ZERO SETS OF HOLOMORPHIC FUNCTIONS IN THE BIDISK

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Abstract. We characterize in geometric terms the zero sets of holomorphic functions f in the bidisk such that $\log|f| \in L^p(\mathbb{D}^2)$, for $1 < p < \infty$.

1. Introduction

In this work, we study some geometrical characterization of the analytic varieties in the bidisk $\mathbb{D}^2 = \{z \in \mathbb{C}^2 : \ |z_1| < 1, \ |z_2| < 1\}$ defined by an holomorphic function with some restriction on its growth. In a strictly pseudo-convex domain, this kind of problems are better understood and, for instance, there is a complete characterization of the zero sets of holomorphic functions in the Nevanlinna class (see [Khe75], [Sko76]).

In the bidisk much less is known. Nevertheless there are some cases where the zero sets have been described. For instance, in the class of holomorphic functions such that $\log|f| \in L^1(\mathbb{D}^2)$ (see [Cha84] and [And85]). In this work we consider a variant of this problem, namely functions such that $\log|f| \in L^p(\mathbb{D}^2)$. We obtain a complete characterization of the zero sets of this class. This problem is closely related to one considered by Beller in one variable (see [Bel75]), where he studied the zero sequences of functions such that $\log^+|f| \in L^p(\mathbb{D})$. This problem in one variable has been further studied in [BO96].

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2. Zeros of functions with $\log|f| \in L^p(\mathbb{D}^2)$

2.1. **Statement of the results.** In this section we will give a complete characterization of the zero sets of holomorphic functions, $f \in H(\mathbb{D}^2)$, such that $\log|f| \in L^p(\mathbb{D}^2)$. Our main tool will be the Poincaré-Lelong theorem [Lel68] that shows that this problem is related to the problem of solving the equation $i\partial\bar{\partial}u = \Theta$ with good estimates on u in terms of Θ. In order to state the theorem we need to introduce some notation.

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Let Θ be a closed positive $(1,1)$-current in the bidisk; for any $z \in \mathbb{D}$ and fixed $0<\varepsilon<1$ let D_z be a small disk

$$D_z = \left\{ \zeta \in \mathbb{D};\ \frac{|\zeta - z|}{|1-\bar{\zeta} z|} < \varepsilon \right\}.$$

If Θ is a closed positive $(1,1)$-current then it can expressed in coordinates as

$$\Theta(z) = i \sum_{i,j=1}^{2} \theta_{ij}(z) dz_i \wedge d\bar{z}_j,$$

and $\theta_{11}(z_1,z_2)$ is a positive measure in the first variable, if we fix z_2, because Θ can always be expressed as $i\partial\bar\partial u$, where u is plurisubharmonic; thus $\theta_{11} = \Delta_{z_1} u \geq 0$. Accordingly $\theta_{22}(z_1,z_2)$ is a positive measure in the second variable if we fix z_1. Therefore we can define $\theta_{11}(D_{z_1}, z_2)$ as

$$\theta_{11}(D_{z_1}, z_2) = \int_{\zeta \in D_{z_1}} d\theta_{11}(\zeta, z_2).$$

Provided with this notation, we can state our main theorem:

Theorem 2.1. *Let Θ be a closed positive $(1,1)$-current on the bidisk, then the equation*

$$i\partial\bar\partial u = \Theta$$

has a solution $u \in L^p(\mathbb{D}^2)$, for a $1 \leq p < \infty$ if, and only if, the function

$$f(z_1,z_2) = \theta_{11}(D_{z_1}, z_2) + \theta_{22}(z_1, D_{z_2}) \tag{1}$$

belongs to $L^p(\mathbb{D}^2)$.

The disks D_{z_1} and D_{z_2} that appear in the statement of condition (1) depend on ε but the condition itself does not. This can be proved in a way completely analogous to the case of one variable in [Lue86]: Let $0<\delta<\varepsilon<1$, we call

$$\kappa_\varepsilon(z) = \theta_{11}(D_{z_1}(\varepsilon), z_2).$$

Then:

Proposition 2.2 (Luecking)**.** *$\kappa_\delta \in L^p(\mathbb{D}^2)$ if, and only if, $\kappa_\varepsilon \in L^p(\mathbb{D}^2)$.*

Theorem 2.1 has an immediate corollary for zero-varieties. Fixed $z \in \mathbb{D}^2$, we consider the cross formed by:

$$C_z = \{\zeta \in \mathbb{D}^2;\ \zeta_1 = z_1,\ \zeta_2 \in D_{z_2}\} \cup \{\zeta \in \mathbb{D}^2;\ \zeta_2 = z_2,\ \zeta_1 \in D_{z_1}\}.$$

Let $n_V(z)$ be the number of points that an analytic variety V meets the cross C_z (counted with multiplicity), then,

Corollary 2.3. *The analytic variety V is the zero set of a function f with $\log|f| \in L^p(\mathbb{D}^2)$ if, and only if,*

$$n_V \in L^p(\mathbb{D}^2).$$

Proof. If we take into account the Lelong-Poincaré theorem, the corollary is in fact a reformulation of theorem 2.1. ♠

Remark. Theorem 2.1 and corollary 2.3 have direct generalizations to the case of the polydisk $\mathbb{D}^n$ with $n > 2$. In this setting, the geometric condition that appears in the theorem is: $f \in L^p(\mathbb{D}^n)$, where

$$f(z) = \sum_{i=1}^{n} \theta_{ii}(z_1, \dots, D_{z_i}, \dots, z_n).$$

For the sake of simplicity in the computations we will give the proof in the case $n = 2$, although the same proof can be carried out in higher dimensions. Moreover we will not consider the case $p = 1$. This case is already known, it has been studied by Charpentier [Cha84, page 58] and Andersson [And85, thm. 1]. The theorem that they prove is the following:

Theorem 2.4 (Charpentier)**.** *If we have a closed positive* $(1,1)$*-current* Θ *in the bidisk, there is a solution* $u \in L^1(\mathbb{D}^2)$ *to the equation*

$$i\partial\bar{\partial}u = \Theta$$

if, and only if,

$$\int_{\mathbb{D}^2} (1 - |z_1|^2)^2 \theta_{11}(z) + (1 - |z_2|^2)^2 \theta_{22}(z) < +\infty.$$

This weighted Blaschke condition is equivalent to condition (1) when $p = 1$. Indeed, if we apply Fubini's theorem:

$$\int_{\mathbb{D}^2} \theta_{11}(D_{z_1}, z_2) + \theta_{22}(z_1, D_{z_2}) dm(z) =$$
$$= \int_{\mathbb{D}^2} |D_{z_1}| \theta_{11}(z_1, z_2) + |D_{z_2}| \theta_{22}(z_1, z_2) \simeq$$
$$\simeq \int_{\mathbb{D}^2} (1 - |z_1|^2)^2 \theta_{11}(z) + (1 - |z_2|^2)^2 \theta_{22}(z).$$

We will divide the proof of theorem 2.1 into two parts. In the first one we will show the necessity of condition (1); in the second one, which is slightly more technical since we need estimates of some integral kernels, we will show the sufficiency.

2.2. **Proof of the necessity of** (1)**.** The scheme of the proof is the following: we start by a Riesz-type decomposition of the plurisubharmonic function u. We evaluate it on the origin, and one gets a new decomposition by composing u with the automorphisms of the bidisk. This new decomposition has better properties than the original for our interests. This technique has been used in one variable, at least by Pascuas in [Pas88], Ahern and Čučković in [AČ95] and Luecking in [Lue96]. The necessity of the condition follows immediately from the new decomposition.

Let $u \in PSH(\mathbb{D}^2) \cap L^p(\mathbb{D}^2)$, $1 < p < \infty$. We consider the decomposition of u into

$$u = \Pi[u] + L[\partial\bar{\partial}u], \tag{2}$$

where $\Pi[u]$ is the orthogonal projection of u onto the pluriharmonic functions with the natural scalar product in $L^2(\mathbb{D}^2)$, i.e.

$$\Pi[u](z) = \frac{1}{\pi} \int_{\mathbb{D}^2} \pi(\zeta, z) u(\zeta) dm(\zeta),$$

where the kernel is

$$\pi(\zeta, z) = \frac{1}{(1-\bar\zeta_1 z_1)^2(1-\bar\zeta_2 z_2)^2} + \frac{1}{(1-\bar z_1\zeta_1)^2(1-\bar z_2\zeta_2)^2} - 1.$$

A priori, this decomposition is valid only in the case $p = 2$, but, since the kernel defines a bounded operator in $L^p(\mathbb{D}^2)$ for $1 < p < \infty$, we can extend the decomposition to all L^p spaces.

The other term in the decomposition (2) has an integral expression of the type

$$L[\partial\bar\partial u](z) = \int_{\zeta\in\mathbb{D}^2} l(\zeta, z) \wedge \partial\bar\partial u(\zeta),$$

and the function $L[\Theta]$ is the minimal solution in $L^2(\mathbb{D}^2)$ of the equation $i\partial\bar\partial v = \Theta$. Note that in view of (2), the operator L is just determined on closed $(1,1)$-currents Θ.

The computation of the kernel $l(\zeta, z)$ and the estimates of its size were carried out by Andersson in [And85]. The expression of the kernel $l(\zeta, z)$ is not unique. There are other kernels that give the same solution. Instead of writing down $l(\zeta, z)$ explicitly, we exhibit L as a linear combination of compositions of explicit operators. In the statement that follows A_1B_2 means the integral operator

$$A_1B_2[\Theta](z) = \int_{\zeta\in\mathbb{D}^2} a(\zeta_1, z_1) \wedge b(\zeta_2, z_2) \wedge \Theta(\zeta),$$

and the kernels appearing are

$$(3)\quad m(\zeta, z) = \frac{1}{2\pi}\left[\log\left|\frac{\zeta - z}{1-\bar\zeta z}\right|^2 + \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-z\bar\zeta|^2} + |z|^2\left(\frac{1-|\zeta|^2}{|1-\bar\zeta z|}\right)^2\right],$$

$$\pi^0(\zeta, z) = \frac{i}{2\pi} d\zeta\wedge d\bar\zeta,$$

$$t(\zeta, z) = \frac{i}{2\pi}\left[\frac{\bar z(1-|\zeta|^2)}{1-\zeta\bar z} + \frac{\bar z(1-|\zeta|^2)}{(1-\zeta\bar z)^2}\right] d\zeta,$$

$$k(\zeta, z) = \frac{i}{2\pi}\frac{(1-|\zeta|^2)}{(1-\bar\zeta z)(\zeta - z)} d\zeta,$$

$$i(\zeta, z) = \frac{i}{2}\delta(\zeta - z) d\zeta\wedge d\bar\zeta,$$

$$p(\zeta, z) = \frac{i}{2\pi}\frac{d\zeta\wedge d\bar\zeta}{(1-\bar\zeta z)^2}.$$

Theorem 2.5 (Andersson)**.** *Let us call*

$$(4)\quad L = \Pi_1^0 M_2 + M_1\Pi_2^0 + \bar T_1 K_2 + \bar K_1 T_2 - \bar T_1 T_2 + M_1 I_2 - M_1\bar P_2 + T_1\bar K_2.$$

Then

$$L[\partial\bar\partial u](z) = \int_{\zeta\in\mathbb{D}^2} l(\zeta, z)\wedge\partial\bar\partial u(\zeta)$$

gives the second term of the decomposition (2).

From the explicit decomposition (2) we can already draw some conclusions. Since $u \in L^p$ by hypothesis, then $\Pi[u] \in L^p$. Therefore, because of decomposition (2), we conclude that $L[\partial\bar\partial u] \in L^p(\mathbb{D}^2)$. This is a necessary condition on $\partial\bar\partial u$ if $u \in L^p$. In fact, since the operator L solves the $\partial\bar\partial$-equation, we can say that $L[\Theta] \in L^p(\mathbb{D}^2)$ is a necessary and sufficient condition for the existence of an L^p solution to the $i\partial\bar\partial u = \Theta$ equation. This condition is poorly handled, since the kernel that defines L is not of constant sign and therefore there are cancelations in $L[\Theta]$ that do not allow to obtain geometric conditions in the variety associated to Θ.

We will find another decomposition of u composing with the automorphisms of the bidisk. In order to avoid some technical difficulties, we will assume that $u \in \mathcal{C}^\infty(\overline{\mathbb{D}^2})$. Afterwards, using an appropriate regularizing process, we will get the general case.

Let us call

$$\tau_z(\zeta) = \left(\frac{\zeta_1 + z_1}{1+\zeta_1\bar z_1}, \frac{\zeta_2 + z_2}{1+\zeta_2\bar z_2}\right) \qquad \zeta, z \in \mathbb{D}^2.$$

We define $u_z(\zeta) = u \circ \tau_z(\zeta)$. We have $u_z(0) = u(z)$ and applying the decomposition (2) to the function u_z at the origin we get

$$u(z) = \int_{\zeta\in\mathbb{D}^2} \pi(\zeta,0)u_z(\zeta)dm(\zeta) + \int_{\zeta\in\mathbb{D}^2} l(\zeta,0)\wedge\partial\bar\partial u_z(\zeta). \tag{5}$$

We take the first integral on the right hand side of (5), and we make the change of variables $\eta = \tau_z(\zeta)$:

$$\int_{\zeta\in\mathbb{D}^2} \pi(\zeta,0)u_z(\zeta) = \int_{\zeta\in\mathbb{D}^2} u_z(\tau_z(\zeta))\,dm(\zeta) =$$
$$= \int_{\eta\in\mathbb{D}^2} u(\eta)\left(\frac{1-|z_1|^2}{|1-\bar z_1\eta_1|^2}\frac{1-|z_2|^2}{|1-\bar z_2\eta_2|^2}\right)^2 dm(\eta).$$

Now we will proof that the operator R defined by

$$R[u](z) = \int_{\eta\in\mathbb{D}^2} u(\eta)\left(\frac{1-|z_1|^2}{|1-\bar z_1\eta_1|^2}\frac{1-|z_2|^2}{|1-\bar z_2\eta_2|^2}\right)^2 dm(\eta),$$

is bounded in $L^p(\mathbb{D}^2)$, i.e. $\|R[u]\|_p \lesssim \|u\|_p$. In order to prove so, we will use Schur's lemma that states

Lemma 2.6 (Schur). *Assume that (X,μ) is a measure space and K a measurable non-negative function defined in $X\times X$. Let T be the integral operator defined by K,*

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y).$$

Let p be such that $1 < p < \infty$ and q be such that $1/p + 1/q = 1$. If there is a constant $C > 0$ and a positive measurable function h such that

$$\int_X K(x,y)h(y)^q d\mu(y) \le Ch(x)^q \qquad \mu - a.e.\ x\in X$$

and

$$\int_X K(x,y)h(x)^p d\mu(y) \le Ch(y)^p \qquad \mu - a.e.\ y\in X,$$

then T is bounded in $L^p(X,d\mu)$ with norm smaller or equal than C.

In our case we take as a function $h(w) = (1-|w_1|^2)^\beta(1-|w_2|^2)^\beta$, with a properly chosen β such that $-1 < \beta < 0$. The estimates needed in the hypothesis of Schur's lemma, come from the following inequality ([Rud80, prop. 1.4.10]):

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-w\bar{z}|^\beta} dm(z) \lesssim (1-|w|^2)^{\alpha-\beta+2} \qquad \text{if } \alpha > -1 \text{ and } \alpha-\beta+2 < 0.$$

We have proved that if $u \in L^p(\mathbb{D}^2)$, then

$$\int_{\zeta\in\mathbb{D}^2} l(\zeta,0) \wedge \partial\bar{\partial} u_z(\zeta) \in L^p(\mathbb{D}^2)$$

with L^p-norm controlled by that of u. Now, considering the expression (4) of $l(\zeta, z)$ and taking into account that $t(\zeta,0) = 0$, $p(\zeta,0) = p(\bar{\zeta},0) = \pi^0(\zeta,0)$, we obtain:

$$\begin{aligned}(6)\ \int_{\zeta\in\mathbb{D}^2} & l(\zeta,0) \wedge \partial\bar{\partial} u_z(\zeta) = \\ &= \int_{\zeta\in\mathbb{D}^2} \frac{i}{2\pi}\left(m(\zeta_2,0)d\zeta_1\wedge d\bar{\zeta}_1 + m(\zeta_1,0)\delta_0(\zeta_2)d\zeta_2\wedge d\bar{\zeta}_2\right) \wedge \partial\bar{\partial} u_z(\zeta).\end{aligned}$$

Let us consider the first term in the right-hand side of (6). For bidegree reasons it is enough to consider

$$\partial_2\bar{\partial}_2 u_z(\zeta) = \theta_{22}(\tau_z(\zeta))\frac{(1-|z_2|^2)^2}{|1+\zeta_2\bar{z}_2|^4} d\zeta_2\wedge d\bar{\zeta}_2,$$

and, therefore,

$$\begin{aligned}\int_{\zeta\in\mathbb{D}^2} & m(\zeta_2,0)d\zeta_1\wedge d\bar{\zeta}_1 \wedge \partial\bar{\partial} u_z(\zeta) = \\ &= \int_{\zeta\in\mathbb{D}^2}\left(\log\left|\frac{z_2-\zeta_2}{1-\bar{\zeta}_2 z_2}\right|^2 + \frac{(1-|\zeta_2|^2)(1-|z_2|^2)}{|1-\bar{\zeta}_2 z_2|^2}\right) \times \\ &\qquad\qquad \times \left(\frac{1-|z_1|^2}{|1-\zeta_1\bar{z}_1|^2}\right)^2 \theta_{22}(\zeta_1,\zeta_2).\end{aligned}$$

From the second term in the right hand side of (6) we only have to consider

$$\partial_1\bar{\partial}_1 u_z(\zeta) = \theta_{11}(\tau_z(\zeta))\frac{(1-|z_1|^2)^2}{|1+\zeta_1\bar{z}_1|^4} d\zeta_1\wedge d\bar{\zeta}_1,$$

and, therefore,

$$\begin{aligned}\int_{\zeta\in\mathbb{D}^2} & m(\zeta_1,0)\delta_0(\zeta_2)d\zeta_2\wedge d\bar{\zeta}_2 \wedge \partial\bar{\partial} u_z(\zeta) = \\ &= \int_{\zeta_1\in\mathbb{D}}\left(\log\left|\frac{z_1-\zeta_1}{1-\bar{\zeta}_1 z_1}\right|^2 + \frac{(1-|\zeta_1|^2)(1-|z_1|^2)}{|1-\bar{\zeta}_1 z_1|^2}\right)\theta_{11}(\zeta_1,z_2).\end{aligned}$$

Both terms of (6) are, therefore, negative and moreover

$$\log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2 + \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar{\zeta}z|^2} \le -\frac{1}{2}\frac{(1-|\zeta|^2)^2(1-|z|^2)^2}{|1-\bar{\zeta}z|^4}.$$

In consequence,

$$\int_{\zeta_1\in\mathbb{D}} \frac{(1-|\zeta_1|^2)^2(1-|z_1|^2)^2}{|1-\bar\zeta_1 z_1|^4}\theta_{11}(\zeta_1,z_2)\in L^p(\mathbb{D}^2) \tag{7}$$

with norm controlled by the norm of u. Obviously, the same happens if we permute the indexes, since the solution $L[\Theta]$ is symmetric in both variables.

Finally, note that for $\zeta_1\in D_{z_1}$,

$$|1-\bar\zeta_1 z_1|\simeq 1-|z_1|^2\simeq 1-|\zeta_1|^2.$$

From (7) we can conclude

$$\|\theta_{11}(D_{z_1},z_2)\|_p\lesssim\|u\|_p.$$

From this inequality, we can obtain the general case (recall that we have only proved the necessity when $u\in\mathcal{C}^\infty(\overline{\mathbb{D}^2})$). For an arbitrary $u\in L^p(\mathbb{D}^2)$ we pick a sequence of $u_n\in\mathcal{C}^\infty(\overline{\mathbb{D}^2})$ such that $u_n\to u$ in L^p. Since the convergence in L^p implies the convergence in the distribution sense, then $\Theta^n=\partial\bar\partial u_n\to\partial\bar\partial u=\Theta$, weakly.

We want to check that $\|\theta_{11}(D_{z_1},z_2)\|_p<\infty$. But, for any function $\psi\in\mathcal{C}^\infty(\mathbb{D}^2)$ positive with compact support, and if $1/p+1/q=1$ the following holds

$$\begin{aligned}\left|\int_{\mathbb{D}^2}\psi(z)\theta_{11}(D_{z_1},z_2)dm(z)\right| &= \left|\lim_{n\to\infty}\int_{\mathbb{D}^2}\psi(z)\theta_{11}^n(D_{z_1},z_2)dm(z)\right|\lesssim\\ &\lesssim\lim_{n\to\infty}\|u_n\|_p\|\psi\|_q\lesssim\|u\|_p\|\psi\|_q,\end{aligned}$$

thus, we have obtained the desired result. ♠

2.3. **Proof of the sufficiency of** (1)**.** We will show that condition (1) is sufficient in order to obtain a solution $u\in L^p(\mathbb{D}^2)$ to $i\partial\bar\partial u=\Theta$. We assume initially that Θ is a closed positive $(1,1)$-form and that $\Theta\in\mathcal{C}^\infty(\overline{\mathbb{D}^2})$. We will show, that under this hypothesis the solution $L[\Theta]\in L^p(\mathbb{D}^2)$ with controlled norm. Later on, we will drop the regularity hypothesis. In order to show this we will make use of the expression of $l(\zeta,z)$ that we have given in (4), i.e.:

$$\begin{aligned}l(\zeta,z)=\pi_1^0\wedge m_2+m_1\wedge\pi_2^0+\bar t_1\wedge k_2+\bar k_1\wedge t_2-\\ -\bar t_1\wedge t_2+m_1\wedge i_2-m_1\wedge\bar p_2+t_1\wedge\bar k_2.\end{aligned} \tag{8}$$

Recall that the definition of the kernels that appear in (8) are given explicitly in (3).

We need the following bounds of the moduli of the respective kernels that can be found in [And85]:

$$\begin{aligned}|m(\zeta,z)|&\lesssim\frac{(1-|\zeta|^2)^2}{|1-\bar\zeta z|^2}\left[1+\log\left|\frac{1-\bar\zeta z}{\zeta-z}\right|\right],\\ |t(\zeta,z)|&\lesssim\frac{1-|\zeta|^2}{|1-\bar\zeta z|^2}.\end{aligned} \tag{9}$$

where $\zeta,z\in\mathbb{D}$. The terms of the type $\bar T_1K_2$ are as follows:

$$\int_{\zeta\in\mathbb{D}^2}\overline{t(\zeta_1,z_1)}d\bar\zeta_1\wedge\frac{1-|\zeta_2|^2}{(1-\bar\zeta_2 z_2)(\zeta_2-z_2)}d\zeta_2\wedge\Theta(\zeta)=D(z_1,z_2).$$

We are going to rewrite $D(z_1,z_2)$ in a more convenient way. Since $t(\zeta_1,z_1)=0$ when $|\zeta_1|=1$, by Stokes theorem

$$\begin{aligned}0=&\int_{\zeta\in\partial\mathbb{D}^2}\overline{t(\zeta_1,z_1)}d\bar\zeta_1\wedge\frac{(1-|\zeta_2|^2)(1-\zeta_2\bar z_2)}{(1-\bar\zeta_2 z_2)(1-|z_2|^2)}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\wedge\Theta(\zeta)=\\ &=\int_{\zeta\in\mathbb{D}^2}\partial_{\zeta_1}\overline{t(\zeta_1,z_1)}\frac{(1-|\zeta_2|^2)(1-\zeta_2\bar z_2)}{(1-\bar\zeta_2 z_2)(1-|z_2|^2)}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\theta_{22}(\zeta)dm(\zeta)+\\ &+\int_{\zeta\in\mathbb{D}^2}\overline{t(\zeta_1,z_1)}\partial_{\zeta_2}\left[\frac{(1-|\zeta_2|^2)(1-\zeta_2\bar z_2)}{(1-\bar\zeta_2 z_2)(1-|z_2|^2)}\right]\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\theta_{12}(\zeta)dm(\zeta)+\\ &+D(z_1,z_2).\end{aligned}$$

An immediate computation yields the following estimates:

$$|\partial_{\zeta_1}\overline{t(\zeta_1,z_1)}|\lesssim\frac{1}{|1-\bar z_1\zeta_1|^2},$$
$$\left|\partial_{\zeta_2}\frac{(1-|\zeta_2|^2)(1-\zeta_2\bar z_2)}{(1-\bar\zeta_2 z_2)(1-|z_2|^2)}\right|\lesssim\frac{1}{1-|z_2|^2}.$$

Hence,

$$\begin{aligned}|D(z_1,z_2)|\lesssim&\int_{\zeta\in\mathbb{D}^2}\frac{1}{|1-\bar\zeta_1 z_1|^2}\frac{1-|\zeta_2|^2}{1-|z_2|^2}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\theta_{22}(\zeta)dm(\zeta)+\\ &+\int_{\zeta\in\mathbb{D}^2}\frac{1-|\zeta_1|^2}{|1-\bar\zeta_1 z_1|^2}\frac{1}{1-|z_2|^2}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2|\theta_{12}(\zeta)|dm(\zeta).\end{aligned}$$

Finally, as

$$\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\simeq\frac{(1-|\zeta_2|^2)(1-|z_2|^2)}{|1-\bar\zeta_2 z_2|^2},\ \text{if}\ \left|\frac{\zeta_2-z_2}{1-\bar\zeta_2 z_2}\right|\ge\frac12,$$

we can estimate $D(z_1,z_2)$ as follows:

$$\begin{aligned}&|D(z_1,z_2)|\lesssim\\ &\lesssim\int_{\zeta\in\mathbb{D}^2}\frac{1}{|1-\bar\zeta_1 z_1|^2}\frac{(1-|\zeta_2|^2)^2}{|1-\bar\zeta_2 z_2|^2}\left(1+\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\right)\theta_{22}(\zeta)dm(\zeta)+\\ &\quad+\int_{\zeta\in\mathbb{D}^2}\frac{1-|\zeta_1|^2}{|1-\bar\zeta_1 z_1|^2}\frac{1-|\zeta_2|^2}{|1-\bar\zeta_2 z_2|^2}\left(1+\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\right)|\theta_{12}|(\zeta)dm(\zeta).\end{aligned}\tag{10}$$

We claim that $|L[\Theta](z)|$ is controlled by a sum of terms of the type

$$\int_{\zeta\in\mathbb{D}^2}\frac{1}{|1-\bar\zeta_1 z_1|^2}\frac{(1-|\zeta_2|^2)^2}{|1-\bar\zeta_2 z_2|^2}\left(1+\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\right)\theta_{22}(\zeta)dm(\zeta),\tag{11}$$

$$\int_{\zeta\in\mathbb{D}^2}\frac{1-|\zeta_1|^2}{|1-\bar\zeta_1 z_1|^2}\frac{1-|\zeta_2|^2}{|1-\bar\zeta_2 z_2|^2}\left(1+\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\right)|\theta_{12}|(\zeta)dm(\zeta),\tag{12}$$

$$\int_{\zeta_1\in\mathbb{D}}\frac{(1-|\zeta_1|^2)^2}{|1-\bar\zeta_1 z_1|^2}\left(1+\log\left|\frac{1-\bar\zeta_1 z_1}{\zeta_1-z_1}\right|^2\right)\theta_{11}(\zeta_1,z_2)dm(\zeta_1).\tag{13}$$

Indeed, in (10) we have already seen that $\bar T_1K_2$ is bounded by (11)+(12). The term in $\bar T_1T_2$ is smaller than (12) and moreover (9) implies that M_1I_2

is bounded by (13) and $\bar{P}_1 M_2$, $\Pi_1^0 M_2$ by (11). They do also appear the same terms changing the indexes. We will show that under the hypothesis (1) all of them belong to $L^p(\mathbb{D}^2)$. The terms of type (11) and (12) will be considered jointly by means of the following two lemmas.

Lemma 2.7. *The condition* (1) *implies*

$$\frac{(1-|\zeta_1|^2)^2\theta_{11}(D_z)+(1-|\zeta_2|^2)^2\theta_{22}(D_z)}{|D_z|}\in L^p(\mathbb{D}^2),$$
$$\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)|\theta_{12}|(D_z)}{|D_z|}\in L^p(\mathbb{D}^2),$$

where $D_z = D_{z_1}\times D_{z_2}$ and $|D_z|$ is the Lebesgue measure of D_z.

Proof. Indeed, consider the operator ψ defined as

$$\psi[f](w)=\frac{1}{|D_w|}\int_{\zeta\in D_w}|f(\zeta)|dm(\zeta),\qquad w\in\mathbb{D}.$$

It is bounded from $L^p(\mathbb{D})$ to $L^p(\mathbb{D})$. If we apply this operator to the function $f(z_1,z_2)=\theta_{11}(D_{z_1},z_2)$ (as function of z_2) and to $g(z_1,z_2)=\theta_{22}(z_1,D_{z_2})$ (as a function of z_1), we have proved the first part of the statement of the claim. The second is an immediate consequence of the first. In fact, by the positivity of the current Θ, one has

$$\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)|\theta_{12}|(D_z)}{|D_z|}\le\frac{(1-|\zeta_1|^2)^2\theta_{11}(D_z)}{|D_z|}+\frac{(1-|\zeta_2|^2)^2\theta_{22}(D_z)}{|D_z|}.$$

And this function belongs to $L^p(\mathbb{D}^2)$. ♠

Lemma 2.8. *If μ is a positive measure in the bidisk such that*

$$\frac{\mu(D_z)}{|D_z|}\in L^p(\mathbb{D}^2),$$

then

$$\int_{\zeta\in\mathbb{D}^2}\frac{d\mu(\zeta)}{|1-\bar\zeta_1 z_1|^2|1-\bar\zeta_2 z_2|^2}\left(1+\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\right)\in L^p(\mathbb{D}^2).$$

Proof. We will split the integral that we want to estimate in two parts

$$\int_{\mathbb{D}^2}\frac{d\mu(\zeta)}{|1-\bar\zeta_1 z_1|^2|1-\bar\zeta_2 z_2|^2}+ \tag{14}$$
$$+\int_{\zeta\in\mathbb{D}\times D_{z_2}}\frac{d\mu(\zeta)}{|1-\bar\zeta_1 z_1|^2(1-|\zeta_2|^2)^2}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2.$$

As the function

$$\frac{1}{|1-\zeta_1\bar z_1|^2|1-\zeta_2\bar z_2|^2}$$

is plurisubharmonic, we have the sub-mean inequality:

$$\frac{1}{|1-\zeta_1\bar z_1|^2|1-\zeta_2\bar z_2|^2}\le\frac{1}{|D_\zeta|}\int_{\eta\in D_\zeta}\frac{dm(\eta)}{|1-\eta_1\bar z_1|^2|1-\eta_2\bar z_2|^2}.$$

If this inequality is inserted in (14) we get that the first integral in (14) is bounded by

$$\begin{aligned}
&\int_{\zeta\in\mathbb{D}^2}\int_{\eta\in D_\zeta}\frac{1}{|D_\zeta|}\frac{dm(\eta)}{|1-\eta_1\bar z_1|^2|1-\eta_2\bar z_2|^2}d\mu(\zeta) = \\
&= \int_{\eta\in\mathbb{D}^2}\int_{\zeta\in D_\eta}\frac{1}{|D_\zeta|}\frac{d\mu(\zeta)}{|1-\eta_1\bar z_1|^2|1-\eta_2\bar z_2|^2}dm(\eta) \lesssim \\
&\lesssim \int_{\eta\in\mathbb{D}^2}\frac{\mu(D_\eta)}{|D_\eta|}\frac{dm(\eta)}{|1-\eta_1\bar z_1|^2|1-\eta_2\bar z_2|^2}.
\end{aligned}$$

Now, Schur's lemma implies that the kernel

$$\frac{1}{|1-\eta_1\bar z_1|^2|1-\eta_2\bar z_2|^2},$$

defines a bounded operator in $L^p(\mathbb{D}^2)$ and hence we can conclude that the first term in (14) is in $L^p(\mathbb{D}^2)$. The second one is a bit more involved.

Due to the subharmonicity of the function

$$\frac{1}{|1-\bar\zeta_1 z_1|^2},$$

we have, as before,

$$\begin{aligned}
&\int_{\zeta\in\mathbb{D}\times D_{z_2}}\frac{d\mu(\zeta)}{|1-\bar\zeta_1 z_1|^2}\frac{1}{(1-|\zeta_2|^2)^2}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2 \lesssim \\
&\lesssim \int_{\zeta\in\mathbb{D}\times D_{z_2}}\int_{\eta\in D_{\zeta_1}}\frac{dm(\eta)}{(1-|\eta|^2)^2|1-\bar\eta z_1|^2}\frac{1}{(1-|\zeta_2|^2)^2}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2 d\mu(\zeta) = \\
&= \int_{\eta\in\mathbb{D}}\frac{1}{|1-\bar\eta z_1|^2}\left(\int_{\substack{\zeta_1\in D_\eta\\ \zeta_2\in D_{z_2}}}\frac{d\mu(\zeta)}{(1-|\eta|^2)^2(1-|\zeta_2|^2)^2}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\right)dm(\eta) = \\
&= A(z_1,z_2).
\end{aligned}$$

We want to prove that $A\in L^p(\mathbb{D}^2)$. Since

$$\frac{1}{|1-\bar\eta z_1|^2}$$

defines a bounded operator from $L^p(\mathbb{D})$ to $L^p(\mathbb{D})$,

$$\begin{aligned}
&\int_{z\in\mathbb{D}^2}|A(z_1,z_2)|^p dm(z) \lesssim \\
&\lesssim \int_{\substack{z_2\in\mathbb{D}\\ \eta\in\mathbb{D}}}\left(\int_{\zeta\in D_\eta\times D_{z_2}}\frac{d\mu(\zeta)}{(1-|\eta|^2)^2(1-|\zeta_2|^2)^2}\log\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2\right)^p dm(\eta)dm(z_2).
\end{aligned}$$

Now we apply Jensen's inequality to estimate this by

$$\int_{\substack{z_2\in\mathbb{D}\\ \eta\in\mathbb{D}}}\int_{\substack{\zeta_1\in D_\eta\\ \zeta_2\in D_{z_2}}}\frac{\mu(D_\eta\times D_{z_2})^{p-1}}{(1-|\zeta_1|^2)^{2p}(1-|\zeta_2|^2)^{2p}}\log^p\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2 d\mu(\zeta)dm(\eta)dm(z_2).$$

If $z_2\in D_{\zeta_2}(\varepsilon)$, then $D_{z_2}\subset D_{\zeta_2}(\delta)$ for some $\delta>\varepsilon$. Since the hypothesis of the lemma is independent of ε, we may assume that $D_{z_2}\subset D_{\zeta_2}$ and $D_\eta\subset D_{\zeta_1}$.

If we apply Fubini's theorem we get the estimate

$$\int_{\zeta\in\mathbb{D}^2}\int_{\substack{\eta\in D_{\zeta_1}\\ z_2\in D_{\zeta_2}}}\frac{\mu(D_\zeta)}{(1-|\zeta_1|^2)^{2p}(1-|\zeta_2|^2)^{2p}}\log^p\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2 dm(\eta)dm(z_2)d\mu(\zeta).$$

Since

$$\int_{\substack{\eta\in D_{\zeta_1}\\ z_2\in D_{\zeta_2}}}\log^p\left|\frac{1-\bar\zeta_2 z_2}{\zeta_2-z_2}\right|^2 dm(z_2)dm(\eta)\lesssim(1-|\zeta_2|^2)^2(1-|\zeta_1|^2)^2,$$

we get as a final bound

$$\lesssim\int_{\zeta\in\mathbb{D}^2}\frac{\mu(D_\zeta)^{p-1}}{(1-|\zeta_1|^2)^{2p-2}(1-|\zeta_2|)^{2p-2}}d\mu(\zeta).$$

It is then enough to show that this last integral is bounded by

$$\int_{z\in\mathbb{D}^2}\frac{\mu(D_z)^p}{(1-|z_1|^2)^{2p}(1-|z_2|)^{2p}}dm(z)$$

In order to check this

$$\begin{aligned}(15)\qquad &\int_{z\in\mathbb{D}^2}\frac{\mu(D_z)^p}{(1-|z_1|^2)^{2p}(1-|z_2|)^{2p}}dm(z)=\\
&=\int_{z\in\mathbb{D}^2}\frac{\mu(D_z)^{p-1}}{(1-|z_1|^2)^{2p}(1-|z_2|)^{2p}}\int_{\zeta\in D_z}d\mu(\zeta)dm(z)\gtrsim\\
&\gtrsim\int_{\zeta\in\mathbb{D}^2}\int_{z\in D_\zeta}\frac{\mu(D_\zeta^2)^{p-1}}{(1-|\zeta_1|^2)^{2p}(1-|\zeta_2|)^{2p}}dm(z)d\mu(\zeta)=\\
&=\int_{\zeta\in\mathbb{D}^2}\frac{\mu(D_\zeta^2)^{p-1}}{(1-|z_1|^2)^{2p-2}(1-|\zeta_2|)^{2p-2}}d\mu(\zeta).\end{aligned}$$

With this, we have proved the lemma. ♠

Now, combining lemmas 2.7 and 2.8, taking

$$d\mu(\zeta)=(1-|\zeta_1|^2)(1-|\zeta_2|^2)|\theta_{12}(\zeta)|dm(\zeta)$$

and

$$d\mu(\zeta)=(1-|\zeta_2|^2)^2\theta_{22}(\zeta)dm(\zeta)$$

respectively we see that the terms (11) and (12) belong to $L^p(\mathbb{D}^2)$.

In order to finish the proof of the theorem it remains to estimate the terms of type (13). In order to control it we split it in two, according to the contribution of $D_{z_1}^c$ and D_{z_1} in the integral. The contribution of $D_{z_1}^c$ is bounded by

$$(16)\qquad \int_{\zeta_1\in\mathbb{D}}\frac{(1-|\zeta_1|^2)^2\theta_{11}(\zeta_1,z_2)}{|1-\bar\zeta_1 z_1|^2}dm(\zeta_1).$$

Due to the subharmonicity of

$$c(\zeta_1,z_1)=\frac{1}{|1-\bar\zeta_1 z_1|^2},$$

we have

$$\int_{\zeta_1\in\mathbb{D}}\frac{(1-|\zeta_1|^2)^2\theta_{11}(\zeta_1,z_2)}{|1-\bar\zeta_1 z_1|^2}dm(\zeta_1)\lesssim\int_{\zeta_1\in\mathbb{D}}\frac{\theta_{11}(D_{\zeta_1},z_2)}{|1-\bar\zeta_1 z_1|^2}dm(\zeta_1).$$

Since $c(\zeta_1, z_1)$ is a kernel that defines an operator bounded from $L^p(\mathbb{D})$ to $L^p(\mathbb{D})$ and, by hypothesis (1), $\theta_{11}(D_{z_1}, z_2) \in L^p(\mathbb{D}^2)$, it follows that (16) belongs to $L^p(\mathbb{D}^2)$.

The contribution of D_{z_1} is bounded by

$$\int_{\zeta_1 \in D_{z_1}} \log \left| \frac{1 - \bar{\zeta}_1 z_1}{\zeta_1 - z_1} \right| \theta_{11}(\zeta_1, z_2) dm(\zeta_1).$$

We want to prove that

$$\int_{z \in \mathbb{D}^2} \left(\int_{\zeta_1 \in D_{z_1}} \log \left| \frac{1 - \bar{\zeta}_1 z_1}{\zeta_1 - z_1} \right| \theta_{11}(\zeta_1, z_2) dm(\zeta_1) \right)^p dm(z) < +\infty.$$

In order to check this, we apply Jensen's inequality and estimate the integral by

$$\int_{z \in \mathbb{D}^2} \int_{\zeta_1 \in D_{z_1}} \theta_{11}(D_{z_1}, z_2)^{p-1} \log^p \left| \frac{1 - \bar{\zeta}_1 z_1}{\zeta_1 - z_1} \right| \theta_{11}(\zeta_1, z_2) dm(\zeta_1) dm(z).$$

We apply Fubini's theorem, integrating first in z_1 and we obtain the estimate

$$\int_{\zeta_1 \in \mathbb{D}} \theta_{11}(D_{\zeta_1}, z_2)^{p-1} (1 - |\zeta_1|^2)^2 \theta_{11}(\zeta_1, z_2) dm(\zeta_1).$$

If we now argue as in (15), this is, in turn, bounded by

$$\int_{z \in \mathbb{D}^2} \theta_{11}(D_{z_1}, z_2)^p dm(z).$$

Therefore we have already proved that there exists a solution u of $i\partial\bar{\partial}u = \Theta$ with

$$\|u\|_p \lesssim \|\theta_{11}(D_{z_1}, z_2)\|_p + \|\theta_{22}(z_1, D_{z_2})\|_p,$$

if $\Theta \in \mathcal{C}^\infty(\overline{\mathbb{D}^2})$. In the general case, by an standard regularizing process (see for instance [Cha84]), we can find a sequence $\Theta^n \to \Theta$ in a distributional sense and with $\Theta^n \in \mathcal{C}^\infty(\overline{\mathbb{D}^2})$, in such a way that

$$\|\theta_{11}^n(D_{z_1}, z_2)\|_p + \|\theta_{22}^n(z_1, D_{z_2})\|_p \lesssim \|\theta_{11}(D_{z_1}, z_2)\|_p + \|\theta_{22}(z_1, D_{z_2})\|_p.$$

We have already proved that there exists a sequence u_n with $i\partial\bar{\partial}u_n = \Theta^n$ and such that

$$\|u_n\|_p \lesssim \|\theta_{11}(D_{z_1}, z_2)\|_p + \|\theta_{22}(z_1, D_{z_2})\|_p.$$

There is a subsequence such that $u_n \to u$ with $i\partial\bar{\partial}u = \Theta$ and

$$\|u\|_p \lesssim \|\theta_{11}(D_{z_1}, z_2)\|_p + \|\theta_{22}(z_1, D_{z_2})\|_p. \spadesuit$$

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