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# On the Jacobian ideal of the binary discriminant

(with an appendix by Abdelmalek Abdesselam)

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Abstract

Let $\Delta$ denote the discriminant of the generic binary $d$-ic. We show that for $d \geq 3$, the Jacobian ideal of $\Delta$ is perfect of height $2$. Moreover we describe its $SL_2$-equivariant minimal resolution and the associated differential equations satisfied by $\Delta$. A similar result is proved for the resultant of two forms of orders $d, e$ whenever $d \geq e-1$. If $\Phi_n$ denotes the locus of binary forms with total root multiplicity $\geq d-n$, then we show that the ideal of $\Phi_n$ is also perfect, and we construct a covariant which characterizes this locus. We also explain the role of the Morley form in the determinantal formula for the resultant. This relies upon a calculation which is done in the appendix by A. Abdesselam.

*Keywords:* Discriminant, resultant, Morley form, transvectant, evectant, classical invariant theory, Hilbert-Burch theorem.
*MSC2000:* 13A50, 13C40.

## 1. Introduction

**1.1** Let

$$\mathbb{F} = a_0\, x_1^d + \cdots + \binom{d}{i} a_i\, x_1^{d-i}\, x_2^i + \cdots + a_d\, x_2^d,$$

denote the generic binary form of order $d$ in the variables $x_1, x_2$. Its discriminant $\Delta = \Delta(a_0, \ldots, a_d)$ is a homogeneous polynomial with the following property: given $\alpha_0, \ldots, \alpha_d \in \mathbb{C}$, the form $F_\alpha = \sum_{i=0}^{d} \binom{d}{i} \alpha_i\, x_1^{d-i} x_2^i$ is divisible by the square of a linear form iff $\Delta(\alpha_0, \ldots, \alpha_d) = 0$. Let $R$ denote the polynomial ring $\mathbb{C}[a_0, \ldots, a_d]$, and let

$$J = \Big(\frac{\partial\Delta}{\partial a_0}, \ldots, \frac{\partial\Delta}{\partial a_d}\Big) \subseteq R,$$

denote the Jacobian ideal of $\Delta$.

Our main result (in §3) is that $J$ is a ***perfect*** ideal of height 2 for $d \geq 3$, with graded minimal resolution

$$0 \leftarrow R/J \leftarrow R \leftarrow R(3-2d)^{d+1} \leftarrow R(2-2d)^3 \oplus R(1-2d)^{d-3} \leftarrow 0. \qquad (1)$$

**1.2** To put this statement into a geometric context, identify the form $F_\alpha$ (distinguished up to a scalar) with the point $[\alpha_0, \ldots, \alpha_d]$ in the projective space $\mathbb{P}^d$. We recall the notion of a Coincident Root locus introduced in [7]. Let

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

be a partition of $d$ into $n$ parts. Now the CR locus associated to $\lambda$ is defined to be

$$X_\lambda = \Big\{F \in \mathbb{P}^d : F = \prod_{i=1}^{n} l_i^{\lambda_i} \text{ for some linear forms } l_i\Big\},$$

which is an irreducible projective subvariety of dimension $n$. Given two partitions $\lambda$ and $\mu$, we have $X_\lambda \subseteq X_\mu$ iff $\mu$ is a refinement of $\lambda$. Now $X_{(2,1^{d-2})}$ is the hypersurface $\{\Delta = 0\}$, and the closed subscheme $Z = \operatorname{Proj}(R/J)$ is supported on its singular locus. By [7, Theorem 5.4], the latter is equal to the union $X_\tau \cup X_\delta$, where

$$\tau = (3, \underbrace{1, \ldots, 1}_{d-3}) \quad \text{and} \quad \delta = (2, 2, \underbrace{1, \ldots, 1}_{d-4}).$$

The result above implies that $Z$ is an arithmetically Cohen-Macaulay scheme. In Proposition 3.5 we show that $Z$ has multiplicities 2 and 1 along $X_\tau$ and $X_\delta$ respectively.

**1.3** The ideas in §3 are based on the 'Cayley method' as explained in [12, Chapter 2]. In §5 we give a précis of this method in the context of binary resultants, and then deduce the following theorem: let $\mathfrak{R}$ denote the resultant of generic binary forms $\mathbb{F}, \mathbb{G}$ of orders $d, e$. If $d \geq e - 1$, then the $\mathbb{F}$-Jacobian ideal of $\mathfrak{R}$ (i.e., the ideal of partial derivatives of $\mathfrak{R}$ with respect to the coefficients of $\mathbb{F}$) is perfect. The Cayley method involves constructing a morphism of vector bundles whose determinant is the resultant. The most interesting ingredient in this morphism is the so-called Morley form $\mathcal{M}$, which encodes the $d_2$-differential of a spectral sequence. Although *a priori* the differential is only well-defined modulo coboundaries, it admits a unique equivariant lifting to a morphism from binary forms of order $e-2$ to those of order $d$. This is explained in §5.6 – 5.7, modulo a calculation which is provided in the appendix by A. Abdesselam. The reader may also consult [18, §3.11] for a very general treatment of multivariate Morley forms.

**1.4** In a slightly different direction, define $\Phi_n = \bigcup_\lambda X_\lambda$, where the union is quantified over all partitions $\lambda$ having $n$ parts. E.g., for $d = 6$ and $n = 3$,

$$\Phi_3 = X_{(4,1,1)} \cup X_{(3,2,1)} \cup X_{(2,2,2)}.$$

Let $I_n \subseteq R$ denote the ideal of $\Phi_n$. In §6 we show that $I_n$ is a determinantal ideal which admits an Eagon-Northcott resolution, in particular it is perfect.

**1.5** Note that the group $SL_2\,\mathbb{C}$ acts on $\mathbb{P}^d$, namely the element

$$g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL_2\,\mathbb{C},$$

sends

$$\sum_i \binom{d}{i} \alpha_i\, x_1^{d-i}\, x_2^i \quad \text{to} \quad \sum_i \binom{d}{i} \alpha_i\,(p\,x_1 + q\,x_2)^{d-i}\,(r\,x_1 + s\,x_2)^i.$$

All the varieties defined above inherit this action, in particular the ideals $I_n, J$ and the Betti modules in their free resolutions are $SL_2$-representations. This equivariance is respected in all of our subsequent constructions. The first syzygy modules occurring in the resolution of $J$ encode the invariant differential equations satisfied by $\Delta$ (and similarly for $\mathfrak{R}$). We write down these equations explicitly using transvectants. The reader is referred to [11, Lecture 11] and [22, §4.2] for basic representation theory of $SL_2$. We will use [13] and [14] as standard references for classical invariant theory and symbolic calculus; more recent accounts of this subject may be found in [8, 9, 19, 20].

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## 2. Preliminaries

Let $V$ be a two-dimensional vector space over $\mathbb{C}$ with basis $\mathbf{x} = \{x_1, x_2\}$. Then $\operatorname{Sym}^m V = S_m\, V$ is the $(m+1)$-dimensional space of binary forms of order $m$ in $\mathbf{x}$. The $\{S_m\, V : m \geq 0\}$ are a complete set of irreducible $SL(V)$-representations. We will omit the $V$ if no confusion is likely, thus $S_m(S_n)$ stands for the plethysm representation $\operatorname{Sym}^m(\operatorname{Sym}^n V)$ etc.

**2.1 Transvectants.** Given integers $m, n \geq 0$, we have a decomposition of $SL_2$-representations

$$S_m \otimes S_n \simeq \bigoplus_{r=0}^{\min\{m,n\}} S_{m+n-2r}. \tag{2}$$

Let $A, B$ denote binary forms of respective orders $m, n$. The $r$-th transvectant of $A$ with $B$, written $(A, B)_r$, is defined to be the image of $A \otimes B$ via the projection map

$$S_m \otimes S_n \longrightarrow S_{m+n-2r}\,.$$

It is given by the formula

$$(A, B)_r = \frac{(m-r)!\,(n-r)!}{m!\,n!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r A}{\partial x_1^{r-i}\,\partial x_2^i} \frac{\partial^r B}{\partial x_1^i\,\partial x_2^{r-i}} \tag{3}$$

By convention $(A, B)_r = 0$ if $r > \min\{m, n\}$. (Some authors choose the scaling factor differently, cf. [20, Chapter 5].) Each $S_m$ is isomorphic to its dual representation $S_m^* = \text{Hom}(S_m, S_0)$ by the map which sends $A \in S_m$ to the functional $B \longrightarrow (A, B)_m$. Two forms $A, B \in S_m$ are said to be *apolar* to each other if $(A, B)_m = 0$. In some of the examples below quite a few complicated transvectants had to be calculated; to this end we programmed formula (3) in MAPLE. If two forms are symbolically expressed, a useful general procedure for calculating their transvectants is given in [13, §3.2.5] (also see [14, §49]).

**2.2** We identify the generic binary $d$-ic $\mathbb{F} = \sum_{i=0}^{d} \binom{d}{i} a_i\, x_1^{d-i}\, x_2^i$ with the natural trace form in $S_d \otimes S_d^*$. Using the self-duality above, this amounts to the identification of $a_i \in S_d^*$ with $1/d!\, x_2^{d-i}\,(-x_1)^i$. Let $R$ be the symmetric algebra

$$\bigoplus_{m \geq 0} S_m(S_d^*) = \bigoplus_{m \geq 0} R_m = \mathbb{C}\,[a_0, \ldots, a_d],$$

and $\mathbb{P}^d = \mathbb{P}\, S_d = \text{Proj}\ R$. Generally $F, G, \ldots$ will denote specific binary forms, as opposed to generic forms $\mathbb{F}, \mathbb{G}, \ldots$.

**2.3** A *covariant* of degree-order $(m, q)$ of binary $d$-ics is by definition a trivial summand in the representation $S_q \otimes R_m$ (cf. [14, §11 et seq.]). An invariant is a covariant of order 0. The most frequently appearing covariants are the Hessian $\mathbb{H} = (\mathbb{F}, \mathbb{F})_2$, and the cubicovariant $\mathbb{T} = (\mathbb{F}, \mathbb{H})_1$, of degree-orders $(2, 2d-4)$ and $(3, 3d-6)$ respectively. The discriminant $\Delta$ is an invariant of order $2(d-1)$. If $I(a_0, \ldots, a_d)$ is an invariant of degree $m$, then its *evectant* is defined to be

$$\mathcal{E}_I = \frac{(-1)^d}{m} \sum_{i=0}^{d} \frac{\partial I}{\partial a_i}\, x_2^{d-i}\,(-x_1)^i.$$

It is a covariant of degree-order $(m-1, d)$. The scaling factor is so chosen that we have an identity $(\mathcal{E}_I, \mathbb{F})_d = I$.

**2.4** The degree of the CR locus $X_\lambda$ is given by a formula due to Hilbert [17]. Let $e_r$ denote the number of parts in $\lambda$ equal to $r$, thus $\sum\limits_{r \geq 1} e_r = n$ and $\sum r\, e_r = d$. Then $\deg X_\lambda = \frac{n!}{\prod\limits_r (e_r!)} \prod\limits_{i=1}^{n} \lambda_i$. For instance, $\deg X_{(3^2,2,1^3)} = \frac{6!}{2!\,1!\,3!}\, 3^2 \times 2 \times 1^3 = 1080$.

## 3. The binary discriminant

Throughout this paper, we will regard $\Delta$ and $\mathfrak{R}$ as well-defined only up to a multiplicative constant.

For a binary $d$-ic $F$, we define its Bezoutiant $\mathbb{B}_F$ as follows: introduce new variables $\mathbf{y} = (y_1, y_2)$, and write $G$ for the form obtained by substituting $y_1, y_2$ for $x_1, x_2$ in $F$. Then

$$\mathbb{B}_F = \Big(\frac{\partial F}{\partial x_1}\frac{\partial G}{\partial y_2} - \frac{\partial G}{\partial y_1}\frac{\partial F}{\partial x_2}\Big)/(x_1\, y_2 - x_2\, y_1),$$

which is a form of order $(d-2, d-2)$ in $\mathbf{x}, \mathbf{y}$. Henceforth we will assume $d \geq 4$ (but see §4.1). In the sequel, $\Bbbk$ will stand for a nonzero rational constant which need not be precisely specified. Define a map

$$\beta_F : S_{d-4} \longrightarrow S_d,$$

by sending $A \in S_{d-4}$ to $[(A, \mathbb{B}_F)_{d-4}]_{\mathbf{y}=\mathbf{x}}$. This is interpreted as follows: take the $(d-4)$-th transvectant of $A$ with $\mathbb{B}_F$ with respect to the $\mathbf{x}$ variables, which gives an $\mathbf{x}\,\mathbf{y}$-form of order $(2, d-2)$. By substituting $\mathbf{x}$ for $\mathbf{y}$ we get an $\mathbf{x}$-form of order $d$. Define another morphism

$$\gamma_F : S_2 \longrightarrow S_d, \quad A \longrightarrow (A, F)_1,$$

and finally let

$$\mathbf{1}_F : S_0 \longrightarrow S_d, \quad 1 \longrightarrow F.$$

Note that $\beta_F$ is quadratic in the coefficients of $F$, whereas $\gamma_F, \mathbf{1}_F$ are linear. Now consider the morphism

$$\underbrace{\beta_F \oplus \gamma_F \oplus \mathbf{1}_F}_{h_F} : S_{d-4} \oplus S_2 \oplus S_0 \longrightarrow S_d.$$

**Proposition 3.1**

*We have an equality*

$$\det\, h_{\mathbb{F}} = \Delta_{\mathbb{F}}$$

*up to a nonzero scalar.*

*Proof.* Let $D_{\mathbb{F}} = \det h_{\mathbb{F}}$. It is an invariant of degree $2(d-3)+3+1 = 2(d-1)$, which is the same as $\deg \Delta_{\mathbb{F}}$. We will show that (i) $D_{\mathbb{F}}$ vanishes whenever $F$ has a repeated linear factor, and (ii) $D_{\mathbb{F}}$ is not identically zero. This will imply that $D_{\mathbb{F}} = \Delta_{\mathbb{F}}$ (up to a scalar).

As to (i), after a change of variables we may assume that $x_1^2$ divides $F$. Then $x_1\, y_1$ divides $\mathbb{B}_F$, and hence $x_1$ divides each form in $\operatorname{im}(\beta_F)$. Similarly, $x_1$ divides each form in $\mathrm{im}(\gamma_F)$ and $\mathrm{im}(\mathbf{1}_F)$, hence $h_F$ is not surjective and $D_F = 0$. Now assume $F = x_1^d + x_2^d$, then

$$\mathbb{B}_F = d^2 \sum_{i=0}^{d-1} (x_1\, y_2)^{d-2-i}\, (x_2\, y_1)^i.$$

By a direct calculation, $\beta_F(x_1^{d-k-4} x_2^k) = \Bbbk\, x_1^{d-k-2}\, x_2^{k+2}$, hence $\mathrm{im}(\beta_F) = \mathrm{Span}\,\{x_1^{d-i} x_2^i : 2 \leq i \leq d-2\}$. Since

$$\gamma_F(x_1^2) = \Bbbk\, x_1\, x_2^{d-1}, \quad \gamma_F(x_1\, x_2) = \Bbbk\, (x_1^d - x_2^d), \quad \gamma_F(x_2^2) = \Bbbk\, x_1^{d-1}\, x_2,$$

we deduce that $h_F$ is surjective. This shows (ii) and completes the proof. □

A similar calculation shows that if $F = x_1^2\,(x_1^{d-2} + x_2^{d-2})$, then $\mathrm{im}(h_{\mathbb{F}}) = \mathrm{Span}\,\{x_1^{d-i}\, x_2^i : 0 \leq i \leq d-1\}$. Hence $h_F$ has rank $d$ for a general $F \in X_{(2,1^{d-2})}$. Let $\mathcal{E}_\Delta$ be the evectant of $\Delta$ (see §2.3), and define the map

$$e_F : S_d \longrightarrow S_0, \quad A \longrightarrow (A, \mathcal{E}_\Delta)_d.$$

**Lemma 3.2**

*The composites*

$$e_F \circ \beta_F : S_{d-4} \longrightarrow S_0, \quad e_F \circ \gamma_F : S_2 \longrightarrow S_0$$

*are zero.*

*Proof.* Since $e_F \circ \beta_F$ is of degree $(2d-1)$ in the coefficients of $F$, it corresponds to an $SL_2$-equivariant map $S_{d-4} \longrightarrow R_{2d-1}$. Said differently, there exists a covariant $C$ of $d$-ics of degree-order $(2d-1, d-4)$ such that $e_F \circ \beta_F(A) = (A, C)_{d-4}$. Similarly, there is a $C'$ of degree-order $(2d-2, 2)$ such that $e_F \circ \gamma_F(A) = (A, C')_2$.

We will show that if $F \in X_{(2,1^{d-2})}$, then $e_F \circ \beta_F = e_F \circ \gamma_F = 0$. This will imply that each coefficient of $C$ or $C'$ vanishes on $X_{(2,1^{d-2})}$, and hence must be divisible by $\Delta_{\mathbb{F}}$. The quotients $C/\Delta_{\mathbb{F}}, C'/\Delta_{\mathbb{F}}$ are of degree-orders $(1, d-4)$ and $(0, 2)$ respectively. Since there are no such nonzero covariants, $C$ and $C'$ must be zero.

Let $x_1^2$ be a factor of $F$. By [12, Chapter 12, formula (1.28)] (also see [21, Article 96]), we have $\mathcal{E}_\Delta = \Bbbk\, x_1^d$. Any form $B$ in the image of $\beta_F$ or $\gamma_F$ is divisible by $x_1$, hence $(B, \mathcal{E}_\Delta)_d = (B, x_1^d)_d = 0$. This completes the proof. □

**3.1** Now consider the map

$$\beta_{\mathbb{F}} \oplus \gamma_{\mathbb{F}} : S_{d-4} \oplus S_2 \longrightarrow S_d,$$

or what is the same, the corresponding map of graded $R$-modules

$$R(-2) \otimes S_{d-4} \oplus R(-1) \otimes S_2 \longrightarrow R \otimes S_d. \tag{4}$$

Let $M$ denote its $d \times (d+1)$ matrix with respect to the natural monomial bases.

**Lemma 3.3**

*The ideal of maximal minors of $M$ equals $J$ (the Jacobian ideal of $\Delta$).*

*Proof.* Let $W$ denote the image of 1 via the map

$$\wedge^d (\beta_{\mathbb{F}} \oplus \gamma_{\mathbb{F}}) : \mathbb{C} \longrightarrow \wedge^d S_d \simeq S_d.$$

By construction $W$ is a covariant of degree-order $(2d-3, d)$ whose coefficients are exactly the maximal minors. Let $\{A_1, \dots, A_d\}$ span $\mathrm{im}(\beta_{\mathbb{F}} \oplus \gamma_{\mathbb{F}})$. On the one hand, $W$ is the Wronskian of the $A_i$, hence it is (up to scalar) the unique $d$-ic which is apolar to all the $A_i$ (see [14, Appendix II]). On the other hand, $(A_i, \mathcal{E}_\Delta)_d = 0$ by the lemma above. Hence $W = \Bbbk\, \mathcal{E}_\Delta$. □

The subvariety of $\mathbb{P}^d$ defined by $J$ is codimension 2, hence the Eagon-Northcott complex (or what is the same in this case, the Hilbert-Burch complex) of the map (4) resolves $J$ (see [6, Chapter 16 F]). We have proved the following:

**Theorem 3.4**

*The ideal $J$ is perfect of height 2 with $SL_2$-equivariant minimal resolution*

$$\begin{aligned} 0 \leftarrow R/J \leftarrow R \leftarrow R(3-2d) \otimes S_d& \\ \leftarrow R(2-2d) \otimes S_2 \oplus R(1-2d) \otimes S_{d-4} \leftarrow 0.& \end{aligned}$$

**3.2** The first syzygy modules $S_2, S_{d-4}$ correspond to systems of $SL_2$-equivariant differential equations for $\Delta$, we proceed to make these equations explicit. For all $A \in S_2$, we have $((A, \mathbb{F})_1, \mathcal{E}_\Delta)_d = 0$. Using classical symbolic calculus (see [14, Chapter I]), let

$$A = \alpha_{\mathbf{x}}^2, \quad \mathbb{F} = f_{\mathbf{x}}^d, \quad \mathcal{E}_\Delta = e_{\mathbf{x}}^d.$$

Then $(A, \mathbb{F})_1 = (\alpha\, f)\, \alpha_{\mathbf{x}}\, f_{\mathbf{x}}^{d-1}$, and

$$\begin{aligned} ((A, \mathbb{F})_1, \mathcal{E}_\Delta)_d &= (\alpha\, f)(\alpha\, e)(f\, e)^{d-1} = (\alpha_{\mathbf{x}}^2, (f\, e)^{d-1}\, f_{\mathbf{x}}\, e_{\mathbf{x}})_2 \\ &= (A, (\mathbb{F}, \mathcal{E}_\Delta)_{d-1})_2 = 0. \end{aligned}$$

Since $(\mathbb{F}, \mathcal{E}_\Delta)_{d-1}$ is apolar to every order 2 form, it must be identically zero.

**3.3** In fact we have an identity $(\mathbb{F}, \mathcal{E}_I)_{d-1} = 0$ for any invariant. This can be informally explained as follows: $I$ is left unchanged by all $g \in SL_2$, hence it is annihilated by the Lie algebra $\mathfrak{sl}_2$. Now observe that $\mathfrak{sl}_2$ (as the adjoint $SL_2$-representation) is isomorphic to $S_2$. The standard generators

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively give the equations (cf. [22, Theorem 4.5.2])

$$\sum_{i=0}^{d} (d-i)\, a_{i+1} \frac{\partial I}{\partial a_i} = \sum_{i=0}^{d} i\, a_{i-1} \frac{\partial I}{\partial a_i} = \sum_{i=0}^{d} (d-2i)\, a_i \frac{\partial I}{\partial a_i} = 0.$$

**3.4** Similarly we have a $(d-3)$-dimensional family of differential equations for $\Delta$ coming from the module $S_{d-4}$. We will express it in a form involving only the quadratic covariants of $\mathbb{F}$. As before,

$$([(A, \mathbb{B}_{\mathbb{F}})_{d-4}]_{\mathbf{y}=\mathbf{x}}, \mathcal{E}_\Delta)_d = 0 \quad \text{for all } A \in S_{d-4}.$$

Let $A = \alpha_{\mathbf{x}}^{d-4}$, $\mathbb{B}_F = b_{\mathbf{x}}^{d-2}\, {b'_{\mathbf{y}}}^{d-2}$ where $b, b'$ are equivalent letters. Then

$$\begin{aligned}([(A, \mathbb{B}_{\mathbb{F}})_{d-4}]_{\mathbf{y}=\mathbf{x}}, \mathcal{E}_\Delta)_d &= ((\alpha\, b)^{d-4}\, {b_{\mathbf{x}}}^2\, {b'_{\mathbf{x}}}^{d-2}, e_{\mathbf{x}}^d)_d \\ &= (\alpha\, b)^{d-4}\, (b\, e)^2 (b'\, e)^{d-2} = (A, b_{\mathbf{x}}^{d-4}\, (b\, e)^2 (b'\, e)^{d-2})_{d-4} = 0,\end{aligned}$$

hence

$$b_{\mathbf{x}}^{d-4}\, (b\, e)^2\, (b'\, e)^{d-2} = 0. \tag{5}$$

Let $\mathbf{x}\, \partial_{\mathbf{y}} = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2}$, usually called the polarization operator. Then $(\mathbf{x}\, \partial_{\mathbf{y}})^2 \circ \mathbb{B}_{\mathbb{F}} = (d-2)\,(d-1)\, b_{\mathbf{x}}^{d-2}\, {b'_{\mathbf{x}}}^2\, {b'_{\mathbf{y}}}^{d-4}$, hence identity (5) is the same as

$$(\mathcal{E}_\Delta, (\mathbf{x}\, \partial_{\mathbf{y}})^2 \circ \mathbb{B}_{\mathbb{F}})_d = 0. \tag{6}$$

Let us write $(\mathbb{F}, \mathbb{F})_{2r} = {\tau_{\mathbf{x}}^{(2r)}}^{2d-4r}$ for the even quadratic covariants. We have a Gordan series (see [14, p. 55])

$$\mathbb{F}(\mathbf{x})\, \mathbb{F}(\mathbf{y}) = \sum_{r=0}^{[\frac{d}{2}]} c_r\, (\mathbf{x}\, \mathbf{y})^{2r}\, {\tau_{\mathbf{x}}^{(2r)}}^{d-2r}\, {\tau_{\mathbf{y}}^{(2r)}}^{d-2r},$$

where $c_r = \frac{\binom{d}{2r}^2}{\binom{2d-2r+1}{2r}}$. Apply the operator

$$\Omega = \frac{\partial^2}{\partial x_1\, \partial y_2} - \frac{\partial^2}{\partial x_2\, \partial y_1},$$

and divide by $(\mathbf{x}\, \mathbf{y})$, then we get an expansion

$$\mathbb{B}_{\mathbb{F}} = \frac{\Omega \circ \mathbb{F}(\mathbf{x})\, \mathbb{F}(\mathbf{y})}{(\mathbf{x}\, \mathbf{y})} = \sum_{r=1}^{[\frac{d}{2}]} c_r\, (2r)\, (2d-2r+1)\, (\mathbf{x}\, \mathbf{y})^{2r-2}\, {\tau_{\mathbf{x}}^{(2r)}}^{d-2r}\, {\tau_{\mathbf{y}}^{(2r)}}^{d-2r}.$$

Apply $(\mathbf{x}\, \partial_{\mathbf{y}})^2$ to each term, which amounts to replacing the expression $(\mathbf{x}\, \mathbf{y})^{2r-2}\, {\tau_{\mathbf{x}}^{(2r)}}^{d-2r}\, {\tau_{\mathbf{y}}^{(2r)}}^{d-2r}$ with

$$(d-2r)(d-2r-1)\, (\mathbf{x}\, \mathbf{y})^{2r-2}\, {\tau_{\mathbf{x}}^{(2r)}}^{d-2r+2}\, {\tau_{\mathbf{y}}^{(2r)}}^{d-2r-2}.$$

Now apply $(\mathcal{E}_\Delta, -)_d$ to each term, then

$$\begin{aligned}&(\epsilon_{\mathbf{x}}^d, (\mathbf{x}\, \mathbf{y})^{2r-2}\, {\tau_{\mathbf{x}}^{(2r)}}^{d-2r+2}\, {\tau_{\mathbf{y}}^{(2r)}}^{d-2r-2})_d = \epsilon_{\mathbf{y}}^{2r-2}\, (\epsilon\, \tau)^{d-2r+2}\, {\tau_{\mathbf{y}}^{(2r)}}^{d-2r-2} \\ &[(\epsilon_{\mathbf{x}}^d, {\tau_{\mathbf{x}}^{(2r)}}^{2d-4r})_{d-2r+2}]_{\mathbf{x}=\mathbf{y}}.\end{aligned}$$

Hence finally we deduce the identity

$$\sum_{r=1}^{[\frac{d-2}{2}]} \xi_r \, (\mathcal{E}_\Delta, (\mathbb{F}, \mathbb{F})_{2r})_{d-2r+2} = 0, \tag{7}$$

where

$$\xi_r = \frac{(2d-4r+1)!}{(2r-1)!(d-2r-2)!(d-2r)!(2d-2r)!}.$$

**3.5 The degree of the Jacobian scheme.** Let $Z = \operatorname{Proj}(R/J)$. It is the scheme-theoretic degeneracy locus where the morphism

$$S_d \otimes \mathcal{O}_{\mathbb{P}^d} \longrightarrow S_2 \otimes \mathcal{O}_{\mathbb{P}^d}(1) \oplus S_{d-4} \otimes \mathcal{O}_{\mathbb{P}^d}(2)$$

has rank $\leq d-1$. Hence, by the Porteous formula (see [3, Chapter II, §4]) the degree of $Z$ is given by the coefficient of $h^2$ in $(1+h)^{-3}(1+2h)^{3-d}$, which is $2\,d\,(d-2)$. By Hilbert's formula in §2.4,

$$\deg X_\tau = 3\,(d-2), \quad \deg X_\delta = 2\,(d-2)\,(d-3).$$

**Proposition 3.5**

*The scheme $Z$ has multiplicities 2 and 1 along $X_\tau$ and $X_\delta$ respectively.*

This means, for instance, that if $\eta_\tau$ is the scheme-theoretic generic point of $X_\tau$, then the ring $\mathcal{O}_{Z,\eta_\tau}$ is of length 2.

*Proof.* If the multiplicities are $a, b$, then $\deg Z = a\,\deg X_\tau + b\,\deg X_\delta$, i.e.,

$$2d\,(d-2) = 3a\,(d-2) + 2b\,(d-2)\,(d-3).$$

We have obvious constraints $a, b \geq 1$, and then it is straightforward to check that $(a, b) = (2, 1)$ is the only possible solution. □

## 4. Examples

In this section we will describe $J$ and its primary decomposition for $d \leq 5$. In each case the minimal system of generators for the ring of covariants was calculated in the nineteenth century (see [14, Chapter V, VII]). If $C$ is a covariant of $d$-ics, then $\mathfrak{I}(C) \subseteq R$ will denote the graded ideal generated by the coefficients of $C$.

**4.1 Cubics and quadratics.** So far we had assumed $d \geq 4$. The case $d = 3$ is a little exceptional, but rather easy. In this case $Z$ is a non-reduced scheme of degree 6 supported on the twisted cubic curve $X_{(3)}$. The minimal system of cubics consists of $\mathbb{F}, \mathbb{H}, \mathbb{T}$, and $\Delta = (\mathbb{T}, \mathbb{F})_3$, i.e., every covariant is a polynomial function in these. It is immediate that $\mathcal{E}_\Delta = \mathbb{T}$, and Theorem 3.4 is true as stated with the convention that $S_{-1} = 0$. Thus we have a resolution

$$0 \leftarrow R/J \leftarrow R \leftarrow R(-3) \otimes S_3 \leftarrow R(-4) \otimes S_2 \leftarrow 0.$$

The ideal of $X_{(3)}$ is $\mathfrak{I}(\mathbb{H})$ (cf. [11, Exercise 11.32]), hence we have an equality $\mathfrak{I}(\mathbb{H}) = \sqrt{\mathfrak{I}(\mathbb{T})}$.

For $d = 2$, we have $\Delta = (\mathbb{F}, \mathbb{F})_2$ and $\mathcal{E}_\Delta = \mathbb{F}$, i.e., $J = (a_0, a_1, a_2)$ is the irrelevant maximal ideal.

**4.2 Quartics.** Define $i = (\mathbb{F}, \mathbb{F})_4$, $j = (\mathbb{F}, \mathbb{H})_4$, which are invariants of degrees $2, 3$. The minimal system for $d = 4$ consists of $\mathbb{F}, \mathbb{H}, \mathbb{T}, i$ and $j$. Let $\mathfrak{P}_\tau, \mathfrak{P}_\delta \subseteq R$ denote the ideals of $X_{(3,1)}$ and $X_{(2,2)}$ respectively.

**Proposition 4.1**

(a1) *We have identities*

$$\Delta_{\mathbb{F}} = i^3 - 6\, j^2, \quad \mathcal{E}_\Delta = i^2\, F - 6\, j\, \mathbb{H}.$$

(a2) $\mathfrak{P}_\tau$ *is the complete intersection ideal* $(i, j)$, *and* $\mathfrak{P}_\delta = \mathfrak{I}(\mathbb{T})$.

(a3) *We have a primary decomposition* $J = (i^2, j) \cap \mathfrak{P}_\delta$.

*Proof.* Since $\Delta$ is of degree 6, it must be a linear combination of $i^3$ and $j^2$, say $c_1\, i^3 + c_2\, j^2$. Specialise to $F = x_1^2\, x_2\, (x_1 + x_2)$, when $\Delta_F$ must vanish. Computing directly, we get the equation $\frac{c_1}{216} + \frac{c_2}{1296} = 0$, hence $c_1 : c_2 = 1 : -6$, i.e., we may take $\Delta = i^3 - 6\, j^2$. Differentiating this identity, we get

$$\mathcal{E}_\Delta = \frac{1}{6}\,(3\, i^2 \times 2\, \mathcal{E}_i - 12\, j \times 3\, \mathcal{E}_j).$$

But $\mathcal{E}_j = \mathbb{H}$ and $\mathcal{E}_i = \mathbb{F}$, hence it equals $i^2\, \mathbb{F} - 6\, j\, \mathbb{H}$. This proves (a1).

Since $X_{(3,1)}$ is exactly the locus of nullforms, it is characterized by the vanishing of all invariants, i.e., $i = j = 0$ at $F \iff F \in X_{(3,1)}$. Since the ideal $(i, j)$ has no embedded primes, it must be $\mathfrak{P}_\tau$-primary. But since it also has degree $6\,(= \deg \mathfrak{P}_\tau)$, we get $(i, j) = \mathfrak{P}_\tau$.

In [1, Theorem 1.4] it is proved that the ideal of every CR-locus of the type $X_{(a,a)}$ is generated in degree 3. It follows from the set-up described there that the degree 3 piece $(\mathfrak{P}_\delta)_3$ is the kernel of the surjective morphism

$$S_3(S_4) \longrightarrow S_3(S_2 \otimes S_2) \longrightarrow S_3(S_2) \otimes S_3(S_2) \longrightarrow S_6 \otimes S_6 \longrightarrow S_2(S_6).$$

We have plethysm decompositions

$$S_3(S_4) = S_{12} \oplus S_8 \oplus S_6 \oplus S_4 \oplus S_0, \quad S_2(S_6) = S_{12} \oplus S_8 \oplus S_4 \oplus S_0,$$

hence $(\mathfrak{P}_\delta)_3 \simeq S_6$. This subrepresentation must correspond to $\mathbb{T}$, since up to scalar it is the only covariant of degree-order $(3, 6)$. This implies that $\mathfrak{P}_\delta = \mathfrak{I}(\mathbb{T})$.

To prove (a3), let $J = \mathfrak{q}_\tau \cap \mathfrak{q}_\delta$ be the (necessarily unique) primary decomposition, such that $\mathfrak{q}_\star$ is $\mathfrak{P}_\star$-primary. (See [4, Chapter 4] for generalities on primary decomposition.) Since $J$ has multiplicity one along $X_{(2,2)}$, we have $\mathfrak{q}_\delta = \mathfrak{P}_\delta$. Note that $(i^2, j)$ is $\mathfrak{P}_\tau$-primary (since it is perfect and its radical is $\mathfrak{P}_\tau$), moreover the expression for $\mathcal{E}_\Delta$ in (a1) shows that $J \subseteq (i^2, j)$. This implies that $\mathfrak{q}_\tau \subseteq (i^2, j)$, and it only remains to show the opposite inclusion. Let $z$ be any of the coefficients of $\mathbb{T}$, then

$$(J : z) = (\mathfrak{q}_\tau : z) \cap (\mathfrak{P}_\delta : z).$$

Now $z \notin \mathfrak{P}_\tau$, hence $(\mathfrak{q}_\tau : z) = \mathfrak{q}_\tau$. Since $(\mathfrak{P}_\delta : z) = R$, we have $(J : z) = \mathfrak{q}_\tau$. From (a1),

$$(\mathcal{E}_\Delta, \mathbb{H})_1 = (i^2\, \mathbb{F} - 6\, j\, \mathbb{H}, \mathbb{H})_1 = i^2\, (\mathbb{F}, \mathbb{H})_1 - 6\, j\, (\mathbb{H}, \mathbb{H})_1 = i^2\, \mathbb{T},$$

and similarly $(\mathcal{E}_\Delta : \mathbb{F})_1 = 6\, j\, \mathbb{T}$. It follows that $i^2\, z, j\, z \in J$, implying $i^2, j \in \mathfrak{q}_\tau$. This completes the proof of the proposition. □

The identity (7) of §3.4 reduces to $(\mathcal{E}_\Delta, \mathbb{H})_4 = 0$, which gives the differential equation

$$(2\, a_0\, a_2 - 2\, a_1^2)\, \frac{\partial \Delta}{\partial a_0} + (a_0\, a_3 - a_1\, a_2)\, \frac{\partial \Delta}{\partial a_1} + \Big(\frac{2}{3}\, a_1\, a_3 - a_2^2 + \frac{1}{3}\, a_0\, a_4\Big)\, \frac{\partial \Delta}{\partial a_2}$$
$$+ (a_1\, a_4 - a_2\, a_3)\, \frac{\partial \Delta}{\partial a_3} + (2\, a_2\, a_4 - 2\, a_3^2)\, \frac{\partial \Delta}{\partial a_4} = 0.$$

**4.3 Quintics.** The invariant theory of the binary $d$-ic rapidly becomes more complicated with increasing $d$, in particular it is progressively harder to calculate $J$ precisely. In this section we will complete the calculation for $d = 5$, making heavy use of machine computations in Maple and Macaulay-2. The minimal system is given on [14, p. 131]. (Since it has 23 members, it will not be reproduced here.) For quintics, the number of linearly independent covariants of degree-order $(m, q)$ is the number of copies of $S_q$ in the plethysm $S_m(S_5)$. We wrote our own set of Maple procedures based on the Cayley-Sylvester formula (see [22, Corollary 4.2.8]) to decompose it into irreducible summands.

In addition to $\mathbb{H}$ and $\mathbb{T}$, we have covariants $i = (\mathbb{F}, \mathbb{F})_4$, $A = (i, i)_2$ of degree-orders $(2, 2), (4, 0)$ respectively. Define

| | degree-order |
|---|---|
| $C_1 = 15\, (i, \mathbb{H})_2 + 2\, i^2$ | $(4, 4)$ |
| $C_2 = 770\, (i, \mathbb{F}\, \mathbb{H})_2 - 675\, (i, (\mathbb{F}, \mathbb{H})_1)_1 + 198\, i^2\, \mathbb{F}$ | $(5, 9)$ |
| $D_1 = -21\, (C_1, \mathbb{F}^2)_4 + 55\, (C_1, \mathbb{H})_2 + 14\, C_1\, i$ | $(6, 6)$ |
| $D_2 = 5\, (C_1, \mathbb{H})_4 + 4\, (C_1, i)_2$ | $(6, 2)$ |

**Proposition 4.2**

(b1) *We have identities*

$$\Delta = 59\, A^2 + 320\, (i^3, \mathbb{H})_6,$$
$$\mathcal{E}_\Delta = \frac{25}{3}\, A\, (i, \mathbb{F})_1 + \frac{3400}{21}\, i\, (i^2, \mathbb{F})_3 - 240\, (i^2, (\mathbb{F}, \mathbb{H})_1)_4.$$

(b2) *If $\mathfrak{P}_\tau, \mathfrak{P}_\delta$ denote the ideals of $X_\tau, X_\delta$ respectively, then*

$$\mathfrak{P}_\tau = \mathfrak{I}(C_1, A), \quad \mathfrak{P}_\delta = \mathfrak{I}(C_2).$$

(b3) *We have a primary decomposition*

$$J = \mathfrak{q}_\tau \cap \mathfrak{P}_\delta,$$

*where $\mathfrak{q}_\tau = \mathfrak{I}(D_1, D_2)$ is $\mathfrak{P}_\tau$-primary.*

*Proof.* The minimal system shows that there are only two independent invariants in degree 8, namely $A^2$ and $(i^3, \mathbb{H})_6$. Hence $\Delta = c_1\, A^2 + c_2\,(i^3, \mathbb{H})_6$ for some $c_i$. Specialise to $F = x_1^2\, x_2\,(x_1+x_2)\,(x_1-x_2)$ (when $\Delta$ must vanish), then we get $320\, c_1 - 59\, c_2 = 0$. Similarly $A\,(i, \mathbb{F})_1, i\,(i^2, \mathbb{F})_3, (i^2, (\mathbb{F}, \mathbb{H})_1)_4$ form a basis of covariants of degree-order $(7, 5)$, hence $\mathcal{E}_\Delta$ must be their linear combination. We can find the coefficients by specialisation as before, and this establishes the formulae in (b1).

First we determine the generators of $\mathfrak{P}_\tau$ using the recipe of [7, §3.1]. Write

$$\sum_{i=0}^{5} \binom{5}{i} a_i\, x_1^{5-i}\, x_2^i = (b_1\, x_1 + b_2\, x_2)^3\, (c_0\, x_1^2 + 2\, c_1\, x_1\, x_2 + c_2\, x_2^2)$$

(where $a, b, c$ are indeterminates), and equate the coefficients. This defines a ring morphism

$$\mathbb{C}[a_0, \dots, a_5] \longrightarrow \mathbb{C}[b_1, b_2, c_0, c_1, c_2],$$

whose kernel is $\mathfrak{P}_\tau$. A computation (done in Macaulay-2) shows that all the ideal generators are in degree 4, and $\dim (\mathfrak{P}_\tau)_4 = 6$. Now $A$ (being an invariant) must vanish on $X_\tau$, hence $(\mathfrak{P}_\tau)_4$ has $S_0$ as a summand. The module $S_4(S_5)$ contains no copies of $S_i$ for $0 < i < 4$, and 2 copies of $S_4$. Hence $(\mathfrak{P}_\tau)_4$ must be isomorphic to $S_0 \oplus S_4$ as an $SL_2$-representation. The order 4 piece (to be called $C_1$) must be a linear combination of $(i, \mathbb{H})_2$ and $i^2$, because the latter form a basis in degree-order $(4, 4)$. Then we determine the actual coefficients as before by specialising $F$ to $x_1^3\, x_2\,(x_1+x_2)$.

A similar computation shows that $\mathfrak{P}_\delta$ is generated by a 10-dimensional vector subspace of $R_5$. Notice that $X_\delta \supseteq X_{(4,1)}$, and by [10], the ideal of $X_{(4,1)}$ equals $\mathfrak{I}(i)$. Thus we have an inclusion $\mathfrak{P}_\delta \subseteq \mathfrak{I}(i)$; this implies that each degree 5 covariant vanishing on $X_\delta$ must be a linear combination of terms of the form $(i, \Phi)_k$ for some degree 3 covariant $\Phi$. (This follows because the vector space $(\mathfrak{I}(i))_5$ is spanned by such terms.) Clearly $0 \le k \le 2$. Now $S_3(S_5) \simeq S_{15} \oplus S_{11} \oplus S_9 \oplus S_7 \oplus S_5 \oplus S_3$, corresponding to the cases

$$\Phi = \mathbb{F}^3,\ \mathbb{F}\,\mathbb{H},\ (\mathbb{F}, \mathbb{H})_1,\ i\,\mathbb{F},\ (i, \mathbb{F})_1,\ (i, \mathbb{F})_2.$$

This allows us to write down all the possibilities for $(i, \Phi)_k$. An exhaustive search shows that $C_2$ is the only linear combination which vanishes on $F = x_1^2\, x_2^2\,(x_1 + x_2)$. This proves (b2).

The $\mathfrak{P}_\delta$-primary component of $J$ is $\mathfrak{P}_\delta$ itself. Let $w$ denote the coefficient of $x_1^9$ in $C_2$, then $\mathfrak{q}_\tau$ (the $\mathfrak{P}_\tau$-primary component) equals the colon ideal $(J : w)$. We calculated the latter in Macaulay-2, and found it to have 10 generators in degree 6, and 12 first syzygies in degree 7. Hence we have a resolution

$$0 \leftarrow R/\mathfrak{q}_\tau \leftarrow R \leftarrow R(-6) \otimes M_{10} \leftarrow R(-7) \otimes M_{12} \leftarrow \dots$$

where $M_r$ denotes an $r$-dimensional $SL_2$-representation. Now

$$S_6\,(S_5) = S_2^{\oplus 2} \oplus S_4 \oplus S_6^{\oplus 4} \oplus S_8^{\oplus 2} \oplus \text{summands } S_i \text{ with } i \geq 10,$$

hence the dimension count forces $M_{10} \simeq S_6 \oplus S_2$. Let $D_1, D_2$ denote the corresponding covariants of orders 6 and 2. Since $\mathfrak{q}_\tau \subseteq \mathfrak{P}_\tau$, each $D_i$ can be written as a sum of terms of the form $(C_1, \Psi)_k, A\,\Psi'$, where $\Psi, \Psi'$ are of degree 2. Thus we may write

$$D_1 = \alpha_1\,(C_1, \mathbb{F}^2)_4 + \alpha_2\,(C_1, \mathbb{H})_2 + \alpha_3\,C_1\,i,$$
$$D_2 = \beta_1\,(C_1, \mathbb{H})_4 + \beta_2\,(C_1, i)_2,$$

for some $\alpha_i, \beta_j \in \mathbb{Q}$. (The terms $A\,\mathbb{H}$ and $A\,i$ are not needed, because a calculation shows that they are respectively equal to

$$\frac{3}{25}\,(C_1, \mathbb{F}^2)_4 - \frac{1}{25}\,(C_1, \mathbb{H})_2 + \frac{162}{875}\,C_1\,i, \quad \frac{18}{25}\,(C_1, \mathbb{H})_4 - \frac{48}{125}\,(C_1, i)_2.)$$

Since $J \subseteq \mathfrak{q}_\tau$, we must have

$$\mathcal{E}_\Delta = \gamma_1\,(D_1, F)_3 + \gamma_2\,(D_2, F)_1$$

for some $\gamma_i \in \mathbb{Q}$. When rewritten in terms of the basis elements $A\,(i, \mathbb{F})_1, i\,(i^2, \mathbb{F})_3$, $(i^2, (\mathbb{F}, \mathbb{H})_1)_4$ for covariants of degree-order $(7, 5)$, this becomes an inhomogeneous system of three linear equations. It turns out that there is a two-dimensional family of solutions, and the general solution can be written as

$$\begin{aligned}&(\gamma_1\,\alpha_1, \gamma_1\,\alpha_2, \gamma_1\,\alpha_3, \gamma_2\,\beta_1, \gamma_2\,\beta_2)\\ &= (-\frac{3}{5} - s + \frac{5}{4}\,t,\ 5 - \frac{5}{3}\,s - \frac{25}{6}\,t, -\frac{2}{7} - \frac{8}{7}\,s + \frac{75}{28}\,t,\ s, t).\end{aligned}$$

In order to determine $s, t$, we need to look at the first syzygies of $\mathfrak{q}_\tau$. Since they are all linear, $M_{12}$ must be a submodule of

$$M_{10} \otimes S_5 \simeq (S_6 \oplus S_2) \otimes S_5 \simeq S_{11} \oplus S_9 \oplus S_7^{\oplus 2} \oplus S_5^{\oplus 2} \oplus S_3^{\oplus 2} \oplus S_1.$$

By a dimension count, there are only four possible choices for $M_{12}$, it can only be $S_{11}, S_5^{\oplus 2}, S_5 \oplus S_3 \oplus S_1$ or $S_7 \oplus S_3$. It cannot be $S_{11}$ since the corresponding covariant is divisible by $\mathbb{F}$, and cancelling the latter would imply the absurdity that there is a first syzygy in degree 6. If $S_5 \subseteq M_{12}$ (i.e., if there were a syzygy in order 5), then there would be a nontrivial identity of the form $\eta_1\,(D_1, \mathbb{F})_3 + \eta_2\,(D_2, \mathbb{F})_1 = 0$. A calculation shows that there is none, this rules out all but the last choice. Thus $S_7 \subseteq M_{12}$, i.e., we have an identity of the form

$$\eta_1\,(D_1, \mathbb{F})_2 + \eta_2\,D_2\,\mathbb{F} = 0.$$

Indeed, it turns out that $(s, t) = (\frac{24}{35}, \frac{96}{175}), \eta_1/\eta_2 = 4$ is the unique nontrivial solution. Finally we choose $\gamma_1 = \frac{1}{35}, \gamma_2 = \frac{24}{175}$, so that $D_1, D_2$ acquire integer coefficients. The proposition is proved. □

It would be of interest to have a general result describing the primary decomposition of $J$ for all $d$, but this appears inaccessible.

**4.4** Not every invariant of binary forms has a perfect Jacobian ideal. E.g., let $d = 4$ (with notation as in §4.2). Let us show that $\mathfrak{b} = \mathfrak{J}(\mathcal{E}_j)$ (the Jacobian ideal of $j$) is not perfect. Since $\mathcal{E}_j$ is a covariant of degree-order $(2, 4)$, it must coincide with $\mathbb{H}$ up to a scalar. The zero locus of $\mathfrak{b} = \mathfrak{J}(\mathbb{H})$ is the rational normal quartic curve, hence $\dim(R/\mathfrak{b}) = 2$. However we have an identical relation $(\mathbb{H}, \mathbb{F})_2 = \frac{1}{6}\, i\, \mathbb{F}$ (see [14, p. 92]), which implies that $i\,(a_0, \dots, a_4) \subseteq \mathfrak{b}$. Consequently $\mathfrak{b}$ is not a saturated ideal, and $\operatorname{depth}(R/\mathfrak{b}) = 0$.

## 5. The binary resultant

We begin with a recapitulation of the Cayley method of calculating the binary resultant (see [12, Chapter 2]). The reader may also consult [2] for variations on this theme. Let

$$\mathbb{F} = \sum_{i=0}^{d} \binom{d}{i} a_i\, x_1^{d-i}\, x_2^i, \quad \mathbb{G} = \sum_{j=0}^{e} \binom{e}{j} b_j\, x_1^{e-j}\, x_2^j,$$

denote generic binary forms of orders $d, e$. Define the product space $Y = \mathbb{P}S_d \times \mathbb{P}S_e \times \mathbb{P}S_1$ with projection maps $\mu_1, \mu_2, \pi$ onto the respective factors. Consider the subvariety

$$\widetilde{\Gamma} = \{(F, G, l) \in Y : l \text{ divides } F, G\} \subseteq Y.$$

Let $f = \mu_1 \times \mu_2$, then $\Gamma = f(\widetilde{\Gamma}) \subseteq \mathbb{P}^d \times \mathbb{P}^e$ is the resultant hypersurface.

For any integers $m, n, p$, let $\mathcal{O}_Y(m, n, p)$ denote the line bundle

$$\mu_1^* \,\mathcal{O}_{\mathbb{P}^d}(m) \otimes \mu_2^* \,\mathcal{O}_{\mathbb{P}^e}(n) \otimes \pi^* \,\mathcal{O}_{\mathbb{P}^1}(p),$$

with similar notation on $\mathbb{P}^d \times \mathbb{P}^e$. There is a tautological global section in

$$H^0(Y, \mathcal{O}_Y(1, 0, d)) = S_d \otimes S_d,$$

corresponding to the trace element $\mathbb{F}$, and similarly for $\mathbb{G}$. Both of these sections simultaneously vanish at $(F, G, l)$ iff $(F, l^d)_d = (G, l^e)_e = 0$, i.e., iff $l$ divides $F, G$. In fact we have a Koszul resolution

$$\begin{aligned} 0 \to \mathcal{O}_Y(-1, -1, -(d+e)) \to \mathcal{O}_Y(-1, 0, -d) \oplus \mathcal{O}_Y(0, -1, -e) \\ \to \mathcal{O}_Y \to \mathcal{O}_{\widetilde{\Gamma}} \to 0. \end{aligned}$$

Now tensor with $\mathcal{O}_Y(0, 0, d)$, and write this complex as

$$0 \to \mathcal{C}^{-2} \to \mathcal{C}^{-1} \to \mathcal{C}^0 \to \mathcal{O}_{\widetilde{\Gamma}}(0, 0, d) \to 0. \tag{8}$$

We have a second quadrant spectral sequence

$$\begin{aligned} &E_1^{p,q} = R^q f_* \,\mathcal{C}^p, \qquad d_r^{p,q} : E_r^{p,q} \to E_r^{p+r, q-r+1}, \\ &E_\infty^{p+q} \Rightarrow R^{p+q} f_* \,\mathcal{O}_{\widetilde{\Gamma}}(0, 0, d) \end{aligned} \tag{9}$$

in the range $p = 0, -1, -2$ and $q = 0, 1$.

**5.1** Now assume $d \geq e - 1$, and $e \geq 2$. The only nonzero $E_1$ terms are

$$E_1^{-2,1} = \mathcal{O}(-1,-1) \otimes S_{e-2}, \qquad E_1^{0,0} = \mathcal{O} \otimes S_d,$$
$$E_1^{-1,0} = \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \otimes S_{d-e}.$$

(Throughout $\mathcal{O}$ stands for $\mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^e}$.) It is immediate that $R^i f_* \mathcal{O}_{\widetilde{\Gamma}}(0,0,d) = 0$ for $i > 1$, moreover we have exact sequences

$$0 \to E_1^{-1,0} \to E_1^{0,0} \to E_2^{0,0} \to 0,$$
$$0 \to E_1^{-2,1} \stackrel{d_2^{-2,1}}{\to} E_2^{0,0} \to f_* \mathcal{O}_{\widetilde{\Gamma}}(0,0,d) \to 0.$$

**Lemma 5.1**

*The map $d_2^{-2,1}$ admits a unique $SL_2$-equivariant lifting (say $\vartheta$) to a map $E_1^{-2,1} \to E_1^{0,0}$.*

*Proof.* Indeed, the obstruction to this lift lies in the group

$$\text{Ext}^1(E_1^{-2,1}, E_1^{-1,0}) = H^1(\mathcal{O}(0,1) \otimes S_{e-2}) \oplus H^1(\mathcal{O}(1,0) \otimes S_{e-2} \otimes S_{d-e})$$

which is zero. Thus we have a surjection of $SL_2$-representations

$$\text{Hom}(E_1^{-2,1}, E_1^{0,0}) \to \text{Hom}(E_1^{-2,1}, E_2^{0,0}). \tag{10}$$

Since the construction of $d_2^{-2,1}$ is equivariant, it spans a copy of $S_0$ in the target of the map (10). By Schur's lemma it must come from an $S_0$ in the source, i.e., we have an equivariant lifting. If there were two such lifts, their difference would lie in

$$\begin{aligned}\text{Hom}(E_1^{-2,1}, E_1^{-1,0}) &= H^0(\mathcal{O}(0,1)) \otimes S_{e-2} \oplus H^0(\mathcal{O}(1,0)) \otimes S_{e-2} \otimes S_{d-e} \\ &= [S_e \otimes S_{e-2}] \oplus [S_d \otimes S_{e-2} \otimes S_{d-e}].\end{aligned}$$

However this is impossible; formula (2) from §2.1 shows that the last module does not contain *any* copy of $S_0$. $\square$

**5.2** Thus we get a map $E_1^{-2,1} \oplus E_1^{-1,0} \stackrel{\eta}{\longrightarrow} E_1^{0,0}$ of vector bundles of rank $d+1$ each, which can be seen as a map

$$S_{e-2} \oplus S_{d-e} \oplus S_0 \stackrel{\eta_{F,G}}{\longrightarrow} S_d \tag{11}$$

parametrised by points $(F, G) \in \mathbb{P}^d \times \mathbb{P}^e$. It fails to be bijective exactly over $\Gamma$. Now

$$\wedge^{d+1}\eta : \wedge^{e-1} E_1^{-2,1} \otimes \wedge^{d-e+2} E_1^{-1,1} \longrightarrow \wedge^{d+1} E_1^{0,0}$$

is the map $\mathcal{O}(-e,-d) \longrightarrow \mathcal{O}$, i.e., $\mathfrak{R} = \det \eta_{\mathbb{F},\mathbb{G}}$ is an invariant of degree $(e,d)$ in the coefficients of $\mathbb{F}, \mathbb{G}$ respectively. Hence $\mathfrak{R}$ must coincide with the resultant of $F, G$ (up to a scalar).

The maps $S_0 \longrightarrow S_d, S_{d-e} \longrightarrow S_d$ are respectively $1 \to F$, and $A \to AG$ for $A \in S_{d-e}$. The map $\vartheta : S_{e-2} \longrightarrow S_d$ is given by the *Morley form* which we describe below. Symbolically write $F = f_{\mathbf{x}}^d$, $G = g_{\mathbf{x}}^e$. Define a joint covariant of $F, G$ by the expression

$$\mathcal{M} = \sum_{i=1}^{e-1} (f\,g)\, f_{\mathbf{x}}^{i-1}\, g_{\mathbf{x}}^{e-i-1}\, f_{\mathbf{y}}^{d-i}\, g_{\mathbf{y}}^{i}.$$

It is of order $e-2$ and $d$ in $\mathbf{x}, \mathbf{y}$ respectively.

**Proposition 5.2**

*For $A = \alpha_{\mathbf{x}}^{e-2} \in S_{e-2}$, the image $\vartheta(A)$ is given by*

$$(-1)^{e-1}\, [(\mathcal{M}, A)_{e-2}]_{\mathbf{y}=\mathbf{x}} = -\sum_{i=1}^{e-1} (f\,g)\,(\alpha\, f)^{i-1}\,(\alpha\, g)^{e-i-1}\, f_{\mathbf{x}}^{d-i}\, g_{\mathbf{x}}^{i}\,. \tag{12}$$

The transvectant on the left hand side is with respect to $\mathbf{x}$-variables, treating the $\mathbf{y}$ as constants. The proof is postponed to §5.6.

**5.3** Now the rest of the argument is very similar to the discriminant case. (At this point we leave the details to the reader.) That is to say, if $l \in S_1$ divides $F, G$, then each form in the image of the map

$$S_{e-2} \oplus S_{d-e} \longrightarrow S_d$$

is divisible by $l$ (see Lemma 5.4 below), and the $\mathbb{F}$-evectant

$$\mathcal{E}_{\mathfrak{R}}^{(\mathbb{F})} = \sum_{i=0}^{d} \frac{\partial \mathfrak{R}}{\partial a_i}\, x_2^i\, (-x_1)^{d-i},$$

reduces to $\Bbbk\, l^d$. In conclusion, we get the following result:

**Theorem 5.3**

*The ideal*

$$J_{\mathbb{F}} = \Big(\frac{\partial \mathfrak{R}}{\partial a_0}, \dots, \frac{\partial \mathfrak{R}}{\partial a_d}\Big) \subseteq Q = \mathbb{C}[a_0, \dots, a_d, b_0, \dots, b_e]$$

*is perfect of height 2, with an equivariant bigraded minimal resolution*

$$\begin{aligned}0 \leftarrow Q/J_{\mathbb{F}} \leftarrow & Q \leftarrow Q(1-e, -d) \otimes S_d \leftarrow \\ & Q(1-e, -d-1) \otimes S_{d-e} \oplus Q(-e, -d-1) \otimes S_{e-2} \leftarrow 0.\end{aligned}$$

**5.4** The syzygy modules $S_{d-e}$ and $S_{e-2}$ respectively correspond to the identities

$$(\mathbb{G}, \mathcal{E}_{\mathfrak{R}}^{(\mathbb{F})})_e = 0, \quad (\mathcal{M}, \mathcal{E}_{\mathfrak{R}}^{(\mathbb{F})}|_{\mathbf{y}=\mathbf{x}})_d^{\mathbf{y}} = 0.$$

In the latter, we have changed $\mathcal{E}$ into a $\mathbf{y}$-form of order $d$. The transvection is with respect to $\mathbf{y}$-variables, leaving an $\mathbf{x}$-form of order $e-2$. We will rewrite this identity non-symbolically, in a form which only involves the joint covariants $(\mathbb{F},\mathbb{G})_r$. First we expand each term of $\mathcal{M}$ into its Gordan series (see [14, p. 55]), i.e., we write

$$f_{\mathbf{x}}^{i-1}\,g_{\mathbf{x}}^{e-i-1}\,f_{\mathbf{y}}^{d-i}\,g_{\mathbf{y}}^{i}=\sum_{s=0}^{e-2}\alpha_s\,(\mathbf{x}\,\mathbf{y})^s\,(\mathbf{y}\partial_{\mathbf{x}})^{d-s}\circ[(f_{\mathbf{x}}^{i-1}\,g_{\mathbf{x}}^{e-i-1},f_{\mathbf{x}}^{d-i}\,g_{\mathbf{x}}^{i})_s], \tag{13}$$

where

$$\alpha_s=\frac{\binom{d}{s}\binom{e-2}{s}}{\binom{d+e-s-1}{s}\binom{d+e-2s-2}{d-s}}.$$

Using the general formalism of [13, §3.2.5],

$$(f_{\mathbf{x}}^{i-1}\,g_{\mathbf{x}}^{e-i-1},f_{\mathbf{x}}^{d-i}\,g_{\mathbf{x}}^{i})_s=\beta_{i,s}\,(f\,g)^{s+1}\,f_{\mathbf{x}}^{d-s-1}\,g_{\mathbf{x}}^{e-s-1},$$

where

$$\beta_{i,s}=\frac{1}{\binom{d}{s}\binom{e-2}{s}s!}\sum_{l=0}^{s}(-1)^l\,l!(s-l)!\binom{i-1}{s-l}\binom{e-i-1}{l}\binom{d-i}{l}\binom{i}{s-l}.$$

Now $(f\,g)^{s+1}\,f_{\mathbf{x}}^{d-s-1}\,g_{\mathbf{x}}^{e-s-1}=(\mathbb{F},\mathbb{G})_{s+1}$, which we write symbolically as $\tau_{\mathbf{x}}^{d+e-2s-2}$. Then

$$(\mathbf{y}\partial_{\mathbf{x}})^{d-s}\circ\tau_{\mathbf{x}}^{d+e-2s-2}=\binom{d+e-2s-2}{d-s}\tau_{\mathbf{x}}^{e-s-2}\,\tau_{\mathbf{y}}^{d-s}.$$

Writing $\mathcal{E}_{\mathfrak{R}}^{(\mathbb{F})}|_{\mathbf{y}=\mathbf{x}}=\epsilon_{\mathbf{y}}^{d}$,

$$\begin{aligned}((\mathbf{x}\,\mathbf{y})^s\,\tau_{\mathbf{x}}^{e-s-2}\,\tau_{\mathbf{y}}^{d-s},\epsilon_{\mathbf{y}}^{d})_d^{\mathbf{y}}&=(-1)^s\,\epsilon_{\mathbf{x}}^{s}\,\tau_{\mathbf{x}}^{e-s-2}\,(\tau\,\epsilon)^{d-s}\\&=(-1)^d\,(\mathcal{E}_{\mathfrak{R}}^{(\mathbb{F})},(\mathbb{F},\mathbb{G})_{s+1})_{d-s}.\end{aligned}$$

Hence, by substituting into (13) we get the required identity

$$\sum_{s=0}^{e-2}\omega_s\,(\mathcal{E}_{\mathfrak{R}}^{(\mathbb{F})},(\mathbb{F},\mathbb{G})_{s+1})_{d-s}=0, \tag{14}$$

where $\omega_s=\binom{d+e-2s-2}{d-s}\,\alpha_s\sum_{i=1}^{e-1}\beta_{i,s}$.

**5.5** If $e=1$, then Theorem 5.3 is true as stated if we take $S_{-1}=0$. If $d-e<-1$, then the spectral sequence (9) has a nonzero term at $E_1^{-1,1}$. We still get a determinantal formula

$$\mathfrak{R}=\det(S_{e-2}\oplus S_0\xrightarrow{\eta'_{\mathbb{F},\mathbb{G}}}S_d\oplus S_{e-d-2}),$$

but $J$ may no longer be perfect. E.g., for $(d,e)=(2,4)$, a Macaulay-2 computation shows that $J$ is of height 2, but $\text{proj-dim}_Q\,(Q/J_{\mathbb{F}})=3$.

**5.6** Now we take up the proof of Proposition 5.2. For $i = 1, 2$, let $U_i = \{l \in S_1 : \frac{\partial l}{\partial x_i} \neq 0\} \subseteq \mathbb{P}^1$, and $\mathcal{U}_i = \pi^{-1}(U_i)$. We will calculate the differential $d_2^{-2,1}$ using a Čech resolution of the complex (8) for the cover $\mathcal{U}_i$. Let us write $\mathcal{S}_k^j$ as an abbreviation for $f_*(\mathcal{C}^j|_{\mathcal{U}_k})$, where $k$ may denote $1, 2$, or $12$. (As usual $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$.) On $\mathbb{P}^d \times \mathbb{P}^e$ we have a double complex of locally free sheaves

$$\begin{array}{ccccc} \mathcal{S}_{12}^{-2} & \xrightarrow{h_1} & \mathcal{S}_{12}^{-1} & \longrightarrow & \mathcal{S}_{12}^{0} \\ \uparrow & & {\scriptstyle h_2}\uparrow & & \uparrow \\ \mathcal{S}_1^{-2} \oplus \mathcal{S}_2^{-2} & \longrightarrow & \mathcal{S}_1^{-1} \oplus \mathcal{S}_2^{-1} & \xrightarrow{h_3} & \mathcal{S}_1^{0} \oplus \mathcal{S}_2^{0}\,. \end{array}$$

It will be convenient to see it as a diagram of morphisms of vector spaces parametrised by the pair $(F, G)$. Since expression (12) is linear in $A$, it is enough to show the proposition for a monomial $A$. Let $A = x_1^r\, x_2^{e-2-r}$.

The isomorphism $S_{e-2} \simeq S_{e-2}^*$ of §2.1 takes the form $A$ to $A' = (-1)^{e-2-r} \binom{e-2}{r} x_2^r\, x_1^{e-2-r}$, since $(A, A')_{e-2} = 1$. This implies that the sequence of isomorphisms

$$S_{e-2} \simeq S_{e-2}^* \simeq H^0(\mathbb{P}^1, \mathcal{O}(e-2))^* \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)) \simeq H^1(\mathbb{P}^1, \mathcal{O}(-e)),$$

takes $A$ to the Čech cocyle

$$\mathbb{A} = \frac{1}{A'} \times \frac{1}{x_1\, x_2} = \frac{(-1)^{e-2-r}}{\binom{e-2}{r}\, x_1^{e-1-r}\, x_2^{r+1}} \in H^0(U_{12}, \mathcal{O}(-e)).$$

Recall that by the usual procedure for calculating the differentials in a spectral sequence (see [5, §14]),

$$d_2^{-2,1}(\mathbb{A}) = h_3 \circ h_2^{-1} \circ h_1(\mathbb{A}).$$

(Throughout, the vector space morphisms over $(F, G)$ are also denoted by $h_i$.)

**5.7** By the construction of the Koszul complex, $h_1(\mathbb{A}) = F\,\mathbb{A} \oplus G\,\mathbb{A}$. To take the pre-image by $h_2$, we need to rewrite each of the summands as a difference $e^{(1)} - e^{(2)}$, where the denominator of $e^{(i)}$ is a power of $x_i$ alone. Write

$$F = \frac{(d-e+1)!}{d!} \left[ \Big( y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} \Big)^{e-1} F \right]_{\mathbf{y}=\mathbf{x}}.$$

Expand and retain only those terms whose power in $x_1$ is at least $e-1-r$, i.e., let

$$\hat{F} = \frac{(d-e+1)!}{d!} \sum_{q \geq e-1-r} \binom{e-1}{q} x_1^q\, x_2^{e-q-1} \frac{\partial^{e-1} F}{\partial x_1^q\, \partial x_2^{e-q-1}}.$$

Multiplying by $\mathbb{A}$, we get

$$\begin{aligned} e^{(2)} &= \widetilde{F}_r \\ &= \frac{(-1)^{e-2-r}\,(d-e+1)!}{x_2^{r+1} \binom{e-2}{r}\, d!} \sum_{q=e-r-1}^{e-1} \binom{e-1}{q} x_1^{q-e+r+1}\, x_2^{e-q-1} \frac{\partial^{e-1} F}{\partial x_1^q\, \partial x_2^{e-q-1}}, \end{aligned} \tag{15}$$

and then $e^{(1)} = F\,\mathbb{A} + \widetilde{F}_r$. Similarly, define

$$\widetilde{G}_r = \frac{(-1)^{e-2-r}}{x_2^{r+1}\binom{e-2}{r}\,e!}\sum_{q=e-r-1}^{e-1}\binom{e-1}{q}\,x_1^{q-e+r+1}\,x_2^{e-q-1}\,\frac{\partial^{e-1}G}{\partial x_1^q\,\partial x_2^{e-q-1}}\,. \tag{16}$$

Now $u = (F\,\mathbb{A} + \widetilde{F}_r, \widetilde{F}_r) \oplus (G\,\mathbb{A} + \widetilde{G}_r, \widetilde{G}_r)$ is an element such that $h_2(u) = F\mathbb{A} \oplus G\mathbb{A}$.

To calculate the image of $u$ by $h_3$, multiply the first summand by $G$, the second by $F$ and subtract the former from the latter. This gives

$$d_2^{-2,1}(\mathbb{A}) = h_3(u) = F\,\widetilde{G}_r - G\,\widetilde{F}_r. \tag{17}$$

(Note that we have used a hidden 'term order' where $F$ comes before $G$. As long as we remain consistent, this should cause no harm.)

It is not *a priori* obvious that the result is invariant under a change of variables, since the Čech cover is clearly not so invariant. On the other hand, expression (12) is entirely in terms of symbolic brackets, hence visibly invariant. Thus, to complete the proof, we have to establish the identity

$$F\,\widetilde{G}_r - G\,\widetilde{F}_r = (-1)^{e-1}\,[(\mathcal{M}, A)_{e-2}]_{\mathbf{y}=\mathbf{x}}. \tag{18}$$

This calculation is done in the appendix.

The following lemma was needed in §5.3.

**Lemma 5.4**

*If $l$ divides $F$ and $G$, then it divides $\vartheta(A)$ for any $A$.*

*Proof.* We may assume that $l = x_1$. Since $\vartheta$ is linear, it suffices to give a proof for a monomial $A$. But then the claim follows because $x_1$ clearly divides the left hand side of (18). □

## 6. The $\Phi_n$ are arithmetically Cohen-Macaulay

Let $\Phi_n \subseteq \mathbb{P}S_d$ be as in the introduction, with ideal $I_n \subseteq R$. We will exhibit $\Phi_n$ as the degeneracy locus of a map of vector bundles and then deduce that $I_n$ is perfect ideal. Along the way we will construct a covariant $\mathcal{A}_n$ of binary $d$-ics such that $F \in \Phi_n \iff \mathcal{A}_n(F) = 0$.

**6.1** Every $F \in S_d$ has a factorization

$$F = l_1^{e_1} \dots l_n^{e_n} \tag{19}$$

where the $l_i$ are pairwise nonproportional, and $e_1 \geq \cdots \geq e_n > 0$. Let

$$g_F = \gcd(F_{x_1}, F_{x_2}).$$

**Lemma 6.1**

*With notation as above, $g_F = \prod_i l_i^{e_i-1}$.*

*Proof.* Evidently $g = \prod l_i^{e_i - 1}$ divides both the $F_{x_i}$, write $F_{x_1} = g\,A, F_{x_2} = g\,B$. Divide Euler's equation $d\,F = x_1\,F_{x_1} + x_2\,F_{x_2}$ by $g$, then $d \prod l_i = x_1\,A + x_2\,B$. If $A, B$ have a common linear factor, it must be one the $l_i$, say $l_1$. But

$$A = \sum_i e_i \frac{\partial l_i}{\partial x_1} \Big( \prod_{j \neq i} l_j \Big),$$

so $l_1 | A$ implies $\frac{\partial l_1}{\partial x_1} = 0$. The same argument on $B$ leads to $\frac{\partial l_1}{\partial x_2} = 0$, so $l_1 = 0$. This is absurd, hence $A, B$ can have no common factor, i.e., $g = g_F$. □

**Corollary 6.2**

*Let $F \in S_d$. Then $F \in \Phi_n$ iff $\operatorname{ord} g_F \geq d - n$.*

**6.2** We have a map $S_d \otimes S_1 \longrightarrow S_{d-1}$ by formula (2), we may see it as a morphism of vector bundles $\mathcal{O}_{\mathbb{P}^d}(-1) \otimes S_1 \longrightarrow S_{d-1}$. Now consider the composite

$$\mathcal{O}_{\mathbb{P}^d}(-1) \otimes S_1 \otimes S_{n-1} \longrightarrow S_{d-1} \otimes S_{n-1} \xrightarrow{\text{mult}} S_{d+n-2},$$

which we denote by $\alpha_n$. On the fibres over $[F] \in \mathbb{P}^d$, this can be thought of as a morphism

$$\begin{aligned} \alpha_{n,F} : \; S_1 \otimes S_{n-1} &\longrightarrow S_{d+n-2}, \\ l \otimes G &\longrightarrow (l, F)_1\, G = \Bbbk\ (l_{x_1} F_{x_2} - l_{x_2} F_{x_1})\, G. \end{aligned}$$

Now $F_{x_1}, F_{x_2}$ have a common factor of order $\geq d - n$, iff there are order $n - 1$ forms $G_1, G_2$ such that $G_2\,F_{x_1} + G_1\,F_{x_2} = 0$. This condition can be rewritten as $\alpha_{n,F}(x_1 \otimes G_1 - x_2 \otimes G_2) = 0$. Hence $\alpha_{n,F}$ fails to be injective iff $F \in \Phi_n$.

Let $\Psi_n$ denote the determinantal scheme $\{\operatorname{rank}(\alpha_n) < 2n\}$ locally defined by the maximal minors of the matrix of $\alpha_{n,F}$. We have shown that $(\Psi_n)_{\text{red}} = \Phi_n$.

**Theorem 6.3**

*The scheme $\Psi_n$ is reduced, hence $\Psi_n = \Phi_n$ as schemes.*

*Proof.* The standard codimension estimate for determinantal loci (see [3, Chapter 2]) takes the form

$$\operatorname{codim} \Psi_n \leq d + n - 1 - (2n - 1) = d - n.$$

Since equality holds, $\Psi_n$ is a Cohen-Macaulay scheme, in particular it has no embedded components. By the Thom-Porteous formula,

$$\deg \Psi_n = (-1)^{d-n} \times \text{coefficient of } h^{d-n} \text{ in } (1 - h)^{d+n-1} = \binom{d + n - 1}{d - n}.$$

If we show that this coincides with $\deg \Phi_n$, then it will follow that $\Psi_n$ is reduced.

Let $\lambda$ be a partition of $d$ with $n$ parts. A moment's reflection will show that $\deg X_\lambda$ as given by Hilbert's formula is the coefficient of the monomial $\prod_{r=1}^{d} {z_r}^{r e_r}$ in the expression

$$(z_1 + 2\,z_2^2 + \cdots + r\,z_r^r + \dots)^n.$$

Now substitute the same letter $z$ for each $z_r$, then $\prod {z_r}^{re_r} = z^d$. Hence the coefficient of $z^d$ in $(z + 2\,z^2 + \cdots + r\,z^r + \dots)^n$ equals

$$\sum_{\lambda \text{ has } n \text{ parts}} \deg X_\lambda = \deg \Phi_n.$$

But

$$(z + 2\,z^2 + \cdots + r\,z^r + \dots) = \frac{z}{(1-z)^2},$$

hence this coefficient is the same as

$$\text{coefficient of } z^{d-n} \text{ in } (1-z)^{-2n} = (-1)^{d-n}\binom{-2n}{d-n} = \binom{d+n-1}{d-n}.$$

This completes the proof of the theorem. □

It follows that the Eagon-Northcott complex of the map

$$R(-1) \otimes S_1 \otimes S_{n-1} \longrightarrow R \otimes S_{d+n-2}$$

gives a resolution of $R/I_n$ (see [6, Chapter 2C]). Its terms are: $\mathcal{E}^0 = R$, and

$$\mathcal{E}^p = \wedge^{2n-p-1}(S_{d+n-2}) \otimes S_{-(p+1)}(S_1 \otimes S_{n-1}) \otimes R(-2n+p+1), \tag{20}$$

for $-(d-n) \leq p \leq -1$.

## 6.3 The covariants $\mathcal{A}_n$.

Consider the map

$$\wedge^{2n} \alpha_{n,\mathbb{F}} : \mathbb{C} \longrightarrow \wedge^{2n} S_{d+n-2}.$$

Let $\mathcal{A}_n$ denote the image of 1 via this map, which is a covariant of degree-order $(2n, 2n(d-n-1))$ of binary $d$-ics. (It is well-defined only up to a multiplicative constant.) By construction, it is the Wronskian of the forms

$$\{x_1^{n-j-1}\, x_2^j\, F_{x_i} : 0 \leq j \leq n-1, i = 1, 2\}, \tag{21}$$

i.e., it is the determinant of the following $2n \times 2n$ matrix:

$$(p,q) \longrightarrow \begin{cases} (x_1^{2n-q-1}\, x_2^q,\, x_1^{n-p-1}\, x_2^p\, \mathbb{F}_{x_1})_{2n-1} & \text{if } 0 \leq p \leq n-1, \\ (x_1^{2n-q-1}\, x_2^q,\, x_1^{2n-p-1}\, x_2^{p-n}\, \mathbb{F}_{x_2})_{2n-1} & \text{if } n \leq p \leq 2n-1, \end{cases} \tag{22}$$

and $0 \leq q \leq 2n-1$. It vanishes at $F$ iff the collection (21) is linearly dependent, hence

**Corollary 6.4**

$F \in \Phi_n \iff \mathcal{A}_n(F) = 0.$

Since $\mathcal{A}_{d-1}$ is an invariant of degree $2(d-1)$ it must coincide with the discriminant. Similarly $\mathcal{A}_1$ is (up to a scalar) the same as the Hessian. Thus the series $\{A_n\}$ can be thought of as an 'interpolation' between the two.

The following lemma will be used in the next section.

**Lemma 6.5**

*Asssume* $[F] \in \Phi_n \setminus \Phi_{n-1}$. *Then*

$$\mathcal{A}_{n-1}(F) = (g_F)^{2n-2}.$$

*Proof.* This is perhaps best proved using the relation between the Wronskian and ramification indices (see [3, pp. 37–43]). By hypothesis, $\alpha_{n-1}$ is of rank $2n-2$ at $[F]$, in fact

$$\mathrm{im}(\alpha_{n-1,F}) = \{F_{x_1}\, G_1 + F_{x_2}\, G_2 : G_i \in S_{n-2}\} = \{g_F\, G : G \in S_{2n-3}\}.$$

This can be seen as a linear series $\Sigma$ on $\mathbb{P}^1$ of degree $d+n-2$ and dimension $2n-3$. Write $F = \prod l_i^{e_i}$, then $\Sigma$ is only ramified at points $p_i \in \mathbb{P}^1$ corresponding to the $l_i$. Its ramification indices at $p_i$ are

$$e_i, e_i + 1, \ldots, e_i + 2n - 3.$$

Hence the Wronskian of $\mathrm{im}(\alpha_{n-1,F})$ is $\prod {l_i}^{(2n-2)e_i} = (g_F)^{2n-2}$. □

### 6.4 The codimension two case.

Assume $n = d-2$. Then in the complex (20) we have

$$\mathcal{E}^{-1} = \wedge^{2d-4} S_{2d-4} \otimes R(-2d+4) = S_{2d-4} \otimes R(-2d+4),$$

i.e., $I_{d-2} = \mathfrak{I}(\mathcal{A}_{d-2})$. Now $J$ (the Jacobian ideal of $\Delta$) is contained in the ideal $I_{d-2}$, hence the image of the natural multiplication map

$$(I_{d-2})_{2d-4} \otimes R_1 \longrightarrow R_{2d-3}$$

must contain the representation $(J)_{2d-3}$. Since the latter is spanned by the coefficients of $\mathcal{E}_\Delta$, we deduce the following:

**Corollary 6.6**

*The covariants* $(\mathcal{A}_{d-2}, \mathbb{F})_{d-2}$ *and* $\mathcal{E}_\Delta$ *are equal up to a nonzero scalar.*

We end this section by constructing covariants which distinguish between the components $X_\tau = X_{(3,1^{d-3})}$ and $X_\delta = X_{(2^2,1^{d-4})}$. A result due to Hilbert [16] says that a binary $d$-ic $F$ lies in $X_{(d)}$ iff $\mathbb{H}(F) = 0$, and it lies in $X_{(d/2,d/2)}$ (assuming $d$ even) iff $\mathbb{T}(F) = 0$.

First assume that $F \in X_\tau \setminus X_\delta$. Then $g_F = l^2$ for some $l \in S_1$, and then $\mathcal{A}_{d-3} = l^{4d-12}$ by Lemma 6.5. If $F \in X_\delta \setminus X_\tau$, then $g_F = l_1\, l_2$ for some nonproportional linear forms, and $\mathcal{A}_{d-3} = (l_1\, l_2)^{2d-6}$. Hence we get the following proposition.

**Proposition 6.7**

*Let* $F$ *be a binary* $d$*-ic. Then*

$$F \in X_\tau \iff \mathcal{A}_{d-2}(F) = \mathbb{H}(\mathcal{A}_{d-3}(F)) = 0,$$
$$F \in X_\delta \iff \mathcal{A}_{d-2}(F) = \mathbb{T}(\mathcal{A}_{d-3}(F)) = 0.$$

*Remark 6.8* Throughout this paper we have used $\mathbb{C}$ as our base field. Note however, that all the irreducible representations of $SL_2\,\mathbb{Q}$ are defined over $\mathbb{Q}$, hence so are all the varieties and schemes defined above. Thus all of our results are valid over an arbitrary field of characteristic zero.

## 7. Appendix: the Morley form

(by A. Abdesselam)

**7.1** We will now prove identity (18) from §5.7. At this point, a brief explanatory remark on the symbolic method should be helpful. We have $f_{\mathbf{x}} = (f_1\,x_1 + f_2\,x_2), g_{\mathbf{x}} = (g_1\,x_1 + g_2\,x_2)$ where $f_i, g_i$ are treated as indeterminates. Introduce the differential operators

$$\mathcal{D}_F = \frac{1}{d!}\,F\Big(\frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2}\Big), \quad \mathcal{D}_G = \frac{1}{e!}\,G\Big(\frac{\partial}{\partial g_1}, \frac{\partial}{\partial g_2}\Big).$$

Then we have identities $F = \mathcal{D}_F\,f_{\mathbf{x}}^d, G = \mathcal{D}_G\,g_{\mathbf{x}}^d$. Moreover, each well-formed symbolic expression in $f, g$ can be evaluated by subjecting it to these operators; this is one way of providing a rigorous justification for the method. Thus the Morley form will be written as

$$\mathcal{M}(\mathbf{x}, \mathbf{y}) = \mathcal{D}_F\,\mathcal{D}_G \sum_{i=1}^{e-1} (f\,g)\,f_{\mathbf{x}}^{i-1}\,g_{\mathbf{x}}^{e-i-1}\,f_{\mathbf{y}}^{d-i}\,g_{\mathbf{y}}^{i}.$$

Now let

$$\vartheta_r = (-1)^{e-1}\,[\,(\mathcal{M}, x_1^r\,x_2^{e-2-r})_{e-2}\,]_{\mathbf{y}:=\mathbf{x}}\,,$$

where the transvection is with respect to $\mathbf{x}$. By definition,

$$\vartheta_r = \frac{(-1)^{e-1}}{(e-2)!^2}\left\{\Big(\frac{\partial^2}{\partial z_1\,\partial x_2} - \frac{\partial^2}{\partial z_2\,\partial x_1}\Big)^{e-2}\mathcal{M}(\mathbf{z}, \mathbf{y})\,x_1^r\,x_2^{e-2-r}\right\}\Bigg|_{\mathbf{y}:=\mathbf{x}}.$$

After a binomial expansion this simplifies to

$$\frac{(-1)^{e-1}}{(e-2)!}\Big(-\frac{\partial}{\partial z_2}\Big)^r\Big(\frac{\partial}{\partial z_1}\Big)^{e-2-r}\mathcal{M}(\mathbf{z}, \mathbf{x}). \tag{23}$$

**7.2** Let us introduce a pair of variables $b = (b_1, b_2)$, which will serve as placeholders. Define the sum

$$\Psi = (-1)^e \sum_{r=0}^{e-2} \binom{e-2}{r} b_1^r\,b_2^{e-2-r}\,(F\,\widetilde{G}_r - G\,\widetilde{F}_r), \tag{24}$$

so that

$$F\,\widetilde{G}_r - G\,\widetilde{F}_r = \frac{(-1)^e}{(e-2)!}\,\frac{\partial^{e-2}\,\Psi}{\partial\,b_1^r\,\partial\,b_2^{e-2-r}}. \tag{25}$$

Now

$$G\,\frac{\partial^{e-1}F}{\partial x_1^q\,\partial x_2^{e-1-q}} = (\mathcal{D}_G\; g_{\mathbf{x}}^e)\left[\Big(\frac{\partial^{e-1}}{\partial x_1^q\,\partial x_2^{e-1-q}}\Big)\,\mathcal{D}_F\; f_{\mathbf{x}}^d\right]$$
$$= \mathcal{D}_F\,\mathcal{D}_G\left[g_{\mathbf{x}}^e\,\Big(\frac{\partial^{e-1}}{\partial x_1^q\,\partial x_2^{e-1-q}}\Big)\,f_{\mathbf{x}}^d\right]$$
$$= \frac{d!}{(d-e+1)!}\,\mathcal{D}_F\,\mathcal{D}_G\left[f_1^q\,f_2^{e-1-q}\,f_{\mathbf{x}}^{d-e+1}g_{\mathbf{x}}^e\right],$$

and similarly

$$F\,\frac{\partial^{e-1}G}{\partial x_1^q\,\partial x_2^{e-1-q}} = e!\,\mathcal{D}_F\,\mathcal{D}_G\,[\,g_1^q\,g_2^{e-1-q}\,f_{\mathbf{x}}^d\,g_{\mathbf{x}}\,].$$

Now substitute these expressions into equations (15) and (16) from §5.7, and then substitute the latter into (24). Then we have $\Psi = \mathcal{D}_F\,\mathcal{D}_G\,\widetilde{\Psi}$, where

$$\widetilde{\Psi} = \sum_{r=0}^{e-2}\Bigg[(-b_1)^r\,b_2^{e-2-r}$$
$$\times \sum_{q=e-r-1}^{e-1}\binom{e-1}{q}\,x_1^{q-e+r+1}\,x_2^{e-q-r-2}\{g_1^q\,g_2^{e-1-q}\,f_{\mathbf{x}}^d\,g_{\mathbf{x}} - f_1^q\,f_2^{e-1-q}\,f_{\mathbf{x}}^{d-e+1}\,g_{\mathbf{x}}^e\}\Bigg].$$

The double sum is over the range $0 \le r \le e-2$, $e-r-1 \le q \le e-1$, which is the same as $1 \le q \le e-1$, $e-q-1 \le r \le e-2$. Therefore, after changing the order of summation,

$$\widetilde{\Psi} = \sum_{q=1}^{e-1}\Bigg[\binom{e-1}{q}b_2^{e-2}x_1^{q-e+1}x_2^{e-q-2}\{g_1^q\,g_2^{e-1-q}\,f_{\mathbf{x}}^d\,g_{\mathbf{x}} - f_1^q\,f_2^{e-1-q}\,f_{\mathbf{x}}^{d-e+1}\,g_{\mathbf{x}}^e\}$$
$$\times \sum_{r=e-1-q}^{e-2}\Big(-\frac{b_1\,x_1}{b_2\,x_2}\Big)^r\Bigg],$$

which we abbreviate to

$$\sum_{q=1}^{e-1}\left[(M_1 - M_2)\times \sum_{r=e-1-q}^{e-2}\Big(-\frac{b_1\,x_1}{b_2\,x_2}\Big)^r\right].$$

The geometric series over $r$ is equal to

$$\frac{(-b_1\,x_1)^{e-1-q}\,(b_2\,x_2)^q - (-b_1\,x_1)^{e-1}}{(b_2\,x_2)^{e-2}\,b_{\mathbf{x}}}$$
$$= (-b_1\,x_1)^{e-1-q}\,(b_2\,x_2)^{q-e+2}\,b_{\mathbf{x}}^{-1} - (-b_1\,x_1)^{e-1}\,(b_2\,x_2)^{-e+2}\,b_{\mathbf{x}}^{-1}$$
$$= N_1 - N_2.$$

Hence, after expansion $\widetilde{\Psi}$ is a sum of four terms

$$\underbrace{\sum M_1\,N_1}_{T_1} + \underbrace{\sum -M_1\,N_2}_{T_2} + \underbrace{\sum -M_2\,N_1}_{T_3} + \underbrace{\sum M_2\,N_2}_{T_4}.$$

Now

$$\begin{aligned} T_1 &= \sum_{q=1}^{e-1} \binom{e-1}{q} (-1)^{e-1-q}\, b_1^{e-1-q}\, b_2^q\, b_{\mathbf{x}}^{-1}\, g_1^q\, g_2^{e-1-q}\, f_{\mathbf{x}}^d\, g_{\mathbf{x}} \\ &= (-1)^{e-1}\, b_1^{e-1}\, b_{\mathbf{x}}^{-1}\, g_2^{e-1}\, f_{\mathbf{x}}^d\, g_{\mathbf{x}} \left\{ \left(1 - \frac{b_2\, g_1}{b_1\, g_2}\right)^{e-1} - 1 \right\} \\ &= (-1)^{e-1}\, (b\, g)^{e-1}\, b_{\mathbf{x}}^{-1}\, f_{\mathbf{x}}^d\, g_{\mathbf{x}} + (-1)^e\, b_1^{e-1}\, b_{\mathbf{x}}^{-1}\, g_2^{e-1}\, f_{\mathbf{x}}^d\, g_{\mathbf{x}}, \end{aligned}$$

and after similar calculations,

$$\begin{aligned} T_2 &= (-1)^e\, b_1^{e-1}\, b_{\mathbf{x}}^{-1}\, x_2^{-e+1}\, f_{\mathbf{x}}^d\, g_{\mathbf{x}}^e + (-1)^{e-1}\, b_1^{e-1}\, b_{\mathbf{x}}^{-1}\, g_2^{e-1}\, f_{\mathbf{x}}^d\, g_{\mathbf{x}}\,, \\ T_3 &= (-1)^e\, (b\, f)^{e-1}\, b_{\mathbf{x}}^{-1}\, f_{\mathbf{x}}^{d-e+1}\, g_{\mathbf{x}}^e + (-1)^{e-1}\, b_1^{e-1}\, b_{\mathbf{x}}^{-1}\, f_2^{e-1}\, f_{\mathbf{x}}^{d-e+1}\, g_{\mathbf{x}}^e\,, \\ T_4 &= (-1)^{e-1}\, b_1^{e-1}\, b_{\mathbf{x}}^{-1}\, x_2^{-e+1}\, f_{\mathbf{x}}^d\, g_{\mathbf{x}}^e + (-1)^e\, b_1^{e-1}\, b_{\mathbf{x}}^{-1}\, f_2^{e-1}\, f_{\mathbf{x}}^{d-e+1}\, g_{\mathbf{x}}^e. \end{aligned}$$

Notice that six of the eight terms cancel in pairs, for instance, the first term of $T_2$ cancels with the first term of $T_4$. We are left with

$$\begin{aligned} \widetilde{\Psi} &= (-1)^{e-1}\, (b\, g)^{e-1}\, b_{\mathbf{x}}^{-1}\, f_{\mathbf{x}}^d\, g_{\mathbf{x}} + (-1)^e\, (b\, f)^{e-1}\, b_{\mathbf{x}}^{-1}\, f_{\mathbf{x}}^{d-e+1}\, g_{\mathbf{x}}^e, \\ &= \frac{(-1)^{e-1}\, f_{\mathbf{x}}^{d-e+1}\, g_{\mathbf{x}}}{b_{\mathbf{x}}} \left[\, (b\, g)^{e-1}\, f_{\mathbf{x}}^{e-1} - (b\, f)^{e-1}\, g_{\mathbf{x}}^{e-1} \right]. \end{aligned}$$

Rewrite $b_{\mathbf{x}}$ using the Plücker syzygy $b_{\mathbf{x}}\,(f\, g) = (b\, g)\, f_{\mathbf{x}} - (b\, f)\, g_{\mathbf{x}}$, and factor the numerator. This gives

$$\widetilde{\Psi} = (-1)^{e-1}\, (f\, g) \sum_{i=1}^{e-1} (b\, f)^{i-1}\, (b\, g)^{e-i-1}\, f_{\mathbf{x}}^{d-i}\, g_{\mathbf{x}}^i\,.$$

Now make a change of variable $(b_1, b_2) = (z_2, -z_1)$. Then $(b\, f) = b_1 f_2 - b_2 f_1 = f_{\mathbf{z}}, (b\, g) = g_{\mathbf{z}}$, and $\mathcal{D}_F\, \mathcal{D}_G\, \widetilde{\Psi} = (-1)^{e-1}\, \mathcal{M}(\mathbf{z}, \mathbf{x})$. By formula (25),

$$F\, \widetilde{G}_r - G\, \widetilde{F}_r = \frac{(-1)^e}{(e-2)!} \Big(\frac{\partial}{\partial z_2}\Big)^r \Big(-\frac{\partial}{\partial z_1}\Big)^{e-2-r} \mathcal{D}_F\, \mathcal{D}_G\, \widetilde{\Psi},$$

which is the same as $\vartheta_r$ by formula (23). This completes the proof of identity (18), and hence that of Proposition 5.2. □

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