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Master's Thesis

Independent Combinatoric Worm Principles for First Order Arithmetic and Beyond

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Γνῶθι Μαθών

Δελφικό Παράγγελμα

Abstract

In this thesis we study Beklemishev's combinatorial principle *Every Worm Dies*, EWD which although true, it is unprovable in Peano Arithmetic (PA). The principle talks about sequences of modal formulas, the finiteness of all of them being equivalent to the one-consistency of PA. We present the elements of proof theory at play here and perform two attempts at generalizing this theorem. One is directed towards its relationship with some known fragments of PA while the other aims to see its connection with fragments of second order arithmetic.

Contents

Chapter 1

Introduction

In this thesis we study Beklemishev's combinatorial principle *Every Worm Dies* (EWD), which was presented in [2] as the *Worm Principle.* This statement is inspired by the Hercules-Hydra game of Kirby and Paris [15] where here instead of hydras, Hercules is is fighting worms. Given a worm, on every step of the game, Hercules chops its head which then regrows based on some combinatorial rules. Every worm dies states that Hercules will eventually win, no matter the given worm he started out with. Beklemishev has shown that *Every Worm Dies* is true but unprovable in PA.

1.1 Combinatorics and provability logics

The elements of the game that we consider in this thesis are worms which are words with their alphabet being that of natural numbers. The rules are such that these combinatoric worms behave similarly to the provability worms –which correspond to formulas consisting of only iterated modalities in the *provability logic.* The provability worms in turn can be interpreted as various fragments of arithmetic, Turing progressions and ordinals –among other things.

One of the main interpretations of worms that we are interested in this thesis is that of iterations of partial uniform reflection principles. For a given theory T, these are schemata of the statement *"if φ is provable by T, then φ is true"* for formulas φ that belong to some class of the arithmetical hierarchy. In essence these are generalizations of consistency statements. Kreisel and Lévy have shown that the uniform reflection principle for *Primitive Recursive Arithmetic* is deductively equivalent to the full induction axiom schema [16]. Leivant had later proven a similar equivalence for partial uniform reflection principles and induction schemata for formulas in some class of the arithmetical hierarchy [17]. We present these results in this thesis in Chapter 3 for a base theory that is weaker than the *Primitive Recursive Arithmetic.* With this slight generalization of the theorem, we follow Beklemishev [2].

1.2 Overview of this thesis

In this thesis we start by presenting the partial uniform reflection principles and their connection to provability logics and some of the fundamental fragments of Peano Arithmetic. We then turn our attention onto the provability logics and give a presentation of the induced ordering of worms with our goal being mostly to provide some intuition on an isomorphism between provability worms and ordinals. The latter forms the basic idea behind *Every Worm Dies* and allows us to intuitively view it as a a kind of well-foundedness claim of ε_0 –which is the proof theoretic ordinal of PA. Our later generalization attempts follow this idea as a vague guideline for choosing how much weaker or stronger we would expect the worm principle to have to be. In the case of smaller theories we fail to prove equivalence, trying two different approaches, one being the proof presented by Beklemishev for the case of PA while the other follows the ideas of a remark of the same paper [2], where a connection of it with a rule of transfinite induction is mentioned. In the final chapters we work on the first step of expanding the worm principle towards theories stronger than PA.

1.3 Layout

In Chapter 2 we present the basic theories, that most of this thesis concerns itself with, and give an introduction of the uniform reflection principle. In Chapter 3 we present the known connections between reflection principles and those basic arithmetics. In Chapter 4 we give a presentation of the provability worms and show various general known properties that we will use in the proof of *Every Worm Dies*. The main focus is the presentation of a well-order of worms (modulo provable equivalence) as well as a way to produce fundamental sequences of "limit" worms under this order. Chapter 5 we start by presenting an equivalence between the consistency of our theories and a transfinite induction rule that was studied in [1]. Then we give a generalization of it for which we credit Pakhomov for the short proof sketch he provided under email correspondence. Chapter 6 presents the equivalence of the worm principle with the 1-consistency of PA which was proven by Beklemishev in [2]. Then, in Chapter 7 we present the case of a weaker version of Beklemishev's *Every Worm Dies* and examine its relationship with the basic known weaker arithmetics. However, we fail to prove an equivalence in that case. In Chapter 8 we show some results from [5] which we need to expand the worm principle into larger theories –though some of the details ought to be resolved in future work. Finally, Chapter 9 is where we present the equivalence of an expansion of the worm principle and the 1-consistency of ACA –a well known theory of second order arithmetic. Furthermore more we examine an alternative natural expansion of the rule-set of *Every Worm Dies* which we find equivalent to the one we chose here at first.

1.4 New results

While most of what is presented here is known, several results are new. These are:

In Chapter 9 where notable are the novelties in the weaker orderings $\trianglelefteq_m$ which are used in the proof, where we split it into many to accommodate for limit ordinals to the worm game. Then in the same chapter there is the connection between the two different ways of expanding the worm principle relating to the novel Definition 4.2.3 and Notation 4.2.4.

The results of Chapter 7 are generalizations of: the main theorem of Chapter 6 –whose proof follows that in [2]– and of an alternative proof of the same theorem which was only implicitly suggested in the same paper.

Remark 6.3.2, Remark 4.3.4 were only implicitly suggested.

For the second part of Corollary 4.2.5, I didn't find it in the literature but it is not completely unlikely of something similar being proven. Proposition 4.1.9 comes as a generalization of part of a lemma used in the proof of the equivalence of EWD and 1-Con(PA) in [2].

Theorems 5.1.1 and 5.1.2 were also implicitly mentioned in a more general equivalence theorem in [1] however only a proof for the case of PA was given there. The proofs of these theorems follow the structure of that proof for the case of PA.

Finally we received great help by Pakhomov who in private correspondence with Joosten helped in further generalizing these results with Theorem 5.2.2. The proof of this theorem should be attributed to Pakhomov the details presented have been worked out by Joosten who has kindly allowed me to incorporate them in this thesis.

Chapter 2

Preliminaries

In this chapter we will present some basic notions of proof theory that we will be using extensively later on. We start with introducing the basic first order theories that we will be dealing with in this paper and we later formally introduce the so-called uniform reflection principle. The structure of this introduction follows that of [2].

2.1 Arithmetic

For the language of arithmetic we will consider the first order language $\{\leq; +, \cdot, S, exp; 0\}$ where exp denotes the unary function $x \mapsto 2^x$. Its standard model has the universe $\mathbb{N}$ and its symbols having the usual interpretation.

We will refer to formulas of this language as arithmetical. The expressions $\forall x{\leq}t\ \varphi(x)$ and $\exists x{\leq}t\ \varphi$ abbreviate the formulas $\forall x\ (x \leq t \to \varphi)$ and $\exists x\ (x{\leq}t \wedge \varphi)$ respectively. Such occurrences of quantifiers we will refer to as bounded. The arithmetical formulas can be classified in what is called the *arithmetical hierarchy.*

Definition 2.1.1. *For every $n \geq 0$, the classes of Σ_n and Π_n- formulas are defined inductively as follows: Σ_0 and Π_0-formulas are those all of whose quantifiers are bounded. The Σ_{n+1}-formulas are those of the form $\exists x_1 \ldots \exists x_m\ \varphi(x_1, \ldots, x_m)$, where φ is a Π_n-formula. The Π_{n+1}-formulas are those of the form $\forall x_1 \ldots \forall x_m\ \varphi(x_1, \ldots, x_m)$, where φ is a Σ_n-formula.*

By the prenex normal form theorem, every arithmetical formula is logically equivalent to some Σ_n and some Π_n-formula for some $n \geq 0$. We will extend the terminology so as to call a formula Σ_n iff it is logically equivalent to a Σ_n-formula as per the above definition and similarly for Π_n. The Δ_n are the formulas that are both Σ_n and Π_n in this equivalence sense. The class of Δ_0-formulas, we will refer to as elementary.

From a computational point of view, a predicate is definable by a Σ_1-formula iff it is computably enumerable (c.e.)[1]. A sentence is elementary iff it is decidable via a decision procedure of an upper bound of 2^x_n, where n is some constant and x is the size of the input. The function 2^x_n is defined as:

$$2^x_0 := x; \quad 2^x_{n+1} := 2^{2^x_n}.$$

We will refer to the binary function 2^x_y as the *super-exponentiation* function and those of the form 2^x_n for a fixed n as *multi-exponential.*

[1] They are also called recursively enumerable (r.e.).

The induction axiom I_φ for a formula φ, as usual denotes the formula

$$\mathrm{I}_\varphi{:=}\varphi(0) \wedge \forall x \ \big(\varphi(x) \to \varphi(S(x))\big) \to \forall x \ \varphi(x),$$

and we will see restricted to various classes of formulas.

The theory EA denotes the Kalmar elementary arithmetic which involves the basic actions describing the non logical symbols, the induction axiom schema with parameters for Δ_0-formulas and the axiom stating that the graph of exponentiation defines a total function[2]. The theory EA^+ is axiomatized over EA by adding the axiom stating that the graph of super-exponentiation defines a total function. For every natural number n, the theories $\mathrm{I}\Sigma_n$ and $\mathrm{I}\Pi_n$ are the extensions of EA via the induction axiom schema with parameters for Σ_n and Π_n-formulas respectively. Hence, in this paper, we make no distinction between $\mathrm{I}\Delta_0$ and EA. PA naturally is the extension of EA by induction on all formulas.

Unlike PA, the theories EA and EA^+ can both be axiomatized by Π_2-formulas in a language without exp. We will later see that all $\mathrm{I}\Sigma_n$ theories are also finitely axiomatizable. For every $n{>}0$, we have that $\mathrm{EA}^+ \subseteq \mathrm{I}\Sigma_n$ as the totality of super-exponentiation is provable via Σ_1-induction.

Kalmar elementary functions are those obtained from $Z(x) \equiv 0, S, +, \cdot, 2^x$, cut-off subtraction

$$x \dot{-} y{:=}\begin{cases} 0, & \text{if } x \leq y \\ x - y, & \text{otherwise} \end{cases},$$

projection functions $I_i^n(x_1, \ldots, x_n){=}x_i$, composition operations and bounded minimization:

$$\mu\, i{\leq}z \ R(i, \vec{x}){:=}\begin{cases} y, & \text{if } y \leq z \text{ and } R(y, \vec{x}) \wedge \forall\, i{<}y \ \neg R(i, \vec{x}) \\ 0, & \text{if } \forall\, i{\leq}z \ \neg R(i, \vec{x}) \end{cases},$$

where R is a predicate of the form $g(i, \vec{x}){=}0$ for some previously defined function g. We denote the class of elementary functions by $\mathcal{E}$. It coincides with the class of functions computable in multi-exponential time on a Turing machine [7].

Now just by the definitions and with cut-off subtraction, we can produce a basic lemma for restricted induction schemata:

Lemma 2.1.2. $\mathrm{I}\Sigma_n \equiv \mathrm{I}\Pi_n$, *for every natural number* n.

Proof. Let φ be a Π_n-formula. We prove the induction instance for φ in $\mathrm{I}\Sigma_n$. Consider the formula $\psi(a, x){:=}\neg\varphi(a \dot{-} x)$ where a is a parameter. Then $\forall x \ (\varphi(x) \to \varphi(x + 1))$ implies $\forall x \ (\psi(a, x) \to \psi(a, x + 1))$, thus by Σ_n-induction, $\psi(a, 0) \to \psi(a, a)$ and so $\varphi(0) \to \varphi(a)$, which by logic, gives us the induction instance for φ. $\square$

If we were to disallow the occurrence of parameters in the induction axioms, the proof wouldn't work.

2.2 Provability predicate

Given a set A, by words of A, we will refer to finite sequences of elements of A. The empty word, we will denote by e and we will refer to A as their alphabet. Arithmetical formulas can be naturally identified with words on a finite alphabet and in turn, they can be one-to-one

[2]In our setting, we get this totality for free since we have included a function symbol for exponentiation in our language.

encoded by numbers by transforming them first into binary words and subsequently using a binary decoding, hence being able to express them within EA. Thus we fix a standard global encoding or Gödel numbering of all expressions of the arithmetical language. Then the Gödel number of an expression τ we will denote by $\ulcorner\tau\urcorner$.

Properties concerned with classifying numbers with codes of formulas, axioms, rules and determining if they belong to a specific class in the arithmetical hierarchy can all be naturally expressed in EA with Δ_0-formulas. Similarly, for finite sequences we can agree on a one-to-one elementary coding. Let $\langle x_0, \ldots, x_n\rangle$ denote the code of a sequence $x_0, \ldots, x_n$, with $(x)_i$ denoting the i-th element of the sequence and $\langle\ \rangle$ denote the code of the empty sequence. Additionally, the relation stating that "x is the code of a sequence" –denoted by $Seq(x)$– as well as the functions returning the length of a sequence $lh(x)$, the last element of it $end(x)$, and the concatenation of two sequences $x * y$, are all elementary representable within EA.

We will not maintain that strict a notation of sequences as later in this paper we will be dealing with words, represented as sequences growing towards the left and denoted as $x_n, \ldots, x_0$.

Up to Chapter 7, we will make the convention that –unless stated otherwise– the theories we will be dealing with, will be first order theories with equality extending EA. Second order theories will be touched on in the final chapter. A theory T is elementary axiomatizable if there is a Δ_0-formula $Ax_T(x)$ that is true iff x is the code of an axiom of T. Similarly, we will say that a theory is finitely axiomatizable if it can be axiomatized by a single formula. By Craig's trick, all c.e. theories have an equivalent that is elementary axiomatizable. We will make the assumption that all theories of this paper are computably enumerable. A theory T is sound if all of its theorems are true in the standard model of arithmetic and Γ-sound, for some complexity class Γ, if all its Γ-theorems are true in the standard model. Between theories, in the same language, writing $T \subseteq U$ will indicate that U proves the theorems of T. Given two theories T, U, by $T + U$ we denote the union of the two theories, axiomatized by $Ax_T(x) \vee Ax_U(x)$. Two theories T, U will be called (deductively) equivalent $T \equiv U$ on a third theory V if $T + V$ and $U + V$ prove the same theorems. Given two theories of possibly different languages T, U we will say that U is a (proof theoretic) conservative extension of T if the theorems of T are also theorems of U and every theorem of U in the language of T is a theorem of T. For a class of formulas Γ and theories in a common language, we will say that U is a Γ-conservative extension of T if all the Γ-theorems of U are also theorems of T. Specifically for Π_{n+1}, we will write $T \equiv_n U$ to denote that U is a Π_{n+1}-conservative extension of T.

Given a c.e. theory T, possibly by using Craig's trick, there is an elementary formula $\mathrm{Prf}_T(x, y)$ expressing *"y codes a proof of a formula x"*. From this, Gödel's provability formula $\Box_T(x)$ is defined as $\exists y\ \mathrm{Prf}_T(y, x)$.

Terms of the form

$$\overline{n} := \underbrace{S(S(\ldots S(0)\ldots))}_{n \text{ times}}$$

are called numerals. For $\varphi(x_1, \ldots, x_n)$ a formula, $\ulcorner\varphi(\dot{x}_1, \ldots, \dot{x}_k)\urcorner$ denotes the natural definable term for the function mapping $n_1, \ldots, n_k$ to the Gödel number $\ulcorner\varphi(\overline{n}_1, \ldots, \overline{n}_k)\urcorner$. The bar on numerals shall be omitted in the cases where n cannot be confused with a variable. So in a simplification of notation, for a sentence σ, we write $\Box_T\sigma$ instead of $\Box_T(\overline{\ulcorner\sigma\urcorner})$ and for a formula φ as above, we write $\Box_T\varphi(\dot{x}_1, \ldots, \dot{x}_k)$ instead of $\Box_T(\ulcorner\varphi(\dot{x}_1, \ldots, \dot{x}_k)\urcorner)$, though we may also not explicitly write out all of its variables.

We denote by $\perp$ the logical falsity and $\mathrm{Con}(T) := \neg\Box_T\perp$. The reading convention of this paper later will be to assume that EA is the operational theory on the provability formula whenever we write $\Box$ without any theory in the subscript. At this point we will state, without a proof, a few theorems involving Gödel's provability formula.

Proposition 2.2.1 (Löb's derivability conditions). *For formulas φ, ψ and T c.e.,*

L1. $T \vdash \varphi \Rightarrow \mathrm{EA} \vdash \Box_T \varphi$;

L2. $\mathrm{EA} \vdash \Box_T(\varphi \to \psi) \to \big(\Box_T \varphi \to \Box_T \psi\big)$;

L3. $\mathrm{EA} \vdash \Box_T \varphi \to \Box_T \Box_T \varphi$;

L4. $\mathrm{EA} \vdash \Box_T \forall x\ \varphi(x) \to \forall x\ \Box_T \varphi(\dot{x})$.

Notice that L4 is derivable from L1, L2 and the axiom $\forall x\ \varphi(x) \to \varphi(x)$. Moreover, *L*3 is a corollary of the more general provable Σ_1-completeness [10]:

Proposition 2.2.2 (Provable Σ_1-completeness). *For any Σ_1-formula $\varphi(x_1, \dots, x_n)$,*

$$\mathrm{EA} \vdash \varphi(x_1, \dots x_n) \to \Box_T \varphi(\ulcorner \dot{x}_1, \dots, \dot{x}_n \urcorner).$$

The following *fixed point lemma*, plays a key role, at first in Gödel's incompleteness proofs and many other results.

Lemma 2.2.3 (Fixed point Lemma). *For any formula $\varphi(x, x_1, \dots, x_n)$, there is a $\psi(x_1, \dots, x_n)$ whose free variables can only be among those of φ, except for x, such that*

$$\mathrm{EA} \vdash \psi(x_1, \dots, x_n) \leftrightarrow \varphi(\ulcorner \psi(\dot{x}_1 \dots \dot{x}_n) \urcorner).$$

What follows, is a generalization of Gödel's second incompleteness theorem, known as Löb's theorem. It will see much use in this paper and for that reason we will present a proof.

Theorem 2.2.4 (Löb's Theorem). *For any c.e. $T \supseteq \mathrm{EA}$ and formula φ,*

$$T \vdash \Box_T \varphi \to \varphi \Leftrightarrow T \vdash \varphi.$$

Proof. Assume that $T \vdash \Box_T \varphi \to \varphi$. By the fixed point lemma, there is a formula ψ such that $\mathrm{EA} \vdash \psi \leftrightarrow (\Box_T \psi \to \varphi)$. Then using Löb's derivability conditions and the fact that T contains EA, we derive:

1. $T \vdash \Box_T(\psi \to (\Box_T \psi \to \varphi))$
2. $T \vdash \Box_T \psi \to \Box_T(\Box_T \psi \to \varphi)$
3. $T \vdash \Box_T \psi \to (\Box_T \Box_T \psi \to \Box_T \varphi)$
4. $T \vdash \Box_T \psi \to \Box_T \varphi$
5. $T \vdash \Box_T \psi \to \varphi$, by the assumption
6. $T \vdash \psi$
7. $T \vdash \Box_T \psi$
8. $T \vdash \varphi$.

The other direction is immediate. $\Box$

Löb's Theorem can be expressed in a more formalized way. The following statement can be inferred from Löb's Theorem and the derivability conditions:

Corollary 2.2.5. *For any formula* φ*,*

$$\mathrm{EA} \vdash \Box_T(\Box_T\varphi \to \varphi) \to \Box_T\varphi.$$

Substituting φ for $\perp$ in Löb's Theorem, we get:

Theorem 2.2.6 (Gödel's Second Incompleteness Theorem)**.** *For any c.e.* $T \supseteq \mathrm{EA}$*,*

(i) If T *is consistent, then* $T \nvdash \mathrm{Con}(T)$*.*

(ii) If in addition, T *is* Σ_1*-sound, then* $T \nvdash \neg\,\mathrm{Con}(T)$*.*

Notice that this clearly implies that any consistent, Σ_1-sound, axiomatizable theory $T \supseteq \mathrm{EA}$ can be extended to $T + \mathrm{Con}(T)$ which is clearly consistent, is axiomatized by $Ax_T(x) \lor x{=}\ulcorner\mathrm{Con}(T)\urcorner$ and is Σ_1-sound. This way, we can expand such a theory by repeating this procedure forming increasing chains of consistent Σ_1-sound theories. This conceptual idea goes by the name of Turing progressions and marks the beginning of this paper.

2.3 Provability logic

Löb's derivability conditions, along with Löb's Theorem, all come to connect in the basic Gödel-Löb provability logic GL. It is the modal logic formulated in the language $\mathcal{L}_\Box$ of propositional logic along with the unary $\Box$ modality. As before, $\Diamond\varphi$ and $\Box^n\varphi$, abbreviate $\neg\Box\neg\varphi$ and $\Box\ldots\Box\varphi$ (n times) respectively. GL can be axiomatized by the propositional axioms on $\mathcal{L}_\Box$, together with the axioms:

A1. $\Box(\varphi \to \psi) \to \big(\Box\varphi \to \Box\psi\big)$, known as the distribution axiom;

A2. $\Box\varphi \to \Box\Box\varphi$, known as the **4** axiom;

A3. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$, known as the Gödel-Löb axiom;

and the inference rules of *modus ponens* and *necessitation*:

$$\frac{\varphi}{\Box\varphi}.$$

If we were to replace the axiom A3 –known as Löb's axiom– with Löb's (inference) rule

$$\frac{\Box\varphi \to \varphi}{\varphi},$$

we would end up with the same logic.

By an arithmetical interpretation of the language of GL, we mean any function $(\cdot)^*$ that maps propositional variables to arithmetical formulas. This can be extended to a function $(\cdot)^*_T$, mapping modal formulas to arithmetical ones by translating $\Box$ into $\Box_T$ and preserving the other logical operations:

$$\begin{aligned}(\varphi \to \psi)^*_T &:= (\varphi)^*_T \to (\psi)^*_T\\ (\perp)^*_T &:= \perp\\ (\Box\varphi)^*_T &:= \Box_T(\varphi)^*_T.\end{aligned}$$

So interpreting $\Box\varphi$ as "φ is provable in the theory T", we have an *arithmetical soundness* theorem as for any formula $\varphi \in \mathcal{L}_\Box$, if $\mathrm{GL} \vdash \varphi$, then $\mathrm{EA} \vdash (\varphi)^*_T$ for any arithmetical realization $(\cdot)^*_T$ of the variables of φ. The opposite implication, proven by Solovay, gives a very important theorem of *arithmetical completeness* and is stated as follows [6]:

Theorem 2.3.1 (Solovay). *Let $T \supseteq \mathrm{EA}$ be an axiomatizable Σ_1-sound theory. Then for every formula φ,*

$$\mathrm{GL} \vdash \varphi \Leftrightarrow T \vdash (\varphi)^*_T, \textit{ for every realization } (\cdot)^*_T \textit{ of the variables of } \varphi.$$

For our purposes in this paper, we will only be using soundness results for GL with respect to various classes of interpretations, so we do not use the difficult part of Solovay's theorem.

2.4 Reflection, n-provability and n-consistency

Formulas of the form $\Box_T\varphi \to \varphi$ we call reflection. They express that if φ is provable in T then it is true. Reflection principles are schemata of formulas bearing this structure that, in essence, expresses the soundness of T. They aren't expressible by a single formula, as we know by Tarski that there is no single arithmetical formula able to express the truth of any arithmetical formula. Thus we need a schema to formulate the statement "every provable sentence is true". Of course by Löb's Theorem, we do not expect reflection principles to be provable.

Definition 2.4.1. *Let T be an axiomatizable theory. The Uniform reflection principle* $\mathrm{RFN}(T)$ *is the schema:*

$$\forall\, x_1, \ldots, x_n \big(\Box_T\varphi(\dot{x}_1, \ldots \dot{x}_n) \to \varphi(x_1, \ldots x_n)\big)$$

for every arithmetical formula $\varphi(x_1, \ldots x_n)$.
For some class of arithmetical formulas Γ, we denote $\mathrm{RFN}_\Gamma(T)$ or $\Gamma\text{-}\mathrm{RFN}(T)$ the reflection schema over formulas in Γ. Usually, Γ will be one of the classes Σ_n or Π_n of the arithmetical hierarchy.

The following lemma relates different partial reflection principles to each other and to the notion of consistency.

Lemma 2.4.2. *For a c.e. theory T, the following equivalences hold over* EA*:*

(i) $\Sigma_n\text{-}\mathrm{RFN}(T) \equiv \Pi_{n+1}\text{-}\mathrm{RFN}(T)$ *for every $n \geq 0$;*

(ii) $\Pi_1\text{-}\mathrm{RFN}(T) \equiv \mathrm{Con}(T)$.

Proof. (i) Let $\forall\, x\ \varphi(x, y) \in \Pi_{n+1}$ with $\varphi(x, y) \in \Sigma_n$. Using the derivability condition $L4$:

$$\begin{aligned}\mathrm{EA} + \Sigma_n\text{-}\mathrm{RFN}(T) \vdash \Box_T\forall\, x\ \varphi(x, \dot{y}) &\to \forall\, x\ \Box_T\varphi(\dot{x}, \dot{y})\\ &\to \forall\, x\ \varphi(x, y).\end{aligned}$$

So $\mathrm{EA} + \Sigma_n\text{-}\mathrm{RFN}(T) \vdash \forall y\ \big(\Box_T\forall\, x\ \varphi(x, \dot{y}) \to \forall\, x\ \varphi(x, y)\big)$.

(ii) Let $\varphi(x) \in \Pi_1$. Then by Σ_1 completeness, $\mathrm{EA} \vdash \neg\varphi(x) \to \Box_T\neg\varphi(\dot{x})$. Thus, we have

$$\begin{aligned}\mathrm{EA} \vdash \Box_T\varphi(\dot{x}) \wedge \neg\varphi(x) &\to \Box_T(\varphi(\dot{x}) \wedge \neg\varphi(\dot{x}))\\ &\to \Box_T\bot.\end{aligned}$$

Therefore, $\mathrm{Con}(T) := \neg\Box_T\bot$ gives $\Box_T\varphi(\dot{x}) \to \varphi(x)$.

$\Box$

Unlike the full uniform reflection principle, its partial versions are finitely axiomatizable. This is because for the classes of Σ_n and Π_n-formulas, there are truth definitions within EA:

Proposition 2.4.3. *For every natural number $n \geq 0$, there is an arithmetical Π_n-formula* $\mathrm{True}_{\Pi_n}(x)$ *–called the* truth definition *of Π_n-formulas in* EA*– such that, for every* $\varphi(x_1, \ldots, x_k) \in \Pi_n$*:*

$$\mathrm{EA} \vdash \varphi(x_1, \ldots, x_k) \leftrightarrow \mathrm{True}_{\Pi_n}(\ulcorner \varphi(\dot{x}_1, \ldots, \dot{x}_k) \urcorner).$$

Similarly for Σ_n classes between the two, it provably holds:

$$\forall z \; (\mathrm{True}_{\Sigma_n}(z) \leftrightarrow \neg \, \mathrm{True}_{\Pi_n}(\ulcorner \neg \urcorner * z)).$$

Moreover, the partial truth predicates provably reflect the expected structural properties of truth, so that, for example for $\varphi, \psi \in \Pi_n$ we have

$$\mathrm{EA} \vdash \mathrm{True}_{\Pi_n}(\ulcorner \varphi \wedge \psi \urcorner) \leftrightarrow \mathrm{True}_{\Pi_n}(\ulcorner \varphi \urcorner) \wedge \mathrm{True}_{\Pi_n}(\ulcorner \psi \urcorner)$$

and likewise for the other connectives.

A proof of this fact can be found in [10], [11] and the appendix of [5]. Now we can formulate the partial uniform reflection principles as single sentences:

Lemma 2.4.4. *Over* EA*, the schema* Π_n*-* $\mathrm{RFN}(T)$ *is equivalent to the universal instance of:*

$$\forall z \; \big(\Box_T \, \mathrm{True}_{\Pi_n}(\dot{z}) \to \mathrm{True}_{\Pi_n}(z)\big). \tag{2.1}$$

A similar statement holds for Σ_n*-*$\mathrm{RFN}(T)$*.*

Proof. Clearly the latter is an instance of the uniform reflection schema. Notice that Proposition 2.4.3 implies that for every $\varphi(x_1, \ldots, x_n) \in \Pi_n$,

$$\mathrm{EA} \vdash \forall x_1, \ldots x_n \; \Box_T \big(\varphi(\dot{x}_1, \ldots, \dot{x}_k) \leftrightarrow \mathrm{True}_{\Pi_n}(\ulcorner \varphi(\dot{x}_1, \ldots, \dot{x}_k) \urcorner)\big).$$

So we infer in EA,

$$\begin{aligned}
\Box_T \varphi(\dot{x}_1, \ldots, \dot{x}_k) &\to \Box_T \, \mathrm{True}_{\Pi_n}(\ulcorner \varphi(\dot{x}_1, \ldots, \dot{x}_k) \urcorner) \\
&\to \mathrm{True}_{\Pi_n}(\ulcorner \varphi(\dot{x}_1, \ldots, \dot{x}_k) \urcorner) \quad \text{(by (2.1) with } z = \ulcorner \varphi(\dot{x}_1, \ldots, \dot{x}_k) \urcorner) \\
&\to \varphi(x_1, \ldots, x_n).
\end{aligned}$$

□

Another way to view partial reflection principles is as analogues of Gödel's consistency assertion. In this case, we will be strengthening the standard consistency statement.

Definition 2.4.5. *For every $n \geq 1$, let $\mathsf{Th}_{\Pi_n}(\mathbb{N})$ be the set of all true Π_n-sentences. Then a theory T is n-consistent, if $T + \mathsf{Th}_{\Pi_n}(\mathbb{N})$ is consistent. Formally, this is expressed by the formula:*

$$n\text{-}\,\mathrm{Con}(T) {:=} \forall z \; \big(\mathrm{True}_{\Pi_n}(z) \to \neg \Box_T \neg \, \mathrm{True}_{\Pi_n}(\dot{z})\big).$$

For true Π_n-sentences, this informally expresses their compatibility with T. Its relationship with reflection is clear as for $T + \mathsf{Th}_{\Pi_n}(\mathbb{N})$ to be inconsistent, there must be a true Π_n-formula that is incompatible with T and therefore, there must be a provable Σ_n-formula that is not true. To produce the formal version of this result, we take the contrapositive of n-$\mathrm{Con}(T)$ and apply Proposition 2.4.3 and Lemma 2.4.4. Therefore we have:

Remark 2.4.6. *For every c.e. theory $T \supseteq$ EA and every $n{\geq}1$, Σ_n-RFN(T) is equivalent to n-Con(T) over* EA.

If we had taken the same definition for n=0, then over EA, we would have had that the 0-consistency of a theory T seems equivalent to Con(T). However, there are subtle issues ([21]) related to collection causing minor differences between the two definitions. As such, we shall work with Con(T).

The dual of the new consistency notions give us the *n-provability formula*

$$[n]_T\varphi := \neg n\text{-}\,\mathrm{Con}(T + \neg\varphi),$$

expressing the provability of φ in $T + \mathsf{Th}_{\Pi_n}(\mathbb{N})$. The following notational step is to let $\langle n\rangle_T\varphi := n\text{-}\,\mathrm{Con}(T+\varphi)$. Similarly to what we did with consistency, we will refrain from using the 0-provability and instead work with $\Box_T$. Finally, by the definitions it should be clear that $\langle n\rangle\varphi$ has arithmetical complexity of Π_{n+1} and therefore, $[n]\varphi$ is of Σ_{n+1}.

Properties we had of the provability formula, we also have for its extensions. For example, we have an analogue of provable completeness.

Proposition 2.4.7 (Σ_n-completeness)**.** *For any Σ_{n+1}-formula $\varphi(x_1,\dots,x_k)$,*

$$\mathrm{EA} \vdash \varphi(x_1,\dots x_k) \to [n]_T\varphi(\ulcorner\dot{x}_1,\dots,\dot{x}_k\urcorner).$$

Moreover, $[n]_T$ preserves Löb's derivability conditions:

Proposition 2.4.8. *For formulas φ,ψ, natural number $n \geq 1$ and T c.e.*

L1^n. $T \vdash \varphi \Rightarrow \mathrm{EA} \vdash [n]_T\varphi$;

L2^n. $\mathrm{EA} \vdash [n]_T(\varphi \to \psi) \to \big([n]_T\varphi \to [n]_T\psi\big)$;

L3^n. $\mathrm{EA} \vdash [n]_T\varphi \to [n]_T[n]_T\varphi$;

L4^n. $\mathrm{EA} \vdash [n]_T\forall x\ \varphi(x) \to \forall x\ [n]_T\varphi(\dot{x})$.

Proof. The first two conditions and the fourth, can be checked by taking into account that

$$\mathrm{EA} \vdash [n]_T\varphi \leftrightarrow \exists z\ \big(\mathrm{True}_{\Pi_n}(z)\ \wedge \Box_T(\mathrm{True}_{\Pi_n}(\dot{z})\ \to \varphi)\big).$$

The third, is derived by the more general fact of Σ_n-completeness since $[n]\varphi$ is a Σ_{n+1}-formula. □

Proposition 2.4.9 (Σ_n-completeness)**.** *For any Σ_{n+1}-formula $\varphi(x_1,\dots,x_k)$,*

$$\mathrm{EA} \vdash \varphi(x_1,\dots x_k) \to [n]_T\varphi(\ulcorner\dot{x}_1,\dots,\dot{x}_k\urcorner).$$

A short proof of the above can be found in [2].

We also have the analogue of the formalized version of Löb's Theorem.

Proposition 2.4.10. *For any formula φ and natural number n,*

$$\mathrm{EA} \vdash [n]_T([n]_T\varphi \to \varphi) \to [n]_T\varphi.$$

Therefore, we can make use of GL-soundness for the n-provability predicates as well by, of course, interpreting $\Box$ as the n-provability for some n. To produce a notational distinction, consider the following:

Notation 2.4.11. *We denote by* GL^n *the variant of* GL *using the modality* $[n]$ *instead of* $\Box$. *Formally, we would have* $([n]\varphi)^*_T = [n]_T(\varphi)^*$.

We will close this chapter by proving a proposition showing us that the uniform reflection formulas cannot be expressed by formulas of lower complexity than the ones we have given.

Proposition 2.4.12. *For* $T \supseteq \mathrm{EA}$ *a c.e. theory, we have that* Π_n-$\mathrm{RFN}(T)$ *is not contained in any consistent extension of* T *of complexity* Σ_n.

Proof. Let U be a Σ_n extension of T and $U \vdash \Pi_n$-$\mathrm{RFN}(T)$. Since Π_n-$\mathrm{RFN}(T)$ can be expressed by a single formula, there exists a finite subtheory $U_0 \subseteq U$ such that $U_0 + T \vdash \Pi_n$-$\mathrm{RFN}(T)$. Without loss of generality, we can express U_0 via a Σ_n-formula θ. So we have

$$\begin{aligned} T + \theta &\vdash \Box_T \neg\theta \to \neg\theta \\ T &\vdash \theta \to (\Box_T \neg\theta \to \neg\theta) \\ T &\vdash \Box_T \neg\theta \to \neg\theta. \end{aligned}$$

Thus we can use Löb's Theorem to produce $T \vdash \neg\theta$ which means that $T + U_0$ and therefore $T + U$ are inconsistent. $\Box$

Finally, this naturally gives us the following for the uniform reflection principle:

Corollary 2.4.13. *For* $T \supseteq \mathrm{EA}$ *a c.e. theory, the axiom scheme* $\mathrm{RFN}(T)$ *is not contained in any consistent extension of* T *of bounded arithmetical complexity.*

Chapter 3

Fragments of Arithmetic

We will begin this chapter with a statement on functions whose graph can be expressed by an elementary formula, which we will immediately use to create our first connection between EA and EA$^+$ via the use of reflection principles.

3.1 Function iteration

Let us first introduce some notation that we will extensively use throughout the thesis

Notation 3.1.1. *For a function f with an elementary graph, we denote by $f^{(y)}(x)$ the function iteratively defined as:*

$$f^{(0)}(x) = x, \quad f^{(S(y))}(x) = f(f^{(y)}(x)).$$

Formally, the function's graph can be given by the formula:

$$f^{(y)}(x) = z \leftrightarrow \exists\, s{\in}Seq\ \big((s)_0{=}x \wedge \forall\, i{<}y\ (s)_{i+1}{=}f((s)_i) \wedge (s)_y{=}z\big),$$

where we point out that the existential quantification on s can be bounded by a function belonging to the closure of $\mathcal{E} + f$ by composition, denoted by $C(f)$. Thus, if f is expressible by a term, the graph of $f^{(y)}(x)$, can be given by a Δ_0-formula. This in turn, implies that the graph of 2^x_y can be given by a Δ_0-formula by substituting f with *exp*.

Now we can state this very important lemma produced by taking two views of Π_2-sentences. We refer to Lemma 3.7 in [2] for a proof.

Lemma 3.1.2. *Let f be a function with elementary graph that is non-decreasing and $f(x) \geq 2^x$. Then it holds that*

$$\mathrm{EA} \vdash \lambda x. f^{(x)}(x){\downarrow} \leftrightarrow \langle 1 \rangle_{\mathrm{EA}}\, f{\downarrow}.$$

If we substitute f with *exp*, we get:

Corollary 3.1.3. $\mathrm{EA}^+ \equiv \mathrm{EA} + \Pi_2\text{-}\,\mathrm{RFN}(\mathrm{EA})$.

3.2 Tait's Calculus

The next natural step at this point is to ask about higher levels of reflection, for which we will need *Tait's calculus.* Formulas in Tait's calculus are constructed from the atomic formulas and their negations with the connectives $\wedge, \vee$ and quantifiers $\forall, \exists$ as in first order logic. Thus negation

here is recursively defined via de Morgan's rules. Sequents are finite sets of formulas, denoted by $\Gamma, \Delta, \ldots$ and are understood as disjunctions. The sequent $\Gamma \cup \{\varphi\}$ is written as Γ, φ.
The axioms of Tait's calculus are sequents of the form $\Gamma, \varphi, \neg\varphi$ with φ being an atomic formula. It consists of the following inference rules:

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}(\wedge) \qquad \frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}(\vee_r) \qquad \frac{\Gamma, \varphi}{\Gamma, \psi \vee \varphi}(\vee_l)$$

$$\frac{\Gamma, \varphi(a)}{\Gamma, \forall x\ \varphi(x)}(\forall), \qquad \text{where } a \text{ does not occur free in } \Gamma$$

$$\frac{\Gamma, \varphi(t)}{\Gamma, \exists x\ \varphi(x)}(\exists) \qquad \frac{\Gamma, \varphi \quad \Gamma, \neg\varphi}{\Gamma}(Cut)$$

It can be shown that a sequent Γ is provable in Tait's calculus iff the formula $\bigvee \Gamma$ is provable in the predicate calculus PC. The *cut-elimination theorem* states that there is an effective procedure transforming a derivation of a sequent Γ in Tait's calculus to a derivation of Γ in which there are no applications of the Cut rule. The advantage of cut-free proofs is that they have the so-called subformula property stating that every formula occurring in the proof is a subformula of a formula in the last sequent. Therefore a cut-free proof of a Π_n-formula consists only of Π_n-formulas. The cut-elimination procedure produces a proof whose size can be roughly bounded by an upper bound of order 2^x_{cn}, where x is the size of the original proof, n is its cut-rank and c is some constant. This allows us to formalize in EA^+ the usual inductive proof of the cut-elimination theorem.

Before we continue, we leave this remark which will let us make use of Tait's Calculus later on:

Remark 3.2.1. *If* $\mathrm{EA} \subseteq T$ *and* $\psi \in \Sigma_n$ *is without free variables, where* $T \vdash \psi$ *and* U *is a subtheory of* T*, then*

$$T \vdash \Pi_n\text{-}\mathrm{RFN}(U + \psi) \leftrightarrow \Pi_n\text{-}\mathrm{RFN}(U)$$

Proof. Let $\varphi \in \Pi_n$ which without loss of generality has at most one free variable, then since $\psi \to \varphi$ is a Π_n formula,

$$\begin{aligned} T \vdash \forall x\ \big(\Box_{U+\psi}\varphi(\dot{x}) \to \varphi(x)\big) &\leftrightarrow \Big(\psi \to \forall x\ \big(\Box_U(\psi \to \varphi(\dot{x})) \to \varphi(x)\big)\Big) \\ &\leftrightarrow \forall x\ \big(\Box_U(\psi \to \varphi(\dot{x})) \to (\psi \to \varphi(x))\big) \end{aligned}$$

And therefore, $T \vdash \Pi_n\text{-}\mathrm{RFN}(U) \to \Pi_n\text{-}\mathrm{RFN}(U + \psi)$. The other direction comes from the fact that if $U_1 \subseteq U_2$, then $T \vdash \Box_{U_1}\varphi(\dot{x}) \to \Box_{U_2}\varphi(\dot{x})$. $\Box$

3.3 Induction schemata and reflection

Now we can safely state the following two main theorems giving us the relationship between EA and the $\mathrm{I}\Sigma_n$ theories via partial reflection principles.

Theorem 3.3.1. *Provably in* EA *we have that* $\mathrm{I}\Sigma_n \subseteq \mathrm{EA} + \langle n+1 \rangle \top$.

Proof. Let $\varphi(x, y_1, \ldots, y_l) \in \Sigma_n$ be a formula of l parameters. By $\psi(y_1, \ldots, y_l)$ we will denote the Π_{n+1} formula:

$$\varphi(0, y_1, \ldots, y_l) \wedge \forall x\big(\varphi(x, y_1, \ldots, y_l) \to \varphi(x+1, y_1, \ldots, y_l)\big).$$

Then, through external induction, we have that for every $k, m_1, \dots m_l \in \mathbb{N}$, there is a proof bounded by an elementary formula; $\mathrm{EA} \vdash \psi(\overline{m}_1, \dots, \overline{m}_l) \to \varphi(\overline{k}, \overline{m}_1, \dots, \overline{m}_l)$. As such, the upper bound allows us to formalize the proof in EA by making use of Δ_0-induction as the replacement of the external one. This results in:

$$\mathrm{EA} \vdash \forall x\, y_1, \dots, y_l \, \Box\big(\psi(\dot{y}_1, \dots, \dot{y}_l) \to \varphi(\dot{x}, \dot{y}_1, \dots, \dot{y}_l)\big),$$

whose proof is also bound by an elementary formula. Now, since $\psi(\dot{y}_1, \dots, \dot{y}_l) \to \varphi(\dot{x}, \dot{y}_1, \dots, \dot{y}_l)$ is a Σ_{n+1} formula, we can make use of the reflection instance for it in $\langle n+1 \rangle \top$. Whence, $\mathrm{EA} + \langle n+1 \rangle \top \vdash \forall x\, y_1, \dots y_l \, \big(\psi(y_1, \dots, y_l) \to \varphi(x, y_1, \dots, y_l)\big)$ which is equivalent to the induction instance for φ.
By parametrizing the formula φ through the use of a truth formula True_{Σ_n}, we finally end up with a proof bound by an elementary formula of $\mathrm{EA} + \langle n+1 \rangle \top \vdash \mathrm{I}\Sigma_n$ which implies that $\mathrm{EA} \vdash \Box\big(\langle n+1 \rangle \top \to \mathrm{I}\Sigma_n\big)$. □

The reverse direction also holds true, however as the proof of this is done with the use of cut-elimination, this means that it is provable in EA^+.

Theorem 3.3.2. *Provably in* EA^+*, we have* $\mathrm{I}\Sigma_n \equiv \mathrm{EA} + \langle n+1 \rangle \top$.

Proof. To prove $\Sigma_{n+1}\text{-}\mathrm{RFN}(\mathrm{EA})$ in $\mathrm{I}\Sigma_n$, we have by Remark 3.2.1 and since EA has a finite Π_2-axiomatization (over PC), that it is sufficient to prove $\Sigma_{n+1}\text{-}\mathrm{RFN}(\mathsf{PC})$. The proof of this fact can be found in [2] and we will omit it here. □

As an immediate consequence we obtain:

Corollary 3.3.3. *Provably in* EA^+*, we have* $\mathrm{PA} \equiv \mathrm{EA} + \mathrm{RFN}(\mathrm{EA})$.

And by using Proposition 2.4.12 and its Corollary,

Corollary 3.3.4. *For every natural number n,*

(i) $\mathrm{I}\Sigma_n$ *is not contained in any consistent extension of* EA *of complexity* Σ_{n+2}.

(ii) PA *is not contained in any consistent extension of* EA *of bounded arithmetical complexity.*

We will end this chapter with a theorem mentioning a connection between partial uniform reflection and the partial *metareflection rule.*

Definition 3.3.5. *Given a c.e. theory T, the metareflection rule over T is the rule*

$$\mathsf{RR}^n(T): \qquad \frac{\varphi(\vec{x})}{\langle n \rangle_T \varphi(\dot{\vec{x}})}.$$

By $\Gamma\text{-}\mathsf{RR}^n(T)$*, where* Γ *is a complexity class of formulas, we denote the above rule with* φ *restricted to* Γ*-formulas.*

The following theorem will play a role in the next paragraph as we will begin with iterating reflection principles. We will leave it here without proof though one can be found in [2].

Theorem 3.3.6. *Let* $T \supseteq \mathrm{EA}$ *be a c.e. theory and* U *be any* Π_{n+2} *extension of* EA*. Then* $U + \Sigma_{n+1}\text{-}\mathrm{RFN}(T)$ *is* Π_{n+1}*-conservative over* $U + \Pi_{n+1}\text{-}\mathsf{RR}^n(T)$.

Chapter 4

Provability logics for ordinal analysis

In previous paragraphs, we hinted at an extension of GL by considering a range of modalities and deciding on axioms expressing the interplay between them. This is, of course, referring to the polymodal provability logic GLP –introduced by Japaridze ([12]). GLP has proven itself very versatile, finding interpretations in Turing progressions, reflection schemata, varying provability interpretations. If we further restrict ourselves to specific elements of GLP –the so-called *worms*– we will also find interpretations as words of an infinite alphabet, special fragments of arithmetic, worlds in a special model for the closed fragment of GLP, and also ordinals. This versatility makes it a very valuable tool to have at our disposal.

4.1 Worms in the Polymodal provability logic

Definition 4.1.1. *For Λ an ordinal, the logic* GLP_Λ *is the propositional modal logic with a modality $[\alpha]$ for every $\alpha < \Lambda$. Each $[\alpha]$ modality satisfies the* GL *identities given by all tautologies, distribution axioms $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$, Löb's axiom scheme $[\alpha]([\alpha]\varphi \to \varphi) \to [\alpha]\varphi$ and the rules modus ponens and necesitation:*

$$\frac{\varphi}{[\alpha]\varphi}.$$

The interaction between modalities is governed by two schemes:

- J1 *Monotonicity: $[\beta]\varphi \to [\alpha]\varphi$, for every $\beta < \alpha < \Lambda$;*
- J2 *Negative introspection: $\langle\beta\rangle\varphi \to [\alpha]\langle\beta\rangle\varphi$, for every $\beta < \alpha < \Lambda$.*

Apart from the customary convention in the literature that GLP denotes GLP_ω we shall sometimes use a different convention where the context should make clear which is meant.

Notation 4.1.2. *By* GLP *we will denote the class-sized logic that has a modality $[\alpha]$ for every ordinal α, along with the corresponding axioms and rules.*

Of course when expressing GLP in EA, the ordinals existing will be restricted by the ordinal notation system that we use. A primary representation of GLP_ω goes as follows:

As with the GL^n, consider an arithmetical interpretation which we extend to a function $(\cdot)^*_T$, mapping formulas of the language of GLP_ω to arithmetical ones by translating $[n]$ to the n-provability $[n]_T$ and preserving the other logical operators as before. Then *arithmetical soundness* follows; with J2 being derived from Σ_n-completeness.

A weaker logic that is related to GLP is the modal logic of the so-called *reflection calculus* RC. The logic RC only allows the logical connectives $\wedge$ and $\langle\alpha\rangle$ and the only other symbols in the alphabet are propositional variables and $\top$. The formulas formed as such are called *strictly positive.* Theorems of RC take the form $\varphi \vdash_{\mathrm{RC}} \psi$ where φ and ψ are strictly positive. The logic RC is related to GLP via the theorem [3], [8]

Theorem 4.1.3. *For strictly positive φ and ψ we have:*

$$\varphi \vdash_{\mathrm{RC}} \psi \textit{ if and only if } \mathrm{GLP} \vdash \varphi \to \psi.$$

The results of GLP that we will be using in later chapters can all be expressed in the reflection calculus via the above theorem.

A closed formula for GLP is simply one without any propositional variables. Therefore, closed formulas are characterized by $\top$ being their only atomic sub-formula. Within the closed fragment of GLP there is a particular class of terms which are called *worms* and they are iterated consistency statements.

Definition 4.1.4 (Worms)**.** *The class of worms of* GLP *is denoted by* $\mathbb{W}$ *and defined inductively as follows:*

- $\top \in \mathbb{W}$*;*
- *If* $A \in \mathbb{W} \;\Rightarrow\; \langle\xi\rangle A \in \mathbb{W}$.

We will be later be putting two restrictions on the classes of worms, one expressing a minimum that the ordinals of worms have to equal or exceed and the other expressing an upper bound on the ordinals. Specifically we will denote by $\mathbb{W}^\Lambda_\alpha$ the class of worms defined inductively as:

- $\top \in \mathbb{W}^\Lambda_\alpha$*;*
- *If* $A \in \mathbb{W}^\Lambda_\alpha \;\Rightarrow\; \langle\xi\rangle A \in \mathbb{W}^\Lambda_\alpha$*, where* $\alpha \leq \xi < \Lambda$.

Naturally, we will omit the subscript or superscript when not considering that restriction. So $\mathbb{W}_0$ will be the same as $\mathbb{W}$. $\overline{\mathbb{W}}^\Lambda_\alpha$ will denote the class of worms $\mathbb{W}^\Lambda_\alpha$ modulo GLP*-provable equivalence.*

Notation 4.1.5. *We will be using the lower-case letters of the Greek alphabet $\alpha, \beta, \gamma, ...$ to denote ordinals. Worms will be denoted by the upper-case Latin characters $A, B, C, \ldots$. Finally, we will be omitting the $\langle\cdot\rangle$ at times, writing $\alpha\varphi$ instead of $\langle\alpha\rangle\varphi$ –for instance.*

We will be treating worms as finite words with the ordinals as their alphabet and will view them as growing towards the left. Then the empty word $\mathrm{e} = \top$ and the concatenation of worms will be defined recursively as $\top A = \mathrm{e}\, A = A$ and $(\xi B)A = \xi(BA)$. The length of a worm will also indicate by this definition the number of modalities present and so by the classical inductive definition we have $|\top| = 0$, and $|\langle\xi\rangle A| = |A| + 1$. The n-times concatenation of a worm A will be denoted by A^n, defined as usual: $A^0 = \top$ and $A^{n+1} = AA^n$.

Definition 4.1.6 ($<, <_\alpha$)**.** *We define the relation $<_\alpha$ on $\mathbb{W} \times \mathbb{W}$ by*

$$A <_\alpha B \;:\Leftrightarrow\; \mathrm{GLP} \vdash B \to \langle\alpha\rangle A.$$

Moreover, $<$ will simply denote $<_0$.

The orderings $<_\alpha$ will be referred to as the *α-consistency orderings*. We know (see [1], [4]) that in each class of worms $\mathbb{W}_\alpha$, the relation $<_\alpha$ is a linear pre-order; so given $A, B \in \mathbb{W}_\alpha$ then either $A <_0 B$, $A \equiv_{\mathrm{GLP}} B$ or $B <_0 A$. Which in turn implies that in $\overline{\mathbb{W}}_\alpha$ –since the relation is well defined as $\overline{A} <_\alpha \overline{B} \Leftrightarrow A <_\alpha B$– the relation $<_\alpha$ is a linear ordering. More specifically, we have by [1] and [4]:

Theorem 4.1.7. *Let α be an ordinal, then $\langle \overline{\mathbb{W}}_\alpha, <_\alpha \rangle$ is isomorphic to the class of all ordinals with the standard $<$ ordering on them.*

Under this ordering, $\top$ is the minimal element in every $\langle \overline{\mathbb{W}}_\alpha, <_\alpha \rangle$ and every worm $A \in \mathbb{W}_\alpha$ has as its immediate $<_\alpha$-successor, the worm αA (see [9]).

Before moving further, we will present some well known results in GLP:

Lemma 4.1.8. *The following formulas are derivable in* GLP*:*

(i) If $\alpha \leq \beta$ and $A \in \mathbb{W}$, then $\mathrm{GLP} \vdash \beta\alpha A \to \alpha A$*;*

(ii) If $\alpha < \beta$, then $\mathrm{GLP} \vdash \beta\varphi \wedge \alpha\psi \leftrightarrow \beta(\varphi \wedge \alpha\psi)$*;*

(iii) If $A \in \mathbb{W}_{\alpha+1}$, then $\mathrm{GLP} \vdash AC \wedge \alpha B \leftrightarrow A(C \wedge \alpha B)$*;*

(iv) If $A \in \mathbb{W}_{\alpha+1}$, then $\mathrm{GLP} \vdash A \wedge \alpha B \leftrightarrow A\alpha B$.

The proof of which follows successively from the axioms of GLP, details for which can be found in [2] and [4]. With this lemma in our tool-belt, we can prove the following proposition which will be of use to us later as we present worm battles.

Proposition 4.1.9. *For every natural number n and every ordinal α, it holds that*

$$\mathrm{GLP} \vdash \langle\alpha\rangle^{n+1}\top \to AB,$$

where $A \in \mathbb{W}^\alpha$, $|B| \leq n$ and $B \in \mathbb{W}^{\alpha+1}$.

Proof. We will prove this fact through two external inductions, first we will show that for every n and B satisfying the above conditions, $\mathrm{GLP} \vdash \langle\alpha\rangle^n\top \to B$. If $n = 0$, then it is clear. Assume now that it holds for $n = k$. Let $B \in \mathbb{W}^{\alpha+1}$ with $|B| \leq k$ and let $\beta \leq \alpha$, then

$$\begin{aligned} \mathrm{GLP} \vdash \langle\alpha\rangle^{k+1}\top &\to \langle\alpha\rangle^{|B|+1}\top, \quad \text{(by at most } k \text{ applications of the } \mathbf{4} \text{ axiom)} \\ &\to \langle\alpha\rangle\langle\alpha\rangle^{|B|}\top \\ &\to \langle\alpha\rangle B \\ &\to \langle\beta\rangle B. \end{aligned}$$

Now we will perform an external induction on $|A|$. If A=β for some $\beta < \alpha$, then we fall in the case of the previous induction. If A=$\langle\beta\rangle C$, where $\beta < \alpha$ and $\mathrm{GLP} \vdash \langle\alpha\rangle^{n+1}\top \to CB$, then

$$\begin{aligned} \mathrm{GLP} \vdash \langle\alpha\rangle^{n+1}\top &\to (CB \wedge \langle\alpha\rangle\top) \\ &\to \langle\alpha\rangle CB, \quad \text{by Lemma 4.1.8.} \end{aligned}$$

□

4.2 Fundamental sequences of worms

Now we will present a way to decompose worms into smaller ones. In [9] this decomposition finds most of its use in the various properties that displays on worms as well as through its ability to present inductive arguments on them. For our purposes, we will make most use of it as a means to produce fundamental sequences of a worm, –typically of $\mathbb{W}^\omega$– on which we can then perform induction with the worm ordering as a basis.

Definition 4.2.1 (head, remainder)**.** *The α-head h_α of A is inductively defined on the length of the worm:*

- $h_\alpha(\top){:=}\top$;
- *if $A = \langle\beta\rangle B$ with $\beta < \alpha$, then $h_\alpha(\langle\beta\rangle B){:=}\top$;*
- *if $A = \langle\beta\rangle B$ with $\beta \geq \alpha$, then $h_\alpha(\langle\beta\rangle B){:=}\langle\beta\rangle h_\alpha(B)$.*

Likewise, the α-remainder r_α of A is defined as:

- $r_\alpha(\top){:=}\top$;
- *if $A = \langle\beta\rangle B$ with $\beta < \alpha$, then $r_\alpha(\langle\beta\rangle B){:=}\langle\beta\rangle B$;*
- *if $A = \langle\beta\rangle B$ with $\beta \geq \alpha$, then $r_\alpha(\langle\beta\rangle B){:=}r_\alpha(B)$.*

The head h and remainder r of a worm A are then defined as:

- $h(\top){=}r(\top){:=}\top$;
- *if $A = \langle\alpha\rangle B$, then $h(\langle\alpha\rangle A){:=}h_\alpha(\langle\alpha\rangle A)$ and $r(\langle\alpha\rangle A){:=}r_\alpha(\langle\alpha\rangle A)$.*

Now the sequences that we mentioned, we will produce with the assistance of the functions between formulas of the language of GLP, Q^α_k. Their notation follows that of [2] and they are defined inductively as:

- $Q^\alpha_0(\varphi){:=}\langle\alpha\rangle\varphi$;
- $Q^\alpha_{k+1}(\varphi){:=}\langle\alpha\rangle\big(\varphi \wedge Q^\alpha_k(\varphi)\big)$.

A first observation we can make is that

$$\text{GLP} \vdash \langle\alpha{+}1\rangle\varphi \to Q^\alpha_k(\varphi) \tag{4.1}$$

for every formula φ and natural number k. This is done by induction on k with the base case following from J1, while on the induction step, we have:

$$\begin{aligned} \text{GLP} \vdash \langle\alpha{+}1\rangle\varphi &\to \big(Q^\alpha_k(\varphi) \wedge \langle\alpha{+}1\rangle\varphi\big), && \text{by the induction hypothesis} \\ &\to \langle\alpha{+}1\rangle(Q^\alpha_k(\varphi) \wedge \varphi), && \text{by Lemma 4.1.8} \\ &\to Q^\alpha_{k+1}(\varphi). \end{aligned}$$

Next up, using the α-head and remainder functions, we can for every $A \in \mathbb{W}$, translate the produce of $Q^\alpha_k(A)$ into worms.

Lemma 4.2.2. *If $A, B \in \mathbb{W}$ and $A = \langle\alpha{+}1\rangle B$ then for every $k \in \mathbb{N}$ we have that,*

$$\text{GLP} \vdash Q^\alpha_k(B) \leftrightarrow (\alpha h_{\alpha+1}(B))^{k+1} r_{\alpha+1}(B).$$

Proof. We reason by induction on k. The base case is trivial as $\alpha h_{\alpha+1}(B)r_{\alpha+1}(B) = \alpha B$. For the induction step we have by the definition of the α+1-head and remainder functions and by the induction hypothesis that:

$$\begin{aligned} \mathrm{GLP} \vdash Q^{\alpha}_{k+1}(B) &\leftrightarrow \alpha\big(h_{\alpha+1}(B)r_{\alpha+1}(B) \wedge (\alpha h_{\alpha+1}(B))^{k+1} r_{\alpha+1}(B)\big) \\ &\leftrightarrow \alpha\big(h_{\alpha+1}(B)(r_{\alpha+1}(B) \wedge (\alpha h_{\alpha+1}(B))^{k+1} r_{\alpha+1}(B))\big), \qquad \text{by Lemma 4.1.8(iii)} \\ &\leftrightarrow (\alpha h_{\alpha+1}(B))^{k+2} r_{\alpha+1}(B), \qquad \text{by Lemma 4.1.8(i).} \end{aligned}$$

□

From what we have so far, we can see that the Q^{α}_{k} function will only be directly useful as a means to describe fundamental sequences of worms whose last modality is a successor ordinal, so of the type $A = \langle\alpha{+}1\rangle B$. However we can easily work around this issue once we fix some fundamental sequences for our ordinals of countable cofinality.

Definition 4.2.3. *For every $A \in \mathbb{W}$ and $k \in \mathbb{N}$, we define $A[\![k]\!]$ as follows:*

- *If $A = \top$, then $A[\![k]\!] = \top$.*
- *If $A = 0B$, then $A[\![k]\!] = B$.*
- *If $A = \langle\alpha{+}1\rangle B$, then $A[\![k]\!] = (\alpha h_{\alpha+1}(B))^{k+1} r_{\alpha+1}(B)$, which is the expression of $Q^{\alpha}_{k}(B)$ from the previous lemma.*
- *If $A = \langle\lambda\rangle B$ where λ is a limit ordinal, then $A[\![k]\!] = \langle\lambda[k]\rangle B$. Here $\lambda[k]$ corresponds to the k-th element of the fundamental sequence of λ.*

For the last case, different choices could be made. Perhaps a more uniform option would be following the structure of the successor stage.

Notation 4.2.4. *For every $A \in \mathbb{W}$ and $k \in \mathbb{N}$, we define $A{\ll}k{\gg}$ as follows:*

- *If $A = \langle\lambda\rangle B$ where λ is a limit ordinal, then $A{\ll}k{\gg} = (\lambda[k]h_{\lambda}(B))^{k+1} r_{\lambda}(B)$. As before, $\lambda[k]$ corresponds to the k-th element of the fundamental sequence of λ.*
- *Otherwise, $A{\ll}k{\gg} = A[\![k]\!]$.*

Either choice does not face any further difficulties in its representation within arithmetical theories $T \supseteq \mathrm{EA}$ for elementary ordinal notation systems.

Corollary 4.2.5. *For any $k \in \mathbb{N}$ and $A \in \mathbb{W}$ with $A \neq \top$, we have $A[\![k]\!] <_0 A$. Additionally, $A{\ll}k{\gg} \leq_0 A[\![k{+}1]\!] \leq_0 A{\ll}k{+}1{\gg}$.*

Proof. We already know from (4.1) that:

$$\mathrm{GLP} \vdash \langle\alpha{+}1\rangle B \to Q^{\alpha}_{k}(B).$$

So by Lemma 4.1.8, for $A = \langle\alpha{+}1\rangle B$,

$$\begin{aligned} \mathrm{GLP} \vdash A &\to \alpha A[\![k]\!] \\ &\to 0A[\![k]\!]. \end{aligned}$$

The limit case follows from J1, while the more complicated alternative is done by also using the result for the successor stages.

Now let $A = \lambda B$ where λ is a limit ordinal and we aim to prove that $A\ll k\gg <_0 A[\![k{+}1]\!]$. Then, $A[\![k{+}1]\!] = \langle\lambda[k{+}1]\rangle B \geq \langle\lambda[k]+1\rangle B$ for which, by the first step of this corollary and by the definition of $A\ll k\gg$,

$$\langle\lambda[k]+1\rangle B > (\langle\lambda[k]+1\rangle B)[\![k]\!] = A\ll k\gg.$$

The proof of $A[\![k{+}1]\!] \leq_0 A\ll k{+}1\gg$ is done by multiple applications of the axioms J1 and the **4** axiom. □

We will return to this distinction in choices for the limit step much later and for now, we will be using the $A[\![k]\!]$ notation for fundamental sequences of worms.

Let us now restrict ourselves to GLP_ω as we will translate worms into arithmetics in order to present the famous reduction property. We remind that the arithmetical interpretation $(\cdot)^*_T$ we will omit. Additionally, we remind that by $U \equiv_n V$ we mean that the theories are mutually Π_{n+1}-conservative. We will use the proof from [2] for the following:

Proposition 4.2.6 (Reduction Property)**.** *Let T be a Π_{n+2}-extension of* EA*. Then, for all φ of the language of* GLP_ω*, over T,*

$$\{\langle n+1\rangle_T\varphi\} \equiv_n \{Q^n_k(\varphi) : k < \omega\}.$$

Proof. We will make use of Theorem 3.3.6 by taking $U = T$ and $T = T + \varphi$. The rule Π_{n+1}-$\mathsf{RR}^n(T+\varphi)$ is equivalent to

$$\frac{\psi}{\langle n\rangle_T\big(\varphi\wedge\psi\big)},\quad \psi{\in}\Pi_{n+1}.$$

Then the theory $T + \{Q^n_k(\varphi) : k < \omega\}$ is the closure of $T + \langle n\rangle\varphi$ under this rule as $T + Q^n_2(\varphi) \vdash \langle n\rangle(\varphi\wedge\psi)$, for $\psi \in \Pi_{n+1}$ with $T + \langle n\rangle\varphi \vdash \psi$. Now, if $T + Q^n_k(\varphi) \vdash \psi$, then

$$\begin{aligned} T + Q^n_{k+1}(\varphi) &\vdash \langle n\rangle_T\big(\varphi\wedge Q^n_k(\varphi)\big)\\ &\vdash \langle n\rangle_T\big(\varphi\wedge\psi\big). \end{aligned}$$

Therefore, all Π_{n+1}-consequences of an element $\langle n+1\rangle\varphi$ of the language of GLP_ω of complexity Π_{n+2}, are generated by Π_{n+1}-elements $Q^n_k(\varphi)$. □

Due to the application of Theorem 3.3.6, this proof can be expressed within EA^+ and so the same hold for this equivalence. A corollary of this, we will also refer to as the reduction property whenever we make use of this later. Once again, the proof is taken from [2].

Corollary 4.2.7 (Reduction Property)**.** *If $m \leq n$, then*

$$\mathrm{EA}^+ \vdash \langle m\rangle\langle n{+}1\rangle\varphi \leftrightarrow \forall k\ \langle m\rangle Q^n_k(\varphi)$$

Proof. EA^+ is a Π_2 extension of EA and therefore, a Π_{n+2} extension of EA for every natural number n. In addition, the reduction property is established within EA^+. Therefore,

$$\mathrm{EA}^+ \vdash \text{“}\{\langle n+1\rangle_{\mathrm{EA}^+}\varphi\} \equiv_m \{Q^n_k(\varphi) : k < \omega\}\text{”},$$

for every $m < n$. So over EA^+, a Π_{m+1} sentence is provable from $\langle n+1\rangle\varphi$ if and only if it is so from $Q^n_k(\varphi)$, for some k. This implies

$$\mathrm{EA}^+ \vdash \langle m\rangle_{\mathrm{EA}^+}\langle n{+}1\rangle_{\mathrm{EA}^+}\varphi \leftrightarrow \forall k\ \langle m\rangle Q^n_k(\varphi),$$

which with the assistance of Remark 3.2.1, completes our proof. □

4.3 More on the ordering relation

Let us now provide a slightly more detailed description of the connection between worms and ordinals. We will do so by following the ideas in [9] where the omitted proofs of the theorems and lemmata that we will presenting can be found.

We start by introducing an operation on worms that shifts all the modalities in a worm by a constant amount. We will define a shift to the right and one to the left. To define the shift to the left, we will need this basic result of ordinal arithmetic.

Lemma 4.3.1. *If $\alpha < \beta$ are ordinals, then there exists a unique γ such that $\alpha + \gamma = \beta$.*

We will denote this unique γ by $-\alpha + \beta$ and this is the operation that we use to define the shift to the left on worms.

Definition 4.3.2 ($\alpha{\uparrow}A$ and $\alpha{\downarrow}A$)**.** *Let $A \in \mathbb{W}$ and α an ordinal. The α-right shift of A, denoted by $\alpha{\uparrow}A$ is the worm that is obtained by simultaneously substituting each β that occurs in A by $\alpha + \beta$.*
For worms $A \in \mathbb{W}_\alpha$, we define the α-left shift of A, denoted as $\alpha{\downarrow}A$, by simultaneously substituting each β that occurs in A with $-\alpha + \beta$.

At first glance, these might seem as somewhat unnatural types of operations to be performing on words. In the case of worms however, they have some really nice properties in relation to the orderings of worms. The proof of the following can be found in [9]:

Lemma 4.3.3. *The map $\alpha{\uparrow}$ is an isomorphism between $\langle \mathbb{W}, < \rangle$ and $\langle \mathbb{W}_\alpha, <_\alpha \rangle$.*

A natural expansion of the right shift is having it apply on all formulas of GLP. Let α be an ordinal. We define the α-right shift inductively on the structure of the formulas of the language of GLP:

- $\alpha{\uparrow}\top := \top$;
- $\alpha{\uparrow}(\varphi \to \psi) := \alpha{\uparrow}\varphi \ \to \ \alpha{\uparrow}\psi$;
- $\alpha{\uparrow}\langle\beta\rangle\varphi := \langle\alpha{+}\beta\rangle\,(\alpha{\uparrow}\varphi)$.

Then GLP is stable under right shift. More specifically, we have the following:

Remark 4.3.4. *Assume that $\mathrm{GLP}_\Lambda \vdash \varphi$ and let $\mathcal{D}$ be the Hilbert proof witnessing this fact. Let β be the greatest ordinal occurring in some modality in a formula of $\mathcal{D}$. Then for every α such that $\alpha + \beta < \Lambda$, $\mathrm{GLP}_\Lambda \vdash \alpha{\uparrow}\varphi$.*

Therefore, given an ordinal Λ, and α such that $\forall\,\beta{<}\Lambda\ (\alpha{+}\beta < \Lambda)$, it holds that if $\mathrm{GLP}_\Lambda \vdash \varphi$, then $\mathrm{GLP}_\Lambda \vdash \alpha{\uparrow}\varphi$.

Proof. We just need to notice that the axioms of GLP_Λ remain axioms of it after any right shift application where the modalities remain smaller than Λ. Same thing applies for the rules. The rest of the proof follows by a simple induction on the proof length. $\square$

Having established these tools, we can direct our attention into defining the basic function for ordering worms. We know by [9] that Theorem 4.1.7 implies that the orderings $<_\alpha$ on $\mathbb{W}_\alpha \times \mathbb{W}_\alpha$ are well founded. So we can consider the following mapping:

Definition 4.3.5. *For every ordinal α, we define $o_\alpha : \mathbb{W}_\alpha \to \mathrm{On}$ as:*

$$o_\alpha(x) = \sup_{y <_\alpha x} (o_\alpha(y) + 1),$$

where $\sup(\emptyset) = 0$*. We will denote o_0 as just o.*

The various o_α can be reduced to o_0 in the following manner:

Lemma 4.3.6. *For $A \in \mathbb{W}_\alpha$ we have $o_\alpha(A) = o(\alpha \downarrow A)$.*

Now we will present set of functions on ordinals, similar to the Veblen function and introduced in [9]. It will help us into making the o mapping more tangible.

Definition 4.3.7 (Worm enumerators σ^α)**.** *We define $\sigma^\alpha : \mathrm{On} \to \mathrm{On}$ as the function that enumerates the $<$-order types of the worms in $\mathbb{W}_\alpha$ in increasing order.*

All the σ^α can then be determined as follows:

Theorem 4.3.8. *For ordinals α and β, the values $\sigma^\alpha(\beta)$ are determined by the following recursion.*

1. $\sigma^\alpha(0) = 0$*, for all* $\alpha \in \mathrm{On}$*;*
2. $\sigma^1(\beta) = -1 + \omega^\beta$*;*
3. $\sigma^{\alpha+\beta} = \sigma^\alpha \sigma^\beta$*;*
4. $\sigma^\alpha(\lambda) = \sup_{\beta<\lambda} \sigma^\alpha(\beta)$ *for limit ordinals* λ*;*
5. $\sigma^\lambda(\beta + 1) = \sup_{\eta<\lambda} \sigma^\eta(\sigma^\lambda(\beta) + 1)$ *for λ an additively indecomposable limit ordinal.*

Finally, we can provide in turn a recursive definition for the o mapping, using the worm enumerators:

Theorem 4.3.9. *Given worms A, B and an ordinal α,*

1. $o(\top) = 0$*;*
2. $o(A0B) = o(B) + 1 + o(A)$*;*
3. $o(\alpha \uparrow A) = \sigma^\alpha o(A)$*.*

Let us finish this chapter with an example:
Consider the worm 210ω. Then $o(210\omega) = o(\omega) + 1 + o(21) = \sigma^\omega(o(0)) + 1 + \sigma^1 o(10)$. Starting with the $\sigma^1 o(10)$, we can determine it via the recursive definition from Theorems 4.3.8 and 4.3.9, as follows:

$$\sigma^1 o(10) = \sigma^1\big(o(\top) + 1 + o(1)\big) = \sigma^1\big(1 + \sigma^1 o(0)\big) = \sigma^1\big(1 + \sigma^1(1)\big) = -1 + \omega^{1+(-1+\omega)} = \omega^\omega.$$

As for $\sigma^\omega(o(0)) = \sigma^\omega(1)$, we know that $\sigma^1(1) = \omega$. Consequently $\sigma^2 = \sigma^1\sigma^1 = \omega^\omega$, $\sigma^3 = \sigma^1\sigma^2 = \omega^{\omega^\omega}$ and so on. Thus $\sigma^\omega 1 = \sup_{\eta<\omega} \sigma^\eta\big(\sigma^\omega(0) + 1\big) = \sup_{\eta<\omega} \sigma^\eta(1) = \varepsilon_0$.
So wrapping things up, we have that $o(210\omega) = \sigma^\omega(o(0)) + 1 + \sigma^1 o(10) = \varepsilon_0 + \omega^\omega$.

Chapter 5

Transfinite induction versus transfinite reflection

One connection between worms and arithmetics can be seen through rules of transfinite induction over the ordering of worms.

Definition 5.0.1. *By* $\mathrm{TI}^R(\Pi_1, <_0\restriction\mathbb{W}^\Lambda)$ *we denote the following inference rule expressing transfinite induction along the ordering of* $<_0$ *for* Π_1*-formulas* φ*:*

$$\frac{\forall\, A{\in}\mathbb{W}^\Lambda \big(\forall\, B{<_0}A\ \varphi(B) \to \varphi(A)\big)}{\forall\, A{\in}\mathbb{W}^\Lambda\ \varphi(A)}.$$

5.1 Consistency and transfinite induction

We shall prove the consistency of $\mathrm{I}\Sigma_n$ for every $n \in \omega$ by one application of the transfinite induction rule $\mathrm{TI}^R(\Pi_1, <_0\restriction\mathbb{W}^{n+1})$ over EA^+. The first proof of the consistency of PA from transfinite induction was obtained by G. Gentzen. Later there were fine tuned results. The proof of the following is based on the proof of

$$[\mathrm{EA}^+, \mathrm{TI}^R(\Pi_1, <_0\restriction\mathbb{W}^\omega)] \vdash \mathrm{Con}(\mathrm{PA}),$$

that is found in [2].

Theorem 5.1.1. $[\mathrm{EA}^+, \mathrm{TI}^R(\Pi_1, <_0\restriction\mathbb{W}^{n+1})] \vdash \mathrm{Con}(\mathrm{I}\Sigma_n)$

Proof. We will denote by A^* the arithmetical interpretation $(A)^*_{\mathrm{EA}}$ of $A \in \mathbb{W}^{n+1}$. The function $(\cdot)^*$, as a mapping between Gödel numbers of formulas, is elementary and hence definable by a (pseudo)-term[1] in EA. Additionally, we shall write $\Diamond$ instead of $\Diamond_{\mathrm{EA}}$.

We have by Theorem 3.3.1 that

$$\mathrm{I}\Sigma_n \subseteq \mathrm{EA} + \langle n+1\rangle\top,$$

a proof of which is formalizable in EA and thus we have

$$\mathrm{EA} \vdash \Diamond\, \langle n+1\rangle\top \to \mathrm{Con}(\mathrm{I}\Sigma_n). \tag{5.1}$$

[1] We shall not distinguish between actual terms of EA and what Boolos calls *pseudo-terms* in [6]. These pseudo terms are functions with an elementary graph that behave exactly as terms were we to add a constant for the function to our language.

We will first prove that $\forall\, A{\in}\mathbb{W}^{n+1}\ \Diamond A^*$ using the transfinite induction rule along $<_0\!\restriction\!\mathbb{W}^{n+1}$ over EA^+ with the Π_1 induction formula being $\Diamond A^*$ (A being the induction variable). We claim that

$$\mathrm{EA}^+ \vdash \forall\, A{\in}\mathbb{W}^{n+1}\ (\forall\, B{<_0}A\ \Diamond B^* \to \Diamond A^*).$$

Reasoning in EA^+, assume that we have $\forall\, B{<_0}A\ \Diamond B^*$ for arbitrary $A \in \mathbb{W}^{n+1}$. This leads us to two cases:

- If $A{=}0B$ then by our assumption $\Diamond B$, which then by use of $\mathrm{RFN}_{\Sigma_1}(\mathrm{EA})$ in EA^+ gives $\Diamond\Diamond B^*$.
- If $A{=}\langle m+1\rangle\, B$ (with $m{<}n$) then, since for any k we have $A[\![k]\!]{<_0}A$, we have $\forall\, k\ \Diamond A[\![k]\!]^*$ by the inductive hypotheses. By the reduction property we conclude

  $$\mathrm{EA}^+ \vdash \Diamond A^* \leftrightarrow \forall\, k\ \Diamond A[\![k]\!]^*.$$

  Hence $\forall\, k\ \Diamond A[\![k]\!]^*$ implies $\Diamond A^*$.

Thus $\forall\, A{\in}\mathbb{W}^{n+1}\ \Diamond A^*$ which means that for all $A{<_0}\langle n+1\rangle\top^*$ we have $\Diamond A^*$ and hence since $A[\![k]\!]{<_0}A$, we have $\forall\, k\ \Diamond A[\![k]\!]$. Once again by use of the reduction property, we conclude with $\Diamond(\langle n+1\rangle\top)^*$. Therefore

$$\begin{aligned}[\mathrm{EA}^+, \mathrm{TI}^R(\Pi_1, <_0\!\restriction\!\mathbb{W}^{n+1})] &\vdash \Diamond(\langle n+1\rangle\top)^* \\ &\vdash \mathrm{Con}(\mathrm{I}\Sigma_n), \qquad \text{by (5.1)}\ .\end{aligned}$$

$\square$

The other inclusion however is of more interest to us. We will start by presenting this slightly more generalized result. The proof is based on the corresponding one of

$$T + \mathrm{Con}(\mathrm{PA}) \supseteq [T, \mathrm{TI}^R(\Pi_1, <_0\!\restriction\!\mathbb{W}^\omega)],$$

found in [1]. This theorem is also mentioned there.

Theorem 5.1.2. *If $T \supseteq \mathrm{EA}$ is axiomatized by a single Π_n-sentence and $T \subseteq \mathrm{I}\Sigma_n$, then we have that $T + n\text{-}\,\mathrm{Con}(\mathrm{I}\Sigma_{n+k})$ contains $[T, \mathrm{TI}^R(\Pi_{n+1}, <_0\!\restriction\!\mathbb{W}^{k+1})]$.*

Proof. Assume that $\varphi \in \Pi_{n+1}$ and

$$T \vdash \forall\, A{\in}\mathbb{W}_n^{n+k+1}\big(\forall\, B{<_n}A\ \varphi(B) \to \varphi(A)\big)$$

We are going to show that $T + n\text{-}\,\mathrm{Con}(\mathrm{I}\Sigma_{n+k}) \vdash \forall\, A{\in}\mathbb{W}_n^{n+k+1}\ \varphi(A)$. Since the n-Con of a theory is equivalent to Π_{n+1}-reflection for the same theory, applying this to $T + A_T$ we obtain

$$\mathrm{EA} \vdash \langle n\rangle_T A_T \to \forall\, B\ (\Box_{T+A_T}\varphi(B) \to \varphi(B)).$$

Thus, we infer

$$\begin{aligned} T \vdash \forall\, B{<_n}A\ \Box_T(\langle n\rangle_T B_T \to \varphi(B)) &\to \big(\langle n\rangle_T A_T \to \forall\, B{<_n}A\ \Box_T(A_T \to \varphi(B))\big) \\ &\to \big(\langle n\rangle_T A_T \to \forall\, B{<_n}A\ \varphi(B)\big) \\ &\to \big(\langle n\rangle_T A_T \to \varphi(A)\big)\end{aligned}$$

Then via reflexive induction on T, we obtain

$$T \vdash \forall\, A \in \mathbb{W}_n^{n+k+1}\ \big(\langle n\rangle_T A_T \to \varphi(A)\big)$$

And therefore $T + \forall A \in \mathbb{W}^{n+k+1}_n \ \langle n \rangle_T A_T \vdash \forall A \in \mathbb{W}^{n+k+1}_n \ \varphi(A)$. Now notice that by Remark 3.2.1 it is clear that since $\mathrm{I}\Sigma_{n+k} \vdash \langle n+k+1 \rangle_{\mathrm{EA}} \top$, then $\mathrm{I}\Sigma_{n+k} \vdash \langle n+k+1 \rangle_T \top$ and henceforth, by Proposition 4.1.9 we have that for every $A{\in}\mathbb{W}^{n+k+1}$, $\mathrm{I}\Sigma_{n+k} \vdash A_T$. Thus $T + n\text{-}\operatorname{Con}(\mathrm{I}\Sigma_{n+k}) \vdash \forall A{\in}\mathbb{W}^{n+k+1}_n \ \varphi(A)$. □

The proof has still some room to generalize the result further however we will not explore that since after private correspondence with Pakhomov, we have been provided with something even more general. The remainder of this chapter is in large written by Joosten after said correspondence with Pakhomov.

5.2 The main theorem

We shall fix a well-behaved ordinal notation system and tacitly assume that all quantification over ordinals is restricted to ordinals from this fixed notation system. Further, we agree that any iteration of reflection contains the base theory:

Definition 5.2.1. *Given an c.e. base theory T, we define*

$$\Pi_n{-}\mathsf{R}^{\alpha}(T) := T + \{\Box_{\bigcup_{\beta<\alpha} \Pi_n-\mathsf{R}^{\beta}(T)} \pi \to \pi \mid \pi \in \Pi_n\}.$$

Here is the main theorem to be proven as stated and proof-sketched by Fedor Pakhomov in private mail correspondence:

Theorem 5.2.2. *Let T be a theory containing* EA *that is axiomatised by a single Π_{n+1} sentence with $n \in \omega$. For $n > 0$ and α from our notation system we have*

$$\Pi_n{-}\mathsf{R}^{\alpha+1}(T) \equiv [T, \mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}(1{+}\alpha)\big)].$$

Moreover, this theorem is formalisable in EA^{+}. In the next chapters we comment on the details.

5.3 Reflection proves induction

We first focus on the inclusion $\Pi_n{-}\mathsf{R}^{\alpha+1}(T) \supseteq [T, \mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}(1{+}\alpha)\big)]$ proving something slightly stronger. First we shall run the proof in an informal setting. Then we see how the proof can be formalised in EA.

5.4 The non-formal inclusion

To see that the statement we prove is slightly stronger we first observe an easy lemma.

Lemma 5.4.1. *For T being any c.e. theory we have for any $\alpha > \beta$ from our notation system that*

$$\Pi_n{-}\mathsf{R}^{\alpha}(T) \vdash \Pi_n{-}\mathsf{R}^{\beta}(T)$$

To prove our inclusion we shall observe that Kleene's rule implies reflection also when the rules and reflection are restricted to the same complexity class.

Lemma 5.4.2. *Given any c.e. theory T containing* EA*, we have for $n{\geq}1$ that restricted reflection $T + \{\forall x \ \big(\Box_T \pi(\dot{x}) \to \pi(x)\big) \mid \pi \in \Pi_n\}$ is equivalent to adding the rule $\dfrac{\forall x \ \Box_T \pi(\dot{x})}{\forall x \, \pi(x)}$ for $\pi \in \Pi_n$ to T.*

Proof. We consider the proof of Proposition 2.1 from [2] and see that this readily can be adapted to our purposes. $\square$

We can also formalise the previous Lemma inside EA:

Lemma 5.4.3. *Given any c.e. theory T containing* EA*, inside* EA*: we have for $n\geq 1$ that restricted reflection $T + \{\forall x\ \big(\Box_T \pi(\dot{x}) \to \pi(x)\big) \mid \pi \in \Pi_n\}$ is equivalent to adding the rule $\dfrac{\forall x\ \Box_T \pi(\dot{x})}{\forall x\, \pi(x)}$ for $\pi \in \Pi_n$ to T.*

Proof. We consider the proof of Proposition 2.1 from [2] and see that this readily can be adapted to our purposes. $\square$

We now state and prove the slight strengthening of first inclusion.

Lemma 5.4.4. *For T being any c.e. theory containing* EA*, we have for any α from our notation system that*

$$\Pi_n{-}\mathsf{R}^\alpha(T) \vdash [T, \mathsf{TI}^\mathsf{R}\big(\Pi_n, \omega{\cdot}\alpha\big)]$$

Proof. The proof goes by transfinite induction on α. Although not strictly necessary, we present the **base case** with quite some detail so that the underlying mechanism of the proof becomes clear. Since for $\alpha = 0$ there is nothing to prove, the base case consists of $\alpha = 1$.

We are to transform a proof π in $[T, \mathsf{TI}^\mathsf{R}\big(\Pi_n, \omega{\cdot}\alpha\big)]$ into a proof in $\Pi_n{-}\mathsf{R}^\alpha(T)$. Since all steps in π are from T and since $\Pi_n{-}\mathsf{R}^\alpha(T)$ contains T we only have to see that the one application of the transfinite induction rule can be mimicked inside $\Pi_n{-}\mathsf{R}^\alpha(T)$. Thus, we assume that for some $\varphi \in \Pi_n$

$$T \vdash \forall n{<}\omega\ \big(\forall m{<}n\, \varphi(m) \to \varphi(n)\big) \tag{5.2}$$

and set out to prove that $\Pi_n{-}\mathsf{R}^1(T) \vdash \forall n{<}\omega\ \varphi(n)$. From (5.2) we obtain

$$T \vdash \Box_T \forall n{<}\omega\ \big(\forall m{<}n\, \varphi(m) \to \varphi(n)\big). \tag{5.3}$$

Consequently, we get inside T that $\Box_T \varphi(0)$. This is tantamount to $\Box_T \forall m{<}1 \varphi(m)$ so that combining that with (5.3) once more we obtain $\Box_T \varphi(1)$ and in particular $\Box_T \forall m{<}2 \varphi(m)$. We can continue doing this and since the so obtained proof of $\Box_T \varphi(\overline{n})$ grows elementary in n and since T contains EA we may conclude $\forall n{<}\omega \Box_T \varphi(\overline{n})$. Now by Kleene's rule we obtain $\forall n{<}\omega \varphi(n)$ as was to be shown for the base case.

We now proceed to the **successor case** $\alpha + 1$. The proof is fairly similar to the base case. Instead of using (5.3) to conclude $\Box_T \varphi(0)$ we now use it together with the inductive hypothesis to conclude $\Box_{\Pi_n - \mathsf{R}^\alpha(T)} \varphi(\omega \cdot \alpha)$ after which we proceed as before. Let us see the details.

So, we assume that

$$T \vdash \forall \beta{<}\omega{\cdot}(\alpha{+}1)\ \big(\forall \gamma{<}\beta\, \varphi(\gamma) \to \varphi(\beta)\big) \tag{5.4}$$

and need to prove that $\Pi_n{-}\mathsf{R}^{\alpha+1}(T) \vdash \forall \beta{<}\omega{\cdot}(\alpha{+}1)\ \varphi(\beta)$.

Our inductive hypothesis tells us

$$\Pi_n{-}\mathsf{R}^\alpha(T) \vdash \forall \beta{<}\omega{\cdot}\alpha\ \varphi(\beta)$$

so that we may conclude

$$T \vdash \Box_{\Pi_n - \mathsf{R}^\alpha(T)} \forall \beta{<}\omega{\cdot}\alpha\ \varphi(\beta) \tag{5.5}$$

From our assumption (5.4) we conclude $T {\vdash} \Box_T \forall \beta{<}\omega{\cdot}(\alpha{+}1)\ \big(\forall \gamma{<}\beta\, \varphi(\gamma) \to \varphi(\beta)\big)$ whence

$$T \vdash \Box_{\Pi_n - \mathsf{R}^\alpha(T)} \forall \beta{<}\omega{\cdot}(\alpha{+}1)\ \big(\forall \gamma{<}\beta\, \varphi(\gamma) \to \varphi(\beta)\big).$$

This can be combined with (5.5) to conclude $T \vdash \Box_{\Pi_n-\mathsf{R}^\alpha(T)}\forall\beta{<}\omega{\cdot}\alpha{+}1\ \varphi(\beta)$. As before, we now grow little by little in an elementary fashion our proof. Since it grows elementary we conclude

$$T \vdash \forall n \Box_{\Pi_n-\mathsf{R}^\alpha(T)}\forall\beta{<}\omega{\cdot}\alpha + \overline{n}\ \varphi(\beta)$$

whence by reflection (Kleene's rule) we conclude the required

$$\Pi_n{-}\mathsf{R}^{\alpha+1}(T) \vdash \forall n \forall\beta{<}\omega{\cdot}\alpha + n\ \varphi(\beta)$$

which concludes the successor case.

The **limit case** is easy but we include it for mere completeness. So, suppose for some limit ordinal λ we have

$$T \vdash \forall\beta{<}\omega{\cdot}\lambda\ \big(\forall\gamma{<}\beta\, \varphi(\gamma) \to \varphi(\beta)\big)$$

and need to prove that $\Pi_n{-}\mathsf{R}^\lambda(T) \vdash \forall\beta{<}\omega{\cdot}\lambda\ \varphi(\beta)$. To this end, we fix some fundamental sequence so that provably in T we have $\{\lambda_i\}_{i\in\omega} < \lambda$ and $\cup_{i\in\omega}\lambda_i = \lambda$. Thus, we reason in $\Pi_n{-}\mathsf{R}^\lambda(T)$ and fix $\beta_0 < \omega{\cdot}\lambda$ arbitrary. We pick $i \in \omega$ so that $\beta_0 < \omega{\cdot}\lambda_i$. Even though β_0 may be non-standard, the λ_i *is* standard so that by the IH we conclude $\Pi_n{-}\mathsf{R}^{\lambda_i}(T) \vdash \forall\beta{<}\omega{\cdot}\lambda_i\ \varphi(\beta)$. In particular $\Pi_n{-}\mathsf{R}^\lambda(T) \vdash \forall\beta{<}\omega{\cdot}\lambda_i\ \varphi(\beta)$ whence $\Pi_n{-}\mathsf{R}^\lambda(T) \vdash \varphi(\beta_0)$ as was to be shown.

□

5.5 The inclusion formalised

In this section we shall see how our previous proof can be mimicked inside EA. It should come as a surprise that the proof can be formalised in a theory as weak as EA since we used transfinite induction in our previous argument. In our formalisation, we will replace transfinite induction by a trick that was first employed by Schmerl in [20]. We call this trick *reflexive induction*. The following lemma and proof is taken from [14]

Theorem 5.5.1 (Reflexive induction). *Let T be any theory capable of coding syntax. If $T \vdash \forall\alpha\Big(\Box_T\big(\forall\beta < \dot{\alpha}\ \varphi(\beta)\big) \to \varphi(\alpha)\Big)$, then $T \vdash \forall\alpha\varphi(\alpha)$.*

Proof. We shall see that from the assumption

$$T \vdash \forall\alpha\Big(\Box_T\big(\forall\,\beta{<}\dot{\alpha}\ \varphi(\beta)\big) \to \varphi(\alpha)\Big)$$

we get $T \vdash \Box_T\forall\alpha\varphi(\alpha) \to \forall\alpha\varphi(\alpha)$ so that the conclusion $T \vdash \forall\alpha\varphi(\alpha)$ follows by Löb's Theorem.

Thus, we reason in T, pick α arbitrary, we assume $\Box_T\forall\alpha\varphi(\alpha)$, or equivalently $\Box_T\forall\theta\varphi(\theta)$, and set out to prove $\varphi(\alpha)$. But using $\Box_T\big(\forall\,\beta{<}\dot{\alpha}\ \varphi(\beta)\big) \to \varphi(\alpha)$ in the last step of the following reasoning, we clearly have

$$\begin{array}{lll}
\Box_T\forall\theta\varphi(\theta) & \to & \Box_T\forall\theta\forall\,\beta{<}\theta\ \varphi(\beta)\\
 & \to & \forall\theta\,\Box_T\forall\,\beta{<}\dot{\theta}\ \varphi(\beta)\\
 & \to & \Box_T\forall\,\beta{<}\dot{\alpha}\ \varphi(\beta)\\
 & \to & \varphi(\alpha).
\end{array}$$

□

The name *reflexive induction* is really a misnomer since the trick also works for non-wellfounded orderings and really boils down to an application of Löb's rule.

Notation 5.5.2. *For formal theories T and U, by "$T \vdash U$" we denote the Π_2-statement $\forall\theta\,\big(\Box_U\theta \to \Box_T\theta\big)$.*

Using this notation, we now state and prove the main theorem.

Theorem 5.5.3. $\mathrm{EA} \vdash \forall\alpha\,``\Pi_n\text{--}\mathsf{R}^{\alpha}(T) \vdash [T, \mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}\alpha\big)]"$.

Proof. By reflexive induction it suffices to show that

$$\mathrm{EA} \vdash \forall\alpha\Big(\Box_{\mathrm{EA}}\big(\forall\beta{<}\dot{\alpha}\ ``\Pi_n\text{--}\mathsf{R}^{\beta}(T) \vdash [T, \mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}\alpha\big)]"\big) \ \longrightarrow \\ ``\Pi_n\text{--}\mathsf{R}^{\alpha}(T) \vdash [T, \mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}\alpha\big)]"\Big).$$

To this end, we reason in EA, fix some arbitrary α, assume (this assumption is called the *Reflexive Inductive Hypothesis*)

$$\Box_{\mathrm{EA}}\big(\forall\beta{<}\dot{\alpha}\ ``\Pi_n\text{--}\mathsf{R}^{\beta}(T) \vdash [T, \mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}\beta\big)]"\big) \tag{5.6}$$

and set out to prove

$$``\Pi_n\text{--}\mathsf{R}^{\alpha}(T) \vdash [T, \mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}\alpha\big)]". \tag{5.7}$$

The first step in proving (5.7) consists of unfolding the notation defined in Notation 5.5.2. Thus, our goal is to prove

$$\forall\theta\ \Big(\Box_{[T,\mathsf{TI}^{\mathsf{R}}(\Pi_n,\omega\cdot\alpha)]}\theta \to \Box_{\Pi_n\text{--}\mathsf{R}^{\alpha}(T)}\theta\Big).$$

Whence, while continuing our reasoning in EA, we fix θ arbitrary and assume

$$\Box_{[T,\mathsf{TI}^{\mathsf{R}}(\Pi_n,\omega\cdot\alpha)]}\theta. \tag{5.8}$$

Our goal is to show $\Box_{\Pi_n\text{--}\mathsf{R}^{\alpha}(T)}\theta$. From (5.8) we find a Hilbert-style proof π that is all in T except from a single application of the rule $\mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}\alpha\big)$. We see how we can replace this to get a π' to witness $\Box_{\Pi_n\text{--}\mathsf{R}^{\alpha}(T)}\theta$. Since (5.7) is trivial in case $\alpha = 0$ we consider successor and limit cases.

In **case** $\alpha + 1$ we reason as follows. We consider some element in π is of the form $\forall\beta{<}\omega{\cdot}(\alpha+1)\ \varphi(\beta)$ with $\varphi \in \Pi_n$ that arose from an earlier entry in π of the form

$$\forall\beta{<}\omega{\cdot}(\alpha{+}1)\ \big(\forall\gamma{<}\beta\,\varphi(\gamma) \to \varphi(\beta)\big) \tag{5.9}$$

via an application of $\mathsf{TI}^{\mathsf{R}}\big(\Pi_n, \omega{\cdot}(\alpha{+}1)\big)$. Since (5.9) appears in π before any rule-application not it T we know that $\Box_T\forall\beta{<}\omega{\cdot}(\alpha{+}1)\ \big(\forall\gamma{<}\beta\,\varphi(\gamma) \to \varphi(\beta)\big)$. By provable Σ_1-completeness we obtain

$$\Box_{\mathrm{EA}}\Box_T\forall\beta{<}\omega{\cdot}(\alpha{+}1)\ \big(\forall\gamma{<}\beta\,\varphi(\gamma) \to \varphi(\beta)\big) \tag{5.10}$$

(at this particular place in the not formalised proof we used necessitation instead) and in particular $\Box_{\mathrm{EA}}\Box_T\forall\beta{<}\omega{\cdot}\alpha\ \big(\forall\gamma{<}\beta\,\varphi(\gamma) \to \varphi(\beta)\big)$. Thus, we conclude $\Box_{\mathrm{EA}}\Box_{[T,\mathsf{TI}^{\mathsf{R}}(\Pi_n,\omega\cdot\alpha)]}\forall\beta{<}\omega{\cdot}\alpha\ \varphi(\beta)$. We can apply the Reflexive Inductive Hypothesis stated in (5.6) to obtain $\Box_{\mathrm{EA}}\Box_{\Pi_n\text{--}\mathsf{R}^{\alpha}(T)}\forall\beta{<}\omega{\cdot}\alpha\ \varphi(\beta)$. We combine this with (5.10) n many times to obtain $\Box_{\mathrm{EA}}\Box_{\Pi_n\text{--}\mathsf{R}^{\alpha}(T)}\forall\beta{<}(\omega{\cdot}\alpha{+}\overline{n})\ \varphi(\beta)$. Moreover, we observe that these combinations can be uniformly elementary bounded so that

$$\Box_{\mathrm{EA}}\forall n\ \Box_{\Pi_n\text{--}\mathsf{R}^{\alpha}(T)}\forall\beta{<}(\omega{\cdot}\alpha{+}\overline{n})\ \varphi(\beta).$$

We now apply Kleene's rule in a formalised setting (see Lemma 5.4.3) to obtain

$$\Box_{\Pi_n\text{--}\mathsf{R}^{\alpha+1}(T)}\forall n\ \forall\beta{<}(\omega{\cdot}\alpha{+}n)\ \varphi(\beta),$$

which implies the required proof witness π^* of $\Box_{\Pi_n-\mathsf{R}^{\alpha+1}(T)}\forall\beta{<}\omega{\cdot}(\alpha{+}1)\ \varphi(\beta)$. We can intertwine this π^* with the original proof π that as we stipulated witnessed $\Box_{[T,\mathsf{TI}^{\mathsf{R}}(\Pi_n,\omega\cdot\alpha)]}\theta$ to obtain the required π' that witnesses $\Box_{\Pi_n-\mathsf{R}^{\alpha}(T)}\theta$. This concludes the successor case.

In **case** $\alpha=\lambda\in\mathsf{Lim}$ we reason similarly. So we consider some element in π is of the form form $\forall\beta{<}\omega{\cdot}(\lambda)\ \varphi(\beta)$ with $\varphi\in\Pi_n$ that arose from an earlier entry in π of the form

$$\forall\beta{<}\omega{\cdot}\lambda\ \big(\forall\gamma{<}\beta\,\varphi(\gamma)\to\varphi(\beta)\big) \tag{5.11}$$

via an application of $\mathsf{TI}^{\mathsf{R}}\big(\Pi_n,\omega{\cdot}\lambda\big)$. So as (5.11) appears in π before any rule applications not in T we similarly have $\Box_T\forall\beta{<}\omega{\cdot}\lambda\ \big(\forall\gamma{<}\beta\,\varphi(\gamma)\to\varphi(\beta)\big)$. To move further, we fix a fundamental sequence so that $\Box_T$"$\{\lambda_i\}_{i\in\omega}<\lambda$" and $\Box_T$"$\cup_{i\in\omega}\lambda_i=\lambda$". Since $\Box_T\forall i\ \forall\beta{<}\omega{\cdot}\lambda_i\ \big(\forall\gamma{<}\beta\,\varphi(\gamma)\to\varphi(\beta)\big)$, it should be clear that then $\Box_{\mathrm{EA}}\forall i\ \Box_T\forall\beta{<}\omega{\cdot}\lambda_{\bar{i}}\ \big(\forall\gamma{<}\beta\,\varphi(\gamma)\to\varphi(\beta)\big)$. Thus we conclude $\Box_{\mathrm{EA}}\forall i\ \Box_{[T,\mathsf{TI}^{\mathsf{R}}(\Pi_n,\omega\cdot\lambda_{\bar{i}})]}\forall\beta{<}\omega{\cdot}\lambda_{\bar{i}}\ \varphi(\beta)$ which using the induction hypothesis (5.6) gives

$$\Box_{\mathrm{EA}}\forall i\ \Box_{\Pi_n\text{-}R^{\lambda_{\bar{i}}}}\forall\beta{<}\omega{\cdot}\lambda_{\bar{i}}\ \varphi(\beta).$$

In the interest of using the fundamental sequence as before, we move over to T, where we summarize our progress as

$$\Box_T\forall\,\beta_0{<}\omega{\cdot}\lambda\ \exists\,i\ \big(\beta_0{<}\lambda_i\wedge\Box_{\Pi_n\text{-}R^{\lambda_{\bar{i}}}(T)}\forall\,\beta{<}\omega{\cdot}\lambda_{\bar{i}}\ \varphi(\beta)\big).$$

Next we simply involve β_0 as in $\Box_T\forall\,\beta_0{<}\omega{\cdot}\lambda\ \exists\,i\ \big(\beta_0{<}\lambda_i\wedge\Box_{\Pi_n\text{-}R^{\lambda_{\bar{i}}}(T)}\varphi(\overline{\beta_0})\big)$ and proceed by applying Kleene's rule in a formalized setting (see Lemma 5.4.3) to arrive at

$$\Box_{\Pi_n\text{-}R^{\lambda}(T)}\forall\,\beta_0{<}\omega{\cdot}\lambda\ \exists\,i\ \big(\beta_0{<}\lambda_i\wedge\varphi(\beta_0)\big),$$

which simplifies into $\Box_{\Pi_n\text{-}R^{\lambda}(T)}\forall\,\beta_0{<}\omega{\cdot}\lambda\ \varphi(\beta_0)$, with a proof witness π^*. As before, we intertwine this π^* with the original proof π to obtain the required π', witnessing $\Box_{\Pi_n\text{-}R^{\alpha}(T)}\theta$.

$\Box$

The inclusion $\Pi_n-\mathsf{R}^{\alpha+1}(T)\subseteq[T,\mathsf{TI}^{\mathsf{R}}\big(\Pi_n,\omega{\cdot}(1{+}\alpha)\big)]$, we omit and we will present it on future work. The lack of its presence doesn't affect the rest of this thesis.

Chapter 6

Worm battles

Here we shall present a simple statement of combinatorial nature that is true yet independent of Peano Arithmetic and is motivated by the corresponding provability point of view. The name of this statement stems from its similarities to the well known Hercules and Hydra game of Kirby and Paris where here worms are the analogues to hydras [15].

6.1 Worm Battle in Action

The game deals with combinatoric worms of $\mathbb{W}^{\Lambda}$, which are finite words in the alphabet of Λ. So, for instance, $\mathbb{W}^{n}$ includes the modalities $0, 1, \ldots n-1$. For the time being, we will assume that $\Lambda \leq \omega$, a restriction which we will later lift as we will expand the game to consider limit ordinals in its rules.

In the original versions, as it can be seen in [2], combinatoric worms grew to the right, contrary to the provability worms which grew towards the left. To avoid the confusion this would create, we shall have the combinatoric worms to be expanding towards the left as well.

Definition 6.1.1. *By the head of a worm $w = x_n \ldots x_1 x_0$ we will refer to its leftmost element x_n.*
We consider the function $\mathrm{next}(w, m)$, *where $w = x_n \ldots x_1 x_0$ is a worm of $\mathbb{W}^{\omega}$ and $m \in \mathbb{N}$ is a step of the game.*

- *If w is empty, then* $\mathrm{next}(w, m) := w$.
- *If x_n=0 then* $\mathrm{next}(w, m) := x_{n-1} \ldots x_0$. *So in this case, the head of the worm is being cut away regardless of the value of m.*
- *If x_n is a successor ordinal, let $k := \max\{i < n : x_i < x_n\}$ where for completeness we may define $\max \varnothing = -1$. Then the worm w, with its head decreased by 1, consists of two parts: the good one $r := x_k \ldots x_0$, which may be empty, and the bad one $s := (x_n - 1)x_{n-1} \ldots x_{k+1}$. Set*

$$\mathrm{next}(w, m) := \underbrace{s * \ldots * s * s}_{m+1 \text{ times}} * r.$$

Finally, the sequence of worms starting from an arbitrary worm $w \in \mathbb{W}^{\omega}$ is defined as follows:

$$w_0 := w \quad \textit{and} \quad w_{n+1} := \mathrm{next}(w_n, n+1).$$

Let us take as an example the worm $w = 2312$. This means that we start with $w_0 = 2312$. For the first step, since the head is a successor, we have $k = 1$, the good part $r = 12$ and $s = 13$. This then gives:

$$\begin{aligned}
w_0 &= 2312 \\
w_1 &= 131312 \\
w_2 &= 031312031312031312 \\
w_3 &= 31312031312031312 \\
w_4 &= 222221312031312031312 \\
w_5 &= (12222)^6 1312031312031312 \\
&\ldots
\end{aligned}$$

Where $(12222)^6$ denotes the 6-fold concatenation of 12222. Notice that w_n is defined by primitive recursion from w and n. In fact, w_n is an elementary function of n and the code of w. This can be seen by the estimate

$$|w_n| \leq (n+1)!\,|w_0|\,,$$

which follows from a simple induction: For $n = 0$, it holds; then assuming it holds for n, the only increase in length for w_{n+1} when compared to w_n can be performed if the leftmost element of w_n is a successor ordinal. Then, no matter what, $w_{n+1} \leq (n+2)w_n$ and by the induction hypothesis, we prove the above estimate.

This shows that the length of each worm only grows at an elementary pace in the course of the game. Also notice that the maximal element of each worm can only decrease, meaning that $\mathrm{next}\restriction(\mathbb{W}^\alpha \times \mathbb{N})$ ranges over $\mathbb{W}^\alpha$ for every $\alpha \leq \omega$. Additionally we can write an elementary formula in three variables expressing that $w_n = u$.

Definition 6.1.2. *By* EWD^Λ*, we will express the statement that every worm of* $\mathbb{W}^\Lambda$ *dies, formally:*

$$\forall\, w{\in}\mathbb{W}^\Lambda\ \exists n\ w_n{=}\mathrm{e}\,.$$

Simply by EWD *we will denote* EWD^ω*.*

Where by e we denote the empty word. Clearly, from the definitions we have followed thus far, there is a natural translation f between combinatoric and the arithmetical interpretation of proof theoretic worms such that $f(x_n \ldots x_0) = (\langle x_n \rangle \ldots \langle x_0 \rangle \top)^*$. Then the functions $\mathrm{next}(w, n)$ and $A[\![n]\!]$ can be used interchangeably as $f(\mathrm{next}(w, n)) = (f(w)[\![n]\!])^*$. This function is clearly definable in EA and as such, we shall notationally use the provability notation with the n-th worm being denoted as A_n. We now ought to point out that we have worms as worms and worms interpreted in arithmetic. For the latter, we will opt to omit the notation of the arithmetical interpretation of worms (A^*) for the sake of clarity and we therefore expect that the context alone should make it clear what reading is intended.
At this point, we are equipped to state the result that forms the starting point of this master thesis.

Theorem 6.1.3 (Beklemishev, 2005)**.** EWD *is equivalent to* $1\text{-}\mathrm{Con}(PA)$ *in* EA.

We will opt for a direct proof of this fact as per [2] that will not make use of the transfinite induction rules found in Chapter 5. As such, the next two sections will closely follow [2]. The proof will be divided into two parts.

6.2 Truth of the worm principle

We will start with the following direction:

Proposition 6.2.1. $\mathrm{EA}+1\text{-Con}(\mathrm{PA}) \vdash \mathrm{EWD}$

The Lemmata that we will use to prove this statement can be seen as split into two categories. The first is to translate the strength of the underlying theory (PA).

Lemma 6.2.2. *For any $A \in \mathbb{W}^\omega$, $\mathrm{PA} \vdash A$.*

Proof. For every $A \in \mathbb{W}^\omega$ there is m such that $A \in \mathbb{W}^m$ and thus, by Proposition 4.1.9,

$$\mathrm{GLP} \vdash \langle m \rangle \top \to A.$$

Therefore by arithmetical soundness of GLP, it holds $\mathrm{EA} \vdash \langle m \rangle \top \to A$ and since $\mathrm{PA} \vdash \langle n \rangle \top$ for every natural number n, the lemma follows and it is formalizable in EA^+ due to Corollary 3.3.3. $\square$

The remaining two Lemmata do not make use of PA and are more focused around the structure of worms themselves. For what follows, we denote by $A^+ := 1{\uparrow}A$ –the 1-right shift of A.

Lemma 6.2.3. *For any $A \in \mathbb{W}^\omega$,*

$$\mathrm{EA} \vdash \forall k \ \big(A_k \neq \mathrm{e} \to \Box(A_k^+ \to \langle 1 \rangle \, A_{k+1}^+)\big).$$

Proof. It is sufficient to prove in EA

$$\forall A \neq \mathrm{e} \ \ \forall k \ \ \mathrm{EA} \vdash A^+ \to \langle 1 \rangle \, A[\![k]\!]^+.$$

For this, we will move over to GLP where we have the size of the following proof is bounded by an elementary function and hence it is formalizable in EA,

$$\mathrm{GLP} \vdash A \to \Diamond A[\![k]\!].$$

As theorems of GLP are stable under right shift,

$$\mathrm{GLP} \vdash A^+ \to \langle 1 \rangle \, A[\![k]\!]^+,$$

which by the arithmetical soundness of GLP, we have for every $A \in \mathbb{W}^\omega$ and every $k \in \mathbb{N}$, a proof of:

$$\mathrm{EA} \vdash A^+ \to \langle 1 \rangle A[\![k]\!]^+.$$

From here, we are of course unable to use Σ_1-induction to prove

$$\mathrm{EA} \vdash \forall k \ \big(A_k \neq \mathrm{e} \to \Box(A_k^+ \to \langle 1 \rangle \, A_{k+1}^+)\big),$$

which is how we would –in principle– expect to complete the proof. Instead we utilize the fact that for a given k,[1] the proof of the claim is bounded by an elementary function of k. The proof itself can be formalized within EA and therefore the formula $\Box(A_k^+ \to \langle 1 \rangle \, A_{k+1}^+)\big)$ can be written as a Δ_0-formula by placing the existential quantifier inside this bound. So we complete the proof with a Δ_0-induction. $\square$

[1]Note that A is external while k is given inside EA.

The Lemma we just proved, we use in turn to prove the Lemma that follows, which allows us to move from combinatoric worms into their provability counterparts. This Lemma is the backbone of the proof that we are providing and its presentation here follows closely the one in [2].

Lemma 6.2.4. *For any $A \in \mathbb{W}^\omega$,*

$$\mathrm{EA} \vdash \langle 1 \rangle\, A_0^+ \to \exists\, m\ A_m{=}\,\mathrm{e}\,.$$

Proof. In particular, this will yield the required $\mathrm{EA} \vdash \forall\, m\ A_m{\neq}\,\mathrm{e} \to [1]\neg A_0^+$. The first part of our reasoning will prepare for an application of Löb's theorem.

$$\begin{aligned}
\mathrm{EA} \vdash \forall\, m\ A_m{\neq}\,\mathrm{e}\ \wedge [1]\,\forall\, m\ [1]\neg A_m^+ &\to [1]\,\forall\, m\ [1]\neg A_{m+1}^+ \\
&\to \forall\, m\ [1][1]\neg A_{m+1}^+ \\
&\to \forall\, m\ [1]\neg A_m^+, \quad \text{by Lemma 6.2.3} \\
\mathrm{EA} \vdash \forall\, m\ A_m{\neq}\,\mathrm{e} \to ([1]\,\forall\, m\ [1]\neg A_m^+ &\to \forall\, m\ [1]\neg A_m^+).
\end{aligned}$$

Then, after necessitation on the [1]-modality and distribution we have,

$$\begin{aligned}
\mathrm{EA} \vdash [1]\,\forall\, m\ A_m{\neq}\,\mathrm{e} &\to [1]([1]\,\forall\, m\ [1]\neg A_m^+ \to \forall\, m\ [1]\neg A_m^+) \\
&\to [1]\,\forall\, m [1]\neg A_m^+, \quad \text{by Löb's Theorem.}
\end{aligned}$$

Now observe that $\forall\, m\ A_m{\neq}\,\mathrm{e}$ is Π_1, so certainly Σ_2 and hence,

$$\begin{aligned}
\mathrm{EA} \vdash \forall\, m\ A_m{\neq}\,\mathrm{e} &\to [1]\,\forall\, m\ A_m{\neq}\,\mathrm{e}, \quad \text{by } \Sigma_2\text{-completeness} \\
&\to [1]\,\forall\, m\ [1]\neg A_m^+ \\
&\to \forall\, m\ [1][1]\neg A_m^+ \\
&\to \forall\, m\ [1]\neg A_m^+ \\
&\to [1]\neg A_0^+.
\end{aligned}$$

By contraposition this proves the lemma. □

Now from Lemmata 6.2.2 and 6.2.4 we obtain that for each $A{\in}\mathbb{W}^\omega$,

$$\begin{aligned}
&\mathrm{PA} \vdash \langle 1 \rangle\, A^+ \\
&\mathrm{EA} \vdash \langle 1 \rangle\, A^+ \to \exists\, m\ A_m{=}\,\mathrm{e}
\end{aligned}$$

Hence, $\forall\, A{\in}\mathbb{W}^\omega\ \ \mathrm{PA} \vdash \exists\, m\ A_m{=}\,\mathrm{e}$. This is formalizable in EA^+, therefore $\mathrm{EA}^+ + 1\text{-}\mathrm{Con}(\mathrm{PA}) \equiv \mathrm{EA} + 1\text{-}\mathrm{Con}(\mathrm{PA})$ implies $\forall\, A{\in}\mathbb{W}^\omega\ \exists\, m\ A_m{=}\,\mathrm{e}$, which is EWD.[2]

6.3 Independence of the worm principle

Now we turn our attention into proving the converse:

Proposition 6.3.1. $\mathrm{EA} + \mathrm{EWD} \vdash 1\text{-}\mathrm{Con}(\mathrm{PA})$.

[2]In [2] there is an error stating that this is formalizable in EA however that is not correct as EA^+ is required to formalize $\mathrm{PA} \vdash \langle n \rangle \top$ for every n, that is found in Lemma 6.2.2. The correction is fairly immediate.

Here we will follow closely the exposition of [2] and start by introducing the following notions found there. Remember we use $A[\![m]\!]$ as an alternative notation for $\text{next}(A,m)$. Let

$$A[\![m\ldots m{+}k]\!] := A[\![m]\!]\ldots[\![m{+}k]\!].$$

In the proof of Proposition 6.3.1, we will use an analogue of the so-called Hardy functions [19]: Let $h_A(m)$ be[3] the smallest k such that $A[\![m\ldots m{+}k]\!]{=}\,\mathrm{e}$.

As we shall see, the function h_v has some good properties in terms of monotonicity, which we will establish via elementary reasoning in EA. Notice that since $A[\![m\ldots m{+}k]\!]$ is defined by bounded recursion in a way similar to A_n, it is elementary and hence there is a natural elementary presentation of $h_A(m)=k$ in EA.

Given worms $B, A \in \mathbb{W}^\omega$, we define the ordering

$$B \trianglelefteq A \text{ iff } B = A[\![0]\!]\ldots[\![0]\!] \text{ for a finite number of iterations.}$$

This is equivalent to the statement: "B is an initial segment of A apart from possibly the first element which should then be smaller". Let us give an explicit proof of this fact:

Remark 6.3.2. *Let A, B be arbitrary worms.*
$\mathrm{EA} \vdash B \trianglelefteq A \;\longleftrightarrow\;$ "$A = D\langle n\rangle C \,\wedge\, B = \langle m\rangle C \,\wedge\, m \leq n$".

Proof. Let $\varphi(B,A)$ be the formula expressing "$\exists\, n\, \exists\, C\, \exists\, D\ (A = D\langle n\rangle C \wedge B = \langle m\rangle C \wedge m \leq n)$". All of these quantifications are bounded since the code of a substring is less than the code of the entire string and so φ is a Δ_0-formula. Additionally, for the purposes of this proof we will denote by,

$$A[\![0]\!]^n := A\,\underbrace{[\![0]\!]\ldots[\![0]\!]}_{n\text{-times}},$$

with $A[\![0]\!]^0 := A$. We will now divide the proof into two parts, proving each of the implications separately.

To start, we prove that $\mathrm{EA} \vdash \forall A \forall n\ \varphi(A[\![0]\!]^n, A)$ in order to prove the $(\rightarrow)$ direction. Reasoning informally within EA, fix some A and we prove this by induction on n.

If $n=0$, then $A[\![0]\!]^0 = A$ and is thus a subsequence of A. Therefore we have $\varphi(A[\![0]\!], A)$.

Assume now that it holds for n, then from the rules of the game and by the definition of φ, it should be clear that $\varphi(A[\![0]\!]^n[\![0]\!], A[\![0]\!]^n)$. By the induction hypothesis, we additionally have $\varphi(A[\![0]\!]^n, A)$. We can then prove within EA that φ has a transitivity property, which leads us to proving the induction step $\varphi(A[\![0]\!]^{n+1}, A)$.

The other direction, we will divide further into three parts.
First we prove

$$\mathrm{EA} \vdash \forall B \forall n\ \big((\langle n\rangle B)[\![0]\!]^{n+1} = B\big), \tag{6.1}$$

by a simple induction on n.

If $n=0$ then by the definition of the next function, $(\langle 0\rangle B)[\![0]\!] = B$.

Assume now that the statement holds for n, then $(\langle n+1\rangle B)[\![0]\!]^{n+2} = (\langle n\rangle B)[\![0]\!]^{n+1} = B$, by the induction hypothesis.

[3] Confusion with the h_α and h function from Definition 4.2.1 is not possible due to different types of arguments.

Now we prove the similar, and in a sense, more generalized statement:

$$\mathrm{EA} \vdash \forall B \, \forall n \, \forall m{\le}n \, \big((\langle n \rangle B)[\![0]\!]^m = \langle n - m \rangle B \big). \tag{6.2}$$

By induction on n, we have:

If $n = 0$, then it clearly holds as $(\langle 0 \rangle B)[\![0]\!]^0 = \langle 0 \rangle B$. So assume that it holds for n. Now if $m = 0$, then the statement clearly holds. Otherwise, let $m = k + 1$ from where we have $(\langle n+1 \rangle B)[\![0]\!]^{k+1} = (\langle n \rangle B)[\![0]\!]^k = \langle n - k \rangle B$ by induction hypothesis, since $k \le n$ and by some basic arithmetic, $\langle n - k \rangle B = \langle n{+}1 - m \rangle B$.

Now for the second implication, we prove

$$\mathrm{EA} \vdash \forall A \, \forall B \, \big(\varphi(B, A) \to \exists n \; B = A[\![0]\!]^n \big).$$

Of course here both the quantifications on B and n are bounded. Fix some A and let us denote $A = \alpha_{|A|-1} \ldots \alpha_0$ and $B = \beta_{|B|-1} \ldots \beta_0$. We will prove the statement by induction on $|A| - |B|$. Notice that this induction is bounded as A is fixed and the B are such that $\varphi(B, A)$.

If $|A| - |B| = 0$, then by (6.2), the claim holds.

Assume now that it holds for all B with $\varphi(B, A)$ and $|A| - |B| = k$. Then let B be such that $|A| - |B| = k + 1$. Let C be such that $C = \alpha_{|C|-1} D$, it is a subsequence of A as $\varphi(C, A)$, and $|A| - |C| = k$. By the way C is defined, we can derive that $\varphi(B, C)$. Then by applying the induction hypothesis on C, we have $\exists n_1 \; C = A[\![0]\!]^{n_1}$. By (6.1), we then have that $\exists n_2 \; D = A[\![0]\!]^{n_2}$ and finally, via (6.2), we conclude with $\exists n_3 \; B = A[\![0]\!]^{n_3}$. $\square$

Therefore, either definition of said relation can be used to provide a natural representation of it in EA. We shall use this relation to prove that much desired monotonicity for the h_B function. Notice that as the h_A functions are defined by a formula, it isn't necessary that they will halt everywhere within EA. For what follows, when we say that $h_A(m)$ is defined, what we mean is, that the entire statement is to be understood within EA.

For worms of $\mathbb{W}^\omega$ we have the following results within EA:

Lemma 6.3.3. *If $h_A(m)$ is defined and $B \trianglelefteq A$, then*

$$\exists k \; A[\![m \ldots m{+}k]\!] = B.$$

Proof. The rules of the game are such that the element of index i in A can only change if all letters to the left of it are deleted. To prove this claim, assume that for a given $i < |A|$, there exists some k' such that the element of index i in $A[\![m \ldots m{+}k_i]\!]$, differs from the element of index i in A. Then let k_i be the least element with that property and call $B_i = A[\![m \ldots m{+}k_i{-}1]\!]$, where if $k_i = 0$ then we instead just take $B_i = A$. By the definition of the next function, we have that $|B_i| = i + 1$ as by the definition of B_i, it cannot be smaller than that. If it's larger, then element with index i of B_i will be unaffected and so again, by the definition of B_i, this cannot be the case either.

In addition, B_i is an initial segment of A. To prove this, assume that this isn't the case and so there is some index $j < i$ on which B_i differs from A. Then there is some $k_j < k_i$ (strictly smaller since only one element is altered per step) and B_j, defined as above for the index j. So $|B_j| = j{+}1 < i{+}1$ which contradicts with the definition of k_i as the least with its properties.

Now, given the assumption that $A[\![m \ldots m{+}s]\!] = \mathrm{e}$ for some given s, and some $B \trianglelefteq A$, we will show that there is some $k < s$ such that $A[\![m \ldots m{+}k]\!] = B$. Clearly, this holds in the trivial case of $B = \top$. Assume now that $B = \langle n \dot{-} l \rangle C$ and $A = D \langle n \rangle C$ for some n, l.

We will perform Δ_0-bounded-induction on l over the formula: "there is some $k < s$ such that $A[\![m \ldots m{+}k]\!] = \langle n \dot{-} l\rangle C$". From what we have proved so far, the case of $l = 0$ should be clear. Assume now that it holds for l. By the induction hypothesis, there is some $k < s$ such that $A[\![m \ldots m{+}k]\!] = \langle n \dot{-} l\rangle C$ and so $A[\![m \ldots m{+}k{+}1]\!] = D'\langle n \dot{-} (l{+}1)\rangle C$ for some D', determined by the next function. Then we can apply what we did at the start of this proof for $D'\langle n \dot{-} (l{+}1)\rangle C$, getting some k_1 such that $A[\![m \ldots m{+}k{+}k_1]\!] = \langle n \dot{-} (l{+}1)\rangle C$. □

This lemma we can then easily expand into the following:

Corollary 6.3.4. *If $h_A(n)$ is defined and $B \trianglelefteq A$, then $\forall\, m{\leq}n\ \exists\, k\ A[\![n \ldots n{+}k]\!]{=}B[\![m]\!]$.*

Proof. By Lemma 6.3.3, there is k_1 such that $A[\![n \ldots n{+}k_1]\!] = B$. Then, $B[\![m]\!]$ is an initial segment of $A[\![n \ldots n{+}k_1{+}1]\!]$ and so once again, the rules of the game dictate via Lemma 6.3.3 that since $h_A(n)$ halts, then there is some k_2 such that $A[\![n \ldots n{+}k_1{+}k_2]\!] = B[\![m]\!]$. □

Using this result, we have the following monotonicity statement:

Lemma 6.3.5. *If $h_A(y)$ is defined, $B \trianglelefteq A$ and $x \leq y$, then $h_B(x)$ is defined and $h_B(x) \leq h_A(y)$.*

Proof. By applying the Corollary 6.3.4 several times, we obtain $s_0, s_1, \ldots$ such that

$$A[\![y \ldots y + s_0]\!] = B[\![x]\!], \quad \text{where } y + s_0 \geq x$$
$$A[\![y \ldots y + s_0 + s_1]\!] = B[\![x]\!][\![x + 1]\!], \quad \text{where } y + s_0 + s_1 \geq x + 1$$
$$\ldots$$

Hence all elements of the sequence starting with B occur in the sequence for A and since $h_A(y)$ is defined, so is $h_B(x)$. We remark that at the end of this procedure, we will have $s_0, \ldots s_n$ with $n \leq h_A(y)$ and $A[\![y \ldots y{+}s_0{+}\ldots{+}s_n]\!] = B[\![x \ldots x{+}n]\!] = \mathrm{e}$. Therefore, $n = h_B(x) \leq h_A(y)$. □

Next we have two monotonicity results on only the natural number ordering that follow from Lemma 6.3.5:

Lemma 6.3.6. *If $h_{B0A}(n)$ is defined, then $h_{B0A}(n) = h_A(n + h_B(n) + 2) + h_B(n) + 1 > h_A(h_B(n))$, with all of those functions on their respective points, defined.*

Proof. Since $0A \trianglelefteq B0A$, by lemma 6.3.3 we have that h_B halts. As $B0A$ first rewrites itself to $0A$ in $h_B(n)$ steps and then begins to rewrite A into e at step $n + h_B(n) + 2$, we have that $h_A(n + h_B(n) + 2)$ is then defined. Finally, by Lemma 6.3.5, $h_A(h_B(n)) \leq h_A(n + h_B(n) + 2)$ and it is also defined. □

Seeing how easy it is to achieve a lower bound based on the composition of functions, we can proceed by trying to get in-series iterations of this. Since the h_A functions are in-general strictly monotonous, we will be getting faster and faster growing functions by following this method.

Corollary 6.3.7. *If $A \in \mathbb{W}_1$ and $h_{1A}(n)$ is defined, then $h_{1A}(n) > h_A^{(n)}(n)$.*

Proof. Since $(1A)[\![n]\!]{=}0A0A \ldots 0A$, we can perform induction on the number of in-series concatenations of $0A$ by applying Lemma 6.3.6. □

As an application of this, we can see how quickly we can reach a growth similar to that of the super-exponentiation function.

Corollary 6.3.8. *If $h_{1111}(n)$ is defined then, $h_{1111}(n) > 2^n_n$ and $h_{111}(n) > 2^n$.*

Proof. We will make use of Corollary 6.3.7 multiple times. Clearly we first have that $h_{1111}(n) > h_{111}^{(n)}(n)$, then $h_{111}(n) > h_{11}^{(n)}(n)$ and $h_{11}(n) > h_1^{(n)}(n)$. We can easily prove by induction on EA that $h_1(n) = n + 1$. So by applying the compositions, $h_{11}(n) > 2n$ and so $h_{111}(n) > 2^n$ and finally $h_{1111} > 2^n_n$. $\Box$

At this point we find ourselves equipped to tackle the main lemma on which the proof of Proposition 6.3.1 rests. In its proof, we closely follow the steps taken in [2]:

Lemma 6.3.9. $\mathrm{EA} \vdash \forall\, A{\in}\mathbb{W}_1\ (h_{A1111}{\downarrow}\ \to \langle 1\rangle\, A)$.

Proof. By Löb's Theorem, this is equivalent to proving

$$\mathrm{EA} \vdash \Box\big(\forall\, A{\in}\mathbb{W}_1\ (h_{A1111}{\downarrow}\ \to \langle 1\rangle\, A)\big) \to \forall\, A{\in}\mathbb{W}_1\ (h_{A1111}{\downarrow}\ \to \langle 1\rangle\, A).$$

We reason in EA. Let us take the antecedent as an additional assumption, which by the monotonicity axiom of GLP interpreted in EA, implies $[1]\big(\forall\, A{\in}\mathbb{W}_1\ (h_{A1111}{\downarrow}\ \to \langle 1\rangle\, A)\big)$. This in turn implies:

$$\forall\, A{\in}\mathbb{W}_1\ [1](h_{A1111}{\downarrow}\ \to \langle 1\rangle\, A). \tag{6.3}$$

We make a case distinction whether $A1111$ starts with a 1 or with an element strictly larger than 1.
If $A1111 = 1B$ then by Corollary 6.3.7, we have $h_{1B}{\downarrow}\ \to \lambda x.h_B^{(x)}(x){\downarrow}$. The function h_B is increasing, has an elementary graph and grows at least exponentially as by Corollary 6.3.8 we know that $h_{111} > 2^x$. So for $A{=}\mathrm{e}$ we have that $h_{1111}{\downarrow}$ implies the totality of 2^x_n and hence EA^+ which by Corollary 3.1.3 implies $\langle 1\rangle\top$. If A is nonempty, we reason as follows:

$$\begin{aligned} \lambda x.h_B^{(x)}{\downarrow} &\to \langle 1\rangle\, h_B{\downarrow}, && \text{by Lemma 3.1.2} \\ &\to \langle 1\rangle\, \langle 1\rangle\, B, && \text{by Assumption (6.3)} \\ &\to \langle 1\rangle\, A. \end{aligned}$$

If $A1111 = C$ starts with $m > 1$, then

$$\begin{aligned} h_C{\downarrow} &\to \lambda x.h_{C[\![x]\!]}(x+1){\downarrow} \\ &\to \forall\, n\ h_{C[\![n]\!]}{\downarrow}. \end{aligned}$$

The last implication is derived by application of Lemma 6.3.5 as for arbitrary n, if $x \leq n$ then $h_{C[\![n]\!]}(x) \leq h_{C[\![n]\!]}(n+1)$ and if $n \leq x$ then $h_{C[\![n]\!]}(x) \leq h_{C[\![x]\!]}(x+1)$. In both cases, the larger value is defined.
Continuing from here:

$$\begin{aligned} \forall\, n\ h_{C[\![n+1]\!]}{\downarrow} &\to \forall\, n\ h_{1C[\![n]\!]}{\downarrow} && (\text{as } 1C[\![n]\!] \trianglelefteq C[\![n+1]\!]) \\ &\to \forall\, n\ \lambda x.h_{C[\![n]\!]}^{(x)}(x){\downarrow} \\ &\to \forall\, n\ \ \langle 1\rangle\, h_{C[\![n]\!]}{\downarrow} && \text{by Lemma 3.1.2.} \end{aligned}$$

Observe that since A starts with something bigger than 1, we have $C[\![n]\!] = A[\![n]\!]1111$, hence we can apply our assumption. Hence the argument continues,

$$\begin{aligned} \forall\, n\ \ \langle 1\rangle\, h_{C[\![n]\!]}{\downarrow} &\to \forall\, n\, \langle 1\rangle\, (\langle 1\rangle\, A[\![n]\!]) && \text{by Assumption (6.3)} \\ &\to \forall\, n (\langle 1\rangle\, A[\![n]\!]) \\ &\to \langle 1\rangle\, A && (\text{by the reduction property -- Corollary 4.2.7}). \end{aligned}$$

The last step is achieved because $h_C{\downarrow}$ implies $h_{1111}{\downarrow}$ which, as per our first step in this proof, implies EA^+, hence allowing the use of the reduction property. $\Box$

Now to prove Proposition 6.3.1, assume that EWD holds. We have:

$$\begin{aligned}\mathrm{EA} \vdash \forall\, A{\in}\mathbb{W}^{\omega}\ \exists\, m\ A_m{=}\mathrm{e} &\to \forall\, A{\in}\mathbb{W}^{\omega}_1\ h_A{\downarrow}\\ &\to \forall\, n\ \ \langle 1\rangle\,\langle n\rangle\top\\ &\to 1\text{-}\,\mathrm{Con}(\mathrm{PA}).\end{aligned}$$

The first implication holds since for every worm A and every number x, there is a worm $A' = 0^x A$ where $A'[\![0 \ldots x-1]\!] = A$ hence $\exists\, m\ A'_m{=}\mathrm{e}$ iff $h_A(x)$ is defined.

Chapter 7

Ramified worm battles

In the previous chapter we have exposed a combinatoric principle unprovable in PA. As it turned out the unprovable principle was equivalent to the 1-consistency of Peano Arithmetic. In this chapter we wish to see if we can obtain similar results for fragments of PA and we shall focus on the theories $\mathrm{I}\Sigma_n$. Some minor adjustment to the worm battle can be made to obtain candidate principles. As we inspect the $\mathrm{I}\Sigma_n$-subtheories of PA, we will maintain the same next function and the only restriction will be done on the worms we will quantify over. One would expect us to end up with:

$$\mathrm{EA} + 1\text{-}\mathrm{Con}(\mathrm{I}\Sigma_n) \equiv \mathrm{EA} + \mathrm{EWD}^{n+1}\,.$$

That is, an equivalence with the worm principle for worms with their alphabet consisting of numbers $m < n{+}1$. Due to this fact, we can easily use results obtained by previous Lemmata in the proof of Theorem 6.1.3 that do not make use of PA. However, as we shall see, in converting the existing proof, we shall end up with a weaker result.

7.1 1-Con($\mathrm{I}\Sigma_n$) proving a worm principle.

As we announced, we cannot prove the expected equivalence between 1-consistency and the corresponding worm principle. In particular, the direction from consistency to the worm principle is in its current form not in phase with what we would have expected. Specifically, following the proof structure of the previous chapter, the result we can prove is:

Proposition 7.1.1. $\mathrm{EA} +$ 1-Con($\mathrm{I}\Sigma_n$) $\vdash \mathrm{EWD}^n$.

As before, we will denote by $A^+ := 1{\uparrow}A$.

Lemma 7.1.2. *For any $A{\in}\mathbb{W}^{n+1}$, $\mathrm{I}\Sigma_n \vdash A$.*

Proof. By Proposition 4.1.9 we have for every $A \in \mathbb{W}^{n+1}$,

$$\mathrm{GLP} \vdash \langle n+1\rangle\top \to A.$$

Therefore, by arithmetical soundness of GLP, it holds $\mathrm{EA} \vdash \langle n+1\rangle\top \to A$ and since $\mathrm{I}\Sigma_n \vdash \langle n+1\rangle\top$, the lemma follows and its proof is formalizable in EA^+. □

Now from the Lemmata 7.1.2 and 6.2.4 we obtain that for each $A \in \mathbb{W}^n$,

$$\begin{aligned}\mathrm{I}\Sigma_n &\vdash \langle 1\rangle\, A^+ \\ \mathrm{EA} &\vdash \langle 1\rangle\, A^+ \to \exists\, m\;A_m{=}\mathrm{e}\,.\end{aligned}$$

Hence, $\forall A \in \mathbb{W}^n \;\; \mathrm{I}\Sigma_n \vdash \exists m \; A_m = \mathrm{e}$. This is formalizable in EA^+, therefore $\mathrm{EA} + $ 1-Con($\mathrm{I}\Sigma_n$) implies $\forall A \in \mathbb{W}^n \; \exists m \; A_m = \mathrm{e}$, which is EWD^n.

Of course, we cannot strengthen Lemma 7.1.2 further to obtain the desired result. This leads us to question whether we can get rid of the 1-right shift on the worms of Lemma 6.2.4. On examining the rest of the proof, we notice that Lemma 6.2.4 makes use of Σ_2-completeness to envelop the Π_1-sentence $\forall m \; A_m \neq \mathrm{e}$ within the 1-provability. In turn, since we make use of Löb's Theorem, this demands from us to use the same modalities and we cannot weaken the resulting sentence into something like $[1]\forall m \Box\neg A_m$. The reason being that then, if we are to follow the proof, we would have:

$$\begin{aligned} \mathrm{EA} \vdash \forall m \; A_m \neq \mathrm{e} \;\wedge\; [1]\,\forall m \; \Box\neg A_m &\to [1]\,\forall m \; \Box\neg A_{m+1} \\ &\to \forall m \; [1]\Box\neg A_{m+1} \\ &\to \forall m \; [1]\neg A_m. \end{aligned}$$

The last step is done via an easy modification of Lemma 6.2.3 and an application of the monotonicity axiom of GLP, (J1) at the end. However the problem becomes that we are left with what is not an instance of Löb's Theorem. Hence why the right shift is required in this proof and we cannot easily strengthen it.

7.2 1-Con($\mathrm{I}\Sigma_n$) proving a worm principle via transfinite induction

If we are to instead turn towards the results we got in Chapter 5, we shall notice that the strength of the theorem will not be altered. By taking this direction, we opt to provide a more indirect proof of the theorem which we can reach whether we choose to use the earlier results of Chapter 5 or the later ones. If we are to utilize theorem 5.1.2, we have that for every $n>0$, $\mathrm{EA} + 1\text{-}\mathrm{Con}(\mathrm{I}\Sigma_n)$ contains $[\mathrm{EA}, \mathrm{TI}^R(\Pi_2, <_0\restriction\mathbb{W}^n)]$. Similarly from Fedor's more generalized suggestion we shall reach a similar result.

Specifically, we will use the following Lemma found in [1] and [13] which states:

Lemma 7.2.1. *Let T be a c.e. extension of EA^+ whose axioms have logical complexity of Π_{n+1}. Then for every worm $A \in \mathbb{W}_n$, we have provably in EA^+ that*

$$T + A_T \equiv_n \Pi_{n+1}\text{-}R^{o(n\downarrow A)}(T).$$

Proof. We will prove this by a direct reflexive induction, as presented in Theorem 5.5.1, on A. So we will prove,

$$\mathrm{EA}^+ \vdash \forall B <_n A \; \Box_{\mathrm{EA}^+}\big(\text{“}T + B_T \equiv_n \Pi_{n+1}\text{-}R^{o(n\downarrow B)}(T)\text{”}\big) \to \text{“}T + A_T \equiv_n \Pi_{n+1}\text{-}R^{o(n\downarrow A)}(T)\text{”}.$$

Reasoning within EA^+, first we will prove the inclusion “$\supseteq$”. By the reflexive induction hypothesis, we have that

$$\Box_{\mathrm{EA}^+}\Big(n\text{-}\mathrm{Con}(T + B_T) \to n\text{-}\mathrm{Con}\left(\Pi_{n+1}\text{-}R^{o(n\downarrow B)}(T)\right)\Big). \tag{7.1}$$

So for all $B <_n A$, unfolding the definition of $<_n$, gives $\Box_T(A_T \to \langle n\rangle_T B_T)$ and then, along with (7.1) it is given that,

$$\Box_T\Big(A_T \to n\text{-}\mathrm{Con}\left(\Pi_{n+1}\text{-}R^{o(n\downarrow B)}(T)\right)\Big)$$

for all $B <_n A$. Therefore by the definition of $\Pi_{n+1}\text{-}R^{o(n\downarrow A)}(T)$,

$$\Box_{T+A}\Pi_{n+1}\text{-}R^{o(n\downarrow A)}(T).$$

For the other inclusion "$\subseteq_n$", let $\Box_{T+A}\varphi$ with $\varphi \in \Pi_{n+1}$. If $A = \langle n\rangle B$, then by (7.1), $\Box_{T+\langle n\rangle_T B}\varphi$ implies $\Box_{T+\Pi_{n+1}\text{-}\mathrm{RFN}\left(\Pi_{n+1}\text{-}R^{o(n\downarrow B)}(T)\right)}\varphi$ and hence we are done.

If instead $A = \langle m+1\rangle B$, with $m \geq n$, then by the reduction property, we have

$$\{\langle m+1\rangle_T B_T\} \equiv_m \{Q_k^m(B_T) : k{<}\omega\}.$$

Of course, for each $Q_k^m(B)$, we can find some equivalent worm $C_k \in \mathbb{W}_n$. Therefore, we have $\Box_{T+(C_k)_T}\varphi$ for some k and since $C_k <_n A$, by the induction hypothesis (7.1) and Lemma 5.4.1, we complete the proof. $\Box$

Now to talk about $\mathrm{I}\Sigma_n$ for $n{\geq}1$, we first have by Remark 3.2.1 the following:

$$\mathrm{EA} + \big(\langle 1\rangle\langle n+1\rangle\top\big)_{\mathrm{EA}} \equiv \mathrm{EA}^+ + \big(\langle 1\rangle\langle n+1\rangle\top\big)_{\mathrm{EA}} \equiv \mathrm{EA}^+ + \big(\langle 1\rangle\langle n+1\rangle\top\big)_{\mathrm{EA}^+}.$$

Using this fact along with Lemma 7.2.1 and Theorem 5.5.3, we land onto the following:

$$\mathrm{EA} + 1\text{-}\mathrm{Con}(\mathrm{I}\Sigma_n) \equiv_1 \Pi_2\text{-}R^{o(0n)}(\mathrm{EA}^+) \supseteq [\mathrm{EA}^+, \mathsf{TI}^{\mathsf{R}}(\Pi_2, \omega{\cdot}o(n))].$$

And so by noting first that $o(n){=}\omega_n$, we finally conclude that[1] for every $n{\geq}\,1$

$$\mathrm{EA} + 1\text{-}\mathrm{Con}(\mathrm{I}\Sigma_n) \supseteq_1 [\mathrm{EA}^+, \mathsf{TI}^{\mathsf{R}}(\Pi_2, \omega{\cdot}\omega_n)]. \tag{7.2}$$

Now to prove EWD^n, we will use the formula $\varphi(A) = \forall\, m\, \exists\, n\ A[\![m, \ldots, m+n]\!]{=}\,\mathrm{e}$ and perform transfinite induction on it. Let $\forall\, B{<_0}A\ \varphi(B)$ be our induction hypothesis –and assume that $A \neq \mathrm{e}$ as otherwise $\varphi(A)$ holds trivially. Since $A[\![n]\!] <_n A$ for any natural number n, we therefore have by the transfinite induction hypothesis that $\exists\, m\ A[\![m]\!][\![m+1, \ldots, m+n]\!] = \mathrm{e}$ which in turn results to $\exists\, m\ A[\![m, \ldots, m+n]\!] = \mathrm{e}$.
Since φ is a Π_2 formula, by (7.2) we can perform transfinite induction on it within $\mathrm{EA} + 1\text{-}\mathrm{Con}(\mathrm{I}\Sigma_n)$. As such, we conclude with

$$\mathrm{EA} + 1\text{-}\mathrm{Con}(\mathrm{I}\Sigma_n) \vdash \forall\, A{\in}\mathbb{W}^n \varphi(A),$$

which gives, EWD^n.

7.3 Independence of EWD

In contrast to the previous direction, there is not that much that has to be done in proving the direction that corresponds to Proposition 6.3.1. Therefore it will look like so:

Theorem 7.3.1. $\mathrm{EA} + \mathrm{EWD}^{n+1} \vdash 1\text{-}\mathrm{Con}(\mathrm{I}\Sigma_n)$.

As before, we use the Hardy functions' analogue $h_A(m)$ which we recall being defined as the smallest k such that $A[\![m \ldots m{+}k]\!] = \mathrm{e}$. We also make use of the ordering $-B \trianglelefteq A$ if and only if, $B = A[\![0]\!] \ldots [\![0]\!]$ for a finite number of iterations– defined as before.

As the next and h_A functions have not been modified for this worm principle and the worm domain doesn't affect the Lemmata of this direction, we obtain the same monotonicity of the h_A function and hence Lemma 6.3.9. This will obviously not remain the same as we later tackle worms that include limit ordinals.

Therefore, we can directly head onto the proof of Theorem 7.3.1.

[1] For natural numbers n, we define ω_n as follows: $\omega_0 = 1$; $\omega_{n+1} = \omega^{\omega_n}$.

Proof. Assume that EWD^{n+1} holds. We have:

$$\begin{aligned}
\mathrm{EA} \vdash \forall\, A{\in}\mathbb{W}^{n+1}\ \exists\, m\ A_m{=}\,\mathrm{e} &\rightarrow \forall\, A{\in}\mathbb{W}^{n+1}_1\ h_A{\downarrow} \\
&\rightarrow \forall\, k\ \ \langle 1\rangle \big(\langle n+1\rangle\top[\![k]\!]\big), \quad \text{by Lemma 6.3.9} \\
&\rightarrow \langle 1\rangle\, \langle n+1\rangle\top \quad \text{(by the reduction property)} \\
&\rightarrow 1\text{-}\,\mathrm{Con}(\mathrm{I}\Sigma_n).
\end{aligned}$$

The first implication holds since for every worm A and every number x, there is a worm $A'{=}0^x A$ where $A'[\![0\ldots x-1]\!]{=}A$ hence $\exists\, m\ A'_m{=}\,\mathrm{e}$ iff $h_A(x)$ is defined. As for the use of the reduction property, notice that here $\langle 1\rangle\,\langle n+1\rangle\top \rightarrow \langle 1\rangle\top$ which in turn implies EA^+. $\square$

Chapter 8

Truth predicates, reflection principles and the hyperarithmetical hierarchy

A different direction we could take to study worms would be turning towards theories beyond PA. However as we take the first step towards that goal, we would want to preserve this vague sense of connection we have established between the game and GLP. More precisely we would want to maintain an arithmetical translation from GLP_Λ, where Λ matches the alphabet of the worms of the expanded game, into our base theory –which in the case of the previous two chapters, has been PA.

8.1 Truth predicates

To this end, we will follow the direction taken in the paper *Reflection algebras and conservation results for theories of iterated truth* by Beklemishev and Pakhomov [5] where we will start by expanding the basic language of arithmetic with a unary predicate T. The expanded language we will denote by $\mathcal{L}(\mathsf{T})$ and by $\Pi_n^{\mathcal{L}}(\mathsf{T})$ we will denote the class of Π_n-formulas in the language of $\mathcal{L}(\mathsf{T})$, while the Π_n-formulas in the language of $\mathcal{L}$ will be denoted by $\Pi_n^{\mathcal{L}}$. We similarly define $\Sigma_n^{\mathcal{L}}(\mathsf{T})$ and $\Delta_n^{\mathcal{L}}(\mathsf{T})$. From here on, the predicate T we will call a *truth predicate* because its purpose will be to express the truth of the formulas in the language $\mathcal{L}$. To get there we will consider two different basic theories whose purpose will be solely to enable the use of T as a truth predicate. The reason we present them as such is to allow us greater freedom in attaching them into any of the theories of arithmetic we have already presented, like $\mathrm{EA}, \mathrm{EA}^+, \mathrm{PA}$ etc. The first theory that we present is the theory that goes by the name of *Uniform Tarski Biconditionals.*

Definition 8.1.1 (UTB). $\mathrm{UTB}_{\mathcal{L}}$*, or simply* UTB*, is the* $\mathcal{L}(\mathsf{T})$*-theory axiomatized by the following axiom schema:*

$$\forall \vec{x}\ \big(\varphi(\vec{x}) \leftrightarrow \mathsf{T}(\ulcorner \varphi(\dot{\vec{x}})\urcorner)\big),$$

For all $\mathcal{L}$*-formulas* φ *and with* $\ulcorner\varphi(\vec{x})\urcorner$ *denoting the Gödel numbering of* $\varphi(\vec{x})$*.*
Every such axiom is called a disquotation axiom.

The importance of UTB and the reason why it is axiomatized by an axiom schema is because we have the following well known lemma about it:

Lemma 8.1.2. *Let $S \supseteq \mathrm{EA}$ be an $\mathcal{L}$-theory. Then $S + \mathrm{UTB}$ is conservative over S for $\mathcal{L}$-formulas.*

The idea of the proof is that every proof of $S + \mathrm{UTB}$ has only finitely many occurrences of disquotation axioms $\forall \vec{x}\ \big(\varphi(\vec{x}) \leftrightarrow \mathsf{T}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)\big)$. Let $\{\varphi_i : i < n\}$ be an enumeration of all the formulas occurring in disquotation axioms in the proof. Let $\tau(y)$ be the formula

$$\bigvee_{i<n} \exists \vec{x}\ (y = \ulcorner \varphi(\dot{\vec{x}}) \urcorner\ \wedge\ \varphi(\vec{x})).$$

Then each occurrence of $\mathsf{T}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)$ can be replaced with $\tau(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)$.

Now the second theory is the theory of *compositional truth.*

Definition 8.1.3 (CT)**.** CT *is the $\mathcal{L}(\mathsf{T})$-theory axiomatized by the following axioms:*

C1. $\forall \vec{x}\ \big(\varphi(\vec{x}) \leftrightarrow \mathsf{T}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)\big)$, *for $\varphi(\vec{x})$ an atomic $\mathcal{L}$-formula;*

C2. $\forall \vec{x}\ \big(\mathsf{T}(\ulcorner \varphi(\dot{\vec{x}}) \wedge \psi(\dot{\vec{x}}) \urcorner) \leftrightarrow (\mathsf{T}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \wedge \mathsf{T}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner))\big)$;

C3. $\forall \vec{x}\ \big(\mathsf{T}(\ulcorner \neg\varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \neg\mathsf{T}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)\big)$;

C4. $\forall \vec{x}\ \big(\mathsf{T}(\ulcorner \forall y\ \varphi(y, \dot{\vec{x}}) \urcorner) \leftrightarrow \forall y\ \mathsf{T}(\ulcorner \varphi(\dot{y}, \dot{\vec{x}}) \urcorner)\big)$,

Where φ and ψ are $\mathcal{L}$-formulas.

An important property of CT over UTB is that it is finitely axiomatized over EA, which grants it significant greater strength over UTB. Proving for instance that PA +CT is conservative over PA is by no means a trivial task [11]. If we additionally allow for full induction over all $\mathcal{L}(\mathsf{T})$-formulas, we lose this conservativity [11].

Therefore we will be focusing more on UTB. Following [5], we further expand $\mathcal{L}(\mathsf{T})$ by iteratively adding a truth predicate and a schema of disquotation axioms for the theory preceding it. Formally, let $\mathcal{L}_\alpha = \mathcal{L} \cup \{\mathsf{T}_\beta : \beta < \alpha\}$. Here we understand that we fix some elementary ordering of ordinals up to α. We use this ordering to determine the Gödel numbering of $\mathcal{L}_\alpha$. For each α, we define the $\mathcal{L}_{\alpha+1}$-theory UTB_α as $\mathrm{UTB}_{\mathcal{L}_\alpha}$. Let $\mathrm{UTB}_{<\alpha} := \bigcup_{\beta<\alpha} \mathrm{UTB}_\beta$.

Typically the language of EA $+\mathrm{UTB}_{<\alpha+1}$ is infinite, however in [5] it is shown how this can be avoided. For every $\varphi \in \mathcal{L}_{\alpha+1}$, let φ^* denote the formula produced after simultaneously substituting $\mathsf{T}_\beta(t)/\mathsf{T}_\alpha(\ulcorner \mathsf{T}_\beta(\dot{t}) \urcorner)$ for all $\beta < \alpha$ where $\mathsf{T}_\beta(t)$ is a subformula of φ. Then if we denote $\mathrm{UTB}^*_{<\alpha+1} = \{\varphi^* : \varphi \in \mathrm{UTB}_{<\alpha+1}\}$, we have that EA $+\mathrm{UTB}_{<\alpha+1}$ is a definitional expansion of EA $+\mathrm{UTB}^*_{<\alpha+1}$.

8.2 Hyperarithmetical reflection

Now we can expand our standard arithmetical hierarchy into the so-called *hyperarithmetical hierarchy* as it is done in [5]. For a given elementary well-ordering $(\Lambda, <)$, we expand it into an ordering of $(\omega(1+\Lambda), <)$ by encoding $\omega\alpha + n$ as pairs $\langle \alpha, n \rangle$ with the expected ordering on them.

Definition 8.2.1 (Hyperarithmetical hierarchy)**.** *For ordinals up to $\omega(1 + \Lambda)$, we define the hyperarithmetical hierarchy as:*

- $\Pi_n := \Pi_n^{\mathcal{L}}$, *for every* $n < \omega$*;*
- $\Pi_{\omega(1+\alpha)+n} := \Pi_{n+1}^{\mathcal{L}_\alpha}(\mathsf{T}_\alpha)$*,*

For λ *a limit ordinal, we denote* $\Pi_{<\lambda} := \bigcup_{\alpha<\lambda} \Pi_\alpha$.

Via the substitution that we mentioned above, the Π_α can be, in general, formulated via only one truth predicate added to our language.
Now the reflection principles can expand into the hyperarithmetical hierarchy. Following the notation of [5], for any theory S and for every $\alpha, \lambda < \omega(1+\Lambda)$, where λ is a limit ordinal, we define:

$$R_\alpha(S) := \Pi_{1+\alpha}\text{- RFN}(S);$$
$$R_{<\lambda}(S) := \Pi_{<\lambda}\text{- RFN}(S).$$

We will present some first results about these expanded reflection principles which are similar to the partial reflection principles that we have presented so far:

Proposition 8.2.2.

(i) *If* $S \supseteq \mathrm{EA} + \mathrm{UTB}_\alpha$*, then over* $\mathrm{EA} + \mathrm{UTB}_\alpha$

$$R_{\omega(1+\alpha)+n} \equiv \Pi_{n+1}^{\mathcal{L}}(\mathsf{T}_\alpha)\text{- RFN}(S);$$

(ii) *If* $S \supseteq \mathrm{EA} + \mathrm{UTB}_\alpha$ *and* $\beta = \omega(1+\alpha)+n$*, then* $R_\beta(S)$ *is finitely axiomatizable over* $\mathrm{EA} + \mathrm{UTB}_\alpha$*;*

(iii) *If* $S \supseteq \mathrm{EA} + \mathrm{UTB}_{<\alpha}$*, then over* $\mathrm{EA} + \mathrm{UTB}_{<\alpha}$

$$R_{<\omega(1+\alpha)}(S) \equiv \mathcal{L}_\alpha\text{- RFN}(S) \equiv \{R_\beta(S) : \beta < \omega(1+\alpha)\}.$$

Proofs can be found in [5]. Of particular interest to us is Item (ii) which allows us to treat them as formulas in the same vein as we have done with the partial reflection principles. Now to expand the notation for conservativity between two theories let $\equiv_\alpha$ and $\equiv_{<\lambda}$ denote conservativity for $\Pi_{1+\alpha}$ and $\Pi_{<\lambda}$-sentences respectively. In [5] the following two conservation results are proven. The first centers around the case for reflection on limit ordinals.

Theorem 8.2.3. *Let* $\lambda = \omega(1+\alpha)$ *and* $S \supseteq \mathrm{EA} + \mathrm{UTB}_\alpha$*. Then over* $\mathrm{EA} + \mathrm{UTB}_{<\Lambda}$*,*

$$R_\lambda(S) \equiv_{<\lambda} R_{<\lambda}(S).$$

The second centers around successors. It can be viewed as an extension of the reduction property to cover all successor ordinals.

Theorem 8.2.4. *Let* V *be a* $\Pi_{1+\alpha+1}$*-axiomatized extension of* $\mathrm{EA} + \mathrm{UTB}_{<\Lambda}$ *and let* $S \supseteq V$*. Then, over* V*,* $R_{\alpha+1}(S) \equiv_\alpha \{R_\alpha(S), R_\alpha(S + R_\alpha(S)), \ldots\}$.

Finally, we have a theorem stating that the structural behaviour of different reflection principles is governed by a simple modal logic. By Σ_1-collection, we refer to the formula

$$\forall x{<}z\, \exists y\ \phi(x,y) \to \exists y_0 \forall x{<}z\, \exists y{<}y_0\ \phi(x,y).$$

The Σ_1-collection rule corresponds to the following rule:

$$\frac{\forall x{<}z\, \exists y\ \phi(x,y)}{\exists y_0 \forall x{<}z\, \exists y{<}y_0\ \phi(x,y)}.$$

We have the following theorem from [5]:

Theorem 8.2.5. *Let $S \supseteq \mathrm{EA} + \mathrm{UTB}_{<\Lambda}$ be such that it satisfies Σ_1-collection rule. For all formulas A, B or RC_Λ, if $A \vdash_{\mathrm{RC}} B$ then*

$$S \vdash \forall x\ (\Box_{B^*}(x) \to \Box_{A^*}(x)).$$

Here, the logic RC is the modal logic that we mentioned in the beginning of Chapter 4

Chapter 9

Independent combinatorial principles beyond Peano Arithmetic

In the interest of understanding the behaviour of the worm principle in relation to theories beyond PA, we will make a modest comparison of it with a relatively small second order extension of PA. This extension serves as a guide and motivation for the choices that we make as we further expand the rules of the worm game.

9.1 On arithmetical comprehension

Second order arithmetics are characterized by them having predicates as variables, essentially allowing quantification over them. Alternatively they can be seen as sets of natural numbers and this is the expression we will follow.

Definition 9.1.1. *The language of second order arithmetic is the extension of the language of first order arithmetic $\mathcal{L}$ by the addition of second order variables and parameters and the predicate symbol $\in$. The expression $t{\in}X$ is an atomic formula where t is a term and X a second order variable. We add no symbol for the second order identity and instead we express it in the language via extensionality $X = Y :\Leftrightarrow \forall x\ (x{\in}X \leftrightarrow x{\in}Y)$.*

We will choose the well known theory ACA, which unlike ACA_0, it isn't conservative over PA and it will help us implement the first limit ordinal into the game as we will see.

Definition 9.1.2 (ACA)**.** *The theory* ACA *is a theory in the language of second order arithmetic that extends* PA *by the induction schema for all second order formulas and the comprehension schema:*

$$\exists Y \forall x\ (x{\in}Y \leftrightarrow \varphi(x)),$$

for every arithmetical formula with possibly both first and second order parameters (excluding Y).

Now, in the first order language $\mathcal{L}(\mathsf{T})$, consider the theory $\mathrm{PA}(\mathsf{T}) := \mathrm{EA} + \mathrm{CT} + I\mathcal{L}(\mathsf{T})$, notated as in [5], where by $I\mathcal{L}(\mathsf{T})$, we denote the induction schema for all first order formulas of the language of $\mathcal{L}(\mathsf{T})$. We have the following well known result from [11]:

Theorem 9.1.3. $\mathrm{PA}(\mathsf{T})$ *and* ACA *are proof theoretically equivalent.*

This is proved by creating two interpretations, one of PA(T) in ACA where a truth predicate is defined in ACA to interpret T. On the interpretation of ACA in PA(T), second order variables are simultaneously translated into first order variables, that are different than the other first order variables. Then $x \in Y$ is written as $\mathsf{Form}(x) \wedge \mathsf{T}(x)$, where $\mathsf{Form}(x)$ is the formula stating that "x is the Gödel number of a formula with one free variable". The full proof can be found in [11] where do note that in this book, CT denotes what we notate here as PA(T).

By [5], we know that:

$$\mathrm{PA}(\mathsf{T}) \equiv \mathrm{EA}^+ + \mathrm{UTB}_{\mathcal{L}(\mathsf{T})} \ + \ R_{<\omega 2}(\mathrm{EA}^+ + \mathrm{UTB}_{\mathcal{L}(\mathsf{T})}),$$

the proof of which, is similar to the proof of the proof we follow to reach Corollary 3.3.3. Then by an analogue of Remark 3.2.1, the latter part is in-turn equivalent to $\mathrm{EA}^+ + \mathrm{UTB}_{\mathcal{L}(\mathsf{T})} \ + R_{<\omega 2}(\mathrm{EA} + \mathrm{UTB}_{\mathcal{L}(\mathsf{T})})$ which then by Corollary 3.1.3 gives us:

$$\mathrm{PA}(\mathsf{T}) \equiv \mathrm{EA} + \mathrm{UTB}_{\mathcal{L}(\mathsf{T})} \ + \ R_{<\omega 2}(\mathrm{EA} + \mathrm{UTB}_{\mathcal{L}(\mathsf{T})}). \tag{9.1}$$

9.2 A disclaimer

In the remainder of this chapter we will investigate how our previous results on the worm principles can be extended to ACA. We will make a methodological assumption in the remainder of this chapter, the correctness of which requires some further checking. Let us make this assumption a bit more precise.

The formulation of systems beyond PA depend on results from [5] where subsystems of second order arithmetic are characterised in terms of hyperarithmetical reflection principles. These reflection principles in turn are related to modal logic via the so-called *reflection calculus* RC and not to the logic GLP.

In this chapter we will re-use our GLP reasoning from Chapter 4 to the setting of ACA. As such, our methodological assumption is that all the reasoning in GLP that we use, can actually be obtained from corresponding reasoning in RC and similarly for statements involving the reduction property. In future work the details of this assumption will be further investigated. However, it seems reasonable to expect few serious hurdles to arise since our GLP reasoning contains no nested implications so that Theorem (4.1.3) will be applicable throughout.

9.3 The worm principle for ACA

Here, we will be using the theory PA(T) as a substitute for ACA to examine its relationship with the worm principle. By (9.1), we are expecting to end up with an equivalence between 1-Con(PA(T)) and $\mathrm{EWD}^{\omega 2}$, in the form that we will present below. We will expand the next function of Definition 6.1.1, first by following the direction we took with the $[\![\cdot]\!]$ operation from Definition 4.2.3 so that it takes into consideration worms with limit ordinals as their heads. As the worms that we are going to concern ourselves with are of $\mathbb{W}^{\omega 2}$, the only limit ordinal consideration shall be the case of ω.

Definition 9.3.1. *Let $w = x_n \ldots x_1 x_0$ be a worm of $\mathbb{W}^{\omega 2}$ and $m \in \mathbb{N}$ a step of the game.*

- *If w is empty, then $\mathrm{next}(w, m) := w$.*
- *If x_n=0 then $\mathrm{next}(w, m) := x_{n-1} \ldots x_0$. So in this case, the head of the worm is being cut away regardless of the value of m.*

- *If x_n is a successor ordinal, let $k := \max\{i < n : x_i < x_n\}$. As before we consider the good part, $r := x_k \dots x_0$, which may be empty, and the bad one $s := (x_n - 1)x_{n-1} \dots x_{k+1}$. Set*

$$\mathrm{next}(w, m) := \underbrace{s * \dots * s * s}_{m+1 \text{ times}} * r.$$

- *If $x_n = \omega$ then* $\mathrm{next}(w, m) := m * x_{n-1} \dots x_0$. *So here the head simply steps down to the mth element of the natural fundamental sequence of ω.*

The sequence of worms starting from an arbitrary worm $w \in \mathbb{W}^{\omega 2}$ is defined in the usual way:

$$w_0 := w \quad \text{and} \quad w_{n+1} := \mathrm{next}(w_n, n+1).$$

As an example, we can consider the worm $w = 1\omega 02$. At the first step we obtain $k = 1$; $r = 02$; $s = 0\omega$ and $w[\![1]\!] = 0\omega 0\omega 02$.
Similarly the game proceeds as:

$$\begin{aligned}
w_0 &= 1\omega 02 \\
w_1 &= 0\omega 0\omega 02 \\
w_2 &= \omega 0\omega 02 \\
w_3 &= 30\omega 02 \\
w_4 &= 222220\omega 02 \\
w_5 &= (12222)^6 0\omega 02 \\
w_6 &= (0222212222122221222212222212222)^7 0\omega 02 \\
&\dots
\end{aligned}$$

The same estimate of $|w_n| \leq (n+1)!\,|w_0|$ applies to this sequence of worms and so w_n is an elementary function of n. Additionally, as before we will be using the notation of provability worms instead, so writing A_n instead of w_n. We will prove (under our disclaimer) that:

Theorem 9.3.2. $\mathrm{EWD}^{\omega 2}$ *is equivalent to* 1-Con(PA(T)) *in* EA +UTB*.*

Of course a hidden additional assumption that we are making is the specific choice of the next function, in which case, we consider $\mathrm{EWD}^{\omega 2}$ to be using the one that we defined above. Since the base theory here is EA +UTB, we will be changing the notational convention we had in the previous chapters. In particular:

Notation 9.3.3. *In this chapter, $[\alpha]\varphi$ in the context of the arithmetics will be a shorthand for $[\alpha]_{\mathrm{EA+UTB}}\varphi$. Similarly, $\langle\alpha\rangle\varphi$ in the context of the arithmetics will be a shorthand for $\langle\alpha\rangle_{\mathrm{EA+UTB}}\varphi$.*

Large part of this chapter has been announced and presented in [18]. Now we will follow the style of the proof for the case of PA (from [2]) that we presented in Chapter 6. Thus we will split the theorem into two parts, starting with:

Proposition 9.3.4. EA +UTB + 1-Con(PA(T)) $\vdash \mathrm{EWD}^{\omega 2}$*.*

As is expected we first have a lemma expressing the consistency strength of PA(T), as is derived from (9.1) that we mentioned above. From now on we will refrain from distinguishing worms from their arithmetical interpretation.

Lemma 9.3.5. *For any $A \in \mathbb{W}^{\omega 2}$,* $\mathrm{PA}(T) \vdash A$*.*

Proof. For every $A \in \mathbb{W}^{\omega 2}$, there is some $m > 0$ such that $A \in \mathbb{W}^{\omega+m}$ and so by Proposition 4.1.9,

$$\mathrm{GLP} \vdash \langle \omega + m \rangle \top \to A.$$

Therefore, by arithmetical soundness[1] of GLP, it holds that $\mathrm{EA} + \mathrm{UTB} \vdash \langle \omega{+}m \rangle \top \to A$ and since $\mathrm{PA}(T) \vdash \langle \omega{+}m \rangle \top$, the lemma follows and its proof is formalizable in EA^+. □

Then we have the two lemmata whose proofs are largely the same and we will be only providing a shorthand of the proof of the first. Remember that we denote $A^+ := 1{\uparrow}A$.

Lemma 9.3.6. *For any $A \in \mathbb{W}^{\omega 2}$,*

$$\mathrm{EA} + \mathrm{UTB} \vdash \forall k \ \big(A_k \neq \mathrm{e} \to \Box(A_k^+ \to \langle 1 \rangle \, A_{k+1}^+)\big).$$

Proof. As before, it is sufficient to prove in EA +UTB

$$\forall A \neq \mathrm{e} \ \ \forall k \ \ \mathrm{EA} + \mathrm{UTB} \vdash A^+ \to \langle 1 \rangle \, A[\![k]\!]^+.$$

For this, we will move over to GLP where we have that the following proof is bounded by a function elementary in A and k and hence it is formalizable in EA,

$$\mathrm{GLP} \vdash A \to \Diamond A[\![k]\!],$$

and as theorems of GLP are stable under right shift,

$$\mathrm{GLP} \vdash A^+ \to \langle 1 \rangle \, A[\![k]\!]^+,$$

which by the arithmetical soundness of GLP, proves that for every $A \in \mathbb{W}^{\omega 2}$ with $A \neq \mathrm{e}$ and for every k,

$$\mathrm{EA} + \mathrm{UTB} \vdash A^+ \to \langle 1 \rangle \, A[\![k]\!]^+.$$

The rest of the proof proceeds as before. □

And now, having proven Lemma 9.3.6, the equivalent of Lemma 6.2.4 follows naturally.

Lemma 9.3.7. *For any $A \in \mathbb{W}^{\omega 2}$,*

$$\mathrm{EA} + \mathrm{UTB} \vdash \langle 1 \rangle \, A_0^+ \to \exists m \ A_m = \mathrm{e}\,.$$

Its proof is exactly the same with nothing of much interest added to it and so we opt to omit it. So now, from Lemmata 9.3.5 and 9.3.7 we obtain that for each $A \in \mathbb{W}^{\omega 2}$,

$$\begin{aligned} \mathrm{PA}(T) &\vdash \langle 1 \rangle \, A^+ \\ \mathrm{EA} + \mathrm{UTB} &\vdash \langle 1 \rangle \, A^+ \to \exists m \ A_m = \mathrm{e} \end{aligned}$$

Hence, $\forall A \in \mathbb{W}^{\omega 2}$ $\mathrm{PA}(T) \vdash \exists m \ A_m = \mathrm{e}$. This is formalizable in EA^+, therefore $\mathrm{EA}^+ + \mathrm{UTB} + 1\text{-}\mathrm{Con}(\mathrm{PA}(T)) \equiv \mathrm{EA} + \mathrm{UTB} + 1\text{-}\mathrm{Con}(\mathrm{PA}(T))$ implies $\forall A \in \mathbb{W}^{\omega 2} \ \exists m \ A_m = \mathrm{e}$, which is $\mathrm{EWD}^{\omega 2}$.

[1]See our disclaimer.

9.4 From the worm principle to ACA

Now we turn into the second direction of Theorem 9.3.2, proving independence of $\mathrm{EWD}^{\omega 2}$.

Proposition 9.4.1. $\mathrm{EA} + \mathrm{UTB} + \mathrm{EWD}^{\omega 2} \vdash 1\text{-}\mathrm{Con}(\mathrm{PA}(\mathsf{T}))$.

Once more, we use the Hardy functions' analogue $h_A(m)$ on which we make no alterations on their definition, that is defining them in the same way –as the smallest k such that $A[\![m \ldots m{+}k]\!]{=}\mathrm{e}$. Notice that in the interest of defining an ordering between worms as before, we cannot use the definition of $A[\![0]\!] \ldots [\![0]\!] = B$ since then this definition would cause ω to be treated similarly to 1 and this is too restrictive. Therefore, the existence of limit ordinals in the alphabet of our words will demand us to use an alternative ordering. For $A, B \in \mathbb{W}^{\omega 2}$, we define the partial ordering $B \trianglelefteq A$ iff B is an initial segment of A apart from possibly the first element which should then be smaller. However the step that we have added for limit ordinals, has the side effect of creating a disconnect between the $\trianglelefteq$ ordering and the h_A function, hence rendering this definition to be too strong. To remedy that, we will consider several restrictions of it.

Definition 9.4.2. *For every natural number m, we define $B \trianglelefteq_m A$ iff $B \trianglelefteq A$ and additionally, if $B = nC$ and $A = D\omega C$, then $n \leq m$.*

Of course, by the definition, we immediately have that if $B \trianglelefteq_m A$ and $m \leq n$ then $B \trianglelefteq_n A$. Additionally, if B is an initial segment of A then $B \trianglelefteq_m A$ for every $m \geq 0$. Over EA, and for worms in $\mathbb{W}^{\omega 2}$, we have the following:

Lemma 9.4.3. *If $h_A(m)$ is defined and $B \trianglelefteq_m A$, then*

$$\exists\, k\ A[\![m \ldots m + k]\!]{=}B.$$

Proof. The rules of the game are such that the letter of index i in A can only change if all letters to the left of it are deleted. This claim is proved as before by defining for each index i, a k_i to be the least such that the element of index i in $A[\![m \ldots m{+}k_i]\!]$, differs from the element of index i in A. We define as before $B_i = A[\![m \ldots m{+}k_i{-}1]\!]$. We additionally have that B_i is an initial segment of A as before. Assume now that the element with index $|B|-1$ of A is $\geq \omega$ and the corresponding in B is $< \omega$ and therefore, by our assumption of $B \trianglelefteq_m A$, it is also some $n \leq m$.

Given the assumption of $A[\![m \ldots m + s]\!]{=}\mathrm{e}$, let the element with index i of A be $\omega + l$. Then following the proof of Lemma 6.3.3, we can show that there is some k_1 such that $A[\![m \ldots m{+}k_1]\!] = \langle\omega\rangle C$ where $B = \langle n\rangle C$ and $A = D\langle\omega + l\rangle C$. Then $A[\![m \ldots m + k_1 + 1]\!] = \langle m + k_1 + 1\rangle C$. The rest of the proof proceeds as before to find some $k < s$ such that $A[\![m \ldots m + k]\!] = B$. ☐

The expansion of this lemma then becomes a bit more complicated, considering the involvement of the $\trianglelefteq_m$ orderings. However it is similar in spirit.

Corollary 9.4.4. *If $h_A(n)$ is defined and $B \trianglelefteq_n A$, then $\forall\, m{\leq}n\ \exists\, k\ A[\![n \ldots n + k]\!]{=}B[\![m]\!]$.*

Proof. By Lemma 9.4.3, there is k_1 such that $A[\![n \ldots n{+}k_1]\!]{=}B$. Then, $B[\![m]\!]$ is an initial segment of $A[\![n \ldots n{+}k_1{+}1]\!]$ and so once again, the rules of the game dictate that since $h_A(n)$ halts, then there is some k_2 such that $A[\![n \ldots n{+}k_1{+}k_2]\!]{=}B[\![m]\!]$. ☐

This we then naturally use to prove the analogue of Lemma 6.3.5.

Lemma 9.4.5. *If $h_A(y)$ is defined, $B \trianglelefteq_y A$ and $x \leq y$, then $h_B(x)$ is defined and $h_B(x) \leq h_A(y)$.*

Proof. By applying the Corollary 9.4.4 several times, we obtain $s_0, s_1, \dots$ such that

$$A[\![y \dots y+s_0]\!]=B[\![x]\!], \quad \text{where } y+s_0 \geq x$$
$$A[\![y \dots y+s_0+s_1]\!]=B[\![x]\!][\![x+1]\!], \quad \text{where } y+s_0+s_1 \geq x+1$$
$$\dots$$

Hence all elements of the sequence starting with B occur in the sequence for A and since $h_A(y)$ is defined, so is $h_B(x)$. $\square$

Finally, we are able to harvest the fruit of this additional complication added to the definition of this ordering.

Lemma 9.4.6. *For every $A, B \in \mathbb{W}^{\omega 2}$, if $h_{B0A}(n)$ is defined, then*

$$h_{B0A}(n)=h_A(n+h_B(n)+2)+h_B(n)+1 > h_A(h_B(n)).$$

Proof. The proof is exactly the same as before, only using Lemma 9.4.5 instead of Lemma 6.3.5. $\square$

Corollary 9.4.7. *If $A \in \mathbb{W}_1^{\omega 2}$ and $h_{1A}(n)$ is defined, then $h_{1A}(n) > h_A^{(n)}(n)$.*

Proof. As with its predecessor, it simply suffices to note that $1A[\![n]\!]=0A0A\dots0A$. $\square$

Due to the lack of changes in these, we will in fact, refer to their correspondents in Chapter 6. Now we move on to the main lemma of this proof to see how the restrictions of the ordering $\trianglelefteq$ manifest to provide us the result we wanted.

Lemma 9.4.8. $\mathrm{EA}+\mathrm{UTB} \vdash \forall A{\in}\mathbb{W}_1^{\omega 2}\ (h_{A1111}{\downarrow} \to \langle 1\rangle A)$.

Proof. By Löb's Theorem, this is once more, equivalent to proving

$$\mathrm{EA}+\mathrm{UTB} \vdash \Box\big(\forall A{\in}\mathbb{W}_1^{\omega 2}\ (h_{A1111}{\downarrow} \to \langle 1\rangle A)\big) \to \forall A{\in}\mathbb{W}_1^{\omega 2}\ (h_{A1111}{\downarrow} \to \langle 1\rangle A).$$

We reason in $\mathrm{EA}+\mathrm{UTB}$. Let us take the antecedent as an additional assumption, which by the monotonicity axiom of GLP interpreted in EA, implies $[1]\big(\forall A{\in}\mathbb{W}_1^{\omega 2}\ (h_{A1111}{\downarrow} \to \langle 1\rangle A)\big)$. This in turn implies:

$$\forall A{\in}\mathbb{W}_1^{\omega 2}\ [1](h_{A1111}{\downarrow} \to \langle 1\rangle A). \tag{9.2}$$

As before, we make a case distinction whether $A1111$ starts with a 1 or with an ordinal strictly larger than 1.
If $A1111 = 1B$ then by Corollary 6.3.7, we have $h_{1B}{\downarrow} \to \lambda x.h_B^{(x)}(x){\downarrow}$. The function h_B is increasing, has an elementary graph and grows at least exponentially as by Corollary 6.3.8, $h_{111} > 2^x$. So for $A{=}\mathrm{e}$ we have that $h_{1111}{\downarrow}$ implies the totality of 2_n^x and hence $\mathrm{EA}^+ +\mathrm{UTB}$, which by Corollary 3.1.3, implies $\langle 1\rangle\top$. If A is nonempty, we reason as follows:

$$\begin{aligned}
\lambda x.h_B^{(x)}{\downarrow} &\to \langle 1\rangle\, h_B{\downarrow}, && \text{by Lemma 3.1.2}\\
&\to \langle 1\rangle\,\langle 1\rangle\, B, && \text{by Assumption (9.2)}\\
&\to \langle 1\rangle\, A.
\end{aligned}$$

If $A1111{=}C$ ends with $\alpha > 1$, then we notice that the interesting case is for α being a limit ordinal, i.e., ω –since $\alpha < \omega 2$. The adjustment we will make for the case of $\alpha = \omega$.

$$\begin{aligned}
h_C{\downarrow} &\to \lambda x.h_{C[\![x]\!]}(x+1){\downarrow}\\
&\to \forall n\ h_{C[\![n]\!]}{\downarrow}.
\end{aligned}$$

The last implication is derived by application of Lemma 9.4.5 as for arbitrary n, if $x \leq n$ then $h_{C[\![n]\!]}(x) \leq h_{C[\![n]\!]}(n+1)$ and if $n \leq x$ then $h_{C[\![n]\!]}(x) \leq h_{C[\![x]\!]}(x+1)$. In both cases, the larger value is defined.
We can perform this line a second time, obtaining:

$$\begin{aligned}\forall n \; h_{C[\![n]\!]}\downarrow &\to \forall n \; \lambda x. h_{C[\![n]\!][\![x]\!]}(x+1)\downarrow \\ &\to \forall n \; h_{C[\![n]\!][\![n+1]\!]}\downarrow.\end{aligned}$$

Now notice that no matter what the α is, we will always have that either $1C[\![n]\!] \trianglelefteq_1 C[\![n+1]\!]$ or $1C[\![n]\!] \trianglelefteq_1 C[\![n+1]\!][\![n+2]\!]$. To prove this, let D be such that $C = \alpha D1111$.
If $\alpha = \omega$, then $1C[\![n]\!] = 1nD1111$ and $C[\![n{+}1]\!] = \langle n+1\rangle D1111$ therefore $C[\![n{+}1]\!][\![n{+}2]\!] = (nh_{n+1}(D1111))^{n+3} r_{n+1}(D1111) = (nh_{n+1}(D1111))^{n+2} nD1111$. So if $n = 0$ then since $D \in \mathbb{W}_1^{\omega 2}$, we have that $r_1(D1111) = \mathrm{e}$ and therefore, $C[\![n+1]\!][\![n+2]\!] = (0D1111)^{0+2}0D1111 = 0D11110D11110D1111 = 0D11110D1111C[\![n]\!]$. If $n > 0$ then clearly $nh_{n+1}(D1111)$ has as its rightmost element something ≥ 1 and so $1C[\![n]\!] \trianglelefteq_1 C[\![n+1]\!][\![n+2]\!]$.
If $\alpha \neq \omega$ then,
$1C[\![n]\!] = 1(\langle \alpha - 1\rangle h_\alpha(D1111))^{n+1} r_\alpha(D1111)$ and $C[\![n{+}1]\!] = (\langle \alpha - 1\rangle h_\alpha(D1111))^{n+2} r_\alpha(D1111)$.
So $1C[\![n]\!] \trianglelefteq_1 C[\![n+1]\!]$.
Therefore from all that, we have:

$$\begin{aligned}\forall n \; h_C\downarrow &\to \forall n \; h_{1C[\![n]\!]}\downarrow \quad \text{(by the above)} \\ &\to \forall n \; \lambda x. h^{(x)}_{C[\![n]\!]}(x)\downarrow \\ &\to \forall n \; \langle 1\rangle \, h_{C[\![n]\!]}\downarrow \quad \text{(by Lemma 3.1.2).}\end{aligned}$$

Again observe that since A starts with something bigger than 1, we have $C[\![n]\!] = A[\![n]\!]1111$, hence we can apply our assumption. Hence the argument continues,

$$\begin{aligned}&\to \forall n \, \langle 1\rangle \, (\langle 1\rangle \, A[\![n]\!]) \quad \text{by Assumption (9.2)} \\ &\to \langle 1\rangle \, A \quad \text{(by the reduction property).}\end{aligned}$$

The last step is achieved because $h_C\downarrow$ implies $h_{1111}\downarrow$ which, as per our first step in this proof, implies EA^+, hence allowing the use of the reduction property. $\square$

The proof of Proposition 9.4.1 proceeds as usual. Assume that $\mathrm{EWD}^{\omega 2}$ holds. We have:

$$\begin{aligned}\mathrm{EA} + \mathrm{UTB} \vdash \forall A{\in}\mathbb{W}^{\omega 2} \; \exists m \; A_m{=}\mathrm{e} &\to \forall A{\in}\mathbb{W}_1^{\omega 2} \; h_A\downarrow \\ &\to \forall n \; \langle 1\rangle \, \langle \omega + n\rangle \top \\ &\to 1\text{-}\mathrm{Con}(\mathrm{PA}(\mathsf{T})).\end{aligned}$$

The first implication holds since for every worm A and every number x, there is a worm $A' = 0^x A$ where $A'[\![0 \ldots x-1]\!] = A$ hence $\exists m \; A'_m{=}\mathrm{e}$ iff $h_A(x)$ is defined.

9.5 Alternative worm rules

When we started giving the definitions of the additional rules to the worm game in order to tackle ACA, we mentioned how the rule we chose for the limit step falls in line to the definition of $[\![\cdot]\!]$ that we presented. Here we will explore what would happen if we are to select $\ll \cdot \gg$ from Notation 4.2.4 as the guide for our next function. So let $\mathrm{next}'(w, m)$ be such that:

If $x_n = \omega$ then let $k := \max\{i < n : x_i < x_n\}$. Consider the good part, $r := x_k \dots x_0$, which may be empty, and the bad one $s := (m)x_{n-1} \dots x_{k+1}$. Let

$$\text{next}'(w,m) := \underbrace{s * \dots * s * s}_{m+1 \text{ times}} * r.$$

Otherwise, $\text{next}'(w,m) = \text{next}(w,m)$.

So the sequence of worms starting from an arbitrary worm $w \in \mathbb{W}^{\omega 2}$ is:

$$w_0 := w \quad \text{and} \quad w_{n+1} := \text{next}(w_n, n+1).$$

We will not be using any notational distinction here between the sequence derived via the next' function and the one from the next function. As before we will be using the notation of the provability worms with the difference being that we write $A\langle\!\langle n\rangle\!\rangle$ interchangeably with $\text{next}'(A,n)$.

Notation 9.5.1. *By* $\text{EWD}^{\omega 2}_{\ll\cdot\gg}$ *we will denote* $\text{EWD}^{\omega 2}$ *where the steps of the game use the* next' *function.*

We are going to prove that this function choice for the next step of the game, is just as good to give us an equivalence with ACA.

Theorem 9.5.2. $\text{EWD}^{\omega 2}_{\ll\cdot\gg}$ *is equivalent to* 1-Con(PA(T)) *in* EA +UTB.

This can be done by following the usual proof of the theorem and making adjustments where necessary or by giving a comparative argument between the two functions. We will aim for the latter and on the way, this will lay the groundwork for the former. We define the Hardy functions' analogue for the $\ll \cdot \gg$ which we denote as $g_A(m)$ and they are the smallest k such that $A\langle\!\langle m \dots m{+}k\rangle\!\rangle = \mathrm{e}$. We have the following for worms of $\mathbb{W}^{\omega 2}$:

Lemma 9.5.3. *Over* EA*, if* $h_A(n)$ *is defined and* $n > m$ *where* m *is greater than any finite element of* A*, then there is some* k *such that* $A[\![n \dots n{+}k]\!] = A\langle\!\langle m\rangle\!\rangle$.

Proof. If the leftmost element of A is a successor ordinal, then $A\langle\!\langle m\rangle\!\rangle = A[\![m]\!]$ and so we are done by Corollary 9.4.4. So assume that the leftmost element of A is ω. So $A = \omega B$ and $A\langle\!\langle m\rangle\!\rangle = (mh_\omega(B))^{m+1}r_\omega(B)$. But by the assumption we have made of m, we have that $h_\omega(B) = h_{n_1}(B)$ and $r_\omega(B) = r_{n_1}(B)$ for every $n_1 \geq m$. Therefore, $A[\![n]\!][\![n{+}1]\!] = (nB)[\![n{+}1]\!] = (\langle n{-}1\rangle h_n(B))^{n+2}r_n(B) = (\langle n{-}1\rangle h_\omega(B))^{n+2}r_\omega(B)$. So as the leftmost element of $A\langle\!\langle m\rangle\!\rangle$ is smaller or equal to the corresponding one from $A[\![n]\!][\![n{+}1]\!]$ and they are both successor ordinals, we have that $A\langle\!\langle m\rangle\!\rangle \trianglelefteq_l A[\![n]\!][\![n+1]\!]$ for every $l \geq 0$. So we can use Lemma 9.4.3 to complete the proof. $\square$

Now we can shape the idea in the proof of Lemma 9.4.5 into the context that we are working in to get:

Lemma 9.5.4. *Over* EA*, if* $h_A(n)$ *is defined and* $n > m$ *where* m *is greater than any finite element of* A*, then* $g_A(m)$ *is defined and* $g_A(m) \leq h_A(n)$.

Proof. By the assumption we have s such that $A[\![n \dots n{+}s]\!] = \mathrm{e}$. We will use Δ_0-induction on the formula $\exists\, k{\leq}s\ (k{\geq}x \wedge A[\![n \dots n{+}k]\!] = A\langle\!\langle m \dots m{+}x\rangle\!\rangle)$.
The case of $x = 0$ is clear by Lemma 9.5.3. Notice that every finite element of $A\langle\!\langle m\rangle\!\rangle$ is smaller or equal to m.

Assume now that the formula holds for x and that every finite element of $A\ll m\gg$ is smaller or equal to $m + x$. So there is some k_1 such that $A[\![n \ldots n{+}k_1]\!] = A\ll m \ldots m{+}x\gg$. Then since $n{+}k_1{+}1 > m{+}x{+}1$, we can once more make use of Lemma 9.5.3 on the worm $A\ll m \ldots m{+}x\gg$. This way we find some $k_2 \geq 1$ such that $A[\![n \ldots n{+}k_1{+}k_2]\!] = A\ll m \ldots m{+}x{+}1\gg$ and $n{+}k_1{+}k_2 > m{+}x{+}1$.
Therefore $A\ll m \ldots m{+}s\gg = \mathrm{e}$ and $g_A(m) \leq h_A(n)$. □

Now, without getting too much into detail we will mention the corresponding monotonicity properties of the g_A functions. The proof of this lemma is, in essence, the same as that of Lemma 9.4.3 and thus we will omit it.

Lemma 9.5.5. *If $g_A(m)$ is defined and $B \trianglelefteq_m A$, then*

$$\exists\, k\ A\ll m \ldots m + k\gg = B.$$

We can then naturally continue into the analogue of Corollary 9.4.4 using effectively the same proof.

Corollary 9.5.6. *If $g_A(n)$ is defined and $B \trianglelefteq_n A$, then $\forall\, m{\leq}n\ \exists\, k\ A\ll n \ldots n + k\gg = B\ll m\gg$.*

From here, all the other monotonicity lemmata follow. Among them is the one corresponding to Lemma 9.4.5.

Lemma 9.5.7. *If $g_A(y)$ is defined, $B \trianglelefteq_y A$ and $x \leq y$, then $g_B(x)$ is defined and $g_B(x) \leq g_A(y)$.*

With these lemmata on the monotonicity of the g_A, we can easily produce the following, final comparative Lemma between the h_A and g_A functions.

Lemma 9.5.8. *Over* EA*, if $g_A(m)$ is defined then $h_A(m)$ is also defined and $h_A(m) \leq g_A(m)$.*

Proof. The proof is similar to that of Lemma 9.5.4. By the assumption, we have s such that $A\ll m \ldots m{+}s\gg = \mathrm{e}$. We will use Δ_0-induction on the formula:
$\exists\, k{\leq}s\ (k{\geq}x \wedge A\ll m \ldots m{+}k\gg = A[\![m \ldots m{+}x]\!])$.
For $x = 0$, we clearly have by each of the rules that $A[\![m]\!] \trianglelefteq_l A\ll m\gg$ for every $l \geq 0$. Then we complete this induction step via Lemma 9.5.5.
Assume now that it holds for x. So there is some $k_1 \geq x$ such that $A\ll m \ldots m{+}k_1\gg = A[\![m \ldots m{+}x]\!]$. Then like in the previous step, $A[\![m \ldots m{+}x{+}1]\!] \trianglelefteq_l A\ll m \ldots m{+}k_1{+}1\gg$ and we complete this step with an application of Lemma 9.5.5. □

Now we have everything we need to prove that in either case, the two functions for the next element are effectively equivalent over EA.

Proposition 9.5.9. $\mathrm{EWD}^{\omega 2}$ *is equivalent to* $\mathrm{EWD}^{\omega 2}_{\ll\cdot\gg}$ *over* EA +UTB.

Proof. Assume first that we have $\mathrm{EWD}^{\omega 2}$. Then for every worm A, there is some m that is larger than all the finite elements of A. By the assumption we know that $h_A{\downarrow}$. Therefore, by Lemma 9.5.4, we know that $g_A(m)$ is defined and therefore by Lemma 9.4.5, we know that A, with the next′ function, eventually dies.
Assume now that $\mathrm{EWD}^{\omega 2}_{\ll\cdot\gg}$. Then by Lemma 9.5.8, we clearly have $\mathrm{EWD}^{\omega 2}$. □

But beyond this point, we could check for any differences in the proof of Theorem 9.5.2 when compared to the standard one. Starting with the first direction:

Proposition 9.5.10. $\mathrm{EA} + \mathrm{UTB} + 1\text{-}\mathrm{Con}(\mathrm{PA}(\mathsf{T})) \vdash \mathrm{EWD}^{\omega 2}_{\ll\cdot\gg}$.

Not much changes since the only requirement of $\ll \cdot \gg$ in the lemmata is only on the step: $\mathrm{GLP} \vdash A \to \Diamond A{\ll}m{\gg}$, which we have from Corollary 4.2.5. Therefore we turn our attention to the other direction, for which we have already done most of the work.

Proposition 9.5.11. $\mathrm{EA} + \mathrm{UTB} + \mathrm{EWD}^{\omega 2}_{\ll\cdot\gg} \vdash 1\text{-}\mathrm{Con}(\mathrm{PA}(\mathsf{T}))$.

In fact, the rest of the proof proceeds as before. The only two exceptions and both are in Lemma 9.4.8. For the first, we just note that $h_C{\downarrow} \to \forall n\ h_{C\ll n\gg}{\downarrow}$ suffices, without needing to perform this step again, as we always have that $1C[\![n]\!] \trianglelefteq_1 C[\![n+1]\!]$. The effect of that is just a simplification of the proof, making it fit more in line to the one for the case of PA as proven by Beklemishev [2]. The second lies with in the reduction property where it has to be adjusted to fit the requirements of $\ll \cdot \gg$, which isn't that hard to do.

Chapter 10

Conclusion and future research

In this paper we presented the worm principle of [2] and made two separate first attempts at generalizing it. For smaller theories than PA, we proved a weaker result than the expected equivalence $-\mathrm{EWD}^{n+1} \equiv 1\text{-}\,\mathrm{Con}(\mathrm{I}\Sigma_n)$ over EA. Though the two approaches from the Sections 7.1 and 7.2 gave the same result, there might still be merit in trying something a bit more involved with them. We have some ideas on strategies to explore here.
For larger theories, we ought to first clear up the details left in our disclaimer in our future work. Provided they do not cause any significant problems, we can then look further into EWD^{λ} for $\lambda > \omega 2$ a limit ordinal. This is likely to complicate the demands of the "next" function and our generalizations of the $\trianglelefteq_m$ relations. Along with this, the range of the various equivalent "next" functions could be explored; in the vein of generalizing the worm principle.

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Bibliography

[1] L. D. Beklemishev. Provability algebras and proof-theoretic ordinals, I. *Annals of Pure and Applied Logic*, 128:103–124, 2004.

[2] L. D. Beklemishev. Reflection principles and provability algebras in formal arithmetic. *Uspekhi Matematicheskikh Nauk*, 60(2):3–78, 2005. In Russian. English translation in: *Russian Mathematical Surveys*, 60(2): 197–268, 2005.

[3] L. D. Beklemishev. Positive provability logic for uniform reflection principles. *ArXiv:1304.4396 [math.LO]*, 2013.

[4] L. D. Beklemishev, D. Fernández-Duque, and J. J. Joosten. On provability logics with linearly ordered modalities. *Studia Logica*, 102:541–566, 2014.

[5] L. D. Beklemishev and F. N. Pakhomov. Reflection algebras and conservation results for theories of iterated truth. *arXiv:1908.10302 [math.LO]*, 2019.

[6] G. S. Boolos. *The Logic of Provability.* Cambridge University Press, Cambridge, 1993.

[7] Nigel Cutland. *Computability, an Introduction to Recursive Function Theory.* Cambridge University Press, 1980.

[8] E. V. Dashkov. On the Positive Fragment of the Polymodal Provability Logic GLP. *Mathematical Notes*, 91(3-4):318–333, 2012.

[9] D. Fernández-Duque and J. J. Joosten. Well-orders in the transfinite japaridze algebra. *Logic Journal of the IGPL*, 22(6):933–963, 2014.

[10] P. Hájek and P. Pudlák. *Metamathematics of First Order Arithmetic.* Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[11] Volker Halbach. *Axiomatic Theories of Truth.* University of Oxford, 2014.

[12] G. Japaridze. The polymodal provability logic. In *Intensional logics and logical structure of theories: material from the Fourth Soviet-Finnish Symposium on Logic*. Metsniereba, Telaviv, 1988. In Russian.

[13] J. J. Joosten. Turing-Taylor expansions of arithmetic theories. *Studia Logica*, 104:1225–1243, 2016.

[14] J. J. Joosten. Münchhausen provability. *Accepted for publication in the Journal of Symbolic Logic*, 2020.

[15] Laurie Kirby and Jeff Paris. Accessible independence results for peano arithmetic. *BULLETIN OF THE LONDON MATHEMATICAL SOCIETY*, 14:725–731, 1982.

[16] G. Kreisel and A. Lévy. Reflection principles and their use for establishing the complexity of axiomatic systems. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 14:97–142, 1968.

[17] D. Leivant. The optimality of induction as an axiomatization of arithmetic. *Journal of Symbolic Logic*, 48:182–184, 1983.

[18] K. Papafilippou and J.J. Joosten. Independent worm battles. In R. Verbrugge and Nicola Olivetti, editors, *Short Papers, Advances in Modal Logic, AiML 2020*, pages 65–69, 2020.

[19] H. E. Rose. *Subrecursion: Functions and Hierarchies.* Oxford University Press, 1984.

[20] U. R. Schmerl. A fine structure generated by reflection formulas over primitive recursive arithmetic. In *Logic Colloquium '78 (Mons, 1978)*, volume 97 of *Stud. Logic Foundations Math.*, pages 335–350. North-Holland, Amsterdam, 1979.

[21] A. Visser. Oracle bites theory. In S. Ghosh and J. Szymanik, editors, *The Facts Matter. Essays on Logic and Cognition in Honour of Rineke Verbrugge*, pages 133–147. College Publications, London, 2015.