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The Stokes second problem and its extension to viscoelastic fluids

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Abstract

This work is focused on the Stokes second problem, a classical problem in fluid mechanics. It is an oscillatory fluid flow problem of great significance both academically and practically, for its relation with industrial and natural processes. This work was inspired by the study of a viscoelastic fluid problem carried on within [4], which possesses a certain similarity with the Stokes second problem. First, we set the fundamental principles of fluid dynamics, which we need to derive the governing equations of motion of the fluid. Next, we introduce the objects used to characterise the viscous properties of fluids and to determine how these affect the motion of the fluid. In addition, a derivation of the Navier-Stokes equation is provided. Finally, we introduce and solve the Stokes second problem, justifying the existence and the uniqueness of the solution. At last, we examine the equations that model the motion of the fluid considered in [4], which can be seen as a generalisation of the Stokes second problem for a specific non-Newtonian fluid.

Resum

Aquest treball se centra en el segon problema de Stokes, un problema clàssic en mecànica de fluids. És un problema de flux oscil·latori de fluid de gran importància acadèmica i pràctica, per la seva relació amb processos industrials. Aquest treball es va inspirar en l'estudi d'un problema de fluid viscoelàstic realitzat en [4], el qual posseeix certa similitud amb el segon problema de Stokes. Primer, establim els principis fonamentals de la dinàmica de fluids, que necessitem per derivar les equacions que governen el moviment del fluid. A continuació, presentem els objectes utilitzats per caracteritzar les propietats viscoses dels fluids i determinar com afecten el moviment del fluid. A més, es proporciona una derivació de l'equació de Navier-Stokes. Finalment, plantegem i resolem el segon problema de Stokes, justificant l'existència i la unicitat de la solució. Finalment, examinem les equacions que regeixen el moviment del fluid considerat en [4], que es pot entendre com una generalització del segon problema de Stokes per a un fluid no-Newtonià en concret.

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Introduction

Oscillatory fluid flows are paramount both in natural and artificial phenomena. We find them in the blood pumping of living organisms (see [21]) and the oil recovery industry (see [14]). A certain family of fluids widely used in the latter is wormlike micellar (WLM) solutions. WLM solutions are complex fluids commonly employed in the enhanced oil recovery business, so to correctly comprehend their flow behaviour is crucial for oil and gas production.

One of the main concepts in the study of oscillatory flows is the Stokes layer. Let us imagine an infinite plate oscillating harmonically in its plane with a certain frequency and a specific oscillation amplitude. This plate has on one of its sides a certain fluid, which extends in a semi-infinite domain. The region of the fluid in contact with the plate that moves because of the oscillatory motion of the plate is what we call the Stokes layer. The problem of obtaining the flow field in the Stokes layer for a Newtonian fluid in this configuration is known as the Stokes second problem. It represents a classical question within the field of fluid mechanics.

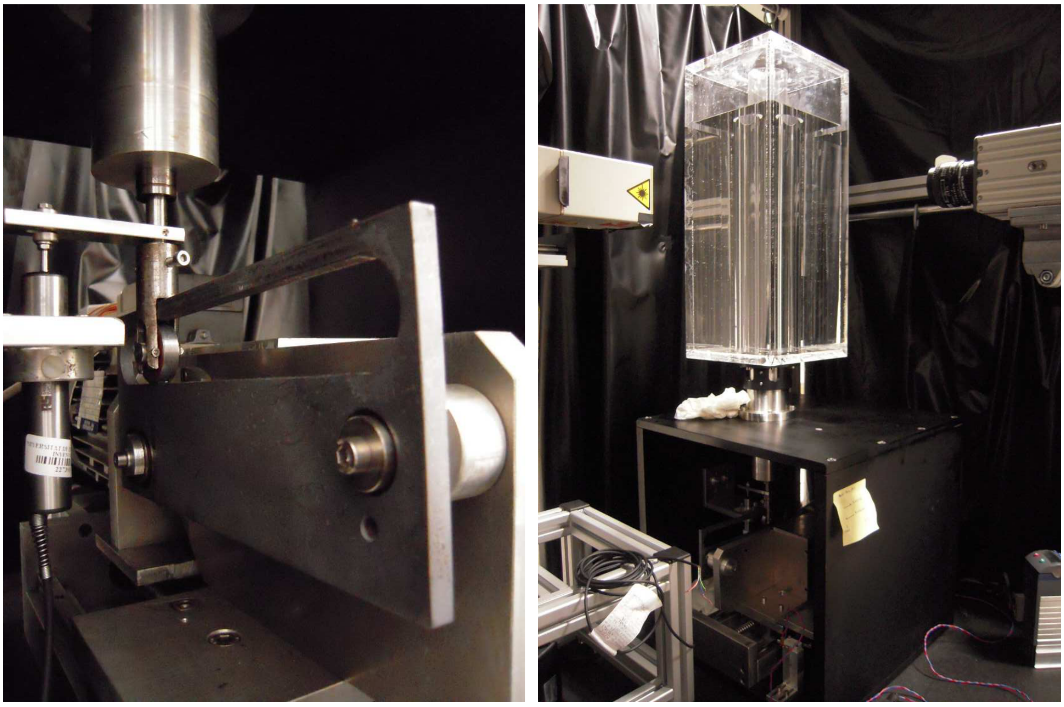

Figure 1: Pictures of the actual experimental setup placed in the Nonlinear Physics Laboratory at the Faculty of Physics. Left: generation of the oscillatory motion; Right: general view of the experimental setup (extracted from [4]).

The previous setting is an oversimplified idealisation since we cannot dispose of an infinite plate in a physics laboratory. The oscillatory flows we are interested in take place in cylindrical structures, may them be the veins of our circulatory system or the injecting pipes in an oil field. This is the setup we find in the experiment carried out in the framework of the PhD thesis of L. Casanellas [4], which is the physical motivation of this work and its original inspiration.

The experimental setup consisted of a WLM solution contained inside a vertical rigid cylindrical tube with an inner radius $a = 2.5$ cm and height $h = 60$ cm. The motion of the fluid was driven by the oscillatory motion of a piston of radius a located at the bottom end of the cylinder. The experimental device allowed to control both the frequency ω_0 and amplitude z_0 of the vertical oscillatory motion of the piston. The top surface of the fluid was covered by a freely moving plastic lid (see Figure 1). For further details of the experimental apparatus, see [4].

The object of study in [4] is the motion of the fluid far from the upper and lower ends of the cylinder, in order to avoid boundary effects at those ends. Thus, translational invariance along the vertical axis is considered to simplify the mathematical formulation. The author looks for steady-periodic stationary solutions, so they consider the regime in which the flow has stabilised after the initial impulse originated by the start-up of the piston. The Stokes second problem sheds light on this other problem in which geometry is not so simple. The WLM solution studied in [4] is incompressible, viscoelastic, and shear-thinning, hence a particular type of non-Newtonian fluid. Even though the Stokes second problem contemplates a Newtonian fluid, it is much illustrating and provides a sensible approach to the experiment in [4], so it will be our object of study.

We embark ourselves on the study of viscous fluid mechanics, a field in which the measure of physical quantities that evolve with time is essential. Therefore, we will need to define how to measure these dynamic magnitudes. How could it be otherwise, the results we shall derive in this work follow from fundamental physical laws and principles. They are deduced from elementary concepts, without the need for prior knowledge of the matter. Our purpose is to solve the Stokes second problem having correctly introduced and explained the fundamental notions of fluid dynamics and all the elements that intervene in the fluid motion and to present the setting for the cylinder problem for a non-Newtonian fluid.

Being this a mathematical work, we indent to carefully immerse the non-physical reader into the disciplines of fluid dynamics and rheology, supplying a scrupulous deduction of all the equations of motion that we need for our purpose. We aim to provide mathematical rigour and formalism to certain relevant results that one may find in the literature. For this purpose, several areas of mathematics will come into play.

In Chapter 1 we introduce the fundamental principles of fluid dynamics in a general perspective. The governing equations of a Eulerian-Lagrangian fluid are derived from the principle of mass conservation and Newton's second law, under the continuum assumption. In Chapter 2 we present the two objects used to describe how the viscosity of a fluid affects its motion: the viscosity stress tensor and the rate of strain tensor. We also give a preliminary view of how they are related by constitutive equations and how these equations determine the type of fluid we are dealing with. Besides, we include a proof of the equivalence between the conservation of angular momentum and the symmetry of the stress tensor. In Chapter 3 we provide a formal derivation of the constitutive equation for a Newtonian fluid that naturally leads us to the Navier-Stokes equation, which we will use to solve the Stokes second problem. Moreover, we examine the existence and the uniqueness of the solution to this problem. Finally, in Chapter 4 we inspect the viscoelastic fluid considered in the aforementioned experiment. We present two different constitutive equations for the viscoelastic fluid, stressing the physical motivation of each one of them and how they represent a more accurate description of the fluid in the experiment than the one arising from the Newtonian constitutive equation. We finish with presenting the equations that determine the equation of motion for this fluid and discussing the particular limitations they introduce with respect to a Newtonian fluid.

Chapter 1

Equations of motion

In this chapter, we provide a derivation of the most general equation of motion for a fluid, considering only its basic characteristics. We start by establishing the elementary definitions and the necessary theorems to proceed, since we intend to give a formal deduction[1]. Afterwards, we will find ourselves in need for some physical assumptions to be made in order to finally consider the dynamics of the fluid and obtain the equation of motion. For further reading, we refer to [6], [10], [12] and [18].

1.1 Preliminary definitions

The Eulerian description of a fluid flow in an open set $V \subset \mathbb{R}^n$ is given by its *velocity field* $u(t,x)$, where $u : \mathbb{R} \times V \to \mathbb{R}^n$ is assumed to be a non-autonomous $\mathcal{C}^1$ vector field. Hence, the velocity of the fluid at point $x \in V$ and time $t \in \mathbb{R}$ is $u(t,x)$. On the other hand, the Lagrangian description of the fluid is given by its deformation $\Phi(t;t_0,x_0)$, where $\Phi : \mathbb{R} \times \mathbb{R} \times V \to V$ is a $\mathcal{C}^1$ evolution operator, which is assumed to be $\mathcal{C}^2$ with respect to t. In this model, $\Phi(t;t_0,x_0)$ is the position at time $t \in \mathbb{R}$ of the particle of fluid that at time $t_0 \in \mathbb{R}$ was at point $x_0 \in V$[2]. Ordinary differential equations theory implies that both descriptions are equivalent and related to one another by the identity

$$\frac{\partial \Phi}{\partial t}(t;t_0,x_0) = u(t,\Phi(t;t_0,x_0)). \tag{1.1}$$

Hereafter we shall write $\varphi(t;x_0) := \Phi(t;0,x_0)$.

[1] A number of the results included in this chapter were seen in the Differential Equations course and have been included and proven here for the sake of completeness.

[2] We consider Φ defined $\forall t \in \mathbb{R}$, which is a natural assumption in this setting.

Remark 1.1. *The velocity field $u(t,x)$ depends on the rheological properties of the fluid (see Chapter 2), thus it will be not completely determined until such properties are established for the fluid at hand.* △

Remark 1.2. *Certain equations that we shall write in this work are prominent laws of fluid mechanics. That is the reason why, eventually, we will use the* nabla, ∇^T, *notation in order to write those equations in the usual physics form. Here we comment on the expression of the corresponding operators when acting on scalar, vector fields and multilinear maps expressed in Cartesian coordinates $(x_1, ..., x_n)$.*

The operator ∇^T is defined as

$$\nabla^T := \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)^T. \tag{1.2}$$

Hence, it acts on scalar, vector fields and multilinear maps as follows:

- *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ scalar field, then its* gradient *is denoted by*

$$\nabla^T f := \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)^T = grad\ f. \tag{1.3}$$

- *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field, its* divergence *can be written as*

$$\nabla^T \cdot u := \sum_i \frac{\partial u_i}{\partial x_i} = tr\,(D_x u) = div\ u, \tag{1.4}$$

  where $\cdot$ is a notation reminiscent of the inner product.

- *Let $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a $\mathcal{C}^1$ map. We define the divergence of σ as the vector operator given by the divergence of each column σ_j (which defines a $\mathcal{C}^1$ vector field). That is,*

$$\nabla^T \cdot \sigma := \left(\sum_i \frac{\partial \sigma_{i1}}{\partial x_i}, \cdots, \sum_i \frac{\partial \sigma_{in}}{\partial x_i}\right)^T = div\ \sigma. \tag{1.5}$$

- *The* Laplace operator *is defined as*

$$\Delta := \left(\nabla^T\right)^2 = \nabla^T \cdot \nabla^T = \sum_i \frac{\partial^2}{\partial x_i^2}. \tag{1.6}$$

  It is an operator that can be applied to either vector or scalar fields. Then, if $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^2$ scalar field and $u : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^2$ vector field, their respective Laplacian operators *are*

$$\Delta f := \sum_i \frac{\partial^2 f}{\partial x_i^2}, \quad \Delta u := \left(\sum_j \frac{\partial^2 u_1}{\partial x_j^2}, \cdots, \sum_j \frac{\partial^2 u_n}{\partial x_j^2}\right)^T. \tag{1.7}$$

△

The evolution of the motion of a fluid is described through the evolution of suitable observables in time. These observables are physical quantities, which we call magnitudes, that can be effectively measured in experiments.

Definition 1.3. Let $g : \mathbb{R} \times V \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function, which we shall name *observable*. We define its *measure* on an open bounded subset $U \subset\subset V$ at time $t \in \mathbb{R}$ as

$$G(t, U) := \int_U g(t, x) dx.$$

Given an open bounded domain U_0, we define $U_t := \varphi(t; U_0)$ as the transported domain by the fluid at time t.

Theorem 1.4. [Transport theorem] *The following equality holds*

$$\frac{d}{dt}\left(G(t, U_t)\right) = \int_{U_t} \left(\frac{\partial g}{\partial t}(t, x) + D_x g(t, x) u(t, x) + div\, u(t, x) g(t, x) \right) dx.$$

Proof. One has that

$$G(t, U_t) = \int_{U_t} g(t, x) dx = \int_{\varphi(t; U_0)} g(t, x) dx.$$

Since $\varphi(t; \cdot) : U_0 \to U_t$ is a $\mathcal{C}^1$ diffeomorphism, it defines a change of variables $x_0 = \varphi(-t; x)$, so that

$$G(t, U_t) = \int_{U_0} g(t, \varphi(t; x_0)) |\det D_{x_0} \varphi(t; x_0))| dx_0.$$

Differentiating both sides of the previous equation one obtains that

$$\frac{d}{dt}\left(G(t, U_t)\right) = \int_{U_0} \frac{d}{dt}\left(g(t, \varphi(t; x_0)) |\det D_{x_0} \varphi(t; x_0)|\right) dx_0. \tag{1.8}$$

On the one hand,

$$\frac{d}{dt}\left(g(t, \varphi(t; x_0))\right) = \frac{\partial g}{\partial t}(t, \varphi(t; x_0)) + D_x g(t, \varphi(t; x_0)) \frac{\partial \varphi}{\partial t}(t; x_0),$$

and since $\varphi(t; x_0)$ verifies (1.1), the previous expression becomes

$$\frac{d}{dt}\left(g(t, \varphi(t; x_0))\right) = \frac{\partial g}{\partial t}(t, \varphi(t; x_0)) + D_x g(t, \varphi(t; x_0)) u(t, \varphi(t; x_0)). \tag{1.9}$$

On the other hand, we know that $D_{x_0} \varphi(t; x_0)$ is a fundamental matrix of solutions of the Cauchy problem associated to the variational equations with respect to initial conditions, that is

$$\begin{cases} \frac{d}{dt} D_{x_0} \varphi(t; x_0) & = D_x u(t, \varphi(t; x_0)) D_{x_0} \varphi(t; x_0), \\ D_{x_0} \varphi(0; x_0) & = \mathrm{Id}, \end{cases} \tag{1.10}$$

where Id is the n-dimensional identity matrix. Then, by the Jacobi-Liouville formula (see [19]), one has

$$\begin{aligned}\det D_{x_0}\varphi(t;x_0)) &= \exp\int_0^t \operatorname{tr}\left(D_x u(s,\varphi(s;x_0))\right)\det D_{x_0}\varphi(0;x_0))ds\\ &= \exp\int_0^t \operatorname{div}\, u(s,\varphi(s;x_0))ds.\end{aligned}$$

In particular, one has that $\det D_{x_0}\varphi(t;x_0) > 0$, $\forall t\in\mathbb{R}$. Taking derivatives, it follows

$$\frac{d}{dt}\left(\det D_{x_0}\varphi(t;x_0)\right) = \operatorname{div}\, u(t,\varphi(t;x_0))\det D_{x_0}\varphi(t;x_0)). \tag{1.11}$$

Joining expressions (1.8) and (1.11) and using (1.9) one gets

$$\begin{aligned}\frac{d}{dt}\left(G(t,U_t)\right) &= \int_{U_0}\frac{\partial g}{\partial t}(t,\varphi(t;x_0))\det D_{x_0}\varphi(t;x_0)\\ &\quad + D_x g(t,\varphi(t;x_0))u(t,\varphi(t;x_0))\det D_{x_0}\varphi(t;x_0)\\ &\quad + g(t,\varphi(t;x_0))\operatorname{div}\, u(t,\varphi(t;x_0))\det D_{x_0}\varphi(t;x_0)dx_0\\ &= \int_{U_0}\left[\frac{\partial g}{\partial t}(t,\varphi(t;x_0)) + D_x g(t,\varphi(t;x_0))u(t,\varphi(t;x_0))\right.\\ &\quad \left. + g(t,\varphi(t;x_0))\operatorname{div}\, u(t,\varphi(t;x_0))\right]\det D_{x_0}\varphi(t;x_0)dx_0.\end{aligned}$$

Reversing the change of variables, we finally obtain

$$\frac{d}{dt}\left(G(t,U_t)\right) = \int_{U_t}\left(\frac{\partial g}{\partial t}(t,x) + D_x g(t,x)u(t,x) + \operatorname{div}\, u(t,x)g(t,x)\right)dx.$$

□

Definition 1.5. Given a fluid described by (1.1) and an open bounded subset $U_t \subset\subset V$, we define the *volume* of the fluid occupying the subset U_t at time $t\in\mathbb{R}$ as

$$\operatorname{vol}(t,U_t) := \int_{U_t} 1dx.$$

Theorem 1.6. [Liouville theorem] *A fluid described by* (1.1) *is* incompressible *(i.e., its volume remains constant) if, and only if, the divergence of u is equal to* 0. *In that case, we say the vector field $u(t,x)$ is conservative.*

Proof. Taking $g(t,x) := 1$ and using the transport theorem 1.4 , one has

$$0 = \frac{d}{dt}\left(\operatorname{vol}(t,U_t)\right) = \int_{U_t}\operatorname{div}\, u(t,x)dx.$$

This expression holds for any arbitrary $U_t \subset\subset V$, therefore

$$\operatorname{div}\, u = 0.$$

□

1.2 Physical hypotheses

We shall assume that there is a $\mathcal{C}^1$ function $\rho : \mathbb{R} \times V \to \mathbb{R}$ such that the observable (recall Definition 1.3) $\rho(t, x)$ gives us the *density* of the fluid at point $x \in V$ at time $t \in \mathbb{R}$. This assumption is called the **continuum assumption**. As explained in [6], this assumption holds only if we do not consider the molecular structure of matter. Since now we are dealing with macroscopic phenomena, the continuum assumption is an accurate hypothesis.

Definition 1.7. Given an open bounded subset $U_t \subset\subset V$, we define the *mass* of the fluid occupying the subset U_t at time $t \in \mathbb{R}$ as

$$\text{mass}(t, U_t) := \int_{U_t} \rho(t, x) dx.$$

The **principle of mass conservation**, a fundamental principle of physics, states that mass is neither created nor destroyed. Therefore, using Definition 1.7 we can express the principle of mass conservation as

$$\frac{d}{dt} (\text{mass}(t, U_t)) = 0. \tag{1.12}$$

Taking $g(t, x) = \rho(t, x)$ and using the transport theorem 1.4 , we see that the latter equation is equivalent to

$$\int_{U_t} \left(\frac{\partial \rho}{\partial t}(t, x) + D_x \rho(t, x) u(t, x) + \text{div } u(t, x) \rho(t, x) \right) dx = 0,$$

a condition that holds for any arbitrary $U_t \subset\subset V$. Hence

$$\frac{\partial \rho}{\partial t}(t, x) + D_x \rho(t, x) u(t, x) + \text{div } u(t, x) \rho(t, x) = 0. \tag{1.13}$$

Moreover, using (1.3), one has

$$\begin{aligned} D_x \rho(t, x) u(t, x) + \text{div } u(t, x) \rho(t, x) &= \left(\frac{\partial \rho(t, x)}{\partial x_1}, \cdots, \frac{\partial \rho(t, x)}{\partial x_n} \right)^T u(t, x) + \text{div } u(t, x) \rho(t, x) \\ &= \text{grad } \rho(t, x) u(t, x) + \text{div } u(t, x) \rho(t, x) \\ &= \text{div } \left(\rho(t, x) u(t, x) \right), \end{aligned}$$

so Equation (1.13) can be written in a more compact form

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho u) = 0, \tag{1.14}$$

known as the **continuity equation**.

Definition 1.8. We define the *linear momentum* of the fluid occupying the subset U_t at time $t \in \mathbb{R}$ as

$$p(t, U_t) := \int_{U_t} \rho(t,x)u(t,x)dx.$$

Taking into account the source of the forces that can act on the portion of fluid that occupies the subset U_t, we can separate them into two types: those originated internally by the fluid and those external to it.

Internal forces refer to the interactions between the fluid occupying U_t and the fluid occupying $V \setminus U_t$. Since any $x \in V$ belongs to a certain ∂U_t (because U_t refers to any open domain in V), these forces define a non-autonomous vector field T on V. We assume that ∂U_t is regular.

Conjecture 1.9. [Cauchy's postulate] *The vector field T depends on the surface ∂U_t at point x only through the normal vector $\tilde{n}$, so we have that*

$$T = T(t, x, \tilde{n})$$

is the force per unit area exerted at time t and point x on surface ∂U_t oriented with outward orthonormal vector $\tilde{n}$ (see [18]). We call the $\mathcal{C}^0$ vector field $T : \mathbb{R} \times V \times \mathbb{R}^n \to \mathbb{R}^n$ the stress vector *(see [12]).*

Then, by definition, T gives us the internal forces that the rest of the fluid $V \setminus U_t$ exerts on U_t at time t through its boundary ∂U_t in the form

$$F_{int}(t, U_t) = \oint_{\partial U_t} T(t, x, \tilde{n})dx, \tag{1.15}$$

where $\tilde{n} \in \mathbb{R}^n$ is the outward orthonormal vector to ∂U_t at point x.

External forces account for all the interactions acting on the the fluid occupying U_t which do not emerge from the fluid itself (e.g., the gravitational force). Since those forces act on every point $x \in U_t$ of the fluid, we accept that the force exerted on U_t at time t by an external force whose intensity is given by the $\mathcal{C}^1$ vector field $f : \mathbb{R} \times V \to \mathbb{R}^n$ can be written as

$$F_{ext}(t, U_t) = \int_{U_t} \rho(t,x)f(t,x)dx. \tag{1.16}$$

Newton's second law states that the rate of change of linear momentum of a portion of a fluid is equal to the sum of the forces acting on it. Hence, considering Equations (1.15) and (1.16), we know that the motion of the portion of fluid occupying subset U_t will be governed by the equation

$$\frac{d}{dt}\left(p(t,U_t)\right) = \int_{U_t} \rho(t,x) f(t,x) dx + \oint_{\partial U_t} T(t,x,\tilde{n}) dx, \tag{1.17}$$

which we may refer to as the **balance of linear momentum**.

Newton's third law declares that whenever a body exerts a force on another, that second body exerts a force equal in magnitude and opposite in direction on the first. This law provides a physical explanation for the following lemma, which shall be used to proof the Cauchy's theorem, as we shall see in Section 1.3.

Lemma 1.10. [Cauchy's lemma] *The stress vector T acting on one side of ∂U_t at time t and point x is equal in magnitude but opposite in direction to the stress vector T acting on the other side of ∂U_t at the same time t and point x; that is,*

$$T(t,x,\tilde{n}) = -T(t,x,-\tilde{n}).$$

1.3 Local equation of motion

In the previous section, we have introduced the stress vector T to characterise the fluid internal forces. We notice that it appears in Equation (1.17) within an integral over ∂U_t, while the external force and the linear momentum time derivative come from integrals over U_t (recall Equation (1.16) and Definition 1.8). This difference prevents us of obtaining a local form of Equation (1.17), which is the purpose of this section. That is why we introduce the following theorem (see [12]).

Theorem 1.11. [Cauchy's theorem] *Let $a : \mathbb{R} \times V \to \mathbb{R}$ be a $\mathcal{C}^1$ function and $b : \mathbb{R} \times V \to \mathbb{R}$ and $c : \mathbb{R} \times V \times \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^0$ functions, all of them defined $\forall t \in \mathbb{R}$, $\forall x \in V$ and $\forall \tilde{n} \in \mathbb{R}^n$ unit vector at x. If, $\forall U_t \subset\subset V$, a, b and c satisfy the balance law*

$$\frac{d}{dt}\int_{U_t} a(t,x) dx = \int_{U_t} b(t,x) dx + \int_{\partial U_t} c(t,x,\tilde{n}) dx,$$

where $\tilde{n}$ is the outward orthonormal vector to ∂U_t at point x, then there exists a unique tensor field $\tilde{c} : \mathbb{R} \times V \to \mathbb{R}^{n\times n}$ such that $c(t,x,\tilde{n}) = \tilde{c}(t,x)\tilde{n}$.

Proof. By the transport theorem 1.4, we can write the balance law as

$$\int_{U_t} \left(\frac{\partial a}{\partial t}(t,x) + D_x a(t,x) u(t,x) + \operatorname{div}\, u(t,x) a(t,x) - b(t,x) \right) dx = \int_{\partial U_t} c(t,x,\tilde{n}) dx. \tag{1.18}$$

Let us consider a fixed time t and Cartesian coordinates $(x_1, ..., x_n)$ with origin at x_0. We draw the standard n-simplex Δ^n which has one of its $n+1$ vertices at x_0. Let Σ be the face of Δ^n that is not normal to any of the coordinate axis and let us label the other faces Σ_i, $i = 1, \ldots, n$. Since x_0 is at the origin of coordinates, it belongs to the Σ_1 face, which is normal to the x_1 coordinate axis, hence it is clear that Σ that does not contain x_0.

Let $\tilde{n}_0$ be the outward orthonormal vector to Σ. Let l be the distance along $\tilde{n}_0$ from x_0 to Σ. Then, the volume of Δ^n is $\text{vol}(\Delta^n) = \alpha l^n$ and the area of Σ is $\text{area}(\Sigma) = \beta l^{n-1}$, where α, $\beta \in \mathbb{R}$ are constants. The area of the other faces can be written as $\text{area}(\Sigma_i) = (\tilde{n}_0 \cdot e_i)\text{area}(\Sigma) = \tilde{n}_0^i \text{area}(\Sigma)$, where e_i denotes the $i-$th vector from the standard basis, $\tilde{n}_0^i$ is the $i-$th component of $\tilde{n}_0$ and $\cdot$ indicates the dot product (see [9]). Thus, one has that

$$\lim_{l \to 0} \frac{\text{vol}(\Delta^n)}{\text{area}(\partial\Delta^n)} = \lim_{l \to 0} \frac{\alpha l^n}{\beta l^{n-1}\left(1 + \sum_{i=1}^n \tilde{n}_0^i\right)} = \lim_{l \to 0} \frac{\alpha l}{\beta\left(1 + \sum_{i=1}^n \tilde{n}_0^i\right)} = 0. \tag{1.19}$$

Since $a(t,x)$ is a $\mathcal{C}^1$ function, we know that its first total derivative is a $\mathcal{C}^0$ function, as so is $b(t,x)$. Moreover, we know that the absolute value is a $\mathcal{C}^0$ function. Thus, the Weierstrass theorem ensures us that $\exists M \in \mathbb{R}_{\geq 0}$ such that

$$M = \max_{x \in \Delta^n} \left| \frac{\partial a}{\partial t}(t,x) + D_x a(t,x) u(t,x) + \text{div }u(t,x) a(t,x) - b(t,x) \right| \geq 0.$$

Hence, we have that

$$\begin{aligned}
\left| \int_{\partial\Delta^n} c(t,x,\tilde{n}) dx \right| &= \left| \int_{\Delta^n} \left(\frac{\partial a}{\partial t}(t,x) + D_x a(t,x) u(t,x) + \text{div }u(t,x) a(t,x) - b(t,x) \right) dx \right| \\
&\leq \int_{\Delta^n} \left| \left(\frac{\partial a}{\partial t}(t,x) + D_x a(t,x) u(t,x) + \text{div }u(t,x) a(t,x) - b(t,x) \right) \right| dx \\
&\leq \int_{\Delta^n} M dx = M \text{vol}(\Delta^n).
\end{aligned}$$

Considering the last inequality and Equation (1.19), we know that

$$\lim_{l \to 0} \frac{1}{\text{area}(\partial\Delta^n)} \int_{\partial\Delta^n} c(t,x,\tilde{n}) dx = 0. \tag{1.20}$$

The mean value theorem for integrals tells us that $\exists p \in \Sigma$, $p_i \in \Sigma_i$ such that

$$\int_{\partial\Delta^n} c(t,x,\tilde{n})dx = c(t,p,\tilde{n}_0)\text{area}(\Sigma) + \sum_{i=1}^{n} c(t,p_i,-\tilde{n}_i)\text{area}(\Sigma_i), \tag{1.21}$$

where $-\tilde{n}_i$ is the outward orthonormal vector to Σ_i at p_i. Besides, we have that

$$\lim_{l\to 0}\frac{\text{area}(\Sigma)}{\text{area}(\partial\Delta^n)} = \lim_{l\to 0}\frac{\beta l^{n-1}}{\beta l^{n-1}\left(1+\sum_{i=1}^{n}\tilde{n}_0^i\right)} = \lim_{l\to 0}\frac{1}{1+\sum_{i=1}^{n}\tilde{n}_0^i} = \frac{1}{1+\sum_{i=1}^{n}\tilde{n}_0^i} =: k,$$
$$\lim_{l\to 0}\frac{\text{area}(\Sigma_i)}{\text{area}(\partial\Delta^n)} = \lim_{l\to 0}\frac{\tilde{n}_0^i\beta l^{n-1}}{\beta l^{n-1}\left(1+\sum_{i=1}^{n}\tilde{n}_0^i\right)} = \lim_{l\to 0}\frac{\tilde{n}_0^i}{1+\sum_{i=1}^{n}\tilde{n}_0^i} = \frac{\tilde{n}_0^i}{1+\sum_{i=1}^{n}\tilde{n}_0^i} = \tilde{n}_0^i k,$$

where we have defined the constant k. Since there is just one face that is not normal to any of the coordinate axis, namely Σ, we realise that $\tilde{n}_i$ is, in fact, e_i. Hence, as l approaches 0, from Equations (1.20) and (1.21) we see that

$$\begin{aligned}
0 &= \lim_{l\to 0}\frac{1}{\text{area }\partial\Delta^n}\int_{\partial\Delta^n} c(t,x,\tilde{n})dx \\
&= \lim_{l\to 0}\left[c(t,p,\tilde{n}_0)\frac{\text{area}(\Sigma)}{\text{area}(\partial\Delta^n)} + \sum_{i=1}^{n} c(t,p_i,-\tilde{n}_i)\frac{\text{area}(\Sigma_i)}{\text{area}(\partial\Delta^n)}\right] \\
&= c(t,x_0,\tilde{n}_0)k + \sum_{i=1}^{n} c(t,x_0,-e_i)\tilde{n}_0^i k.
\end{aligned}$$

Since $k \neq 0$, the last expression implies that

$$0 = c(t,x_0,\tilde{n}_0) + \sum_{i=1}^{n} c(t,x_0,-e_i)\tilde{n}_0^i.$$

By the Cauchy's lemma 1.10, the last equation becomes

$$c(t,x_0,\tilde{n}_0) = \sum_{i=1}^{n} c(t,x_0,e_i)\tilde{n}_0^i = \sum_{i=1}^{n} c(t,x_0,e_i)(\tilde{n}_0\cdot e_i).$$

Both the choice of point x_0 as vertex and the choice of the outward orthonormal vector $\tilde{n}_0$ are arbitrary, the above relation holds $\forall x \in V$ and for all orthonormal vectors $\tilde{n}$. We realise that this equation shows that $c(t,x,\tilde{n})$ is linear in $\tilde{n}$, therefore $\forall (t,x) \in \mathbb{R}\times V$ we can write $c(t,x,\tilde{n}) = \tilde{c}(t,x)\tilde{n}$, where $\tilde{c} : \mathbb{R}\times V \to \mathbb{R}^{n\times n}$ is a linear application. $\square$

Corollary 1.12. **[Stress tensor]** *Let $t \in \mathbb{R}$ and consider $U_t \subset\subset V$, as before. There exists a unique tensor field σ which only depends on $t \in \mathbb{R}$ and $x \in \partial U_t$, such that the stress vector T at $x \in \partial U_t$ along direction $\tilde{n}$ is given by*

$$T(t, x, \tilde{n}) = \sigma(t, x)\tilde{n}.$$

Since σ associates to each $(t, x) \in \mathbb{R} \times V$ a matrix $\sigma(t, x) \in \mathbb{R}^{n \times n}$, we can identify $\sigma(t, x)$ with a second-order tensor. That is why σ is called the stress tensor *(see Section 2.2 and Annex B),*

Proof. Considering Definition 1.8, Equation (1.17) becomes

$$\frac{d}{dt} \int_{U_t} \rho(t, x)u(t, x)v dx = \int_{U_t} \rho(t, x)f(t, x)v dx + \oint_{\partial U_t} T(t, x, \tilde{n})v dx.$$

Then, since $a \equiv \rho u$ and $b = \rho f$ are $\mathcal{C}^1$ functions because ρ, u and f are so and $c \equiv T$ is a $\mathcal{C}^0$ function, the Cauchy's theorem 1.11 tells us that there exists a unique tensor field σ such that $T(t, x, \tilde{n}) = \sigma(t, x)\tilde{n}$. $\square$

An immediate result of Corollary 1.12 is that we can rewrite Equation (1.17) as

$$\frac{d}{dt}\left(p(t, U_t)\right) = \int_{U_t} \rho(t, x)f(t, x)dx + \oint_{\partial U_t} \sigma(t, x)\tilde{n}(x)dx. \tag{1.22}$$

Using the transport theorem 1.4 on the left-hand side of Equation (1.22), considering $g(t, x) = \rho(t, x)u(t, x)$, we get that

$$\begin{aligned}
\frac{d}{dt}\left(p(t, U)\right) = \int_{U_t} \Bigg(& \frac{\partial \rho}{\partial t}(t, x)u(t, x) + \frac{\partial u}{\partial t}(t, x)\rho(t, x) \\
& + \left[D_x\rho(t, x)u(t, x) + D_x u(t, x)\rho(t, x)\right] u(t, x) \\
& + \operatorname{div} u(t, x)\rho(t, x)u(t, x)\Bigg) dx \\
= \int_{U_t} \Bigg(& u(t, x)\left[\frac{\partial \rho}{\partial t}(t, x) + D_x\rho(t, x)u(t, x) + \operatorname{div} u(t, x)\rho(t, x)\right] \\
& + \rho(t, x)\left[\frac{\partial u}{\partial t}(t, x) + D_x u(t, x)u(t, x)\right]\Bigg) dx \\
= \int_{U_t} \Bigg(& u(t, x)\left[\frac{\partial \rho}{\partial t}(t, x) + \operatorname{div}\,(\rho u)(t, x)\right] + \\
& \rho(t, x)\left[\frac{\partial u}{\partial t}(t, x) + D_x u(t, x)u(t, x)\right]\Bigg) dx.
\end{aligned}$$

By Equation (1.14), the first term of the integral vanishes, so we have that

$$\frac{d}{dt}\left(p(t, U_t)\right) = \int_{U_t} \left(\rho(t,x) \left[\frac{\partial u}{\partial t}(t,x) + D_x u(t,x) u(t,x) \right] \right) dx. \tag{1.23}$$

Now we employ the divergence theorem on the second integral on the right-hand side of Equation (1.22)

$$\oint_{\partial U_t} \sigma(t,x)\tilde{n}(x) dx = \int_{U_t} \operatorname{div} \sigma(t,x) dx. \tag{1.24}$$

If we substitute the linear momentum time derivative and the stress tensor integral in Equation (1.22) for the expressions we have found in Equations (1.23) and (1.24), we obtain that

$$\int_{U_t} \left(\rho(t,x) \left[\frac{\partial u}{\partial t}(t,x) + D_x u(t,x) u(t,x) \right] \right) dx = \int_{U_t} \rho(t,x) f(t,x) dx + \int_{U} \operatorname{div} \sigma(t,x) dx.$$

The equation above is valid for any arbitrary $U_t \subset\subset V$, therefore

$$\rho(t,x) \left[\frac{\partial u}{\partial t}(t,x) + D_x u(t,x) u(t,x) \right] = \rho(t,x) f(t,x) + \operatorname{div} \sigma(t,x),$$

which can also be written as

$$\frac{\partial u}{\partial t}(t,x) + D_x u(t,x) u(t,x) = f(t,x) + \frac{1}{\rho(t,x)} \operatorname{div} \sigma(t,x). \tag{1.25}$$

Remark 1.13. *We will write the second term on the left-hand side of Equation* (1.25) *using conventional physical notation to obtain the best known form of it.*

$$\begin{aligned} D_x u(t,x) u(t,x) &= \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial u_1}{\partial x_1} u_1 & \cdots & \frac{\partial u_1}{\partial x_n} u_n \\ \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} u_1 & \cdots & \frac{\partial u_n}{\partial x_n} u_n \end{pmatrix} = \begin{pmatrix} u_1 \frac{\partial u_1}{\partial x_1} & \cdots & u_n \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ u_1 \frac{\partial u_n}{\partial x_1} & \cdots & u_n \frac{\partial u_n}{\partial x_n} \end{pmatrix} \\ &= \left((u \cdot \nabla^T) u_1, \cdots, (u \cdot \nabla^T) u_n \right)^T =: (u \cdot \nabla^T) u. \end{aligned}$$

$\triangle$

Therefore, Equation (1.25) becomes

$$\rho \frac{\partial u}{\partial t} + \rho (u \cdot \nabla^T) u = \rho f + \nabla^T \cdot \sigma. \tag{1.26}$$

Equation (1.25) and Equation (1.26) are two different forms of the **local equation of motion for a fluid particle**. Notice that these two equations are local forms of the balance of linear momentum expression (Equation (1.17)).

In this chapter, we have established the elemental mathematical results we need for our work and the basic physical concepts of fluid dynamics. This has led us to the derivation of the local equation of motion for a fluid particle, which is the equation that governs the dynamics of any fluid admitting an Eulerian-Lagrangian description under the continuum assumption.

To obtain a local equation of motion we have been forced to introduce the stress tensor, a second-order tensor of which, at the moment, we only know what it is called. Nevertheless, it is the cornerstone of rheology since it contains all the necessary information to describe the motion of non-ideal fluids, as we shall see in Chapter 2.

Chapter 2

Fluid rheology

Rheology is the branch of physics that deals with the flow and deformation of matter under the continuum assumption. In this chapter, we immerse ourselves in the study of the dynamics of viscous fluids[1]. Considering the hypotheses established in Chapter 1, we first see how the deformation rate (strain rate) of a fluid can be quantified, without worrying about its origin. This matter is treated in the second section, through the analysis of the viscous effects. In the third section we introduce angular momentum in the description of fluid dynamics and we see which are the mathematical consequences of that. Finally, we briefly present how the strain rate is related to the own viscosity of the fluid. We refer the interested reader to [2], [10] and [12].

2.1 Deformations in the fluid flow: the rate of strain tensor

In this section, we discuss how the deformation rate of a fluid is mathematically described. To this end, suitable tensors (see Annex B) will be defined. For a more detailed physical explanation we refer to [10].

Let us consider a particle of fluid that at time $t_0 \in \mathbb{R}$ is located at point $x_0 \in V$ and has velocity $u(t_0, x_0)$. We want to study the velocity field nearby this fluid particle. For a particle located at point $x_0 + \delta x$ at the same time t_0 its velocity would be $u(t_0, x_0) + \delta u$. A Taylor approximation to first order gives us that

$$u(t_0, x_0 + \delta x) = u(t_0, x_0) + D_x u(t_0, x_0)\delta x + O(\delta x^2),$$

thus the velocity increment δu would be

$$\delta u = D_x u(t_0, x_0)\delta x,$$

[1]In this chapter, we begin the study of real-world fluids, so hereafter we will work in $n = 3$ dimensions.

regarded from the Eulerian description of the fluid. Let us see that one can also obtain this result from the Lagrangian description of the fluid.

In Section 1.1 we saw that, in the Lagrangian formulation, a fluid is described by its deformation, which is characterised by the evolution operator Φ (recall Equation (1.1)). Since we are interested in how the fluid becomes deformed in the vicinity of $(t_0, x_0) \in \mathbb{R} \times V$, we should consider the variational equations with respect to initial conditions, which are given by (recall Equation (1.10))

$$\begin{cases} \frac{d}{dt} D_{x_0}\Phi(t; t_0, x_0) &= D_x u(t, \Phi(t; t_0, x_0)) D_{x_0}\Phi(t; t_0, x_0), \\ D_{x_0}\Phi(t_0; t_0, x_0) &= \mathrm{Id}, \end{cases} \tag{2.1}$$

Then, by Equation (1.1), we see that the deformation rate is given by

$$\begin{aligned} \frac{d}{dt} D_{x_0}\Phi(t_0; t_0, x_0) &= D_x u(t_0, \Phi(t_0; t_0, x_0)) D_{x_0}\Phi(t_0; t_0, x_0) \\ &= D_x u(t_0, x_0)\mathrm{Id} = D_x u(t_0, x_0). \end{aligned} \tag{2.2}$$

Definition 2.1. We define the *tensor of the deformation rates* for a fluid described by (1.1) as the tensor field $G : \mathbb{R} \times V \to \mathbb{R}^{3\times 3}$ such that

$$G(t, x) := D_x u(t, x) \in \mathbb{R}^{3\times 3}.$$

Notice that the tensor field G associates the matrix $D_x u(t, x)$ to each $(t, x) \in \mathbb{R} \times V$, which can be thought of as a second-order tensor. This is why G is also called the *velocity gradient tensor*.

Definition 2.2. Considering a fluid described by (1.1), we define the *rate of strain tensor* as the symmetric part of the velocity gradient tensor

$$e(t, x) := \frac{1}{2}\left(D_x u(t, x) + D_x u(t, x)^T\right) \in \mathbb{R}^{3\times 3}. \tag{2.3}$$

The antisymmetric part of the velocity gradient tensor is given by

$$\omega(t, x) := \frac{1}{2}\left(D_x u(t, x) - D_x u(t, x)^T\right) \in \mathbb{R}^{3\times 3}, \tag{2.4}$$

so that we have

$$G = e + \omega. \tag{2.5}$$

Therefore, the scalar field div $u : \mathbb{R} \times V \to \mathbb{R}$ is given by

$$\operatorname{div} u(t, x) = \operatorname{tr}\left(D_x u(t, x)\right) = \operatorname{tr}\left(e(t, x)\right), \tag{2.6}$$

since the antisymmetric part has trace equal to 0.

2.2 Surface forces in a fluid: the stress tensor

In this section, we review the different forces which act on a fluid. We shall see the physical reason behind the necessity of the stress tensor, which has already been introduced in Section 1.3, in order to completely describe the dynamics of a fluid.

Definition 2.3. A fluid described by (1.1) is *ideal* if it has no viscosity. Even though no real-world fluid can be considered truly ideal, the ideal fluid description is a good approximation for the vast majority of them. That is the reason why the stress tensor σ is divided into an ideal part and into a viscous part (see Definition 2.12).

Let us consider the fluid occupying an open bounded subset $U_t \subset\subset V$. We want to inspect the stress T (which is defined as the force per unit area, recall Conjecture 1.9) that the fraction of fluid occupying U_t exerts on the portion of fluid in $V \setminus U_t$ through its boundary ∂U_t. For an ideal fluid at rest or in uniform rectilinear motion, T is **normal** to ∂U_t and its magnitude is independent of the orientation of ∂U_t, i.e., T is isotropic. Therefore, a real number is sufficient to characterise the value of T at each point of the fluid.

Definition 2.4. We define the *pressure* P as the normal to ∂U_t and isotropic stress acting on an ideal fluid at rest or moving with uniform rectilinear motion. It can be written as

$$P = P_0 + P_h + P_d,$$

where in our context one may consider that

- P_0 is a constant pressure externally acting on the fluid, e.g., the atmospheric pressure.
- $P_h = P_h(x)$ is the *hydrostatic pressure*, which is the pressure originated by the own mass of the fluid for being itself in a gravitational field. This term can also incorporate body forces of inertial origin (see comments in Section 4.1 for the pipe flow problem).
- $P_d = P_d(t, x)$ is the *dynamic pressure*, originated by the linear velocity of the fluid.

Observe that the pressure defines a scalar field $P : \mathbb{R} \times V \to \mathbb{R}$. Note that the pressure P has been defined up to sign, since one can choose $\tilde{n}$ or $-\tilde{n}$ as the unitary normal vector to ∂U_t. We shall consider the proper sign according to the physical framework (see below).

If pressure was the only stress acting on a fluid, the stress vector T would simply be $T(t,x,\tilde{n}) = \pm P(t,x)\tilde{n}$. However, if we are not working with an ideal fluid, its viscosity will generate frictional forces between layers of fluid sliding between them, which lead to stresses that are **tangential** to ∂U_t. Thus, we ought to find a way to determine the total stress exerted on ∂U_t along all directions. Since the orientation of ∂U_t is given by the orthonormal vector $\tilde{n}$, we need an object who can connect $\tilde{n}$ to the total stress T. As we saw in Section 1.3, this is accomplished by the stress tensor σ.

Definition 2.5. We define the *stress tensor* $\sigma : \mathbb{R} \times V \to \mathcal{L}\left(\mathbb{R}^3, \mathbb{R}^3\right)$, where $\mathcal{L}\left(\mathbb{R}^3, \mathbb{R}^3\right)$ denotes the set of linear maps of $\mathbb{R}^3$, as the $\mathcal{C}^1$-transformation such that, given $(t,x) \in \mathbb{R} \times V$, relates the stress in the direction $\tilde{n} \in \mathbb{R}^n$ with the direction itself. That is, the stress acting on $x \in V$ in the direction given an unitary vector $\tilde{n} \in \mathbb{R}^3$ is

$$T(t,x,\tilde{n}) := \sigma(t,x)\tilde{n}.$$

Remark 2.6. *We notice that $\forall x \in V$ and $\forall \tilde{n} \in \mathbb{R}^3$ we can construct an open bounded subset $U_t \subset\subset V$ such that $\tilde{n}$ is the outward orthonormal vector to ∂U_t at $x \in \partial U_t$. Then, the stress tensor σ is well defined, as we have introduced it in Corollary 1.12* $\triangle$

By Definition 2.5, we see that the matrix element σ_{ii} is the i-component of the force exerted on a unit area with orthonormal vector $\tilde{n}$ parallel to the same i-direction. Hence, all diagonal elements of σ are **normal stresses**. On the other hand, the element σ_{ij} is the i-component of the force exerted on a unit area with orthonormal vector $\tilde{n}$ parallel to the j-direction, so all non-diagonal elements of σ are **tangential** or **shear stresses**.

Remark 2.7. *Note that σ is diagonal whenever there are no velocity gradients that may generate shear stresses. For instance, σ is diagonal for a fluid at rest, $\sigma(t,x) = \pm P(t,x) = \pm\left(P_0 + P_h(x)\right)$, and for an ideal fluid at rest or moving with uniform rectilinear motion, $\sigma(t,x) = \pm P(t,x) = \pm\left(P_0 + P_h(x) + P_d(t,x)\right)$.* $\triangle$

2.3 Angular momentum: stress tensor symmetry

In this section, we study the balance of angular momentum within the fluid and how it relates to the symmetry of the stress tensor. In addition, we introduce the viscosity stress tensor, an also symmetric tensor which responds to the aforementioned intention to split the stress tensor in order to achieve a clearer view of the fluid behaviour (see Section 2.2, in particular Definition 2.3).

Definition 2.8. We define the *centre of mass* of the fluid occupying the subset U_t at time $t \in \mathbb{R}$ as

$$x_{CM}(t, U_t) := \frac{1}{\text{mass}(t, U_t)} \int_{U_t} \rho(t, x) x dx.$$

Definition 2.9. We define the *angular momentum* of the fluid occupying the subset U_t at time $t \in \mathbb{R}$ as

$$L(t, U_t) := x_{CM}(t, U_t) \times p(t, U_t) := \int_{U_t} x \times (\rho(t, x) u(t, x))\, dx,$$

where $\times$ indicates the vector product and $p(t, U_t)$ is the linear momentum of the fluid occupying the subset U_t at time t (recall Definition 1.8).

Definition 2.10. We define the *moment of force* or *torque* of a force F acting on the fluid occupying the subset U_t at time $t \in \mathbb{R}$ as

$$M(t, U_t) := x_{CM}(t, U_t) \times F(t, U_t).$$

As we have seen in Section 1.2, forces acting on a fluid at any given time t can be either internal or external. We define the *internal torque* that the rest of the fluid $V \setminus U_t$ exerts on U_t at time t through its boundary ∂U_t as

$$M_{int}(t, U_t) = \oint_{\partial U_t} x \times (\sigma(t, x) \tilde{n}(x))\, dx,$$

and the *external torque* exerted on U_t at time t by a force whose intensity is given by the $\mathcal{C}^1$ vector field f as

$$M_{ext}(t, U_t) = \int_{U_t} \rho(t, x)\, (x \times f(t, x))\, dx.$$

Newton's second law tells us that the rate of change of angular momentum of a portion of a fluid is equal to the sum of the torques acting on it, that is

$$\frac{d}{dt} (L(t, U_t)) = \int_{U_t} \rho(t, x)\, (x \times f(t, x))\, dx + \oint_{\partial U_t} x \times (\sigma(t, x) \tilde{n}(x))\, dx, \tag{2.7}$$

which we may refer to as the **balance of angular momentum**.

Theorem 2.11. *If the principle of mass conservation (Equation* (1.12)*) and the balance of linear momentum (Equation* (1.25)*) hold, then the balance of angular momentum (Equation* (2.7)*) is fulfilled if and only if the stress tensor is symmetric, that is*

$$\sigma_{ij} = \sigma_{ji}, \quad \forall i, j \in \{1, 2, 3\}.$$

Proof. We encourage the reader to visit Appendices A and B prior to reading this proof. Let us rewrite Equation (2.7) using Definition 2.9

$$\frac{d}{dt}\int_{U_t} x \times (\rho u)\, dx = \int_{U_t} \rho\, (x \times f)\, dx + \oint_{\partial U_t} x \times (\sigma \tilde{n})\, dx, \tag{2.8}$$

where we have omitted the dependence on $(t, x) \in \mathbb{R} \times V$. Cauchy's theorem 1.11 tells us that there exists a unique tensor field $\tilde{c} : \mathbb{R} \times V \to \mathbb{R}^{3\times 3}$ such that Equation (2.8) is equivalent to

$$\frac{d}{dt}\int_{U_t} x \times (\rho u)\, dx = \int_{U_t} \rho\, (x \times f)\, dx + \oint_{\partial U_t} \tilde{c}\tilde{n}\, dx, \tag{2.9}$$

By Lemma A.1, we know that $\tilde{c}$ is of the form $x \times \sigma$, hence Equation (2.9) can be rewritten as

$$\frac{d}{dt}\int_{U_t} x \times (\rho u)\, dx = \int_{U_t} \rho\, (x \times f)\, dx + \oint_{\partial U_t} (x \times \sigma)\, \tilde{n}\, dx, \tag{2.10}$$

Using the divergence theorem, we can write Equation (2.10) as

$$\frac{d}{dt}\int_{U_t} x \times (\rho u)\, dx = \int_{U_t} \rho\, (x \times f)\, dx + \int_{U_t} \operatorname{div}\ (x \times \sigma)\, dx. \tag{2.11}$$

By the transport theorem 1.4, the left-hand side of Equation (2.11) becomes

$$\begin{aligned}
\frac{d}{dt}\int_{U_t} x \times (\rho u)\, dx &= \int_{U_t} \left(\frac{\partial\, (x \times (\rho u))}{\partial t} + D_x\, (x \times (\rho u))\, u + \operatorname{div}\, u\, (x \times (\rho u)) \right) dx \\
&= \int_{U_t} \left(\frac{d\, (x \times (\rho u))}{dt} + \operatorname{div}\, u\, (x \times (\rho u)) \right) dx \\
&= \int_{U_t} \left(\frac{dx}{dt} \times (\rho u) + x \times \frac{d\rho}{dt} u + x \times \rho \frac{du}{dt} + \operatorname{div}\, u\, \rho\, (x \times u) \right) dx \\
&= \int_{U_t} \left(\rho \frac{dx}{dt} \times u + \frac{d\rho}{dt}\, (x \times u) + x \times \rho \frac{du}{dt} + \operatorname{div}\, u\, \rho\, (x \times u) \right) dx \\
&= \int_{U_t} \left(\left[\frac{\partial \rho}{\partial t} + D_x \rho\, u + \operatorname{div}\, u\, \rho \right] (x \times u) + x \times \rho \frac{du}{dt} \right) dx \\
&= \int_{U_t} x \times \left[\rho \frac{\partial u}{\partial t} + \rho D_x u\, u \right] dx = \int_{U_t} x \times [\rho f + \operatorname{div}\, \sigma]\, dx.
\end{aligned}$$

Notice that we have used the result (1.9) twice in this derivation and that $\rho\frac{dx}{dt} \times u$ vanishes because $u := \frac{dx}{dt}$. In addition, we have used Equation (1.13), which is a direct result of Equation (1.12), and finally Equation (1.25) at the last step of the derivation. Thus, we can rewrite Equation (2.11) as

$$\int_{U_t} [\rho\,(x \times f) + x \times \operatorname{div}\,\sigma]\,dx = \int_{U_t} \rho\,(x \times f)\,dx + \int_{U_t} \operatorname{div}\;(x \times \sigma)\,dx,$$

which leads us to the expression

$$\int_{U_t} x \times \operatorname{div}\,\sigma dx = \int_{U_t} \operatorname{div}\;(x \times \sigma)\,dx. \tag{2.12}$$

By Lemma B.9, Equation (2.12) leads us to the result

$$\varepsilon_i^{jl}\sigma_{lj} = 0, \quad \forall i \in \{1, 2, 3\},$$

which is equivalent to the symmetry of σ:

$$\begin{aligned}
\varepsilon_1^{jl}\sigma_{lj} &= \varepsilon_1^{23}\sigma_{32} + \varepsilon_1^{32}\sigma_{23} = \sigma_{32} - \sigma_{23} = 0 \implies \sigma_{32} = \sigma_{23},\\
\varepsilon_2^{jl}\sigma_{lj} &= \varepsilon_2^{31}\sigma_{13} + \varepsilon_2^{13}\sigma_{31} = \sigma_{13} - \sigma_{31} = 0 \implies \sigma_{13} = \sigma_{31},\\
\varepsilon_3^{jl}\sigma_{lj} &= \varepsilon_3^{12}\sigma_{21} + \varepsilon_3^{21}\sigma_{12} = \sigma_{21} - \sigma_{12} = 0 \implies \sigma_{21} = \sigma_{12}.
\end{aligned}$$

□

In Section 2.2 we have seen that the structure of the stress tensor σ is closely related to the different kinds of stresses. Normal stresses are the diagonal components of σ, and tangential stresses are the non-diagonal ones. From Definition 2.4 we know that pressure is normal and isotropic, hence it is diagonal and all its elements have the same value, $-P = -P(t, x)$, that is, $\operatorname{diag}(\sigma) = -P$. The negative sign before P indicates that the pressure stress is acting opposite to the outward orthonormal vector $\tilde{n}$. This, along with the fact that pressure is the only stress acting on an ideal fluid (recall Remark 2.7), is a strong motivation to detach pressure from the stress tensor.

Definition 2.12. We define the *viscosity stress tensor* τ as

$$\tau := \sigma - \operatorname{diag}(\sigma) = \sigma + P\,\mathrm{Id},$$

where Id is the 3-dimensional identity matrix. τ is also called the *deviatoric part of the stress tensor*.

Since the viscosity stress tensor τ only differs from the stress tensor σ in terms the diagonal, by theorem 2.11 we know that τ is also symmetric, that is

$$\tau_{ij} = \tau_{ji}, \quad \forall i, j \in \{1, 2, 3\}. \tag{2.13}$$

Regarding Equation (1.25) we calculate the divergence of Equation (2.12) in order to rewrite the local equation of motion

$$\text{div } \sigma = \text{div } \tau - \text{div } (P\text{Id}) = \text{div } \tau - \left(\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \frac{\partial P}{\partial x_3} \right)^T = \text{div } \tau - \text{grad } P,$$

where we have used (1.5) and (1.3). Therefore Equation (1.25) becomes

$$\frac{\partial u}{\partial t}(t,x) + D_x u(t,x)u(t,x) = f(t,x) - \frac{1}{\rho(t,x)}\text{grad } P + \frac{1}{\rho(t,x)}\text{div } \tau(t,x). \tag{2.14}$$

Using the nabla notation introduced in Remark 1.2, the last equation becomes

$$\rho \frac{\partial u}{\partial t} + \rho(u \cdot \nabla^T)u = \rho f - \nabla^T P + \nabla^T \cdot \tau. \tag{2.15}$$

Equation (2.14) and Equation (2.15) are two equivalent forms of the **equation of motion for any fluid admitting an Eulerian-Lagrangian description under the continuum assumption**.

2.4 Newtonian and non-Newtonian fluids: constitutive equations

So far no assumptions on the form of the viscosity stress tensor have been made, for we needed to first define the rate of strain tensor. In this work, we consider as a fluid a system of particles for which the **Stokes hypothesis** holds. This assumes that the viscosity stress tensor τ is a function of the rate of strain tensor e, see [10]. This relationship, which we shall see that determines the different kinds of fluids, is called **constitutive equation**. Actually, a constitutive equation is any relation between two physical magnitudes that is specific to a certain substance and aims to describe its response to stimuli. In fluid dynamics constitutive equations aim to describe how fluids react to stresses originated by viscous or other physical effects (e.g., elasticity or magnetism), as we shall see in Section 3.1 and Section 4.2.

The original constitutive equation proposed by Newton is $\tau = \mu G$, where $\mu \in \mathbb{R}$ is just a proportionality constant between both tensors. Being this relationship introduced for only one-directional motion, other relations have been developed since then in order to provide an accurate description of multidimensional rates of strain in fluids. In this day and age, one says that a **Newtonian fluid** is any fluid for which each component of the viscosity stress tensor τ is a linear combination of the rate of strain tensor e coefficients. That is, a Newtonian fluid is a fluid for which the viscosity stress tensor field τ depends linearly on the rate of strain tensor field e (see Definition 3.3). We will derive the expression for the constitutive equation for a Newtonian fluid in Chapter 3, which is specially dedicated to this kind of fluids. Despite its restrictive definition, a great number of fluids can be considered to be Newtonian, e.g., water, the most abundant liquid in nature. That is why they receive particular attention.

Nevertheless, there are numerous cases in which the Newtonian approach is insufficient and we need to find a better relation between stresses and rates of strain. This is the case of **non-Newtonian fluids**. Experimentally, non-Newtonian behaviours are detected through rheometers, highly sophisticated laboratory devices whose design varies depending on the type of stress we wish to measure. Rheometers help us classify fluids beyond the Newtonian criterion, assisting us in their categorisation into one of the non-Newtonian subclasses and in the consequent proposal of constitutive equations which intend to describe them.

There are many non-Newtonian types of fluid, both dependent and independent of time; we encourage the interested reader to consult [10] and, specially, [3] and [15] for extensive and detailed information about rheology and rheometry and the different kinds of non-Newtonian fluids. Among their variety, **viscoelastic fluids** exhibit an intriguing behaviour, a combination between that of an elastic solid and that of a liquid. They represent a really interesting subject in fluid mechanics and we shall dedicate Chapter 4 to a specific viscoelastic fluid family (recall WLM solutions from the Introduction).

In this chapter, we have presented all the relevant tensors we need to define fluid dynamics, being the viscosity stress tensor τ and the rate of strain tensor e the most important ones for our purposes. We have seen that both of them are symmetric, the latter by definition and the former as a consequence of angular momentum conservation, and we have introduced the concept of the constitutive equation, in particular, in the rheology framework.

We have not given any specific constitutive equation yet. This shall be our purpose for the next section in Chapter 3, to derive the constitutive equation for a Newtonian fluid. This will lead us to the equation of motion for Newtonian fluids and to the Navier-Stokes equation, being the latter a particular case of the former, which we will use to solve the Stokes second problem.

Chapter 3

Newtonian fluid

Throughout this chapter we work under the hypothesis that we are dealing with a Newtonian fluid, also conserving the assumptions and definitions introduced in Chapter 1. In order to derive the equation of motion for a Newtonian fluid, we first give a form of the constitutive equation for this particular kind of fluid. This leads us to the celebrated Navier-Stokes equation, which we use to solve a representative question in fluid mechanics: the Stokes second problem. We refer to [2], [6], [10] and [12] for further reading.

3.1 Newtonian constitutive equation

In this section, we derive the relation between the viscosity stress tensor and the rate of strain tensor for a homogeneous and isotropic Newtonian fluid, that is, the so-called constitutive equation (see Section 2.4).

Definition 3.1. One says that a fluid is *homogeneous* if its characteristic properties (e.g., density, viscosity or surface tension) are the same within the entire fluid, i.e., they do not depend on the position $x \in \mathbb{R}^3$. By extension, we shall say that a scalar field f, a vector field u, a tensor field σ or a tensor T is homogeneous if it is independent of $x \in \mathbb{R}^3$.

Definition 3.2. One says that a fluid is *isotropic* if its characteristic properties (e.g., density, viscosity or surface tension) are the same within any direction of space, i.e., they do not depend on the particular orientation of our coordinates. By extension, we shall say that a scalar field f, a vector field u, a tensor field σ or a tensor T is isotropic if it is independent of the orientation of the coordinates.

Definition 3.3. One says that a fluid is *Newtonian* if the relation between the viscosity stress tensor τ and the rate of strain tensor e is of the form

$$\tau = A \otimes e,$$

where A is the fourth-order proportionality tensor and $\otimes$ denotes the tensor product. Since we consider a Newtonian fluid which is homogeneous and isotropic, so will be the proportionality tensor.

Remark 3.4. *Note that the relation between τ and e can be viewed as a generalised Hooke's law, in the sense that the viscosity stress tensor (which represents the force) is proportional to the rate of strain tensor (which relates to the deformation). Moreover, the stress tensor σ is separated into an spherical part and a deviatoric part (recall Definition 2.12). The spherical part, i.e., the pressure P, induces an equal deformation to the fluid in all three axis (hence that we refer to it as "spherical"). The deviatoric part, i.e., τ, generates deformations in the fluid which differ from the previous uniform behaviour.* △

Remark 3.5. *In Definition 3.3 we have considered a general proportionality tensor A_{ijkl}. However, the requirement of it being homogeneous and isotropic imposes certain constraints to it. Particularly, the isotropy condition implies that both the relation established in Definition 3.3 and the proportionality tensor itself cannot depend on the coordinates orientation. Thus, we ought to consider the most general fourth-order isotropic tensor instead.* △

Theorem 3.6. *The constitutive equation for a homogeneous and isotropic Newtonian fluid can be written as*

$$\tau = \eta \left[2e - \frac{2}{3}(div\ u)Id \right] + \zeta(div\ u)Id,$$

where $\eta \in \mathbb{R}_{\geq 0}$ is called the dynamic shear viscosity *and $\zeta \in \mathbb{R}_{\geq 0}$ the* bulk viscosity.

Proof. We encourage the reader to visit Appendix B prior to reading this proof. Let us consider the scalar S we obtain when we perform the interior product (see Definition B.5) between A and four arbitrary unit vectors $u, v, w, t \in \mathbb{R}^3$

$$S = A^{ijkl} u_i v_j w_k t_l,$$

which linearly depends on each one of the vectors. Since A is isotropic, the scalar S should only depend on the relative orientations between u, v, w and t, which remain invariant under coordinate rotations. In other words, the scalar S depends only on the cosines between the vectors u, v, w and t, that is

$$\begin{aligned} A^{ijkl}u_i v_j w_k t_l &= \alpha(u \cdot v)(w \cdot t) + \beta(u \cdot w)(v \cdot t) + \gamma(u \cdot t)(v \cdot w) \\ &= \alpha u_i v^i w_j t^j + \beta u_i w^i v_j t^j + \gamma u_i t^i v_j w^j \\ &= \alpha u_i \delta^{ij} v_j w_k \delta^{kl} t_l + \beta u_i \delta^{ik} w_k v_j \delta^{jl} t_l + \gamma u_i \delta^{il} t_l v_j \delta^{jk} w_k \\ &= \left(\alpha \delta^{ij}\delta^{kl} + \beta \delta^{ik}\delta^{jl} + \gamma \delta^{il}\delta^{jk}\right) u_i v_j w_k t_l, \end{aligned}$$

Then, since u, v, w, t are arbitrary vectors, the last expression holds $\forall u, v, w, t \in \mathbb{R}^n$. Therefore, we have that the proportionality tensor takes the form

$$A^{ijkl} = \alpha \delta^{ij}\delta^{kl} + \beta \delta^{ik}\delta^{jl} + \gamma \delta^{il}\delta^{jk}, \tag{3.1}$$

as we find in [10].

By the theorem 2.11 we know that the stress tensor σ and, consequently, the viscosity stress tensor τ are symmetric, as we have already stated in Equation (2.13). Then, since $\tau_{ij} = \tau_{ji}$, we must also require that the proportionality tensor is symmetric under the (i, j) index permutation, that is $A^{ijkl} = A^{jikl}$. Imposing this condition to Equation (3.1) we obtain that

$$\begin{aligned} A^{ijkl} &= A^{jikl} \\ \alpha \delta^{ij}\delta^{kl} + \beta \delta^{ik}\delta^{jl} + \gamma \delta^{il}\delta^{jk} &= \alpha \delta^{ji}\delta^{kl} + \beta \delta^{jk}\delta^{il} + \gamma \delta^{jl}\delta^{ik} \\ \beta \delta^{ik}\delta^{jl} + \gamma \delta^{il}\delta^{jk} &= \beta \delta^{jk}\delta^{il} + \gamma \delta^{jl}\delta^{ik} \\ (\beta - \gamma)\, \delta^{ik}\delta^{jl} &= (\beta - \gamma)\, \delta^{jk}\delta^{il}, \end{aligned}$$

which implies that $\beta = \gamma$ since the last expression must hold $\forall i, j, k, l \in \{1, \ldots, n\}$. Thus, Equation (3.1) becomes

$$A^{ijkl} = \alpha \delta^{ij}\delta^{kl} + \beta \left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right), \tag{3.2}$$

as one can find in [10] and [12]. Applying Definition 3.3, we obtain an expression for the viscosity stress tensor components

$$\begin{aligned} \tau^{ij} = A^{ijkl} e_{kl} &= \left[\alpha \delta^{ij}\delta^{kl} + \beta \left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)\right] e_{kl} \\ &= \alpha \delta^{ij} e_{kk} + \beta \left(e_{ij} + e_{ji}\right) = \alpha \delta^{ij} e_{kk} + 2\beta e_{ij}, \end{aligned}$$

so that the viscosity stress tensor can be written as

$$\tau = 2\eta e + \lambda (\text{div } u)\text{Id},$$

where we have redefined $\lambda = \alpha$ and $\eta = \beta$, following the usual convention (see [6], [10] and [12]). Regarding the expression found in the literature (see [2], [6] and [10]), we define the coefficient $\zeta := \lambda + \frac{2}{3}\eta$ to finally obtain the desired expression

$$\tau = \eta \left[2e - \frac{2}{3}(\text{div } u)\text{Id}\right] + \zeta (\text{div } u)\text{Id}.$$

□

Remark 3.7. *For an ideal fluid one has that $\eta = 0$ and $\zeta = 0$. We note that the bulk viscosity is not relevant for the fluids considered in this work by Liouville theorem 1.6, since they are all assumed to be incompressible. Moreover, notice that for incompressible fluids the constitutive equation becomes $\tau = 2\eta e$, i.e., the viscosity stress tensor τ is directly proportional to the rate of strain tensor e (the symmetric part of the velocity gradient tensor G), which is the original proposal for constitutive equation introduced by Newton (recall Section 2.4).* △

Remark 3.8. *By Definition 2.2 we know that the rate of strain tensor e is symmetric, so the proportionality tensor must be also symmetric under the (k, l) index permutation, that is $A^{ijkl} = A^{ijlk}$. However, we see from Equation* (3.2) *that A already fulfils this condition. Nevertheless, it implies another reduction in the number of independent components of the proportionality tensor. We started with a fourth-order tensor A with $3^4 = 81$ independent components. The (i, j) symmetry imposed by σ reduced the number of independent components to $2 \cdot 3^3 = 54$, and the (k, l) symmetry lowered it to $2^2 \cdot 3^2 = 36$. At last, the required symmetry $A^{ijkl} = A^{klij}$ reduces the number of independent components of A to* 21 *(see [13]).* △

3.2 Navier-Stokes equations

Once we have obtained an specific expression for the Newtonian fluid constitutive equation, we calculate the divergence[1] of the viscosity stress tensor to introduce it in Equation (2.14). By theorem 3.6, one has that

$$\mathrm{div}\ \tau = 2\eta \mathrm{div}\ e - \frac{2}{3}\eta \mathrm{div}\ [(\mathrm{div}\ u)\mathrm{Id}] + \zeta \mathrm{div}\ [(\mathrm{div}\ u)\mathrm{Id}]. \tag{3.3}$$

First, we assess the term div $[(\mathrm{div}\ u)\mathrm{Id}]$:

$$\begin{aligned} \mathrm{div}\ [(\mathrm{div}\ u)\mathrm{Id}] &= \left(\sum_j \frac{\partial(\mathrm{div}\ u)}{\partial x_j}\delta_{1j}, \cdots, \sum_j \frac{\partial(\mathrm{div}\ u)}{\partial x_j}\delta_{nj} \right)^T \\ &= \left(\frac{\partial(\mathrm{div}\ u)}{\partial x_1}, \cdots, \frac{\partial(\mathrm{div}\ u)}{\partial x_n} \right)^T = \mathrm{grad}\ (\mathrm{div}\ u), \end{aligned} \tag{3.4}$$

[1] From now one, we require $u(t, x)$ to be a $\mathcal{C}^2$ vector field.

where we have used (1.3) and (1.5). Now we evaluate the divergence of the rate of strain tensor

$$\begin{aligned}
\text{div } e =& \frac{1}{2}\text{div } \left(D_x u + D_x u^T\right) \\
=& \frac{1}{2}\left(\sum_j \frac{\partial (D_x u)_{1j}}{\partial x_j}, \cdots, \sum_j \frac{\partial (D_x u)_{nj}}{\partial x_j}\right)^T + \frac{1}{2}\left(\sum_j \frac{\partial (D_x u^T)_{1j}}{\partial x_j}, \cdots, \sum_j \frac{\partial (D_x u^T)_{nj}}{\partial x_j}\right)^T \\
=& \frac{1}{2}\left(\sum_j \frac{\partial^2 u_1}{\partial x_j \partial x_j}, \cdots, \sum_j \frac{\partial^2 u_n}{\partial x_j \partial x_j}\right)^T + \frac{1}{2}\left(\sum_j \frac{\partial^2 u_j}{\partial x_j \partial x_1}, \cdots, \sum_j \frac{\partial^2 u_j}{\partial x_j \partial x_n}\right)^T \\
=& \frac{1}{2}\left(\sum_j \frac{\partial^2 u_1}{\partial x_j^2}, \cdots, \sum_j \frac{\partial^2 u_n}{\partial x_j^2}\right)^T + \frac{1}{2}\left(\frac{\partial}{\partial x_1}\sum_j \frac{\partial u_j}{\partial x_j}, \cdots, \frac{\partial}{\partial x_n}\sum_j \frac{\partial u_j}{\partial x_j}\right)^T \\
=& \frac{1}{2}\Delta u + \frac{1}{2}\left(\frac{\partial(\text{div } u)}{\partial x_1}, \cdots, \frac{\partial(\text{div } u)}{\partial x_n}\right)^T = \frac{1}{2}\Delta u + \frac{1}{2}\text{grad } (\text{div } u),
\end{aligned} \tag{3.5}$$

where we have applied (1.3), (1.5) and (1.7). Replacing Equations (3.4) and (3.5) in Equation (3.3), one has that

$$\begin{aligned}
\text{div } \tau =& \eta\Delta u + \eta \text{grad } (\text{div } u) - \frac{2}{3}\eta \text{grad } (\text{div } u) + \zeta \text{grad } (\text{div } u) \\
=& \eta\Delta u + \left(\zeta + \frac{\eta}{3}\right) \text{grad } (\text{div } u).
\end{aligned} \tag{3.6}$$

If we substitute the previous expression in Equation (2.14) we finally get

$$\frac{\partial u}{\partial t} + D_x u\, u = f - \frac{1}{\rho}\text{grad } P + \eta\frac{1}{\rho}\Delta u + \left(\zeta + \frac{\eta}{3}\right)\frac{1}{\rho}\text{grad } (\text{div } u), \tag{3.7}$$

which is the **equation of motion for a Newtonian compressible fluid**.

If the fluid happens to be incompressible, by Liouville theorem 1.6 we obtain the **equation of motion for a Newtonian incompressible fluid**

$$\frac{\partial u}{\partial t} + D_x u\, u = f - \frac{1}{\rho}\text{grad } P + \eta\frac{1}{\rho}\Delta u, \tag{3.8}$$

which is better known as the **Navier-Stokes equation**.

If the fluid is defined as incompressible and ideal, which means that there are no viscosity effects and consequently $\eta = 0$, Equation (3.8) becomes

$$\frac{\partial u}{\partial t} + D_x u\, u = f - \frac{1}{\rho}\text{grad } P, \tag{3.9}$$

which is the **Euler equation**.

Remark 3.9. *Using notation introduced in Remark 1.2, Equations* (3.7), (3.8) *and* (3.9) *can be rewritten, respectively, as*

$$\rho\frac{\partial u}{\partial t}+\rho(u\cdot\nabla^T)u=\rho f-\nabla^T p+\eta\Delta u+\left(\zeta+\frac{\eta}{3}\right)\nabla^T\left(\nabla^T\cdot u\right), \tag{3.10}$$

$$\rho\frac{\partial u}{\partial t}+\rho(u\cdot\nabla^T)u=\rho f-\nabla^T P+\eta\Delta u, \tag{3.11}$$

$$\rho\frac{\partial u}{\partial t}+\rho(u\cdot\nabla^T)u=\rho f-\nabla^T P. \tag{3.12}$$

△

3.3 Stokes second problem

In his paper [20] published in 1880, Sir George Gabriel Stokes proposes an academic problem especially addressed to analyse the diffusion of linear momentum in the transverse direction to an oscillating wall induced by the no-slip condition of the fluid at the wall (recall the experiment carried out [4] described in the Introduction and its relation with the Stokes second problem).

Stokes poses an infinite plate oscillating harmonically in its own plane $y-z$, along the z-axis, with angular frequency ω_0 and oscillation amplitude z_0. This plate has on one of its sides an incompressible viscous fluid, which extends indefinitely, i.e., the fluid occupies the semi-infinite domain $V=[0,+\infty)\times(-\infty,+\infty)\times(-\infty,+\infty)\in\mathbb{R}^3$ (see Figure 3.1).

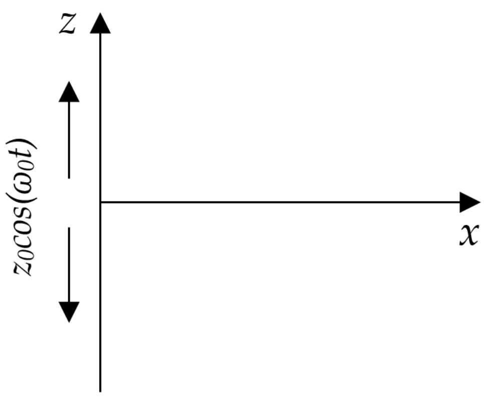

Figure 3.1: Semi-infinite domain V in which the Stokes second problem is defined. The oscillation of the infinite plate is also indicated.

On a general approach, $u=(u_x,u_y,u_z)=u(t,x,y,z)$ is the velocity field of the fluid in Cartesian coordinates, which is limited by the following boundary conditions

$$u(t,0,y,z) = (0,0,z_0\omega_0\cos(\omega_0 t)) \quad \forall t > 0, \forall (0,y,z) \in V, \tag{3.13}$$
$$\lim_{x\to\infty} u = (0,0,0). \tag{3.14}$$

The oscillating plate extends all over the $y - z$ plane, thus the y and z coordinates are not relevant to the behaviour of the fluid. Then, the actual velocity field u depends only on position x and time t, i.e., $u = u(t,x)$. Since the fluid is incompressible, by Liouville theorem 1.6 we find that

$$0 = \text{div } u(t,x) = \frac{\partial u_x(t,x)}{\partial x} + \frac{\partial u_y(t,x)}{\partial y} + \frac{\partial u_z(t,x)}{\partial z} = \frac{\partial u_x(t,x)}{\partial x},$$

which implies that $u_x(t,x) = C \in \mathbb{R}\, \forall x$. However, from boundary condition (3.13) we know $u_x(t,0) = 0$, $\forall t > 0$, hence

$$u_x(t,x) = 0 \quad \forall t > 0, \forall x \in \mathbb{R}_{\geq 0}. \tag{3.15}$$

The motion of the plate only takes place along the z-axis, so there is no momentum in the y-axis and, consequently, no momentum transference in that coordinate. Moreover, we know $u_y(t,0) = 0$, $\forall t > 0$, from the plate boundary condition (3.13). Therefore, we infer that

$$u_y(t,x) = 0 \quad \forall t > 0, \forall x \in \mathbb{R}_{\geq 0}. \tag{3.16}$$

Considering Equations (3.15) and (3.16), we observe that the actual velocity field for the Stokes second problem fluid is

$$u = u(t,x) = (0,0,u_z(t,x)). \tag{3.17}$$

Let us write now the Navier-Stokes equations for this fluid. Gravity is the only external force present in the Stokes second problem, being $g : \mathbb{R} \times V \to \mathbb{R}^3$ its intensity, which we suppose constant. Having made this assumption, Equation (3.8) takes the form

$$\frac{\partial u}{\partial t} + D_x u\, u = g - \frac{1}{\rho}\text{grad } P + \eta\frac{1}{\rho}\Delta u. \tag{3.18}$$

We notice that, for the particular form of velocity field described in Equation (3.17), one has that

$$D_x u\, u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial x} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_z \end{pmatrix} = 0,$$

so Equation (3.18) becomes

$$\rho\frac{\partial u}{\partial t} = \rho g - \text{grad } P + \eta\Delta u. \tag{3.19}$$

Remark 3.10. *We know that the density ρ of the fluid is constant, for the fluid is considered to be incompressible. Therefore, as it is shown in [2], we can write the pressure P (recalling Definition 2.4) as*

$$P = P_0 + P_h + P_d = P_0 + \rho(g \cdot r) + P_d,$$

where $r = (x, y, z)$. Therefore, by (1.3)*, we have that the pressure gradient is*

$$grad\ P = \rho g + grad\ P_d.$$

△

As a result, Equation (3.19) turns into the expression

$$\rho \frac{\partial u}{\partial t} = -\text{grad } P_d + \eta \Delta u. \tag{3.20}$$

Definition 3.11. Let be a Newtonian fluid described by (1.1) with constant density ρ and dynamic shear viscosity η. Then, its *kinematic viscosity ν* is defined by

$$\nu := \frac{\eta}{\rho}.$$

Definition 3.11 allows us to rewrite Equation (3.20) as

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \text{grad } P_d + \nu \Delta u, \tag{3.21}$$

which leads us to the component-wise equations

$$0 = -\frac{1}{\rho} \frac{\partial P_d}{\partial x}, \tag{3.22}$$

$$0 = -\frac{1}{\rho} \frac{\partial P_d}{\partial y}, \tag{3.23}$$

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho} \frac{\partial P_d}{\partial z} + \nu \Delta u_z. \tag{3.24}$$

Equations (3.22) and (3.23) imply that the dynamic pressure P_d is constant in both x and y directions. Aside from that, we know from Definition 2.4 that dynamic pressure arises from the velocity of the fluid. Equation (3.17) tells us that the fluid velocity u only depends on the x Cartesian coordinate. Then, the dynamic pressure that u develops shall only depend on the x Cartesian coordinate too, and consequently we can say that

$$\frac{\partial P_d}{\partial z} = 0,$$

which means dynamic pressure is constant also in the z direction. Hence, from the component-wise equations written from Equation (3.21), the only one that is relevant to us in order to solve the Stokes second problem is Equation (3.24), which we have deduced that it actually takes the form

$$\frac{\partial u_z}{\partial t} = \nu \frac{\partial^2 u_z}{\partial x^2}, \tag{3.25}$$

that is nothing but the equation for the diffusion of linear momentum. Therefore, recalling the boundary conditions given by Equations (3.13) and (3.14), we deduce that in order to solve the Stokes second problem we must find a one-dimensional velocity $u_z(t,x)$ that satisfies

$$\frac{\partial u_z(t,x)}{\partial t} = \nu \frac{\partial^2 u_z(t,x)}{\partial x^2}, \tag{3.26}$$

$$u_z(t,0) = z_0\omega_0\cos(\omega_0 t) \quad \forall t > 0, \tag{3.27}$$

$$\lim_{x\to\infty} u_z(t,x) = 0. \tag{3.28}$$

Equations (3.26), (3.27) and (3.28) define a one-dimensional Stokes second problem. In the following we discuss about the existence and uniqueness of solution of this problem.

3.3.1 Existence of a solution of the Stokes second problem

We notice that the boundary condition (3.27) forbids the existence of stationary solutions and because of that we must seek an oscillatory solution periodic in t. Let us see first that there is no solution of the form

$$\tilde{u}(t,x) = AX(x)T(t). \tag{3.29}$$

Substituting the latter expression in Equation (3.26), one has that

$$AX(x)\dot{T}(t) = A\nu X''(x)T(t),$$

where $\dot{T}(t)$ denotes the first temporal derivative of $T(t)$ and $X''(x)$ the second spatial derivative of $X(x)$. The expression above leads to

$$\frac{\dot{T}(t)}{T(t)} = \nu\frac{X''(x)}{X(x)} = k \in \mathbb{C},$$

which implies that

$$\begin{cases} \dot{T}(t) = kT(t), \quad \forall t > 0, \\ X''(x) = \dfrac{k}{\nu}X(x), \quad \forall x \in \mathbb{R}_{\geq 0}. \end{cases} \tag{3.30}$$

From the latter equation one has

$$X(x) = \alpha \cos\left(\sqrt{\frac{k}{\nu}}x\right) + \beta \sin\left(\sqrt{\frac{k}{\nu}}x\right).$$

Then, if we impose that the function $\tilde{u}(t,x)$ satisfies the boundary condition (3.27), we obtain the equality

$$z_0\omega_0\cos(\omega_0 t) = \tilde{u}(t,0) = AX(0)T(t) = A\alpha T(t),$$

which means that

$$T(t) = \cos(\omega_0 t),$$

where we have defined $A\alpha := z_0\omega_0$. However, we notice that the previous function does not satisfy (3.30), since $\dot{T}(t) = -\omega_0\sin(\omega_0 t)$ and that is not equal to $kT(t) = k\cos(\omega_0 t)$ for any time $t > 0$. Therefore, we conclude that there are no solutions of the form (3.29).

The previous considerations motivate that we now look for a solution of the following form

$$\tilde{u}(t,x) = A\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)}. \tag{3.31}$$

Substituting the latter expression in Equation (3.26), one has

$$\begin{aligned}
A(i\omega)\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)} &= A\nu\frac{\partial}{\partial x}\left[\alpha\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)} + (i\beta)\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)}\right] \\
&= A\nu\left[\alpha^2\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)} + (i\beta)\alpha\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)} + \alpha(i\beta)\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)} + (i\beta)^2\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)}\right] \\
&= A\nu\left[\alpha^2 + 2\alpha(i\beta) - \beta^2\right]\mathrm{e}^{\alpha x}\mathrm{e}^{i(\beta x+\omega t)},
\end{aligned}$$

which leads to the equality

$$i\omega = \nu\left(\alpha + 2\alpha\beta i - \beta^2\right) = \nu\left(\alpha + i\beta\right)^2,$$

that is equivalent to

$$(\alpha + i\beta)^2 = \frac{i\omega}{\nu}.$$

Thus, we find that

$$\alpha + i\beta = \pm\sqrt{\frac{i\omega}{\nu}} = \pm\sqrt{\frac{\omega}{\nu}}\sqrt{i} = \pm\sqrt{\frac{\omega}{\nu}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \pm\sqrt{\frac{\omega}{2\nu}} \pm \sqrt{\frac{\omega}{2\nu}}i,$$

which implies that Equation (3.31) takes the form

$$\tilde{u}(t,x) = A\mathrm{e}^{\pm\sqrt{\frac{\omega}{2\nu}}x}\mathrm{e}^{i\left(\pm\sqrt{\frac{\omega}{2\nu}}x+\omega t\right)}. \tag{3.32}$$

Since we want Equation (3.32) to fulfill the far-field boundary condition (3.28), we choose the minus sign. Hence, one has that

$$\begin{aligned}\tilde{u}(t,x) &= A\mathrm{e}^{-\sqrt{\frac{\omega}{2\nu}}x}\mathrm{e}^{i\left(-\sqrt{\frac{\omega}{2\nu}}x+\omega t\right)} = A\mathrm{e}^{-\sqrt{\frac{\omega}{2\nu}}x}\left[\cos\left(\omega t-\sqrt{\frac{\omega}{2\nu}}x\right)+i\sin\left(\omega t-\sqrt{\frac{\omega}{2\nu}}x\right)\right]\\ &= A\mathrm{e}^{-\sqrt{\frac{\omega}{2\nu}}x}\cos\left(\omega t-\sqrt{\frac{\omega}{2\nu}}x\right)+i\,A\mathrm{e}^{-\sqrt{\frac{\omega}{2\nu}}x}\sin\left(\omega t-\sqrt{\frac{\omega}{2\nu}}x\right)\\ &= \Re\left(\tilde{u}(t,x)\right)+i\,\Im\left(\tilde{u}(t,x)\right),\end{aligned}$$

where $\Re\left(\tilde{u}(t,x)\right)$ and $\Im\left(\tilde{u}(t,x)\right)$ respectively denote the real and imaginary parts of $\tilde{u}(t,x)$. Thus, if we define $u_1(t,x) = \Re\left(\tilde{u}(t,x)\right)$ and $u_2(t,x) = \Im\left(\tilde{u}(t,x)\right)$, considering Equation (3.26), by the linearity of the derivatives one has

$$\frac{\partial u_1(t,x)}{\partial t}+i\,\frac{\partial u_2(t,x)}{\partial t}=\frac{\partial\tilde{u}(t,x)}{\partial t}=\nu\frac{\partial^2\tilde{u}(t,x)}{\partial x^2}=\nu\left[\frac{\partial^2 u_1(t,x)}{\partial x^2}+i\,\frac{\partial^2 u_2(t,x)}{\partial x^2}\right],$$

from which we infer that

$$\frac{\partial u_1(t,x)}{\partial t}=\nu\frac{\partial^2 u_1(t,x)}{\partial x^2},$$

meaning that the function

$$u_1(t,x)=z_0\,\omega_0\exp\left(-\frac{x}{\delta}\right)\cos\left(\omega_0 t-\frac{x}{\delta}\right),\qquad \delta=\sqrt{\frac{2\nu}{\omega_0}}, \tag{3.33}$$

where we have defined $A := z_0\omega_0$ and $\omega := \omega_0$, fulfills Equation (3.26). Moreover, we observe that it satisfies the boundary conditions (3.27) and (3.28).

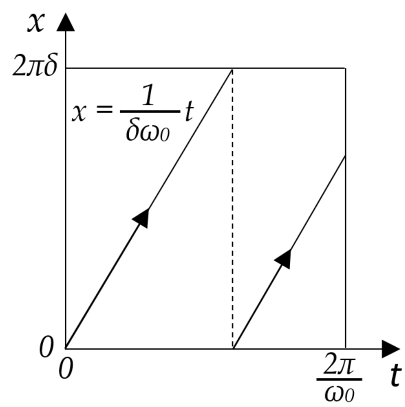

Figure 3.2: Representation of the fundamental torus and the filling of it done by solution (3.33).

Remark 3.12. *The oscillatory part of the solution* (3.33) *defines a doubly periodic function*

$$\begin{aligned} g:\ \mathbb{T}^2 &\longrightarrow \mathbb{R} \\ (t,x) &\longmapsto cos\left(\omega_0 t - \frac{x}{\delta}\right), \end{aligned}$$

with fundamental periods $\left(\frac{2\pi}{\omega_0}, 2\pi\delta\right)$. *If the ratio of these periods is rational, the solution is quasi-periodic and fills up the fundamental torus. Otherwise, the solution is periodic and revolves periodically around the fundamental torus (see Figure 3.2).* △

3.3.2 Uniqueness of the Stokes second problem solution

In the previous section we have shown that

$$u(t,x) = z_0\omega_0 \mathrm{e}^{-\sqrt{\frac{\omega_0}{2\nu}}x}\cos\left(\omega_0 t - \sqrt{\frac{\omega_0}{2\nu}}x\right) \tag{3.34}$$

is a solution of the one-dimensional Stokes second problem determined by the Equations (3.26), (3.27) and (3.28). Now we want to prove that $u(t,x)$ is the only solution for the above stated problem.

Clearly, this problem has no stationary solutions due to the boundary condition (3.27). Then, we look for steady-periodic stationary solutions. This means that we consider a sufficiently large transient time so that the initial time condition is irrelevant. Thus, we can assume that

$$u(0,x) = h(x), \tag{3.35}$$

where $h(x)$ is an arbitrary regular function (e.g., $h(x) \equiv 0$). Let us suppose that there exist two solutions $v_1(t,x)$ and $v_2(t,x)$ of Equations (3.26), (3.27), (3.28) and (3.35). Then, $w(t,x) := v_1(t,x) - v_2(t,x)$ fulfills

$$\begin{aligned} \frac{\partial w(t,x)}{\partial t} = \frac{\partial v_1(t,x)}{\partial t} - \frac{\partial v_2(t,x)}{\partial t} &= \nu\left[\frac{\partial^2 v_1(t,x)}{\partial x^2} - \frac{\partial^2 v_2(t,x)}{\partial x^2}\right] = \nu\frac{\partial^2 w(t,x)}{\partial x^2}, \\ w(t,0) &= 0, \\ \lim_{x\to\infty} u_z(t,x) &= 0, \\ w(0,x) &= 0. \end{aligned}$$

We want to see that the previous problem has $w(t,x) \equiv 0$ as the unique solution. This result follows from the uniqueness theory of solutions for the initial value problem for the diffusion equation, see [7] (p. 58, Theorem 7). For reader's convenience, we adapt the statement of the mentioned theorem to our setting. Fixed

$T > 0$, let $W := (0, T] \times [0, \infty)$ and consider the following initial value problem

$$\frac{\partial w(t,x)}{\partial t} = \nu \frac{\partial^2 w(t,x)}{\partial x^2},$$
$$w(0,x) = h(x),$$

for $(t, x) \in \overline{W}$. Then, the following result holds.

Theorem 3.13. *There exists (at most) one solution $w \in \mathcal{C}^2(W) \cap \mathcal{C}\left(\overline{W}\right)$ of the initial value problem for the diffusion equation such that $\forall K, a > 0$,*

$$|w(t,x)| \leq K\,exp\left(a\,x^2\right), \quad \forall t \in [0,T], \forall x \in [0,\infty).$$

The proof is a consequence of the maximum principle for the diffusion equation, see [7] (p. 54). We note that the condition (3.28) implies, in particular, the upper bound of the growth of the solution stated in the theorem. Hence, the solution (3.34) is the unique bounded solution (thus, with physical meaning) of the one-dimensional Stokes second problem. For an explanation of the growth requirement and examples of solutions with no physical meaning, we refer to [11] (Chapter 7).

Therefore, we conclude that the unique solution to the Stokes second problem initially proposed at the beginning of the present section, defined by the Navier-Stokes Equation (3.8) and the boundary conditions (3.13) and (3.14), is

$$u(t,x) = \left(0, 0, z_0 \omega_0 \mathrm{e}^{-x/x_0} \cos\left(\omega_0 t - \frac{x}{x_0}\right)\right), \tag{3.36}$$

where we have defined

$$x_0 := \sqrt{\frac{2\nu}{\omega_0}}, \tag{3.37}$$

where x_0 denotes the typical length for the exponential decay (see Figure 3.3).

Remark 3.14. *If instead of a single wall we consider two parallel walls in synchronous oscillation, we arrive to a problem that shares many similarities with our oscillatory vertical cylinder, but has a simpler geometry. If we assume rotational symmetry then we have a Newtonian problem which has some analogies to the Newtonian oscillatory pipe flow problem. The solution of this two-wall oscillatory problem can be obtained by linear superposition of solutions of the Stokes second problem. In that case the oscillatory part of both solutions (one for each oscillating wall) can interact and give rise to resonances. Even though, for a Newtonian fluid, the amplitude of the transverse waves decays exponentially on the wavelength scale (see Equation* (3.36)*).* △

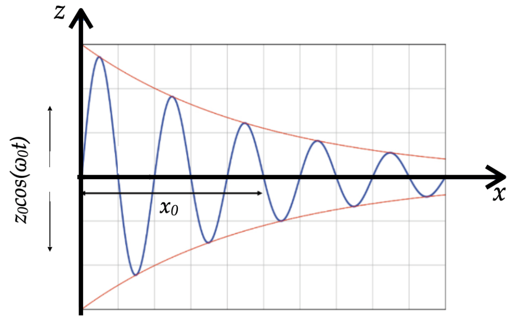

Figure 3.3: Solution for the Stokes second problem, i.e., Equation (3.36), in blue. Also, the enveloping exponential, in red (see [14]).

In this chapter, Newtonian fluids have finally made their appearance. We have shown how one can obtain the constitutive equation for this kind of fluids parting from its simple definition and some basic principles. This has led us to the renowned Navier-Stokes equation and to the resolution of the Stokes second problem, at last.

Having now solved this classical problem, we would like to know if a solution can also be achieved in the viscoelastic, shear-thinning problem we exposed in the Introduction. Its interest relies on not only its several real-world applications but also in the fact that the fluid we are about to inspect in Chapter 4 is a non-Newtonian fluid, and that supposes great difference from what we have seen until now.

Chapter 4

WLM solutions: an example of non-Newtonian fluid

In this chapter, we finally consider the viscoelastic fluid we mentioned in the Introduction. We consider a wormlike micellar (WLM) solution contained in a vertical cylinder that is forced to oscillate periodically in time by the sinusoidal motion of a piston placed at the bottom of the cylinder, which is the main object of study in [4]. As it has been found in this work, the nature of this fluid is not Newtonian, so we cannot model its velocity field with the results we have achieved in Chapter 3. Thus, we introduce new constitutive equations which intend to approximate the intricate behaviour of the considered fluid, which is far from the Newtonian behaviour. We refer the reader eager to know more about the specific non-Newtonian that this chapter is about to [4] and [5], and to [14] for a review.

4.1 Velocity field of the WLM solution

As we explained in the Introduction, the viscoelastic fluid is placed inside a rigid tube of circular cross section (see the Introduction and [4] for a detailed description of the experimental setup). Hence, the geometry of the problem is better described in cylindrical coordinates (r, θ, z), where $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$. In these coordinates the fluid occupies the domain $V = [0, a] \times [0, 2\pi) \times [0, z_{max}] \in \mathbb{R}^3$ of interest, where we have imposed the restrictions of the experimental apparatus (radius a) and z_{max} is chosen accordingly to the oscillation amplitude (see Figure 4.1). One may think of an infinite cylinder in the z coordinate due to the translational invariance considered in the experiment. We note that the cylindrical coordinates are singular at $r = 0$ but the rotational symmetry guarantees that the velocity field is well behaved and can be extended regularly to the singular axis.

As a first attempt to mathematically model this problem, we focus on the flow in a central region of the cylinder where we suppose that the flow oscillates accordingly to the periodic forcing driven by the piston. From the observer reference frame, the domain considered oscillates periodically with the same frequency as the piston. If we considered an oscillating reference frame synchronous with the considered domain, then the domain remains fixed and the cylindrical side-wall oscillates. Note, however, that this new oscillatory reference frame is no longer inertial, meaning that inertial body forces appear. Nevertheless, for an incompressible fluid, these body forces are conservative and consequently can be written in gradient form and thus included in the pressure gradient [16]. In the oscillating reference frame, we assume that the flow can be modelled by a system of partial differential equations with no-slip periodic boundary conditions at the cylindrical sidewall and mass conservation in the fluid domain, which implies zero vertical net flux.

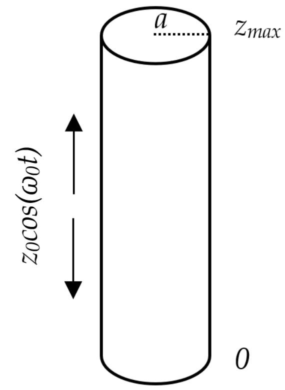

Figure 4.1: Finite domain V in which the viscoelastic, shear-thinning fluid problem is defined. The oscillation of the walls is also indicated.

Let us consider then a viscoelastic fluid occupying the domain V. We denote by (u_r, u_θ, u_z) the components of the three-dimensional velocity field of the fluid $u = u(t, r, \theta, z)$.

Remark 4.1. *For the following derivation it can be useful to express the operators introduced in Remark 1.2 in cylindrical coordinates. The nabla operator, ∇^T, in cylindrical coordinates acts as follows:*

- *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ scalar field, then its* gradient *is denoted by*

$$\nabla^T f := \left(\frac{\partial f}{\partial r} \quad \frac{1}{r}\frac{\partial f}{\partial \theta} \quad \frac{\partial f}{\partial z} \right)^T = grad\ f. \tag{4.1}$$

- *Let $u : \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field, its* divergence *can be written as*

$$\nabla^T \cdot u := \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = div\ u, \tag{4.2}$$

where $\cdot$ is a notation reminiscent of the inner product.

△

The vector field must satisfy the following boundary condition

$$u(t, a, \theta, z) = (0, 0, z_0\omega_0\cos(\omega_0 t)) \quad \forall t > 0, \forall(a, \theta, z) \in V, \tag{4.3}$$

and must flow through V with a constant total net flux.

Remark 4.2. *We refer to* base flow *as to the one-dimensional velocity field u, which is present when the velocity is relatively small. As the velocity increases, instabilities of the base flow occur, which yields to symmetry breaking. Thus the translational invariance along z is lost first, followed by the azimuthal invariance and the temporal periodicity, at which point the flow becomes turbulent (see [4] and [14]).* △

The geometry and constraints of the problem at hand impose some restrictions on the base flow u. Since we have considered the oscillating wall of the cylinder to be infinite, we know that the z coordinate is not relevant to the behaviour of the fluid. Moreover, the azimuthal symmetry of the cylinder tells us that the θ coordinate neither needs to be considered in the description of the velocity of the fluid. Then, the actual velocity field of the base flow u depends only on radial distance r and time t, i.e., $u = u(t, r)$. Since the fluid is incompressible, by Liouville theorem 1.6 we find that

$$0 = \text{div}\ u(t,r) = \frac{1}{r}\frac{\partial(ru_r(t,r))}{\partial r} + \frac{1}{r}\frac{\partial u_\theta(t,r)}{\partial \theta} + \frac{\partial u_z(t,r)}{\partial z} = \frac{1}{r}u_r(t,r) + \frac{\partial u_r(t,r)}{\partial r},$$

which leads us to

$$\frac{\partial}{\partial r}u_r(t,r) = -\frac{1}{r}u_r(t,r),$$

and hence $u_r(t, r) = Cr$, $C \in \mathbb{R}$. However, from the boundary condition (4.3) we know $u_r(t, a) = Ca = 0$, $\forall t > 0$, which can only be possible if $C = 0$, since $a > 0$. Hence

$$u_r(t,r) = 0 \quad \forall t > 0, \forall r \in [0, a]. \tag{4.4}$$

The motion of the cylinder wall only takes place along the z-axis, so there is no momentum in the θ direction and, consequently, no momentum transference in that coordinate. Moreover, we know that $u_\theta(t,a) = 0$, $\forall t > 0$, from boundary condition (4.3). Therefore, we infer that

$$u_\theta(t,r) = 0 \quad \forall t > 0, \forall r \in [0,a]. \tag{4.5}$$

Considering Equations (4.4) and (4.5), we conclude that the velocity field for the base flow of an incompressible fluid placed inside the cylinder has the form

$$u = u(t,r) = (0, 0, u_z(t,r)). \tag{4.6}$$

4.2 Constitutive equations for the WLM solution

The Newtonian constitutive equation that we used in Chapter 3 is insufficient to describe the response of the viscoelastic fluid we are dealing with now. Several modifications of the Newtonian constitutive equation have been proposed in the literature. A common one for this kind of fluids is the so-called Upper-Convected Maxwell (UCM) constitutive equation. This modification adds an extra term to the Newtonian constitutive equation which modifies the viscosity stress tensor τ by adding a Hookean elastic contribution. Rigorously this must be done in such a way that the rate of strain tensor e is objective (independent of the coordinate system). To this aim, one defines the following derivative, which is a correction of the total time derivative of $\tau(t,x)$ along a solution $x = x(t)$ of $\dot{x} = u(t,x)$ describing the fluid (see (1.1)).

Definition 4.3. The *upper-convected time derivative* is defined as

$$\tau_{(1)} := \frac{d}{dt}\tau - [D_x u \tau + \tau D_x u^T],$$

which was introduced by Oldroyd (see [3]).

The upper-convected time derivative is a correction that accounts for the rate of change of τ in the coordinate system moving along with the fluid, thus also experimenting the deformations of the fluid. Therefore, it gives a more precise derivative for any tensor describing the fluid.

Remark 4.4. *We shall rewrite the total time derivative in Definition 4.3 to obtain an expression of the upper-convected time derivative which will be easier to work with. Let us consider a fluid described by* (1.1). *Then, one has*

$$\frac{d}{dt}\tau(t,x) = \frac{\partial \tau(t,x)}{\partial t} + D_x\tau(t,x)\frac{\partial x}{\partial t} = \frac{\partial \tau(t,x)}{\partial t} + D_x\tau(t,x)u(t,x),$$

so that we have a new expression for the upper-convected time derivative

$$\tau_{(1)} = \frac{\partial \tau}{\partial t} + D_x \tau u - [D_x u \tau + \tau D_x u^T]. \tag{4.7}$$

$\triangle$

Definition 4.5. The constitutive equation for an *Upper-Convected Maxwell (UCM) fluid* is given by

$$\tau + \lambda \tau_{(1)} = 2\eta_0 e,$$

where $\lambda \in \mathbb{R}_{\geq 0}$ is called the *characteristic relaxation time of the fluid* and $\eta_0 \in \mathbb{R}_{\geq 0}$ the *zero-shear viscosity*.

Remark 4.6. *The UCM constitutive equation is thus a linear superposition of viscous and elastic stresses; elastic stresses relax on time scales $t \simeq \lambda$, so that viscous stresses govern the long-time behaviour of the fluid, i.e., when $t >> \lambda$. Nevertheless, experimental results presented in [4] show that the UCM constitutive equation does not model accurately enough the behaviour of our viscoelastic fluid. In physical terms, it is observed that the UCM model does not reflect the shear-thinning[1] nature of the fluid..* $\triangle$

Definition 4.7. The constitutive equation for a *Giesekus fluid* is given by

$$\tau + \lambda \tau_{(1)} + \alpha \frac{\lambda}{\eta_0} (\tau \cdot \tau) = 2\eta_0 e,$$

where $\alpha \in \mathbb{R}_{\geq 0}$ is called the *mobility factor*.

Remark 4.8. *The quadratic term in τ is a first-order nonlinear correction to the UCM constitutive equation. It was proposed to account for the decrease of fluid viscosity with shear rate that results from flow alignment of the micelles within the WLM solution (see [4]).* $\triangle$

In order to gain insight in the how the constitutive equations above presented determine the motion of our fluid, let us calculate its specific form for the viscosity stress tensor τ particular to our problem. We recall from Equation (2.13) that τ is symmetric. We have deduced that the velocity field u is actually a one-dimensional vector field parallel to the z-axis which only depends on r, by the symmetry of the experimental configuration. This leads us to assume that the only non-diagonal components of τ that are different from zero, i.e., the only tangential stresses the fluid is experimenting, are the ones involving the r and z coordinates.

[1]Shear-thinning is the most common type of non-Newtonian behaviour. Shear-thinning fluids are those whose viscosity decreases under shear stresses. For further details, see [3], [10] and [15].

Furthermore, in the same way that the velocity u of the fluid does not depend on the coordinate z because of the infinity of the oscillating wall of the cylinder and neither depends on the azimuthal angle θ because of the symmetry of the cylinder itself, we infer that these symmetries inherent in the description of the problem also imply that the the viscosity stress tensor τ does not depend neither on z nor θ. Therefore, we suppose that the viscosity stress tensor takes the form

$$\tau = \tau(t,r) = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \tau_{\theta\theta} & \tau_{z\theta} \\ \tau_{rz} & \tau_{z\theta} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{rz} & 0 & \tau_{zz} \end{pmatrix}. \tag{4.8}$$

Let us compute the upper-convected time derivative of the viscosity stress tensor. For starters, we have that

$$D_x \tau u = \frac{\partial}{\partial z} \begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{rz} & 0 & \tau_{zz} \end{pmatrix} u_z = 0,$$

where we have used the fact the tensor τ does no depend on z. Proceeding with the derivation, one has that

$$D_x u = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial r} & 0 & 0 \end{pmatrix},$$

where we have considered the form of $u(t,r)$ given in Equation (4.6). Hence

$$D_x u \tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial r} & 0 & 0 \end{pmatrix} \begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{rz} & 0 & \tau_{zz} \end{pmatrix} = \frac{\partial u_z}{\partial r} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tau_{rr} & 0 & \tau_{rz} \end{pmatrix},$$

$$\tau D_x u^T = \begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{rz} & 0 & \tau_{zz} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{\partial u_z}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\partial u_z}{\partial r} \begin{pmatrix} 0 & 0 & \tau_{rr} \\ 0 & 0 & 0 \\ 0 & 0 & \tau_{rz} \end{pmatrix},$$

so that we finally obtain the upper-convected time derivative of τ

$$\tau_{(1)} = \begin{pmatrix} \frac{\partial \tau_{rr}}{\partial t} & 0 & \frac{\partial \tau_{rz}}{\partial t} \\ 0 & \frac{\partial \tau_{\theta\theta}}{\partial t} & 0 \\ \frac{\partial \tau_{rz}}{\partial t} & 0 & \frac{\partial \tau_{zz}}{\partial t} \end{pmatrix} - \frac{\partial u_z}{\partial r} \begin{pmatrix} 0 & 0 & \tau_{rr} \\ 0 & 0 & 0 \\ \tau_{rr} & 0 & 2\tau_{rz} \end{pmatrix}.$$

Let us now calculate the quadratic term in the Giesekus constitutive equation

$$\tau \cdot \tau = \begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{rz} & 0 & \tau_{zz} \end{pmatrix} \begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{rz} & 0 & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \tau_{rr}^2 + \tau_{rz}^2 & 0 & \tau_{rr}\tau_{rz} + \tau_{rz}\tau_{zz} \\ 0 & \tau_{\theta\theta}^2 & 0 \\ \tau_{rr}\tau_{rz} + \tau_{rz}\tau_{zz} & 0 & \tau_{rz}^2 + \tau_{zz}^2 \end{pmatrix}.$$

By Definition 2.2 we know the expression for the rate of strain tensor

$$e = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial r} & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & \frac{\partial u_z}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \frac{\partial u_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial r} & 0 & 0 \end{pmatrix},$$

so that the Giesekus constitutive equation (recall Definition 4.7) is

$$\begin{aligned} &\begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{rz} & 0 & \tau_{zz} \end{pmatrix} + \lambda \begin{pmatrix} \frac{\partial \tau_{rr}}{\partial t} & 0 & \frac{\partial \tau_{rz}}{\partial t} \\ 0 & \frac{\partial \tau_{\theta\theta}}{\partial t} & 0 \\ \frac{\partial \tau_{rz}}{\partial t} & 0 & \frac{\partial \tau_{zz}}{\partial t} \end{pmatrix} - \lambda \frac{\partial u_z}{\partial r} \begin{pmatrix} 0 & 0 & \tau_{rr} \\ 0 & 0 & 0 \\ \tau_{rr} & 0 & 2\tau_{rz} \end{pmatrix} \\ &+ \alpha \frac{\lambda}{\eta_0} \begin{pmatrix} \tau_{rr}^2 + \tau_{rz}^2 & 0 & \tau_{rr}\tau_{rz} + \tau_{rz}\tau_{zz} \\ 0 & \tau_{\theta\theta}^2 & 0 \\ \tau_{rr}\tau_{rz} + \tau_{rz}\tau_{zz} & 0 & \tau_{rz}^2 + \tau_{zz}^2 \end{pmatrix} = \eta_0 \begin{pmatrix} 0 & 0 & \frac{\partial u_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial r} & 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.9}$$

which gives us the following equations

$$\tau_{rr} + \lambda \frac{\partial \tau_{rr}}{\partial t} + \alpha \frac{\lambda}{\eta_0}(\tau_{rr}^2 + \tau_{rz}^2) = 0, \tag{4.10}$$

$$\tau_{rz} + \lambda \frac{\partial \tau_{rz}}{\partial t} + \alpha \frac{\lambda}{\eta_0}\tau_{rz}(\tau_{rr} + \tau_{zz}) - \lambda \frac{\partial u_z}{\partial r}\tau_{rr} = \eta_0 \frac{\partial u_z}{\partial r}, \tag{4.11}$$

$$\tau_{zz} + \lambda \frac{\partial \tau_{zz}}{\partial t} + \alpha \frac{\lambda}{\eta_0}(\tau_{rz}^2 + \tau_{zz}^2) - 2\lambda \frac{\partial u_z}{\partial r}\tau_{rz} = 0, \tag{4.12}$$

$$\tau_{\theta\theta} + \lambda \frac{\partial \tau_{\theta\theta}}{\partial t} + \alpha \frac{\lambda}{\eta_0}\tau_{\theta\theta}^2 = 0. \tag{4.13}$$

Setting $\alpha = 0$, we recover the UCM equations

$$\tau_{rr} + \lambda \frac{\partial \tau_{rr}}{\partial t} = 0, \tag{4.14}$$

$$\tau_{rz} + \lambda \frac{\partial \tau_{rz}}{\partial t} - \lambda \frac{\partial u_z}{\partial r}\tau_{rr} = \eta_0 \frac{\partial u_z}{\partial r}, \tag{4.15}$$

$$\tau_{zz} + \lambda \frac{\partial \tau_{zz}}{\partial t} - 2\lambda \frac{\partial u_z}{\partial r}\tau_{rz} = 0, \tag{4.16}$$

$$\tau_{\theta\theta} + \lambda \frac{\partial \tau_{\theta\theta}}{\partial t} = 0. \tag{4.17}$$

We realise that we have two coupled systems of nonlinear partial differential equations which, together with Equation (2.14), describe the motion of our viscoelastic fluid in two different paradigms. In other words, the equation of motion of the WLM solution is determined by a system of five coupled partial differential equations: Equation (2.14) plus either Equations (4.10)-(4.13) or Equations (4.14)-(4.17). In contrast, in the Stokes second problem, there only were two equations (Equation (2.14) and theorem 3.6) and just one of them was a linear partial differential equation.

The UCM case (i.e., Equations (4.14)-(4.17)) is an exactly solvable one (see [4] and [5]). Equation (4.14) and Equation (4.17) show that normal stresses decay exponentially in a λ time, and therefore when the flow has stabilised (recall the Introduction) the different stress components become decoupled.

In contrast, in the Giesekus case, the differential equations are coupled and cannot be analytically solved. Despite our efforts, we have not been able to obtain a solution to this problem analogous to the one we derived in Section 3.3 (recall Equation (3.36)).

4.3 Discussion of the time domain of the solution

In this section, we study in detail the constitutive equation for the Giesekus fluid contained in a domain V. We shall see that, while the UCM allows for a solution which exists indefinitely in time, the Giesekus consitutive equation imposes a limit to the time domain of the solution. For the sake of clarity, we shall redefine the magnitudes in Giesekus constitutive equation.

Definition 4.9. Recalling Equation (4.9), we define

$$X := \tau_{rr}, \quad Y := \tau_{rz}, \quad Z := \tau_{zz}, \quad t := -\lambda s, \quad \varepsilon := \alpha \frac{\lambda}{\eta_0}.$$

Then, Equations (4.10), (4.11) and (4.12) can be written as follows

$$X - \frac{\partial X}{\partial s} + \varepsilon(X^2 + Y^2) = 0, \tag{4.18}$$

$$Y - \frac{\partial Y}{\partial s} + \varepsilon Y(X + Z) - \lambda X \frac{\partial u_z}{\partial r} = \eta_0 \frac{\partial u_z}{\partial r}, \tag{4.19}$$

$$Z - \frac{\partial Z}{\partial s} + \varepsilon(Y^2 + Z^2) - 2\lambda Y \frac{\partial u_z}{\partial r} = 0. \tag{4.20}$$

Remark 4.10. *Assuming X, Y and Z only depend on the new time s, we can rewrite the previous equations as*

$$\dot{X} = X + \varepsilon(X^2 + Y^2), \tag{4.21}$$

$$\dot{Y} = Y + \varepsilon Y(X + Z) - (\lambda X + \eta_0) \frac{\partial u_z}{\partial r}, \tag{4.22}$$

$$\dot{Z} = Z + \varepsilon(Y^2 + Z^2) - 2\lambda Y \frac{\partial u_z}{\partial r}. \tag{4.23}$$

In [4] one can find the following typical values for the parameters intervening in the Giesekus constitutive equation

$$\lambda = 1.9\ s, \quad \alpha = 0.8, \quad \eta_0 = 60\ Pa \cdot s, \tag{4.24}$$

where λ is given in seconds, η_0 in Pascal seconds and the mobility factor α is a dimensionless parameter. Then the new mobility factor ε parameter has a value

$$\varepsilon = \alpha \frac{\lambda}{\eta_0} = 0.8 \frac{1.9\ s}{60\ Pa \cdot s} \approx 0.0253\ Pa^{-1} = 0.0253 \frac{m \cdot s}{kg} \tag{4.25}$$

which is sufficiently small to suggest that we can study the problem using a perturbative approach. △

For an UCM fluid, i.e. when $\varepsilon = 0$, Y has a form $Y = A\cos(\omega s)$, as we see in [8]. Substituting this expression in Equation (4.21) one has

$$\dot{X} = X + \varepsilon X^2 + \varepsilon A^2 \cos^2(\omega s) =: \varepsilon X^2 + X + \kappa \cos^2(\omega s), \tag{4.26}$$

where we have defined $\kappa := \varepsilon A^2$. Notice that Equation (4.26) is a quadratic non-autonomous ordinary differential equation (see [19]). Delimiting the trigonometric term, one has that

$$\dot{X} = \varepsilon X^2 + X + \kappa \cos^2(\omega s) \leq \varepsilon X^2 + X + \kappa. \tag{4.27}$$

In [8] it is shown that for a low value of κ the X coefficient of the viscosity stress tensor oscillates with a small amplitude, and for a high value it diverges in a finite time. We want to estimate the value of κ separating both regimes, κ_{crit}, and relate it to the critical finite time, s_{crit}. From Equation (4.27), one has that

$$\frac{dX}{ds} \leq \varepsilon X^2 + X + \kappa,$$

which leads us to

$$\frac{dX}{\varepsilon X^2 + X + \kappa} \leq ds.$$

Thus, the critical time s_{crit} will be given by the expression

$$\begin{aligned} s_{crit} = \int_0^{s_{crit}} ds \geq \int_0^{\infty} \frac{dX}{\varepsilon X^2 + X + \kappa} &= \frac{2}{\sqrt{4\varepsilon\kappa - 1}} \left[\arctan\left(\frac{2\varepsilon X + 1}{\sqrt{4\varepsilon\kappa - 1}} \right) \right]_0^{\infty} \\ &= \frac{2}{\sqrt{4\varepsilon\kappa - 1}} \left[\frac{\pi}{2} - \arctan\left(\frac{1}{\sqrt{4\varepsilon\kappa - 1}} \right) \right]. \end{aligned} \tag{4.28}$$

For ε fixed, Equation (4.28) provides a lower bound of s_{crit} as a function of κ. On the other hand, the Taylor series around 0 of $\arctan(x)$ has radius of convergence equal to 1. Hence, the lower bound in (4.28) is analytic provided

$$\frac{1}{\sqrt{4\varepsilon\kappa - 1}} < 1.$$

This condition means that the lower bound is finite whenever $\kappa > \frac{1}{2\varepsilon}$. In other words, the critical value of κ that separates the two regimes is given by

$$\kappa_{crit} = \frac{1}{2\varepsilon}. \tag{4.29}$$

When $\kappa < \kappa_{crit}$, one expects the solution to be bounded for all times, while for $\kappa > \kappa_{crit}$, the previous reasoning implies that there is a finite time at which the solution escapes to infinity. This result agrees with the numerical result found in [8], where it was observed an oscillatory behaviour of the solution for $\kappa < \kappa_{crit}$. Note that since ε is supposed to be relatively small, we expect κ_{crit} to be a high value, as it is also found in [8].

Conclusions

This work was motivated by a real experiment whose object of study was the oscillatory flow of a certain viscoelastic fluid, one of the several non-Newtonian fluids families. Being the problem with a Newtonian fluid instead, namely the Stokes second problem, of great interest from both a historical perspective and the physical and the mathematical contexts, we made it our objective.

With the purpose of providing a formal resolution of the Stokes second problem, we initiated the study of fluid dynamics parting from the Eulerian and the Lagrangian descriptions of fluids. From that starting point, we derived and proved every result and explained each element that we needed to correctly define the Stokes second problem, for the sake of rigour, clarity, and completeness.

In the pursuit of this goal, we have presented and analysed crucial concepts of fluid mechanics, such as the stress tensor or constitutive equations. Moreover, we have written formal proofs for relevant results in fluid mechanics, to wit, Cauchy's theorem, the symmetry of the stress tensor, or the deduction of the Newtonian constitutive equation. Some of these are results unclearly proven or even not completely proven in the literature. Furthermore, we have fully derived the Navier-Stokes equation, one of the cornerstones of fluid dynamics.

Finally, we have considered the Stokes second problem and discussed the existence and uniqueness of its solution. In addition, we have addressed an analogous oscillatory flow problem but with a non-Newtonian fluid instead. We have raised the equations that describe the problem and, even though we have been struck with the fact that a rigorous analytical treatment of the Giesekus equations appears to be extremely challenging, we have grasped some insight into the complexity of non-Newtonian fluids.

Appendix A

Algebraic results

In this appendix, we present a purely algebraic lemma that we need to proof the theorem 2.11.

Lemma A.1. *Let $u, v \in \mathbb{R}^3$ be two arbitrary vectors and $a \in \mathbb{R}^{3\times 3}$ be an arbitrary squared matrix. Then, the following equality holds*

$$u \times (av) = (u \times a)\, v,$$

where $\times$ denotes the vector product.

Proof. The matrix a can be written as

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =: \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}, \tag{A.1}$$

where we have defined a_1, a_2 and a_3 as the first, second and third column of a, respectively. Then, we have that the product av takes the form of

$$av = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{pmatrix}. \tag{A.2}$$

Let us now compute the vector product of u and av:

$$\begin{aligned}
u \times (av) &= \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \times \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{pmatrix} \\
&= \begin{pmatrix} u_2a_{31}v_1 + u_2a_{32}v_2 + u_2a_{33}v_3 - u_3a_{21}v_1 - u_3a_{22}v_2 - u_3a_{23}v_3 \\ u_3a_{11}v_1 + u_3a_{12}v_2 + u_3a_{13}v_3 - u_1a_{31}v_1 - u_1a_{32}v_2 - u_1a_{33}v_3 \\ u_1a_{21}v_1 + u_1a_{22}v_2 + u_1a_{23}v_3 - u_2a_{11}v_1 - u_2a_{12}v_2 - u_2a_{13}v_3 \end{pmatrix} \\
&= \begin{pmatrix} u_2a_{31} - u_3a_{21} & u_2a_{32} - u_3a_{22} & u_2a_{33} - u_3a_{23} \\ u_3a_{11} - u_1a_{31} & u_3a_{12} - u_1a_{32} & u_3a_{13} - u_1a_{33} \\ u_1a_{21} - u_2a_{11} & u_1a_{22} - u_2a_{12} & u_1a_{23} - u_2a_{13} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\
&=: \begin{pmatrix} u \times a_1 & u \times a_2 & u \times a_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} =: (u \times a)\, v,
\end{aligned} \tag{A.3}$$

where we have used the linearity of the inner product and defined $u \times a$ as the matrix formed by the three column vectors $u \times a_1$, $u \times a_2$ and $u \times a_3$. □

Appendix B

Tensors

This appendix aims to be a brief introduction to tensors. We define what do we understand as a tensor, certain operations that we need for this work and some particular tensors which we will use. To this end, we have closely followed [1].

Let $E_1, \dots, E_m, F$ be $m+1$ vector spaces over $\mathbb{R}$. Then, $\mathcal{L}^m(E_1, \dots, E_m; F)$ denotes the vector space of m-multilinear maps from $E_1 \times \cdots \times E_m$ to F. The particular case $\mathcal{L}(E; \mathbb{R})$ is called the *dual space* of E and is denoted by E^*. If E has finite dimension n and $\{e_1, \dots, e_n\}$ is an ordered basis of E, there is a unique ordered basis of E^*, the *dual basis* $\{e^1, \dots, e^n\}$, such that

$$\langle e^j, e_i \rangle = \delta_i^j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{B.1}$$

Furthermore, $\forall v \in E$ and $\forall \alpha \in E^*$ we have that

$$v = \sum_{i=1}^{n} \langle e^i, v \rangle e_i = \langle e^i, v \rangle e_i, \quad \alpha = \sum_{i=1}^{n} \langle \alpha, e_i \rangle e^i = \langle \alpha, e_i \rangle e^i, \tag{B.2}$$

where $\langle , \rangle$ denotes the pairing between E and E^* and we have used *Einstein summation convention*, which establishes that summation is implied whenever we have a single term in which an index appears repeated on upper and lower levels.

Definition B.1. Let E be a vector space, we define $T_s^r(E) := \mathcal{L}^{r+s}(E^*, \overset{r}{\dots}, E^*, E, \overset{s}{\dots}, E; \mathbb{R})$. The elements of $T_s^r(E)$ are called *tensors on E, contravariant of order r and covariant of order s*, or *(r,s) tensors*, or *$r+s$-th-order tensors*.

Proposition B.2. *Let E be an n-dimensional vector space. If $\{e_1, \dots, e_n\}$ is a basis of E and $\{e^1, \dots, e^n\}$ is the dual basis, then*

$$\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} | i_1, \dots, i_r, j_1, \dots, j_s \in \{1, \dots, n\}$$

is a basis of $T^r_s(E)$ and thus dim $(T^r_s(E)) = n^{r+s}$.

Proof. See [1]. □

Definition B.3. Let $t \in T^r_s(E)$ be a (r,s) tensor. The coefficients

$$t^{i_1 \dots i_r}_{j_1 \dots j_s} = t\left(e^{i_1}, \cdots, e^{i_r}, e_{j_1}, \cdots, e_{j_s}\right)$$

are called the *components of t relative to the basis* $\{e_1, \dots, e_n\}$.

Definition B.4. Let $t_1 \in T^{r_1}_{s_1}(E)$ and $t_2 \in T^{r_2}_{s_2}(E)$ be two tensors, we define the *tensor product* of t_1 and t_2 as the tensor $t_1 \otimes t_2 \in T^{r_1+r_2}_{s_1+s_2}(E)$ given by

$$\begin{aligned}(t_1 \otimes t_2)&\left(\alpha^1, \dots, \alpha^{r_1}, \beta^1, \dots, \beta^{r_2}, v_1, \dots, v_{s_1}, u_1, \dots, u_{s_2}\right)\\ &= t_1\left(\alpha^1, \dots, \alpha^{r_1}, v_1, \dots, v_{s_1}\right) t_2\left(\beta^1, \dots, \beta^{r_2}, u_1, \dots, u_{s_2}\right),\end{aligned}$$

or, component-wise,

$$(t_1 \otimes t_2)^{i_1 \dots i_{r_1} i_{r_1+1} \cdots i_{r_1+r_2}}_{j_1 \dots j_{s_1} j_{s_1+1} \dots j_{s_1+s_2}} = t_1{}^{i_1 \dots i_{r_1}}_{j_1 \dots j_{s_1}} t_2{}^{i_{r_1+1} \cdots i_{r_1+r_2}}_{j_{s_1+1} \dots j_{s_1+s_2}},$$

where $\alpha^i, \beta^j \in E^*$, $\forall i \in \{1, \dots, r_1\}$, $\forall j \in \{1, \dots, r_2\}$ and $v_k, u_l \in E$, $\forall k \in \{1, \dots, s_1\}$, $\forall l \in \{1, \dots, s_2\}$. The tensor product, which is denoted by $\otimes$, is associative, bilinear and continuous, and it is not commutative.

Definition B.5. The *interior product* of a vector $v \in E$ (resp., a form $\alpha \in E^*$) with a tensor $t \in T^r_s(E;F)$ is the $(r, s-1)$ (resp., $(r-1, s)$) type F-valued tensor defined by

$$\begin{aligned}(i_v t)\left(\alpha^1, \dots, \alpha^r, v_1, \dots, v_{s-1}\right) &= \left(\alpha^1, \dots, \alpha^r, v, v_1, \dots, v_{s-1}\right),\\ (i^\alpha t)\left(\alpha^1, \dots, \alpha^r, v_1, \dots, v_{s-1}\right) &= \left(\alpha, \alpha^1, \dots, \alpha^r, v_1, \dots, v_{s-1}\right).\end{aligned}$$

$i_v : T^r_s(E;F) \to T^r_{s-1}(E;F)$ and $i_\alpha : T^r_s(E;F) \to T^{r-1}_s(E;F)$ are linear continuous maps, as are $v \mapsto i_v$ and $\alpha \mapsto i^\alpha$. If $F = \mathbb{R}$ and $\dim(E) = n$, these operations can be written component-wise as follows

$$\begin{aligned}i_{e_k}\left(e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}\right) &= \delta^{j_1}_k e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_2} \otimes \cdots \otimes e^{j_s},\\ i^{e^k}\left(e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}\right) &= \delta^k_{i_1} e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes e^{j_2} \otimes \cdots \otimes e^{j_s},\end{aligned}$$

where $\{e_1, \dots, e_n\}$ is a basis of E and $\{e^1, \dots, e^n\}$ is the dual basis in E^*. By Proposition B.2 we see that with the above expressions together with linearity we can compute any interior product.

Definition B.6. Let E be an n-dimensional vector space over $\mathbb{R}$ with an inner product with a basis $\{e_1, \ldots, e_n\}$ and corresponding dual basis $\{e^1, \ldots, e^n\}$ in E^*. Using the inner product, with matrix denoted by $[g_{ij}]$, so $g_{ij} = e_i \cdot e_j$, where $\cdot$ denotes the inner product, we get the isomorphism

$$^\flat : E \to E^* \text{ given by } v \mapsto v\cdot, \quad \text{and its inverse } ^\sharp : E^* \to E.$$

The matrix of $^\flat$ is $[g_{ij}]$; that is,

$$\left(v^\flat\right)_i = g_{ij}v^j,$$

and of $^\sharp$ is its inverse, $[g^{ij}]$; that is,

$$\left(\alpha^\sharp\right)^i = g^{ij}\alpha_j,$$

where v^j and α_j are the components of v and α, respectively. We call $^\flat$ the *index lowering operator* and $^\sharp$ the *index raising operator*. These operators can be applied to tensors to produce new ones. The later are called *associated tensors*, and are (in general) different from the original ones.

Definition B.7. The *Kronecker delta* is the tensor $\delta \in T^1_1(E)$ defined by $\delta(\alpha, e) = \langle \alpha, e \rangle$. If E has finite dimension n, δ corresponds to the n-dimensional identity matrix $\mathrm{Id} \in \mathcal{L}(E; E)$ under the canonical isomorphism $T^1_1(E) \cong \mathcal{L}(E; E)$. Relative to any basis, the components of δ are the usual Kronecker symbols δ^i_j, that is, $\delta = \delta^i_j e_i \otimes e^j$.

The associated tensors to δ, that is, the ones given by the δ_{ij} and δ^{ij} components (which are equal to the Kronecker symbols), are equivalent to δ, in the sense that they too correspond to the n-dimensional identity matrix.

Definition B.8. We define the *Levi-Civita symbol* ε_i^{jk} as

$$\varepsilon_i^{jk} = \begin{cases} +1, & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1, & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0, & \text{if any index is repeated.} \end{cases}$$

Lemma B.9. *Let $x \in \mathbb{R}^3$ be an arbitrary position and $a \in \mathbb{R}^{3\times 3}$ be an arbitrary squared matrix. Then, the following equality holds*

$$div\ (x \times a)_i = \varepsilon_i^{jk} a_{kj} + (x \times div\ a)_i\,,$$

where $(x \times div\ a)_i$ denotes the i-th component of the vector $(x \times div\ a)$, as $(x \times a)_i$ indicates the i-th column of the matrix $x \times a$ (recall Lemma A.1).

Proof. From Lemma A.1 we observe that the components of the matrix $x \times a$ can be written in the following way using the Levi-Civita symbol

$$(x \times a)_{ij} = \varepsilon_i^{kl} x_k a_{lj}, \tag{B.3}$$

as we also see in [12]. Let us calculate the divergence of $(x \times a)_i$ using the expression above

$$\begin{aligned} \text{div } (x \times a)_i =& \frac{\partial}{\partial x_j} \left(\varepsilon_i^{kl} x_k a_{lj} \right) = \varepsilon_i^{kl} \frac{\partial x_k}{\partial x_j} a_{lj} + \varepsilon_i^{kl} x_k \frac{\partial a_{lj}}{\partial x_j} \\ =& \varepsilon_i^{kl} \delta_k^j a_{lj} + \varepsilon_i^{kl} x_k \left[\text{div } a \right]_l = \varepsilon_i^{jl} a_{lj} + \left[x \times \text{div } a \right]_i . \end{aligned} \tag{B.4}$$

□

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