a sequence $\{\zeta_j : j \geq 1\} \subseteq \mathbb{C}^n$ such that $\Lambda + \zeta_j \xrightarrow{w} \Gamma$ and $\phi_{\zeta_j} \to \psi$ uniformly on compact sets where the ϕ_{ζ_j}'s are the translated weights introduced in Section 4.1. In what follows, we invoke Theorem 3.4 several times. This is applicable to ϕ, to the translated weights ϕ_{ζ_j}, and to its locally uniform limits ψ, because, by Proposition 4.1, they all satisfy bounds similar to (1).

5.1. **Stability of sampling and interpolation under weak limits.**

Proposition 5.1. *Assume that ϕ satisfies (1) and (13). Let $p \in [1, \infty]$, Λ be a sampling set for A_ϕ^p, and suppose that $(\Gamma, \psi) \in W(\Lambda, \phi)$. Then Γ is a sampling set for A_ψ^p.*

Proof. By Theorem 3.4, we can restrict the problem to L^2 norms. We use of Hörmander's $\bar{\partial}$ estimates, and proceed as in [19].

We argue by contradiction and assume that Γ is not a sampling set for A_ψ^2. Let $\{\zeta_j : j \geq 1\} \subseteq \mathbb{C}^n$ be a sequence such that $\Lambda + \zeta_j \xrightarrow{w} \Gamma$ and $\phi_{\zeta_j} \to \psi$ uniformly on compact sets. Then for fixed $\varepsilon \in (0, 1/2)$, we can find $f \in A_\psi^2$ such that $\|f\|_{\psi,2} = 1$ and $\|f_{|\Lambda}\|_{\psi,2}^2 \leq \varepsilon$. Take $R > 0$ so large that

$$\int_{|z| \geq R-3} |f(z)|^2 e^{-2\psi(z)} dm(z) \leq \varepsilon.$$

We also take a smooth and positive cut-off function χ such that $\chi = 1$ on $B_{R-1}(0)$, $\chi = 0$ on $\mathbb{C}^n \setminus B_R(0)$ and $|\bar{\partial}\chi| \lesssim 1$. We define $h = \chi f$. Let $j \geq 1$ be such that $\|h_{|\Lambda + \zeta_j}\|_{\phi_{\zeta_j},2}^2 \leq 2\varepsilon$, $\|h\|_{\phi_{\zeta_j},2}^2 \geq 1/2$ and $e^{-2\phi_{\zeta_j}(z)} \leq 2e^{-2\psi(z)}$ on $B_R(0)$.

We will produce an analytic function having properties comparable to those of h, thus giving the contradiction that we seek. By Hörmander's estimate for the $\bar{\partial}$-operator we can find $u_j \in L^2(e^{-2\phi_{\zeta_j}})$ solving $\bar{\partial}u_j = \bar{\partial}h = f\bar{\partial}\chi$ such that

$$\begin{aligned}
\|u_j\|_{\phi_{\zeta_j},2}^2 &= \int_{\mathbb{C}^n} |u_j(z)|^2 e^{-2\phi_{\zeta_j}(z)} dm(z) \\
&\leq \frac{1}{(2m)^n} \int_{\mathbb{C}^n} |f(z)|^2 |\bar{\partial}\chi(z)|^2 e^{-2\phi_{\zeta_j}(z)} dm(z) \\
&\lesssim \int_{|z| \geq R-1} |f(z)|^2 e^{-2\psi(z)} dm(z) \lesssim \varepsilon,
\end{aligned}$$

where the constant m is from (1). Let us consider the sets

$$\Lambda_j^1 := (\Lambda + \zeta_j) \cap (B_{R+1}(0) \setminus B_{R-2}(0)),$$
$$\Lambda_j^2 := (\Lambda + \zeta_j) \setminus \Lambda_j^1,$$

and the holomorphic function $h_j^* := h - u_j$, which satisfies

(27) $$\|h_j^*\|_{\phi_{\zeta_j},2}^2 \geq 1/2 - C\varepsilon,$$

for some constant C.

Note that u_j is holomorphic outside $B_R(0) \setminus B_{R-1}(0)$, and

$$\left(\Lambda_j^2 + B_1(0)\right) \cap (B_R(0) \setminus B_{R-1}(0)) = \emptyset,$$

while $\mathrm{rel}(\Lambda_j^2) \leq \mathrm{rel}(\Lambda + \zeta_j) = \mathrm{rel}(\Lambda)$. Thus, by Corollary 3.2,

$$\|u_{j|\Lambda_j^2}\|_{\phi_{\zeta_j},2}^2 \lesssim \int_{\mathbb{C}^n} |u_j(z)|^2 e^{-2\phi_{\zeta_j}(z)} dm(z) \lesssim \varepsilon.$$

Hence, $\|h_{j|\Lambda_j^2}^*\|_{\phi_{\zeta_j},2}^2 \lesssim \varepsilon$.

To estimate the values of h_j^* on the set Λ_j^1, we note that $\left(\Lambda_j^1 + B_1(0)\right) \cap B_{R-3}(0) = \emptyset$, $\mathrm{rel}(\Lambda_j^1) \leq \mathrm{rel}(\Lambda + \zeta_j) = \mathrm{rel}(\Lambda)$, and

$$\int_{\mathbb{C}^n \setminus B_{R-3}(0)} |h_j^*(z)|^2 e^{-2\phi_{\zeta_j}(z)} dm(z) \lesssim \|u_j\|_{\phi_{\zeta_j},2}^2 + \int_{\mathbb{C}^n \setminus B_{R-3}(0)} |f(z)|^2 e^{-2\psi(z)} dm(z) \lesssim \varepsilon.$$

Hence, Corollary 3.2 implies that $\|h_{j|\Lambda_j^1}^*\|_{\phi_{\zeta_j},2}^2 \lesssim \varepsilon$.

Since Λ is a sampling set for A_ϕ^2, the sets $\Lambda + \zeta_j$ are sampling sets for $A_{\phi_{\zeta_j}}^2$ with the same stability constants for all j. However, we have produced an analytic function h_j^* such that $\|h_{j|\Lambda+\zeta_j}^*\|_{\phi_{\zeta_j},2}^2 \lesssim \varepsilon$ and (27), where the constants are independent of ε. This contradiction concludes the proof. $\qquad \square$

Proposition 5.2. *Assume that ϕ satisfies* (1) *and* (13). *Let $p \in [1, \infty]$ and Λ be an interpolating set for A_ϕ^p, and suppose that $(\Gamma, \psi) \in W(\Lambda, \phi)$. Then Γ is an interpolating set for A_ψ^p.*

Proof. We proceed as in [14]. By Theorem 3.4, we can restrict the problem to L^1-norms. Let $\{\zeta_j : j \geq 1\} \subseteq \mathbb{C}^n$ be a sequence such that $\Lambda + \zeta_j \xrightarrow{w} \Gamma$ and that $\phi_{\zeta_j} \to \psi$ uniformly on compact sets.

We first show that for given $\gamma_0 \in \Gamma$ the interpolation problem

(28) $$f(\gamma_0) = e^{\psi(\gamma_0)},$$
$$f(\gamma) = 0, \quad \gamma \in \Gamma, \gamma \neq \gamma_0$$

has a solution in A_ψ^1.

Let $\gamma_j \in \Lambda + \zeta_j$ be such that $\gamma_j \to \gamma_0$. Because $\Lambda + \zeta_j$ is an interpolating set for $A^1_{\phi_{\zeta_j}}$ with the same stability constant as Λ has for A^1_ϕ, we can find functions $f_j \in A^1_{\phi_{\zeta_j}}$ such that

$$(29) \qquad\qquad\qquad f_j(\gamma_j) = e^{\phi_{\zeta_j}(\gamma_j)},$$

$$(30) \qquad\qquad\qquad f_j(\gamma) = 0, \qquad \gamma \in \Lambda + \zeta_j, \gamma \neq \gamma_j,$$

$$(31) \qquad\qquad\qquad \|f_j\|_{\phi_{\zeta_j},1} \leq C,$$

where C is the stability constant of interpolation related to Λ in A^1_ϕ. This, together with Lemma 3.1 and Montel's theorem, implies the existence of a subsequence of $\{f_j : j \geq 1\}$ that converges to a holomorphic function $f = f_{\gamma_0}$ uniformly on compact sets. It is readily verified that $f \in A^1_\psi$ and $\|f\|_{\psi,1} \leq C$. Since $\gamma_j \to \gamma_0$ and $\phi_{\zeta_j} \to \psi$, we obtain $f(\gamma_0) = e^{\psi(\gamma_0)}$. Second, since $\Lambda + \zeta_j \xrightarrow{w} \Gamma$, given $\gamma' \in \Gamma \setminus \{\gamma_0\}$, there exist $\gamma'_j \in \Lambda + \zeta_j$ such that $\gamma'_j \to \gamma'$. For $j \gg 1$, $\gamma'_j \neq \gamma_j$ and, therefore, $f(\gamma') = \lim_j f_j(\gamma'_j) = 0$. Hence, f_{γ_0} solves the interpolation problem (28).

The general interpolation problem is now easily solved. Given a sequence $a \in \ell^1_\psi(\Gamma)$, the series $f = \sum_{\gamma \in \Gamma} a_\gamma e^{-\psi(\gamma)} f_\gamma$ converges in A^1_ψ by (31) and therefore uniformly on compact sets. This implies that $f(\gamma_j) = a_j$ for all j, as desired. $\qquad\square$

5.2. Characterization of sampling sets.

Theorem 5.3. *Assume that ϕ satisfies (1) and (13). Then a set Λ is a sampling set for A^p_ϕ if and only if every pair $(\Gamma, \psi) \in W(\Lambda, \phi)$ has the property that Γ is a uniqueness set for A^∞_ψ.*

Proof. One implication is settled by Theorem 3.4 and Proposition 5.1.

For the converse, suppose that Λ is not a sampling set for A^p_ϕ. We will show that there exists $(\Gamma, \psi) \in W(\Lambda, \phi)$, such that Γ is not a uniqueness set for A^∞_ψ.

By Theorem 3.4, Λ is not a sampling set for A^∞_ϕ. This means that, for every $j \in \mathbb{N}$, there exists $f_j \in A^\infty_\phi$ such that $\|f_j\|_{\phi,\infty} = 1$ and $\sup_{\lambda \in \Lambda} |f_j(\lambda)| e^{-\phi(\lambda)} \leq 1/j$. We select a sequence $\{\zeta_j : j \geq 1\} \subseteq \mathbb{C}^n$ such that

$$[f_j(\zeta_j)| e^{-\phi(\zeta_j)} = |T_{-\zeta_j} f_j(0)| e^{-\phi_{-\zeta_j}(0)} \geq 1/2,$$

where we have used property (18) for the translation operator. We also have

$$\sup_{\tau \in \Lambda - \zeta_j} |T_{-\zeta_j} f_j(\tau)| e^{-\phi_{-\zeta_j}(\tau)} \leq 1/j.$$

By Montel's theorem, the growth bound (17) and Proposition 4.1, we can pass to a subsequence and assume that the following hold: (i) $\Lambda - \zeta_j \xrightarrow{w} \Gamma$, (ii) $\phi_{-\zeta_j} \to \psi$ uniformly on compact sets for some plurisubharmonic ψ satisfying (1), and (iii) $T_{-\zeta_j} f_j \to f$ uniformly on compact sets for some holomorphic function f.

Clearly $f \in A_\psi^\infty$, $f(0) \neq 0$ and $f_{|\Gamma} = 0$. This shows that Γ is not a uniqueness set for A_ψ^∞.   □

5.3. **Interpolation and uniqueness.** Interpolating sets can also be characterized with weak limits. For our purposes, we will need the following technical variation of [14, Lemma 5.6].

Lemma 5.4. *Assume that ϕ satisfies (1) and (13). Let $\{\Lambda_j : j \geq 1\}$ be a family of separated sets with a uniform separation constant, i.e.*

$$\inf_j \text{sep}(\Lambda_j) = \inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda_j, \lambda \neq \lambda', j \geq 1\} > 0.$$

Let $c^j \in \ell^\infty(\Lambda_j)$, $j \geq 1$, be sequences with $\|c^j\|_\infty = 1$ such that

$$(32) \qquad \|\sum_{\lambda \in \Lambda_j} c_\lambda^j e^{-\phi(\lambda)} K_{\phi,\lambda}\|_{\phi,\infty} \to 0$$

as $j \to \infty$. Then there exists a subsequence $\{j_k : k \geq 1\} \subseteq \mathbb{N}$; points $\lambda_{j_k} \in \Lambda_j$; a separated set $\Gamma \subseteq \mathbb{C}^n$; a nonzero sequence $c \in \ell^\infty(\Gamma)$; and a plurisubharmonic function ψ satisfying the bounds (1) such that (i) $\phi_{\lambda_{j_k}} \to \psi$ uniformly on compact sets; (ii) $\Lambda_{j_k} - \lambda_{j_k} \xrightarrow{w} \Gamma$; and (iii) the following relation holds

$$(33) \qquad \sum_{\gamma \in \Gamma} c_\gamma e^{-\psi(\gamma)} K_{\psi,\gamma} = 0.$$

Proof. For each j, we select $\lambda_j \in \Lambda_j$ such that $|c_{\lambda_j}^j| \geq 1/2$. Using (20), we can rewrite the condition (32) as

$$\sup_{w \in \mathbb{C}^n} \left| \sum_{\lambda \in \Lambda_j - \lambda_j} c_{\lambda+\lambda_j}^j e^{i\text{Im}(q(\lambda,-\lambda_j))} K_{\phi_{-\lambda_j}}(w,\lambda) e^{-\phi_{-\lambda_j}(\lambda)-\phi_{-\lambda_j}(w)} \right|$$

$$(34) \qquad = \|\sum_{\lambda \in \Lambda_j - \lambda_j} c_{\lambda+\lambda_j}^j e^{i\text{Im}(q(\lambda,-\lambda_j))} K_{\phi_{-\lambda_j},\lambda} e^{-\phi_{-\lambda_j}(\lambda)}\|_{\phi_{-\lambda_j},\infty}$$

$$= \|\sum_{\lambda \in \Lambda_j} c_\lambda^j e^{-\phi(\lambda)} K_{\phi,\lambda}\|_{\phi,\infty} \to 0.$$

Since the sets $\Lambda^j - \lambda_j$ are uniformly separated, by passing to a subsequence, we may find a separated set Γ such that $\Lambda^j - \lambda_j \xrightarrow{w} \Gamma$ - see e.g. [14, Section 4]. Now define the sequences

$$d^j := \left(c_{\lambda+\lambda_j}^j e^{i\text{Im}(q(\lambda,-\lambda_j))} \right)_{\lambda \in \Lambda_j - \lambda_j} \in \ell^\infty(\Lambda_j - \lambda_j)$$

and consider the associated measure

$$\mu_j := \sum_{\lambda \in \Lambda_j - \lambda_j} d_\lambda^j \delta_\lambda.$$

These measures satisfy $\|\mu_j\|_{W(\mathcal{M},L^\infty)} \lesssim \mathrm{rel}(\Lambda_j - \lambda_j)\|d\|_\infty \lesssim 1$. Thus, by passing to a subsequence, there exists a measure $\mu \in W(\mathcal{M}, L^\infty)$ such that $\mu_j \to \mu$ in the $\sigma(W(\mathcal{M}, L^\infty), W(C_0, L^1))$-topology. As shown in [14, Lemma 4.3], it follows that $\mathrm{supp}(\mu) \subseteq \Gamma$, so that we may write

$$\mu = \sum_{\gamma \in \Gamma} c_\gamma \delta_\gamma.$$

In addition, $\|c\|_\infty \lesssim \|\mu\|_{W(\mathcal{M},L^\infty)} < \infty$ by [14, Lemma 4.6]. By Proposition 4.1, we may pass to a further subsequence such that $\phi_{-\lambda_j} \to \psi$ for some plurisubharmonic ψ satisfying (1) and (17). By construction $0 \in \Gamma$ and $c(0) = \lim_{j\to\infty} d_0^j \neq 0$, therefore μ is not identically zero.

Let

$$f_{j,w}(z) := K_{\phi_{-\lambda_j}}(z, w)e^{-\phi_{-\lambda_j}(z)-\phi_{-\lambda_j}(w)}$$

and

$$f_w(z) := K_\psi(z, w)e^{-\psi(z)-\psi(w)}$$

be the modified reproducing kernels of $A^2_{\phi_{-\lambda_j}}$ and A^2_ψ. The kernels $K_{\phi_{-\lambda_j}}$ and K_ψ satisfy the off-diagonal estimate (11) with uniform constants. This fact implies that $f_{j,w}$ and f_w belong to $W(C_0, L^1)(\mathbb{R}^{2n})$.

With this notation, (33) can be recast in terms of the measure μ as the statement that $\mu(f_w) = 0$ for all w. We now show that this is indeed the case.

Let $w \in \mathbb{C}^n$ and write

$$\mu(f_w) = \mu_j(f_{j,w}) + \mu_j(f_w - f_{j,w}) + (\mu - \mu_j)(f_w).$$

The first term tends to zero by our assumption (34), and the third term tends to zero by the weak convergence of μ_j to μ.

For the second term, we use Proposition 4.1 which says that $f_{j,w} \to f_w$ uniformly on compacts, possibly after passing to a further subsequence. In addition, the uniform off-diagonal decay of the reproducing kernels (11) implies that

$$|f_w(z) - f_{j,w}(z)| \leq Ce^{-c|z-w|},$$

for some constants $c, C > 0$ that are independent of j. This localization estimate and the uniform convergence on compact sets imply that $f_{j,w} \to f_w$ in $W(C_0, L^1)$ as $j \to \infty$. Since the weakly convergent sequence $\{\mu_j : j \geq 1\}$ is bounded, we obtain that

$$|\mu_j(f_w - f_{w,j})| \leq \sup_{j'}\|\mu_{j'}\|_{W(\mathcal{M},L^\infty)} \|f_w - f_{w,j}\|_{W(C_0,L^1)} \to 0, \quad \text{as } j \to \infty.$$

We have proved that $\mu(f_w) = 0$ for all $w \in \mathbb{C}^n$, which is (33). $\qquad\square$

As in [14, Theorem 5.4], this lemma can also be used to prove a sharp necessary density condition for interpolating sets.

6. Stability of sampling and interpolation under Lipschitz deformations

We now prove the main result on deformation of sampling and of interpolating sets.

Proof of Theorem 1.4. By Proposition 2.6 we may assume, without loss of generality, that ϕ satisfies the growth condition (13).

Part (a). If Λ is a sampling set for A_ϕ^p, then Λ is also a sampling set for A_ϕ^∞ by Theorem 3.4. We will prove that Λ_j is a sampling set for A_ϕ^∞ for $j \geq j_0$. Applying Theorem 3.4 once more, Λ_j is then a sampling set for A_ϕ^p for all $j \geq j_0$.

To prove the claim, we argue by contradiction and, by passing to a subsequence, assume that none of the Λ_j is a sampling set for A_ϕ^∞. Then there exists a sequence of functions $f_j \in A_\phi^\infty$ such that $\|f_j\|_{\infty,\phi} = 1$ and $\sup_{z \in \Lambda_j}[f_j(z)|e^{-\phi(z)} \leq 1/j$. Let $\zeta_j \in \mathbb{C}$ be such that $|f_j(\zeta_j)|e^{-\phi(\zeta_j)} \geq 1/2$. By Proposition 4.1, we pass to a subsequence, and assume that (i) the translates $\phi_{-\zeta_j}$ converge to a plurisubharmonic function ψ uniformly on compact sets, and that (ii) there exists a separated set Γ such that $\Lambda_j - \zeta_j \xrightarrow{w} \Gamma$. See, e.g., [14, Section 4]. Since Λ is relatively dense by Corollary 2.3 and Theorem 3.4, Lemma 3.5 is applicable and implies that $\Gamma \in W(\Lambda)$.

By construction, the translates $T_{-\zeta_j} f_j(z)$ satisfy the following:

$$\|T_{-\zeta_j} f_j\|_{\phi_{-\zeta_j},\infty} = \|f_j\|_{\phi,\infty} \,,$$

$$(35) \qquad |T_{-\zeta_j} f_j(0)|e^{-\phi_{-\zeta_j}(0)} = |f_j(\zeta_j)|e^{-\phi(\zeta_j)} \geq 1/2 \,,$$

$$(36) \qquad \sup_{z \in \Lambda_j - \zeta_j} |T_{-\zeta_j} f_j(z)|e^{-\phi_{-\zeta_j}(z)} = \sum_{z \in \Lambda_j} |f_j(z)|^{-\phi(z)} \leq 1/j \,.$$

It follows from the growth estimate (13) that the sequence $T_{-\zeta_j} f_j$ is uniformly bounded on compact sets. Therefore, Montel's theorem guarantees the existence of a subsequence converging uniformly on compact sets to a holomorphic function f, which is not identically zero by (35). Clearly, the function f belongs to A_ψ^∞, and (36) implies that f vanishes on $\Gamma \in W(\Lambda)$. By Proposition 5.3, Λ is not a sampling set, which contradicts the assumption. Consequently, Λ_j must be a sampling set for A_ϕ^∞, for all for sufficiently large j.

Part (b). If Λ is an interpolating set for A_ϕ^p, then Λ is also an interpolating set for A_ϕ^1 by Theorem 3.4. We will prove that Λ_j is an interpolating set for A_ϕ^1 for $j \geq j_0$. Applying Theorem 3.4 once more, Λ_j is then an interpolating set for A_ϕ^p for $j \geq j_0$.

Again, we argue by contradiction, and assume, without loss of generality, that none of the Λ_j is an interpolating set for A_ϕ^1. This means that, for every $j \in \mathbb{N}$, the operator

$$A_\phi^1 \longrightarrow \ell^1(\Lambda_j), \qquad f \longmapsto f|_{\Lambda_j}$$

fails to be surjective. By duality, it follows that the operator

$$\ell^\infty(\Lambda_j) \longrightarrow A_\phi^\infty, \qquad c \longmapsto \sum_\lambda c_\lambda e^{-\phi(\lambda)} K_{\phi,\lambda}$$

is not bounded below. Therefore there exist sequences $c^j \in \ell^\infty(\Lambda_j)$ such that $\|c^j\|_\infty = 1$ and

$$\|\sum_{\lambda \in \Lambda_j} c_\lambda^j e^{-\phi(\lambda)} K_{\phi,\lambda}\|_{\phi,\infty} \to 0.$$

Note that Λ is separated by Corollary 3.3. Since $\Lambda_j \xrightarrow{Lip} \Lambda$, by [14, Lemma 6.7], we can pass to a further subsequence, and assume that $\{\Lambda_j : j \geq 1\}$ is uniformly separated. Therefore the assumptions of Lemma 5.4 are satisfied. Using Lemma 5.4 and passing to a further subsequence, we find points $\lambda_j \in \Lambda_j$; a limiting weight $\phi_{\lambda_j} \longrightarrow \psi$; a separated set Γ such that $\Lambda_j - \lambda_j \xrightarrow{w} \Gamma$; and a nonzero sequence $c \in \ell^\infty(\Gamma)$ such that

$$\sum_{\gamma \in \Gamma} c_\gamma e^{-\psi(\gamma)} K_{\psi,\gamma} = 0.$$

Thus, for every $f \in A_\psi^1$,

$$0 = \sum_{\gamma \in \Gamma} c_\gamma e^{-\psi(\gamma)} \langle f, K_{\psi,\gamma} \rangle = \sum_{\gamma \in \Gamma} c_\gamma e^{-\psi(\gamma)} f(\gamma).$$

Since $c \not\equiv 0$, it follows that Γ is not an interpolating set for A_ψ^1. Since Λ is separated, Lemma 3.5 is applicable to Λ and asserts that $\Gamma \in W(\Lambda)$. By Proposition 5.2, Λ is not an interpolating set, which contradicts the assumption. Consequently, Λ_j must be an interpolating set for A_ϕ^1 for all sufficiently large j. $\qquad\square$

As a corollary of the stability with respect to Lipschitz deformations we show the result announced in the introduction: sampling and interpolating sets cannot attain the critical density.

6.1. **Proof of Theorem 1.2.** By Theorem 3.4, we can restrict our attention to $p = 2$. Suppose that Λ is a sampling set for A_ϕ^2 with $D_\phi^-(\Lambda) \leq 1$, and consider $\Lambda_j := (1 + 1/j)\Lambda$. Then $\Lambda_j \xrightarrow{Lip} \Lambda$, and by Theorem 1.4, Λ_j is sampling set for A_ϕ^2 for sufficiently large j. The bounds on the diagonal of the reproducing kernel (10) and (11) imply that $D_\phi^-(\Lambda_j) < 1$ - see Lemma 6.1 below - which contradicts Theorem 2.1. The statement about interpolation follows similarly. $\quad\square$

Lemma 6.1. *Assume that ϕ satisfies* (1). *Let $\Lambda \subseteq \mathbb{C}^n$ and $a > 1$. Then $D_\phi^-(a\Lambda) < D_\phi^-(\Lambda)$ and $D_\phi^+(a\Lambda) < D_\phi^+(\Lambda)$.*

Proof. Using the bounds on the diagonal of the reproducing kernel (10) and (11), we estimate

$$\int_{B_{ar}(z)\setminus B_r(z)} K(w,w)e^{-2\phi(w)}dm(w) \asymp m(B_{ar}(z)\setminus B_r(z))$$

$$= (a^{2n}-1)m(B_r(z)) \asymp (a^{2n}-1)\int_{B_r(z)} K(w,w)e^{-2\phi(w)}dm(w).$$

Hence,

$$\int_{B_{ar}(z)} K(w,w)e^{-2\phi(w)}dm(w) \geq (1+c)\int_{B_r(z)} K(w,w)e^{-2\phi(w)}dm(w),$$

for some $c = c(a) > 0$. As a consequence,

$$D_\phi^-(a\Lambda) = \liminf_{r\to\infty} \inf_{z\in\mathbb{C}^n} \frac{\#(a\Lambda \cap B_r(z))}{\int_{B_r(z)} K(w,w)e^{-2\phi(w)}dm(w)}$$

$$= \liminf_{r\to\infty} \inf_{z\in\mathbb{C}^n} \frac{\#(\Lambda \cap B_{r/a}(z))}{\int_{B_r(z)} K(w,w)e^{-2\phi(w)}dm(w)}$$

$$= \liminf_{r\to\infty} \inf_{z\in\mathbb{C}^n} \frac{\#(\Lambda \cap B_r(z))}{\int_{B_{ar}(z)} K(w,w)e^{-2\phi(w)}dm(w)}$$

$$\leq \frac{1}{1+c}D_\phi^-(\Lambda) < D_\phi^-(\Lambda).$$

The statement about $D_\phi^+(\Lambda)$ follows similarly.   □

7. Localizable reproducing kernel Hilbert spaces

This section treats the problem of sampling and interpolation in reproducing kernel Hilbert spaces with a localized reproducing kernel. The results extend considerably those of the theory of localized frames [1, 3, 4, 13, 24] and may be of independent interest.

7.1. **Wiener's lemma with localized subspaces.**

Theorem 7.1. *Let* $\Lambda, \Gamma \subseteq \mathbb{R}^d$ *be relatively separated, and let* $P : \ell^2(\Gamma) \to \ell^2(\Gamma)$ *and* $A : \ell^2(\Gamma) \to \ell^2(\Lambda)$ *be bounded operators, represented by matrices* $A = (A_{\lambda,\gamma})_{\lambda\in\Lambda,\gamma\in\Gamma}$ *and* $P = (P_{\gamma,\gamma'})_{\gamma,\gamma'\in\Gamma}$. *Assume that* $P^2 = P$ *and that* A *and* P *satisfy the localization estimates:*

(37) $$|A_{\lambda,\gamma}| \leq \Theta(\lambda - \gamma), \qquad \lambda \in \Lambda, \gamma \in \Gamma,$$

(38) $$|P_{\gamma,\gamma'}| \leq \Theta(\gamma - \gamma'), \qquad \gamma, \gamma' \in \Gamma,$$

for some $\Theta \in W(C_0, L^1)(\mathbb{R}^d)$.

Let $p \in [1, \infty]$ *and assume that* A *is p-bounded below on the range of* P, *i.e., there exists* $C_p > 0$ *such that*

(39) $$\|APc\|_p \geq C_p\|Pc\|_p, \qquad c \in \ell^p(\Gamma).$$

Then there exist $C' > 0$ such that for all $q \in [1, \infty]$:

$$\|APc\|_q \geq C'\|Pc\|_q, \qquad c \in \ell^q(\Gamma). \tag{40}$$

Moreover, C' is independent of q and depends only on Θ, C_p, and upper bounds for the relative separation of Λ and Γ.

Proof. When $P = I$ (identity) the result is (a slight extension of) Sjöstrand's version of Wiener's lemma [23] in the precise formulation of [14, Prop. A.1]. To prove the result for general P, we consider the operator

$$\tilde{A} : \ell^2(\Gamma) \to \ell^2(\Lambda) \oplus \ell^2(\Gamma), \qquad \tilde{A}c = ((AP)c, (I - P)c). \tag{41}$$

The matrices representing AP and $I - P$ satisfy enveloping conditions similar to (37), (38). To ease the notation, we keep the same envelope and write:

$$|(AP)_{\lambda,\gamma}| \leq \Theta(\lambda - \gamma), \qquad \lambda \in \Lambda, \gamma \in \Gamma, \tag{42}$$

$$|(I - P)_{\gamma,\gamma'}| \leq \Theta(\gamma - \gamma'), \qquad \gamma, \gamma' \in \Gamma. \tag{43}$$

In order to apply Sjöstrand's Wiener-type lemma to $\tilde{A}$, we consider the following augmented sets:

$$\Lambda^* := \{(\lambda, 0) : \lambda \in \Lambda\} \subseteq \mathbb{R}^{d+1},$$

$$\Gamma^* := \{(\gamma, 1) : \gamma \in \Gamma\} \subseteq \mathbb{R}^{d+1},$$

$$\Upsilon^* := \Lambda^* \cup \Gamma^*.$$

Then $\Lambda^*, \Gamma^*, \Upsilon^*$ are relatively separated. Under the identifications $\ell^2(\Gamma^*) \cong \ell^2(\Gamma)$ and $\ell^2(\Upsilon^*) \cong \ell^2(\Lambda) \oplus \ell^2(\Gamma)$, the operator $\tilde{A}$ can be identified with the operator $B : \ell^2(\Gamma^*) \to \ell^2(\Upsilon^*)$ with matrix entries:

$$B_{(\lambda,0),(\gamma,1)} := (AP)_{\lambda,\gamma}, \tag{44}$$

$$B_{(\gamma',1),(\gamma,1)} := (I - P)_{\gamma',\gamma}. \tag{45}$$

Let $\eta \in C^\infty(\mathbb{R})$ be a cut-off function supported on $[-2, 2]$ such that $\eta \equiv 1$ on $[-1, 1]$, and consider the augmented envelope $\tilde{\Theta} \in W(L^\infty, L^1)(\mathbb{R}^{d+1})$ defined by

$$\tilde{\Theta}(x, t) := \Theta(x)\eta(t).$$

Using (42) and (43) we see that B satisfies the enveloping condition:

$$\left|B_{(\lambda,0),(\gamma,1)}\right| = |(AP)_{\lambda,\gamma}| \leq \Theta(\lambda - \gamma) = \Theta(\lambda - \gamma)\eta(0 - 1) = \tilde{\Theta}((\lambda, 0) - (\gamma, 1)),$$

$$\left|B_{(\gamma',1),(\gamma,1)}\right| = |(I - P)_{\gamma',\gamma}| \leq \Theta(\gamma' - \gamma) = \Theta(\gamma' - \gamma)\eta(1 - 1) = \tilde{\Theta}((\gamma', 1) - (\gamma, 1)).$$

Let us assume (39) holds for a certain value of $p \in [1, \infty]$. We now estimate for $c \in \ell^p(\Gamma)$

$$\|\tilde{A}c\|_{\ell^p \oplus \ell^p} = \|APc\|_{\ell^p} + \|(I - P)c\|_{\ell^p} \geq C_p\|Pc\|_{\ell^p} + \|(I - P)c\|_{\ell^p} \gtrsim \|c\|_p;$$

see Remark 7.2.

Hence $\tilde{A}$ is p-bounded below, and therefore so is B. By Sjöstrand's Wiener-type lemma [23], [14, Prop. A.1], B and therefore also $\tilde{A}$ are q-bounded below for all $q \in [1, \infty]$. Finally, for $c \in \ell^q(\Gamma)$,

$$\|APc\|_q = \|\tilde{A}Pc\|_q \geq C'\|Pc\|_q,$$

as desired. The dependence of the constant C' is discussed in [14]. In particular, C' is independent of q.   □

Remark 7.2. The idempotent P in Theorem 7.1 need not be an orthogonal projection. The operator P acts on all ℓ^p due to the enveloping condition (38), and it is still idempotent on ℓ^p by density. In addition, $\|c\|_p \asymp \|Pc\|_p + \|(I - P)c\|_p$.

7.2. Localizable reproducing kernel Hilbert spaces.

Definition 7.3. *We say that a (closed) subspace $V \subseteq L^2(\mathbb{R}^d)$ is a* localizable reproducing kernel Hilbert space *(localizable RKHS), if there exist: (a) a relatively separated $\Gamma \subseteq \mathbb{R}^d$ (nodes); (b) a function $\Theta \in W(L^\infty, L^1)(\mathbb{R}^d)$ (envelope); and (c) a frame for V, $\mathrm{F} \equiv \{F_\gamma : \gamma \in \Gamma\}$, consisting of continuous functions that satisfy the localization estimate*

$$(46) \qquad\qquad |F_\gamma(x)| \leq \Theta(x - \gamma), \qquad \gamma \in \Gamma.$$

We now briefly describe some consequences of the definition. These are part of the abstract theory of localized frames [4, 11, 13]. The concrete application of this theory to the present setting can be found in first sections of [20] (where localizable RKHS are called *spline-type spaces*). See also [24] for closely related estimates.

The *canonical dual frame* $\widetilde{F} \equiv \{\tilde{F}_\gamma : \gamma \in \Gamma\} \subseteq V$ satisfies a similar localization estimate

$$(47) \qquad\qquad \left|\tilde{F}_\gamma(x)\right| \leq \tilde{\Theta}(x - \gamma), \qquad \gamma \in \Gamma,$$

for some $\tilde{\Theta} \in W(L^\infty, L^1)(\mathbb{R}^d)$. As a consequence of the localization estimates (46) and (47) all operators associated to the frame F obey the expected mapping properties. Specifically, the *coefficient maps* $f \mapsto Cf := (\langle f, F_\gamma \rangle)_{\gamma \in \Gamma}$ and $f \mapsto \tilde{C}f := (\langle f, \tilde{F}_\gamma \rangle)_{\gamma \in \Gamma}$, which are bounded from $L^2(\mathbb{R}^d) \to \ell^2(\Gamma)$ by the frame property, extend to bounded operators $C, \tilde{C} : L^p(\mathbb{R}^d) \to \ell^p(\Gamma)$. Likewise the adjoint operators

$$(48) \qquad c \mapsto C^*c := \sum_{\gamma \in \Gamma} c_\gamma F_\gamma \quad \text{and} \quad c \mapsto \tilde{C}^*c := \sum_{\gamma \in \Gamma} c_\gamma \tilde{F}_\gamma$$

can be extended to bounded operators $C^*, \tilde{C}^* : \ell^p(\Gamma) \to L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. Here both series in (48) converge unconditionally in $L^p(\mathbb{R}^d)$ for $p < \infty$ and in the weak*-topology for $p = \infty$.

As a consequence, the range-space

$$V^p := C^*(\ell^p(\Gamma)) = \left\{ \sum_{\gamma \in \Gamma} c_\gamma F_\gamma : c \in \ell^p(\Gamma) \right\} \subseteq L^p(\mathbb{R}^d)$$

is a well-defined closed, complemented subspace of $L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$. It follows that the dual space of V^p is $V^{p'}$ where $1 \le p < \infty$ and $1/p + 1/p' = 1$. These spaces are independent of the particular choice of the frame F in the sense that if two such frames satisfy (46) and expand the same Hilbert space V, then they will produce the same range spaces V^p. Each function $f \in V^p$ admits the two expansions

$$(49) \qquad f = C^* \tilde{C} f = \sum_{\gamma \in \Gamma} \langle f, F_\gamma \rangle \, \tilde{F}_\gamma = \tilde{C}^* C f = \sum_{\gamma \in \Gamma} \left\langle f, \tilde{F}_\gamma \right\rangle F_\gamma \,,$$

and the continuity of C^* and $\tilde{C}$ implies that $C : V^p \to \ell^p(\Gamma)$ is bounded below:

$$(50) \qquad \|Cf\|_p \asymp \|f\|_p, \qquad f \in V^p \,.$$

Furthermore, $P = C\tilde{C}^*$ is a projection onto its closed range in $\ell^p(\Lambda)$ and possesses a matrix representation with entries

$$P_{\gamma',\gamma} = \langle \tilde{F}_\gamma, F_{\gamma'} \rangle, \qquad \gamma, \gamma' \in \Gamma \,.$$

Finally, we remark that, on the subspace V^p, the L^p and $W(L^\infty, L^p)$ norms are equivalent. Since the frame elements are continuous, every function in V^p is continuous, and therefore

$$(51) \qquad V^p \subseteq W(C_0, L^p),$$

and the following sampling estimate holds:

$$(52) \qquad \|f|\Lambda\|_{\ell^p(\Lambda)} \lesssim \mathrm{rel}(\Lambda)\|f\|_p, \qquad f \in V^p.$$

In particular, the *sampling operator*

$$(53) \qquad S : V^p \to \ell^p(\Lambda), \qquad Sf := (f(\lambda))_{\lambda \in \Lambda} \,,$$

is bounded for every relatively separated set Λ. (See e.g. [20, 24] for proofs.)

The following lemma follows easily from the definitions.

Lemma 7.4. *Let $V \subseteq L^2(\mathbb{R}^d)$ be a localizable RKHS. Suppose that $f_n \longrightarrow f$ in $\sigma(V^\infty, V^1)$, and $x_n \longrightarrow x$, with f_n, $f \in V^\infty$ and x_n, $x \in \mathbb{R}^d$. Then $f_n(x_n) \longrightarrow f(x)$.*

7.3. **Universality of sampling and interpolating sets.** The universality of sampling and interpolation in weighted Fock spaces in Theorem 3.4 is a special case of the following much more general statement about localizable RKHSs.

Theorem 7.5. *Let $V \subseteq L^2(\mathbb{R}^d)$ be a localizable RKHS and let $\Lambda \subseteq \mathbb{R}^d$ be relatively separated.*

(i) If Λ is a sampling set for V^p for some $p \in [1, \infty]$, then Λ is a sampling set for V^q for all $q \in [1, \infty]$.

(ii) If Λ is an interpolating set for V^p for some $p \in [1, \infty]$, then Λ is an interpolating set for V^q for all $q \in [1, \infty]$.

Proof. Let $F \equiv \{F_\gamma : \gamma \in \Gamma\}$ be a localized frame, as in Definition 7.3.

(i) Assume that Λ is a sampling set for V^p for some $p \in [1, \infty]$ and let $q \in [1, \infty]$. We must show that Λ is a sampling set for V^q. In addition to the orthogonal projection $P = C\tilde{C}^*$ and the sampling operator S from (53), we define $A = S\tilde{C}^*$. Written as a matrix, A has the entries

$$A_{\lambda,\gamma} = \widetilde{F_\gamma}(\lambda), \qquad \lambda \in \Lambda, \gamma \in \Gamma.$$

By (46) and (47) A and P satisfy the decay estimates (37) and (38).

In order to apply Theorem 7.1, we will verify the stability condition (39). Given $c \in \ell^p(\Gamma)$, let $f = \tilde{C}^*c \in V^p$. Since Λ is a sampling set for V^p, we have $\|Sf\|_p \asymp \|f\|_p$. Using the properties of the operators $C, \tilde{C}$, we obtain the estimate

$$\|APc\|_p = \|S\tilde{C}^*C\tilde{C}^*c\|_p = \|S\tilde{C}^*Cf\|_p = \|Sf\|_p$$

(54)
$$\asymp \|f\|_p \asymp \|Cf\|_p = \|C\tilde{C}^*c\|_p = \|Pc\|_p.$$

At this point we can invoke Theorem 7.1 and conclude that $\|APc\|_q \gtrsim \|Pc\|_q$ for all $q \in [1, \infty]$.

To show that Λ is a sampling set for V_q, let $f \in V_q$ and $c := Cf$, so that $f = \tilde{C}^*c$. We repeat the estimates (54) and obtain

$$\|Sf\|_q = \|S(\tilde{C}^*C)f\|_q = \|S(\tilde{C}^*C)\tilde{C}^*c\|_q$$
$$= \|(S\tilde{C}^*)(C\tilde{C}^*)c\|_q = \|APc\|_q$$
$$\gtrsim \|Pc\|_q = \|C\tilde{C}^*Cf\|_q = \|Cf\|_q \asymp \|f\|_q,$$

as desired.

(ii) By definition, Λ is an interpolating set for V^p, if the sampling operator $S : V^p \to \ell^p(\Lambda)$ is surjective. This is the case if and only if $S\tilde{C}^* : \ell^p(\Gamma) \to \ell^p(\Lambda)$ is surjective, which in turn holds if and only if $\tilde{C}S^* : \ell^{p'}(\Lambda) \to \ell^{p'}(\Gamma)$ is bounded below, where $1/p + 1/p' = 1$.

(For the case $p = \infty$, we use the fact the the operators are weak*-continuous.) The operator $\tilde{C}S^*$ is represented by the matrix with entries

$$A^*_{\gamma,\lambda} = \overline{\tilde{F_\gamma}(\lambda)}, \qquad \lambda \in \Lambda, \gamma \in \Gamma.$$

We invoke again Theorem 7.1 — this time with $P = I$ — and conclude that if $S^*\tilde{C}$ is p-bounded below for some $p \in [1, \infty]$, then it is p-bounded below for all $p \in [1, \infty]$. This concludes the proof. $\qquad\square$

7.4. **Sets close to being sampling and interpolating.** We say that a localizable RKHS is uniformly localizable if

(55)
$$\sup_{\substack{x,y \in \mathbb{R}^d \\ |x-y| \leq \delta}} \sup_{\substack{f \in V^\infty \\ \|f\|_\infty \leq 1}} \left| |f(x)| - |f(y)| \right| \longrightarrow 0, \text{ as } \delta \longrightarrow 0^+.$$

Note that the uniformity property concerns the *absolute values* of the functions in V^∞. It is thus a weaker condition than the uniform equicontinuity of the ball of V^∞. For comparison, the equicontinuity of the ball of V^∞ amounts to the condition

(56)
$$\sup_{\substack{x,y \in \mathbb{R}^d \\ |x-y| \leq \delta}} \|K_x - K_y\|_1 \longrightarrow 0, \text{ as } \delta \longrightarrow 0^+$$

for the reproducing kernel of V^∞. In several examples, in particular in spaces of analytic functions, the uniformity property (55) is easier to verify than (56). See Proposition 7.7 below.

The following is an abstract version of the construction in [16]. Sampling sets with close to critical density can also be obtained from the general result in [3].

Theorem 7.6. *Let $V \subseteq L^2(\mathbb{R}^d)$ be a uniformly localizable RKHS. Then there exists a separated set $\Lambda \subseteq \mathbb{R}^d$ with the following properties.*

(i) *($\ell^1 - \ell^\infty$ interpolation.) There exist a collection of functions $\{l_\lambda : \lambda \in \Lambda\} \subseteq V^\infty$ such that $\|l_\lambda\|_\infty = 1$, and*

(57)
$$l_\lambda(\lambda') = \delta_{\lambda,\lambda'}, \qquad \lambda, \lambda' \in \Lambda.$$

In particular, given $a \in \ell^1(\Lambda)$, $f = \sum_{\lambda \in \Lambda} a_\lambda l_\lambda \in V^\infty$ and satisfies $f(\lambda) = a_\lambda$ for all $\lambda \in \Lambda$.

(ii) *($\ell^1 - \ell^\infty$ sampling.) For all $f \in V^1$*

(58)
$$\|f\|_\infty \leq \sum_{\lambda \in \Lambda} |f(\lambda)|.$$

Proof. Step 1 (Construction of Fekete points and the set Λ). Without loss of generality we assume that $\dim V = \infty$. Let $\{P_n : n \geq 1\} \subseteq V^1$ be a nested sequence of subspaces

such that $\dim P_n = n$ and $\bigcup_n P_n$ is dense in V^1. Let $\{p_1^n, \ldots, p_n^n\}$ be an orthogonal basis of P_n. Consider the functional

$$\Delta(x_1, \ldots, x_n) = \left| \det \left(p_i^n(x_j) \right)_{i,j=1\ldots n} \right|, \qquad x_j \in \mathbb{R}^d,$$

and let $(x_1^n, \ldots, x_n^n)$ be a maximizer of Δ. We denote the corresponding set of points by $\Lambda^n = \{x_1^n, \ldots, x_n^n\} \subseteq \mathbb{R}^d$. A maximizer of Δ always exists because, by (51), every function in V^1 vanishes at infinity, and therefore it is enough to maximize Δ on a suitable compact set.

We consider the Lagrange functions $l_1^n, \cdots, l_n^n$ defined as

$$l_{x_i^n}(x) = \frac{\begin{vmatrix} p_1^n(x_1^n) & \cdots & p_1^n(x) & \cdots & p_1^n(x_n^n) \\ \vdots & & \vdots & & \vdots \\ p_n^n(x_1^n) & \cdots & p_n^n(x) & \cdots & p_n^n(x_n^n) \end{vmatrix}}{\det \left(p_i^n(x_j^n) \right)_{i,j=1\ldots n}}.$$

The functions $\{l_\lambda^n : \lambda \in \Lambda^n\}$ form a basis of P_n, and satisfy $l_\lambda^n(\mu) = \delta_{\lambda,\mu}$, for all $\lambda, \mu \in \Lambda^n$. In addition, since $(x_1^n, \ldots, x_n^n)$ is a maximizer for Δ, we know that $\|l_\lambda^n\|_\infty = 1$.

Next, for distinct $\lambda, \mu \in \Lambda^n$, $1 = \big| |l_\lambda(\lambda)| - |l_\lambda(\mu)| \big|$. By the uniformity property (55), it follows that the sets Λ^n are uniformly separated, i.e.,

$$\inf_{n \in \mathbb{N}} \inf \{ |\lambda - \mu| : \lambda, \mu \in \Lambda^n, \lambda \neq \mu \} > 0.$$

By passing to a subsequence, we may assume that $\Lambda^n \xrightarrow{w} \Lambda$, for some separated set Λ, and because of the uniform separation, the associated measures converge in the following manner:

$$(59) \qquad \mu_n := \sum_{\lambda \in \Lambda^n} \delta_\lambda \longrightarrow \mu := \sum_{\lambda \in \Lambda} \delta_\lambda, \qquad \text{in } \sigma(\mathcal{M}, W(C_0, L^1)),$$

see, e.g., [14, Section 4].

Step 2 (Construction of the dual system). Let $\lambda \in \Lambda$, then there exists a sequence $\lambda^n \in \Lambda^n$ such that $\lambda^n \longrightarrow \lambda$. By passing to a subsequence we may assume that $l_{\lambda^n}^n \longrightarrow l_\lambda$ in $\sigma(V^\infty, V^1)$, for some $l_\lambda \in V^\infty$. Note that $\|l_\lambda\|_\infty \leq \liminf_n \|l_{\lambda^n}^n\|_\infty = 1$. Using Lemma 7.4, it follows that (57) holds. (See also the proof of Proposition 5.2.)

Step 3 (Interpolation). Given $a \in \ell^1(\Lambda)$ we let $f := \sum_\lambda a_\lambda l_\lambda$. Then the series converges absolutely, $\|f\|_\infty \leq \|a\|_1$, and $f(\lambda) = a_\lambda$, for all $\lambda \in \Lambda$.

Step 4 (Sampling). Fix $n \geq 1$ and let $f \in P_n$. For all $N \geq n$, since $f \in P_N$, $f(x) = \sum_{\lambda \in \Lambda^N} f(\lambda) l_\lambda^N(x)$. By (51), $f \in V^1 \subseteq W(C_0, L^1)$, and, by (59),

$$\|f\|_\infty \leq \sum_{\lambda \in \Lambda^N} |f(\lambda)| = \int |f| \, d\mu_N \longrightarrow \int |f| \, d\mu = \sum_{\lambda \in \Lambda} |f(\lambda)|.$$

Since $\bigcup_n P_n$ is dense in V^1 and (52) holds, the full sampling estimate (58) now follows. $\square$

7.5. **Application to Fock spaces.** We now apply the results about localizable RKHSs to weighted Fock spaces. First we need to show that every weighted Fock space A_ϕ^2 is a localizable RKHS.

Proposition 7.7. *Let* $\phi : \mathbb{C}^n \to \mathbb{R}$ *be a plurisubharmonic function satisfying* (1) *and consider the weighted Fock space* A_ϕ^2. *Then the space*

$$V_\phi^2 := \left\{ f = g e^{-\phi} : g \in A_\phi^2 \right\}$$

is a uniformly localizable RKHS in $L^2(\mathbb{R}^{2n})$.

Proof. Let $d = 2n$ and let $P : L^2(\mathbb{R}^d) \to V_\phi^2$ be the orthogonal projection. If K_ϕ is the kernel of A_ϕ^2, then the reproducing kernel of V_ϕ^2 is given by

$$(60) \qquad \widetilde{K_\phi}(z, w) = K_\phi(z, w) e^{-\phi(z) - \phi(w)}$$

Therefore the off-diagonal decay estimate for the producing kernel of A_ϕ^2 in (11) reads as

$$(61) \qquad \left| \widetilde{K_\phi}(z, w) \right| \lesssim e^{-c|z-w|}, \qquad z, w \in \mathbb{R}^d,$$

for some constant $c > 0$. Let $\delta \in (0, 2/\sqrt{d})$, $\Gamma := \delta \mathbb{Z}^d$, $I := [-1/2, 1/2]^d$ (so that $\delta I \subseteq B_1(0)$ for $\delta < 2/\sqrt{d}$), and

$$F_\gamma := P(\delta^{-d} 1_{\gamma + \delta I}), \qquad \gamma \in \Gamma.$$

By (61),

$$|F_\gamma(z)| \lesssim \delta^{-d} \int_{\mathbb{R}^d} 1_{\gamma + \delta I}(w) e^{-c|z-w|} dm(w) \leq C_\delta e^{-c|z-\gamma|},$$

for some constant $C_\delta > 0$. Hence the family $F \equiv \{F_\gamma : \gamma \in \Gamma\}$ satisfies the localization estimate (47).

Let us show that for suitably small $\delta \in (0, 2/\sqrt{d})$, F forms a frame of V_ϕ^2. Let $f \in V_\phi^2$ and define

$$\tilde{f} := \sum_{\gamma \in \Gamma} \delta^{-d} \left(\int_{\delta I + \gamma} f \right) 1_{\delta I + \gamma} = \sum_{\gamma \in \Gamma} \langle f, F_\gamma \rangle 1_{\delta I + \gamma}.$$

Using the mean value theorem, (15) of Lemma 3.1 with $p = r = 2$, and the fact that $\delta < 2/\sqrt{d}$, we obtain the pointwise estimate

$$\left| f - \tilde{f} \right|^2 1_{\delta I + \gamma} \lesssim \delta^2 \int_{B_2(\gamma)} |f(z)|^2 \, dm(z),$$

and consequently

$$(62) \qquad \|f - \tilde{f}\|_2^2 \lesssim \delta^2 \delta^d \sum_{\gamma \in \Gamma} \int_{B_2(\gamma)} |f|^2 \lesssim \delta^2 \|f\|_2^2.$$

Choosing $\delta \ll 1$ small enough, we conclude that

$$\|f\|_2 \le 2\|\tilde{f}\|_2 = 2\delta^{d/2} \left(\sum_{\gamma \in \Gamma} |\langle f, F_\gamma \rangle|^2 \right)^{1/2}.$$

The converse inequality $\sum_{\gamma \in \Gamma} |\langle f, F_\gamma \rangle|^2 = \delta^d \|\tilde{f}\|_2^2 \le \frac{3}{2}\|f\|_2^2$ follows also from (62). Hence F is a frame of $V^2(\mathbb{C}^n)$, as claimed.

Finally, if $f \in V_\phi^\infty$, i.e., $f = f_0 e^{-\phi}$ for $f_0 \in A_\phi^\infty$, then by Lemma 3.1, with $r = p = 1$,

$$|\nabla (|f(z)|)| \lesssim \left[\int_{B_1(z)} |f(w)| dm(w) \right] \lesssim \|f\|_\infty.$$

This implies the property of uniform localization (55).

$\square$

7.6. **Proof of Theorem 3.4.** By Proposition 7.7, V_ϕ^2 is a uniformly localizable RKHS. Theorem 7.5 then assert the universality of sampling and interpolation for V_ϕ^p for all $p \in [1, \infty]$, which is the same as universality for the weighted Fock spaces A_ϕ^p. $\square$

7.7. **Proof of Theorem 1.1.**

Step 1. Again we argue in terms of the reweighted spaces

$$V_\phi^p := \{f = ge^{-\phi} : g \in A_\phi^p\} \subseteq L^p(\mathbb{R}^d).$$

According to (10) and (11), the reproducing kernel $\widetilde{K_\phi}$ satisfies the estimates $\widetilde{K_\phi}(z, z) \asymp 1$ and $\widetilde{K_\phi}(z, w) \lesssim e^{-c|z-w|}$ for some constant $c > 0$. These estimates hold, of course, also for $\widetilde{K_{a\phi}}$ for all $a > 0$ with possibly different constants.

Fix $\varepsilon > 0$. To prove Theorem 1.1, we need to produce a set $\Lambda \subseteq \mathbb{C}^n$ that is interpolating for $A_{(1+\varepsilon)\phi}^p$ and a sampling set for $A_{(1-\varepsilon)\phi}^p$ with $1 \le p \le \infty$. By Proposition 7.7 and Theorem 7.5, it suffices to produce Λ that is an interpolating set for $V_{(1+\varepsilon)\phi}^1$ and a sampling set for $V_{(1-\varepsilon)\phi}^\infty$.

We now show that the set Λ constructed in Theorem 7.6 as a weak limit of Fekete points does the job. Theorem 7.6 yields a set Λ and interpolating functions $\{l_\lambda : \lambda \in \Lambda\} \subseteq V_\phi^\infty$ such that $\|l_\lambda\|_\infty \le 1$ and $l_\lambda(\mu) = \delta_{\lambda,\mu}$ for all $\lambda, \mu \in \Lambda$.

(i) *Interpolation.* We improve the localization of the functions l_λ in the following way: let

$$\tilde{l}_\lambda(z) := l_\lambda(z) \frac{\widetilde{K_{\varepsilon\phi}}(z, \lambda)}{\widetilde{K_{\varepsilon\phi}}(\lambda, \lambda)}, \qquad z \in \mathbb{C}^n.$$

Then clearly $\tilde{l}_\lambda(\mu) = \delta_{\lambda,\mu}$, for all $\lambda, \mu \in \Lambda$, and, due to the decay of $\widetilde{K_{\varepsilon\phi}}$,

(63) $\left| \tilde{l}_\lambda(z) \right| \lesssim e^{-c|z-\lambda|}.$

Consequently, we have

$$\|\widetilde{l_\lambda}\|_1 \lesssim 1\,.$$

Furthermore, since $l_\lambda = he^{-\phi}$ for some $h \in A_\phi^\infty$ and $\widetilde{K_{\varepsilon\phi}}(z,\lambda) = e^{-\varepsilon\phi(\lambda)}K_{\varepsilon\phi}(z,\lambda)e^{-\varepsilon\phi(z)}$ with $K_{\varepsilon\phi,\lambda} \in A_{\varepsilon\phi}^1$, the product $\widetilde{l_\lambda} = l_\lambda\widetilde{K_{\varepsilon\phi,\lambda}}$ is in $V_{(1+\varepsilon)\phi}^1$.

Thus, for $a \in \ell^1(\Lambda)$ the function $f(z) := \sum_{\lambda\in\Lambda} a_\lambda\widetilde{l_\lambda}$ is in $V_{(1+\varepsilon)\phi}^1$ and satisfies $f(\lambda) = a_\lambda$, as desired.

(ii) *Sampling.* Here we want to verify the inequality $\sup_{z\in\mathbb{C}^n} |f(z)| \leq \sup_{\lambda\in\Lambda} |f(\lambda)|$ for every $f \in V_{(1-\varepsilon)\phi}^\infty$. To this end, fix $z_0 \in \mathbb{C}^n$ and define

$$g(z) := f(z)\frac{\widetilde{K_{\varepsilon\phi}}(z,z_0)}{\widetilde{K_{\varepsilon\phi}}(z_0,z_0)}.$$

As in (i) we see that $g \in V_\phi^1$. By the ℓ^1-ℓ^∞-sampling part of Theorem 7.6 applied to g we obtain

$$|f(z_0)| = |g(z_0)| \leq \sum_{\lambda\in\Lambda} |g(\lambda)| \lesssim \sum_{\lambda\in\Lambda} |f(\lambda)|\, e^{-c|z_0-\lambda|} \leq C_\Lambda \sup_{\lambda\in\Lambda} |f(\lambda)|\,,$$

where $C_\Lambda := \sup_{z\in\mathbb{C}^n} e^{-c|z-\lambda|}$ is finite because Λ is separated.

Step 2. Let us finally show that (7) holds. We consider only the supremum $\sup_{\Lambda\in SI} D^-(\Lambda) = 1$ over all interpolating sets for A_ϕ^2; the argument for the infimum is analogous. By Theorem 2.1, the supremum in (7) is at most 1. To show that it is indeed 1, we fix $\delta \in (0,1)$ and use Proposition 4.2 to select $\varepsilon > 0$ such that

$$\widetilde{K}_{(1-\varepsilon)/(1+\varepsilon)\phi}(z,z) \geq (1-\delta)\widetilde{K_\phi}(z,z),$$

for all $z \in \mathbb{C}^n$. We now apply the construction from Step 1 to the weight $(1+\varepsilon)^{-1}\phi$ and obtain a separated set $\Lambda \subseteq \mathbb{C}^n$ that is interpolating for A_ϕ^2 and sampling for $A_{(1-\varepsilon)\phi/(1+\varepsilon)}^2$. By Theorem 2.1, we conclude that $D^-_{(1-\varepsilon)\phi/(1+\varepsilon)}(\Lambda) \geq 1$. Hence, for $R \gg 1$, and all $z \in \mathbb{C}^n$,

$$\#\left(\Lambda \cap B_R(z)\right) \geq (1-\delta)\int_{B_R(z)} \widetilde{K}_{(1-\varepsilon)\phi/(1+\varepsilon)}(w,w)dm(w)$$

$$\geq (1-\delta)^2\int_{B_R(z)} \widetilde{K_\phi}(w,w)dm(w).$$

This means that $D^-_\phi(\Lambda) \geq (1-\delta)^2$. Since $\delta \in (0,1)$ was arbitrary, the conclusion follows. $\square$

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