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Bounds on multisequant lines

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To the memory of Fernando Serrano

ABSTRACT

The purpose of this paper is twofold. First, we give an upper bound on the order of a multisequant line to an integral arithmetically Cohen-Macaulay subscheme in \mathbb{P}^n of codimension two in terms of the Hilbert function. Secondly, we give an explicit description of the singular locus of the blow up of an arbitrary local ring at a complete intersection ideal. This description is used to refine a standard linking theorem. These results are tied together by the construction of sharp examples for the bound, which uses the linking theorems.

0. Introduction

There are many questions one can ask about the geometry of a curve $C \subset \mathbb{P}^3$ and its embedding. The question of interest here is, what is the maximal order of a multisequant line to C in terms of natural invariants of C , such as the degree d and the genus g ? Gruson and Peskine have shown that a smooth connected space curve is expected to have a nonzero finite number of 3-secant and 4-secant lines (and have used intersection theory to give formulas for these numbers in terms of d and g , see [10], Theorem 2.5) but no secant lines of higher order. In another paper by Gruson, Lazarsfeld and Peskine, it is shown that the ideal sheaf of such a curve is $(d - 1)$ -regular [8], which implies that the order of a multisequant line (for nondegenerate curves) is bounded above by $d - 1$. Further, they show that if the ideal sheaf is not $(d - 2)$ -regular, then C is rational and has a $(d - 1)$ -secant line.

From the above, we know that the maximal order of a multiseccant line of a curve is between 4 and $d - 1$ and that both occur. In the present paper, we give an upper bound on the order of a multiseccant line to an integral arithmetically Cohen-Macaulay subscheme in \mathbb{P}^n of codimension two in terms of its Hilbert function and show that the bound is sharp by example. The bound on multiseccant order follows immediately from work of Campanella (see [3], Theorem 2.1(c)), although we give a new proof here. The sharp examples are achieved by first understanding the singularities of \mathbb{P}^n blown up at a local complete intersection, and then using this to prove a Bertini theorem which produces the examples. Usually the bound obtained is strictly better than that given by the Castelnuovo-Mumford regularity.

In the first section, we set up our notation and conventions for dealing with arithmetically Cohen-Macaulay subschemes in \mathbb{P}^n of codimension two. We use the γ -character of Martin-Deschamps and Perrin [15] to describe the Hilbert function. In terms of the γ -character, we recover the upper bounds of Campanella on highest degree of a minimal generator for the total ideal, which in turn gives an upper bound on the order of a multiseccant line.

In section two, we develop linking theorems for producing integral subschemes with small singular locus by refining the standard linking theorem of Peskine and Szpiro ([20], Proposition 4.1). The key ingredient is a description of the singular locus of the spectrum of a local ring blown up at a complete intersection ideal (see Theorem 2.1).

In the last section, we use the Bertini theorems to produce sharp examples for the upper bound on multiseccant order. This strengthens the usual smoothing result for ACM curves in \mathbb{P}^3 (see Theorem 2.5 of [9], Theorem 4 of [14] or the main theorem of [21]), in that we do not assume characteristic zero and obtain the smooth connected curves with maximal order multiseccants. While our construction gives subvarieties which are smooth in codimension ≤ 2 , we also recover the fact that general determinantal subvarieties are smooth in codimension ≤ 3 (see [4], Example 2.1).

This paper is dedicated to the memory of Ferran Serrano, who introduced me to algebraic geometry at Berkeley when he was a visiting professor there. I appreciate the hospitality of the University of Barcelona during the writing of this paper and the comments of the referee, who helped me to give proper credit where it was due.

1. ACM subvarieties

In this section we prove our results on multiseccants to integral arithmetically Cohen-Macaulay (henceforth ACM) subschemes in \mathbb{P}^n of codimension two. To state the

results, we need to use some invariant which describes the Hilbert function of the subscheme. For convenience, we use the γ -character used by Martin-Deschamps and Perrin in [15].

DEFINITION 1.1. Let $X \subset \mathbb{P}^n$ be a subscheme of codimension two. The γ -character γ_X of X is given by

$$\gamma_X(l) = -\Delta^n H_X(l)$$

for $l \in \mathbb{Z}$, where H_X is the Hilbert function of X .

For $X \subset \mathbb{P}^n$, we make a few more definitions that will be useful. Let $I = H_*^0(\mathcal{I}_X)$ be the saturated homogeneous ideal which defines X . Then we let $s_0(X) = \min \{l : I_l \neq 0\}$ and $s_1(X) = \min \{l : I_{\leq l} \text{ is not principal}\}$. We set $t_1(X) = \min \{l : V(I_l) \text{ contains no hypersurface}\}$, which is the same the least twist of the ideal that cuts out a subscheme of codimension two. For the maximal degree generator, we use the notation $\omega(X) = \min \{l : I_{\leq l} \text{ generates } I\}$. Clearly we have the inequalities $s_0(X) \leq s_1(X) \leq t_1(X) \leq \omega(X)$. The following definition reflects some necessary conditions on γ_X .

DEFINITION 1.2. Let $\gamma : \mathbb{Z} \rightarrow \mathbb{Z}$ be a function. We say that γ is a *character* if γ is not identically zero, has finite support and $\sum \gamma(l) = 0$. We say that γ is an *admissible character* if there exist integers $0 < s_0(\gamma) \leq s_1(\gamma)$ such that

- (a) $\gamma(l) = 0$ for $l < 0$
- (b) $\gamma(l) = -1$ for $0 \leq l < s_0(\gamma)$
- (c) $\gamma(l) = 0$ for $s_0(\gamma) \leq l < s_1(\gamma)$
- (d) $\gamma(s_1(\gamma)) > 0$.

In considering a minimal free graded resolution for $I = I_X$, it is easy to check that γ_X is an admissible character with $s_0(\gamma_X) = s_0(X)$ and $s_1(\gamma_X) = s_1(X)$. There are two more properties of the γ -character that we will consider. We say that γ is *positive* if $\gamma(l) \geq 0$ for $l \geq s_1(\gamma)$. We will say that γ is *connected* if for each $a < b$ satisfying $\gamma(a) > 0$ and $\gamma(b) > 0$ we have that $\gamma(l) > 0$ for all $a \leq l \leq b$.

Proposition 1.3

If $X \subset \mathbb{P}^n$ is an ACM subscheme of codimension two, then γ_X is a positive admissible character. Conversely, each positive admissible character arises as γ_X for some such X .

Proof. In [15] V, Theorem 1.3, this is proved for curves in \mathbb{P}^3 , but the same proof holds in general. \square

Lemma 1.4

Let $X, Y \subset \mathbb{P}^n$ be ACM subschemes of codimension two which are linked by hypersurfaces S, T of degrees s, t . Then

$$\gamma_X(l) + \gamma_Y(s + t - 1 - l) = \binom{l-s}{0} + \binom{l-t}{0} - \binom{l}{0} - \binom{l-s-t}{0}$$

for $l \in \mathbb{Z}$.

Proof. In [6], Theorem 3(b) it is shown that if A is a graded Gorenstein ring of dimension m contains ideal I and J which are linked, then

$$\Delta^m H(A, t) = \Delta^m H(A/I, t) + \Delta^m H(A/J, N - t)$$

where $N = \max\{t \in \mathcal{N} : \Delta^m H(A, t) \neq 0\}$. We apply this when A is the homogeneous coordinate ring of the complete intersection $S \cap T$, noting that $N = s + t - 1$. Taking one further difference function and using the Koszul resolution to compute $\Delta^n H(A, t)$ gives the formula of the lemma. \square

Next we give a linkage-theoretic proof of the upper bound on the degree of a minimal generator for the total ideal of codimension two ACM subscheme given by Campanella, who further shows that the bound is sharp [3].

Proposition 1.5

Let $X \subset \mathbb{P}^n$ be an ACM subscheme of codimension two satisfying $s = s_0(X)$ and $t = s_1(X) = t_1(X)$. Then one of the following statements holds.

- (a) $\gamma_X(l) \leq 1$ for all l and X is a complete intersection of hypersurfaces of degrees s and t . In this case $\gamma_X(l) > 0$ if and only if $l \in [t, s + t - 1]$ and $\omega(X) = t$.
- (b) $\gamma_X(l) > 1$ for some $l \in \mathbb{Z}$ and X links to an ACM subscheme Y via hypersurfaces of degrees s and t . In this case $\gamma_X(l) > 0$ if and only if $l \in [t, s + t - 1 - s_0(Y)]$ and $\omega(X) \leq \max\{l : \gamma_X(l) > 1\} = s + t - 1 - s_1(Y)$.

Proof. Since $s_1(X) = t_1(X)$, X is contained in a complete intersection $S \cap T$ formed by hypersurfaces of degrees s and t . If $X = S \cap T$, then all the statements in (a) are clear. The Koszul resolution shows that

$$\gamma_X(l) = \binom{l-s}{0} + \binom{l-t}{0} - \binom{l}{0} - \binom{l-s-t}{0}.$$

If X is not a complete intersection, then $S \cap T$ links X to another subscheme Y . Now consider the formula of Lemma 1.4. In view of Definition 1.2 and the comment following it, we see that $\gamma_X(l)$ is given by the binomial part of the formula for $l < t$, thus $\gamma_Y(p) = 0$ for $p \geq s$ and hence γ_Y is supported on $[0, s - 1]$. It follows that $\gamma_X(l) = 0$ for $l \geq s + t$ and that $\gamma_X(l) = \gamma_Y(s + t - 1 - l) + 1$ on $[t, s + t - 1]$. Since Y is ACM, γ_Y is a positive character and it follows that γ_X is connected. In fact, $\gamma_X > 0$ precisely on the interval $[t, s + t - 1 - s_0(Y)]$ and $M = \max \{m : \gamma_X(m) > 1\} = s + t - 1 - s_1(Y)$.

To finish the proof, we must show that I_M generates $I_{\geq M}$. Since Y is ACM of codimension two, it has a minimal resolution for the form

$$0 \rightarrow \oplus \mathcal{O}(-l)^{a(l)} \rightarrow \oplus \mathcal{O}(-l)^{b(l)} \oplus \mathcal{O}(-s_0(Y)) \rightarrow \mathcal{I}_Y \rightarrow 0$$

in which $b(l) = 0$ for $l < s_1(Y)$ and $a(l) = 0$ for $l \leq s_1(Y)$. Applying the cone construction ([20], Proposition 2.5) we obtain a resolution for X of the form

$$0 \rightarrow \oplus \mathcal{O}(-s - t + l)^{b(l)} \rightarrow \oplus \mathcal{O}(-s - t + l)^{a(l)} \oplus \mathcal{O}(-s) \oplus \mathcal{O}(-t) \rightarrow \mathcal{I}_X \rightarrow 0.$$

In taking global sections, we see that $\omega(X) \leq M$. \square

Corollary 1.6

Let $X \subset \mathbb{P}^n$ be an ACM subscheme of codimension two which satisfies $s_1(X) = t_1(X)$ and let $L \not\subset X$ be a line with $o_L(X) = \text{length}(X \cap L)$.

- (a) If $\gamma(l) \leq 1$ for each $l \in \mathbb{Z}$, then $o_L(X) \leq t_1(X) = \min \{l : \gamma(l) > 0\}$.
- (b) If $\gamma(l) > 1$ for some $l \in \mathbb{Z}$, then $o_L(X) \leq \max \{l : \gamma(l) > 1\}$.

Proof. If $o_L(X)$ exceeds the bound B given, then L is contained in every hypersurface of degree $\leq B$ which contains X . Since $I_X(B)$ is generated by global sections, it follows that $L \subset X$, a contradiction. \square

Remark 1.7. The bound on $\omega(X)$ given in Proposition 1.5 is stronger than that given by Castelnuovo-Mumford regularity. For example, the complete intersection of two quadrics in \mathbb{P}^3 is 3-regular, although its total ideal is generated in degree 2. On the other hand, if we drop the condition $s_1(X) = t_1(X)$, then the Castelnuovo-Mumford regularity bound is sharp. For example, the complete intersection of quadrics mentioned above degenerates to the union $C \subset \mathbb{P}^3$ of a plane cubic and a line whose ideal is generated in degree 3. In this case, C has a resolution of the form

$$0 \rightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-3) \rightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2)^2 \rightarrow \mathcal{I}_C \rightarrow 0$$

in which the summand $\mathcal{O}(-3)$ cannot be canceled (clearly it can be canceled when C is deformed to the complete intersection of quadrics).

Remark 1.8. We note in either case (a) or (b) of Proposition 1.5 that γ_X is a connected positive admissible character. This condition is equivalent (via the formula of [15], Proposition 2.9) to the numerical character of X having no gaps. In particular, the conclusion that an integral ACM subscheme of codimension two in \mathbb{P}^n has numerical character with no gaps ([9], Corollary 2.2) holds under the weaker hypothesis that $s_1(X) = t_1(X)$. This statement generalizes to any even linkage class of codimension two subschemes in \mathbb{P}^n ([18], Theorem 2.4), however there are stronger necessary conditions for integral subschemes ([18], Theorem 3.4).

There is another interpretation of the condition that X have a connected positive character which stems from a definition of Sauer [21]. For a comparison of Sauer’s results with those of Gruson-Pesquine [9] and Maggioni-Ragusa [14], we refer the reader to Geramita and Migliore’s paper [7]. Since X is ACM of codimension two, it has a resolution of the form

$$0 \rightarrow P \rightarrow Q \rightarrow \mathcal{I}_X \rightarrow 0$$

where P and Q are direct sums of line bundles. Gathering summands of the same degree, this can be written

$$0 \rightarrow \bigoplus_{i=1}^{t-1} \mathcal{O}(-\alpha_i) \oplus B \rightarrow \bigoplus_{i=1}^t \mathcal{O}(-\beta_i) \oplus B \rightarrow \mathcal{I}_X \rightarrow 0$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are nondecreasing sequences and $\alpha_i \neq \beta_j$ for each (i, j) . Here Sauer defines $m(X) = \min \{i : \alpha_k < \beta_{k+i} \text{ for some } 1 \leq k < t\}$ (if $\alpha_i > \beta_j$ for all (i, j)), we will set $m(X) = \infty$). The resolution above gives the formula

$$\gamma_X(l) = \# \{i : \beta_i \leq l\} - \# \{i : \alpha_i \leq l\} - 1$$

for $l \geq 0$, which allows us to interpret the integer $m(X)$ as a condition on γ_X . We find the following:

Lemma 1.9

Let $X \subset \mathbb{P}^n$ be ACM of codimension two. Then $m(X) \geq 2$. Moreover, $m(X) \geq r \geq 2$ if and only if for each $0 \leq k \leq r-2$, the following condition holds: if $\gamma_X(a) \geq k$ and $\gamma_X(b) \geq k$ for some $0 < a < b$, then $\gamma_X(l) \geq k$ for $a \leq l \leq b$. In particular, $m(X) \geq 3$ if and only if γ_X is connected.

Proof. In view of the formula for $\gamma = \gamma_X$ in terms of the $\{\alpha_i\}$ and $\{\beta_i\}$ for $l \geq 0$, we see that γ strictly increases from $\beta_i - 1$ to β_i , strictly decreases from $\alpha_i - 1$ to α_i and is constant elsewhere. In particular, for $l \geq s_0(\gamma)$, the local minimum values of γ occur among the $\gamma(\alpha_i)$. Thus we see that $m(X) = \infty$ if and only if γ is nondecreasing on $[0, \beta_t]$ and nonincreasing on $[\beta_t, \infty]$ if and only if the condition of Lemma 1.9 holds for all $k \geq 0$. If $m(X) < \infty$, then γ has at least one local minimum, and the smallest such local minimum value is precisely $m(X) - 2 = \gamma(\alpha_K)$, where K is chosen with $\alpha_K < \beta_{K+m(X)}$. \square

In [21], Sauer was interested in smooth ACM curves in \mathbb{P}^3 . He proved that if $m(X) = 2$, then X is necessarily singular. In [14], Chang shows that if $m(X) = 2$, then X must be reducible (hence singular, since any ACM subscheme is connected). In Proposition 1.5 above, we see further that if $m(X) = 2$, then X does not lie on hypersurfaces of minimal possible degrees (with respect to the Hilbert function) which meet properly. On the other hand, if $m(X) \geq 3$ and X is a space curve, then X is smoothable, as has been shown using various notations [4, 9, 14, 17, 21].

2. Singularities in blow ups

In this section we give a precise description of the singular locus of \mathbb{P}^n blown up along a local complete intersection ideal. This is obtained by first carrying out a local analysis, which generalizes some other results. We conclude the section with some geometric applications, which will be used in the existence theorems of the last section.

Theorem 2.1

Let (A, \mathfrak{m}, k) be a local ring. Let $I \subset A$ be an ideal generated by a regular sequence of r elements with corresponding quotient ring $(\overline{A}, \overline{\mathfrak{m}}, k)$. Let $\pi : P \rightarrow \text{Spec } A$ be the blow up of $\text{Spec } A$ at I , and let $x \in \text{Spec } A$ denote the closed point

with fibre $f = \pi^{-1}(x) \cong \mathbb{P}_k^{r-1}$. Then the singular locus of P along f can be described as follows:

- (1) If A is not regular, then $f \subset \text{Sing}(P)$.
- (2) Assume that A is regular.
 - (a) If $\dim_k \bar{m}/\bar{m}^2 = \dim \bar{A}$, then $f \cap \text{Sing} P = \emptyset$.
 - (b) If $\dim_k \bar{m}/\bar{m}^2 = \dim \bar{A} + 1$, then $f \cap \text{Sing} P$ is a hyperplane in f .
 - (c) If $\dim_k \bar{m}/\bar{m}^2 > \dim \bar{A} + 1$, then $f \subset \text{Sing} P$.

Proof. First note that in the special case $r = 1$, the map π is an isomorphism and the theorem holds (here case (c) cannot occur and the “hyperplane” of part 2(b) is empty because the fibre is \mathbb{P}^0). Thus we assume $r \geq 2$ and that $I = (f_1, \dots, f_r)$ is generated by a regular sequence. Because this is a regular sequence, we have an isomorphism $\bigoplus_{d \geq 0} I^d \cong A[T_1, \dots, T_r]/(T_i f_j - T_j f_i)$. and P is covered by standard affines $\text{Spec } B_k$ where $B_k = A[T_1, \dots, \hat{T}_k, \dots, T_r]/(T_j f_k - f_j)$ ([12], Proposition 14.1).

We consider maximal ideals $\mathfrak{n} \subset A[T_1, \dots, \hat{T}_k, \dots, T_r]$ which contain \mathfrak{m} and whose image in B_k we denote $\bar{\mathfrak{n}}$. Let p be the dimension of the subvector space $V \subset \mathfrak{n}/\mathfrak{n}^2$ generated by the $r - 1$ equations $T_j f_k - f_j$. Then we have

$$\dim_k \bar{\mathfrak{n}}/\bar{\mathfrak{n}}^2 = \dim_k \mathfrak{n}/\mathfrak{n}^2 - p = \dim_k \mathfrak{m}/\mathfrak{m}^2 + (r - 1) - p.$$

This is always $\geq \dim A$ because $p \leq r - 1$ (V is generated by $r - 1$ elements) and $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$. Thus \mathfrak{n} corresponds to a regular point in P if and only if $p = r - 1$ and A is regular. This already covers case (1) of the theorem, so we now assume that A is regular.

By the standard inductive proof that $A[T_1, \dots, \hat{T}_k, \dots, T_r]$ is a regular ring, a system of parameters for \mathfrak{m} extends to a system of parameters for \mathfrak{n} . Thus we may write $\mathfrak{m} = (u_1, \dots, u_d)$ and $\mathfrak{n} = (u_1, \dots, u_d, g_1, \dots, g_{r-1})$. If A/I is regular, then f_i extends to a regular system of parameters for \mathfrak{m} , and we may take $u_i = f_i$ for $1 \leq i \leq r$. In this case, we certainly have that $p = r - 1$, for if the relations $\{T_j f_k - f_j\}$ are not linearly independent in $\mathfrak{n}/\mathfrak{n}^2$, then there is a relation $\sum a_j (T_j f_k - f_j) = 0$, which can be rewritten $(\sum a_j X_j) f_k - \sum a_j f_j = 0$, hence we deduce that $a_j \in \mathfrak{n}$ for all j because the $\{f_j\}$ were part of a system of parameters for \mathfrak{n} . This finishes case 2(a).

In case 2(b), only $r - 1$ of the generators for I will be independent in $\mathfrak{m}/\mathfrak{m}^2$, so we may take $u_i = f_i$ for $1 \leq i \leq r - 1$ and $f_r \in \mathfrak{m}^2$. Now we consider regularity at \mathfrak{n} . First, I claim that the affine patch ring B_r is regular. Indeed, since $f_r \in \mathfrak{m}^2 \subset \mathfrak{n}^2$, the relations $T_j f_r - f_j$ have the same images in $\mathfrak{n}/\mathfrak{n}^2$ as $-f_j$, which were part of a regular sequence of parameters for \mathfrak{n} . Thus $p = r - 1$ and B_r is regular. It remains to consider a patch ring B_k with $k < r$ and a maximal ideal \mathfrak{n} which contains T_r .

In this case the relation $X_r f_k - f_r$ lies in \mathfrak{n}^2 , so $p < r - 1$ and \mathfrak{n} corresponds to a singular point. Thus the singular locus is given by the hyperplane $\{T_r = 0\}$.

Finally, in case 2(c) we may choose generators for I in such a way that $f_{r-1}, f_r \in \mathfrak{m}^2$. In this case the relations $T_j f_k - f_j$ are contained in the vector space generated by the f_j , which has dimension at most $r - 2 < r - 1$, hence each maximal ideal \mathfrak{n} corresponds to a singular point. \square

Corollary 2.2

With the notation of Theorem 2.1, P is smooth if and only if \overline{A} is regular.

Proof. This is immediate from the theorem. This characterization is also proved by O’Carroll and Valla in [19], Theorem 2.1, where these conditions are also shown to be equivalent to smoothness of the exceptional divisor. \square

EXAMPLE 2.3: Consider the special case of two lines meeting at a point in \mathbb{P}^3 . Locally the ideal at the singularity can be written $I = (xy, z) \subset A = k[x, y, z]$ and the blow up at I is given by $\text{Proj } A[X, Y]/(xyY - zX)$. This is covered by two affines whose coordinate rings are $B_1 = k[x, y, z, X]/(xy - zX)$ and $B_2 = k[x, y, z, Y]/(xyY - z)$. Clearly B_2 is a regular ring, while B_1 is the ring of the affine cone over the nonsingular quadric surface in \mathbb{P}^3 , which has exactly one singularity at the vertex.

EXAMPLE 2.4: A planar double line in \mathbb{P}^3 is locally described by the ideal $I = (x^2, y) \subset A = k[x, y, z]$. The blow up at I is $\text{Proj } A[X, Y]/(x^2Y - yX)$, which is covered by two affines with coordinate rings $B_1 = k[x, y, z, X]/(x^2 - yX)$ and $B_2 = k[x, y, z, Y]/(x^2Y - y)$. Clearly B_2 is a regular ring, while B_1 is the ring of the affine cone over the quadric cone in \mathbb{P}^3 . Since the quadric cone has one singularity (at the vertex), the affine cone has singular locus $Z(x, y, X)$, which is a (set-theoretic) section of the line $Z(x, y)$ in $\text{Spec } A$.

Remark 2.5. One can give a geometric description of the singular locus in the interesting case 2(b) of Theorem 2.1. If $X \subset \text{Spec } A$ is the closed subscheme corresponding to I of dimension d , then the condition 2(b) says that locally X is contained in a smooth subvariety Y of dimension $d + 1$ which has the same Zariski tangent space as X . Moreover, X is locally the intersection of Y and a singular hypersurface Z . The blow up \tilde{Z} of Z at X is a closed subscheme of $P = \widetilde{\text{Spec } A}$, and the exceptional divisor E_Z is a \mathbb{P}^{r-2} -bundle over X , which sits inside the exceptional divisor E on P as the hyperplane of singularities.

Proposition 2.6

Let $Y \subset \mathbb{P}^n$ be a subscheme of pure codimension two such that $\mathcal{I}_Y(s)$ is generated by global sections. Assume that Y is reduced, has embedding dimension $\leq n - 1$ in codimension ≤ 1 , is a local complete intersection in codimension ≤ 2 , and is locally generated by ≤ 3 elements in codimension ≤ 3 . Then the general scheme X linked to Y by hypersurfaces F, G of degrees $> s$ enjoys the following properties.

- (a) The hypersurfaces F and G are smooth in codimension ≤ 2 , and both X and Y are Cartier on these hypersurfaces in codimension ≤ 1 (on X and Y).
- (b) The scheme X is integral, smooth in codimension ≤ 2 , and a local complete intersection in codimension ≤ 3 .

Proof. We follow the proof of Peskine and Szpiro ([20], Proposition 4.1), taking into account the refinement offered by Theorem 2.1. Let $\pi : \widetilde{\mathbb{P}^n} \rightarrow \mathbb{P}^n$ be the blow up of \mathbb{P}^n at Y with exceptional divisor E . For $d \geq s$, the invertible sheaf $\mathcal{I}_E \otimes \mathcal{O}(d)$ is generated by global sections, giving a morphism $\widetilde{\mathbb{P}^n} \rightarrow \mathbb{P}^n \times \mathbb{P}I_d \rightarrow \mathbb{P}I_d$, where the first map is a closed immersion and the second map is the second projection (here $I_d = H^0(\mathcal{I}_Y(d))$). Let σ_d denote the composite morphism. For $f \in I_d$, f represents hypersurfaces $H_f \subset \mathbb{P}I_d$ and $Z(f) \subset \mathbb{P}^n$, which are isomorphic away from the base locus Y (that is, $\sigma_d^{-1}(H_f) = \pi^{-1}(Z(f))$ away from $\pi^{-1}(Y)$). The map σ_d is a closed immersion when $d > s$.

Let $W \subset \mathbb{P}^n$ denote the closed set (of dimension $\leq n - 5$) on which Y is not a local complete intersection and let U be its complement. The restricted exceptional divisor $E \cap \pi^{-1}(U)$ is Cartier on $\pi^{-1}(U)$, which is smooth in codimension ≤ 2 by 2.1. By Jouanolou's Bertini theorem ([13], Theorem 6.10), $\sigma_d^{-1}(H_f)$ is integral (because σ_d is a closed immersion) and $\sigma_d^{-1}(H_f) \cap \pi^{-1}(U)$ is smooth in codimension ≤ 2 . Moreover, $E \cap \sigma_d^{-1}(H_f)$ is Cartier on $\sigma_d^{-1}(H_f)$. For $e > s$, the local complete intersection $\sigma_d^{-1}(H_f) \cap \sigma_e^{-1}(H_g)$ is integral, smooth in codimension ≤ 2 , and is Cartier on $\sigma_d^{-1}(H_f)$.

The fibres $\pi^{-1}(u) \cong \mathbb{P}^1$ for $u \in U \cap Y$ map to straight lines in $\mathbb{P}I_d$ (the restriction of $\mathcal{I}_E \otimes \pi^*(\mathcal{O}(d))$ to a fibre is $\mathcal{O}_{\mathbb{P}^1}(1)$ because \mathcal{I}_E is the relative $\mathcal{O}(1)$ in the proj construction of $\widetilde{\mathbb{P}^n}$). A dimension count shows that a general hyperplane $H_f \subset \mathbb{P}I_d$ contains a family of these image lines of dimension $\leq n - 4$, hence $\sigma_d^{-1}(H_f)$ contains an $(n - 4)$ -dimensional family of the fibres $\pi^{-1}(u)$ parametrized by $W_1 \subset U$. Similarly, $\sigma_d^{-1}(H_f) \cap \sigma_e^{-1}(H_g)$ contains a family of these fibres parametrized by $W_2 \subset W_1$ of dimension $\leq n - 6$.

It is easy to prove that if $A \rightarrow B$ is a closed immersion, $B \rightarrow C$ is projective, and A meets the fibres of $B \rightarrow C$ in schemes of length ≤ 1 , then the composite map $A \rightarrow C$ is a closed immersion. Using this, we see that $\sigma_d^{-1}(H_f)$ (resp. $E \cap$

$\sigma_d^{-1}(H_f)$ maps isomorphically onto its image $Z(f)$ (resp. Y) on $U - W_1$ and that $\sigma_d^{-1}(H_f) \cap \sigma_e^{-1}(H_g)$ maps isomorphically onto its image X on $U - W_2$. This suffices to deduce all conclusions of the proposition except that X is a complete intersection in codimension ≤ 3 .

For this last assertion, we consider the the locally closed subset W_3 of dimension $\leq n-5$ where \mathcal{I}_Y is locally generated by exactly 3 elements. Since $\mathcal{I}_Y(s)$ is generated by global sections, it is clear for general $f \in I_d, g \in I_e$ that $\mathcal{I}_Y/(f, g)$ is principal at the generic points of W_3 . In this situation we can apply [20], Lemma 3.5 to see that \mathcal{I}_X is minimally generated by 2 elements at these generic points, and hence that the subset of $W_4 \subset W_3$ where X is not a local complete intersection has dimension $\leq n - 6$. It follows that X is a local complete intersection in codimension ≤ 3 . \square

Remark 2.7. When $\text{char } k = 0$, one can deduce the same smoothness results while considering hypersurfaces of degrees $= s$ instead of $> s$. For curves in \mathbb{P}^3 , these statements can also be found in [16], Lemma 5.2 and Theorem 5.1 or [17], Theorems 4.3.1 and 4.3.3. However, in this case we cannot expect to take X integral, as is seen by taking Y to be a pair of skew lines in \mathbb{P}^3 and $s = 2$.

Remark 2.8. Regarding part (a) of the proposition, more general statements about the smoothness of hypersurfaces containing projective schemes have been proven by Altman and Kleiman [1]. For the case of curves in \mathbb{P}^3 , the results are the same.

Corollary 2.9

Let $Y \subset \mathbb{P}^n$ be a subscheme of pure codimension two such that $\mathcal{I}_Y(s)$ is generated by global sections. Assume that Y has generic embedding dimension $\leq n - 1$ and that Y is a local complete intersection in codimension ≤ 1 . Then

- (a) The general hypersurface F of degree $> s$ containing Y is smooth in codimension ≤ 1 .
- (b) The general scheme X linked to Y by hypersurfaces of degrees $> s$ is integral and smooth in codimension ≤ 1 .

Proof. Almost the same as the proof of Corollary 2.6, except that $\pi^{-1}(U)$ is smooth in codimension ≤ 1 . \square

Corollary 2.10

Let $Y \subset \mathbb{P}^n$ be a subscheme of pure codimension two such that $\mathcal{I}_Y(s)$ is generated by global sections. Assume that Y is smooth in codimension ≤ 2 and a local complete intersection in codimension ≤ 3 . Then the general scheme X linked to Y by hypersurfaces of degrees $> s$ is integral and smooth in codimension ≤ 3 .

Proof. We use the same notation as in the proof of Proposition 2.6. Letting $W = \text{Sing}(Y)$ and $U = \mathbb{P}^n - W$, we see that $\pi^{-1}(U)$ is smooth and hence so is $\tilde{X} = \sigma_d^{-1}(H_f) \cap \sigma_e^{-1}(H_g) \cap \pi^{-1}(U)$. This maps isomorphically onto its image X off the set of dimension $\leq n - 6$ corresponding to the fibres contained in \tilde{X} . On the other hand, consider the locally closed set $W_2 \subset Y - U$ (of dimension $\leq n - 5$) where Y is a local complete intersection. For general $w_2 \in W_2$, the intersection $\sigma_d^{-1}(H_f) \cap \sigma_e^{-1}(H_g) \cap \pi^{-1}(w_2)$ is empty, so the possible singularities of X occurring on W_2 is a closed set of dimension at most $n - 6$. \square

Remark 2.11. Similarly, one can prove that if $Y \subset \mathbb{P}^n$ is a smooth subvariety of codimension r and $\mathcal{I}_Y(s)$ is generated by global sections, then the general subscheme X linked to Y by hypersurfaces of degrees $> s$ is smooth in codimension $\leq 2r - 1$ (using the same sort of dimension count, one sees that the general intersection $\cap_{i=1}^r \sigma_{d_i}^{-1}(H_{f_i})$ fails to meet the fibres in schemes of length ≤ 1 off a set of dimension $\leq n - 3r$). Thus we can link a smooth subvariety $Y \subset \mathbb{P}^n$ to a smooth subvariety X if $\dim Y < \frac{2}{3}n$. On the other hand, if $\dim Y \geq \frac{2}{3}n$, then Hartshorne's conjecture predicts that the only smooth subvarieties are complete intersections.

3. Existence results

In this section we prove that the bound of Corollary 1.6 is sharp and that examples can be found which are smooth in codimension ≤ 2 . In particular, we obtain smooth connected curves in \mathbb{P}^3 and smooth surfaces in \mathbb{P}^4 . On the other hand, our construction gives varieties which are typically singular in codimension 3 rather than the optimal codimension of 4. This reflects the expectation that maximal order multiscanted varieties should occur on some proper closed subset of the moduli. On the other hand, if $m(X) \geq 4$, then X deforms to a subvariety which is smooth in codimension ≤ 3 .

Lemma 3.1

Let $Y \subset \mathbb{P}^n$ be a subscheme of pure codimension two contained in a hypersurface S of degree s . Let H be a plane meeting Y properly and consider the basic double link $Z = Y \cup (S \cap H)$. Then

(a) The total ideal of Z is given by $I_Z = (f, hI_Y)$, where f is the equation of S and h is the equation of H .

(b) The γ -characters of Y and Z are related by the formula

$$\gamma_Z(l) = \gamma_Y(l - 1) - \binom{l}{0} + \binom{l - 1}{0} + \binom{l - s}{0} - \binom{l - s - 1}{0}$$

for $l \in \mathbb{Z}$.

Proof. This is standard linkage theory. Part (a) can be proven using Schwartau's linkage addition (see [2], page 358). The formula for change in γ -character is given in [15], III, Proposition 3.4(2). \square

Proposition 3.2

Let γ be a positive admissible character. Then there exists an ACM subscheme $Y = \mathbb{P}^{n-2} \cup Z \subset \mathbb{P}^n$ of codimension two with $\gamma_Y = \gamma$ which satisfies the hypothesis of Proposition 2.6 and the following conditions.

- (a) Setting $A = s_1(\gamma) = \min \{s : \gamma(s) > 0\}$, there is a linear $\mathbb{P}^3 \subset \mathbb{P}^n$ meeting Y properly such that the genus of the curve restrictions of Y and Z satisfy $p_a(Y \cap \mathbb{P}^3) - p_a(Z \cap \mathbb{P}^3) = A - 2$.
- (b) Setting $B = \max \{s : \gamma(s) > 0\}$, the sheaf $\mathcal{I}_Y(r)$ is generated by its global sections.

Proof. We proceed by induction on $s = s_0(\gamma)$. For the induction base $s = 1$ we see that $\gamma(r) = 1$ (with r as in part (a)) and $\gamma(n) = 0$ for $1 \leq n, n \neq r$. In this case, choose linear subspaces \mathbb{P}^3 and \mathbb{P}^{n-2} in \mathbb{P}^n which intersect in a line L . Let H be a hyperplane containing \mathbb{P}^{n-2} and take Y to be the union of \mathbb{P}^{n-2} with a general complete intersection of H and a hypersurface of degree $r - 1$ (The scheme Z is empty when $r = 1$. In this case we must interpret $p_a(\emptyset) = \chi(\mathcal{I}_\emptyset) = 1$ for the empty curve, and condition (a) holds). Then Y is a complete intersection of H and a hypersurface of degree r , hence satisfies conditions (a) and (b).

Now assume that $s > 1$. Define γ' by

$$\gamma'(n - 1) = \gamma(n) + \binom{n - 1}{0} - \binom{n}{0} - \binom{n - r}{0} + \binom{n - r - 1}{0}.$$

Then γ' is a positive admissible character with $s_0(\gamma') = s - 1$ and it is clear from the formula above that $\gamma_{Y'}(l) = 0$ for $l > r - 1$. By induction hypothesis, there exists $Y' = \mathbb{P}^{n-2} \cup Z'$ satisfying (a) and (b). In particular, $\mathcal{I}_{Y'}(r - 1)$ is generated by global sections. Applying Proposition 2.6 to Y' , we can find a hypersurface F of degree r containing Y' which is smooth in codimension ≤ 2 and such that Y' is Cartier on F in codimension ≤ 1 .

Let W_1 be the closed subset of Y' on which Y' is not Cartier on F and let W_2 be the closed set on which Y' is not a local complete intersection. We can choose a general hyperplane H such that $H \cap F$ is integral and H meets each irreducible component of Y', F, W_1, W_2 and $\text{Sing}(F)$ properly. Now let $Y = Y' \cup (F \cap H)$ be the basic double link obtained from Y' by F and H (see Lemma 3.1). It is clear that $Y = \mathbb{P}^{n-2} \cup Z$, where $Z = Z' \cup (F \cap H)$. Since Y was obtained from Y' by a

basic double link on F , the induction hypothesis and Lemma 3.1 shows that $\mathcal{I}_Y(r)$ is generated by its global sections.

We check that the conditions of Proposition 2.6 hold for Y . By construction Y is reduced. Because $F \cap H$ is a complete intersection on the smooth hyperplane H , the conditions of 2.6 automatically hold away from the intersection $Y' \cap (F \cap H) = Y' \cap H$. H was chosen so that F is generically smooth along this intersection, which has codimension 1 in Y , so we conclude that Y has embedding dimension $\leq n - 1$ in codimension ≤ 1 . Since H meets the components of W_1 properly, the dimension of $H \cap W_1$ is $\leq n - 5$. Away from this set, both Y' and $H \cap F$ are Cartier on F along $Y' \cap H$, hence their union is also. It follows that Y is a local complete intersection in codimension ≤ 2 . Finally, the dimension of $H \cap W_2$ is $\leq n - 6$, hence \mathcal{I}_Y is at most 3-generated away from $H \cap W_2$ (use the local version of the exact sequence from Lemma 3.1 to see this), and so Y is 3-generated in codimension ≤ 3 .

Finally, we check that part (a) holds for Y . We consider the restriction of Y' and Y to the linear \mathbb{P}^3 to get two curves $C' = L \cup D'$ and $C = L \cup D$, where L is the line $\mathbb{P}^3 \cap \mathbb{P}^{n-2}$. Since the plane H meets L in a single point, we see that $\#\{L \cap D\} = \#\{L \cap D'\} + 1$ and hence

$$p_a(C) - p_a(D) = p_a(C') - p_a(D') + 1.$$

On the other hand, $s_1(\gamma_Y) = s_1(\gamma_{Y'}) + 1$ as well, so the formula of part (a) continues to hold. \square

Theorem 3.3

Let γ be a connected, positive admissible character. Then there exists an integral ACM subscheme $X \subset \mathbb{P}^n$ of codimension two with $\gamma_X = \gamma$ and having a multisection line achieving the bound of Corollary 1.6. Further,

- (a) X can be taken to be smooth in codimension ≤ 2 and a local complete intersection in codimension ≤ 3 .
- (b) The general hypersurface of minimal degree containing X is smooth in codimension ≤ 2 .

Proof. First suppose that $\gamma(l) \leq 1$ for all $l \in \mathbb{Z}$. Letting $s = s_0(\gamma)$ and $t = s_1(\gamma)$, the connectedness of γ shows that $\gamma = 1$ precisely on $[t, t + s - 1]$, and hence γ is the γ -character of a complete intersection of hypersurfaces of degrees s and t . By Proposition 2.6, the general hypersurface F of degree s containing a linear $\mathbb{P}^{n-2} \subset \mathbb{P}^n$ is smooth in codimension ≤ 2 (in the special case $s = 1$ take F to be a hyperplane). If G is a general hypersurface of degree t , then $F \cap G$ is smooth in codimension ≤ 2 , and a general line $L \subset \mathbb{P}^{n-2}$ serves as a t -secant line.

Now we consider the general case in which $\gamma(l) > 1$ for some l , and we let R be the maximal such l . Defining s and t as in the previous paragraph, we define the function δ by

$$\gamma(l) = \delta(s + t - 1 - l) - \binom{l}{0} + \binom{l - s}{0} + \binom{l - t}{0} - \binom{l - s - t - 1}{0}$$

as suggested by Lemma 1.4. It is easy to see that δ is a positive admissible character such that $\delta(l) = 0$ for $l \geq s$ and $s_1(\delta) = s + t - 1 - R$.

Applying Proposition 3.2, there exists an ACM subscheme $Y \subset \mathbb{P}^n$ of codimension two such that $\gamma_Y = \delta$ and Y satisfies the hypotheses of Proposition 2.6. In particular, $\mathcal{I}_Y(s - 1)$ is generated by global sections so we may apply Proposition 2.6 to link Y to a subscheme X by hypersurfaces F and G of degrees s and t such that X is integral, smooth in codimension ≤ 2 , is a local complete intersection in codimension ≤ 3 , and lies on a hypersurface F which is smooth in codimension ≤ 2 . By Lemma 1.4 we have $\gamma_X = \gamma$, and in particular F is a hypersurface of minimal degree containing X . Since these hypersurfaces are general, we can also choose them meeting the linear $\mathbb{P}^3 \subset \mathbb{P}^n$ and the line $\mathbb{P}^3 \cap \mathbb{P}^{n-2}$ properly.

It remains to show that X has an R -secant line. Restricting to the linear $\mathbb{P}^3 \subset \mathbb{P}^n$ from Proposition 3.2, the curve $D = Y \cap \mathbb{P}^3$ contains the line $L = \mathbb{P}^{n-2} \cap \mathbb{P}^3$, and we have that $p_a(D) - p_a(D - L) = s_1(\delta) - 2 = s + t - 1 - R$ (from 3.2). Noting that the surfaces $S = F \cap \mathbb{P}^3$ and $T = G \cap \mathbb{P}^3$ link D to $C = X \cap \mathbb{P}^3$ and $D - L$ to $C \cup L$, we get the formulas

$$\begin{aligned} p_a(C \cup L) - p_a(D - L) &= \left(\frac{s + t}{2} - 2\right)(d(C \cup L) - d(D - L)) \\ p_a(C) - p_a(D) &= \left(\frac{s + t}{2} - 2\right)(d(C) - d(D)). \end{aligned}$$

Subtracting the bottom from the top yields and recalling that the difference in genre for D and $D - L$ was $s + t - 1 - R$, we find that

$$p_a(C \cup L) - p_a(C) = R - 1$$

which shows that L is an R -secant line for C . \square

Remark 3.4. This strengthens the usual existence result [9, 14, 21] by removing the characteristic zero hypothesis and obtaining maximal order multiseccants. We take a moment to compare the methods used in these papers (see also [7] for a more detailed comparison). Gruson and Peskine [9] inductively produced the smooth curves by performing basic double links on surfaces of high degree, and then using a

characteristic zero Bertini theorem to smooth these double links. Sauer's method [21] was to work with the singular locus of the matrix defining the curves (although this also has a linkage-theoretic aspect - see the account in [7]). Maggioni and Ragusa [14] first constructed an ACM stick figure, and then did a direct link with surfaces of high degree (which could be taken smooth). The method used here is similar to that of Maggioni and Ragusa, except that special care was taken to obtain the multisequant line.

Theorem 3.5

Suppose that $X \subset \mathbb{P}^n$ is an ACM subscheme of codimension two and that $m(X) \geq 4$. Then X deforms to an integral ACM subscheme of codimension two which is smooth in codimension ≤ 3 .

Proof. First suppose that $\gamma(l) \leq 1$ for all $l \in \mathbb{Z}$. Letting $s = s_0(\gamma)$ and $t = s_1(\gamma)$, the connectedness of γ shows that $\gamma = 1$ precisely on $[t, t + s - 1]$, and hence γ is the γ -character of a complete intersection of hypersurfaces of degrees s and t . Clearly we can obtain such a smooth such complete intersection.

Now we consider the general case where $\gamma(l) > 1$ for some l and let R be the largest such l . Defining s and t as in the previous paragraph, we define the function δ by

$$\gamma(l) = \delta(s + t - 1 - l) - \binom{l}{0} + \binom{l-s}{0} + \binom{l-t}{0} - \binom{l-s-t-1}{0}$$

as in Lemma 1.4. In this case, we use the condition $m(X) \geq 4$ and its interpretation in Lemma 1.9 to see that δ is a *connected* positive admissible character such that $\delta(l) = 0$ for $l \geq s$ and $s_1(\delta) = s + t - 1 - R$.

According to Theorem 3.3. We can find an ACM subscheme Y of codimension two such that $\gamma_Y = \delta$, Y is smooth in codimension ≤ 2 and Y is a local complete intersection in codimension ≤ 3 . Applying Corollary 2.10 to Y with hypersurfaces of degrees s and t , we produce $X' \subset \mathbb{P}^n$ which is smooth in codimension ≤ 3 . the formula of Lemma 1.4 shows that $\gamma_{X'} = \gamma_X$. Since these subschemes are both ACM, this implies that X deforms to X' . \square

Remark 3.6. Theorem 3.5 can also be proved with Chang's Filtered Bertini theorem (see [4], Example 2.1). Bolondi and Migliore have shown that this holds on any smooth projective Gorenstein variety of dimension five (see [2], Theorem 4.2), and that $m(X) \geq 4$ is a necessary condition for smoothness.

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