ON PROJECTIVE VARIETIES OF MINIMAL DEGREE

by

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DEFINITIONS AND NOTATIONS

Let $\mathbb{P}^n$ be the n-dimensional projective space over an algebraically closed field. We consider reduced equidimensional projective algebraic varieties $V = V_1 \cup \ldots \cup V_r \subseteq \mathbb{P}^n$, not contained in any hyperplane (unless otherwise stated) and set $g = \deg(V)$, $d = \dim(V)$. We will say that this variety is *connected in codimension one* if it is possible to arrange its components in such a way that

$$\operatorname{codim}_{V_j} V_j \cap (V_1 \cup \ldots \cup V_{j-1}) = 1$$

for $j = 2, \ldots, r$. For equidimensional varieties this definition coincides with the analogous definition given by Hartshorne [H2]. In fact, for any non-negative integer k the following two conditions (a) and (b) are equivalent: (a) For any closed set $W \subset V$ such that $\operatorname{codim}_V W > k$, $V-W$ is connected; (b) For $j = 2, \ldots, r$ (possibly after rearranging), $\operatorname{codim}_{V_j} V_j \cap (V_1 \cup \ldots \cup V_{j-1}) \leqslant k$. Varieties satisfying these conditions are said to be connected in codimension k.

For any subset $S \subseteq \mathbb{P}^n$ we write $< S >$ to denote the linear span of S. In particular we will set $L_i = < V_i >$, and $n_i = \dim(L_i)$. The degree of V_i will be denoted g_i.

By a *normal rational scroll* (or just a scroll) we understand an irreducible variety obtained as the image of the projectivized bundle of $\mathcal{O}(n_1) + \ldots + \mathcal{O}(n_d)$ over $\mathbb{P}^1$ under the complete linear system $|\mathcal{O}(1)|$. Such a scroll will be denoted $S = S(n_1, \ldots, n_d)$. Here $d \geqslant 1$ and we may assume, without loss of generality, that $n_1 \geqslant \ldots \geqslant n_d$. The embedding dimension of S is $n = n_1 + \ldots + n_d + d-1$ and its degree $n-d+1$. For $d = 1$, $S = S(n)$ is a *nonrmal rational curve* in $\mathbb{P}^n$.

The scroll $S(n_1, \ldots, n_d)$ admits a more down to earth description (see [H1]). Take independent linear spaces $L_1, \ldots, L_d$ in $\mathbb{P}^n$, say of dimensions

$n_1, \ldots, n_d$, such that $L_1 + \ldots + L_d = \mathbb{P}^n$. For each i select a normal rational curve C_i of degree n_i in L_i and isomorphisms $h_i: \mathbb{P}^1 \to C_i$ whenever $n_i \geqslant 1$. If $n_i = 0$ then L_i, and hence C_i, is a point and h_i will denote the constant map $\mathbb{P}^1 \to C_i$. Then the linear spaces $< h_1(t), \ldots, h_d(t) >$, when t varies in $\mathbb{P}^1$, sweep out an $S(n_1, \ldots, n_d)$, and conversely, any scroll can be obtained in this way.

A surface in $\mathbb{P}^5$ will be called a *Veronese surface* V_2^4 if it is projectively equivalent to the image of $\mathbb{P}^2$ under the complete linera system $|\mathcal{O}(2)|$.

A variety is said to be *ruled* when it is the closure in the Zariski topology of $\mathbb{P}^n$ of an ∞^1 family of codimension one linear spaces, which will be called *generators* or *rulings* of the variety. A set of linear spaces is said to be an ∞^1 *family* if they are the linear spaces corresponding to the points of a curve on a Grassmannian variety.

OVERVIEW

Assume V is irreducible. Then it is well known that $g \geqslant n - d + 1$ (see the first paragraph of the proof of theorem 1). When $g = n - d + 1$ we get *irreducible minimal degree varieties*. The classification of these varieties, up to projective equivalence, is the content of what I will call Del Pezzo/Bertini/Harris theorem (theorem 2 below).

In this paper we first look at varieties V (in the sense explained above) which are connected in codimension one and such that $g \leqslant n - d + 1$. These appear to have a quite simple structure (theorem 1). In particular it turns out that $g = n - d + 1$ always, so that for these varieties the minimum value of g is in fact the same as for irreducible varieties.

Next we apply theorems 1 and 2 to describe the set-theoretic structure of equidimensional linear sections of scrolls (theorem 3). In particular this theorem says that *any* irreducible linear section of a scroll is itself a scroll. It also implies that an irreducible variety is a scroll if and only if it is the set of common zeroes of the 2×2 minors of a $2 \times q$ matrix of homogeneous linear forms.

We also apply theorems 1-3 to study some aspects of the geometry of surface scrolls.

Finally we show how theorem 1 and a few simple properties of Veronese surfaces V_2^4 can be used to simplify somewhat the arguments currently involved in the proof of theorem 2.

CONNECTED IN CODIMENSION ONE MINIMAL DEGREE VARIETIES

1. Let V be such that $g \leqslant n - d + 1$. Assume also that V is connected in codimension one. Then we have.

(i) $g_i = n_i - d + 1$

(ii) $g = n - d + 1$, and (possibly after rearranging)

(iii) $V_j \cap (V_1 \cup \ldots \cup V_{j-1}) = L_j \cap (L_1 + \ldots + L_{j-1})$, which is a *linear* space of dimension $d - 1$ $(j = 2, \ldots, r)$.

Proof: If V is irreducible then $g \geqslant n - d + 1$. This is easily seen by induction on d: if H is a general hyperplane then $H \cap V$ has dimension $d - 1$, is irreducible ([W1], p. 300), has the same degree as V ([S2], p. 106), and its linear span is H; if $d = 1$ then $H \cap V$ will contain at least n points, so that $g \geqslant n$, and hence the inequality is true for curves; if $d \geqslant 2$, then by induction $H \cap V$ satisfies the inequality, so that $g \geqslant (n - 1) - (d - 1) + 1 = n - d + 1$.

Now let us return to a connected in codimension one variety such that $g \leqslant n - d + 1$. Set $e_j = \dim L_j \cap (L_1 + \ldots + L_{j-1})$, $j = 2, \ldots, r$. Then the dimension formula tells us that $e_j = n_j + \dim (L_1 + \ldots + L_{j-1}) - \dim (L_1 + \ldots + L_j)$. Adding up all these equalities we see that $e_2 + \ldots + e_r = n_1 + \ldots + n_r - \dim (L_1 + \ldots + L_r)$. By the first paragraph of this proof we see that $n_i \leqslant g_i + d - 1$. On the other hand $L_1 + \ldots + L_r$ is equal to $\mathbb{P}^n$, since V is not contained in any hyperplane. Moreover, $g \leqslant n - d + 1$ by hypothesis, and $g = g_1 + \ldots + g_r$. Finally we may assume, possibly after rearranging the components, that $e_j \geqslant d - 1$, since V is connected in codimension one. Combining all these relations we deduce the inequalities

$$(r-1)(d-1) \leqslant e_2 + \ldots + e_r = n_1 + \ldots + n_r - n$$
$$\leqslant g + r(d-1) - n \leqslant (r-1)(d-1).$$

From thse inequalities we infer that all inequalities used before must be equalities, and in particular (i) and (ii) follow. We also get $e_j = d - 1$. Since $V_j \cap (V_1 \cup \ldots \cup V_{j-1})$ also has dimension $d - 1$, (iii) follows as well. QED.

Remark: Theorem 1 admits the following converse. Suppose that $L_1, \ldots, L_r$ are linear spaces in $\mathbb{P}^n$ such that $e_j = d - 1$, where e_j is defined as in the proof above. Assume also that $L_1 + \ldots + L_r = \mathbb{P}^n$. For each j choose and algebraic irreducible subvariety $V_j \subseteq L_j$ whose degree g_j satisfies $g_j = n_j - d + 1$, where $n_j = \dim L_j$, and such that (iii) is true (take for instance a scroll of dimension d and degree n_j such that one of its rulings is $L_j \cap (L_1 + \ldots + L_{j-1})$; notice that the condition on the degree already implies that $< V_j > = L_j$). Then $V = V_1 \cup \ldots \cup V_r$ is connected in codimension one, of degree $n - d + 1$, and not contained in any hyperplane.

We thus see that theorem 1 reduces the knowledge of connected in codimension one minimal degree varieties to the knowledge of irreducible minimal degree varieties. And the classification of the latter, up to projective equivalence, is given by the following theorem of Del Pezzo/Bertini/Harris ([P], [B1], [H1]):

2. If V is a irreducible minimal degree variety of P^n then V belongs to precisely one of the following three classes:

(i) Scrolls
(ii) Quadrics of rank not less than five
(iii) A Veronese surface or a cone over a Veronese surface.

Note: For the case of curves theorem 1 is already contained in [A]. As far as theorem 2 goes, the surface case is due to Del Pezzo [P]. Later Bertini [B1] found a generalization which can be stated as follows: An irreducible minimal degree variety which is not a quadric, nor a Veronese surface, nor a cone over a Veronese surface, is a rationally ruled variety, i.e., the locus of a rational ∞^1 family of (d 1)-dimensional linear spaces. Finally J. Harris [H1] proved that the ∞^1 family ocurring in these varieties actually turns them into scrolls. The theorem of Del Pezzo has been proved by other authors a number of times (cf. for instance [E1], [B1], or [N]). On the other hand, Saint-Donat [S1] states a theorem similar to theorem 2 above, although he apparently gets more cases due to his restricted use of the word scroll. However he does not provide a proof, nor can such a proof be found in the references he quotes. Finally J. Harris uses in his proof the Lefschetz hyperplane section theorem, which can be avoided and substituted by a direct argument. In the last section we write down a rather detailed proof of theorem 2. In one of the main steps we use teorem 1.

LINEAR SECTIONS OF SCROLLS

In [B2] it is shown that if $S_{1,n}$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$ in $\mathbb{P}^{2n+1}$ and if L is a linear space which cuts each (n-dimensional) ruling of $S_{1,n}$ in exactly one point then $L \cap S_{1,n}$ is a normal rational curve (loc. cit., Satz 2). In this section we generalize this result. In fact we describe the set-teoretic structure of any equidimensional linear section of a scroll. We also point out a couple of applications.

3. Let $S = S(n_1, \ldots, n_d)$ be a scroll in $\mathbb{P}^n$ and let L be a linear space such that $L \cap S$ is equidimensional. Then

$$L \cap S = \bar{S} \cup F_1 \cup \ldots \cup F_s,$$

where $\bar{S}$ is a scroll (possibly empty) and where each F_i is a linear space contained in a ruling of S. If $\bar{S}$ is non-empty, each F_i meets $\bar{S}$ along a ruling. In any case $L \cap S$ is a connected in codimension one minimal degree variety in its linear span.

Proof: If d = 1 the theorem is true in a trivial fashion for in this case $L \cap S$

is finite and this is connected in codimension one, and has minimal degree in its linear span (see descending induction below). Assume thus that $d \geqslant 2$.

Let p be the common dimension of the components of $L \cap S$. It is clear that for any ruling G of S we have $\dim (L \cap G) \leqslant p$. If $\dim (L \cap G) = p$ for all generators G then $L \cup G$ is independent of G and hence it is a p-dimensional linear space $\overline{S}$ contained in all rulings of S. In this case $L \cap S = \overline{S}$ and this satisfies the conditions of the statement. Assume now that $\dim (L \cap G) < p$ for a least one ruling G. Then $\dim (L \cap G) = p$ can be satisfied by only finitely many rulings $G_1, \ldots, G_s$ of S (of course s may be zero). Set $F_i = L \cap G_i$. It may happen that $L \cap S = F_1 \cup \ldots \cup F_s$, in which case the result also holds. Otherwise let $\overline{S}$ denote the closure of the union of the intersections $L \cap G$, where G runs over all rulings of S such that $L \cap G \not\subseteq F_1 \cup \ldots \cup F_s$. We claim that for any such G we have $\dim (L \cap G) = p - 1$. In fact, if $\inf \dim (L \cap G) = p'$ were less than $p - 1$, then $\dim (L \cap G) = p'$ for all rulings G of S but a finite number, and $L \cap S$ would contain a component $\overline{S}'$ of lower dimension than p, namely the closure of the union of all $L \cap G$ where G runs over all rulings G of S such that $\dim (L \cap G) = p'$. Consequently we may assume that $\overline{S}$ is an irreducible component of $L \cap S$; the other components are $F_1, \ldots, F_s$. It is clear that $\overline{S}$ is ruled.

Next we observe that $p \leqslant \dim (L) \leqslant g + p - 1$, where $g = \deg (S)$. The first inequality is clear. To see the second, let G be a generic ruling of S. Then $\dim (L) = \dim (L + G) + \dim (L \cap G) - \dim (G) \leqslant n + p - 1 - (d - 1) = g + p - 1$, since $g = n - d + 1$.

We also observe that $L \cap S$ is connected in codimension one. To see this it is enough to show that $F_i \cap \overline{S}$ is $(p - 1)$-dimensional. And since $\overline{S} \subseteq L$ it is enough to see that $G_i \cap \overline{S}$ is $(p - 1)$-dimensional, or equivalently, that $G \cap \overline{S}$ is $(p - 1)$-dimensional for *all* rulings G of S. If $n_d \geqslant 1$ then any two rulings are disjoint and we have a projection map $u: S \to \mathbb{P}^1$ whose fibers are the rulings of S. Let v be the restriction of u to $\overline{S}$, so that if G is the fiber of u over $t \in \mathbb{P}^1$ then $G \cap \overline{S}$ is the fiber of v over t. But by construction the generic fiber of v is $(p - 1)$-dimensional. It follows that all fibers of v are $(p - 1)$-dimensional. This proves the claim when $n_d \geqslant 1$. If $n_d = 0$ then S is a cone over an $S(n_1, \ldots, n_{d-1})$ with vertex the point L_d and the claim follows easily by induction.

To proceed with the proof we may restrict ourselves to consider only linear subspaces L such that $L = < L \cap S >$. In fact, if $L' = < L \cap S >$ then $L' \cap S = L \cap S$.

Now suppose first that $\dim (L) = g + p - 1$. Then since $\deg (L \cap S) \leqslant \deg (L \cdot S) = g = \dim (L) - p + 1$, and since $L \cap S$ is connected in codimension one, we can apply theorem 1 to conclude that $\overline{S}$ has minimal degree in $< \overline{S} >$, that $\deg(L \cap S) = g$, that $F_i \cap \overline{S}$ is a ruling of $\overline{S}$, and that $L \cap S$ has minimal degree in its linear span, namely L. Notice also that $\overline{S}$ is a scroll, by theroem 2.

Therefore we can assume that $p \leqslant \dim(L) < g + p - 1$. In this case we are going to use a descending induction argument on $\dim(L)$. We see that $\dim(L + G_i) = \dim(L) + d - 1 - p$, since $\dim(L \cap G_i) = p$. Hence $\dim(L + G_i) < g + p - 1 + d - 1 - p = g + d - 2 = n - 1$, and $L + G_i$ is a proper subspace, so that it can contain at most a finite number of rulings of S. Thus there exists a ruling G_0 and a point $P_0 \in G_0$ such that $P_0 \in L + G_i$ for $i = 1, \ldots, s$. If $G' \neq G_i$, a similar computation as above shows that $\dim(L+G') < n$, so that $L+G'$ will not contain a generic point P_0 of a generic ruling G_0, if G' is itself generic. In other words, we can select G_0 so that $L + G'$ does not contain G_0 but for finitely many $G' \neq G_i$. Let $G_{s+1}, \ldots, G_r$ denote the exceptional rulings G' such that $L + G'$ contains G_0. Define $L^* = L + P_0$. Then L^* is the linear span of $L^* \cap S$. Moreover, a straightforward computation shows that

$$\begin{aligned}\dim(L^* \cap G) &= p \text{ if } G = G_1, \cdots, G_s, G_{s+1}, \cdots, G_r,\\ &= p - 1 \text{ otherwise.}\end{aligned}$$

Since $\dim(L^*) = \dim(L) + 1$, by descending induction we may assume that $L^* \cap S$ satisfies the theorem. But $L^* \cap S = \overline{S} \cup F_1 \cup \ldots \cup F_s \cup F_{s+1} \cup \ldots \cup F_r$, where $F_i = L \cap G_i$ also for $i = s + 1, \ldots, r$. From this the theorem follows immediately. QED.

As a corollary we have.

4. If the set-theoretic intersection of a scroll and a linear space is irreducible, then this intersection is itself a scroll. The intersection is irreducible if the linear space cuts the rulings of the scroll in linear spaces that have constant dimension. In particular, a hyperplane that does not contain any ruling cuts a scroll in an irreducible variety that is itself a scroll.

To simplify terminology we will say that a variety is a *crown* if it satisfies the conclusions of therorem 3, that is, if it consists of finitely many linear spaces going through rulings of a scroll and in such a way that it is minimal degree in its linear span.

Now recall that scrolls have equations given by the vanishing of the 2×2 minors of a $2 \times g$ matrix of linear forms. In fact given $S = S(n_1, \ldots, n_d)$ there exists a system of projective coordinates X_{ij} in $\mathbb{P}^n$, where $n = n_1 + \ldots + n_d + d - 1$, $1 \leqslant i \leqslant d$, $0 \leqslant j \leqslant n_i$, such that S is the set of common zeroes of the 2×2 minors of a matrix of the form $(A_1 | \ldots | A_d)$, where

$$A_i = \begin{pmatrix} X_{i0} & X_{i1} & \cdots & X_{i,n_i-1} \\ X_{i1} & X_{i2} & \cdots & X_{i,n_i} \end{pmatrix}$$

(see [B] for the surface case; the general case follows just as easily). Also, as

D. Eisenbud points out to me, the homogeneous ideal of the unique normal rational curve going through n + 3 points in general position in $\mathbb{P}^n$ is generated by the 2 X 2 minors of a matrix

$$\begin{pmatrix} X_0 & X_1 / \cdots & X_{n-1} \\ Y_0 & Y_1 & \cdots & Y_{n-1} \end{pmatrix}$$

where the X_i's are the homogeneous coordinate functions of $\mathbb{P}^n$ and $Y_i = (a_n X_i \quad a_i X_n)/(a_n \cdot a_i)$, and where we normalize the points so that the first n + 1 are the vertices of the reference pyramid, the (n + 2)-th is the unit point, and the last is $(a_0, \ldots, a_n)$. In the presence of these facts one may ask what kind of varieties V we can get by the vanishing of the 2 X 2 minors of a 2 X q matrix of linear forms

$$(*) \qquad \begin{pmatrix} u_1 & \cdots & u_q \\ v_1 & \cdots & v_q \end{pmatrix}$$

5. If the 2 X 2 minors of (*) cut out en equidimensional variety V, then V is a crown. If V is irreducible then V is a scroll.

Proof: Let m = dim $<u_1, \ldots, u_q, v_1, \ldots, v_q>$. Let us take new homogeneous variables $X_{n+1}, \ldots, X_{n+p}$, $p = 2q - m$, and let us replace p of the forms $u_1, \ldots, u_q, v_1, \ldots, v_q$ by $X_{n+1}, \ldots, X_{n+p}$ in such a way that the 2 q components of the matrix

$$(*') \qquad \begin{pmatrix} u'_1 & \cdots & u'_q \\ v'_1 & \cdots & v'_q \end{pmatrix}$$

which we get in this way are linearly independent. Let V' be the variety in $\mathbb{P}^{n+p}$ given by the vanishing of the 2 X 2 minors of the matrix (*'). It is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{q-1}$ in $\mathbb{P}^{n+p}$ and in particular it is a scroll. Now our variety V is a linear section of V', $V = V' \cap L$, where L is the linear space given by the equations $X_j \quad w_j = 0, j = 1, \ldots, p$, and where $w_1, \ldots, w_p$ denote those entries of (*) which have been substituted by the new variables. Therefore V is a crown. QED.

6. With the same notations as in 5, if V is not contained in any hyperplane of $\mathbb{P}^n$ and if its codimension equals the generic codimension for these kind of varieties then deg(V) = q.

Proof: In this case V is a crown spanning $\mathbb{P}^n$ and hence deg(V) = n − d + 1 = codim(V) + 1 = codim(V') + 1 = q. QED.

The last corollary to theorem 3 involves the polar variety L^* of a linear space $L \subset \mathbb{P}^n$ with respecto to a pencial of quadrics $Q = \{Q_t\}$, $t \in \mathbb{P}^1$. Suppose that the polar space L_t^* of L with respect to Q_t has constant dimension p. Then

7. L^* is a scroll of degree $\bar{n} - p$, where $\bar{n} = \dim L^*$.

Proof: It is an straightforward computation to see that L^* has equations given by the vanishing of the 2×2 minors of a $2 \times (m + 1)$ matrix of linear forms, where m is the dimension of L. If $\dim(L_t^*)$ is independent of t then L^* is irreducible and consequently it is a scroll. QED.

EXAMPLES COMING FROM RULED SURFACES

We are going to apply theorem 2 to some ruled surfaces whose construction we describe presently. Let L_1 and L_2 be proper linear subspaces in $\mathbb{P}^n$ and set $m_i = \dim(L_i)$, $i = 1,2$. We will assume that $m_1 \geqslant m_2 \geqslant 1$. For $i = 1,2$, let $V_i \subsetneq L_i$ be a curve such that $< V_i > = L_i$ and assume we have a birrational isomorphism $h: V_1 \to V_2$. Let S be the ruled surface swept out by the line that joins pairs of corresponding points, that is, S is the closure of the union of lines $< t, h(t) >$, where t is a point on V_1 at which h is defined and which is not a fixed point for the correspondence. Let $Q_1, \ldots, Q_s$ be the fixed points of h, which we will assume to be simple both on V_1 and on V_2. Then if $\deg(V_i) = g_i$, $i = 1,2$, we have a formula for the degree of S, namely

8. The degree of S is $g_1 + g_2 - s$. Moreover, S has minimal degree in $\mathbb{P}^n$ (assuming that $L_1 + L_2 = \mathbb{P}^n$) if and only if V_i is a normal rational curve in L_i, and $s = m + 1$, where $m = \dim(L_1 \cap L_2)$.

Proof: Let L be a generic linear space of dimension $n - 2$. We want to find out the number of points in $L \cap S$. To this end, consider the pencil Z of hyperplanes H that contain L and the correspondence f from Z to Z whose graph $G_f \subset Z \times Z$ is formed with pairs (H, H') such that $H \cap V_1$ contains a point which corresponds under h to a point in $H' \cap V_2$. This is an algebraic correspondence, since G_f is the image of $G_h \subset V_1 \times V_2$ under the morphism $p_1 \times p_2 : V_1 \times V_2 \to Z \times Z$, where $p_i: V_i \to Z$ is given by $P \to P + L$. Next it happens that the correspondence f has type (g_1, g_2). In fact grom the definition it turns out that $f(H) = p_2(h(H \cap V_1))$, $f^{-1}(H') = p_1(h^{-1}(H' \cap V_2))$. By Chasles principle, f has $g_1 + g_2$ fixed points. Our hypothesis on the fixed points Q_i imply that the hyperplanes $Q_i + L$ are fixed points for the correspondence f and that they have multiplicity one. Thus f has, aside from the hyperplanes $L + Q_i$, $g_1 + g_2 - s$ fixed hyperplanes, which all count with multiplicity one due to the fact that L is generic. This means that there are $g_1 + g_2 - s$ hyperplanes in Z which contain a ruling of S, and that the remaining members of Z cut all rulings of S at a single point. Fron this we infer that $L \cap S$ contains exactly $g_1 + g_2 - s$ points.

To see the second statement, notice that $m_1 + m_2 = n + m$, so that S has minimum degree if and only if $g_1 + g_2 - s = m_1 + m_2 - m - 1$, that is, if and only if

$$(g_1 \quad m_1) + (g_2 - m_2) + m + 1 = s.$$

On the other hand if we take $m_i - m \cdot 1$ general points of V_i, these points and the s fixed points Q_i are contained in a hyperplane of L_i, so that $s \leqslant (g_i - m_i) + m + 1$. Since this relation is true for $i = 1, 2$, we get, together with the previous relation, that actually $g_i = m_i$ and as a result also $s = m + 1$. QED.

We can apply some of the previous results to give a rather weak characterization of the so called directrices of a surface scroll. A curve D on a ruled variety V is called a *directrix* if it cuts each ruling in exactly one point and $<D> \cap V = D$.

9. Let D be an irreducible curve on $S = S(n_1, n_2)$ which is not a ruling. Assume that $n_1 \geqslant n_2 = 1$ and set $n = n_1 + n_2 + 1$, $g = n - 1$. Then we have:

(a) The following conditions are equivalent:

(i) D is a directrix,

(ii) $< D > \neq \mathbb{P}^n$,

(iii) $\deg(D) = g$.

(b) If D satisfies these conditions then D is a normal rational curve.

(c) If D_1 and D_2 are two distinct directrices of S then they meet in exactly $m_1 + m_2 - g$ points, $m_i = \deg(D_i)$.

Proof (cf. also [B1]): That (i) implies (ii) is obvious. Assume (ii). Then if $L = < D >$, $L \cap S$ contains D as a component, and possibly contains also a finite number of rulings. By theorem 3, D is a normal rational curve and hence $\deg(D) = \dim < D > \leqslant n - 1 = g$. Thus (ii) implies (iii). Now assume (iii). Then $\dim < D > \leqslant \deg(D) \leqslant g = n \cdot 1$, and so D is a normal rational curve by the same argument as above. If $< D > \cap S \neq D$ then $< D >$ would contain a ruling G of S. Take $g - m$ generic points on S, where $m = \dim < D >$, say $P_1, \ldots, P_{g-m}$. Then $< D > + P_1 + \ldots + P_{g-m}$ is contained in a hyperplane H of $\mathbb{P}^n$ which cuts S at least along D and $g - m + 1$ rulings, which contradicts the fact that S has degree g. Therefore $< D > \cap S = D$ and D is a directrix. Statement (b) has already been proved. And (c) follows immediately from theorem 8. QED.

A PROOF OF THE DEL PEZZO/BERTINI/HARRIS THEOREM

We first prove a preparatory lemma which is a slight improvement of a similar lemma in [H1].

10. Let S be an irreducible minimal degree surface in $\mathbb{P}^n$, x and y two distinct points of S, and L the line joining x and y. If L contains a third point of

S, or if x or y is singular for S, or if L is tangent to S at x or y, then L is contained in S.

Proof: Project $\mathbb{P}^n$ onto $\mathbb{P}^{n-2}$ with center L and let S' be the projection of S. If L were not contained in S, and S' were a surface, then deg(S') would be deg(S) (namely $n-1$) diminished in the number of points, counted with multiplicities, that a general codimension two linear space through L has in common with S. If any of the assumptions in the statement is true then this number is at least three, so that $\deg(S') \leqslant n-4$. But this is a contradiction because S' spans $\mathbb{P}^{n-2}$ and the minimal degree of such a surface is $n-3$. Therefore if L is not contained in S, then S' must be a curve. If x or y is simple, then S' contains a line, namely the line corresponding to the tangent space at the simple point, and consequently S' *is* a line. This is only possible if $n-2=1$, which implies that $n=3$, and hence that S is a quadric, which *does* satisfy the lemma. And if both x and y are singular on S, then by what we have already proved the plane joining x, y and z, for any simple z, is contained in S, hence also L is. QED.

Next theorem is due to J. Harris [11]. It appears to be a natural complement to Bertini's generalization of Del Pezzo's theorem. We prove it using theorem 1, but the idea is already contained in Bertini's proof of Del Pezzo's theorem (surface case).

11. Let V be an irrducible minimal degree variety of $\mathbb{P}^n$. If V is ruled, then V is a scroll.

Proof: It is clear that we may assume $d \geqslant 2$. Set $p = \left[\frac{n}{d}\right]$ and pick out p generators $L_1, \ldots, L_p$ of V. Then since $\dim(L_1 + \ldots + L_p) \leqslant p(d-1) + p-1 = pd-1 \leqslant n-1$ we see that there exists a hyperplane H which contains $L_1 + \ldots + L_p$. Then $H \cap V$ is the union of a finite number of rulings $L_1, \ldots, L_m$ (thus $m \geqslant p$) plus a component V_0 such that $V_0 \cap L \supseteq H \cap L$ for all rulings $L \neq L_i$, $i=1,\ldots,m$. In fact V_0 is the closure of the union of the linear spaces $H \cap L$, where L runs through the rulings such that $L \neq L_1, \ldots, L_m$. Thus $V_0 \cap L$ contains a linear space of dimension $d-2$ *for all* rulings L. In fact let C be the curve on $Gr_{n,d-1}$ (the Grassmannian variety of $(d-1)$-planes in $\mathbb{P}^n$) whose points correspond to the rulings of V, let $h: X \dashrightarrow C$ be a desingularization of C, and consider the rational map $s: X \to Gr_{n,d-2}$ given by $s(x) = H \cap L_{h(x)}$, where $L_{h(x)}$ denotes the $(d-1)$-dimensional linear space corresponding to h(x). Then s is regular everywhere, because X is non-singular and $Gr_{n,d-2}$ is projective. Since $L_{s(x)} \subseteq L_{h(x)}$ for generic $x \in X$, it turns out that $L_{s(x)} \subseteq L_{h(x)}$ for all $x \in X$. From this the claim follows immediately.

If V_0 were contained in $L_1 \cup \ldots \cup L_m$, then either V_0 has dimension $d-2$, in which case it must be a linear space contained in any generator L, or else V_0 has dimension $d-1$, in which case $V_0 = L_i$ for some i, say $V_0 = L_1$. In the first case V is a cone over a normal rational curve with vertex a $(d-2)$-dimensional linear space, hence a scroll. The second case can not occur, for such a property is stable under general hyperplane sections and also under projections form a general point of the variety; since the hypotheses in the theorem are also preserved under such operations, we would find that an irreducible quadric in $\mathbb{P}^3$ would satisfy the property, which does not.

Therefore we may assume that V_0 is not contained in $L_1 \cup \ldots \cup L_m$. We can also assume that the previous construction has been carried out so that m is maximum (in any case m is bounded form above by the degree of V).

Now $< H \cap V > = H$, for otherwise $H \cap V$ would be contained in a hyperplane H' of H and then if Q is a general point of V, H' + Q would be a hyperplane that would contain at least m + 1 rulings of V. On the other hand, $\deg(H \cap V) \leqslant n - d + 1 = (n - 1) - (d - 1) + 1$. Since $H \cap V$ is connected in codimension one, by theorem 1 we conclude that V_0 has minimal degree in $<V_0>$, that $\deg(H \cap V) = n - d + 1$, and also that $V_0 \cap L_i$ is a linear space of dimension $d - 2$, Thus $\deg(V_0) = n - d + 1 - m$ and $\dim < V_0 > = n - m - 1$. Since V_0 is ruled, by induction it is a scroll, say of type $S(n_1, \ldots, n_{d-1})$, where this time we will assume $n_1 \leqslant \ldots \leqslant n_{d-1}$. Set C_1 to denote the normal rational curve of V_0 corresponding to the summand $\mathcal{O}(n_1)$ of the bundle which defines V_0. Set $E_1 = < C_1 >$. If $n_1 = 0$, then V_0 is a cone with vertex E_1, form which it follows that V itself is a cone with vertex E_1 (by 10). The directrix of this cone is a general hyperplane section of V, which by induction is a scroll, so V itself is a scroll. We may thus assume that $n_1 \geqslant 1$. Notice that by construction any ruling of V_0, and hence any ruling of V, cuts C_1 at a unique point. We have the following bound for n_1:

$$n_1 = \deg(C_1) \leqslant (\deg H \cap V)/(d - 1) = (n - d + 1 - m)/(d - 1).$$

But $m + 1 \geqslant p + 1 > n/d$, so that $dm > n - d$ and consequently

$$\begin{aligned} n_1 &\leqslant (dn - d(d - 1) - dm)/d(d - 1) \\ &< (dn - d(d - 1) - n + d)/d(d - 1) \\ &= (n(d - 1) - d(d - 1) + d)/d(d - 1) \\ &= n/d - 1 + 1/(d - 1) \\ &\leqslant n/d. \end{aligned}$$

This shows that if we take rulings $L'_1, \ldots, L'_{n_1}$ of V such that they cut C_1 in n_1 distinct points $P_1, \ldots, P_{n_1}$ then

$$\dim(L'_1 + \ldots + L'_{n_1}) \leqslant n_1(d - 1) + n_1 - 1 = n_1 d - 1 < n - 1$$

In particular there exist hyperplanes H' which contain $L'_1 + \ldots + L'_{n_1}$. Since this last linear space intersects E_1 along $P_1 + \cdots + P_{n_1}$, which is a proper subspace of E_1, we may select H' in such a way that it does not contain E_1. In this fashion H' intersects C_1 exactly at the points $P_1, \ldots, P_{n_1}$ and by an argument similar to one used above, H' can contain anly the rulings $L'_1, \ldots, L'_{n_1}$ of V. Again as before $H' \cap V = V'_0 \cup L'_1 \cup \ldots \cup L'_{n_1}$, where V'_0 is a scroll, say $S(m_1, \ldots, m_{d-1})$, which cuts every ruling of V along a linear $(d - 2)$-dimensional subspace. Moreover, V'_0 does not cut C_1. In fact, E_1 and V'_0 are supplementary subspaces of $\mathbb{P}^n$, for $< V'_0 > + E_1 = < V > = \mathbb{P}^n$ and $\deg(V_0) = n - d + 1 - n_1$, so that $\dim < V'_0 > = n - n_1 + 1 = n - \dim(E_1) + 1$. This implies immediately that V itself is a scroll of type $S(n_1, m_1, \ldots, m_{d-1})$. QED.

Now we proceed to the proof of theorem 2. If $d = 1$, then V is an irreducible curve of degree n in $\mathbb{P}^n$ and hence it is a normal rational curve, that is, a $S(n)$.

Next assume that $d = 2$. If $n = 3$, V is a quadric of rank 3 or 4, hence a scroll of type $S(2,0)$, or of type $S(2,1)$. If $n = 4$, V is a cubic surface in $\mathbb{P}^4$. If V is singular then, by 10, V is of type $S(3,0)$. And if V is non-singular then we can show that V is of type $S(2,1)$ as follows (cf. [XXX]). Let x_1, x_2 be to general points of V and let Q_1, Q_2 be the quadrics of $\mathbb{P}^4$ formed taking the cone of vertex x_i and directrix V, $i = 1,2$. Since V is non-singular, they are rank 4 quadrics. Then $Q_1 \cap Q_2 = V \cup L$, where L is a plane. Take coordinates so that L is given by $X_0 = X_1 = 0$. Then we can arrange the equation of Q_i as $X_0 G_i - X_1 F_i = 0$, where F_i, G_i are linear forms. We clearly can assume tha $F_1 = X_2$ and $G_1 = X_3$. Moreover, one of the forms F_2, G_2 must be linearly independent of $X_0, \ldots, X_3$, given that their vertices are different. Thus we can suppose that $F_2 = X_4$. We conclude that V is the variety given by the vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} X_0 & X_2 & X_4 \\ X_1 & X_3 & G \end{pmatrix}$$

where $G = G_2$ is a linear form in $X_0, \ldots, X_4$. Now it is easy to see, by row and column opperations, that in fact G may be assumed to be X_0. Thus V is indeed of type $S(2,1)$.

So we may assume that $n \geqslant 5$. Assume also that V does not contain ∞^2 conics, so that through a general point x on V do not pass ∞^1 conics contained in V. Let V' be the projection of V in $\mathbb{P}^{n-1}$ from x. Then V' has minimal degree in $\mathbb{P}^{n-1}$ and contains a line (corresponding to the tangent of V at x), say L, so that in particular V' is not V_2^4. By induction we can assume that V' is a scroll. It happens that L is a ruling of V', for otherwise the rulings of V' would cut L and therefore they would be projections of conics through x on

V, against our assumption. We see then that V itself is ruled. By 11 it is a scroll. And if V contains ∞^2 conics, then it is a fact that V must be a Veronese surface V_2^4 in $\mathbb{P}^5$ (cf. [B1]; however, see Note 1 at the end).

So assume that $3 \leqslant d \leqslant n - 2$ (for $d = n - 1$, V is a quadric). Take a generic linear space L of dimension $n - d + 2$. Then $L \cap V$ is a minimal degree surface in L. We distinguish two cases: (i) $L \cap V$ is a scroll; (ii) $L \cap V$ is a Veronese surface V_2^4.

In case (i) V is ruled, hence a scroll; for if V_x is the union of lines contained in V that go through a generic point x of $L \cap V$, then $L \cap V_x$ is the unique line of $L \cap V$ that goes through x, hence V_x is a $(d - 1)$-dimensional linear space and V must be ruled.

In case (ii) it is enough to see that if $n \geqslant 6$ then V is singular, since then it will be a cone over a generic hyperplane section V' (by 10), which has again propery (ii), so that by induction V' is a cone over a Veronese surface V_2^4, thus V itself is also a cone over a V_2^4. To see that V is singular when $n \geqslant 6$, assume first that $n = 6$. Then under the assumption (ii) the degree of V is 4, hence $d = 3$. In this case (ii) says that the generic hyperplane section of V is a Veronese surface V_2^4. Suppose that V were non-singular. Then we derive a contradiction. Let x be a generic point on V and le W be the projection of V from x into $\mathbb{P}^5$. Then W is a non-singular cubic threefold in $\mathbb{P}^5$ (again by 10), which, by what we have already proved, will be a scroll. It therefore contains an ∞^1 family of disjoint planes. This and the hypothesis on V imply that V_2^4 contains lines, which is the desired contradiction. If $n > 6$, the fact that V is singular follows immediately by induction taking a generic hyperplane section, which will satisfy (ii).

NOTES

1. We do not need the general result stated in [B1] according to which any surface with ∞^2 conics on it is either a V_2^4 or a projection of a V_2^4. We only need to prove that if V is a minimal degree surface in $\mathbb{P}^n$, $n \geqslant 5$, that contains ∞^2 conics generically irreducible, then V is a V_2^4 in $\mathbb{P}^5$. And this can be proved easily using 10. Indeed, let $V_n = V$ and define V_j recursively, $n-1 \geqslant j \geqslant 4$, by taking the projection of V_{j+1} to $\mathbb{P}^j$ from a general point of V_{j+1}. Then V_j is a minimal degree surface and the projection $V_{j+1} \to V_j$ is a birrational isomorphism (by 10). Each V_j contains ∞^2 conics. In particular V_4 is a cubic surface in $\mathbb{P}^4$ that contains ∞^2 conics. Therefore V_4 is a surface of type S(2,1), since S(3,0) does not contain conics. Now S(2,1) is also the projection of a V_2^4 from a point, so that it exists a birrational map f: $\mathbb{P}^2 \to V_4$ which corresponds to the linear system of conics which go through a fixed point P. This map sends lines through P to rulings of V_4 and all other lines to conics. Consider the birrational map g: $\mathbb{P}^2 \to V_5$ given by composing f with the inverse of the projection $V_5 \to V_4$. This projection sends conics on V_5 that pass through the center of the projection $V_5 \to V_4$ (there are ∞^1 of them) to rulings of V_4 and so g: $\mathbb{P}^2 \to V_5$ sends lines of $\mathbb{P}^2$ to conics. This implies that g is given by 6 linearly independent homogeneous quadratic polynomials, so that V_5 is a V_2^4. Since a V_2^4 is normal we conclude that actually $n = 5$ and hence that $V = V_5$ is a V_2^4.

2. For the structure of the homogeneous coordinate ring of a minimal degree variety, see [E2] and [E3].

3. Theorem 2 can be applied to give an "enumeration" of the quartic varieties somewhat more explicit than Swinnerton-Dyer's [S3]. See [X1] or [X2].

REFERENCES

A Artin, M. Deformations of Singularities. Institute of Fundamental Research (Bombay, 1976).

B1 Bertini, E. Introduzione a la Geometria Proiettiva degli Iperspazi. Giuseppe Principato (Messina, 1923).

B2 Burau, W.; Zeuge, J. Über den Zusammenhang zwischen den Partitionen einer natürlichen Zahl und den linearen Schnitten der einfachsten Segremanigfaltigkeiten. Crelle J. 274/275 (1975) 104-111.

E1 Enriques/Conforto. Le superficie rationali. Zanichelli (1945).

E2 Eisenbud, D. Introduction to algebras with straightening laws. Ring Theory and Albebra III, Lect. N. in Pure and Appl. Math. 55. Dekker (1980).

E3 De Concini/Eisenbud/Procesi. Hodge Algebras (preprint).

H1 Harris, J. A bound on the geometric genus of projective varieties. Thesis, Harvard University (1976).

H2 Hartshorne, R. Complete intersections and connectedness. Amer. J. Math. 84 (1962) 497-508.

N Nagata, M. On rational surfaces, I Mem. Coll. Sci. Kyoto 32 (1960) 351-270.

P Del Pezzo. Sulle superficie di ordine n immerse nello spazio di n+1 dimensioni. Rend. dell'Acad. di Napoli, 1885.

S1 Saint-Donat. Projective models of K-3 surfaces. Amer. J. Math. 96 (1974) 602-639.

S2 Samuel, P. Méthodes d'Algebre Abstraite en Géométrie Algébrique. Springer (1967, 2nd).

S3 Swinnerton-Dyer. An enumeration of all varieties of degree 4. Amer. J. Math. 95 (1973) 403-418.

W Weil, A. Foundations of Algebraic Geometry. Amer. Math. Soc. Colloq. Publ. 29 (1962, 2nd).

XXX X. X. X. Correspondence. Amer. J. Math. 79(1957) 951-952.

X1 Xambó, S. Variedades Proyectivas de Grado Mínimo. Tesis. Barcelona (1981).

X2 Xambó, S. Scrolls and quartics. (Forthcoming).

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