HARMONIC MEASURE AND UNIFORM DENSITIES

JOAQUIM ORTEGA-CERDÀ AND KRISTIAN SEIP

Abstract. We study two problems concerning harmonic measure on certain "champagne subdomains" of the unit disk $\mathbb{D}$. The domains that we consider are obtained by removing from $\mathbb{D}$ little disks around sequences of points with a uniform distribution with respect to the pseudohyperbolic metric of $\mathbb{D}$. We find (I) a necessary and sufficient condition on the decay of the radii of the little disks for the exterior boundary to have positive harmonic measure, and (II) describe sampling and interpolating sequences for Bergman spaces in terms of the harmonic measure on such "champagne subdomains".

1. Introduction

This paper presents two theorems concerning harmonic measure on certain "champagne subdomains" of the unit disk. Our first result solves a problem posed in a recent paper by Akeroyd [Ake02], while our second result gives a Bergman space counterpart of a result of Garnett, Gehring, and Jones [GGJ83] for interpolation by bounded analytic functions.

The setting is as follows. Let Λ be a sequence of distinct points in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and define

$$\rho(z,\zeta) = \left|\frac{z-\zeta}{1-\zeta\bar{z}}\right|,$$

which is the pseudohyperbolic distance between $z, \zeta \in \mathbb{D}$. For $z \in \mathbb{D}$ and $0 < r < 1$ we set

$$D(z,r) = \{\zeta \in \mathbb{D} : \rho(z,\zeta) \le r\}.$$

We say that Λ is a *uniformly dense sequence* if

(i) Λ is separated, i.e., $\inf_{\lambda\neq\lambda'} \rho(\lambda,\lambda') > 0$, $\lambda, \lambda' \in \Lambda$.
(ii) There exists an $r < 1$ such that $\mathbb{D} = \bigcup_{\lambda\in\Lambda} D(\lambda, r)$.

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Uniformly dense sequences appear naturally in the study of Bergman spaces, as sampling or interpolating sequences (see below).

We are interested in studying harmonic measure on "champagne subdomains" with "bubbles" around the points $\lambda \in \Lambda$, i.e., we consider infinitely connected domains of the form

$$\Omega = \mathbb{D} \setminus \bigcup_{\lambda\in\Lambda} D(\lambda, r_\lambda), \tag{1}$$

where the r_λ ($0 < r_\lambda < 1$) are such that the closed disks $D(\lambda, r_\lambda)$ are pairwise disjoint. The notation $\omega(z, A, \Omega)$ stands for the value at z of the harmonic measure on Ω of a set $A \in \partial\Omega$. (See below for a formal definition.) We will be concerned with the following two problems:

(I) Find a necessary and sufficient condition on the decay of r_λ for the exterior boundary $\partial\mathbb{D}$ to have positive harmonic measure, i.e., for $\omega(z, \partial\mathbb{D}, \Omega) > 0$ to hold when $z \in \Omega$.

(II) Characterize sampling and interpolating sequences in terms of harmonic measure on such "champagne subdomains".

We were led to Problem (I) by a question in [Ake02]. The issue was whether for any uniformly dense sequence Λ one may pick the r_λ such that the exterior boundary $\partial\mathbb{D}$ has zero harmonic measure and

$$\sum_{\lambda\in\Lambda} \text{length}(\partial D(\lambda, r_\lambda)) < \infty. \tag{2}$$

Our solution shows that there is ample room for such constructions.

When dealing with Problem (I), we assume that $r_\lambda = \varphi(|\lambda|)$ with φ a nonincreasing function bounded by some constant less than 1. In Theorem 1 below, we arrive at the following necessary and sufficient condition for $\partial\mathbb{D}$ to have positive harmonic measure:

$$\int_0^1 \frac{dt}{(1-t)\log(1/\varphi(t))} < \infty. \tag{3}$$

Thus, for example, $\varphi(t) = c(1-t)^\gamma$ for arbitrary $\gamma > 0$ yields zero harmonic measure of $\partial\mathbb{D}$ as well as (2). Note that the result is independent of the particular choice of Λ.

Both problems (I) and (II) can be seen as originating from a paper by Garnett, Gehring, and Jones [GGJ83], dealing with similar considerations when Λ satisfies the Blaschke condition

$$\sum_{\lambda\in\Lambda}(1-|\lambda|) < \infty. \tag{4}$$

On the one hand, our work contrasts the following elementary fact: Suppose Λ is merely separated and $r_\lambda = r < 1$ for all $\lambda \in \Lambda$. In this case, if Λ satisfies (4), then the exterior boundary of Ω has positive harmonic measure. This can be seen as a statement about the sparsity of Blaschke sequences, while our condition (3) reflects the density of uniformly dense sequences. On the other hand, our work parallels

[GGJ83] in that we give a description of interpolating sequences for Bergman spaces in terms of the harmonic measure (see Theorem 2 below), in a similar way as is obtained for classical interpolating sequences in [GGJ83].

It is interesting to note that the integral condition (3) has appeared before, in a different context. In [LS97], a sequence Z of distinct points in $\mathbb{D}$ is said to be a separated non-Blaschke sequence if Z is separated and

$$\sum_{z\in Z}(1-|z|)=\infty.$$

Also, φ (again a nonincreasing function bounded by some constant less than 1) is an *essential minorant for* H^∞ if the inequality

$$|f(z)|\le\varphi(|z|)\quad\text{for all } z\in Z,$$

f a bounded analytic function and Z some separated non-Blaschke sequence, implies that $f\equiv 0$. The theorem proved in [LS97] says that φ is an essential minorant for H^∞ if and only if (3) holds. We are not able to give a direct proof of the link between essential minorants and our "champagne subdomains" whose exterior boundaries have positive harmonic measure, but we will offer a heuristic argument for the connection.

The following notation will be used repeatedly below: We write $A\lesssim B$ to signify that $A\le CB$ for some constant $C>0$, independent of whatever arguments are involved. If both $A\lesssim B$ and $B\lesssim A$, then we write $A\simeq B$.

2. Positive harmonic measure of the exterior boundary

We begin by noting that the domains Ω defined by (1) are Dirichlet domains because they satisfy the exterior cone condition. Thus every continuous function on $\partial\Omega$ can be extended continuously to a harmonic function in the interior of Ω. The maximum principle shows that the evaluation at any point $z\in\Omega$ of this extension is a bounded linear functional on $\mathcal{C}(\partial\Omega)$ with norm less than 1. By the Riesz representation theorem, there is a probability measure ω_z supported on $\partial\Omega$ such that the action of this functional on f can be represented as an integral against ω_z. The measure ω_z is called the harmonic measure of Ω at z, and the harmonic measure of a set $A\subset\partial\Omega$ is denoted $\omega(z,A,\Omega)$.

Note that the function $z\mapsto\omega(z,A,\Omega)$ is a nonnegative harmonic function on Ω. The maximum principle implies that this function is either identically 0 or strictly positive on Ω.

There exist several equivalent definitions of harmonic measure. One such definition is given in probabilistic terms: The harmonic measure $\omega(z,A,\Omega)$ coincides with the probability that a Brownian motion starting at the point z exits the open set Ω for the first time at one of the points in A. We refer to [Bas95] for a proof of this fact and for some examples of estimates of the harmonic measure using this probabilistic interpretation.

Fix a uniformly dense sequence Λ, and assume that φ is a nonincreasing function on $(0,1)$ such that the closed disks $D(\lambda, \varphi(|\lambda|))$, $\lambda \in \Lambda$, are pairwise disjoint. Set

$$\Omega(\Lambda, \varphi) = \mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} D(\lambda, \varphi(|\lambda|)),$$

and assume for convenience that $0 \in \Omega(\Lambda, \varphi)$.

Theorem 1. *The exterior boundary of* $\Omega(\Lambda, \varphi)$ *has positive harmonic measure, i.e.*

$$\omega(0, \partial\mathbb{D}, \Omega(\Lambda, \varphi)) > 0,$$

if and only if

$$\int_0^1 \frac{dt}{(1-t)\log(1/\varphi(t))} < \infty. \tag{5}$$

Note that the condition may be written equivalently as

$$\sum_{j=1}^{\infty} \frac{1}{\log(1/\varphi(1-K^{-j}))} < \infty \tag{6}$$

for some $K > 1$.

Theorem 1 reflects the following dichotomy: Either the little disks are so small that the contribution to the harmonic measure from each of them can be viewed as independent of the contributions from the others, or the disks are so large that their contributions to the harmonic measure interact in a profound way. The first case corresponds to positive harmonic measure of the exterior boundary, the second case to zero harmonic measure of the exterior boundary.

The proof of the sufficiency of (5) illuminates this point: We begin by observing that we may safely disregard a finite number of points; thus we may consider instead ($r < 1$)

$$\Lambda_r = \Lambda \cap \{z : \ |z| > r\}.$$

It is immediate that

$$\omega(0, \partial D(\zeta, s), \mathbb{D} \setminus D(\zeta, s)) = \frac{\log|\zeta|}{\log s}. \tag{7}$$

Then

$$1 - \omega(0, \partial\mathbb{D}, \Omega(\Lambda_r, \varphi)) \leq \sum_{|\lambda| \geq r} \frac{\log \frac{1}{|\lambda|}}{\log \frac{1}{\varphi(|\lambda|)}} \lesssim \int_r^1 \frac{dt}{(1-t)\log(1/\varphi(t))},$$

where the latter inequality follows from the fact that Λ is a separated sequence. We are done because the integral can be made smaller than 1 by choosing r sufficiently close to 1.

Before proving the necessity of (5), we comment on the relation to essential minorants. As explained in [LS97], the result describing essential minorants is really a statement about the size of the exceptional set on which a positive superharmonic function $u(z)$ exceeds $\log(1/\varphi(|z|))$. It was proved in [LS97] that with m denoting Lebesgue area measure on $\mathbb{D}$, we have

$$\int_{u(z)>\log(1/\varphi(|z|))} \frac{dm(z)}{1-|z|} < \infty \tag{8}$$

for each positive superharmonic function u if and only if (5) holds. By Harnack's inequality, we may recast the integral in (8) as a sum:

$$\sum_{u(\lambda)>\log(1/\varphi(|\lambda|))} (1-|\lambda|) < \infty. \tag{9}$$

Now if the little disks can be seen as acting independently of each other, then again by Harnack's inequality as well as the Riesz representation formula and (7),

$$u(0) \gtrsim \sum_{u(\lambda)>\log(1/\varphi(|\lambda|))} (1-|\lambda|),$$

and it follows that φ is an essential minorant.

We now turn to the necessity of (5). We will estimate the probability that a Brownian motion starting at 0 and moving in $\Omega(\Lambda,\varphi)$ will reach $\partial\mathbb{D}$. (We assume the motion is stopped once the particle exits $\Omega(\Lambda,\varphi)$.) Define

$$C_j = \{z : |z| = 1 - K^{-j}\},$$

$j = 0, 1, \dots$ and K is some large constant chosen such that for every $z \in C_{j-1}$ there is a nearby point $\lambda_z \in \Lambda$ in the annulus bounded by C_{j-1} and C_j such that

$$\sup_j \sup_{z\in C_j} \rho(z, \lambda_z) < 1.$$

Let P_j denote the probability that our Brownian motion hits C_j. If Q_j denotes the supremum of the probabilities that a Brownian motion starting from some point at C_{j-1} hits C_j, then we get

$$P_j \le Q_j P_{j-1},$$

and so by induction

$$P_n \le \prod_{j=1}^{n} Q_j.$$

Thus it is necessary that $\prod_{j=1}^{\infty} Q_j > 0$. Equivalently, we have

$$\sum_{j=1}^{\infty} (1 - Q_j) < \infty. \tag{10}$$

Note that $1-Q_j$ is the infimum of the probabilities that a Brownian motion starting from some point on C_{j-1} hits a disk $D(\lambda,\varphi(|\lambda|))$ before reaching C_j. For any point on C_{j-1} we may discard all disks except $D_{\lambda_z} = D(\lambda_z,\varphi(|\lambda_z|))$ lying in the annulus bounded by C_{j-1} and C_j, because we are thus diminishing the probability of hitting the disks. Therefore, if we denote by D_j the disk bounded by C_j, then

$$1-Q_j \geq \inf_{z\in C_{j-1}} \omega(z,\partial D_{\lambda_z}, D_j \setminus D_{\lambda_z}).$$

This harmonic measure can be estimated because $\sup \rho(z,\lambda_z) < 1$:

$$\omega(z,\partial D_{\lambda_z}, D_j \setminus D_{\lambda_z}) \gtrsim \frac{1}{\log(1/\varphi(1-K^{-j}))} \qquad \forall z \in C_{j-1}.$$

Combining this estimate with (10), we arrive at (6).

3. Lower and upper uniform densities

In the previous section, the particular choice of uniformly dense sequence Λ was inessential. However, such sequences may have different densities, and a natural question is whether these densities can be captured in terms of harmonic measure. We will now show how this can be done.

Let Λ be a separated sequence. Following [Sei93], we define the *lower uniform density* of Λ as

$$D^-(\Lambda) = \liminf_{r\to 1} \inf_{z\in\mathbb{D}} \frac{\sum_{\rho(\lambda,z)<r}(1-\rho(\lambda,z))}{\log\frac{1}{1-r}}$$

and *the upper uniform density* of Λ as

$$D^+(\Lambda) = \limsup_{r\to 1} \sup_{z\in\mathbb{D}} \frac{\sum_{\rho(\lambda,z)<r}(1-\rho(\lambda,z))}{\log\frac{1}{1-r}}.$$

Note that we always have $D^-(\Lambda) \leq D^+(\Lambda) < \infty$, and that $D^-(\Lambda) > 0$ if and only if Λ is a uniformly dense sequence.

To see the significance of these densities, we cite the main results of [Sei93]. Let $A^{-\alpha}$ ($\alpha > 0$) be the space of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f(z)| < \infty.$$

We say that Λ is a *sampling sequence* for $A^{-\alpha}$ if there is a positive constant C such that

$$\|f\|_\alpha \leq C \sup_{\lambda\in\Lambda} (1-|\lambda|^2)^\alpha |f(\lambda)|$$

for every function $f \in A^{-\alpha}$. On the other hand, we say that Λ is an *interpolating sequence* for $A^{-\alpha}$ if the interpolation problem

$$f(\lambda) = a_\lambda$$

has a solution $f \in A^{-\alpha}$ whenever $\{(1-|\lambda|^2)^\alpha a_\lambda\}$ is a bounded sequence. In [Sei93], it was proved that a separated sequence Λ is a sampling sequence for $A^{-\alpha}$ if and only if

$$D^-(\Lambda) > \alpha,$$

and that Λ is an interpolating sequence for $A^{-\alpha}$ if and only if

$$D^+(\Lambda) < \alpha.$$

These density conditions also describe similar sequences of sampling and interpolation for weighted Bergman L^p spaces.

Before stating our second theorem, we mention the result on which it is modelled. A sequence Λ of distinct points in $\mathbb{D}$ is an interpolating sequence for H^∞ if the interpolation problem

$$f(\lambda) = a_\lambda$$

has a solution $f \in H^\infty$ whenever $\{a_\lambda\}_{\lambda\in\Lambda}$ is a bounded sequence. We let Λ_λ be the sequence obtained from Λ by removing the one element λ. In [GGJ83], the following re-interpretation of Carleson's theorem [Car58] was given:

Theorem B. *A separated sequence Λ is an interpolating sequence for H^∞ if and only if*

$$\inf_{\lambda\in\Lambda} \omega(\lambda, \partial\mathbb{D}, \Omega(\Lambda_\lambda, c)) > 0$$

for some $0 < c < 1$.

To obtain a counterpart of this result, we define the following densities. Set

$$\Omega(z, r) = \Omega(\Lambda; z, r) = \mathbb{D} \setminus \bigcup_{1/2<\rho(\lambda,z)<r} D(\lambda, 1-r),$$

which is a finitely connected domain. We see that the uniform pseudo-hyperbolic radius of the little disks tends to 0 as $r \to 1$. This decay is tuned with the growth of r in such a way that the numbers

$$D_h^-(\Lambda) = \liminf_{r\to 1^-} \inf_{z\in\mathbb{D}} \log \frac{1}{\omega(z, \partial\mathbb{D}, \Omega(z, r))}$$

and

$$D_h^+(\Lambda) = \limsup_{r\to 1^-} \sup_{\lambda\in\Lambda} \log \frac{1}{\omega(\lambda, \partial\mathbb{D}, \Omega(\lambda, r))}$$

are positive when Λ is uniformly dense. In fact, we have the following precise characterization.

Theorem 2. *For a separated sequence Λ in $\mathbb{D}$ we have*

$$D^-(\Lambda) = D_h^-(\Lambda) \quad \textit{and} \quad D^+(\Lambda) = D_h^+(\Lambda).$$

The proof of Theorem 2 combines probabilistic arguments with certain precise function theoretic constructions, to be given in the next section.

4. Growth of analytic functions vanishing on Λ

We will now see how $D^-(\Lambda)$ and $D^+(\Lambda)$ are related to the growth of analytic functions vanishing on Λ. The growth estimates to be established rely on a basic approximation result for subharmonic functions.

We require some notation. If f is analytic in $\mathbb{D}$, we denote by $Z(f)$ its sequence of zeros. If f has a zero of order n at z, then this is recorded by letting z appear n times in $Z(f)$. On the other hand, we also think of $Z(f)$ as a subset of the disk. In particular, when we say that $Z(f)$ is separated, we mean that $Z(f)$ consists of distinct points and that

$$\inf_{z\neq z'} \rho(z,z') > 0, \quad z,z' \in Z(f).$$

We will rely on the following approximation result from [Sei95].

Theorem A. *Let Ψ be subharmonic in $\mathbb{D}$ so that its Laplacian $\Delta\Psi$ satisfies*

$$\Delta\Psi(z) \simeq \frac{1}{(1-|z|^2)^2} \tag{11}$$

for all $z \in \mathbb{D}$. Then there exists a function g analytic in $\mathbb{D}$, with $Z(g)$ a uniformly dense sequence, and

$$|g(z)| \simeq \rho(z, Z(g))e^{\Psi(z)}$$

for all $z \in \mathbb{D}$.

We deduce two lemmas from Theorem A.

Lemma 1. *Let Λ be a uniformly dense sequence satisfying $D^-(\Lambda) > \alpha > 0$, and let f be an analytic function on $\mathbb{D}$ with $Z(f) = \Lambda$. Then there exists a uniformly dense sequence Σ and an analytic function g on $\mathbb{D}$ with $Z(g) = \Sigma$ such that*

$$\frac{|f(z)|}{|g(z)|} \simeq \frac{\rho(z,\Lambda)}{\rho(z,\Sigma)}(1-|z|^2)^{-\alpha}$$

for all $z \in \mathbb{D}$.

Proof. Set $u = \log|f|$. Its Laplacian is

$$\Delta u(z) = 2\pi \sum_{\lambda\in\Lambda} \delta_\lambda.$$

To be able to apply Theorem A, we need to smooth this Laplacian. We do this using a slight variation of an idea from [BOC95]. We intend to replace $\Delta u(z)$ by

$$\Delta u_r(z) = \frac{2\pi}{c_r} \sum_{\rho(z,\lambda)<r} \frac{1-\rho^2(z,\lambda)}{(1-|z|^2)^2},$$

where

$$c_r = \int_{|z|<r} \frac{dm(z)}{1-|z|^2} = \pi \log \frac{1}{1-r^2}.$$

We claim that for each $0 < r < 1$, we can find an appropriate u_r that behaves like u outside the singular points λ. Before proving this claim, let us note that we may then set

$$\Psi(z) = u_r(z) - \alpha \log \frac{1}{1-|z|^2}$$

so that

$$\Delta\Psi(z) = \frac{4}{(1-|z|^2)^2}\Big(\sum_{\rho(\lambda,z)<r} \frac{1-\rho^2(\lambda,z)}{2\log\frac{1}{1-r^2}} - \alpha\Big).$$

It follows that

$$D^-(\Lambda) - o(1) - \alpha \le \frac{1}{4}(1-|z|^2)^2 \Delta\Psi(z) \le C$$

as $r \nearrow 1$, where C is a constant depending only on Λ. Here the first inequality follows from the definition of $D^-(\Lambda)$, while the second follows from the separation of Λ. We see that Theorem A applies if r is sufficiently close to 1.

To see that a suitable u_r approximates u, we argue as follows. By the change of variables

$$t = \frac{\lambda - \zeta}{1-\bar{\lambda}\zeta},$$

we get

$$\frac{1}{c_r}\int_{\rho(\lambda,\zeta)<r} \log\Big|\frac{z-\zeta}{1-\bar{z}\zeta}\Big| \frac{1-\rho^2(\lambda,\zeta)}{(1-|\zeta|^2)^2} dm(\zeta) = \log\Big|\frac{z-\lambda}{1-\bar{z}\lambda}\Big| + \frac{1}{c_r}\int_{|t|<r} \log \frac{\rho\big(\frac{\lambda-t}{1-\bar{\lambda}t}, z\big)}{\rho(\lambda,z)} \frac{dm(t)}{1-|t|^2}.$$

If we set

$$w = \frac{\lambda - z}{1-\bar{\lambda}z},$$

then we may write

$$\rho\Big(\frac{\lambda-t}{1-\bar{\lambda}t}, z\Big) = \rho(t,w) = \rho(0,w)\Big|\frac{1-t/w}{1-t\bar{w}}\Big| = \rho(\lambda,z)\Big|\frac{1-t/w}{1-t\bar{w}}\Big|,$$

so that

$$\frac{1}{c_r}\int_{|t|<r} \log \frac{\rho\big(\frac{\lambda-t}{1-\bar{\lambda}t}, z\big)}{\rho(\lambda,z)} \frac{dm(t)}{1-|t|^2} = \frac{1}{c_r}\int_{|t|<r} \log|1-t/w| \frac{dm(t)}{1-|t|^2}.$$

In particular, this integral vanishes when $|w| > r$. Since Λ is a separated sequence, we therefore obtain

$$\sup_{\rho(z,\Lambda)\ge\varepsilon} \sum_{\lambda\in\Lambda} \frac{1}{c_r}\int_{|t|<r} \log \frac{\rho\big(\frac{\lambda-t}{1-\bar{\lambda}t}, z\big)}{\rho(\lambda,z)} \frac{dm(t)}{1-|t|^2} < \infty$$

for every $\varepsilon > 0$. We conclude that we may set

$$u_r(z) = u(z) + \sum_{\lambda\in\Lambda}\Big(\frac{1}{c_r}\int_{\rho(\zeta,\lambda)<r}\log\Big|\frac{z-\zeta}{1-\bar z\zeta}\Big|\frac{1-\rho^2(\zeta,\lambda)}{(1-|\zeta|^2)^2}dm(\zeta) - \log\Big|\frac{z-\lambda}{1-\bar z\lambda}\Big|\Big).$$

□

Acting similarly as above, but considering instead

$$\Psi(z) = \alpha\log\frac{1}{1-|z|^2} - u_r(z),$$

we arrive at the following lemma.

Lemma 2. *Let Λ be a separated sequence satisfying $D^+(\Lambda) < \alpha$, and let f be an analytic function on $\mathbb{D}$ with $Z(f) = \Lambda$. Then there exists a uniformly dense sequence Σ and an analytic function g on $\mathbb{D}$ with $Z(g) = \Sigma$ such that*

$$|f(z)g(z)| \simeq \rho(z,\Lambda)\rho(z,\Sigma)(1-|z|^2)^{-\alpha}$$

for all $z \in \mathbb{D}$.

We note that the construction in [Sei95] can be adjusted so that $\Lambda \cup \Sigma$ becomes a separated sequence in both cases. However, we will not need this separation in what follows.

5. Proof of Theorem 2

We will need the following elementary fact.

Lemma 3. *Suppose Λ is a separated sequence and let B be the finite Blaschke product with zeros $\lambda \in \Lambda$ such that $|\lambda| < r$. Given $0 < \varepsilon < 1$, there exists a constant C depending only on Λ and ε, but not on r, such that*

$$|B(z)| \geq C$$

whenever $|z| > r$ and $\rho(z,\Lambda) > \varepsilon$.

Proof. We want to prove that under the hypothesis we have

$$\log|B(z)| = \sum_{\lambda\in\Lambda}\log\rho(z,\lambda) \geq \log C.$$

For each of the terms we have $\rho(z,\lambda) > \varepsilon$, and therefore $-\log\rho(z,\lambda) \lesssim 1-\rho^2(z,\lambda)$. Thus we want to estimate

$$\sum_{\lambda\in\Lambda}\big(1-\rho^2(z,\lambda)\big) = \sum_{\lambda\in\Lambda}\frac{(1-|\lambda|^2)(1-|z|^2)}{|1-\bar z\lambda|^2}.$$

Since Λ is a separated sequence and $|z| \geq |\lambda|$, the latter sum is bounded by a constant times the integral

$$\int_{|w|\leq|z|} \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{z}w|^2} \frac{dm(w)}{(1-|w|^2)^2}.$$

Computing this integral by means of polar coordinates, we find that it equals $\pi \ln(1+|z|^2)$. Hence it is bounded independently of z. □

We turn to the proof of Theorem 2. Set $\alpha = D^-(\Lambda)$. We begin by showing that $\alpha \leq D_h^-(\Lambda)$. We prefer to give a general argument and define

$$\Omega_\delta(z,r) = \mathbb{D} \setminus \bigcup_{1/2<\rho(\lambda,z)<r} D(\lambda, \delta(r)),$$

where we only assume $\delta(r) \to 0$. Pick some small $\varepsilon > 0$. Let $h = f/g$ be the function with zeros Λ and poles Σ given by Lemma 1 such that

$$|h(z)| \simeq (1-|z|)^{-\alpha+\varepsilon}$$

far from Λ and Σ; the constants involved here will depend on ε.

By conformal invariance, we may assume $z = 0$. We will give a probabilistic argument, estimating the probability that a Brownian motion starting at 0 and moving in $\Omega_\delta(0,r)$ will reach $\partial D(0,r)$. To this end, choose some function $\eta(r) \to 0$, such that

$$\log\frac{1}{\eta(r)} = o\Big(\log\frac{1}{\delta(r)}\Big), \qquad \log\frac{1}{\eta(r)} = o\Big(\log\frac{1}{1-r}\Big).$$

We also require that $\eta(r)$ is such that

$$\log\frac{1}{1-r} = n \log\frac{1}{\eta(r)}$$

for some positive integer $n = n(r)$. We define

$$C_j(r) = \{z : |z| = 1 - \eta^j(r)\},$$

$j = 0, 1, \ldots, n(r)$. Note that these circles split the disk $\{z \in \mathbb{C};\ |z| \leq r\}$ into concentric annuli. We may assume the sequence $\Sigma \cup \Lambda$ is bounded away from the circles $C_j(r)$ by slightly perturbing a finite number of points λ. (Alternatively, we may replace the circles $C_j(r)$ by circles with small detours around the points from Σ and Λ.) Let P_j denote the probability that our Brownian motion hits $C_j(r)$. If Q_j denotes the supremum of the probabilities that a Brownian motion starting from some point at $C_{j-1}(r)$ hits $C_j(r)$, then we get

$$P_j \leq Q_j P_{j-1}$$

and so by induction

$$P_n \leq \prod_{j=1}^{n} Q_j.$$

To estimate Q_j, we disregard the points from Λ on the inside of $C_{j-1}(r)$. We may also disregard the points from Λ close to $C_j(r)$

(correspondingly, we divide out these zeros from h, but still call the function h). This will increase the probability of hitting $C_j(r)$. On $\mathbb{D} \setminus \cup_{\lambda\in\Lambda} D(\lambda, \delta(r))$, we define the subharmonic function

$$U_j(z) = \frac{\log \frac{1}{|h(z)|} + j(\alpha - \varepsilon) \log \frac{1}{\eta(r)} - C}{\log \frac{1}{\delta(r)} + (\alpha - \varepsilon) \log \frac{1}{\eta(r)}},$$

where the constant C (independent of r) is such that U_j is bounded above by 0 on $C_j(r)$ and by 1 on $\partial D(\lambda, \delta(r))$ for $\lambda \in \Lambda$ between $C_{j-1}(r)$ and $C_j(r)$. Also, on $C_{j-1}(r)$ we have

$$U_j(z) \geq \frac{(\alpha - \varepsilon) \log \frac{1}{\eta(r)} - 2C}{\log \frac{1}{\delta(r)} + (\alpha - \varepsilon) \log \frac{1}{\eta(r)}}.$$

It follows that

$$Q_j \leq \frac{\log \frac{1}{\delta(r)} + 2C}{\log \frac{1}{\delta(r)} + (\alpha - \varepsilon) \log \frac{1}{\eta(r)}}.$$

Thus

$$\log \frac{1}{\omega(\lambda, \partial\mathbb{D}, \Omega(\lambda, r))} \geq n \log \frac{\log \frac{1}{\delta(r)} + (\alpha - \varepsilon) \log \frac{1}{\eta(r)}}{\log \frac{1}{\delta(r)} + 2C}$$
$$= n \frac{(\alpha - \varepsilon) \log \frac{1}{\eta(r)} - 2C}{\log \frac{1}{\delta(r)} + 2C}(1 + o(1)) = (\alpha - \varepsilon) \frac{\log \frac{1}{1-r}}{\log \frac{1}{\delta(r)}}(1 + o(1)).$$

We now prove that $\alpha \geq D_h^-(\Lambda)$. This time we cannot allow $\delta(r)$ to decrease too slowly. Let B_ζ denote the finite Blaschke product with zeros $\lambda \in \Lambda$ such that $1/2 < \rho(\zeta, \lambda) < r$. Define

$$c(r) = \sup_z \log |B_z(z)|$$

and pick ζ such that

$$\log |B_\zeta(\zeta)| > c(r) - 1. \tag{12}$$

By conformal invariance, we may assume $\zeta = 0$. We set

$$c(r) - 1 = -(\alpha - \varepsilon) \log \frac{1}{1 - r}$$

and note that by our definition of α, $\varepsilon = \varepsilon(r) \to 0$ when $r \to 1$.

We introduce a function $\eta(r)$ as above and a similar partition: Let B_j be the Blaschke product with zeros $\lambda \in \Lambda$ such that

$$1 - \eta^{j-1}(r) \leq |\lambda| < 1 - \eta^j(r).$$

We define

$$U_j(z) = \log \frac{1}{|B_j(z)|}.$$

We now build a harmonic function which exceeds the harmonic measure of the inner boundary of $\Omega_\delta(0,r)$. This function will be of the form

$$U = \sum_{j=1}^{n} w_j U_j,$$

with appropriate positive weights w_j such that $U(z) \geq 1$ on $\partial D(\lambda, \delta(r))$ for $|\lambda| < r$. To determine the w_j, we begin by noting that

$$U_j(z) \geq \log \frac{1}{\delta(r)} =: a$$

on the boundary of the "bubbles" corresponding to the zeros of B_j. Moreover, U_j is a superharmonic function with the following lower bound for $|z| \leq 1 - \eta^{j-1}(r)$:

$$U_j(z) \geq (\alpha - \xi) \log \frac{1}{\eta(r)},$$

with $\xi = \xi(r) \to 0$ as $\eta(r) \to 0$. This estimate is first proved for $|z| = 1 - \eta^{j-1}(r)$ by $|z|$ using the definition of $D^-(\Lambda)$ and Lemma 3. The estimate for $|z| \leq 1 - \eta^{j-1}(r)$ then follows by the minimum principle for superharmonic functions.

We now set

$$w_n = \frac{1}{a}.$$

Next observe that on the boundary of the "bubbles" corresponding to the zeros of $B_{n-1}(z)$, we get

$$w_{n-1} U_{n-1}(z) + w_n U_n(z) \geq w_{n-1} a + w_n (\alpha - \xi) \log \frac{1}{\eta(r)}.$$

We set

$$b := (\alpha - \xi) \log \frac{1}{\eta(r)}$$

and then

$$w_{n-1} = \frac{a - b}{a^2}.$$

Inductively, we get

$$w_{n-j} = \frac{1}{a} \Big(\frac{a - b}{a} \Big)^j.$$

To estimate $U(0)$, we argue as follows. The worst case is that $U_n(0)$ is maximal because w_n is the largest weight. Combining our upper estimate (12), i.e.,

$$\sum_{j=1}^{n} U_j(0) \leq \log \frac{1}{|B_0(0)|} \leq (\alpha - \varepsilon) \log \frac{1}{1 - r}$$

with the lower estimates $U_j(0) \geq (\alpha - \xi) \log(1/\eta(r))$, we get

$$U_n(0) \leq (\alpha - \xi) \log \frac{1}{\eta(r)} + n(\xi - \varepsilon) \log \frac{1}{\eta(r)}.$$

If $U_n(0)$ attains this upper bound, then $U_j(0) = (\alpha - \xi)\log(1/\eta(r)) = b$ for $1 \le j < n$, and we arrive at the estimate

$$U(0) \le n(\xi - \varepsilon)\frac{\log\frac{1}{\eta(r)}}{\log\frac{1}{\delta(r)}} + 1 - \Big(\frac{a-b}{a}\Big)^n + o(1),$$

and so

$$\omega(0, \partial\mathbb{D}, \Omega_\delta(0,r)) \ge \Big(\frac{a-b}{a}\Big)^n - (\xi - \varepsilon)\frac{\log\frac{1}{1-r}}{\log\frac{1}{\delta(r)}};$$

We now require the second term to be small "oh" of the first term. This is certainly the case if $\delta(r) = 1 - r$ (the first term is bounded and the second tends to 0). Thus

$$\log\frac{1}{\omega(0, \partial\mathbb{D}, \Omega_\delta(0,r))} \le n\frac{b}{a}(1 + o(1)) = \alpha\frac{\log\frac{1}{1-r}}{\log\frac{1}{\delta(r)}}(1 + o(1)),$$

and we are done.

We next set $\alpha = D^+(\Lambda)$. The scheme is very similar. We first show that $\alpha \ge D_h^+(\Lambda)$. Again

$$\Omega_\delta(z,r) = \mathbb{D} \setminus \bigcup_{\substack{1/2<\rho(\lambda,z)<r\\ \lambda\in\Lambda}} D(\lambda, \delta(r)),$$

where we only assume $\delta(r) \to 0$. Pick some small $\varepsilon > 0$. Let $h = fg$ be the function given by Lemma 2 with zeros Λ and Σ such that

$$|h(z)| \simeq (1 - |z|)^{-\alpha-\varepsilon}$$

far from Λ and Σ; the constants involved here will depend on ε.

For arbitrary $\lambda \in \Lambda$ we wish to prove that

$$\log\frac{1}{\omega(\lambda, \partial\mathbb{D}, \Omega(\lambda,r))} \le (\alpha + \varepsilon)\frac{\log\frac{1}{1-r}}{\log\frac{1}{\delta(r)}}(1 + o(1)).$$

By conformal invariance, we may assume $\lambda = 0$. We estimate again the probability that a Brownian motion starting at 0 and moving in $\Omega_\delta(0,r)$ will reach $\partial D(0,r)$. To this end, choose some function $\eta(r) \to 0$, such that

$$\log\frac{1}{\eta(r)} = o\Big(\log\frac{1}{\delta(r)}\Big),$$

and

$$\log\frac{1}{1-r} = n\log\frac{1}{\eta(r)}$$

and define

$$C_j(r) = \{z : |z| = 1 - \eta^j(r)\}$$

$j = 0, 1, \ldots, n$. We may assume the sequence $\Lambda \cup \Sigma$ is bounded away from the circles $C_j(r)$. (Alternatively, we may make small detours around the points.) Let P_j denote the probability that our Brownian motion hits $C_j(r)$. If R_j denotes the infimum of the probabilities that a

Brownian motion starting from some point at $C_{j-1}(r)$ hits $C_j(r)$, then we get

$$P_j \geq R_j P_{j-1}$$

and so by induction

$$P_n \geq \prod_{j=1}^{n} R_j.$$

We estimate R_j. For some constant C (independent of r) the superharmonic function

$$U_j(z) = \frac{\log \frac{1}{|h(z)|} + j(\alpha+\varepsilon)\log\frac{1}{\eta(r)} + C}{\log\frac{1}{\delta(r)} + (\alpha+\varepsilon)\log\frac{1}{\eta(r)}}$$

is bounded from below by 0 on $C_j(r)$ and above by 1 on $\partial D(\lambda, \delta(r))$ for $\lambda \in \Lambda$ on the inside of $C_j(r)$. Also, on $C_{j-1}(r)$ we have

$$U_j(z) \geq \frac{(\alpha+\varepsilon)\log\frac{1}{\eta(r)} + 2C}{\log\frac{1}{\delta(r)} + (\alpha+\varepsilon)\log\frac{1}{\eta(r)}}.$$

It follows that

$$R_j \geq \frac{\log\frac{1}{\delta(r)} - 2C}{\log\frac{1}{\delta(r)} + (\alpha+\varepsilon)\log\frac{1}{\eta(r)}}.$$

Thus

$$\begin{aligned}\log\frac{1}{P_n} &\leq n\log\frac{\log\frac{1}{\delta(r)} + (\alpha+\varepsilon)\log\frac{1}{\eta(r)}}{\log\frac{1}{\delta(r)} - 2C}\\ &= n\frac{(\alpha+\varepsilon)\log\frac{1}{\eta(r)} + 2C}{\log\frac{1}{\delta(r)} - 2C}(1+o(1)) = (\alpha+\varepsilon)\frac{\log\frac{1}{1-r}}{\log\frac{1}{\delta(r)}}(1+o(1)).\end{aligned}$$

We finally have to estimate the infimum of the probabilities that a particle starting from $C_n(r)$ hits $\partial\mathbb{D}$. Then take B to be the Blaschke product with zeros $\lambda \in \Lambda$, $|\lambda| < r$, and set

$$U(z) = \frac{\log\frac{1}{|B(z)|}}{\log\frac{1}{\delta(r)}}.$$

By Lemma 3,

$$U(z) \leq \frac{C}{\log\frac{1}{\delta(r)}}$$

on $C_n(r)$, and so we get

$$\log\frac{1}{\omega(\lambda, \partial\mathbb{D}, \Omega(\lambda, r))} \leq (\alpha+\varepsilon)\frac{\log\frac{1}{1-r}}{\log\frac{1}{\delta(r)}}(1+o(1)).$$

We now prove $\alpha \leq D_h^+(\Lambda)$. Let B_ζ denote the finite Blaschke product with zeros $\lambda \in \Lambda$ such that $1/2 < \rho(\zeta, \lambda) < r$. Define

$$d(r) = \inf_{\lambda\in\Lambda} \log|B_\lambda(\lambda)|$$

and pick λ^* such that

$$\log |B_{\lambda^*}(\lambda^*)| < d(r) + 1.$$

By conformal invariance, we may assume $\lambda^* = 0$. We set

$$d(r) + 1 = -(\alpha + \varepsilon) \log \frac{1}{1 - r}$$

and note that by our definition of α, $\varepsilon = \varepsilon(r) \to 0$ when $r \to 1$.

We introduce a function $\eta(r)$ as above and let B_j and U_j be as before. We now build a harmonic function

$$U = \sum_{j=1}^{n} w_j U_j$$

such that $U(z) \leq 1$ on $\partial D(\lambda, \delta(r))$. First note that

$$\sum_{j=1}^{n} U_j(z) \leq \log \frac{1}{\delta(z)} + (\alpha + \xi) \log \frac{1}{\eta(r)}$$

on the boundary of the "bubbles" corresponding to the zeros of B_n, with $\xi(r) \to 0$ as $\eta(r) \to 0$. Thus we set

$$w_n = \frac{1}{a + b},$$

where

$$a = \log \frac{1}{\delta(r)}, \quad b = (\alpha + \xi) \log \frac{1}{\eta(r)}.$$

Next we observe that on the boundary of the "bubbles" corresponding to the zeros of $B_{n-1}(z)$, we get

$$w_{n-1} \sum_{j=1}^{n-1} U_j(z) + w_n U_n(z) \leq w_{n-1}(a + b) + w_n b$$

and so we set

$$w_{n-1} = \frac{a}{(a + b)^2}.$$

Inductively, we get

$$w_{n-j} = \frac{1}{(a + b)} \Big(\frac{a}{a + b} \Big)^j.$$

Note that the desired estimate on the boundaries of the "bubbles" is achieved because w_j decreases when j decreases.

To estimate $U(0)$, we argue in a similar way as above. The worst case is that $U_n(0)$ is minimal because w_n is the largest weight. By our lower estimate

$$\sum_{j=1}^{n} U_j(0) \geq (\alpha + \varepsilon) \log \frac{1}{1 - r}$$

and the upper estimates $U_j(0) \le (\alpha+\xi)\log(1/\eta(r))$, we get

$$U_n(0) \ge (\alpha+\xi)\log\frac{1}{\eta(r)} - n(\xi-\varepsilon)\log\frac{1}{\eta(r)}.$$

This leads us to the estimate

$$U(0) \ge 1 - \Big(\frac{a}{a+b}\Big)^n - n(\xi+\varepsilon)\frac{\log\frac{1}{\eta(r)}}{\log\frac{1}{\delta(r)}+\log\frac{1}{\eta(r)}},$$

and so

$$\omega(0,\partial\mathbb{D},\Omega_\delta(0,r)) \ge \Big(\frac{a}{a+b}\Big)^n + (\xi+\varepsilon)\frac{\log\frac{1}{1-r}}{\log\frac{1}{\delta(r)}+\log\frac{1}{\eta(r)}}.$$

We now require the second term to be "small oh" of the first term. This is certainly the case if $\delta(r) = 1-r$. Thus

$$\log\frac{1}{\omega(0,\partial\mathbb{D},\Omega_\delta(0,r))} \ge n\frac{b}{a}(1+o(1)) = \alpha\frac{\log\frac{1}{1-r}}{\log\frac{1}{\delta(r)}}(1+o(1)),$$

and we are done.

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Dept. Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

Email address: jortega@ub.edu

Dept. of Mathematical Sciences, Norwegian University of Science and Technology, N–7491 Trondheim, Norway

Email address: seip@math.ntnu.no