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On the Gorenstein property of the diagonals of the Rees algebra

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Dedicated to the memory of Fernando Serrano

Abstract

Let Y be a closed subscheme of $\mathbb{P}_k^{n-1}$ defined by a homogeneous ideal $I \subset A = k[X_1,\ldots,X_n]$, and X obtained by blowing up $\mathbb{P}_k^{n-1}$ along Y. Denote by I_c the degree c part of I and assume that I is generated by forms of degree $\leq d$. Then the rings $k[(I^e)_c]$ are coordinate rings of projective embeddings of X in $\mathbb{P}_k^{N-1}$, where $N = \dim_k(I^e)_c$ for $c \geq de+1$. The aim of this paper is to study the Gorenstein property of the rings $k[(I^e)_c]$. Under mild hypothesis we prove that there exist at most a finite number of diagonals (c,e) such that $k[(I^e)_c]$ is Gorenstein, and we determine them for several families of ideals.

1. Introduction

Let Y be a closed subscheme of $\mathbb{P}_k^{n-1}$ defined by a homogeneous ideal $I \subset A = k[X_1,\ldots,X_n]$, and X obtained by blowing up $\mathbb{P}_k^{n-1}$ along Y. Denote by I_c the degree c part of I and assume that I is generated by forms of degree $\leq d$. Then the rings $k[(I^e)_c]$ are coordinate rings of projective embeddings of X in $\mathbb{P}_k^{N-1}$, where $N = \dim_k(I^e)_c$ for $c \geq de+1$ (see [3], [2], [9]).

Among the projective varieties obtained in this way we have the Room surfaces, which have been studied in detail by A. Geramita and A. Gimigliano in [5]. These

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surfaces are obtained by blowing-up $\mathbb{P}^2_k$ along $\binom{d+1}{2}$ points, $d \geq 2$, which do not lie on any curve of degree $d-1$, and then embedding in $\mathbb{P}^{2d+2}_k$. See also [6] and [7] for other results about embedded rational surfaces obtained by blowing up a set of points in $\mathbb{P}^2$.

Recently, the study of the Cohen-Macaulay property of the rings $k[(I^e)_c]$ has received much attention. Considering the Rees algebra $R_A(I) = \bigoplus_{n\geq 0} I^n t^n \subset A[t]$ endowed with a natural bigrading, one can obtain the above rings as diagonals of $R_A(I)$. A useful strategy consists in assuming the Cohen-Macaulay property of $R_A(I)$ and then to look for which diagonals inherit this property, see for instance A. Simis, N.V. Trung and G. Valla [19], A. Conca, J. Herzog, N.V. Trung and G. Valla [2] and O. Lavila–Vidal [17]. In particular it is known that if $R_A(I)$ is Cohen-Macaulay there are infinitely many pairs (c,e) such that $k[(I^e)_c]$ is Cohen-Macaulay ([17], Theorem 4.5).

Here we are interested in the (quasi) Gorenstein property of the rings $k[(I^e)_c]$. Recall that the a-invariant of a positively graded ring T over a local ring T_0 is defined as $a(T) = \max\{i \mid [H^d_{\mathcal{M}}(T)]_i \neq 0\}$, where $\mathcal{M}$ is the maximal homogeneous ideal of T and $d = \dim T$. Assuming that T has a canonical module K_T, T is said to be quasi-Gorenstein if there exists a graded isomorphism $K_T \cong T(a)$ with $a = a(T)$, and Gorenstein if in addition T is Cohen-Macaulay.

Under appropriate hypothesis we are able to determine for which pairs (c,e) the ring $k[(I^e)_c]$ is quasi-Gorenstein. In order to state the result assume that I is minimally generated by forms $f_1, ..., f_r$ of degrees $d_1, ..., d_r$ respectively, and put $d = d_r \geq ... \geq d_1$. Suppose $n \geq r \geq 2$ and $c \geq de+1$. Let $G_A(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ be the form ring of I. Then we prove the following:

Theorem (Theorem 2.8)

Assume $ht(I) \geq 2$, $\dim(A/I) > 0$, and $G_A(I)$ is Gorenstein. Set $a = -a(G_A(I))$. Then $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.

This result can be applied to several families of ideals. In particular, to any complete intersection ideal (extending in this way a result by A. Conca et al. in [2] for the case $r = 2$) and to the ideal generated by the maximal minors of a generic matrix. Note also that under the assumptions of the above theorem there are at most a finite number of rings $k[(I^e)_c]$ which are quasi-Gorenstein. We show that this holds in general:

Proposition (Proposition 3.1)

There exist at most a finite number of diagonals (c,e) such that $k[(I^e)_c]$ is quasi-Gorenstein.

For a real number x, let us denote by $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$. Assuming that the Rees algebra is Cohen-Macaulay we can give upper bounds for the diagonals (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein:

Proposition (Proposition 3.2)

Assume that $ht(I) \geq 2$ and $R_A(I)$ is Cohen-Macaulay. Let $a = -a(G_A(I))$. If $k[(I^e)_c]$ is quasi-Gorenstein, then $e \leq a-1$ and $c \leq n$. If $\dim(A/I) > 0$ then $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{c} = l \in \mathbb{Z}$. In particular, if $a = 1$ there are no diagonals (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein.

We also prove a converse of Theorem 2.8 by showing that, under some restrictions, the existence of a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein implies that $G_A(I)$ is Gorenstein. Denoting by $l(I)$ the analytic spread of an ideal I, we have:

Theorem (Theorem 3.3)

Assume that $R_A(I)$ is Cohen-Macaulay, $ht(I) \geq 2$, $l(I) < n$ and I is equigenerated. If there exists a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein then $G_A(I)$ is Gorenstein.

Finally, by using a variation of Proposition 3.2, we study the case of the Room surfaces. We show that the only Room surface which is Gorenstein is the del Pezzo sestic surface in $\mathbb{P}^6$, so recovering that well known result (see [5], Example 1).

Throughout the paper we shall use the following notation: $A = k[X_1, ..., X_n]$ will denote the usual polynomial ring with coefficients in a field k, and $I \subset A$ a homogeneous ideal minimally generated by forms $f_1, ..., f_r$ of degrees $d_1, ..., d_r$. We put $d = d_r \geq ... \geq d_1$, $u = \sum_{j=1}^{r} d_j$. If $d_1 = d_2 = \ldots = d_r$ we say that I is *equigenerated*. Let us consider the Rees algebra of I: $R_A(I) = \bigoplus_{n \geq 0} I^n t^n \subset A[t]$ endowed with the $\mathbb{N}^2$-grading given by $R_A(I)_{(i,j)} = (I^j)_i t^j$. Let $S = k[X_1, ..., X_n, Y_1, ..., Y_r]$ be the polynomial ring with the $\mathbb{N}^2$-grading obtained by giving deg $X_i = (1, 0)$ for $i = 1, ..., n$, deg $Y_j = (d_j, 1)$ for $j = 1, ..., r$. Then $R_A(I)$ can be seen in a natural way as a bigraded S-module.

For any pair of positive integers $\Delta = (c, e)$ and any bigraded S-module $L = \bigoplus_{(i,j)} L_{(i,j)}$ we may consider $L_\Delta := \bigoplus_{s \in \mathbb{Z}} L_{(cs,es)}$ which is a graded module over the graded ring $S_\Delta := \bigoplus_{s \geq 0} S_{(cs,es)}$. We call these modules the *diagonals of L and S along* Δ. We shall always assume that $e > 0, c \geq de + 1$. It is then known ([2], Section 1) that S_Δ is Cohen-Macaulay with dim $S_\Delta = n + r - 1$, $R_A(I)_\Delta \cong k[(I^e)_c]$ and dim $k[(I^e)_c] = n$.

Let T be a positively bigraded d-dimensional ring defined over a local ring, and denote by $\mathcal{M}$ the maximal homogeneous ideal of T. The bigraded a-invariant of T is then defined by $\mathbf{a}(T)=(a_1,a_2)$, where $a_j=\max\{n_j \mid \mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2\,,[H^d_{\mathcal{M}}(T)]_{\mathbf{n}}\neq 0\}$.

2. The case of ideals whose form ring is Gorenstein

Let $S = k[X_1,...,X_n,Y_1,...,Y_r]$ be the polynomial ring introduced before and $\Delta = (c,e)$. Applying the diagonal functor, S_Δ is always a Cohen-Macaulay ring. We begin this section by showing that, on the contrary, S_Δ is Gorenstein only for a finite number of diagonals. Furthermore, we may determine them.

Proposition 2.1

S_Δ is Gorenstein if and only if $\frac{r}{e}=\frac{n+u}{c}=l\in\mathbb{Z}$. Then $a(S_\Delta)=-l$.

Proof. Let $T=S_\Delta=\bigoplus_{s\geq 0}U_s$, where U_s is the k-vector space generated by the monomials $X_1^{\alpha_1}\dots X_n^{\alpha_n}Y_1^{\beta_1}\dots Y_r^{\beta_r}$ with $\alpha_i,\beta_j\geq 0$ satisfying the equations $(\star)$

$$\sum_{i=1}^n \alpha_i+\sum_{j=1}^r d_j\beta_j = cs$$

$$\sum_{j=1}^r \beta_j = es\,.$$

By [2], Lemma 3.1 and local duality, $K_T=\bigoplus_{s\geq 1}V_s$ with V_s the k-vector space generated by the monomials $X_1^{\alpha_1}\dots X_n^{\alpha_n}Y_1^{\beta_1}\dots Y_r^{\beta_r}$, and $\alpha_i>0,\beta_j>0$ which satisfy $(\star)$. Since T is Cohen-Macaulay, T is Gorenstein if and only if $K_T\cong T(a(T))$. Assume first that $\frac{r}{e}=\frac{n+u}{c}=l\in\mathbb{Z}$. Then, multiplication by $X_1\dots X_nY_1\dots Y_r\in T_l$ induces an isomorphism $T\cong K_T(l)$ and so T is Gorenstein with $a(T)=-l$.

To prove the converse set $(\alpha,\beta)=(\alpha_1,...,\alpha_n,\beta_1,...,\beta_r)$ with $\alpha_i,\beta_j>0$ and assume the contrary. This means that $(\mathbf{1},\mathbf{1})$ is not a solution of $(\star)$ for any s. On the other hand, the set of solutions of $(\star)$ for some s is partially ordered by means of $(\alpha,\beta)\leq(\gamma,\rho)\iff\alpha_i\leq\gamma_i,\ \beta_j\leq\rho_j,\ \forall i,j$. Then one can easily check that for any i,j there exists a solution of $(\star)$ for some s such that $\alpha_i=\beta_j=1$. This implies the existence of at least two minimal solutions, and so T is not Gorenstein. $\square$

Remark 2.2. Note that the number of minimal elements in the set of solutions of the system $(\star)$ coincides with the type of S_Δ. It is not difficult to see that if S_Δ is not Gorenstein, then its type is $\geq r$.

This result leads to the question of when there exist diagonals (c, e) such that $k[(I^e)_c]$ be quasi-Gorenstein, and how one can determine them.

Our answer will be partially based on the following proposition which links the diagonal of the canonical module of $R_A(I)$ to the canonical module of the diagonal of $R_A(I)$. It is stated and proved for complete intersection ideals in [2], Proposition 4.5 but in fact the same statement and proof are valid in general. We include the proof for completeness.

Proposition 2.3

$$K_{R_A(I)_\Delta} = (K_{R_A(I)})_\Delta.$$

Proof. Let us denote by $T = S_\Delta$ and $R = R_A(I)$. Consider a presentation of R as S-module

$$0 \to C \to S \to R \to 0$$

which leads to the bigraded exact sequence of local cohomology modules

$$0 \to H^{n+1}_{m_S}(R) \to H^{n+2}_{m_S}(C) \to H^{n+2}_{m_S}(S) \to 0\,,$$

where m_S is the maximal homogeneous ideal of S.

Similarly, we get the graded exact sequence

$$0 \to H^{n}_{m_T}(R_\Delta) \to H^{n+1}_{m_T}(C_\Delta) \to H^{n+1}_{m_T}(T) \to 0\,,$$

where m_T is the maximal homogeneous ideal of T.

On the other hand, by [2], Theorem 3.6 we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H^{n+1}_{m_S}(R)_\Delta & \to & H^{n+2}_{m_S}(C)_\Delta & \to & H^{n+2}_{m_S}(S)_\Delta & \to & 0 \\ & & \varphi^n_R \uparrow & & \varphi^{n+1}_C \uparrow & & \varphi^{n+1}_S \uparrow & & \\ 0 & \to & H^{n}_{m_T}(R_\Delta) & \to & H^{n+1}_{m_T}(C_\Delta) & \to & H^{n+1}_{m_T}(T) & \to & 0 \end{array}$$

where φ^{n+1}_C, φ^{n+1}_S are isomorphisms, and so φ^n_R also is an isomorphism. Therefore $H^n_{m_T}(R_\Delta) \cong H^{n+1}_{m_S}(R)_\Delta$ and we get

$$\begin{aligned} K_{R_\Delta} &= \mathrm{Hom}_k\big(H^n_{m_T}(R_\Delta), k\big) = \mathrm{Hom}_k\big(H^{n+1}_{m_S}(R)_\Delta, k\big) \\ &= \mathrm{Hom}_k\big(H^{n+1}_{m_S}(R), k\big)_\Delta = (K_R)_\Delta. \ \square \end{aligned}$$

Remark 2.4. The hypothesis $n \geq r \geq 2$ fixed in the introduction is only used in this paper to prove Proposition 2.3, and of course its applications. Nevertheless, the isomorphism $K_{R_A(I)_\Delta} = (K_{R_A(I)})_\Delta$ is also valid if $n, r \geq 2$, I is equigenerated and $R_A(I)$ is Cohen-Macaulay. To prove this, set $R = R_A(I)$ and assume $r > n$ (if $n \geq r$ we may apply Proposition 2.3). Let

$$0 \to D_{r-1} \to \ldots \to D_1 \to D_0 = S \to R_A(I) \to 0$$

be the $\mathbb{Z}^2$-graded minimal free resolution of R over S. For every p, D_p is a direct sum of S-modules of the type $S(a, b)$. Denote by $\overline{b}$ the maximum of the $-b$'s which appear in the resolution. Since R is Cohen-Macaulay, we get from [17], Lemmas 3.6 and 3.7 that $\overline{b} = -1+r$. On the other hand, from [2], Lemmas 3.1 and 3.3 (note that hypothesis $n \geq r$ is not used there) we have that $H^r_{m_S}(S(a,b)_\Delta)_s \neq 0$ if and only if $\frac{(b+r)d-u-a}{c-ed} \leq s \leq \frac{-b-r}{e}$, hence $s < 0$. Also by [17], Proposition 4.1 $-a \geq -bd$ and so $(\overline{b}+r)d - u - a = bd - a \geq 0$. So we get $H^r_{m_T}((D_p)_\Delta) = 0$ for all p, and by [2], Lemma 3.1 that $H^i_{m_T}((D_p)_\Delta) = 0$ for all $n < i < n + r - 1$ and that $\varphi^{n+r-1}_{D_p}$ is an isomorphism for all p. By [2], Lemma 1.7 we then have φ^i_R, φ^i_C are isomorphisms for all $i > n$, and the same proof as in Proposition 2.3 shows that $K_{R_\Delta} = (K_R)_\Delta$.

This means that all the results we are going to prove are also valid if $n, r \geq 2$, I is equigenerated and $R_A(I)$ is Cohen-Macaulay.

In view of Proposition 2.3 any information on the bigraded structure of $K_{R_A(I)}$ will be of interest. Let B be a d-dimensional local ring, $d \geq 1$, which has a canonical module K_B and $I \subset B$ an ideal of positive height such that $R_B(I)$ is Cohen-Macaulay. In [21], Theorem 2.2 it is given a description of $K_{R_B(I)}$ in terms of a filtration of submodules of K_B. Assume now that $B = \bigoplus_{n\geq 0} B_n$ is a positively graded ring of positive dimension over a local ring B_0, which has a canonical module K_B. Let $I \subset B$ be a homogeneous ideal of positive height. Then, the Rees algebra $R_B(I)$ has a bigraded structure by means of $[R_B(I)]_{(i,j)} = (I^j)_i t^j$ for all $i, j \geq 0$. We also have a bigraded structure on the form ring by means of $[G_B(I)]_{(i,j)} = (I^j)_i/(I^{j+1})_i$ for all $i, j \geq 0$.

Then, the proof of [21], Theorem 2.2 may be "bigraded" and we thus obtain a description of the bigraded structure of $K_{R_B(I)}$. Namely, we get:

Theorem 2.5

With the notation above assume that $R_B(I)$ is Cohen-Macaulay. Then there exists a homogeneous filtration $\{K_m\}_{m\geq 0}$ of K_B and isomorphisms of bigraded modules such that

$$K_{R_B(I)} \cong \bigoplus_{(l,m),\, m\geq 1} [K_m]_l\,,$$

$$K_{G_B(I)} \cong \bigoplus_{(l,m),\, m\geq 1} [K_{m-1}]_l/[K_m]_l\,.$$

Several other results of [2] may also be "bigraded". In particular [21], Lemma 4.1 which makes precise when the canonical module of the Rees algebra has the expected form. Recall that $K_{R_B(I)}$ has the expected form if

$$K_{R_B(I)} \cong Bt \oplus Bt^2 \oplus \ldots \oplus Bt^l \oplus It^{l+1} \oplus I^2t^{l+2} \oplus \ldots ,$$

for some $l \geq 0$. This definition was introduced by J. Herzog, A. Simis and W. Vasconcelos in [14]. We still use the same notation and again omit the proof.

Corollary 2.6

Assume $R_B(I)$ is Cohen-Macaulay and $G_B(I)$ is quasi-Gorenstein. Let $a(G_B(I)) = (-b, -a)$ be the bigraded a-invariant of $G_B(I)$. Then $K_B \cong B(-b)$ and

$$K_{R_B(I)} = \bigoplus_{(l,m),\, m\geq 1} [I^{m-a+1}]_{l-b},$$

where $I^n = B$ if $n \leq 0$.

Note that $-a$ coincides with the usual a-invariant of $G_B(I)$. By Ikeda-Trung's criterion [16] it is always negative if $R_B(I)$ is Cohen-Macaulay, and it has been calculated in many cases (see for instance [13], [10]). As for b, it is clear that under the hypothesis of Corollary 2.6 we get $-b = a(B)$. It is then also easy to compute the bigraded a-invariant of $R_B(I)$. Namely, we get that if $a = 1$ then $a(R_B(I)) = (-d_1 + a(B), -1)$, and if $a > 1$ then $a(R_B(I)) = (a(B), -1)$.

Remark 2.7. Assume that $B = A = k[X_1, \ldots, X_n]$ and I is a complete intersection ideal. Then, the Eagon-Northcott complex provides a $\mathbb{Z}^2$-graded minimal free resolution of $R_A(I)$. Following the proof of Yoshino [24] it is possible to see that

$$K_{R_A(I)} = J\big((r-2)d_1 - n, -1\big)$$

with $J = (f_1^{r-2}, f_1^{r-2}\, t, \ldots, f_1^{r-2}\, t^{r-2})R_A(I)$.

Observe that in this case $a(G_A(I)) = (-n, -r)$ and by Corollary 2.6

$$K_{R_A(I)} = \bigoplus_{(l,m),\, m\geq 1} [I^{m-r+1}]_{l-n}.$$

A straightforward computation shows that, in fact, multiplication by f_1^{r-2} provides an explicit isomorphism

$$\bigoplus_{(l,m),\, m\geq 1} [I^{m-r+1}]_{l-n} \cong J\big((r-2)d_1 - n, -1\big).$$

Let us now assume that $I \subset A = k[X_1, \dots, X_n]$ is a homogeneous ideal whose form ring is Gorenstein. We are now ready to prove the main result of this section determining the possible quasi-Gorenstein diagonals of $R_A(I)$. We use the same notation as before, and note that in this case $b = -a(A) = n$. Then we get:

Theorem 2.8

Assume $ht(I) \geq 2$, $\dim(A/I) > 0$, and $G_A(I)$ is Gorenstein. Then $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.

Proof. Let $R = R_A(I)$. Recall that $R_\Delta = k[(I^e)_c] = \bigoplus_{l \geq 0} [I^{le}]_{lc}$. Note that R is Cohen-Macaulay by using a result of Lipman [18, Theorem 5]. By now applying Corollary 2.6, $K_R = \bigoplus_{(l,m), m \geq 1} [I^{m-a+1}]_{l-n}$, so that by Proposition 2.3 we get $K_{R_\Delta} = (K_R)_\Delta = \bigoplus_{l \geq 1} [I^{el-a+1}]_{cl-n}$. Let $l_0 = \min\{l \in \mathbb{Z} \mid l \geq \frac{n}{c}\}$, $s = a - 1 - el_0$. We shall now distinguish three cases.

If $s = 0$, then the first non-zero component of K_{R_Δ} is $[I^{el_0-a+1}]_{cl_0-n} = A_{cl_0-n}$, so that if R_Δ is quasi-Gorenstein $cl_0 - n = 0$ and we get that $l_0 = \frac{n}{c} = \frac{a-1}{e}$ and $a(R_\Delta) = -l_0$. Conversely, if $l_0 = \frac{n}{c} = \frac{a-1}{e}$ then $[K_{R_\Delta}]_{l_0+m} = [I^{el_0-a+1+em}]_{cl_0+cm-n} = [I^{em}]_{cm} = [R_\Delta]_m$ for all m and so R_Δ is quasi-Gorenstein.

If $s < 0$, let $l_1 = \min\{l \mid el - a + 1 > 0, cl - n \geq d_1(el - a + 1)\}$. Then $l_1 \geq l_0$ and the first non-zero component of K_{R_Δ} is $[K_{R_\Delta}]_{l_1} = [I^{el_1-a+1}]_{cl_1-n}$. In particular, $a(R_\Delta) = -l_1$. Assume R_Δ is quasi-Gorenstein. Then $K_{R_\Delta} \cong R_\Delta(-l_1)$ and so $[K_{R_\Delta}]_{l_1} \cong k$. This implies that $cl_1 - n = d_1(el_1 - a + 1)$: If $cl_1 - n - d_1(el_1 - a + 1) = r > 0$ we may choose two linearly independent forms $g, h \in A_r$ such that $gf_1^{el_1-a+1}, hf_1^{el_1-a+1} \in [I^{el_1-a+1}]_{cl_1-n} \cong k$, which is a contradiction. From the isomorphism one gets that K_{R_Δ} is generated by $f_1^{el_1-a+1}$ as R_Δ-module. Now let $f_j \notin rad(f_1)$ (it exists because $ht(I) \geq 2$), and choose m such that $m(c - d_j e) > d_j - d_1$ and there exists $f \in A_{d_1+cm-d_j(em+1)}$ such that $(f, f_1) = 1$. Then $f_1^{el_1-a} f_j^{em+1} f \in [I^{el_1-a+1+em}]_{d_1(el_1-a+1)+cm} = f_1^{el_1-a+1}[I^{em}]_{cm}$, and we get $f_j^{em+1} f \in (f_1)$ which is a contradiction.

If $s > 0$, the first non-zero component of K_{R_Δ} is $[I^{el_0-a+1}]_{cl_0-n} = A_{cl_0-n}$, so if R_Δ is quasi-Gorenstein we get $cl_0 - n = 0$. Furthermore, for all $m \geq 1$ we have $[K_{R_\Delta}]_{l_0+m} = [I^{-s+em}]_{cl_0-n+cm} = [I^{-s+em}]_{cm} \cong [I^{em}]_{cm}$. Since $s > 0$ and $[I^{em}]_{cm} \subset [I^{-s+em}]_{cm}$ this isomorphism is possible if and only if $[I^{em}]_{cm} = [I^{-s+em}]_{cm}$. Now choose X_i such that $X_i \notin rad(I)$ (it always exists because $\dim(A/I) > 0$) and m with $em - s \geq 1$. For any j consider $F_j = X_i^{\alpha_j} f_j^{em-s}$ where $\alpha_j = cm - d_j(em - s) = (c - d_j e)m + d_j s \geq 1$, and assume $[I^{em}]_{cm} = [I^{-s+em}]_{cm}$. Then $F_j \in [I^{em-s}]_{cm}$ and so $X_i^{\alpha_j} f_j^{em-s} \in I^{em}$. Now let $f_1^{c_1} \dots f_r^{c_r}$ such that

$c_1 + \ldots + c_r \geq r(em - s)$. This implies that there exists l with $c_l \geq em - s$ and so $X_i^{\alpha_1} f_1^{c_1} \ldots f_r^{c_r} = X_i^{\alpha_1} f_l^{em-s} f_1^{c_1} \ldots f_l^{c_l - em + s} \ldots f_r^{c_r} \in I^{c_1+\ldots+c_r+s}$, since $\alpha_1 \geq \alpha_i$ for all i. Thus we get $X_i^{\alpha} I^h \subset I^{h+s}$ for $h >> 0$, which implies that $X_i^{\alpha} \in I^s \subset I$ since $R_A(I)$ is Cohen-Macaulay. But this contradicts $X_i \notin rad(I)$ and so R_Δ cannot be quasi-Gorenstein. □

The remaining cases $ht(I) = 1, n$ in the above theorem are studied separately in the following remarks.

Remark 2.9. If $ht(I) = 1$ then $k[(I^e)_c]$ is never quasi-Gorenstein. In fact, by [21] Proposition 4.6, $a(G_A(I)) = -1$ and so $a = 1$. Following the same proof as in Theorem 2.8 we have that $s = -el_0 < 0$. On the other hand, since $ht(I) = 1$ we may write $I = gJ$, with $ht(J) \geq 2$, $J = (\overline{f}_1, \ldots, \overline{f}_r)$ and $f_j = \overline{f}_j g$ for all j. The same argument as in Theorem 2.8 for the case $s < 0$ but taking $\overline{f}_j \notin rad(\overline{f}_1)$ and $f \in A_{d_1+cm-d_j(em+1)}$ such that $(f, \overline{f}_1) = 1$ leads to $\overline{f}_j^{em+1} f \in (\overline{f}_1)$, which is a contradiction.

Remark 2.10. When $\dim(A/I) = 0$, the condition $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$ is sufficient but not necessary for $k[(I^e)_c]$ to be quasi-Gorenstein. For instance, let $A = k[X_1, X_2, X_3]$ and $I = (X_1, X_2, X_3)$. Set $R = R_A(I)$. Note that $a = -3$ and by Corollary 2.6 $K_R = \bigoplus_{(l,m), m\geq 1} [I^{m-2}]_{l-3}$. By taking the $(3,1)$-diagonal, $K_{R_\Delta} = \bigoplus_{l\geq 1} [I^{l-2}]_{3(l-1)} = \bigoplus_{l\geq 1} A_{3(l-1)} = (\bigoplus_{l\geq 0} A_{3l})(-1) = R_\Delta(-1)$ and so $R_\Delta = k[I_3]$ is quasi-Gorenstein. In this case, $n = a = 3 = c$, $e = 1$ and $\frac{n}{c} = 1 \neq 2 = \frac{a-1}{e}$.

As a consequence of Theorem 2.8 we obtain the following result for the case of complete intersection ideals. It generalizes [2], Corollary 4.7 where the case of ideals generated by two elements was considered.

Corollary 2.11

Let $I \subset k[X_1, \ldots, X_n]$ be a homogeneous complete intersection ideal minimally generated by r forms of degrees $d_1 \leq \ldots \leq d_r = d$, with $r < n$. Then for $c \geq de + 1$, $k[(I^e)_c]$ is Gorenstein if and only if $\frac{n}{c} = \frac{r-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.

Proof. Since $a(G_A(I)) = -r$ we get by Theorem 2.8 that $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{n}{c} = \frac{r-1}{e} = l_0 \in \mathbb{Z}$. But then $\sum_{j=1}^{r} d_j + (e-1)d - n \leq rd + ed - d - n = (r-1)d + de - n = e\frac{n}{c}d + de - n = n(\frac{ed-c}{c}) + de \leq de < c$, and by [2], Theorem 4.3, $k[(I^e)_c]$ is also Cohen-Macaulay and so Gorenstein. □

We may also study the ideals generated by the maximal minors of a generic matrix. We thank A. Conca for suggesting we consider this case.

EXAMPLE 2.12: Let $X = (X_{ij}), 1 \le i \le n, 1 \le j \le m$ be a generic matrix, with $m \le n$. Let us consider $I \subset A = k[X_{ij}, 1 \le i \le n, 1 \le j \le m]$ the ideal generated by the maximal minors of X, where k is a field of arbitrary characteristic. It is well-known (see [4]) that the Rees algebra $R_A(I)$ is Cohen-Macaulay and the form ring $G_A(I)$ is Gorenstein. Moreover, it has been proved by A. Conca (personal communication) that all the diagonals of $R_A(I)$ are Cohen-Macaulay. Now we want to study the Gorenstein property of these rings. Note that I is an equigenerated ideal whose Rees algebra is Cohen-Macaulay, so one can apply Theorem 2.8. From the fact that I is generically a complete intersection, one can easily see that $a(G_A(I)) = -ht(I) = -(n-m+1)$. We shall distinguish two cases.

If $m < n$, then $k[(I^e)_c]$ is Gorenstein if and only if $\frac{nm}{c} = \frac{n-m}{e} \in \mathbb{Z}$. So there is always at least one diagonal which is Gorenstein by taking $c = nm, e = n-m$.

If $m = n$, note that I is a principal ideal and so the Rees algebra is isomorphic to a polynomial ring. Then it is easy to prove that the only diagonal which is Gorenstein occurs when $c = n(n+1), e = 1$.

3. Restrictions to the existence of Gorenstein diagonals. Applications

In the previous section we proved that under the assumptions of Theorem 2.8 there exist at most a finite number of diagonals (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein. Our next result shows that this holds in general.

Proposition 3.1

There exist at most a finite number of diagonals (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein.

Proof. Let $R = R_A(I)$ and $w_1, ..., w_m \in K_R$ a system of generators of K_R as R-module with $\deg w_i = (\alpha_i, \beta_i)$ for all i, and so $K_R = \sum_{i=1}^m Rw_i$. Note that since R is a domain K_R is torsion free. For $\Delta = (c, e)$ we then have by Proposition 2.3 that for all $l \ge 1$

$$[K_{R_\Delta}]_l = \sum_{i=1,\ldots,m, el-\beta_i \ge 0} [I^{el-\beta_i}]_{cl-\alpha_i} w_i.$$

If R_Δ is quasi-Gorenstein there exists an integer l such that $[K_{R_\Delta}]_l = k$ and so $[I^{el-\beta_i}]_{cl-\alpha_i} \neq 0$ for some i $(\star)$. We shall distinguish two cases.

Assume first that I is an equigenerated ideal of degree d. Then condition $(\star)$ implies that $el-\beta_i=0$ and $cl-\alpha_i\geq 0$ or $el-\beta_i>0$ and $cl-\alpha_i\geq d(el-\beta_i)$. If $el-\beta_i=0$, then $k=[K_{R_\Delta}]_l\supset A_{cl-\alpha_i}w_i$ and since K_R is torsion-free we get $cl-\alpha_i=0$. Hence (c,e) satisfies $\frac{\beta_i}{e}=\frac{\alpha_i}{c}=l\in\mathbb{Z}$ and the statement holds. If $el-\beta_i>0$ then $k=[K_{R_\Delta}]_l\supset[I^{el-\beta_i}]_{cl-\alpha_i}w_i$ which is impossible since K_R is torsion free and $cl-\alpha_i\geq d(el-\beta_i)$.

Assume now that I is not equigenerated. Condition $(\star)$ implies that $el-\beta_i=0$ and $cl-\alpha_i\geq 0$ or $el-\beta_i>0$ and $cl-\alpha_i\geq d_1(el-\beta_i)$. In the first case we may proceed as before to get the statement. In the second case we have that $k=[K_{R_\Delta}]_l\supset[I^{el-\beta_i}]_{cl-\alpha_i}w_i$ and so $cl-\alpha_i=d_1(el-\beta_i)$ and $d_1<d_2$. Then $\alpha_i-d_1\beta_i=cl-d_1el\geq c-d_1e\geq(d-d_1)e$ since $l\geq 1$ and $c\geq de+1>de$. Thus we obtain the inequality $e\leq\frac{\alpha_i-d_1\beta_i}{d-d_1}$ and for each e, we have $c\leq d_1e+\alpha_i-d_1\beta_i$. In any case, these inequalities hold for at most a finite number of diagonals and so we get the result. $\square$

If the Rees algebra $R_A(I)$ is Cohen-Macaulay we can also give bounds for the diagonals (c,e) such that $k[(I^e)_c]$ is quasi-Gorenstein.

Proposition 3.2

Assume that $ht(I)\geq 2$ *and* $R_A(I)$ *is Cohen-Macaulay. Let* $a=-a(G_A(I))$. *If* $k[(I^e)_c]$ *is quasi-Gorenstein, then* $e\leq a-1$ *and* $c\leq n$. *Moreover, if* $\dim(A/I)>0$ *then* $\lceil\frac{a}{e}\rceil-1=\frac{n}{c}=l\in\mathbb{Z}$. *In particular, if* $a=1$ *there are no diagonals* (c,e) *such that* $k[(I^e)_c]$ *is quasi-Gorenstein.*

Proof. Set $R=R_A(I)$ and $G=G_A(I)$. By Theorem 2.5, there exists a homogeneous filtration $\{K_m\}_{m\geq 0}$ of K_A such that $K_R\cong\bigoplus_{m\geq 1}K_m$ and $K_G\cong\bigoplus_{m\geq 1}K_{m-1}/K_m$. Bigrading the proof of [21], Corollary 2.5, we have that $K_m=Hom_A(I,K_{m+1})$ for every $m\geq 0$. Note that K_A may be viewed as an ideal of A. Assume that R_Δ is quasi-Gorenstein. Then there is an integer l_0 such that $[K_{R_\Delta}]_{l_0}=k$. By Proposition 2.3 we may find an element $f\in[K_{el_0}]_{cl_0}=[K_R]_{(cl_0,el_0)}$, $f\neq 0$, $K_{R_\Delta}=R_\Delta f$.

Claim. $K_{el_0}=Af$.

To prove the claim we first show that for any $g\in K_{el_0}$, $g\neq 0$, then $\deg g\geq cl_0$. Assume the contrary: $\deg g=k<cl_0$. Then $[Ag]_{cl_0}=A_{cl_0-k}g\subset[K_{el_0}]_{cl_0}\cong k$. But since $cl_0-k>0$, $\dim_k A_{cl_0-k}>1$, so we get a contradiction.

Now let $g\in K_{el_0}$. If $\deg g=cl_0$, then $g\in Af$ because $[K_{el_0}]_{cl_0}\cong k$. Let us assume that $\deg g=k>cl_0$. Then, for each $l>0$, $[I^{el}]_{cl}f+[I^{el}]_{c(l_0+l)-k}g\subset$

$[K_{e(l_0+l)}]_{c(l_0+l)} \cong [I^{el}]_{cl}$ as k-vector spaces, and so $[I^{el}]_{c(l_0+l)-k}g \subset [I^{el}]_{cl}f$. Now let $I^{el} = (F_1, \dots, F_t)$ where F_i is a homogeneous polynomial of degree $\leq del$ for all i, and set $\alpha = c(l_0+l) - k - \deg F_i$. Note that for $l >> 0$, $\alpha \geq c(l_0+l) - k - del = (c-de)l + cl_0 - k > 0$ and we can find $h \in A_\alpha$ such that $(h, f) = 1$. Then $hgF_i \in [I^{el}]_{c(l_0+l)-k}g \subset [I^{el}]_{cl}f \subset Af$ and we have that $gF_i \in Af$ for all i. Thus $I^{el}g \subset (f)$ and writing $g = d\overline{g}$, $f = d\overline{f}$ with $(\overline{f}, \overline{g}) = 1$ we get $I^{el}\overline{g} \subset A\overline{f}$. If $g \notin Af$, then $\overline{f} \notin k$ and so $I^{el} \subset (\overline{f})$ which is absurd because $ht(I) \geq 2$.

Now, as grade$(I) \geq 2$ we have $K_m = K_{el_0}$ for all $m \leq el_0$, which implies that $K_A = K_{el_0}$ and so $c \leq cl_0 = n$. Furthermore, $e \leq el_0 \leq \min\{m \mid K_m \subsetneq K_{m-1}\} - 1 = a - 1$.

Finally assume that $\dim(A/I) > 0$. We shall distinguish two cases. If $e = 1$ we have that $K_{l_0+1} \subsetneq K_{l_0}$: If not, then $I_c \cong [K_{l_0+1}]_{c(l_0+1)} = [Af]_{c(l_0+1)} \cong A_c$ which is absurd if $\dim(A/I) > 0$. Therefore $a = l_0 + 1 = \frac{n}{c} + 1$. If $e > 1$, let $\widetilde{\Delta} = (c, 1)$ and $\widetilde{R} = R(I^e)$. Note that $\widetilde{R}_{\widetilde{\Delta}} = R_\Delta$ which is quasi-Gorenstein. Applying the case before we obtain that $-a(G_A(I^e)) = \frac{n}{c} + 1$. By [15], $a(G_A(I^e)) = [\frac{-a}{e}] = -\lceil \frac{a}{e} \rceil$ and so $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{c} = l \in \mathbb{Z}$. □

Let us denote by $\mathfrak{m}$ the maximal homogeneous ideal of A. Given a homogeneous ideal $I \subset A$ we define the fiber cone of I as $F_\mathfrak{m}(I) = \bigoplus_{n\geq 0} I^n/mI^n$. Then $l(I) = \dim F_\mathfrak{m}(I)$ is called the *analytic spread* of I. Note that if I is equigenerated in degree d the fiber cone of I is nothing but $k[I_d]$.

Our next result shows that in some cases the existence of a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein forces the form ring to be Gorenstein. It may be seen as a converse of Theorem 2.8 for those cases.

Theorem 3.3

Assume that $R_A(I)$ is Cohen-Macaulay, $ht(I) \geq 2$, $l(I) < n$ and I is equigenerated. If there exists a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein then $G_A(I)$ is Gorenstein.

Proof. Let $R = R_A(I)$, $G = G_A(I)$ and $\Delta = (c, e)$. Assume first that $e = 1$. We have seen in the proof of Proposition 3.2 that there is a homogeneous filtration $\{K_m\}_{m\geq 0}$ of K_A such that $K_R \cong \bigoplus_{m\geq 1} K_m$ and $K_G \cong \bigoplus_{m\geq 1} K_{m-1}/K_m$, and an integer $l_0 = -a(R_\Delta)$ such that $K_0 = \ldots = K_{l_0} = Af$, with $f \in K_R$ and deg $f = cl_0$. It is then clear that for all $m \geq 0$, $I^m f \subset K_{l_0+m}$ and so $[I^m]_{cm} f \subset [K_{l_0+m}]_{c(l_0+m)} \cong [I^m]_{cm}$ since R_Δ is quasi-Gorenstein. This implies that $[K_{l_0+m}]_{c(l_0+m)} = [I^m]_{cm} f$.

We want to show that $K_{l_0+m} = I^m f$ for all $m \geq 0$. Suppose that there exists m_0 such that $I^{m_0} f \subsetneq K_{l_0+m_0}$. Then let $g \in K_{l_0+m_0}$, $g \notin I^{m_0} f$ be a homogeneous element of degree k. Note that from the inclusion $K_{l_0+m_0} \subset K_{l_0} = Af$ one has $g = f\overline{g}$ with $\overline{g} \notin I^{m_0}$.

If $k \geq c(l_0 + m_0)$ then for all $m > m_0$ we have $I^m f + I^{m-m_0} g \subset K_{l_0+m}$ and so $[I^m]_{cm} f + [I^{m-m_0}]_{c(l_0+m)-k} g \subset [K_{l_0+m}]_{c(l_0+m)} \cong [I^m]_{cm}$. Hence $[I^{m-m_0}]_{c(l_0+m)-k} g \subset [I^m]_{cm} f$ and we get that $[I^{m-m_0}]_{c(l_0+m)-k}\overline{g} \subset [I^m]_{cm}$. Let $\lambda = c(l_0+m) - k - d(m-m_0) = (c-d)m + cl_0 + dm_0 - k$. For $m >> 0$ we have that $\lambda > 0$. Then, if $A_\lambda \overline{g} \in I^{m_0}$ we would have that $\overline{g} \in (I^{m_0})^* = \{p \in A \mid p\,\mathfrak{m}^k \subset I^{m_0}, \text{for some}\, k\}$, the saturation of I^{m_0}. It is well-known that if $G_A(I)$ is Cohen-Macaulay then $\inf\{\text{depth}(A/I^n)\} = \dim A - l(I)$ [4]. As $l(I) < n$, we then get $\overline{g} \in I^{m_0}$ which is a contradiction. So there exist $\lambda > 0$, $h \in A_\lambda$ such that $\overline{g}h \notin I^{m_0}$. On the other hand, $\overline{g}h[I^{m-m_0}]_{d(m-m_0)} \subset \overline{g}[I^{m-m_0}]_{c(l_0+m)-k} \subset [I^m]_{cm}$. So by using that I is equigenerated we have that $\overline{g}h \in (I^m : I^{m-m_0}) = I^{m_0}$, since R is Cohen-Macaulay. This is a contradiction.

If $k < c(l_0 + m_0)$, let us write $k = c(l_0+m_0) - s$ with $s > 0$. Then $A_s g \subset [K_{l_0+m_0}]_{c(l_0+m_0)} = [I^{m_0}]f$, and $g \in (I^{m_0})^* = I^{m_0}$ which, as before, is a contradiction.

Hence we have proved that $K_{l_0+m} = I^m f$ for all $m \geq 0$ and $K_R = f(At \oplus \ldots \oplus At^{l_0} \oplus It^{l_0+1} \oplus \ldots)$, i.e. K_R has the expected form. By [21], Theorem 4.2 this implies that both $R_A(I^{l_0})$ and $G_A(I)$ are Gorenstein.

Finally assume $e > 1$, and denote by $\widetilde{\Delta} = (c, 1)$ and $\widetilde{R} = R(I^e)$. Then $\widetilde{R}_{\widetilde{\Delta}} = R_\Delta$ is quasi-Gorenstein and so there exists l_0 such that $R_A(I^{el_0})$ is Gorenstein. By [21], Theorem 4.2 this implies again that $G_A(I)$ is Gorenstein. □

Example 3.4 *(Room surfaces)*: Let k be an algebraically closed field. Set $t = \binom{d+1}{2}$, with $d \geq 2$. We are going to study the rational projective surfaces which arise as embeddings of blowing ups of $\mathbb{P}^2_k$ at a set of t distinct points $P_1, \ldots, P_t$ not contained in any curve of degree $d-1$.

Let I be the ideal defining the set of points $P_1, \ldots, P_t$. It can be easily seen that I is a homogeneous ideal equigenerated in degree d. For each $c \geq d+1$, we obtain a surface by the embedding associated to I_c. For $c = d+1$ the resulting surfaces are called Room surfaces. It has been proved by A. Geramita and A. Gimigliano that they are arithmetically Cohen-Macaulay. Assume $d \geq 3$. Gimigliano [8] proved that I_d also defines an embedding of this blow up in the projective space $\mathbb{P}^d_k$ with defining ideal given by the 3×3 minors of a $3 \times d$ matrix of linear forms, and that this ideal has a linear resolution that comes from the Eagon-Northcott complex. From this fact and applying [1], Example 3.6.15, one obtains that $a(k[I_d]) = -1$ and so by [20] the reduction number of I is $r(I) = a(k[I_d]) + l(I) = -1 + 3 = 2$. Moreover the analytic deviation of I is $ad(I) = l(I) - ht(I) = 1$ and I is generically a complete intersection ideal. So we may conclude by [11] that $G_A(I)$ is Cohen-Macaulay and hence by [10], Proposition 2.4, $a(G_A(I)) = r(I) - ht(I) - 1 = -1$. By Ikeda-Trung's criterion, $R_A(I)$ is also Cohen-Macaulay. From Proposition 3.2 we get that there are not diagonals (c, e) such that $k[(I^e)_c]$ is Gorenstein. In particular, $k[I_{d+1}]$ is not Gorenstein for $d \geq 3$.

If $d = 2$, by choosing the points to be [1:0:0], [0:1:0] and [0:0:1], we have $I = (X_1X_2, X_1X_3, X_2X_3)$. Note that I is an almost complete intersection ideal such that A/I is Cohen-Macaulay. Moreover, it is easy to check that $\mu(I_{\mathfrak{p}}) \leq ht(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$. So one knows from [13] that $G_A(I)$ is Gorenstein and $a(G_A(I)) = -ht(I) = -2$. By Theorem 2.8, $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{3}{c} = \frac{1}{e} \in \mathbb{Z}$. So $(3, 1)$ is the only diagonal with the Gorenstein property. This corresponds to the del Pezzo sestic surface in $\mathbb{P}^6$.

References

1. W. Bruns and J. Herzog, *Cohen Macaulay rings*, Cambridge Stud. Adv. Math. **39**, 1993.
2. A. Conca, J. Herzog, N.V. Trung and G. Valla, Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces, *Amer. J. Math.* **119** (1997), 859–901.
3. S.D. Cutkosky and J. Herzog, Cohen-Macaulay coordinate rings of blowup schemes, *Comment. Math. Helv.* **72** (1997), 605–617.
4. D. Eisenbud and C. Huneke, Cohen-Macaulay Rees algebras and their specializations, *J. Algebra* **81** (1983), 202–224.
5. A. Geramita and A. Gimigliano, Generators for the defining ideal of certain rational surfaces, *Duke Math. J.* **62**(1) (1991), 61–83.
6. A. Geramita, A. Gimigliano and B. Harbourne, Projectively Normal but Superabundant Embeddings of Rational Surfaces in Projective Space, *J. Algebra* **169** (1994), 791–804.
7. A. Geramita, A. Gimigliano and Y. Pitteloud, Graded Betti numbers of some embedded rational n-folds, *Math. Ann.* **301** (1995), 363–380.
8. A. Gimigliano, On Veronesean surfaces, *Proc. Kon. Nederl. Akad. Wetensch.* **92** (1989), 71–85.
9. A. Gimigliano and A. Lorenzini, On the ideal of Veronesean surfaces, *Can. J. Math.* **45** (1993), 758–777.
10. S. Goto and S. Huckaba, On graded rings associated to analytic deviation one ideals, *Amer. J. Math.* **116** (1994), 905–919.
11. S. Goto and Y. Nakamura, On the Gorensteinness of graded rings associated to ideals of analytic deviation one, *Contemp. Math.*, **159** (1994), 51–72.
12. M. Herrmann, E. Hyry and J. Ribbe, On the Cohen-Macaulay and Gorenstein properties of multigraded Rees algebras, *Manuscripta Math.* **79** (1993), 343–377.
13. M. Herrmann, J. Ribbe and S. Zarzuela, On the Gorenstein property of Rees and form rings of powers ideals, *Trans. Amer. Math. Soc.* **342** (1994), 631–643.
14. J. Herzog, A. Simis and W.V. Vasconcelos, On the canonical module of the Rees algebra and the associated graded ring of an ideal, *J. Algebra* **105** (1987), 285–302.

15. L.T. Hoa and S. Zarzuela, Reduction number and a-invariant of good filtrations, *Comm. Alg.* **22**(4) (1994), 5635–5656.
16. S. Ikeda and N.V. Trung, When is the Rees algebra Cohen-Macaulay?, *Comm. Alg.* **17**(12) (1989), 2893–2922.
17. O. Lavila–Vidal, On the Cohen-Macaulay property of diagonal subalgebras of the Rees algebra, *Manuscripta Math.* **95** (1998), 47–58.
18. J. Lipman, Cohen-Macaulayness in graded algebras, *Math. Res. Letters* **1** (1994), 149–157.
19. A. Simis, N.V. Trung and G. Valla, The diagonal subalgebra of a blow-up algebra, *J. Pure Appl. Algebra* **125** (1998), 305–328.
20. N.V. Trung, Reduction exponent and degree bound for the defining equations of graded rings, *Proc. Amer. Math. Soc.* **101** (1987), 229–236.
21. N.V. Trung, D.Q. Viêt and S. Zarzuela, When is the Rees Algebra Gorenstein?, *J. Algebra* **175** (1995), 137–156.
22. K. Watanabe, Certain invariant subrings are Gorenstein I, *Osaka J. Math.* **11** (1974), 1–8.
23. K. Watanabe, Certain invariant subrings are Gorenstein II, *Osaka J. Math.* **11** (1974), 379–388.
24. Y. Yoshino, The canonical modules of graded rings defined by generic matrices, *Nagoya Math. J.* **81** (1981), 105–112.