Quasinormal modes of black holes

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Abstract: Quasinormal modes are characteristic oscillatory modes that appear when a black hole is perturbed. In this work we focus our attention on computing them with the WKB approximation method. First, we start by revising some of the pioneering methods to calculate them using this approach. We then present a recently developed technique that allows us to achieve very high-order calculations of WKB series in perturbation theory and automates the computation of such modes.

I. INTRODUCTION

When a black hole is subject to external perturbations, characteristic vibration modes appear. These modes decay in time due to gravitational radiation emission. That is the reason why they are called quasinormal modes. They are characterized by complex frequencies. The real part of these frequencies corresponds to the frequency of vibration, and the imaginary part corresponds to the rate at which each mode is damped as a result of the emission of radiation. See [1] for a review.

Since the recent detection of gravitational waves by LIGO [2], black hole quasinormal modes are no longer merely theoretical objects: their measurement is crucial to pin down the mass and angular momentum of the final black hole after a binary merger.

In some exceptional cases they can be computed exactly. Nevertheless, in most cases these calculations require approximations or numerical methods. We will follow the first alternative and use the WKB approximation, widely used in Quantum Mechanics, to compute them.

In the next section we present one of the pioneering techniques in black hole quasinormal modes computation developed by Schutz and Will [3] in the mid 80's, that involves a first order WKB approximation. Next we show succinctly how one could push the computation to higher orders following this path. In the following we will not be interested in pursuing these ideas. We will present a recent method based on the idea of Blome *et al.* [4] to connect black hole quasinormal modes with bound state energies of anharmonic oscillators. This relation will allow us to use a technique due to Bender and Wu [5] to compute quasinormal modes, and will enable us to automate their calculation. Throughout this work we will consider only the Schwarzschild black hole.

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II. WKB APPROXIMATION: FIRST-ORDER CALCULATION

The main motivation to use the WKB approximation is the similarity between the equations of black hole perturbation theory and the one-dimensional time-independent Schrödinger equation for a potential barrier. In both cases the master equation is of the form [1, 3]

$$\frac{d^2\Psi(x)}{dx^2} + Q(x)\Psi(x) = 0. \tag{1}$$

In the black hole case, Ψ represents the radial part of the perturbation variable, x is a *tortoise coordinate* r_* which ranges from $-\infty$ at the event horizon to $+\infty$ at spatial infinity, and

$$-Q(x) := V(x) - \omega^2, \tag{2}$$

where ω corresponds to the quasinormal mode frequency and V is the radial potential, is a complex function that takes constant values at $x = \pm\infty$ (not necessarily equal), and has a maximum at $x = 0$.

In quantum mechanics this function is defined as $-Q(x) := \frac{2m}{\hbar^2}[V(x) - E]$, where E is the energy of the particle of mass m, and $V(x)$ is the potential barrier, assumed to tend to constant values as $x \to \pm\infty$

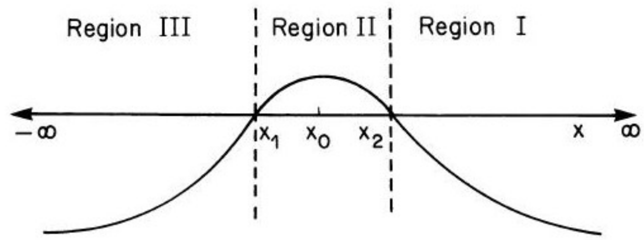

FIG. 1: The function $-Q(x)$. Figure taken from [3].

To determine the quasinormal modes we must impose physically appropriate boundary conditions at the event horizon ($x \to -\infty$) and at spatial infinity ($x \to +\infty$). From now on, we will use the convention that *outgoing* modes refers to the ones moving away from the potential barrier. With this convention *outgoing as* $x \to -\infty$

corresponds to waves crossing the horizon into the black hole.

First, since $-Q(x)$ tends to a constant both at the horizon and at spatial infinity we obtain the behavior $\Psi \sim e^{-i\omega \pm x}$. Now, we know that classically nothing should leave the horizon, so at the horizon only outgoing modes should be considered, therefore $\Psi \sim e^{-i\omega x}$. On the other hand, keeping in mind that a normal mode is a free oscillation of the black hole itself, with no incoming radiation driving it, we need to discard ingoing modes at spatial infinity, hence $\Psi \sim e^{i\omega x}$. Now to apply the WKB approximation we have to relate two WKB solutions across a *matching region* (region II in Fig.1) whose limits are the classical turning points, i.e. where $-Q(x) = 0$. Outside the matching region the WKB functions are given by (see [6])

$$\Psi_I(x) \approx Q^{-\frac{1}{4}} exp\left\{ \pm i \int_{x_2}^{x} [Q(s)]^{\frac{1}{2}} ds \right\}$$

$$\Psi_{III}(x) \approx Q^{-\frac{1}{4}} exp\left\{ \pm i \int_{x}^{x_1} [Q(s)]^{\frac{1}{2}} ds \right\}. \qquad (3)$$

In region II, we approximate $Q(x)$ by a parabola. This is justified provided the turning points are closely spaced, i.e., provided $[-Q(x)]_{max} \ll |Q(\pm\infty)|$. Under this assumption we can expand $Q(x)$ in a Taylor series around its extremum, obtaining

$$Q(x) \approx Q_0 + \frac{1}{2} Q_0''(x - x_0)^2 \qquad (4)$$

where $Q_0 \coloneqq Q(x_0) < 0$, and $Q_0'' \coloneqq \left.\frac{d^2Q}{dx^2}\right|_{x_0} > 0$.

Performing the changes of variable $k \coloneqq \frac{1}{2}Q_0''$, $t \coloneqq (4k)^{\frac{1}{4}} e^{i\frac{\pi}{4}}(x - x_0)$ and $\nu + \frac{1}{2} \coloneqq -i\frac{Q_0}{(2Q_0'')^{\frac{1}{2}}}$, we bring equation (1) into the form

$$\frac{d^2\Psi(t)}{dt^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}t^2\right)\Psi(t) = 0, \qquad (5)$$

whose solutions are parabolic cylinder functions. For large $|t|$ the asymptotic forms of these solutions yield (see [6])

$$\begin{aligned}
\Psi \approx & B e^{\frac{-3i\pi(\nu+1)}{4}} (4k)^{-\frac{(\nu+1)}{4}} (x - x_0)^{-(\nu+1)} e^{ik^{\frac{1}{2}} \frac{(x-x_0)^2}{2}} \\
& + \left[A + \frac{B(2\pi)^{\frac{1}{2}} e^{-\frac{i\nu\pi}{2}}}{\Gamma(\nu+1)} \right] e^{\frac{i\pi\nu}{4}} (4k)^{\frac{\nu}{4}} \\
& \times (x - x_0)^{\nu} e^{-\frac{ik^{\frac{1}{2}}(x-x_0)^2}{2}}, \\
x \gg & x_2 \\
\Psi \approx & A e^{\frac{-3i\pi\nu}{4}} (4k)^{\frac{\nu}{4}} (x_0 - x)^{\nu} e^{-ik^{\frac{1}{2}} \frac{(x-x_0)^2}{2}} \\
& + \left[B - \frac{iA(2\pi)^{\frac{1}{2}} e^{-\frac{i\pi\nu}{2}}}{\Gamma(-\nu)} \right] e^{\frac{i\pi(\nu+1)}{4}} \\
& \times (4k)^{-\frac{(\nu+1)}{4}} (x_0 - x)^{-(\nu+1)} e^{\frac{ik^{\frac{1}{2}}(x-x_0)^2}{2}}, \\
x \ll & x_1, \qquad (6)
\end{aligned}$$

where $\Gamma(\nu)$ is a gamma function. Now it is easily verified that the $e^{-\frac{ik^{\frac{1}{2}}(x-x_0)^2}{2}}$ parts of both solutions in equation (6) match to the outgoing wave of the WKB solutions of equation (3). To satisfy the quasinormal mode boundary conditions, the coefficients of the $e^{\frac{ik^{\frac{1}{2}}(x-x_0)^2}{2}}$ parts must vanish. We achieve this only if $B = 0$ and $\Gamma(-\nu) = \infty$. The latter condition implies that ν must be an integer. This leads us to the following quantization rule for quasinormal modes

$$\frac{Q_0}{(2Q_0'')^{\frac{1}{2}}} = i\left(n + \frac{1}{2}\right), \qquad \forall n \in \mathbb{N}. \qquad (7)$$

Remembering the definition of Q given in (2) (which shows us that it is frequency dependent), we see that (7) allows us to find the quasinormal modes complex frequencies.

Now we can apply the preceding to determine the quasinormal modes for the Schwarzschild black hole. The master equation for such a black hole is given by

$$\frac{d^2\Psi}{dr_*^2} + \left\{ \sigma^2 - \left[1 - \frac{2}{r}\right]\left[\frac{\lambda}{r^2} + \frac{2\beta}{r^3}\right] \right\}\Psi = 0, \qquad (8)$$

where $\lambda = l(l+1)$, where l is the angular harmonic index; $\beta = 1, 0, -3$ for the three types of perturbation, respectively; and $\sigma = M\omega$, where M is the mass of the black hole. The radial coordinates have been expressed in units of M, and r_* is related to r by $\frac{dr}{dr_*} = 1 - \frac{2}{r}$. Note that the Q function corresponds to the quantity in braces in (8). Computing the pertinent derivatives of Q we find that the $-Q$ peak occurs at

$$r = r_0 = \frac{3}{2}\lambda^{-1}\left\{ \lambda - \beta + \left[\lambda^2 + \frac{14\lambda\beta}{9} + \beta^2 \right]^{\frac{1}{2}} \right\}. \qquad (9)$$

Then from equation (7) we obtain

$$\sigma^2 = \left[1 - \frac{2}{r_0}\right]\left[\frac{\lambda}{r_0^2} + \frac{2\beta}{r_0^3}\right] + i(2Q_0'')^{\frac{1}{2}}\left(n + \frac{1}{2}\right), \qquad (10)$$

which allows us to compute the quasinormal modes in this case.

III. PUSHING THE WKB COMPUTATION TO HIGHER ORDERS

Higher order corrections to equation (4) have been computed. In [7], Iyer and Will computed the third order correction, and Konoplya extended it to the sixth order [8]. Recently, the computation up to the 13th order has been done by Matyajasek and Opala [9]. Nevertheless, this showed us that getting to higher orders using this procedure involves harder and harder calculations. We will not follow this path in the following. We just illustrate briefly how the third order WKB expansion could be performed.

First we would rewrite the master equation (1) in the form

$$\epsilon^2 \frac{d^2\Psi}{dx^2} + Q(x)\Psi(x) = 0, \tag{11}$$

where the perturbation parameter ϵ allows us to keep track of orders in the WKB approximation. Then we would define the asymptotic approximation

$$\Psi \sim exp\left[\frac{S(x)}{\epsilon}\right], \tag{12}$$

where S is expanded in powers of ϵ:

$$S(x) = \sum_{n=0}^{\infty} \epsilon^n S_n(x). \tag{13}$$

Substituting (13) in (11), and equating equal powers of ϵ we would then determine $S_n(x)$ until the third order. Now to determine Ψ in the matching region we would have to expand $Q(x)$ in a Taylor series around its extremum up to sixth order. Substituting this expansion in (11) and performing the appropriate changes of variable we would arrive to a new second order linear differential equation. Solving it involves heavier calculations than in the previous section. We finally would arrive at a quantization rule such as (7), namely

$$\begin{gathered}\frac{iQ_0}{(2Q_0'')^{\frac{1}{2}}} - \Lambda(n) - \Omega(n) = n + \frac{1}{2},\\ n = \begin{cases} 0,1,2,... & \text{, if } Re\omega > 0 \\ -1,-2,... & \text{, if } Re\omega < 0 \end{cases},\end{gathered} \tag{14}$$

where

$$\begin{aligned}\Lambda(n) = \frac{1}{(2Q_0'')^{\frac{1}{2}}}\Bigg\{&\frac{1}{8}\left[\frac{Q_0^{(4)}}{Q_0''}\right]\left[\frac{1}{4} + \alpha^2\right] \\ &- \frac{1}{288}\left[\frac{Q_0'''}{Q_0''}\right]^2 (7 + 60\alpha^2)\Bigg\},\end{aligned}$$

$$\begin{aligned}\Omega(n) = \frac{\alpha}{2Q_0''}\Bigg\{&\frac{5}{6912}\left[\frac{Q_0'''}{Q_0''}\right]^4 (77 + 188\alpha^2) \\ &- \frac{1}{384}\left[\frac{Q_0'''^2 Q_0^{(4)}}{Q_0''^3}\right](51 + 100\alpha^2) \\ &+ \frac{1}{2304}\left[\frac{Q_0^{(4)}}{Q_0''}\right]^2 (67 + 68\alpha^2) \\ &+ \frac{1}{288}\left[\frac{Q_0''' Q_0^{(5)}}{Q_0''^2}\right](19 + 28\alpha^2) \\ &- \frac{1}{288}\left[\frac{Q_0^{(6)}}{Q_0''}\right](5 + 4\alpha^2)\Bigg\},\end{aligned} \tag{15}$$

where $\alpha := n + \frac{1}{2}$.

IV. ANHARMONIC OSCILLATOR

We now apparently change the subject and turn our attention to the problem of finding the ground state energy of an anharmonic oscillator [5]. This will provide a very powerful technique to compute the quasinormal modes, once established the connection between black hole quasinormal modes and bound states energies of anharmonic oscillators .

The system under consideration is described by a Hamiltonian of the form

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{g}{4}q^4, \tag{16}$$

where $W(q) = \frac{1}{2}q^2 + W_{int}(q)$ is the potential energy and where $W_{int}(q)$ is an interaction term assumed to be quartic. We assume that the ground energy has the following perturbative expansion

$$E_0(g) = \frac{1}{2} + \sum_{k=1}^{\infty} a_k \left(\frac{g}{4}\right)^k, \tag{17}$$

for some coefficients a_k. Now we write down the Schrödinger equation for our system

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2} + \frac{gx^4}{4}\right)\Psi(x) = E_0(g)\Psi(x). \tag{18}$$

For $g = 0$, we know that the solution to this equation is the ground state of the harmonic oscillator, that is, the Gaussian $e^{-\frac{x^2}{2}}$. This motivates an ansatz for the solution of the form

$$\Psi(x) = e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \left(\frac{g}{4}\right)^n B_n(x), \quad B_0(x) := 1. \tag{19}$$

Plugging this ansatz in equation (18), an recalling that the energy has the form (17), we find the following recursive equation for the $B_k(x)$ and the a_k

$$xB_k'(x) - \frac{1}{2}B_k''(x) + x^4 B_{k-1}(x) = \sum_{p=0}^{k} a_{k-p} B_p(x). \tag{20}$$

In order to solve this recursion, we further assume

$$B_i(x) = \sum_{j=1}^{2i} x^{2j}(-1)^i B_{i,j}. \tag{21}$$

Looking at the term of degree zero in (20) and equating coefficients, we find that

$$a_k = (-1)^{k+1} B_{k,1}. \tag{22}$$

Now substituting (21) and (22) in (20) we obtain the following recursion relation for $B_{i,j}$

$$2jB_{i,j} = (j+1)(2j+1)B_{i,j+1} + B_{i-1,j-2} - \sum_{p=1}^{i-1} B_{i-p,1}B_{p,j}. \tag{23}$$

This recursion can be easily solved and allows us to find the coefficients a_k, and consequently the ground state energy of the system.

V. RECURSIVE APPROACH TO QUASINORMAL MODES

In this section we present a method to compute the quasinormal modes due to Y. Hatsuda [10].

We are now equipped with the tools developed in the previous section. The focus will be put on the study of spherically symmetric black holes. For this type of black holes the radial master equation has the form

$$\left(\epsilon^2 \frac{d^2}{dr_*^2} + \omega^2 - V(r_*)\right)\phi(r_*) = 0, \tag{24}$$

where the notation is as in section (III). The potential $V(r_*)$ is assumed to have a global maximum. Now let us examine the Schrödinger equation with the inverted potential $-V(r_*)$,

$$\left(-\hbar^2 \frac{d^2}{dr_*^2} - V(r_*)\right)\Psi(r_*) = E\Psi(r_*). \tag{25}$$

The inverted potential usually has bound states for $E < 0$. We denote this bound energy states by $E_n^{BS}(\hbar)$ for all $n \in \mathbb{N}$. Now we realize that we can identify equations (24) and (25) by making an analytic continuation of $\hbar$. If we set $\hbar = i\epsilon$, equation (25) with $E = -\omega^2$ is formally identical to (24). We then expect the quasinormal frequency ω_n^{QNM} at $\epsilon = 1$ to be related to the bound state energy E_n^{BS} at $\hbar = i$ by

$$(\omega_n^{QNM})^2 = -E_n^{BS}(\hbar = i). \tag{26}$$

This equation will give us the connection between our problem and the one described in section (IV). Note that we do not provide a rigorous proof of relation (26). We will check its validity by comparing the results obtained by using it to the ones already present in the literature.

A. Steps of the method

First we compute the Taylor expansion of the potential present in the master equation (24), namely

$$-V(r_*) = V_0 + \sum_{k=2}^{\infty} V_k (r_* - r_{0*})^k, \tag{27}$$

where $r_* = r_{0*}$ corresponds to the minimum of the inverted potential. Now it is more convenient to define $r_* - r_{0*} := \sqrt{\hbar}x$, which substituted in (25) gives

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{V_2}{2} + V_{int}\right)\Psi(x) = \epsilon\Psi(x) \tag{28}$$

where

$$\epsilon := \frac{E - V_0}{2\hbar}, \qquad V_{int}(x) := \frac{1}{2}\sum_{k=3}^{\infty} \hbar^{\frac{k}{2}-1} V_k x^k. \tag{29}$$

Now regarding equation (28) as the one of an anharmonic oscillator we can relate our problem to the one presented in section (IV). Nevertheless, note that now our problem is harder since our anharmonic oscillator has an infinite number of interaction terms. At this point, we proceed by using a *Mathematica* package [11] that allows us to compute the perturbative expansion of the energy for a given potential employing the technique described in section (IV). Looking at equation (29) it is important to remark that the interaction term x^k leads to the contribution with order $\hbar^{\frac{k}{2}-1}$, that is, if we want to know the energy perturbative series up to the lth order, we need to compute the Taylor expansion up to the $2(l+1)$th order.

B. Schwarzschild black hole

Now we show how powerful this method is by applying it to the Schwarzschild black hole. We know that for this type of black hole the potential has the form

$$V(r_*) = \left(1 - \frac{1}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{1-s^2}{r^3}\right), \tag{30}$$

where $r_* = r + log(r-1)$, the black hole mass has been set to $2M = 1$, and $s = 0, 1, 2$ corresponds respectively to scalar, electromagnetic and gravitational perturbations. We focus our attention now on the case $s = 2$ and $l = 2$. First, we need to find the minimum of the inverted potential. It attains its minimum value at

$$r_0 = \frac{1}{8}(9 + \sqrt{17}). \tag{31}$$

We define $r_{0*} = r_0 + log(r_0 - 1)$. Now we need to compute the Taylor expansion of the inverted potential around $r_* = r_{0*}$. To do so we use *Mathematica*. We obtain

$$\begin{aligned} -V(r_*) = &-0.60514679 + 0.0793552t^2 - 0.0134244t^3 \\ &- 0.0063813t^4 + 0.0026341t^5 + 0.0001602t^6 \\ &- 0.0003001t^7 + 0.0000425t^8 + \mathcal{O}(t^9), \end{aligned} \tag{32}$$

where $t := r_* - r_{0*}$ (we just show the calculation up to eight order for space reasons). Now the *Mathematica* package [11] automatically computes the perturbative series

$$\epsilon_n^{pert} = \sum_{k=0}^{\infty} \epsilon_n^{(k)} \hbar^k. \tag{33}$$

For the ground state energy to the tenth order we obtain

$$\begin{aligned} \epsilon_0^{pert} = &\, 0.14085 - 0.0399931\hbar + 0.00768009\hbar^2 \\ &- 0.000617727\hbar^3 - 0.000324705\hbar^4 + 0.000181561\hbar^5 \\ &+ 0.000134855\hbar^6 - 0.000396636\hbar^7 + 0.000290997\hbar^8 \\ &+ 0.000510482\hbar^9 - 0.00165866\hbar^{10} + \mathcal{O}(\hbar^{11}) \end{aligned} \tag{34}$$

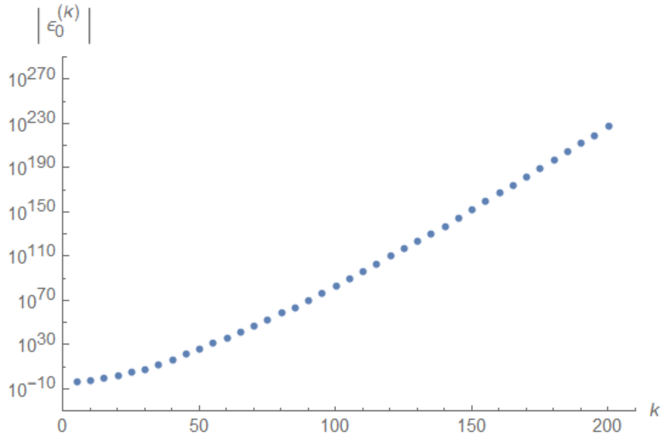

FIG. 2: The large order behavior of $|\epsilon_0^{(k)}|$ for $(s, l) = (2, 2)$.

To get the quasinormal modes we use the relation (26), that is, $\omega_n^2 = -(V_0 + 2\hbar\epsilon_n)$, and set $\hbar = i$. The perturbative series of ϵ_n up to $\hbar^{10}$ gives us the following value:

$$\omega_0(s = 2, l = 2) \approx 0.748934 - 0.179803i. \quad (35)$$

Using *Mathematica* we have computed $\epsilon_0^{(k)}$ up to $k = 200$. In Fig. 2 we show the large order behavior of the absolute values of the ground state coefficients. The seriously growing behavior we can appreciate makes us suspect that (33) is an asymptotic series.

VI. CONCLUSIONS

We have verified that the method proposed by Y. Hatsuda to compute the quasinormal modes works, since it reproduces the values present in the literature. Using this technique, the problem encountered when using the prior methods, namely that to increase an order in the WKB expansion has a very high computational cost, becomes obsolete. Now we can push the calculation to any desired order just by running a simple code using the BenderWu package [11]. In a certain way, the problem has been trivialized.

An interesting question remains open. Is the WKB series expansion of the energy asymptotic or not? And if it is asymptotic, as we suspect, what new physics is behind that fact?

We have limited our verification of the method by Y. Hatsuda to the Schwarzschild black hole. An interesting next step would be to extend this work to the more realistic case of a rotating black hole.

Acknowledgments

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