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Semipurity of tempered Deligne cohomology

José Ignacio Burgos Gil

Facultad de Matemáticas, Universidad de Barcelona
Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain

E-mail: burgos@ub.edu

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Abstract

In this paper we define the formal and tempered Deligne cohomology groups, that are obtained by applying the Deligne complex functor to the complexes of formal differential forms and tempered currents respectively. We then prove the existence of a duality between them, a vanishing theorem for the former and a semipurity property for the latter. The motivation of these results comes from the study of covariant arithmetic Chow groups. The semi-purity property of tempered Deligne cohomology implies, in particular, that several definitions of covariant arithmetic Chow groups agree for projective arithmetic varieties.

1. Introduction

The aim of this note is to study some properties of formal and tempered Deligne cohomology (with real coefficients). These cohomology groups are defined by applying the Deligne complex functor to the complexes of formal differential forms and tempered currents respectively.

Let X be a complex projective manifold and let W be a Zariski locally closed subset of X. Let $i : W \longrightarrow X$ denote the inclusion and let $i^*, i^!, i_*, i_!$ be the induced functors in the derived category of abelian sheaves. Then the complex of formal differential forms of W computes the cohomology of W with compact supports. That is, it computes the groups $H^*(X, i_!i^*\mathbb{R})$. The complex of tempered currents on W compute the cohomology of X with supports on W, that is, it computes the groups

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$H^*(X, i_* i^! \mathbb{R})$. Following Deligne, the previous groups have a mixed Hodge structure, hence a Hodge filtration that we will call the Deligne-Hodge filtration. The complexes of formal differential forms and tempered currents are examples of Dolbeault complexes (see [6]). Therefore they have a Hodge filtration obtained from the bigrading of differential forms. In general, this Hodge filtration does not induce the Deligne-Hodge filtration in cohomology. Moreover, the spectral sequence associated to this Hodge filtration does not degenerate at the E^1-term.

This implies that formal and tempered Deligne cohomology groups with real coefficients will not have, in general, the same properties as Deligne-Beilinson cohomology. For instance they do not need to be finite dimensional. They have a structure of topological vector spaces, but they may be non-separated.

Note however that, in the particular case when $W = X$, the formal and tempered Deligne cohomology groups with real coefficients, agree with the usual real Deligne cohomology groups.

In this note we will construct a (Poincaré like) duality between formal Deligne cohomology and tempered Deligne cohomology, that induce a perfect pairing between the corresponding separated vector spaces. In particular, applying this duality to the case $W = X$ we obtain an exceptional duality for real Deligne Beilinson cohomology (Corollary 2.28) of smooth projective varieties that, to my knowledge, is new. The shape of this exceptional duality reminds very much the functional equation of L-functions. It would be interesting to know whether this duality has any arithmetic meaning.

The second result is a vanishing result for formal Deligne cohomology. Thanks to the previous duality, the vanishing result of formal Deligne cohomology implies a semipurity property of tempered Deligne cohomology (Corollary 2.34).

The motivation for these results comes from the study of covariant arithmetic Chow groups introduced in [3] and [6]. The covariant arithmetic Chow groups are a variant of the arithmetic Chow groups defined by Gillet and Soulé, that are covariant for arbitrary proper morphism. By contrast, the groups defined by Gillet and Soulé are only covariant for proper morphisms between arithmetic varieties that induce smooth maps between the corresponding complex varieties. The covariant arithmetic Chow groups do not have a product structure, but they are a module over the contravariant arithmetic Chow groups (see [6] for more details). Similar definitions of covariant Chow groups have been given by Kawaguchi and Moriwaki [13] and by Zha [16]. These two definitions are equivalent except for the fact that Zha neglects the structure of real manifold induced on the complex manifold associated to an arithmetic variety.

Although not explicitly stated, in the paper [6], the covariant arithmetic Chow groups are defined by means of tempered Deligne cohomology. The semi-purity property of tempered Deligne cohomology was announced and used in [6]. Hence this paper can be seen as a complement of [6]. A new consequence of the semipurity property is that, for an arithmetic variety that is generically projective, the covariant Chow groups introduced in [3] and [6] are isomorphic to the covariant Chow groups introduced by Kawaguchi and Moriwaki.

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2. Complexes of forms and currents

By a complex algebraic manifold we will mean the analytic manifold associated to a smooth scheme over $\mathbb{C}$. Let X be a projective complex algebraic manifold. We will consider the following situation: let $Z \subset Y$ be closed subvarieties of X, let U and V be the open subsets $U = X \setminus Y$, $V = X \setminus Z$ and let W be the locally closed subset $W = Y \setminus Z$.

2.1 Flat forms and Whitney forms

THE COMPLEX OF WHITNEY FORMS. Let $\mathscr{E}_X^*$ denote the sheaf of smooth differential forms on X. We will denote by $E^*(U)$ the complex of global differential forms over U and by $E_c^*(U)$ the complex of differential forms with compact support.

Let $\mathscr{E}_X^*(\operatorname{flat} Y)$ denote the ideal sheaf of differential forms that are flat along Y. Recall that a differential form on X is called flat along Y if its Taylor expansion vanishes at all points of Y. We write

$$\mathscr{E}_{Y^\infty}^* = \mathscr{E}_X^* / \mathscr{E}_X^*(\operatorname{flat} Y).$$

The sections of this complex of sheaves are called Whitney forms on Y. Whitney's extension theorem ([15, IV Theorem 3.1]), gives us a precise description of the space of Whitney forms in terms of jets over Y. For instance, if Y is the smooth subvariety of $\mathbb{C}^n$ defined by the equations $z_1 = \cdots = z_k = 0$, then the germ of the sheaf of Whitney functions on Y at the point $x = (0, \ldots, 0)$ is

$$\mathscr{E}_{Y^\infty,x}^0 = \mathscr{E}_{Y,x}^0[[z_{k+1}, \ldots, z_n, \bar{z}_{k+1}, \ldots, \bar{z}_n]].$$

We will write

$$\mathscr{E}_{Y^\infty}^*(\operatorname{flat} Z) = \mathscr{E}_X^*(\operatorname{flat} Z)/\mathscr{E}_X^*(\operatorname{flat} Y).$$

Observe that $\mathscr{E}_{Y^\infty}^*(\operatorname{flat} Z)$ can also be defined as the kernel of the morphism

$$\mathscr{E}_{Y^\infty}^* \longrightarrow \mathscr{E}_{Z^\infty}^*.$$

The sheaf $\mathscr{E}_{Y^\infty}^*(\operatorname{flat} Z)$ agrees with the sheaf denoted $\mathbb{C}_W \overset{W}{\otimes} \mathcal{C}_X^\infty$ in [12].

The complex $\mathscr{E}_{Y^\infty}^*(\operatorname{flat} Z)$ is a complex of fine sheaves. We will denote the corresponding complex of global sections by $E_{X^W}^*(W) := \Gamma(X, \mathscr{E}_{Y^\infty}^*(\operatorname{flat} Z))$. Note that the complex $E_{X^W}^*(W)$ depends only on the locally closed subspace $W \subset X$ and not on a particular choice of closed subsets Y and Z. Observe also that $E_{X^W}^*(X) = E^*(X)$ is the usual complex of smooth differential forms on X.

We will denote by $E_{X^W,\mathbb{R}}^*(W)$ the real subcomplex underlying $E_{X^W}^*(W)$.

By the acyclicity of fine sheaves, there is a diagram of short exact sequences

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & E^*_{X^W}(W) \arrow[d] & \\
0 \arrow[r] & E^*_{X^W}(U) \arrow[r] \arrow[d] & E^*(X) \arrow[r] \arrow[d] & E^*_{X^W}(Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^*_{X^W}(V) \arrow[r] \arrow[d] & E^*(X) \arrow[r] \arrow[d] & E^*_{X^W}(Z) \arrow[r] \arrow[d] & 0 \\
 & E^*_{X^W}(W) \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}
\tag{2.1}
$$

The complex $E^*(X)$ is a topological vector space with the C^∞ topology. With this topology $E^*(X)$ is a Fréchet topological vector space ([1, III p. 9]). Moreover $E^*_{X^W}(U)$ is a closed subspace. In fact, by [15, V Corollaire 1.6], it is the closure of the complex of differential forms that have compact support contained in U, that we denote $E^*_c(U)$. More generally, all the monomorphisms in diagram (2.1) are closed immersions.

The following result states that, being U an algebraic open subset of X, the complex $E^*_{X^W}(U)$ does not depend on X but only on U.

Proposition 2.2

Let $\pi : \widetilde{X} \longrightarrow X$ be a proper birational morphism with $D = \pi^{-1}(Y)$, that induces an isomorphism between $\widetilde{X} \setminus D$ and U, then the natural map

$$\pi^* : E^*(X) \longrightarrow E^*(\widetilde{X})$$

induces an isomorphism $\pi^* : \Gamma(X, \mathscr{E}^*_X(\mathrm{flat}\, Y)) \longrightarrow \Gamma(\widetilde{X}, \mathscr{E}^*_{\widetilde{X}}(\mathrm{flat}\, D))$.

Proof. By [14] the morphism

$$\pi^* : E^*(X) \longrightarrow E^*(\widetilde{X})$$

is a closed immersion. Since $\Gamma(X, \mathscr{E}^*_X(\mathrm{flat}\, Y))$ and $\Gamma(\widetilde{X}, \mathscr{E}^*_{\widetilde{X}}(\mathrm{flat}\, D))$ are the closure of $E^*_c(U)$ in $E^*(X)$ and $E^*(\widetilde{X})$ respectively, then they are identified by π^*. $\square$

THE COHOMOLOGY OF THE COMPLEX OF WHITNEY FORMS. By [14] (see also [2] for a more general statement) we have

Proposition 2.3

The complex $\mathscr{E}^*_{Y^\infty}$ is a resolution of the constant sheaf $\underline{\mathbb{C}}$ on Y by fine sheaves. Therefore

$$H^*(E^*_{X^W}(W)) = H^*_c(W, \mathbb{C}),$$

where H^*_c denotes cohomology with compact supports. □

2.2 Currents with support in a subvariety

THE COMPLEX OF CURRENTS. We first recall the definition of the complex of currents and we fix the sign convention and some normalizations. We will follow the conventions of [6, § 5.4] but with the homological grading.

Let $\mathscr{D}^X_n$ be the sheaf of degree n currents on X. That is, for any open subset V of X, the group $\mathscr{D}^X_n(V)$ is the topological dual of the group of sections with compact support $E^n_c(V)$. The differential

$$\mathrm{d} : \mathscr{D}^X_n \longrightarrow \mathscr{D}^X_{n-1}$$

is defined by

$$\mathrm{d}\, T(\varphi) = (-1)^n T(\mathrm{d}\, \varphi);$$

here T is a current and φ a test form. Note that we are using the sign convention of, for instance [11], instead of the sign convention of [9].

The bigrading $\mathscr{E}^n_X = \bigoplus_{p+q=n} \mathscr{E}^{p,q}_X$ induces a bigrading

$$\mathscr{D}^X_n = \bigoplus_{p+q=n} \mathscr{D}^X_{p,q},$$

with $\mathscr{D}^X_{p,q}(V)$ the topological dual of $\Gamma_c(V, \mathscr{E}^{p,q}_X)$.

The real structure of $\mathscr{E}^n_X$ induces a real structure

$$\mathscr{D}^{X,\mathbb{R}}_n \subset \mathscr{D}^X_n.$$

We will denote

$$\mathscr{D}^{X,\mathbb{R}}_n(p) = \frac{1}{(2\pi i)^p} \mathscr{D}^{X,\mathbb{R}}_n \subset \mathscr{D}^X_n.$$

If X is equidimensional of dimension d we will write

$$\mathscr{D}^n_X = \mathscr{D}^X_{2d-n}, \quad \mathscr{D}^{p,q}_X = \mathscr{D}^X_{d-p,d-q}, \quad \text{and} \quad \mathscr{D}^n_{X,\mathbb{R}}(p) = \mathscr{D}^{X,\mathbb{R}}_{2d-n}(d-p). \tag{2.4}$$

We will use all the conventions of [6, § 5.4]. In particular, if y is an algebraic cycle of X of dimension e, we will write $\delta_y \in \mathscr{D}^X_{e,e} \cap \mathscr{D}^{X,\mathbb{R}}_{2e}(e)$ for the current

$$\delta_y(\eta) = \frac{1}{(2\pi i)^e} \int_y \eta.$$

Furthermore, there is an action

$$\begin{array}{ccc} \mathscr{E}^n_X \otimes \mathscr{D}^X_m & \longrightarrow & \mathscr{D}^X_{m-n}, \\ \omega \otimes T & \longmapsto & \omega \wedge T \end{array}$$

where the current $\omega \wedge T$ is defined by

$$(\omega \wedge T)(\eta) = T(\eta \wedge \omega).$$

This action induces actions

$$\mathscr{E}_X^{p,q} \otimes \mathscr{D}_{r,s}^X \longrightarrow \mathscr{D}_{r-p,s-q}^X, \quad \text{and} \quad \mathscr{E}_{X,\mathbb{R}}^n(p) \otimes \mathscr{D}_m^{X,\mathbb{R}}(q) \longrightarrow \mathscr{D}_{m-n}^{X,\mathbb{R}}(q-p).$$

Finally, if X is equidimensional of dimension d, there is a fundamental current $\delta_X \in \mathscr{D}_{d,d}^X \cap \mathscr{D}_{2d}^{X,\mathbb{R}}(d)$, and a morphism

$$\mathscr{E}_X^* \longrightarrow \mathscr{D}_{2d-*}^X = \mathscr{D}_X^*, \quad \omega \longmapsto [\omega] = \omega \wedge \delta_X. \tag{2.5}$$

This morphism sends $\mathscr{E}_{X\mathbb{R}}^n(p)$ to $\mathscr{D}_{2d-n}^{X,\mathbb{R}}(d-p) = \mathscr{D}_{X,\mathbb{R}}^n(p)$.

CURRENTS WITH SUPPORT ON A SUBVARIETY AND TEMPERED CURRENTS. As in the previous section let $Z \subset Y$ denote two closed subvarieties of X and put $U = X \setminus Y$, $V = X \setminus Z$ and $W = Y \setminus Z$. We denote by $\mathscr{D}_*^{Y^\infty}$ the subcomplex of $\mathscr{D}_*^X$ formed by currents with support on Y. In other words, for any open subset U' of X we have

$$\mathscr{D}_n^{Y^\infty}(U') = \{T \in \mathscr{D}_n^X(U') \mid T(\eta) = 0,\ \forall \eta \in \Gamma_c(U' \cap U, \mathscr{E}_X^n)\}.$$

Observe that, by continuity, the sections of $\mathscr{D}_n^{Y^\infty}(U')$ vanish on the subgroup $\Gamma_c(U', \mathscr{E}_X^*(\operatorname{flat} Y))$.

We write $\mathscr{D}_n^{X/Y^\infty} = \mathscr{D}_n^X \Big/ \mathscr{D}_n^{Y^\infty}$ and $\mathscr{D}_n^{Y^\infty/Z^\infty} = \mathscr{D}_n^{Y^\infty} / \mathscr{D}_n^{Z^\infty}$.

As in the case of differential forms, the complex $\mathscr{D}_n^{Y^\infty/Z^\infty}$ can also be defined as the kernel of the morphism

$$\mathscr{D}_n^{X/Z^\infty} \longrightarrow \mathscr{D}_n^{X/Y^\infty}.$$

All the above sheaves inherit a bigrading and a real structure.

Observe that, except for the fact that we are using here the homological grading, the complex of sheaves $\mathscr{D}_n^{X/Y^\infty}$ agrees with the complex denoted by $\mathcal{T}\mathcal{H}om(\mathbb{C}_W, \mathcal{D}b_X)$ in [12].

The complex $\mathscr{D}_n^{Y^\infty/Z^\infty}$ is a complex of fine sheaves. We will denote the complex of global sections by $D_*^{X^\mathcal{T}}(W^\infty) = \Gamma(X, \mathscr{D}_*^{Y^\infty/Z^\infty})$. Thus the complex $D_*^{X^\mathcal{T}}(W^\infty)$ is defined for any Zariski locally closed subset $W \subset X$. The corresponding real complex will be denoted by $D_*^{X^\mathcal{T},\mathbb{R}}(W^\infty)$.

By [14], the complex $D_*^{X^\mathcal{T}}(U)$ can be identified with the image of the morphism

$$D^*(X) \longrightarrow D^*(U).$$

That is, it is the complex of currents on U that can be extended to a current on the whole X. The elements of $D_*^{X^\mathcal{T}}(U)$ will be called tempered currents. In the literature they are called also moderate, temperate or extendable currents. Moreover, as was the case with the complex $E_{X^\mathcal{W}}^*(U)$, being U a Zariski open subset, the complex $D_*^{X^\mathcal{T}}(U)$ only depends on U and not on X.

THE PAIRING BETWEEN FORMS AND CURRENTS. We have already introduced an action

$$E^n(X) \otimes D_m(X) \longrightarrow D_{m-n}(X), \quad \omega \otimes T \longmapsto \omega \wedge T, \tag{2.6}$$

where the current $\omega \wedge T$ is defined by

$$(\omega \wedge T)(\eta) = T(\eta \wedge \omega).$$

The subspace $D_*^{X^{\mathcal{T}}}(Y)$ is invariant under this action and annihilates the subspace $E_{X^{\mathcal{W}}}^*(U)$. Therefore we obtain induced actions

$$E_{X^{\mathcal{W}}}^n(Y) \otimes D_m^{X^{\mathcal{T}}}(Y) \longrightarrow D_{m-n}^{X^{\mathcal{T}}}(Y), \qquad E_{X^{\mathcal{W}}}^n(U) \otimes D_m^{X^{\mathcal{T}}}(U) \longrightarrow D_{m-n}^{X^{\mathcal{T}}}(U) \tag{2.7}$$

and, more generally, an action

$$E_{X^{\mathcal{W}}}^n(W) \otimes D_m^{X^{\mathcal{T}}}(W) \longrightarrow D_{m-n}^{X^{\mathcal{T}}}(W). \tag{2.8}$$

Since X is proper, there is a canonical morphism

$$\deg : D_0(X) \longrightarrow \mathbb{C}$$

given by $\deg(T) = T(1)$. Observe that $\deg(D_0^{\mathbb{R}}(X)) \subset \mathbb{R}$.

Combining the degree and the above actions, we recover the pairing

$$E^n(X) \otimes D_n(X) \longrightarrow \mathbb{C},$$

that identifies $D_n(X)$ with the topological dual of $E^n(X)$. Under this identification, the subspace $E_{X^{\mathcal{W}}}^n(U)$ is the orthogonal to the subspace $D_n^{X^{\mathcal{T}}}(Y)$. Therefore $D_n^{X^{\mathcal{T}}}(U)$ is the topological dual of $E_{X^{\mathcal{W}}}^n(U)$ and $D_n^{X^{\mathcal{T}}}(Y)$ is the topological dual of $E_{X^{\mathcal{W}}}^n(Y)$. More generally $D_n^{X^{\mathcal{T}}}(W)$ is the topological dual of $E_{X^{\mathcal{W}}}^n(W)$. Note that here, the key point is the fact that $E_{X^{\mathcal{W}}}^n(U)$ is the closure of $\Gamma_c(U, \mathscr{E}_X^n)$ and hence a closed subspace.

The above pairings induce a pairing

$$E_{\mathbb{R}}^n(X)(p) \otimes D_n^{\mathbb{R}}(X)(p) \longrightarrow \mathbb{R},$$

and similar pairings for the other complexes of forms and currents.

Finally, observe that there is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & D_*^{X^{\mathcal{T}}}(W) \arrow[d] & \\
0 \arrow[r] & D_*^{X^{\mathcal{T}}}(Z) \arrow[r] \arrow[d] & D_*(X) \arrow[r] \arrow[d] & D_*^{X^{\mathcal{T}}}(V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_*^{X^{\mathcal{T}}}(Y) \arrow[r] \arrow[d] & D_*(X) \arrow[r] \arrow[d] & D_*^{X^{\mathcal{T}}}(U) \arrow[r] \arrow[d] & 0 \\
 & D_*^{X^{\mathcal{T}}}(W) \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd} \tag{2.9}$$

that is the topological dual of the diagram (2.1).

THE HOMOLOGY OF THE COMPLEXES OF CURRENTS. By [14] we have

Proposition 2.10

The homology of the complexes $D_*^{X^{\mathcal{T}}}(W)$ *is given by*

$$H_*(D_*^{X^{\mathcal{T}}}(W)) = H_*^{BM}(W, \mathbb{C}),$$

where H_*^{BM} *denote Borel-Moore homology. In particular, since we are assuming* Y *proper,*

$$H_*(D_*^{X^{\mathcal{T}}}(Y)) = H_*(Y, \mathbb{C}). \qquad \square$$

2.3 Formal and tempered Deligne cohomology

FORMAL DELIGNE COHOMOLOGY. The complex $E^*_{X^{\mathcal{W}},\mathbb{R}}(W)$ is an example of a Dolbeault algebra (see [6]). Recall that, following Deligne, the cohomology of any complex variety has a mixed Hodge structure. We will call the Hodge filtration of this mixed Hodge structure the Deligne-Hodge filtration.

From the structure of Dolbeault algebra of $E^*_{X^{\mathcal{W}}}(W)$ we can define a Hodge filtration. It is the filtration associated to the bigrading. In general, this Hodge filtration does not induce the Deligne-Hodge filtration in cohomology. Moreover, the spectral sequence associated to this Hodge filtration does not need to degenerate at the E_1 term. Therefore, the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(E^*(Y^\infty))$ are not, in general, direct summands of $H^{p+q}(Y, \mathbb{C})$. In fact, they can be infinite dimensional as can be seen in the easiest example: Put $X = \mathbb{P}^1_{\mathbb{C}}$. Let t be the absolute coordinate and let Y be the point $t = 0$. Then $H^{0,0}_{\bar{\partial}}(E^*(Y^\infty)) = \mathbb{C}[[t]]$, the ring of formal power series in one variable.

Following [4] and [6], to every Dolbeault algebra we can associate a Deligne algebra. We refer the reader to [4] and [6, § 5] for the definition and properties of Dolbeault algebras, Dolbeault complexes and the associated Deligne complexes. We will use freely the notation therein. In particular the Deligne algebra associated to the above Dolbeault algebra will be denoted $\mathcal{D}^*(E^*_{X^{\mathcal{W}}}(W), *)$.

DEFINITION 2.11 The real formal Deligne cohomology of W (with compact supports) is defined by

$$H^*_{\mathcal{D}^f,c}(W^\infty, \mathbb{R}(p)) = H^*(\mathcal{D}^*(E_{X^{\mathcal{W}}}(W), p)).$$

When W is proper we will just write $H^*_{\mathcal{D}^f}(W^\infty, \mathbb{R}(p))$.

The notation W^∞ is a reminder that this cohomology depends, not only on W but on an infinitesimal neighborhood of infinite order of W in X.

Remark 2.12 Since we are assuming that X is smooth and proper, the formal Deligne cohomology of X, $H^*_{\mathcal{D}^f}(X^\infty, \mathbb{R}(p))$, given in the previous definition, agrees with the usual Deligne cohomology of X. Nevertheless, by the discussion before the definition, the formal Deligne cohomology with compact supports of U or the formal Deligne cohomology of Y, do not agree, in general, with the usual Deligne-Beilinson cohomology. For instance the groups $H^*_{\mathcal{D}^f}(U, \mathbb{R}(p))$ can be infinite dimensional.

Homological Dolbeault complexes and homological Deligne complexes. In order to define formal Deligne homology we first translate the notions of [6, § 5.2] to the homological grading.

Definition 2.13 A *homological Dolbeault complex* $A = (A^{\mathbb{R}}_*, \mathrm{d}_A)$ is a graded complex of real vector spaces, which is bounded from above and equipped with a bigrading on $A^{\mathbb{C}} = A^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, i.e.,

$$A^{\mathbb{C}}_n = \bigoplus_{p+q=n} A_{p,q},$$

satisfying the following properties:

(i) The differential d_A can be decomposed as the sum $\mathrm{d}_A = \partial + \bar{\partial}$ of operators ∂ of type $(-1,0)$, resp. $\bar{\partial}$ of type $(0,-1)$.

(ii) It satisfies the symmetry property $\overline{A_{p,q}} = A_{q,p}$, where $\overline{}$ denotes complex conjugation.

Notation 2.14 Given a homological Dolbeault complex $A = (A^{\mathbb{R}}_*, \mathrm{d}_A)$, we will use the following notations. The Hodge filtration F of A is the increasing filtration of $A^{\mathbb{C}}_*$ given by

$$F_p A_n = F_p A^{\mathbb{C}}_n = \bigoplus_{p' \leq p} A_{p',n-p'}.$$

The filtration $\overline{F}$ of A is the complex conjugate of F, i.e.,

$$\overline{F}_p A_n = \overline{F}_p A^{\mathbb{C}}_n = \overline{F_p A^{\mathbb{C}}_n}.$$

For an element $x \in A^{\mathbb{C}}$, we write $x_{i,j}$ for its component in $A_{i,j}$. For $k, k' \in \mathbb{Z}$, we define an operator $F_{k,k'} : A^{\mathbb{C}} \longrightarrow A^{\mathbb{C}}$ by the rule

$$F_{k,k'}(x) := \sum_{l \leq k, l' \leq k'} x_{l,l'}.$$

We note that the operator $F_{k,k'}$ is the projection of $A^*_{\mathbb{C}}$ onto the subspace $F_k A_* \cap \overline{F}_{k'} A_*$. This subspace will be denoted $F_{k,k'} A_*$. We will also denote by F_k the operator $F_{k,\infty}$.

We denote by $A^{\mathbb{R}}_n(p)$ the subgroup $(2\pi i)^{-p} \cdot A^{\mathbb{R}}_n \subseteq A^{\mathbb{C}}_n$, and we define the operator

$$\pi_p : A^{\mathbb{C}} \longrightarrow A^{\mathbb{R}}(p)$$

by setting $\pi_p(x) := \frac{1}{2}(x + (-1)^p \bar{x})$.

To any homological Dolbeault complex we can associate a homological Deligne complex.

Definition 2.15 Let A be a homological Dolbeault complex. We denote by $A_*(p)^{\mathcal{D}}$ the complex $s(A^{\mathbb{R}}(p) \oplus F_p A \xrightarrow{u} A^{\mathbb{C}})$, where $u(a, f) = -a + f$ and $s(\ )$ denotes the simple complex of a morphism of complexes.

DEFINITION 2.16 Let A be a homological Dolbeault complex. Then, the *(homological) Deligne complex* $(\mathcal{D}^*(A,*), \mathrm{d}_{\mathcal{D}})$ *associated to* A is the graded complex given by

$$\mathcal{D}_n(A,p) = \begin{cases} A^{\mathbb{R}}_{n+1}(p+1) \cap F_{n-p,n-p}A^{\mathbb{C}}_{n+1}, & \text{if} \quad n \geq 2e+1, \\ A^{\mathbb{R}}_n(p) \cap F_{p,p}A^{\mathbb{C}}_n, & \text{if} \quad n \leq 2p, \end{cases}$$

with differential given, for $x \in \mathcal{D}_n(A,p)$, by

$$\mathrm{d}_{\mathcal{D}}\, x = \begin{cases} -F_{n-p+1,n-p+1}\, \mathrm{d}_A\, x, & \text{if} \quad n > 2p+1, \\ -2\partial\bar{\partial}x, & \text{if} \quad n = 2p+1, \\ \mathrm{d}_A\, x, & \text{if} \quad n \leq 2p. \end{cases}$$

For instance, let A be a Dolbeault complex satisfying $A_{p,q} = 0$ for $p < 0$, $q < 0$, $p > n$, or $q > n$. Then, for $p \geq n$, the complex $\mathcal{D}(A,p)$ agrees with the real complex $A^{\mathbb{R}}_*(p)$. For $0 \leq p < n$, we have represented $\mathcal{D}(A,p)$ in Figure 1, where the upper right square is shifted by one; this means in particular that $A_{n,n}$ sits in degree $2n-1$ and $A_{p+1,p+1}$ sits in degree $2p+1$. For $p < 0$ the complex $\mathcal{D}(A,p)$ agrees with the real complex $A^{\mathbb{R}}_*(p+1)[1]$.

$$\begin{array}{ccc}
 & & \left(\begin{array}{ccccc} A_{p+1,n} & \leftarrow & \cdots & \leftarrow & A_{n,n} \\ \downarrow & & & & \downarrow \\ \vdots & & & & \vdots \\ \downarrow & & & & \downarrow \\ A_{p+1,p+1} & \leftarrow & \cdots & \leftarrow & A_{n,p+1} \end{array}\right)_{\mathbb{R}}(p+1) \\
 & \swarrow{\scriptstyle -2\partial\bar{\partial}} & \\
\left(\begin{array}{ccccc} A_{0,p} & \leftarrow & \cdots & \leftarrow & A_{p,p} \\ \downarrow & & & & \downarrow \\ \vdots & & & & \vdots \\ \downarrow & & & & \downarrow \\ A_{0,0} & \leftarrow & \cdots & \leftarrow & A_{p,0} \end{array}\right)_{\mathbb{R}}(p) & &
\end{array}$$

Figure 1: $\mathcal{D}(A,p)$

Remark 2.17 It is clear from the definition that, for all $p \in \mathbb{Z}$, the functor $\mathcal{D}(\cdot,p)$ is exact.

The main property of the Deligne complex is expressed by the following proposition; for a proof in the cohomological case see [4].

Proposition 2.18

The complexes $A_*(p)^{\mathcal{D}}$ and $\mathcal{D}_*(A,p)$ are homotopically equivalent. The homotopy *equivalences* $\psi : A_n(p)^{\mathcal{D}} \longrightarrow \mathcal{D}_n(A,p)$, *and* $\varphi : \mathcal{D}_n(A,p) \longrightarrow A_n(p)^{\mathcal{D}}$ *are given by*

$$\psi(a,f,\omega) = \begin{cases} \pi(\omega), & \text{if} \quad n \geq 2p+1, \\ F_{p,p}a + 2\pi_p(\partial\omega_{p+1,n-p-1}), & \text{if} \quad n \leq 2p, \end{cases}$$

where $\pi(\omega) = \pi_{p+1}(F_{n-p,n-p}\omega)$, *i.e.,* π *is the projection of* $A_{\mathbb{C}}$ *over the cokernel of* u, *and*

$$\varphi(x) = \begin{cases} (\partial x_{p+1,n-p} - \bar{\partial} x_{n-p,p+1}, 2\partial x_{p+1,n-p}, x), & \text{if} \quad n \geq 2p+1, \\ (x,x,0), & \text{if} \quad n \leq 2p. \end{cases}$$

Moreover, $\psi \circ \varphi = \mathrm{id}$, *and* $\varphi \circ \psi - \mathrm{id} = \mathrm{d}\, h + h\, \mathrm{d}$, *where* $h : A_n(p)^{\mathcal{D}} \longrightarrow A_{n+1}(p)^{\mathcal{D}}$ *is given by*

$$h(a,f,\omega) = \begin{cases} (\pi_p(\overline{F}_p\omega + \overline{F}_{n-p}\omega), -2F_p(\pi_{p+1}\omega), 0), & \text{if} \quad n \geq 2p+1, \\ (2\pi_p(\overline{F}_{n-p}\omega), -F_{p,p}\omega - 2F_{n-p}(\pi_{p+1}\omega), 0), & \text{if} \quad n \leq 2p. \end{cases} \qquad \square$$

Tempered Deligne homology. Applying the above discussion to the complex of currents $D_*^{X^{\mathcal{T}},\mathbb{R}}(W)$ we define the homological Deligne complex $\mathcal{D}_*(D_*^{X^{\mathcal{T}}}(W), *)$.

Definition 2.19 *The tempered Deligne (Borel-Moore) homology of* W is defined by

$$H_*^{\mathcal{D}^{\mathcal{T}}}(W^\infty, \mathbb{R}(p)) = H_*(\mathcal{D}_*(D_*^{X^{\mathcal{T}}}(W), p)).$$

Remark 2.20

(i) Again, since X is smooth and proper, the tempered Deligne homology of X agrees with the Deligne homology of X. In particular, the group $H_n^{\mathcal{D}}(X, \mathbb{R}(p))$ agrees with the group denoted ${}'H_{\mathcal{D}}^{-n}(X, \mathbb{R}(-p))$ in [11]. But, since the Hodge filtration of the complex of currents with support on Y does not induce the Deligne-Hodge filtration in the homology of Y, the tempered Deligne homology does not agree in general with Deligne-Beilinson homology.

(ii) As in the case of formal cohomology, the notation $H_*^{\mathcal{D}^{\mathcal{T}}}(W^\infty, \mathbb{R}(p))$ reminds us that these groups do not depend only on W but on an infinitesimal neighborhood of W of infinite order.

Equidimensional manifolds. If X is equidimensional of dimension d the morphism (2.5) induces morphisms

$$\mathcal{D}^n(E^*(X), p) \longrightarrow \mathcal{D}_{2d-n}(D_*(X), d-p), \quad p \in \mathbb{Z}, \tag{2.21}$$

that, in turn, induce the Poincaré duality isomorphisms

$$H_{\mathcal{D}}^n(X, \mathbb{R}(p)) \longrightarrow H_{2d-n}^{\mathcal{D}}(X, \mathbb{R}(d-p)), \quad n,p \in \mathbb{Z}. \tag{2.22}$$

By analogy, we can define tempered Deligne cohomology groups as follows

$$H_{\mathcal{D}^{\mathcal{T}}}^n(U, \mathbb{R}(p)) = H_{2d-n}^{\mathcal{D}^{\mathcal{T}}}(U, \mathbb{R}(d-p)),$$
$$H_{\mathcal{D}^{\mathcal{T}},W}^n(V, \mathbb{R}(p)) = H_{2d-n}^{\mathcal{D}^{\mathcal{T}}}(W^\infty, \mathbb{R}(d-p)).$$

In general, if X is a disjoint union of equidimensional algebraic manifolds, then we define the tempered Deligne cohomology of X as the direct sum of the tempered Deligne cohomology of its components.

THE MODULE STRUCTURE OF TEMPERED DELIGNE HOMOLOGY. The notion of Dolbeault module over a Dolbeault algebra introduced in [6] can be easily modified to define homological Dolbeault modules over a Dolbeault algebra. The actions (2.6), (2.7) and (2.8) provide the basic examples. Modifying the construction of [6, 5.17, 5.18] we obtain

Proposition 2.23

There is a pseudo-associative action

$$\mathcal{D}^n(E_{X^{\mathcal{W}}}(W),p)\otimes\mathcal{D}_m(D_*^{X^{\mathcal{T}}}(W),q)\longrightarrow\mathcal{D}_{m-n}(D_*^{X^{\mathcal{T}}}(W),q-p)$$

that induces an associative action

$$H^n_{\mathcal{D}^f,c}(W^\infty,\mathbb{R}(p))\otimes H^{\mathcal{D}^{\mathcal{T}}}_m(W^\infty,\mathbb{R}(q))\longrightarrow H^{\mathcal{D}^{\mathcal{T}}}_{m-n}(W^\infty,\mathbb{R}(q-p)). \qquad \square$$

THE EXCEPTIONAL DUALITY. In general, Poincaré duality for Deligne cohomology is not given by a bilinear pairing, but by the isomorphism (2.22) between Deligne cohomology and Deligne homology (see for instance [11]). Nevertheless, in the case of real Deligne cohomology, there is an exceptional duality that comes from the symmetry of the Deligne complex associated with a Dolbeault complex. This duality can be generalized to a pairing between formal Deligne cohomology and tempered Deligne homology.

Proposition 2.24

For every pair of integers n,p, there is a pairing

$$\mathcal{D}^n(E_{X^{\mathcal{W}}}(W),p)\otimes\mathcal{D}_{n-1}(D^{X^{\mathcal{T}}}(W),p-1)\longrightarrow\mathbb{R}$$

given by $\omega\otimes T\longmapsto T(\omega)$. This pairing identifies $\mathcal{D}_{n-1}(D^{X^{\mathcal{T}}}(W),p-1)$ with the topological dual of $\mathcal{D}^n(E_{X_{\mathcal{W}}}(W),p)$. Moreover, it is compatible, up to the sign, with the differential in the Deligne complex:

$$T(\mathrm{d}_{\mathcal{D}}\,\omega)=\begin{cases}(-1)^{n+1}(\mathrm{d}_{\mathcal{D}}\,T)(\omega), & \text{if } n\leq 2p-1,\\ (-1)^{n}(\mathrm{d}_{\mathcal{D}}\,T)(\omega), & \text{if } n\geq 2p.\end{cases}$$

It is also compatible, up to the sign, with the action of $\mathcal{D}^*(E_{X^{\mathcal{W}}}(W),*)$. That is, if the forms $\omega\in\mathcal{D}^n(E_{X^{\mathcal{W}}}(W^\infty),p)$ and $\eta\in\mathcal{D}^l(E_{X^{\mathcal{W}}}(W),r)$, and the current $T\in\mathcal{D}_m(D^{X^{\mathcal{T}}}(W),q)$, with $n-m+l=1$ and $p-q+r=1$ then

$$(\omega\bullet T)(\eta)=\begin{cases}(-1)^nT(\eta\bullet\omega), & \text{if } m>2q,\ l\geq 2r,\\ T(\eta\bullet\omega), & \text{if } m\leq 2q,\ l<2r,\\ (-1)^{m-1}T(\eta\bullet\omega), & \text{if } m>2q,\ l<2r,\\ (-1)^{l}T(\eta\bullet\omega), & \text{if } m\leq 2q,\ l\geq 2r.\end{cases}$$

Proof. Assume that $n < 2p$. Put $q = p-1$ and $m = n-1$. Then

$$
\begin{aligned}
&\mathcal{D}^n(E_{X^{\mathcal{W}}}(W), p)\\
&\quad = E^{n-1}_{X^{\mathcal{W}},\mathbb{R}}(W)(p-1) \Big/ (F^p E^{n-1}_{X^{\mathcal{W}}}(W) + \bar{F}^p E^{n-1}_{X^{\mathcal{W}}}(W)) \cap E^{n-1}_{X^{\mathcal{W}}}(W)_{\mathbb{R}}(p-1)\\
&\quad = E^{n-1}_{X^{\mathcal{W}},\mathbb{R}}(W)(p-1) \cap \bar{F}^{n-p} E^{n-1}_{X^{\mathcal{W}}}(W)) \cap F^{n-p} E^{n-1}_{X^{\mathcal{W}}}(W),\\
&\mathcal{D}_m(D^{X^{\mathcal{T}}}(W), q)\\
&\quad = D^{X^{\mathcal{T}},\mathbb{R}}_m(W^{\infty})(q) \cap F_q D^{X^{\mathcal{T}}}_m(W) \cap \bar{F}_q D^{X^{\mathcal{T}}}_m(W))\\
&\quad = D^{X^{\mathcal{T}},\mathbb{R}}_{n-1}(W)(p-1) \cap F_{p-1} D^{X^{\mathcal{T}}}_{n-1}(W) \cap \bar{F}_{p-1} D^{X^{\mathcal{T}}}_{n-1}(W)).
\end{aligned}
$$

Therefore, the first statement follows from the duality between $E_{X^{\mathcal{W}}}(W)$ and $D^{X^{\mathcal{T}}}(W)$ and the fact that, under this duality, $D^{X^{\mathcal{T}},\mathbb{R}}_{n-1}(W)(p-1)$ is identified with the dual of $E^{n-1}_{X^{\mathcal{W}},\mathbb{R}}(W)(p-1)$ and $F_{p-1}D^{X^{\mathcal{T}}}_{n-1}(W)$ is identified with the dual of $\bar{F}^{n-p}E^{n-1}_{X^{\mathcal{T}}}(W)$.

The compatibility with the differential is a straightforward computation using the formulas for the differential given in [4, Theorem 2.6]. For instance, if $\omega \in \mathcal{D}^n(E_{X^{\mathcal{W}}}(W), p)$, with $n < 2p-1$ and $T \in \mathcal{D}_m(D^{X^{\mathcal{T}}}(W), q)$, with $m = n$ and $q = p-1$, then we have

$$
\begin{aligned}
(\mathrm{d}_{\mathcal{D}} T)(\omega) &= (\mathrm{d}\, T)(\omega)\\
&= (-1)^n T(\mathrm{d}\,\omega)\\
&= (-1)^n T(F^{n-p+1,n-p+1}\,\mathrm{d}\,\omega)\\
&= (-1)^n T(-\,\mathrm{d}_{\mathcal{D}}\,\omega).
\end{aligned}
$$

In the third equality we have used that $T \in F_q \cap \bar{F}_q = F_{p-1,p-1}$, which implies that, for any form η, we have $T(\eta) = T(F^{n-p+1,n-p+1}\eta)$. The other cases are analogous.

Similarly, the compatibility with the product follows from [4, Theorem 2.6]. For instance, let $\omega \in \mathcal{D}^n(E_{X^{\mathcal{W}}}(W), p)$, $T \in \mathcal{D}_m(D^{X^{\mathcal{T}}}(W), q)$ and $\eta \in \mathcal{D}^l(E_{X^{\mathcal{W}}}(W), r)$, with $n - m + l = 1$ and $p - q + r = 1$. Assume that $n < 2p$, $m > 2q$, $l \geq 2r$, then

$$
(\omega \bullet T)(\eta) = ((-1)^n r_p(\omega) \wedge T + \omega \wedge r_q(T))(\eta),
$$

where $r_p(\omega) = 2\pi_p(F^p\,\mathrm{d}\,\omega)$ and $r_q(T) = 2\pi_q(F_q\,\mathrm{d}\,T)$.

But

$$
(-1)^n r_p(\omega) \wedge T(\eta) = (-1)^n T(\eta \wedge r_p(\omega)),
$$

and

$$
\begin{aligned}
(\omega \wedge r_q(T))(\eta) &= r_q(T)(\eta \wedge \omega)\\
&= 2\pi_q F_q(\mathrm{d}\,T)(\eta \wedge \omega)\\
&= 2F_q(\mathrm{d}\,T)(\eta \wedge \omega)\\
&= 2\partial T_{q+1,m-q}(\eta \wedge \omega)\\
&= T\left(2(-1)^{m-1}\partial(\eta \wedge \omega)^{q,m-q}\right)\\
&= T\left(2(-1)^{n+l}\partial(\eta \wedge \omega)^{p+r-1,n+l-p-r}\right).
\end{aligned}
$$

On the other hand

$$T(\eta \bullet \omega) = T\left(\eta \wedge r_p(\omega) + (-1)^l 2\partial(\omega \wedge \eta)^{p+r-1,n+l-p-r}\right).$$

The other cases are analogous. □

DUALITY. We summarize in the next proposition the basic properties of formal Deligne cohomology and tempered Deligne homology that follow from the previous discussions.

Proposition 2.25

For every pair of integers n and p, by applying the exact functors $\mathcal{D}^*(\_,p)$ and $\mathcal{D}_*(\_,p-1)$ to the diagrams (2.1) and (2.9) respectively, we obtain the corresponding diagrams of Deligne complexes that are the topological dual of each other. In particular we obtain long exact sequences

$$\begin{aligned} H^n_{\mathcal{D}^f,c}(W^\infty,\mathbb{R}(p)) \to H^n_{\mathcal{D}^f}(Y^\infty,\mathbb{R}(p)) \to H^n_{\mathcal{D}^f}(Z^\infty,\mathbb{R}(p)) \\ \to H^{n+1}_{\mathcal{D}^f,c}(W^\infty,\mathbb{R}(p)) \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} \leftarrow H^{\mathcal{D}^\mathcal{T}}_{n-1}(W^\infty,\mathbb{R}(p-1)) \leftarrow H^{\mathcal{D}^\mathcal{T}}_{n-1}(Y^\infty,\mathbb{R}(p-1)) \\ \leftarrow H^{\mathcal{D}^\mathcal{T}}_{n-1}(Z^\infty,\mathbb{R}(p-1)) \leftarrow H^{\mathcal{D}^\mathcal{T}}_{n}(W^\infty,\mathbb{R}(p-1)) \end{aligned} \tag{2.27}$$

and pairings

$$\begin{gathered} H^n_{\mathcal{D}^f}(Y^\infty,\mathbb{R}(p)) \otimes H^{\mathcal{D}^\mathcal{T}}_{n-1}(Y^\infty,\mathbb{R}(p-1)) \longrightarrow \mathbb{R}, \\ H^n_{\mathcal{D}^f,c}(W^\infty,\mathbb{R}(p)) \otimes H^{\mathcal{D}^\mathcal{T}}_{n-1}(W^\infty,\mathbb{R}(p-1)) \longrightarrow \mathbb{R}, \\ H^n_{\mathcal{D}^f}(Z^\infty,\mathbb{R}(p)) \otimes H^{\mathcal{D}^\mathcal{T}}_{n-1}(Z^\infty,p-1) \longrightarrow \mathbb{R}. \end{gathered}$$

that are compatible with the above sequences.

Moreover, the topologies of the space of differential forms and of the space of currents induce structures of topological vector spaces on the real formal Deligne cohomology groups and the tempered Deligne homology groups. The above pairings induce a perfect pairing of the corresponding separated vector spaces.

Proof. This is a direct consequence of the exactness of the functors $\mathcal{D}^*(\_,p)$ and $\mathcal{D}_*(\_,p-1)$ and Proposition 2.24. □

The image of $\mathrm{d}_\mathcal{D}$ in the complex $\mathcal{D}^*(E_\flat(U),p)$ does not need to be closed. Therefore the pairing between formal cohomology and tempered homology do not need to be perfect. Only the induced pairing in the corresponding separated vector spaces is perfect. Nevertheless, in the case of a proper algebraic complex manifold X, by Hodge theory, we obtain a perfect pairing between Deligne-Beilinson cohomology and homology.

Corollary 2.28 (Exceptional duality for Deligne cohomology)

Let X be a proper complex algebraic manifold, equidimensional of dimension d. Then there is a perfect duality

$$H^n_{\mathcal{D}}(X, \mathbb{R}(p)) \otimes H^{2d-n+1}_{\mathcal{D}}(X, \mathbb{R}(d-p+1)) \longrightarrow \mathbb{R}$$

which is compatible, up to a sign, with the product in Deligne cohomology.

Proof. By Poincaré duality in Deligne cohomology (cf. [11, 1.5]) there is a natural isomorphism

$$H^{2d-n+1}_{\mathcal{D}}(X, \mathbb{R}(d-p+1)) \cong H^{\mathcal{D}}_{n-1}(X, \mathbb{R}(p-1)).$$

By Hodge theory we know that

$$H^n_{\mathcal{D}}(X, \mathbb{R}(p)) = \begin{cases} H^{n-1}(X, \mathbb{R}(p-1)) \cap \overline{F}^{n-p} \cap F^{n-p}, & \text{if} \quad n < 2p, \\ H^n(X, \mathbb{R}(p)) \cap \overline{F}^p \cap F^p, & \text{if} \quad n \geq 2p. \end{cases}$$

Moreover, the pairing is given, up to a sign, by the wedge product of differential forms followed by the integral along X. Therefore, by Serre's duality, the pairing of Proposition 2.25 is perfect. □

2.4 Semi-purity of tempered Deligne cohomology

VANISHING THEOREMS. The aim of this section is to prove the following result

Theorem 2.29 (Semi-purity of tempered Deligne homology)

Let X be a projective complex algebraic manifold, W a locally closed subvariety, of dimension at most p. Then

$$H^{\mathcal{D}^{\mathcal{T}}}_n(W^\infty, \mathbb{R}(e)) = 0, \text{ for all } n > \max(e+p, 2p-1).$$

Proof. We will prove the result by ascending induction over p. The result is trivially true for $p < 0$. Then, by the exact sequence (2.27) and induction, one is reduced to the case W closed.

We will deduce the theorem by duality from the following proposition

Proposition 2.30

Let Y be a closed subvariety of a projective complex algebraic manifold. Let p be the dimension of Y. Then

$$H^{n+1}_{\mathcal{D}^f}(Y^\infty, \mathbb{R}(e+1)) = 0, \text{ for all } n > \max(e+p, 2p-1)$$

Proof. Let $\mathscr{I}_Y$ be the ideal of holomorphic functions on X vanishing at Y. We denote

$$\Omega^q_{Y^\infty} = \varprojlim_k \Omega^q_X \Big/ \mathscr{I}^k_Y \Omega^q_X.$$

By [12, Theorem 5.12] we have

Lemma 2.31

The complex of sheaves $\mathscr{E}^{q,*}_{Y,\mathbb{R}}$ is a fine resolution of $\Omega^q_{Y^\infty}$.

Since, by [14], the sheaf $\mathscr{E}^*_{Y^\infty,\mathbb{R}}$ is an acyclic resolution of the constant sheaf $\underline{\mathbb{R}}_Y$, from Lemma 2.31 and the techniques of [4], we deduce that $H^*_{\mathcal{D}^f}(Y^\infty, \mathbb{R}(e+1))$ is isomorphic to the hypercohomology of the complex of sheaves

$$\mathbb{R}_{\mathcal{D}^f,Y^\infty}(e) := \mathbb{R}_Y(f) \longrightarrow \Omega^0_{Y^\infty} \longrightarrow \ldots \longrightarrow \Omega^e_{Y^\infty}. \tag{2.32}$$

Lemma 2.33

If $n > p$ then $H^n(Y, \Omega^q_{Y^\infty}) = 0$.

Proof. By [10, Proposition I.6.1]

$$H^n(Y, \Omega^q_{Y^\infty}) = H^n(Y^{\mathrm{alg}}, \hat{\Omega}^q_Y),$$

where Y^{alg} is the corresponding algebraic variety and $\hat{\Omega}^q_Y$ is the completion of the sheaf of algebraic differentials. But now Y^{alg} is a noetherian topological space of dimensionVp, hence the lemma. □

Using Lemma 2.33 we obtain that the $E_1^{s,t}$ term of the spectral sequence of the hypercohomology of the complex (2.32) can be non zero only for $s = 0$, $0 \leq t \leq 2p$ and $1 \leq s \leq e+1$, $0 \leq t \leq p$, which implies Proposition 2.30. □

We finish now the proof of the theorem. By Proposition 2.30, for every $n > \max(p+e, 2p-1)$, the morphism

$$\mathrm{d}^n_{\mathcal{D}} : \mathcal{D}^n(E_{X^W}(Y), e+1) \longrightarrow \mathcal{D}^{n+1}(E_{X^W}(Y), e+1)$$

satisfies $\mathrm{Im}(\mathrm{d}^n_{\mathcal{D}}) = \mathrm{Ker}(\mathrm{d}^{n+1}_{\mathcal{D}})$, hence the image of $\mathrm{d}^n_{\mathcal{D}}$ is a closed subspace. Therefore, by [1, IV. 2 Theorem 1], we have that the dual morphism

$$\mathrm{d}_{\mathcal{D}} : \mathcal{D}_n(D^{X^{\mathcal{T}}}(Y), e) \longrightarrow \mathcal{D}_{n-1}(D^{X^{\mathcal{T}}}(Y), e)$$

has closed image. This implies that, for $n \geq \max(p+e, 2p-1)$, the vector space $H_n^{\mathcal{D}^{\mathcal{T}}}(Y^\infty, \mathbb{R}(e))$ is separated. Therefore, by Proposition 2.25, for $n > \max(p+e, 2p-1)$ the pairing

$$H^{n+1}_{\mathcal{D}^f}(Y^\infty, \mathbb{R}(e+1)) \otimes H_n^{\mathcal{D}^{\mathcal{T}}}(Y^\infty, \mathbb{R}(e)) \longrightarrow \mathbb{R}$$

is perfect. Hence by Proposition 2.30 we obtain the theorem. □

SEMI-PURITY OF TEMPERED DELIGNE COHOMOLOGY. The semi-purity theorem can be stated in terms of tempered Deligne cohomology as follows.

Corollary 2.34

Let X be a complex quasi-projective manifold and Y a closed subvariety of codimension at least p. Then

$$H^n_{\mathcal{D}^{\mathcal{T}},Y}(X, \mathbb{R}(e)) = 0, \text{ for all } n < \min(e+p, 2p+1),$$

In particular

$$H^n_{\mathcal{D}^{\mathcal{T}},Y}(X, \mathbb{R}(p)) = 0, \text{ for all } n < 2p.$$

This is the weak purity property used in [6, 6.4].

3. Arithmetic Intersection Theory

3.1 Definition of Covariant arithmetic Chow groups

In [3], the author introduced a variant of the arithmetic Chow groups that are covariant with respect to arbitrary proper morphisms. In the paper [6] these groups are further studied as an example of cohomological arithmetic Chow groups. These groups are denoted by $\widehat{\mathrm{CH}}^*(X,\mathcal{D}_{\mathrm{cur}})$. The semi-purity property (Corollary 2.34) was announced in [6] and has consequences in the behavior of the covariant arithmetic Chow groups. On the other hand, Kawaguchi and Moriwaki [13] have given another definition of covariant arithmetic Chow groups called D-arithmetic Chow groups. A consequence of Corollary 2.34 is that, when X is equidimensional and generically projective, both definitions of covariant arithmetic Chow groups agree. We note that Zha [16] has also introduced a notion of covariant arithmetic Chow groups that only differs from the definition of [13] on the fact that he neglects the anti-linear involution F^{∞}.

In this section we will summarize the properties of the covariant arithmetic Chow groups. We will follow the notations and terminology of [6], but we will use the grading by dimension that is more natural when dealing with covariant Chow groups.

Arithmetic rings and arithmetic varieties. Let A be an arithmetic ring (see [7]) with fraction field F. In particular A is provided with a non empty set of complex embeddings Σ and a conjugate linear involution F_∞ of $\mathbb{C}^\Sigma$ that commutes with the diagonal embedding of A in $\mathbb{C}^\Sigma$. Since we will be working with dimension of cycles, following [8] we will further impose that A is equicodimensional and Jacobson. Let $S=\operatorname{Spec} A$ and let $e=\dim S$.

An arithmetic variety X is a flat quasi-projective scheme over A, that has smooth generic fiber X_F. To every arithmetic variety X we can associate a complex algebraic manifold X_Σ and a real algebraic manifold $X_{\mathbb{R}}=(X_\Sigma,F_\infty)$.

The arithmetic complex of tempered Deligne homology. To every pair of integers n,p, and every open Zariski subset U of $X_{\mathbb{R}}$ we assign the group

$$\mathcal{D}_n^{\mathrm{cur},X}(U,p)=\mathcal{D}_n\Big(D_*^{X_\Sigma^{\mathcal{T}}}(U),p\Big)^{\sigma},$$

where σ is the involution that acts as complex conjugation on the space and on the currents. That is, if $T\in D_n(X_{\mathbb{C}})$ then $\sigma(T)=\overline{(F_\infty)_*T}$. And $(\quad)^\sigma$ denote the elements that are fixed by σ. Then $\mathcal{D}_n^{\mathrm{cur},X}(\_,p)$ is a totally acyclic sheaf (in the sense of [6]) for the real scheme underlying $X_{\mathbb{R}}$. When X is fixed, $\mathcal{D}_*^{\mathrm{cur},X}$ will be denoted by $\mathcal{D}_*^{\mathrm{cur}}$.

If U is a Zariski open subset of $X_{\mathbb{R}}$ and $Y=X\setminus U_{\mathbb{R}}$ we write

$$H_*^{\mathcal{D}^{\mathcal{T}}}(U,\mathbb{R}(p))=H_*(\mathcal{D}^{\mathrm{cur}}(U,p)),\tag{3.1}$$

$$H_*^{\mathcal{D}^{\mathcal{T}},Y}(X_{\mathbb{R}},\mathbb{R}(p))=H_*(s(\mathcal{D}^{\mathrm{cur}}(U,p),\mathcal{D}^{\mathrm{cur}}(X_{\mathbb{R}},p))),\tag{3.2}$$

$$\widetilde{\mathcal{D}}_{2p-1}^{\mathrm{cur}}(X_{\mathbb{R}},p)=\mathcal{D}_{2p-1}^{\mathrm{cur}}(X_{\mathbb{R}},p)\,/\operatorname{Im}\mathrm{d}_{\mathcal{D}},\tag{3.3}$$

$$\mathrm{Z}\mathcal{D}_{2p}^{\mathrm{cur}}(X_{\mathbb{R}},p)=\operatorname{Ker}(\mathrm{d}_{\mathcal{D}}:\mathcal{D}_{2p}^{\mathrm{cur}}(X_{\mathbb{R}},p)\longrightarrow\mathcal{D}_{2p+1}^{\mathrm{cur}}(X_{\mathbb{R}},p)).\tag{3.4}$$

Let $\mathcal{Z}_p = \mathcal{Z}_p(X_{\mathbb{R}})$ be the set of dimension p Zariski closed subsets of $X_{\mathbb{R}}$ ordered by inclusion. Then we will write

$$\begin{aligned}
\mathcal{D}^{\mathrm{cur}}_*(X_{\mathbb{R}} \setminus \mathcal{Z}_p, p) &= \varinjlim_{Y \in \mathcal{Z}^p} \mathcal{D}^{\mathrm{cur}}_*(X_{\mathbb{R}} \setminus Y, p),\\
\widetilde{\mathcal{D}}^{\mathrm{cur}}_*(X_{\mathbb{R}} \setminus \mathcal{Z}_p, p) &= \mathcal{D}^{\mathrm{cur}}_*(X_{\mathbb{R}} \setminus \mathcal{Z}_p, p) \,/ \operatorname{Im} \mathrm{d}_{\mathcal{D}}\ ,\\
H^{\mathcal{D}^{\mathcal{T}},\mathcal{Z}_p}_*(X_{\mathbb{R}}, \mathbb{R}(p)) &= H_*(s(\mathcal{D}^{\mathrm{cur}}(X_{\mathbb{R}} \setminus \mathcal{Z}_p, p), \mathcal{D}^{\mathrm{cur}}(X_{\mathbb{R}}, p))).
\end{aligned}$$

Green objects. We recall the definition of Green object for a cycle given in [6] but adapted to the grading by dimension. Let y be a dimension p algebraic cycle of $X_{\mathbb{R}}$. Let Y be the support of y. The class of y in $H^{\mathcal{D}^{\mathcal{T}},Y}_{2p}(X_{\mathbb{R}}, \mathbb{R}(p))$, denoted $\mathrm{cl}(y)$, is represented by the pair $(\delta_y, 0) \in s(\mathcal{D}^{\mathrm{cur}}(X_{\mathbb{R}}, p), \mathcal{D}^{\mathrm{cur}}(U_{\mathbb{R}}, p))$. We denote also by $\mathrm{cl}(y)$ the image of this class in $H^{\mathcal{D}^{\mathcal{T}},\mathcal{Z}_p}_{2p}(X_{\mathbb{R}}, \mathbb{R}(p))$.

In this setting, the truncated homology classes can be written as

$$\widehat{H}^{\mathcal{D}^{\mathcal{T}},\mathcal{Z}_p}_*(X_{\mathbb{R}}, \mathbb{R}(p)) = \{(\omega_y, \widetilde{g}_y) \in \mathrm{Z}\mathcal{D}^{\mathrm{cur}}_{2p}(X, p) \oplus \widetilde{\mathcal{D}}^{\mathrm{cur}}_{2p-1}(X_{\mathbb{R}} \setminus \mathcal{Z}_p, p) \mid \mathrm{d}_{\mathcal{D}}\, \widetilde{g}_y = \omega_y\}.$$

There is an obvious class map

$$\mathrm{cl} : \widehat{H}^{\mathcal{D}^{\mathcal{T}},\mathcal{Z}_p}_*(X_{\mathbb{R}}, \mathbb{R}(p)) \longrightarrow H^{\mathcal{D}^{\mathcal{T}},\mathcal{Z}_p}_*(X_{\mathbb{R}}, \mathbb{R}(p)).$$

Then a Green object for y is an element

$$\mathfrak{g}_y = (\omega_y, \widetilde{g}_y) \in \widehat{H}^{\mathcal{D}^{\mathcal{T}},\mathcal{Z}_p}_{2p}(X_{\mathbb{R}}, \mathbb{R}(p))$$

such that $\mathrm{cl}(\mathfrak{g}_y) = \mathrm{cl}(y)$.

The following result follows directly from the definition

Lemma 3.5

An element $\mathfrak{g}_y = (\omega_y, \widetilde{g}_y) \in \widehat{H}^{\mathcal{D}^{\mathcal{T}},\mathcal{Z}_p}_{2p}(X_{\mathbb{R}}, \mathbb{R}(p))$ is a Green object for y if and only if there exists a current $\widetilde{\gamma} \in \widetilde{\mathcal{D}}^{\mathrm{cur}}_{2p-1}(X_{\mathbb{R}}, p)$ such that

$$\begin{aligned}
\widetilde{g}_y &= \widetilde{\gamma}|_{X \setminus \mathcal{Z}_p}\\
\mathrm{d}_{\mathcal{D}}\, \widetilde{\gamma} + \delta_y &= \omega_y.
\end{aligned}$$

Arithmetic Chow groups. Every dimension p algebraic cycle y on X defines a dimension $(p-e)$ algebraic cycle $y_{\mathbb{R}}$ on $X_{\mathbb{R}}$, where e is the dimension of the base scheme S.

Definition 3.6 The group of arithmetic cycles of dimension p is defined as

$$\widehat{\mathrm{Z}}_p(X, \mathcal{D}^{\mathrm{cur}}) = \{(y, \mathfrak{g}_y) \in \mathrm{Z}_p(X) \oplus \widehat{H}^{\mathcal{D}^{\mathcal{T}},\mathcal{Z}_{p-e}}_{2p-2e}(X_{\mathbb{R}}, \mathbb{R}(p-e)) \mid \mathrm{cl}(y_{\mathbb{R}}) = \mathrm{cl}(\mathfrak{g}_y)\}.$$

Let W be a dimension $p+1$ irreducible subvariety of X and $f \in K(W)^*$ be a rational function. Let $\widetilde{W_{\mathbb{R}}}$ be a resolution of singularities of $W_{\mathbb{R}}$ and let $\iota : \widetilde{W_{\mathbb{R}}} \longrightarrow X_{\mathbb{R}}$ be the induced map. Then we write

$$\widehat{\operatorname{div}}\, f = \left(\operatorname{div} f, \Big(0, \iota_*\big(-\frac{1}{2} \log f\bar{f}\big)\Big)\right).$$

The group of cycles rationally equivalent to zero is the subgroup

$$\widehat{\mathrm{Rat}}_p(X,\mathcal{D}^{\mathrm{cur}}) \subset \widehat{\mathrm{Z}}_p(X,\mathcal{D}^{\mathrm{cur}})$$

generated by the elements of the form $\widehat{\operatorname{div}} f$. The *homological arithmetic Chow groups* of X are defined as

$$\widehat{\mathrm{CH}}_p(X,\mathcal{D}^{\mathrm{cur}}) = \widehat{\mathrm{Z}}_p(X,\mathcal{D}^{\mathrm{cur}}) \Big/ \widehat{\mathrm{Rat}}_p(X,\mathcal{D}^{\mathrm{cur}}).$$

There are well-defined maps

$$\begin{aligned}
&\zeta : \widehat{\mathrm{CH}}_p(X,\mathcal{D}^{\mathrm{cur}}) \longrightarrow \mathrm{CH}_p(X), && \zeta[y,\mathfrak{g}_y]=[y],\\
&\rho : \mathrm{CH}_{p,p+1}(X) \longrightarrow H^{\mathcal{D}^{\mathcal{T}}}_{2p-2e+1}(X,p-e) \subseteq \widetilde{\mathcal{D}}^{\mathrm{cur}}_{2p+1}(X,p), && \rho[f]=\mathrm{cl}(f),\\
&\mathrm{a} : \widetilde{\mathcal{D}}_{2p-2e+1}(X,p-e) \longrightarrow \widehat{\mathrm{CH}}_p(X,\mathcal{D}^{\mathrm{cur}}), && \mathrm{a}(\widetilde{a})=[0,\mathrm{a}(\widetilde{a})],\\
&\omega : \widehat{\mathrm{CH}}_p(X,\mathcal{D}^{\mathrm{cur}}) \longrightarrow \mathrm{Z}\mathcal{D}^{\mathrm{cur}}_{2p-2e}(X,p-e), && \omega[y,\mathfrak{g}_y]=\omega(\mathfrak{g}_y),\\
&h : \mathrm{Z}\mathcal{D}^{\mathrm{cur}}_{2p}(X,p) \longrightarrow H^{\mathcal{D}^{\mathcal{T}}}_{2p}(X,p), && h(\alpha)=[\alpha].
\end{aligned}$$

3.2 Properties of Covariant arithmetic Chow groups

Basic properties. Recall that in [6], there are defined contravariant arithmetic Chow groups denoted by $\widehat{\mathrm{CH}}^*(X,\mathcal{D}_{\log})$. The following result follows from the theory developed [6] and Corollary 2.34 (semi-purity property).

Theorem 3.7

With the above notations, we have the following statements:

(i) *There are exact sequences*

$$\mathrm{CH}_{p,p+1}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{\mathrm{cur}}_{2p-2e+1}(X,p-e) \xrightarrow{\mathrm{a}} \widehat{\mathrm{CH}}_p(X,\mathcal{D}^{\mathrm{cur}}) \xrightarrow{\zeta} \mathrm{CH}_p(X) \longrightarrow 0.$$

$$\mathrm{CH}_{p,p+1}(X) \xrightarrow{\rho} H^{\mathcal{D}^{\mathcal{T}}}_{2p-2e+1}(X_{\mathbb{R}},\mathbb{R}(p-e)) \xrightarrow{\mathrm{a}} \widehat{\mathrm{CH}}_p(X,\mathcal{D}^{\mathrm{cur}}) \xrightarrow{(\zeta,-\omega)}$$
$$\mathrm{CH}_p(X) \oplus \mathrm{Z}\mathcal{D}^{\mathrm{cur}}_{2p-2e}(X,p-e) \xrightarrow{\mathrm{cl}+h} H^{\mathcal{D}^{f}}_{2p-2e}(X_{\mathbb{R}},\mathbb{R}(p-e)) \longrightarrow 0.$$

In particular, if X_F is projective, then there is an exact sequence

$$\mathrm{CH}_{p,p+1}(X) \xrightarrow{\rho} H^{\mathcal{D}}_{2p-2e+1}(X_{\mathbb{R}},\mathbb{R}(p-e)) \xrightarrow{\mathrm{a}} \widehat{\mathrm{CH}}_p(X,\mathcal{D}^{\mathrm{cur}}) \xrightarrow{(\zeta,-\omega)}$$
$$\mathrm{CH}_p(X) \oplus \mathrm{Z}\mathcal{D}^{\mathrm{cur}}_{2p-2e}(X,p-e) \xrightarrow{\mathrm{cl}+h} H^{\mathcal{D}}_{2p-2e}(X_{\mathbb{R}},\mathbb{R}(p-e)) \longrightarrow 0.$$

(ii) *For any regular arithmetic variety X over A there are defined contravariant arithmetic Chow groups $\widehat{\mathrm{CH}}^p(X,\mathcal{D}_{\log})$. Furthermore, if X is equidimensional of dimension d, then there is a morphism of arithmetic Chow groups*

$$\widehat{\mathrm{CH}}^p(X,\mathcal{D}_{\log}) \longrightarrow \widehat{\mathrm{CH}}_{d-p}(X,\mathcal{D}^{\mathrm{cur}}).$$

When X_F is projective this morphism is a monomorphism. Moreover, if X_F has dimension zero, this morphism is an isomorphism.

(iii) *For any proper morphism $f : X \longrightarrow Y$ of arithmetic varieties over A, there is a morphism of covariant arithmetic Chow groups*

$$f_* : \widehat{\mathrm{CH}}_p(X, \mathcal{D}_{\mathrm{cur}}) \longrightarrow \widehat{\mathrm{CH}}_p(Y, \mathcal{D}_{\mathrm{cur}}).$$

If $g : Y \longrightarrow Z$ is another such morphism, the equality $(g \circ f)_* = g_* \circ f_*$ holds. Moreover, if X and Y are regular and $f_F : X_F \longrightarrow Y_F$ is a smooth proper morphism of projective varieties, then f_* is compatible with the direct image of contravariant arithmetic Chow groups.

(iv) *If $f : X \longrightarrow Y$ is a flat morphism, equidimensional of relative dimension d, and such that f_F is smooth, then there is a pull-back map*

$$f^* : \widehat{\mathrm{CH}}_p(Y, \mathcal{D}^{\mathrm{cur}}) \longrightarrow \widehat{\mathrm{CH}}_{p+d}(X, \mathcal{D}^{\mathrm{cur}}).$$

If X and Y are regular and equidimensional, this map is equivalent with the pullback map defined in the contravariant Chow groups.

(v) *Let $f : X \longrightarrow Y$ be a flat map between arithmetic varieties, which is smooth over F and let $g : P \longrightarrow Y$ be a proper map. Let Z be the fiber product of X and P over Y, with $p : Z \longrightarrow P$ and $q : Z \longrightarrow X$ the two projections. Thus p is flat and smooth over F and q is proper. Then for any $x \in \widehat{\mathrm{CH}}_*(P, \mathcal{D}^{\mathrm{cur}})$, it holds*

$$q_* p^*(x) = f^* g_*(x) \in \widehat{\mathrm{CH}}_*(X, \mathcal{D}^{\mathrm{cur}}).$$

Proof. Part (i) follows from the standard exact sequences of [6, Theorem 4.13] adapted to the grading by dimension and Corollary 2.34.

For (ii) we first note that, if M is an equidimensional complex algebraic manifold, $D \subset X$ is a normal crossing divisor, ω is a differential form with logarithmic singularities along D and η is a form that is flat along D, then $\eta \wedge \omega$ is flat along D. In particular, if M is proper and $U = M \setminus D$, then the associated current $[\omega]$ belongs to $D^{\mathrm{extd}}_*(U)$. Therefore, if y is a codimension p cycle on X then, by the assumptions on X and on the arithmetic ring, y is a dimension $d-p$ algebraic algebraic cycle. Moreover, if $(\omega_y, \widetilde{g}_y)$ is a Green form for y (i.e. a $\mathcal{D}_{\log}$-Green object for y) then, by Lemma 3.5 and [6, Proposition 6.5] we have that $([\omega_y], [\widetilde{g}_y])$ is a $\mathcal{D}^{\mathrm{cur}}$-Green object for y. Thus we have a well defined map

$$\widehat{\mathrm{Z}}^p(X, \mathcal{D}_{\log}) \longrightarrow \widehat{\mathrm{Z}}_{d-p}(X, \mathcal{D}^{\mathrm{cur}}).$$

By definition this map is compatible with rational equivalence, hence we obtain a map at the level of Chow groups.

To prove (iii) we first observe that, if $Z \subset X_\Sigma$ is a closed subset, then $f_* D^{X^{\mathcal{I}}_\Sigma}_*(Z) \subset D^{X^{\mathcal{I}}_\Sigma}(f(Z))$. Therefore, the push-forward of currents define a covariant f-morphism

$$f_\# : f_* \mathcal{D}^{\mathrm{cur},X}_* \longrightarrow \mathcal{D}^{\mathrm{cur},Y}_*.$$

Here we are using the terminology of [6, 3.67] but adapted to the grading by dimension. Therefore applying [6, § 4.5] we obtain the push-forward map for covariant arithmetic Chow groups. More concretely this map is defined as

$$f_*(y, (\omega_y, \widetilde{g}_y)) = (f_* y, (f_* \omega_y, (f_* g_y)^{\widetilde{\ }})).$$

It is straightforward to check that it is compatible with the direct image of $\mathcal{D}_{\log}$-arithmetic Chow groups when Y is projective and f_F smooth.

We now prove (iv). Since f_F is smooth, for any Zariski closed subset $Z \subset Y_{\mathbb{R}}$ equidimensional of dimension p, there is a well defined morphism $f^* D_n(Y_\Sigma) \longrightarrow D_{n+2d}(X_\Sigma)$ that sends $D_n^{Y_\Sigma^{\mathcal{T}}}(Z)$ to $D_{n+2d}^{X_\Sigma^{\mathcal{T}}}(f^{-1}(Z))$. Therefore we obtain well defined morphisms

$$\begin{array}{rcl} f^\# : \mathcal{D}_n^{\mathrm{cur}}(Y_{\mathbb{R}}, p) & \longrightarrow & \mathcal{D}_{n+2d}^{\mathrm{cur}}(X_{\mathbb{R}}, p+d), \\ f^\# : \mathcal{D}_n^{\mathrm{cur}}(Y_{\mathbb{R}} \setminus Z, p) & \longrightarrow & \mathcal{D}_{n+2d}^{\mathrm{cur}}(X_{\mathbb{R}} \setminus f^{-1}Z, p+d), \end{array}$$

that send T to $f^*T/(2\pi i)^d$. Then the proof of (iv) is straightforward using the theory of [6, 4.4] adapted to the grading by dimension.

(v) Follows as [8, Lemma 11]. □

Multiplicative properties. In the next result we state the multiplicative properties between covariant and contravariant Chow groups. The proofs are simple modification of [8, Theorem 3]. First, for a form $\eta \in \widetilde{\mathcal{D}}_{\log}^{2p-1}(X_{\mathbb{R}}, p)$ and an element $x \in \widehat{\mathrm{CH}}_q(X, \mathcal{D}^{\mathrm{cur}})$ we define

$$\eta \cap x = \mathrm{a}(\eta \bullet \omega(x)) = \mathrm{a}(\eta \wedge \omega(x)).$$

Theorem 3.8

Given a map $f : X \longrightarrow Y$ of arithmetic varieties, with Y regular, there is a cap product

$$\begin{array}{ccc} \widehat{\mathrm{CH}}^p(Y, \mathcal{D}_{\log}) \otimes \widehat{\mathrm{CH}}_q(X, \mathcal{D}^{\mathrm{cur}}) & \longrightarrow & \widehat{\mathrm{CH}}_{q-p}(X, \mathcal{D}^{\mathrm{cur}})_{\mathbb{Q}} \\ y \otimes x & \longmapsto & y \cdot_f x \end{array}$$

which is also denoted $y \cap X$ if $X = Y$. This product satisfies the following properties

(i) *$\omega(y \cdot_f x) = f^*\omega(y) \wedge \omega(x)$, and, for any $\eta \in \widetilde{\mathcal{D}}_{\log}^{2p-1}(Y_{\mathbb{R}}, p)$, it holds $\mathrm{a}(\eta) \cdot_f x = \mathrm{a}(f^*(\eta)) \cap x$.*

(ii) *$\widehat{\mathrm{CH}}_*(X, \mathcal{D}^{\mathrm{cur}})_{\mathbb{Q}}$ is a graded $\widehat{\mathrm{CH}}^*(Y, \mathcal{D}_{\log})$-module.*

(iii) *If $g : Y \longrightarrow Y'$ is a map of arithmetic varieties with Y' also regular, $y' \in \widehat{\mathrm{CH}}^p(Y', \mathcal{D}_{\log})$ and $x \in \widehat{\mathrm{CH}}_q(X, \mathcal{D}^{\mathrm{cur}})$, then $y' \cdot_{gf} x = g^*(y') \cdot_f x$.*

(iv) *If $h : X' \longrightarrow X$ is a projective morphism, $x' \in \widehat{\mathrm{CH}}_q(X', \mathcal{D}^{\mathrm{cur}})$ and $y \in \widehat{\mathrm{CH}}^p(Y, \mathcal{D}_{\log})$, then $y \cdot_f (h_*(x')) = h_*(y \cdot_{fh} x')$.*

(v) *If $h : X' \longrightarrow X$ is flat and smooth over F, $x \in \widehat{\mathrm{CH}}_q(X, \mathcal{D}^{\mathrm{cur}})$, $y \in \widehat{\mathrm{CH}}^p(Y, \mathcal{D}_{\log})$, then $h^*(y \cdot_f x) = y \cdot_f (h^*(x))$.*

(vi) *Let $f : X \longrightarrow Y$ be a flat map between arithmetic varieties, with Y regular and projective, and let $g : P \longrightarrow Y$ be a proper smooth map of arithmetic varieties of relative dimension d. Let Z be the fiber product of X and P over Y, with $p : Z \longrightarrow P$ and $q : Z \longrightarrow X$ the two projections. Then, for all $x \in \widehat{\mathrm{CH}}_p(X, \mathcal{D}^{\mathrm{cur}})$ and $\gamma \in \widehat{\mathrm{CH}}^q(P, \mathcal{D}_{\log})$, it holds the equality*

$$q_*(\gamma \cdot_p q^*(x)) = g_* \gamma \cdot_f \alpha.$$

Proof. To define $y._f x$ we follow closely [8]. We may assume that Y is equidimensional, that $x = (V, \mathfrak{g}_V)$ with V a prime algebraic cycle and $y = (W, \mathfrak{g}_W)$ with each component of W meeting V properly on the generic fiber X_F. As in [8] we can define a cycle $[V]._f[W] \in CH_{q-p}(V \cap f^{-1}(|W|))_{\mathbb{Q}}$ that gives us a well defined cycle $([V]._f[W])_F \in \mathrm{Z}_{q-p}(X_F)$. Our task now is to construct the Green object for this cycle. Let $\mathfrak{g}_W = (\omega_W, \widetilde{g}_W)$ and $\mathfrak{g}_V = (\omega_V, \widetilde{g}_V)$. We write $U_V = X_{\mathbb{R} \setminus |V|}$, $U_W = X_{\mathbb{R} \setminus f^{-1}|W|}$ and $r = q - p$.

We now define, in analogy with [6, Theorem 3.37],

$$
\begin{aligned}
\mathfrak{g}_W *_f \mathfrak{g}_V &= f^*\mathfrak{g}_W * \mathfrak{g}_V \\
&= \Big(f^*(\omega_W) \bullet \omega_V, ((f^*(g_W) \bullet \omega_V, f^*(\omega_W) \bullet g_V), -f^*(g_W) \bullet g_V)^{\sim}\Big) \\
&= (f^*(\omega_W) \wedge \omega_V, ((f^*(g_W) \wedge \omega_V, f^*(\omega_W) \wedge g_V), \\
&\quad \partial f^*(g_W) \wedge g_V - \bar{\partial} f^*(g_W) \wedge g_V - f^*(g_W) \wedge \partial g_V + f^*(g_W) \wedge \bar{\partial} g_V)^{\sim} \\
&\in \widehat{H}_{2e}(\mathcal{D}^{\mathrm{cur}}_*(X_{\mathbb{R}}, e), s(\mathcal{D}^{\mathrm{cur}}_*(U_W, e) \oplus \mathcal{D}^{\mathrm{cur}}_*(U_V, e) \to \mathcal{D}^{\mathrm{cur}}_*(U_W \cap U_V, e))) \\
&\cong \widehat{H}_{2e}(\mathcal{D}^{\mathrm{cur}}_*(X_{\mathbb{R}}, e), \mathcal{D}^{\mathrm{cur}}_*(U_W \cup U_V, e)).
\end{aligned}
$$

Now the proof follows as in [8, Theorem 3] and Lemma 12. □

Remark 3.9

(i) The main difference between the arithmetic Chow groups introduced here and the arithmetic Chow groups used in [8] is that, if $x \in \widehat{\mathrm{CH}}_*(X, \mathcal{D}^{\mathrm{cur}})$ then $\omega(x)$ is an arbitrary current instead of a smooth differential form. This allows us to define direct images for arbitrary proper morphisms. But the price we have to pay is that there ire defined inverse images only for morphisms that are smooth over F.

(ii) The fact that the compatibility of direct images for the covariant Chow groups and direct images for the contravariant Chow groups in Theorem 3.7 is stated only for varieties that are generically projective, is due to the fact that the latter is only defined when the base is proper. There are two ways to overcome this difficulty. One is to allow arbitrary singularities at infinity in the spirit of [5, 3.5], but then, one will have to allow also arbitrary singularities at infinity for currents. This means that we will have to consider currents that are tempered in some components of the boundary but are not tempered in the other. The second option would be to use a different notion of logarithmic singularities that has better properties with respect to direct images.

RELATIONSHIP WITH OTHER ARITHMETIC CHOW GROUPS. Let us assume now that X_F is projective and let $\widehat{\mathrm{CH}}^*(X)$ denote the arithmetic Chow groups introduced in [7] and $\widehat{\mathrm{CH}}_*(X)$ denote the arithmetic Chow groups introduced in [8]. In [6] it is shown that there is an isomorphism

$$\psi : \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\log}) \longrightarrow \widehat{\mathrm{CH}}^*(X),$$

that is compatible with products, inverse images with respect to arbitrary morphisms and direct images with respect to proper morphism that are smooth over F. We shall state the analogous result for covariant arithmetic Chow groups.

Proposition 3.10

Let X be an arithmetic variety with X_F projective. Then there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow \widehat{\mathrm{CH}}_*(X) \xrightarrow{\phi} \widehat{\mathrm{CH}}_*(X,\mathcal{D}^{\mathrm{cur}}) \\ \longrightarrow \bigoplus_p \mathrm{Z}\mathcal{D}^{\mathrm{cur}}_{2p}(X_{\mathbb{R}},p) \Big/ \mathrm{Z}\mathcal{D}^{\mathrm{smooth}}_{2p}(X_{\mathbb{R}},p) \longrightarrow 0 \,, \end{aligned}$$

where $\mathcal{D}^{\mathrm{smooth}}_{2p}(X_{\mathbb{R}},p)$ denotes the subspace of currents that can be represented by smooth differential forms. Moreover ϕ satisfies the following properties

(i) *If $f : X \longrightarrow Y$ is a proper morphism of arithmetic varieties that is smooth over F and with Y_F projective, then $f_* \circ \phi = \phi \circ f_*$.*

(ii) *If $f : X \longrightarrow Y$ is a flat morphism of arithmetic varieties that is smooth over F, with X_F and Y_F projective, then $f^* \circ \phi = \phi \circ f^*$.*

(iii) *If $f : X \longrightarrow Y$ is a morphism of arithmetic varieties, with X_F and Y_F projective and Y regular then, for $y \in \widehat{\mathrm{CH}}^p(Y,\mathcal{D}_{\log})$ and $x \in \widehat{\mathrm{CH}}_q(Y)$, it holds the equality*

$$y._f\phi(x) = \psi(y)._f x.$$

Proof. Let y be a dimension p algebraic cycle of X and let g_y be a Green current for y in the sense of [8]. Recall that the normalization used here for the current δ_y differs with the normalization used in [8] by a factor $\frac{1}{(2\pi i)^p}$. Then, by 3.5, the pair

$$\left(\frac{1}{2(2\pi i)^{p+1}} g_y|_{X_{\mathbb{R}}\setminus\mathcal{Z}_p}, \frac{1}{2(2\pi i)^{p+1}}(-2\partial\bar{\partial})g_y + \delta_y\right)$$

is a $\mathcal{D}^{\mathrm{cur}}$-Green object for y. Therefore we obtain a well defined morphism $\widehat{\mathrm{Z}}_p(X) \longrightarrow \widehat{\mathrm{Z}}_p(X,\mathcal{D}^{\mathrm{cur}})$. It is straightforward to check that this map preserves rational equivalence, the exactness of the above exact sequence and properties (i), (ii) and (iii). □

Corollary 3.11

With the hypothesis of the proposition, every element $x \in \widehat{\mathrm{CH}}_p(X,\mathcal{D}^{\mathrm{cur}})$ can be represented as

$$x = \phi(x_1) + \mathrm{a}(\eta)$$

where $x_1 \in \widehat{\mathrm{CH}}_p(X)$ and $\eta \in \widetilde{\mathcal{D}}^{\mathrm{cur}}_{2p+1}(X_{\mathbb{R}},p)$. Moreover, if

$$x = \phi(x_1) + \mathrm{a}(\eta) = \phi(x_1') + \mathrm{a}(\eta')$$

are two such representations, then $\eta - \eta' \in \widetilde{\mathcal{D}}^{\mathrm{smooth}}_{2p+1}(X_{\mathbb{R}},p)$.

Proof. This follows from the previous proposition and the fact that the map

$$\mathrm{d}_{\mathcal{D}} : \widetilde{\mathcal{D}}^{\mathrm{cur}}_{2p+1}(X_{\mathbb{R}},p) \longrightarrow \mathrm{Z}\mathcal{D}^{\mathrm{cur}}_{2p}(X_{\mathbb{R}},p) \Big/ \mathrm{Z}\mathcal{D}^{\mathrm{smooth}}_{2p}(X_{\mathbb{R}},p)$$

is surjective due to the projectivity of X. The last statement follows from [7, Theorem 1.2.2]. □

The following result follows now easily from the previous corollary.

Corollary 3.12

Assume furthermore that X is equidimensional of dimension d and let $\widehat{\mathrm{CH}}^{*}_{D}(X)$ denote the D-arithmetic Chow groups introduced in [13]. Then there is a natural isomorphism

$$\bigoplus_{p} \widehat{\mathrm{CH}}^{p}_{D}(X) \longrightarrow \bigoplus_{p} \widehat{\mathrm{CH}}_{d-p}(X, \mathcal{D}^{\mathrm{cur}}).$$

Moreover this isomorphism is compatible with push-forwards and the structure of module over the contravariant arithmetic Chow groups. □

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