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Inconsistency lemmas: an algebraic approach

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Introduction

One of the distinguishing features of classical propositional logic **CPL** is that it admits proofs by contradiction. Formally speaking, this means that for every set of formulas $\Gamma \cup \{\alpha\}$,

$$\Gamma \vdash \alpha \quad \text{iff} \quad \Gamma \cup \{\neg\alpha\} \text{ is inconsistent,} \tag{1}$$

where $\vdash$ is the consequence relation naturally associated with **CPL**. While it is well known that intuitionistic propositional logic **IPL** rejects proofs by contradiction, still it satisfies the following weaker principle: for every set of formulas $\Gamma \cup \{\alpha\}$,

$$\Gamma \vdash \neg\alpha \quad \text{iff} \quad \Gamma \cup \{\alpha\} \text{ is inconsistent,} \tag{2}$$

where $\vdash$ is the consequence relation naturally associated with **IPL**. Notice that (1) and (2) can be rewritten as follows:

$$\Gamma \vdash \{\alpha_1, ..., \alpha_n\} \quad \text{iff} \quad \Gamma \cup \{\neg(\alpha_1 \wedge ... \wedge \alpha_n)\} \text{ is inconsistent} \tag{3}$$

and

$$\Gamma \vdash \neg(\alpha_1 \wedge ... \wedge \alpha_n) \quad \text{iff} \quad \Gamma \cup \{\alpha_1, ..., \alpha_n\} \text{ is inconsistent,} \tag{4}$$

respectively. Here $\Gamma \vdash \{\alpha_1, ..., \alpha_n\}$ means that $\Gamma \vdash \alpha_i$ for all $i \in \{1, ..., n\}$.

Within the framework of Abstract Algebraic Logic, a logic is determined by its formal consequence relation $\vdash$ that is seen as a consequence operation on the algebra of formulas [15, 16]. A *(sentential) logic* or a *deductive system* of type L is any pair $\langle \mathbf{Fm}, \vdash \rangle$ where $\mathbf{Fm}$ is the formula algebra of type L with a denumerable set of variables Var and $\vdash \subseteq \mathcal{P}(Fm) \times Fm$ is a relation that satisfies that for all sets of formulas Γ, Δ and all formulas φ, ψ

1.- if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$ (identity),
2.- if $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for all $\psi \in \Gamma$, then $\Delta \vdash \varphi$ (cut),
3.- If $\Gamma \vdash \varphi$ and h is a *substitution*, then $h[\Gamma] \vdash h(\varphi)$ (substitution invariance),

where a substitution h is just an endomorphism of $\mathbf{Fm}$.

A deductive system $\vdash$ is *finitary* if whenever we have $\Gamma \vdash \varphi$, there is a finite subset Δ of Γ in such a way that $\Delta \vdash \varphi$. Throughout this work we shall be concerned with arbitrary deductive systems that we always assume to be finitary.

In this context we say that a given deductive system $\vdash$ has a *classical inconsistency lemma* if there exists a sequence of sets of formulas $\{\Psi_n : n \in \mathbb{N}\}$, where for each n, Ψ_n is a set of formulas whose variables are among $v_1, ..., v_n$, such that for every set of formulas $\Gamma \cup \{\alpha_1, ..., \alpha_n\}$,

$$\Gamma \vdash \{\alpha_1, ..., \alpha_n\} \quad \text{iff} \quad \Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n) \text{ is inconsistent.}$$

And we say that $\vdash$ has an *inconsistency lemma* if there exists a sequence of sets of formulas $\{\Psi_n : n \in \mathbb{N}\}$ as above that satisfies

$$\Gamma \vdash \Psi_n(\alpha_1, ..., \alpha_n) \quad \text{iff} \quad \Gamma \cup \{\alpha_1, ..., \alpha_n\} \text{ is inconsistent}$$

for every set of formulas $\Gamma \cup \{\alpha_1, ..., \alpha_n\}$. Every deductive system with a classical inconsistency lemma has an inconsistency lemma, while the converse is not true in general.

Observe that from (3) we obtain that **CPL** has a classical inconsistency lemma and from (4) it follows that **IPL** has an inconsistency lemma.

One of the most important ideas leading the algebraic approach to the study of sentential logics is that the consequence relation that determines a deductive system can be translated into a consequence relation defined by means of algebraic structures. A deductive system $\vdash$ is *algebraizable* when it is equivalent, in the sense of [3], to the equational consequence $\vDash_{\mathsf{K}}$ relative to a *quasivariety* K of the suitable similarity type. In this case, the quasivariety K is uniquely determined by $\vdash$ and is called the *equivalent algebraic semantics* of $\vdash$ [6]. When a deductive system $\vdash$ is algebraizable with equivalent algebraic semantics K, it is often the case that the metalogical properties of $\vdash$ are reflected by purely algebraic properties of the quasivariety K. For instance, it is well known that $\vdash$ has a *deduction detachment theorem* (DDT) iff K has equationally definable principal relative congruences (EDPC) [7]. It is therefore sensible to wonder whether an algebraic formulation of the inconsistency lemma can be given (as in the case of DDT) and what are the conditions under which a given deductive system has an inconsistency lemma.

The purpose of this work is to investigate what is the algebraic counterpart of the inconsistency lemma. This question already has an answer in [29]. We have only put together in this monograph all the results and concepts that are necessary to understand the whole building.

We will state the inconsistency lemma (together with its classical version) in abstract terms and see that when a given deductive system $\vdash$, that is algebraized by a quasivariety K, has an inconsistency lemma, the set of *compact relative congruences* of any algebra $\mathbf{A}$ in K has the algebraic structure of a *lattice* that is *dually pseudo-complemented* and conversely (Theorem 2.11). Besides, if $\vdash$ has an inconsistency lemma that is classical, then this structure is a *Boolean sublattice* of the lattice of all relative congruences of $\mathbf{A}$ (Theorem 2.26). These are some of the central results the reader will find in this work.

In oder to get the above conclusions it is necessary to establish an important characterization of a quasivariety whose members have *equationally definable principal relative congruence meets* (EDPRM). This characterization theorem says that the quasivarieties with EDPRM are exactly those which are relatively congruence distributive (that is, the set of relative congruences of all the algebras in the quasivariety has the structure of a *distributive* lattice) and whose *finitely relatively subdirectly irreducible* members form a universal class (Theorem 2.22). This result was proved in [11] and it allows to prove that when an algebra $\mathbf{A}$ belongs to a *relatively semisiple* quasivariety with *equationally definable principal relative congruences* (EDPRC), then the compact relative congruences of $\mathbf{A}$ form a sublattice of the lattice of all relative congruences of it (Corollary 2.24).

We also show that when $\vdash$ is a *strongly* algebraizable deductive system (i.e., its equivalent algebraic semantics is a class of algebras that is a *variety*) with a greatest *compact theory*, then $\vdash$ has a classical inconsistency lemma iff it is algebraizable by a *filtral* variety, i.e., semisimple with EDPC (Corollary 2.27). This result has been proven in a more general way. It is true that for every algebraizable logic $\vdash$ with a greatest compact theory it holds that $\vdash$ has a classical inconsistency lemma iff $\vdash$ is algebraized by a filtral quasivariety. In this work we do not give a proof of this fact, we refer the reader to [9].

This is all for what concerns the study of inconsistency lemmas in the setting of algebraizable deductive systems. On the other hand, it is well known that not every deductive system is algebraizable. For instance, the local consequence of the modal system $\mathbf{K}$ is not algebraizable ([6], Corollary 5.6). However, most of the familiar deductive systems have the following two properties:

1. $\vdash \alpha \to \alpha$

and

2. $\{\alpha, \alpha \to \beta\} \vdash \beta$,

for every $\alpha, \beta \in \mathrm{Fm}$. This motivates the following definition. A deductive system $\vdash$ is said to be *protoalgebraic* when there exists a set of formulas $\Lambda(v_1, v_2)$, whose elements have been built with at most two variables v_1, v_2, such that:

3. $\vdash \Lambda(v_1, v_1)$,

4. $v_1, \Lambda(v_1, v_2) \vdash v_2$.

It is well known that all algebraizable deductive systems are protoalgebraic, but the converse is not true in general as, for instance, the local consequence of the modal system $\mathbf{K}$ is easily seen to be protoalgebraic just by taking $\Lambda(v_1, v_2) = \{v_1 \to v_2\}$.

For the case of *protoalgebraic* logics we also have characterization theorems. A protoalgebraic deductive system has an inconsistency lemma in the general sense if and only if its join semilattice of compact theories is dually pseudo-complemented (Theorem 2.8); it has a classical inconsistency lemma if and only if this semilattice is a Boolean lattice (Theorem 2.14). In both cases, the properties of the compact theories are shared by the compact deductive filters of all algebras. In the classical case, we obtain that the set of compact filters is closed under finite intersections. From this fact we deduce that a protoalgebriac logic with a classical inconsistency lemma has a deduction-detachment theorem (Corollary 2.15). On the other hand, if a logic has a deduction-detachment theorem and a greatest *compact theory*, then it must have an inconsistency lemma (Corollary 2.10). We will see that the converse of this results are false (Examples 2.30 and 2.34, respectively).

Finally, at the end of this work, we discuss some examples of deductive systems with inconsistency lemmas and find its corresponding formulation through its own language. Some systems may have an inconsistency lemma even if they lack a DDT (Example 2.34). Also, an algebraizable system with a classical inconsistency lemma can admit as its equivalent algebraic semantics a class of algebras that is not a variety, a different situation from the classical propositional logic (Example 2.35).

Chapter 1

Preliminaries

This chapter is an overview of the principal concepts and results that are the background of our work. Some facts are just presented without proof. At the beginning of each section we refer the reader to our main sources in the literature. The key concepts are those of *compact congruence, quasivariety* and *logic* among others. It is important to keep in mind our presentation of these notions throughout the work.

1.1 Lattices

For a comprehensive study of lattices we refer the reader to [2, 12].

A partially ordered set $\mathbf{L} = \langle L, \leq \rangle$ is a *lattice* if for every $a, b \in L$, $\inf\{a, b\}$ and $\sup\{a, b\}$ exists.

We will use the notations

$$a \wedge b := \inf\{a, b\},$$
$$a \vee b := \sup\{a, b\}.$$

The binary operation $\wedge$ is the *meet* operation of $\mathbf{L}$ and $a \wedge b$ is the meet of a and b. Similarly, the binary operation $\vee$ is the *join* operation of $\mathbf{L}$ and $a \vee b$ is the join of a and b. It follows from the definition that the operations of join and meet obey the following laws:

(L1) $a \wedge a = a$ $\qquad$ $a \vee a = a,$
(L2) $a \wedge b = b \wedge a$ $\qquad$ $a \vee b = b \vee a,$
(L3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ $\qquad$ $a \vee (b \vee c) = (a \vee b) \vee c.$

Moreover, it holds that

$$a \leq b \Leftrightarrow a \wedge b = a \quad \text{and} \quad a \leq b \Leftrightarrow a \vee b = b.$$

They also obey the *absorption laws*:

(L4) $a \wedge (a \vee b) = a$ $\qquad$ $a \vee (a \wedge b) = a.$

A lattice $\mathbf{L}$ is *distributive* if it satisfies the *distributive laws*:

(D1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
(D2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

and it is *complete* if for every subset $X \subseteq L$ both sup X and inf X exist. The elements sup X and inf X will be denoted by $\bigvee X$ and $\bigwedge X$, respectively.

If $\mathbf{L}$ is a lattice and L' is a non-empty subset of L such that for every pair of elements a, b in L' both $a \vee b$ and $a \wedge b$ are in L' (where $\vee$ and $\wedge$ are the lattice operations of $\mathbf{L}$), then we say that L' together with the same operations (restricted to L') is a *sublattice* of $\mathbf{L}$.

Recall that a binary relation R on a set A is a subset of $A \times A$. An *equivalence relation* R on A is a binary relation such that for any a, b, c in A it holds:

(E1) $\langle a, a\rangle \in R$,
(E2) $\langle a, b\rangle \in R$ implies $\langle b, a\rangle \in R$,
(E3) $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ imply $\langle a, c\rangle \in R$.

The poset $\mathrm{Eq}(A)$ of all equivalence relations on A with $\subseteq$ as the partial ordering is a complete lattice. For an arbitrary family $\{\theta_i : i \in I\}$ of elements in $\mathrm{Eq}(A)$ the meet of the family, denoted by $\bigwedge_{i\in I} \theta_i$, is just $\bigcap_{i\in I} \theta_i$ and the join of the family, denoted by $\bigvee_{i\in I} \theta_i$, is $\bigcap\{\theta \in \mathrm{Eq}(A) : \bigcup_{i\in I} \theta_i \subseteq \theta\}$. In particular, for θ_1 and θ_2 in $\mathrm{Eq}(A)$ we have that $\theta_1 \wedge \theta_2 := \theta_1 \cap \theta_2$ and $\theta_1 \vee \theta_2 := \bigcap\{\theta \in \mathrm{Eq}(A) : \theta_1 \cup \theta_2 \subseteq \theta\}$.

Let $\mathbf{L}$ be a lattice. An element $a \in L$ is *compact* if whenever $\bigvee X$ exists and $a \leq \bigvee X$ for $X \subseteq L$, then $a \leq \bigvee Y$ for some finite $Y \subseteq X$. We say that $\mathbf{L}$ is *algebraic* if it is complete and every element in L is a join of compact elements.

If $\langle L, \wedge, \vee\rangle$ is a complete lattice, an element $a \in L$ is said to be *meet irreducible* if it is not the greatest element of L and, whenever $b, c \in L$, if $a = b \wedge c$, then $a = b$ or $a = c$. Besides, $a \in L$ is *completely meet-irreducible* if for every $X \subseteq L$, if $a = \bigwedge X$, then $a \in X$. We say that $a \in L$ is a *co-atom*, if $a < 1$ and there is no $b \in L$ such that $a < b < 1$, where 1 is the greatest element in L. It is an *atom* if $0 < a$ and there is no $b \in L$ such that $0 < b < a$, where 0 is the least element of L. The greatest element 1 is not completely meet-irreducible.

1.2 Elements of Universal Algebra

Our main references for universal algebra are [1] and [8]. For the concept of a quasivariety the reader also can see [21, 27, 28].

An *algebraic language* or *algebraic similarity type* L can be described as a similarity type for a first-order language without relation symbols. It is a set of finitary function and (or) constant symbols. We assume that the constant symbols are function symbols of arity 0. The fists order formulas of an algebraic similarity type L are the formulas that are built with the aid of the equality symbol $\approx$, the usual connectives $\dot\wedge, \dot\vee, \dot\rightarrow, \dot\neg$ and the quantifiers $\forall$ and $\exists$ of a first order language.

First we define the *terms* of L over a set of variables Va, which is assumed to be countably infinite, as

1.- Any variable x is a term of L.
2.- If $f \in L$ is a 0-ary function symbol (i.e., a constant), then f is a term of L.
3.- If f is an n-ary function symbol of L with $n > 0$ and $t_0, ..., t_{n-1}$ are terms of L, then $ft_0...t_{n-1}$ is a term of L.

Then we define the *formulas* of L, or L-formulas over Va, as follows

1.- If t_1, t_2 are terms of L, then $t_1 \approx t_2$ is a formula of L.
2.- If φ is a formula of L, then $\dot{\neg}\varphi$ is a formula of L.
3.- If φ, ψ are formulas of L, so are $\varphi \dot{\wedge} \psi, \varphi \dot{\vee} \psi, \varphi \dot{\to} \psi$.
4.- If φ is a formula of L and x is a variable, then $\forall x\varphi, \exists x\varphi$ are formulas of L.

From the perspective of universal algebra, the most important formulas of an algebraic language L are the *equations*, that is, formulas of the form

$$t \approx t'$$

where t and t' are terms. Other formulas that play an important role in universal algebra are the *quasiequations*, namely the formulas of the form

$$(t_1 \approx s_1 \dot{\wedge} ... \dot{\wedge} t_n \approx s_n) \dot{\to} t_{n+1} \approx s_{n+1}$$

where $t_1, ..., t_n, s_1, ..., s_n$ are terms. The antecedent of the above expression can be empty, in which case we have an equation. Thus, equations are also quasiequations.

Definition 1.1 *Let L be an algebraic similarity type. An L-*algebra *is a structure of type L, that is, a tuple*

$$\mathbf{A} = \langle A, \langle f^{\mathbf{A}} : f \in L \rangle \rangle,$$

where
1.- *A is a non-empty set, called the* universe *of the algebra.*
2.-*For each f of arity 0, $f^{\mathbf{A}}$ is an element of A.*
3.-*For each $f \in L$ of arity $n > 0$, $f^{\mathbf{A}}$ is a n-ary function on A, namely a map from A^n to A.*

When the similarity type L is finite, say $L = \{f_1, ..., f_n\}$, we refer to L-algebras in the following way:

$$\langle A, f_1^{\mathbf{A}}, ..., f_n^{\mathbf{A}} \rangle.$$

If the interpretations of the function symbols are clear we just write

$$\langle A, f_1, ..., f_n \rangle.$$

An L-algebra $\mathbf{A}$ is *finite* if A is, and it is *trivial* if $|A| = 1$.

For instance, if we consider the algebraic similarity type $L = \{f, c\}$, where f is a binary function symbol and c a constant, the structure $\mathbf{A} = \langle \mathbb{N}, f^{\mathbf{A}}, c^{\mathbf{A}} \rangle$, where $\mathbb{N}$ is the set of natural numbers, $f^{\mathbf{A}}$ is the operation of addition and $c^{\mathbf{A}}$ is the number zero, is an algebra of type L. Another algebra of type L is $\mathbf{B} = \langle \mathbb{N}, f^{\mathbf{B}}, c^{\mathbf{B}} \rangle$, where $f^{\mathbf{B}}$ is the product operation and $c^{\mathbf{B}}$ is the number one.

Let L be an algebraic similarity type and let X be a nonempty set of variables disjoin from L. The set of L-terms over X, $Ter(X)$, is the least set of finite sequences of elements of $X \cup L$ such that,

1.- every variable in X belongs to $Ter(X)$;
2.- every constant symbol $c \in L$ belongs $Ter(X)$;
3.- fro every function symbol $f \in L$ of arity $n > 0$ and every $t_1, ...t_n \in Ter(X)$, $ft_1...t_n \in Ter(X)$.

An *L-term* over X is any element of $Ter(X)$.

We can regard the set of L-terms over X from an algebraic perspective and turn it into an L-algebra where the operation that interprets an n-ary function symbol f is the operation that applies the function symbol f to n terms to obtain a new term. In this way we obtain the L-algebra $\mathbf{Ter}_X = \langle Ter(X), \langle f^{\mathbf{Ter}_X} : f \in L\rangle\rangle$, where

1.- if f is a constant symbol, $f^{\mathbf{Ter}_X} = f$,
2.- if f is an n-ary function symbol for $n > 0$, then for every $t_1, ..., t_n \in Ter(X)$, $f^{\mathbf{Ter}_X}(t_1, ..., t_n) = ft_1...t_n$.

The algebra $\mathbf{Ter}_X$ is the *algebra of L-terms over X.*

Given two L-algebras $\mathbf{A}$ and $\mathbf{B}$, a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a function $h : A \to B$ such that

1.- for every constant symbol $c \in L$,

$$h(c^{\mathbf{A}}) = c^{\mathbf{B}},$$

2.- for every n-ary function symbol $f \in L$,

$$h(f^{\mathbf{A}}(a_1, ..., a_n)) = f^{\mathbf{B}}(h(a_1), ..., h(a_n)),$$

for every $a_1, ..., a_n \in A$.

An injective homomorphism is called an *embedding*, and a surjective embedding is called an *isomorphism.* If h is a homomorphism from $\mathbf{A}$ onto $\mathbf{B}$ we say that $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$ and that h is a *surjective homomorphism* or an onto homomorphism. The set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ will be denoted by $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$.

The next theorem shows that the usual procedure to obtain the interpretation of a term in an algebra $\mathbf{A}$ under a fixed interpretation of the variables defines a homomorphism from $\mathbf{Ter}_X$ to $\mathbf{A}$. The theorem will be used frequently without any explicit mention

Theorem 1.2 *Let* $\mathbf{A}$ *be an L-algebra. Any function h from the set of variables X into the universe of* $\mathbf{A}$ *can be extended to a unique homomorphism* $\hat{h}$ *from the set Ter(X) of the L-terms over X to A such that*

1.- $\hat{h}(c) = c^{\mathbf{A}}$, *for every constant symbol c,*
2.- $\hat{h}(ft_1...t_n) = f^{\mathbf{A}}(\hat{h}(t_1), ..., \hat{h}(t_n))$, *for every function symbol of arity* $n > 0$ *and arbitrary terms* $t_1, ..., t_n$.

This theorem says that the algebra $\mathbf{Ter}_X$ is the absolutely free algebra with X as a set of generators.

If we consider a function h from the set of variables X to the universe of an L-algebra $\mathbf{A}$ as an interpretation of the variables, then the homomorphism $\hat{h}$ that the theorem shows to exists is the *denotation function* in $\mathbf{A}$ for the L-terms over X under h. In order to simplify the notation, and due to the uniqueness property, we will write h instead of $\hat{h}$.

Let $\mathbf{A}$ be an L-algebra and let t be a term. With $t(x_1, ..., x_n)$ we indicate that the variables of the term t are among $x_1, ..., x_n$. Given $a_1, ..., a_n \in A$, $t^{\mathbf{A}}(a_1, ..., a_n)$ is the denotation in $\mathbf{A}$ of the term $t(x_1, ..., x_n)$ under any interpretation h such that $h(x_i) = a_i$ for every $1 \leq i \leq n$.

Given an L-algebra $\mathbf{A}$, an equation $t \approx t'$ with $t, t' \in Ter(X)$

(i) is *satisfied* in $\mathbf{A}$ under an interpretation $h : X \to A$, if $h(t) = h(t')$,
(ii) is *valid* in $\mathbf{A}$ if it is satisfied under every interpretation in $\mathbf{A}$.

A quasiequation $t_1 \approx s_1 \dot{\wedge} ... \dot{\wedge}\, t_n \approx s_n \dot{\to} t_{n+1} \approx s_{n+1}$

(i) is *satisfied* in $\mathbf{A}$ under an interpretation $h : X \to A$ if $t_{n+1} \approx s_{n+1}$ is satisfied in $\mathbf{A}$ under h or there is i with $1 \leq i \leq n$ such that $t_i \approx s_i$ is not satisfied in $\mathbf{A}$ under h,
(ii) is *true* or *valid* in $\mathbf{A}$ if it is satisfied under every interpretation in $\mathbf{A}$.

We recall the definition of a first-order formula that is satisfied by an interpretation on an algebra.

Let $\mathbf{A}$ be an L-algebra and $h : Va \to A$ an interpretation. Given $a \in A$ and a variable $x \in Va$ by h^x_a we refer to the map $h^x_a : Va \to A$ such that $h^x_a(x) = a$ and $h^x_a(y) = h(y)$ for every $y \in Va$ different from x. So,

• $t \approx t'$ is *satisfied* in $\mathbf{A}$ under h if $h(t) = h(t')$,
• $\dot{\neg}\varphi$ is *satisfied* in $\mathbf{A}$ under h if φ is not satisfied in $\mathbf{A}$ under h,
• $(\varphi \dot{\wedge} \psi)$ is *satisfied* in $\mathbf{A}$ under h if both φ and ψ are satisfied in $\mathbf{A}$ under h,
• $(\varphi \dot{\vee} \psi)$ is *satisfied* in $\mathbf{A}$ under h if at least one of φ and ψ is satisfied in $\mathbf{A}$ under h,
• $(\varphi \dot{\to} \psi)$ is *satisfied* in $\mathbf{A}$ under h if φ is not satisfied in $\mathbf{A}$ under h or ψ is satisfied in $\mathbf{A}$ under h,
• $\exists x\varphi$ is *satisfied* in $\mathbf{A}$ under h if there exists $a \in A$ such that φ is satisfied in $\mathbf{A}$ under h^x_a,
• $\forall x\varphi$ is *satisfied* in $\mathbf{A}$ under h if for every $a \in A$, φ is satisfied in $\mathbf{A}$ under h^x_a.

We write $\mathbf{A} \vDash \varphi[h]$ to say that the formula φ is satisfied in $\mathbf{A}$ under h.

A formula φ is *valid* in an L-algebra $\mathbf{A}$ if it is satisfied by every interpretation $h : Var \to A$.

Given a class of L-algebras K and a set of equations $\Phi \cup \{\varphi \approx \psi\}$ in variables Var, we define

$\Phi \vDash_{\mathsf{K}} \varphi \approx \psi$ iff for every $\mathbf{A} \in \mathsf{K}$ and every homomorphism $h : \mathbf{Fm}(X) \to \mathbf{A}$, if $h(\sigma) = h(\delta)$ for all $\sigma \approx \delta \in \Phi$, then $h(\varphi) = h(\psi)$.

The relation $\vDash_{\mathsf{K}}$ is called the *equational consequence relative* to K.

We say that $\models_{\mathsf{K}}$ is *finitary* if for every set of variables X and any set of equations $\Phi \cup \{\varphi \approx \psi\}$ in variables X, whenever $\Phi \models_{\mathsf{K}} \varphi \approx \psi$, there exists a finite subset $\Delta \subseteq \Phi$ such that $\Delta \models_{\mathsf{K}} \varphi \approx \psi$.

Any set of sentences of the first-order language of a given algebraic similarity type defines a class of algebras, the class of all the algebras where the sentences are true. In universal algebra and in algebraic logic the classes of algebras defined by equations and by quasiequations are central.

For instance, an algebra of algebraic similarity type $\{+, i, e\}$, where $+$ is a binary function symbol, i is a unary function symbol, and e is a constant symbol, is a *group* if the following equations are valid in it:

(i) $x + (y + z) \approx (x + y) + z$
(ii) $x + e \approx x$ and $e + x \approx x$
(iii) $x + ix \approx ix + x$ and $ix + x \approx e$.

A class K of L-algebras is said to be an *equational class* if there exists a set of L-equations Φ such that K is the class of the algebras where all the equations in Φ are valid. Similarly, a class of L-algebras K is a *quasi-equational class* if there exists a set of L-quasiequations Φ such that K is the class of the algebras where the quasiequations in Φ are valid.

Let $\mathbf{A}$ be an L-algebra and $X \subseteq A$. We say that X is an *L-closed subset* of $\mathbf{A}$ if

1.- for every constant symbol $c \in L$, $c^{\mathbf{A}} \in X$,
2.- for every function symbol $f \in L$ with arity $n > 0$, and any $b_1, ..., b_n \in X$, $f^{\mathbf{A}}(b_1, ..., b_n) \in X$.

An L-algebra $\mathbf{B}$ is a *subalgebra* of an L-algebra $\mathbf{A}$, in symbols $\mathbf{B} \leq \mathbf{A}$, if

1.- B is an L-closed subset of A,
2.- $c^{\mathbf{B}} = c^{\mathbf{A}}$ for every constant symbol $c \in L$,
3.- $f^{\mathbf{B}}(b_1, ..., b_n) = f^{\mathbf{A}}(b_1, ..., b_n)$ for every function symbol $f \in L$ of arity $n > 0$ and every $b_1, ..., b_n \in B$.

Given any subset Y of an L-algebra $\mathbf{A}$, if the least L-closed subset Z of A that contains Y is nonempty, then it can be turned into a subalgebra of $\mathbf{A}$ by interpreting the function symbols as in $\mathbf{A}$ but restricting the interpretation to the elements of Z. The algebra so obtained is called *the subalgebra of* $\mathbf{A}$ *generated by* Y.

Proposition 1.3 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two L-algebras such that* $\mathbf{B} \leq \mathbf{A}$. *Then, every quasiequation that holds in* $\mathbf{A}$, *holds in* $\mathbf{B}$.

A *congruence* of an L-algebra $\mathbf{A}$ is an equivalence relation θ on A where for every n-ary ($n > 0$) function symbol $f \in L$ satisfies the following *compatibility condition*: for every $a_1, ..., a_n, b_1, ..., b_n \in A$,

if $\langle a_i, b_i \rangle \in \theta$ for every $i < n + 1$, then $\langle f^{\mathbf{A}}(a_1, ..., a_n), f^{\mathbf{A}}(b_1, ..., b_n) \rangle \in \theta$.

The set of congruences of an algebra $\mathbf{A}$ will be denoted by Con$\mathbf{A}$.

For instance, if $\mathbf{A}$ is a group and $\mathbf{B}$ is one of its normal subgrups, then the binary relation on A defined by

$$\langle a,b\rangle \in \theta \quad \text{iff} \quad a\cdot b^{-1}\in B$$

is a congruence of $\mathbf{A}$.

Every homomorphism is associated with a congruence in a very natural way. Let h be a homomorphism from an L-algebra $\mathbf{A}$ into an L-algebra $\mathbf{B}$. The *kernel* of h is the relation

$$\ker h = \{\langle a,b\rangle \in A\times A : h(a)=h(b)\}.$$

This relation is a congruence of $\mathbf{A}$.

Given an algebra $\mathbf{A}$, the identity relation $\mathrm{id}_{\mathbf{A}}$ on A and the total relation $A\times A$ are congruences of $\mathbf{A}$. Moreover, given a family $\{\theta_i : i\in I\}$ of congruences of $\mathbf{A}$, its intersection $\bigcap\{\theta_i : i\in I\}$ is also a congruence. Therefore, for any family $\{\theta_i : i\in I\}$ of congruences of $\mathbf{A}$ there is a least congruence $\theta\in \mathrm{Con}\,\mathbf{A}$, namely $\theta=\bigcap\{\gamma\in \mathrm{Eq}(A) : \bigcup_{i\in I}\theta_i\subseteq\gamma\}$, which contains every congruence in the family, it is the join of the family and is denoted by $\bigvee_{i\in I}\theta_i$. This shows that the set of congruences of $\mathbf{A}$ is closed under the meet and join operations of $\mathrm{Eq}(A)$ and hence when it is ordered under the inclusion relation is a complete sublattice of $\mathrm{Eq}(A)$.

If $X\subseteq A\times A$, then we have the set of congruences $\{\theta\in\mathrm{Con}\,\mathbf{A} : X\subseteq\theta\}$, which is nonempty because $A\times A$ is one of its elements. Therefore, we have the congruence $\bigcap\{\theta\in\mathrm{Con}\,\mathbf{A} : X\subseteq\theta\}$. This is the least congruence of $\mathbf{A}$ that includes X. It is called the *congruence generated* by X an it is denoted by $\theta(X)$. If $a,b\in A$, then we denote the congruence generated by $\{\langle a,b\rangle\}$ as $\theta(a,b)$, the congruences of this form are called the *principal congruences*. A congruence $\theta\in\mathrm{Con}\,\mathbf{A}$ is finitely generated if there exists a finite $X\subseteq A\times A$ such that $\theta=\theta(X)$.

Theorem 1.4 *Let* $\mathbf{A}$ *be an algebra and suppose* $a_1,...,a_n,b_1,...,b_n\in A$. *Let* $\theta\in\mathrm{Con}\,\mathbf{A}$. *Then*

1.- $\theta(a_1,b_1)=\theta(b_1,a_1)$,
2.- $\theta(\{\langle a_1,b_1\rangle,...,\langle a_n,b_n\rangle\})=\theta(a_1,b_1)\vee...\vee\theta(a_n,b_n)$,
3.- $\theta=\bigcup\{\theta(a,b) : \langle a,b\rangle\in\theta\}=\bigvee\{\theta(a,b) : \langle a,b\rangle\in\theta\}$.

This shows that Con $\mathbf{A}$ is an algebraic lattice and its compact members are the finitely generated elements $\theta(\{\langle a_1,b_1\rangle,...,\langle a_n,b_n\rangle\})$. Indeed, let $\theta\in\mathrm{Con}\,\mathbf{A}$. If θ is compact, as $\theta=\bigvee\{\theta(a,b) : \langle a,b\rangle\in\theta\}$ by 3 of the above theorem, we have that there must be a finite set $\{\langle a_i,b_i\rangle : 1\leq i\leq n\}$ such that $\theta=\bigvee\{\theta(a_i,b_i) : \langle a_i,b_i\rangle\in\theta\,\text{and}\, i\in\{1,...,n\}\}$. Then, by 2 of the same theorem, $\theta=\theta(\{\langle a_1,b_1\rangle,...,\langle a_n,b_n\rangle\})$ and so θ is finitely generated. Conversely, suppose $\theta=\theta(X)$ for some finite $X\subseteq A\times A$ and assume $\theta\leq\bigvee\{\gamma_i : i\in I\}$, for a family $\{\gamma_i : i\in I\}\subseteq\mathrm{Con}\,\mathbf{A}$. Thus, for every $\langle x,y\rangle\in X$, there is a finite subset $I_{\langle x,y\rangle}=\{i_1,...,i_n\}$ of I such that $\langle x,y\rangle\in(\gamma_{i_1}\circ...\circ\gamma_{i_n})$ (because $\bigvee\{\gamma_i : i\in I\}=\bigcap\{\theta\in\mathrm{Con}\,\mathbf{A} : \bigcup_{i\in I}\gamma_i\subseteq\theta\}$). Take $J=\bigcup_{\langle x,y\rangle\in X} I_{\langle x,y\rangle}$. It is clear that J is finite. Therefore, $\theta=\theta(X)\leq\bigvee\{\gamma_j : j\in J\}$. Consequently, θ is compact.

Recall that, given a set A, an equivalence relation R on A and $a \in A$, the equivalence class of a with respect to R is the set $a/R = \{b \in A : \langle a, b\rangle \in R\}$. The quotient set of A modulo R is the set $A/R = \{a/R : a \in A\}$.

Let $\mathbf{A}$ be an L-algebra and θ a congruence of $\mathbf{A}$. The *quotient algebra* $\mathbf{A}/\theta$ is the L-algebra with universe the quotient set A/θ together with the operations that interpret the function symbols defined as follows:

1.- for every constant symbol $c \in L$,

$$c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta,$$

2.-for every function symbol $f \in L$ with $n \leq 1$,

$$f^{\mathbf{A}/\theta}(a_1/\theta, ..., a_n/\theta) = f^{\mathbf{A}}(a_1, ..., a_n)/\theta.$$

Let $\mathbf{A}$ be an L-algebra and θ one of its congruences. We have an onto homomorphism $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ defined by $\pi_\theta(a) = a/\theta$ for every $a \in A$. It is called the *natural homomorphism* from $\mathbf{A}$ onto $\mathbf{A}/\theta$.

Note that for any algebra $\mathbf{A}$ and any one of its congruences θ, every equation valid in $\mathbf{A}$ is also valid in $\mathbf{A}/\theta$.

Theorem 1.5 (First Homomorphism Theorem) *Suppose $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism onto B. Then there exists an isomorphism $f : \mathbf{A}/\ker h \to \mathbf{B}$ given by $f(a/\ker h) = h(a)$ for all $a \in A$.*

For a closed interval $[a, b]$ of a lattice $\mathbf{L}$, where $a \leq b$, let $\mathbf{[}a, b\mathbf{]}$ be the corresponding sublattice of $\mathbf{L}$.

Theorem 1.6 (Correspondence Theorem) *Let $\mathbf{A}$ be an algebra and let $\theta \in \mathrm{Con}\,\mathbf{A}$. Then, there exists a lattice isomorphism from the interval $\mathbf{[}\theta, A \times A\mathbf{]}$ in $\mathrm{Con}\ \mathbf{A}$ to $\mathrm{Con}\,\mathbf{A}/\theta$.*

Given a class of L-algebras K and an L-algebra $\mathbf{A}$, the set of congruences θ of $\mathbf{A}$ whose quotient $\mathbf{A}/\theta$ belongs to K will be denoted by $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. These congruences are the so called K-*congruences* or K-*relative congruences* of $\mathbf{A}$. When K is a quasi-equational class, this set $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ is closed under arbitrary intersections and so it forms a complete lattice. Hence, for any pair of elements $a, b \in A$ we can form a least congruence on $\mathbf{A}$ that contains $\{\langle a, b\rangle\}$ and belongs to $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$, this congruence will be denoted by $\theta_{\mathsf{K}}(a, b)$. Moreover, the lattice $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ of K-congruences of $\mathbf{A}$ is also algebraic and the compact elements are the finitely generated K-congruences. If K is an equational class, for every $\mathbf{A} \in \mathsf{K}$ each congruence of $\mathbf{A}$ is a K-congruence of $\mathbf{A}$. Therefore, in this case $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A} = \mathrm{Con}\,\mathbf{A}$.

Let $\{A_i : i \in I\}$ be a family of sets. A choice function for $\{A_i : i \in I\}$ is a map $a : I \to \bigcup_{i\in I} A_i$ such that $a(i) \in A_i$ for every $i \in I$. The *Cartesian product* of $\{A_i : i \in I\}$ is the set

$$\prod_{i\in I} A_i := \{a : a \text{ is a choice function for } \{A_i : i \in I\}\}.$$

Let us fix an algebraic similarity type L. Given a family $\{\mathbf{A}_i : i \in I\}$ of L-algebras, the *direct product* is the L-algebra $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$ with universe the

Cartesian product $\prod_{i\in I} A_i$ whose interpretations of the function symbols is as follows.

1.- If c is a constant symbol, $c^{\mathbf{A}}$ is the choice function defined by

$$c^{\mathbf{A}}(i) = c^{\mathbf{A}_i} \text{ for each } i \in I.$$

2.- If f is a n-ary function symbol with $n \geq 1$, $f^{\mathbf{A}}$ is the choice function defined by

$$f^{\mathbf{A}}(a_1, ..., a_n)(i) = f^{\mathbf{A}_i}(a_1(i), ..., a_n(i))$$

for each $i \in I$ and all $a_1, ..., a_n \in \prod_{i\in I} A_i$.

The empty product $\prod \emptyset$ is the trivial algebra with universe $\{\emptyset\}$.

Let $\{\mathbf{A}_i : i \in I\}$ be a family of L-algebras and let $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$. For every L-term $t(x_1, ..., x_n)$ and every $a_1, ..., a_n \in \prod_{i\in I} A_i$ we have

$$t^{\mathbf{A}}(a_1, ..., a_n)(i) = t^{\mathbf{A}_i}(a_1(i), ..., a_n(i)).$$

An algebra $\mathbf{A}$ is a *subdirect product* of a family $\{\mathbf{A}_i : i \in I\}$ of L-algebras if $\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$ and $\pi_i(\mathbf{A}) = \mathbf{A}_i$ for each $i \in I$, where for all $j \in I$, $\pi_j : \prod_{i\in I} A_i \to A_j$ is the *projection map* defined as $\pi_j(a) = a(j)$ which give us a surjective homomorphisms $\pi_j : \prod_{i\in I} \mathbf{A}_i \to \mathbf{A}_j$.

Proposition 1.7 *Let $\{\mathbf{A}_i : i \in I\}$ be a family of L-algebras. Then an L-equation $\varphi \approx \psi$ is valid in every algebra $\mathbf{A}_i$ if and only if it is valid in the direct product $\prod_{i\in I} \mathbf{A}_i$.*

Let I be a set. A *filter over* I is a non-empty family $\mathcal{F}$ of subsets of I such that

1.- if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.
2.- If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$.

A filter $\mathcal{F}$ is proper if it is different from $\mathcal{P}(I)$. An *ultrafilter over* I is a proper filter $\mathcal{F}$ over I such that for every $X \subseteq I$, $X \in \mathcal{F}$ or $I - X \in \mathcal{F}$.

Let $\{\mathbf{A}_i : i \in I\}$ be a family of L-algebras and suppose $\mathcal{F}$ is a filter over I. We define the binary relation $\theta_{\mathcal{F}}$ on $\prod_{i\in I} A_i$ by

$$\langle a, b\rangle \in \theta_{\mathcal{F}} \quad \text{iff} \quad \{i \in I : a(i) = b(i)\} \in \mathcal{F},$$

for each $a, b \in \prod_{i\in I} A_i$.

This relation $\theta_{\mathcal{F}}$ is a congruence relation of the algebra $\prod_{i\in I} \mathbf{A}_i$.

The quotient algebra $\prod_{i\in I} \mathbf{A}_i/\theta_{\mathcal{F}}$, denoted also by $\prod_{i\in I} \mathbf{A}_i/\mathcal{F}$, is the *reduced product* of the family $\{\mathbf{A}_i : i \in I\}$ modulo $\mathcal{F}$. If all the algebras are the same, say $\mathbf{A}$, then $\prod_{i\in I} \mathbf{A}_i/\mathcal{F}$ is called the *reduced power* of $\mathbf{A}$ modulo $\mathcal{F}$. If $\mathcal{F}$ is a ultrafilter over I, $\prod_{i\in I} \mathbf{A}_i/\mathcal{F}$ is called the *ultraproduct* of $\{\mathbf{A}_i : i \in I\}$ modulo $\mathcal{F}$. We denote by $a/\mathcal{F}$ the equivalence class of $a \in \prod_{i\in I} A_i$ by the congruence relation $\theta_{\mathcal{F}}$.

Theorem 1.8 (Łos) *Let $\{\mathbf{A}_i : i \in I\}$ be a family of L-algebras and let $\mathcal{U}$ be an ultrafilter over I. For every formula $\psi(x_1, ..., x_n)$ and each $a_1, ..., a_n \in \prod_{i\in I} A_i$,*

$$\prod_{i\in I} \mathbf{A}_i/\mathcal{U} \vDash \psi(a_1/\mathcal{U}, ..., a_n/\mathcal{U}) \quad \text{iff} \quad \{i \in I : \mathbf{A}_i \vDash \psi(a_1(i), ..., a_n(i))\} \in \mathcal{U}.$$

If $\{\mathbf{A}_i : i \in I\}$ is a family of L-algebras and $\mathcal{F}$ a filter over I, every equation and quasiequation valid in all the algebras $\mathbf{A}_i$ is valid in the reduced product $\prod_{i\in I}\mathbf{A}_i/\mathcal{F}$.

Let L be an algebraic similarity type. A *variety* of L-algebras is a class of L-algebras which is closed under subalgebras, homomorphic images and direct products. Note that it is therefore closed under isomorphic copies.

Given a class of algebras K of a given similarity type L, the *variety generated* by K, denoted as $\mathbf{V}(\mathsf{K})$, is the least variety of L-algebras which includes K, that is, the least class of L-algebras which includes K and is closed under subalgebras, homomorphic images and direct products.

We introduce the following operators mapping classes of algebras to classes of algebras of the same type.

$\mathbf{A} \in \mathbf{I}(\mathsf{K})$ iff $\mathbf{A}$ is isomorphic to some member of K.
$\mathbf{A} \in \mathbf{S}(\mathsf{K})$ iff $\mathbf{A}$ is a subalgebra of some member of K.
$\mathbf{A} \in \mathbf{H}(\mathsf{K})$ iff $\mathbf{A}$ is a homomorphic image of some member of K.
$\mathbf{A} \in \mathbf{P}(\mathsf{K})$ iff $\mathbf{A}$ is a direct product of a nonempty family of algebras in K.
$\mathbf{A} \in \mathbf{P}_S(\mathsf{K})$ iff $\mathbf{A}$ is a subdirect product of a family of algebras in K.
$\mathbf{A} \in \mathbf{P}_U(\mathsf{K})$ iff $\mathbf{A}$ is an ultraproduct of a family of algebras in K.

If O_1 and O_2 are two operators on classes of algebras we write O_1O_2 for the composition of the two operators. A class of algebras K is closed under an operator O if $O(\mathsf{K}) \subseteq \mathsf{K}$.

Theorem 1.9 (Tarski) *For every class of algebras* K, $\mathbf{V}(\mathsf{K}) = \mathbf{HSP}(\mathsf{K})$.

Another characterization of varieties is the following:

Theorem 1.10 (Birkhoff) *A class of L-algebras is a variety if and only if it is an equational class.*

Recall that a *trivial algebra* is an algebra whose universe is a unitary set. All trivial algebras of the same type are isomorphic. Let us denote a canonical one, for example the one whose universe is $\{0\}$, by **Triv**.

Note that a variety contains all the trivial algebras of its similarity type since the trivial algebras are homomorphic images of every algebra.

A *quasivariety* of L-algebras is a class of L-algebras which is closed under isomorphism, subalgebras, direct products, ultraproducts and contains a trivial algebra. The *quasivariety generated* by a class of L-algebras K is the least quasivariety of L-algebras that includes K.

Theorem 1.11 *The quasivariety generated by a class of L-algebras K is the class of algebras* $\mathbf{ISPP}_U(\mathsf{K} \cup \{\mathbf{Triv}\})$.

And as in the case of varieties and equations we have a similar situation for quasivarieties and quasiequations.

Theorem 1.12 *A class of L-algebras is a quasivariety if and only if it is a quasi-equational class.*

Remark 1.13 *Let* $\mathbf{A}$ *be an algebra in a quasivariety* K *and* θ *a congruence of* $\mathbf{A}$*. Then,* $\theta \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ *if and only if for every quasiequation*

$$\bigwedge_{i<m} t_j(x_1,...,x_k) \approx s_j(x_1,...,x_k) \to t(x_1,...,x_k) \approx s(x_1,...,x_k)$$

valid in K *and all* $a_1,...,a_k \in A$*, if for every* $j < m$ $\langle t_j^{\mathbf{A}}(a_1,...,a_k), s_j^{\mathbf{A}}(a_1,...,a_k)\rangle \in \theta$*, then* $\langle t^{\mathbf{A}}(a_1,...,a_k), s^{\mathbf{A}}(a_1,...,a_k)\rangle \in \theta$.

In a quasivariety K, an algebra $\mathbf{A} \in \mathsf{K}$ is *finitely subdirectly irreducible* if the identity relation on $\mathbf{A}$ is a finitely meet irreducible element of the complete lattice $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. It is *sudirectly irreducible* in K if the identity relation on $\mathbf{A}$ is a completely meet irreducible element of $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. We denote by $\mathsf{K}_{\mathrm{RFSI}}$ the class of elements of K that are finitely subdirectly irreducible in K and by $\mathsf{K}_{\mathrm{RSI}}$ the class of elements in K that are subdirectly irreducible in K.

Lemma 1.14 *For every non-trivial* $\mathbf{A}$ *in a quasivariety* K *and every* $a, b \in A$ *such that* $a \neq b$*, there exists a* K*-congruence* $\theta_{(a,b)}$ *of* $\mathbf{A}$ *such that* $\langle a, b\rangle \notin \theta_{(a,b)}$ *and is maximal with respect to inclusion with this property and* $\mathbf{A}/\theta_{(a,b)} \in \mathsf{K}_{\mathrm{RSI}}$.

Proof. Let $\mathbf{A} \in \mathsf{K}$ be a non-trivial algebra and $a, b \in A$ such that $a \neq b$. We first prove that there exists a K-congruence $\theta_{(a,b)}$ of $\mathbf{A}$ such that $\langle a, b\rangle \notin \theta_{(a,b)}$ and is maximal with respect to inclusion.

Consider the set

$$\Gamma = \{\theta \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A} : \langle a, b\rangle \notin \theta\}.$$

Since $\mathbf{A} \in \mathsf{K}$, we have that the identity relation $\mathrm{Id}_{\mathbf{A}}$ on $\mathbf{A}$ belongs to $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. It is easy to see, using Remark 1.13, that Γ is closed under unions of chains. By Zorn's lemma, there exists a maximal element with respect to inclusion in Γ. Thus we have attained our goal.

Now, we prove that $\mathbf{A}/\theta_{(a,b)} \in \mathsf{K}_{\mathrm{RSI}}$. Let Δ be the identity relation on $\mathbf{A}/\theta_{(a,b)}$. We prove that it is completely meet irreducible element of $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}/\theta_{(a,b)}$. Since $\theta_{(a,b)} \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ we have $\mathbf{A}/\theta_{(a,b)} \in \mathsf{K}$. Therefore, $\Delta \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}/\theta_{(a,b)}$. Let now $\{\theta_i : i \in I\}$ be a family of K-congruences of $\mathbf{A}/\theta_{(a,b)}$ such that

$$\Delta = \bigcap_{i\in I} \theta_i.$$

Consider the quotient homomorphism $\pi : \mathbf{A} \to \mathbf{A}/\theta_{(a,b)}$ which is onto. Then, $\theta_{(a,b)} = \ker \pi = \pi^{-1}[\Delta]$. Therefore,

$$\theta_{(a,b)} = \pi^{-1}[\Delta] = \pi^{-1}[\bigcap_{i\in I} \theta_i] = \bigcap_{i\in I} \pi^{-1}[\theta_i].$$

Using Remark 1.13 we have that for every $i \in I$, $\pi^{-1}[\theta_i]$ is a K-congruence of $\mathbf{A}$. Now, since $\langle a, b\rangle \notin \theta_{(a,b)}$, there exists $i \in I$ such that $\langle a, b\rangle \notin \pi^{-1}[\theta_i]$ and since $\theta_{(a,b)} \subseteq \pi^{-1}[\theta_i]$, the maximality of $\theta_{(a,b)}$ in Γ implies that $\theta_{(a,b)} = \pi^{-1}[\theta_i]$. The surjectivity of π implies the second equality in

$$\pi[\theta_{(a,b)}] = \pi[\pi^{-1}[\theta_i]] = \theta_i.$$

Hence, since $\pi[\theta_{(a,b)}] = \Delta$, we have $\Delta = \theta_i$. Thus, Δ is completely meet irreducible in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}/\theta_{(a,b)}$. Therefore, $\mathbf{A}/\theta_{(a,b)} \in \mathsf{K}_{\mathrm{RSI}}$. ■

We have that a quasivariety K is generated by its finitely subdirectly irreducible elements in the following sense.

Proposition 1.15 *Let* K *be a quasivariety. Every* $\mathbf{A} \in \mathsf{K}$ *is isomorphic to a subdirect product of elements in* $\mathsf{K}_{\mathrm{RSI}}$ *and therefore in* $\mathsf{K}_{\mathrm{RFSI}}$.

Proof. Let $\mathbf{A} \in \mathsf{K}$. If $\mathbf{A}$ is trivial, then it holds because $\mathbf{A}$ is isomorphic to a subdirect product of the empty family. Assume now that $\mathbf{A}$ is not trivial and consider the set $I = \{\langle a, b\rangle \in A \times A : a \neq b\}$. For every $\langle a, b\rangle \in I$, let, using Lemma 1.14, $\theta_{(a,b)} \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ such that $\langle a, b\rangle \notin \theta_{(a,b)}$, which is maximal with respect to inclusion with this property, and $\mathbf{A}/\theta_{(a,b)} \in \mathsf{K}_{\mathrm{RSI}}$. We prove that $\mathbf{A}$ is subdirectly embeddable in $\prod_{\langle a,b\rangle\in I} \mathbf{A}/\theta_{(a,b)}$ by the map $g : A \to \prod_{\langle a,b\rangle\in I} A/\theta_{(a,b)}$ defined by setting for every $c \in A$

$$g(c) = \langle c/\theta_{(a,b)} : \langle a, b\rangle \in I\}.$$

It is easy to see that g is a homomorphism. It is injective because if $c, d \in A$ are different, then $\langle c, d\rangle \notin \theta_{(c,d)}$ and therefore $c/\theta_{(c,d)} \neq d/\theta_{(c,d)}$; hence $g(c) \neq g(d)$. Let for every $\langle a, b\rangle \in I$, $\pi_{(a,b)} \prod_{\langle a,b\rangle\in I} \mathbf{A}/\theta_{(a,b)} \to \mathbf{A}/\theta_{(a,b)}$ be the projection homomorphism. We prove that $\pi_{(a,b)} \circ g$ is onto $\mathbf{A}/\theta_{(a,b)}$ for every $\langle a, b\rangle \in I$. Fix $\langle a, b\rangle \in I$. Let $c/\theta_{(a,b)} \in A/\theta_{(a,b)}$. Then, $\pi_{(a,b)}(g(c)) = c/\theta_{(a,b)}$. Therefore, $\pi_{(a,b)} \circ g$ is onto $\mathbf{A}/\theta_{(a,b)}$. Now, since as we proved, that $\mathbf{A}$ is subdirectly embeddable in $\prod_{\langle a,b\rangle\in I} \mathbf{A}/\theta_{(a,b)}$ by g, it follows that $\mathbf{A}$ is isomorphic to a subdirect product of $\{\mathbf{A}/\theta_{(a,b)} : \langle a, b\rangle \in I\}$. ■

This proposition implies the following corollary.

Corollary 1.16 *The quasivariety generated by* $\mathsf{K}_{\mathrm{RFSI}}$ *is* K.

Proof. Since $\mathsf{K}_{\mathrm{RFSI}} \subseteq \mathsf{K}$, it is clear that the quasivariety generated by $\mathsf{K}_{\mathrm{RFSI}}$ is included in K. To prove the other inclusion, let $\mathbf{A} \in \mathsf{K}$. By Proposition 1.15 there is a family $\{\mathbf{A}_i : i \in I\}$ of elements in $\mathsf{K}_{\mathrm{RFSI}}$ such that $\mathbf{A}$ is isomorphic to a subalgebra of $\prod_{i\in I} \mathbf{A}_i$. Therefore, $\mathbf{A}$ belongs to the quasivariety generated by $\mathsf{K}_{\mathrm{RFSI}}$. ■

An analogous situation holds for varieties.

Corollary 1.17 *If* K *is a variety, then every member of* K *is isomorphic to a subdirect product of subdirectly irreducible members of* K.

Let K be a class of L-algebras, $\mathbf{B}$ an L-algebra and $X \subseteq B$. We say that $\mathbf{B}$ has *the universal mapping property for* K *over* X if for every $\mathbf{A} \in \mathsf{K}$ it holds that every map $f : X \to A$ can be extended to a homomorphism $h : \mathbf{B} \to \mathbf{A}$. If X generates $\mathbf{B}$ we say that $\mathbf{B}$ is *free for* K *over* X.

Let K be a class of L-algebras and let X be a nonempty set of variables. We can consider the set

$$Eq_X(\mathsf{K}) := \{\langle t, t' \rangle \in Ter(X) \times Ter(X) : t \approx t' \text{ is valid in every algebra in } \mathsf{K}\}.$$

It is easy to see that this set is a congruence relation on $\mathbf{Ter}_X$. Let us consider the quotient algebra $\mathbf{Ter}_X/Eq_X(\mathsf{K})$. We denote it by $F_{\mathsf{K}}(X)$. This algebra is generated by the set $X/Eq_X(\mathsf{K}) := \{x/Eq_X(\mathsf{K}) : x \in X\}$ and has the universal mapping property for K over $X/Eq_X(\mathsf{K})$.

Theorem 1.18 *If* K *is a class of* L*-algebras with at least one non trivial algebra an* X *is a nonempty set of variables, then* $F_{\mathsf{K}}(X) \in \mathbf{SP}(\mathsf{K})$.

1.3 The abstract concept of logic

For a general background on Abstract Algebraic Logic and for the concept of logic we work with we refer the reader to [6, 15, 16, 17]. For the concept of protoalgebraic logic and related topics see [5, 10].

Given an algebraic similarity type L we consider the free algebra of type L with a denumerable set of generators. This algebra is the algebra of L-terms. In this context we call it *the formula algebra of type L* and is denoted by $\mathbf{Fm}_L$. The generators of the algebra are called the *propositional variables* and the set of all of them is denoted by Var. We assume that the generators are enumerated by a fixed enumeration $x_1, x_2, ..., x_n, ...$ and all are distinct. The function symbols are called *connectives* and the constant symbols *propositional variables.*

Throughout this monograph we shall work with the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ (without 0). The reason of this convention will become clear later on. Also, if no confusion is likely to arise we will delete the subscript L from **Fm**.

A *substitution* is any function from Var to Fm. Since $\mathbf{Fm}_L$ is the absolutely free algebra of type L, any substitution h can be extended in a unique way to a homomorphism from $\mathbf{Fm}_L$ into $\mathbf{Fm}_L$ that we denote also by h. We therefore identify substitutions with homomorphisms from the formula algebra to itself. Accordingly, we will indistinctly write $h : Var \to \mathbf{Fm}$ and $h : \mathbf{Fm} \to \mathbf{Fm}$.

Since we are working with an infinite set of variables, we may assume another denumerable sequence of distinct variables $v_1, v_2, ...$ (different from the generators) that we shall use to avoid ambiguities in some substitutions.

For each $n \in \mathbb{N} = \{1, 2, 3, ...\}$ we define

$$\mathrm{Fm}(n) = \{\beta \in \mathrm{Fm} : \text{the variables occurring in } \beta \text{ are among } v_1, ..., v_n\}.$$

If $\sigma \in \mathrm{Fm}(n)$ and $\mathbf{A}$ is an algebra, with $a_1, ..., a_n \in A$, then $\sigma^{\mathbf{A}}(a_1, ..., a_n)$ denotes $h(\sigma)$, where $h : \mathbf{Fm} \to \mathbf{A}$ is any homomorphism such that $h(v_i) = a_i$ for $i = 1, ..., n$. If $\Xi \subseteq \mathrm{Fm}(n)$, then $\Xi^{\mathbf{A}}(a_1, ..., a_n)$ abbreviates $\{\xi^{\mathbf{A}}(a_1, ..., a_n) : \xi \in \Xi\}$.

A set of formulas Δ is *closed under substitutions* if for every substitution h and every $\varphi \in \Delta$, $h(\varphi) \in \Delta$, i.e., $h[\Delta] \subseteq \Delta$.

A *(sentential) logic* or a *deductive system* of type L is any pair $\langle \mathbf{Fm}, \vdash \rangle$ where $\mathbf{Fm}$ is the formula algebra of type L and $\vdash \subseteq \mathcal{P}(Fm) \times Fm$ is a *consequence* *relation* on Fm, that is, it satisfies that for all sets of formulas Γ, Δ and all formulas φ, ψ

1.- if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$ (identity),
2.- if $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for all $\psi \in \Gamma$, then $\Delta \vdash \varphi$ (cut),

and in addition it also satisfies for every set of formulas Γ and each formula φ the following condition:

3.- If $\Gamma \vdash \varphi$ and h is a substitution, then $h[\Gamma] \vdash h(\varphi)$.

This latter condition is called *substitution invariance.* From (1) and (2) it follows

4.- if $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$ (monotonicity).

A logic $\langle \mathbf{Fm}, \vdash \rangle$ is *finitary* if $\vdash$ satisfies

5.- if $\Gamma \vdash \varphi$, then there is a finite set $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$ (finitarity).

The *finitary companion* $\vdash^f$ of a deductive system $\vdash$ is defined by requiring

$$\Gamma \vdash^f \varphi \quad \text{iff} \quad \text{there exists a finite } \Delta \subseteq \Gamma \text{ such that } \Delta \vdash \varphi,$$

for every set of formulas $\Gamma \cup \{\varphi\}$.

We shall work along this monograph only with finitary deductive systems. In order to simplify the notation we just use $\vdash$ to denote a given logic $\langle \mathbf{Fm}, \vdash \rangle$.

Let $\vdash$ be a logic and Γ a set of formulas. We say that a formula φ is *deducible* from Γ if $\Gamma \vdash \varphi$. The *theorems* of a logic $\vdash$ are the formulas φ that are deducible from the emptyset. In this case we write $\vdash \varphi$.

A logic $\vdash$ is *inconsistent* if every formula φ is a theorem, or equivalently, if $\Gamma \vdash \varphi$ for every set of formulas Γ and every formula φ. A logic $\vdash$ is *almost inconsistent* if it has no theorems and $\Gamma \vdash \varphi$ for every nonempty set of formulas Γ and every formula φ.

The *identity logic* of type L is the logic whose consequence relation $\vdash_{id}$ is defined by setting for every set of formulas Γ and every formula φ

$$\Gamma \vdash_{id} \varphi \quad \text{iff} \quad \varphi \in \Gamma.$$

This consequence relation will be called the *identity consequence relation.*

We will use the notation $\Gamma, \varphi \vdash \psi$ instead of $\Gamma \cup \{\varphi\} \vdash \psi$ and similar abbreviations like $\varphi_1, ..., \varphi_n \vdash \psi$, which is a shorthand for $\{\varphi_1, ..., \varphi_n\} \vdash \psi$. In particular $\varphi \vdash \psi$ abbreviates $\{\varphi\} \vdash \psi$. Also we write $\Gamma \vdash \Delta$, where Γ and Δ are sets of formulas, to indicate that for every $\delta \in \Delta$, $\Gamma \vdash \delta$. If also $\Delta \vdash \Gamma$, we just write $\Gamma \dashv\vdash \Delta$.

Let L be an algebraic similarity type and let $\vdash$ be a logic of type L. A logic $\vdash^*$ of type L is an *extension* of $\vdash$ if for any set of formulas Γ and any formula φ,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \Gamma \vdash^* \varphi.$$

Let L be an algebraic language. The set of all consequence relations on Fm_L that are invariant under substitutions ordered under the relation of extension is a complete lattice, where the top element is the inconsistent consequence relation, the bottom element is the identity consequence relation and the infimum of a nonempty family is the intersection.

Let $\vdash$ be a logic of type L. We define for every $\Gamma \subseteq \mathrm{Fm}$

$$\mathrm{Cn}_{\vdash}(\Gamma) = \{\varphi \in \mathrm{Fm} : \Gamma \vdash \varphi\}.$$

Hence, for any set of formulas Γ and Δ we have:

1.- $\Gamma \subseteq \mathrm{Cn}_{\vdash}(\Gamma)$,
2.- if $\Gamma \subseteq \Delta$, then $\mathrm{Cn}_{\vdash}(\Gamma) \subseteq \mathrm{Cn}_{\vdash}\Delta$,
3.- $\mathrm{Cn}_{\vdash}(\mathrm{Cn}_{\vdash}(\Gamma)) = \mathrm{Cn}_{\vdash}(\Gamma)$.

For any set of fomulas Γ and any substitution h it holds, by substitution invariance, that $\mathrm{Cn}_{\vdash}(h[\Gamma]) = \mathrm{Cn}_{\vdash}(h[\mathrm{Cn}_{\vdash}(\Gamma)])$.

We have that a logic $\vdash$ is finitary if and only if for every set of formulas Γ, $\mathrm{Cn}_{\vdash}(\Gamma) = \bigcup\{\mathrm{Cn}_{\vdash}(\Delta) : \Delta \subseteq \Gamma \text{ and } \Delta \text{ is finite}\}$.

Let L be an algebraic similarity type. An *L-matrix* is a pair $\langle \mathbf{A}, F\rangle$ such that $\mathbf{A}$ is an algebra of type L and F is a subset (possibly empty) of its domain.

Given an L-matrix $\langle \mathbf{A}, F\rangle$ the set F is known as the *filter* or the *truth set* of the matrix and $\mathbf{A}$ as the algebra of the matrix. A *g-sequent* of $\vdash$ is a pair $\langle \Gamma, \varphi\rangle$ where Γ is a set of L-formulas, φ an L-formula and $\Gamma \vdash \varphi$.

An L-matrix is a *model of* a g-sequent $\langle \Gamma, \varphi\rangle$, in symbols $\langle \mathbf{A}, F\rangle \vDash \langle \Gamma, \varphi\rangle$, if for every homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ such that $h[\Gamma] \subseteq F$, it holds that $h(\varphi) \in F$. If a matrix $\langle \mathbf{A}, F\rangle$ is a model of a g-sequent $\langle \Gamma, \varphi\rangle$, then it is a model of its substitution instances, that is, for every substitution h it holds that $\langle \mathbf{A}, F\rangle \vDash \langle h[\Gamma], h(\varphi)\rangle$.

Let $\vdash$ be a logic in the algebraic similarity type L. An L-matrix $\langle \mathbf{A}, F\rangle$ is a *model of* $\vdash$ if it is a model of every g-sequent of $\vdash$.

Any L-matrix can be used to define a logic of type L. Let $\mathcal{M} = \langle \mathbf{A}, F\rangle$ be an L-matrix. Consider the set of all g-sequents $\langle \Gamma, \varphi\rangle$ such that $\langle \mathbf{A}, F\rangle \vDash \langle \Gamma, \varphi\rangle$. This set is a logic, although not necessarily finitary. In other words, it is the consequence relation $\vdash_{\mathcal{M}}$ defined as follows:

$$\Gamma \vdash_{\mathcal{M}} \varphi \text{ iff } \forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}), \text{ if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F,$$

for every set of formulas Γ and every formula φ.

Given a logic $\vdash$ of type L and an L-algebra $\mathbf{A}$ we say that $F \subseteq A$ is a *$\vdash$-filter of* $\mathbf{A}$ if the matrix $\langle \mathbf{A}, F\rangle$ is a model of the logic $\vdash$. The set of all $\vdash$ filters of an algebra $\mathbf{A}$ will be denoted by $\mathcal{F}i_{\vdash}(\mathbf{A})$.

Proposition 1.19 *Let* $\mathbf{A}$ *be an algebra.*

1.- A *is a* $\vdash$*-filter of* $\mathbf{A}$.

2.- *The intersection of any family of* $\vdash$*-filters of* $\mathbf{A}$ *is a* $\vdash$*-filter of* $\mathbf{A}$.

3.- *If* $\vdash$ *has theorems, then any* $\vdash$*-filter is non-empty.*

4.- *If* $\vdash$ *does not have theorems, then* $\emptyset$ *is a* $\vdash$ *filter of* $\mathbf{A}$.

These conditions imply that the set $\mathcal{F}i_{\vdash}(\mathbf{A})$ ordered by inclusion is a complete lattice where the infimum of a set of $\vdash$-filters is its intersection. For any $X \subseteq A$ the least $\vdash$-filter of $\mathbf{A}$ containing X will be called the *$\vdash$-filter of* $\mathbf{A}$ *generated by* X and it is denoted by $\mathrm{Fg}_{\vdash}^{\mathbf{A}}(X)$. If $\vdash$ is finitary, the lattice of $\vdash$-filters is algebraic and its compact elements are just the finitely generated $\vdash$-filters of $\mathbf{A}$, that is, the filters of the form $\mathrm{Fg}_{\vdash}^{\mathbf{A}}(X)$ where X is a finite set.

For every algebra $\mathbf{A}$ and every two sets X, Y we have

1.- $X \subseteq \mathrm{Fg}_{\vdash}^{\mathbf{A}}(X)$,
2.- if $X \subseteq Y$, then $\mathrm{Fg}_{\vdash}^{\mathbf{A}}(X) \subseteq \mathrm{Fg}_{\vdash}^{\mathbf{A}}(Y)$,
3.- $\mathrm{Fg}_{\vdash}^{\mathbf{A}}(\mathrm{Fg}_{\vdash}^{\mathbf{A}}(X)) = \mathrm{Fg}_{\vdash}^{\mathbf{A}}(X)$.

A *theory* of a logic $\vdash$ is any set of formulas Γ such that for any formula φ, if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$. So, a theory of a logic $\vdash$ is any set of formulas Γ such that $\Gamma = \mathrm{Cn}_{\vdash}(\Gamma)$. The theories of $\vdash$ are exactly the $\vdash$-filters of the formula algebra. If Γ is a theory of a logic $\vdash$ and there exists a finite set $\Delta \subseteq \Gamma$ such that $\mathrm{Cn}_{\vdash}(\Delta) = \Gamma$, we say that Γ is a *compact* theory.

Let $\mathbf{A}$ be an algebra and $F \subseteq A$ a subset of its universe. A congruence θ of $\mathbf{A}$ is said to be *compatible with* F if for every $\langle a, b\rangle \in \theta$, if $a \in F$, then $b \in F$.

Observe that every congruence of $\mathbf{A}$ is compatible with $\emptyset$ and A. Also, a congruence θ is compatible with a set $F \subseteq A$ if and only if F is the union of a set of equivalence classes of elements of A.

Proposition 1.20 *Let* $\mathbf{A}$ *be an algebra. For every* $F \subseteq A$ *there exists the largest congruence of* $\mathbf{A}$ *which is compatible with* F.

We denote this congruence by $\Omega^{\mathbf{A}}(F)$ and refer to it as the *Leibniz congruence* of F.

Note that $\theta \in \mathrm{Con}\,\mathbf{A}$ is compatible with F if and only if $\theta \subseteq \Omega^{\mathbf{A}}(F)$. In other words, the set of congruences of $\mathbf{A}$ compatible with F is the interval $[\mathrm{id}_{\mathbf{A}}, \Omega^{\mathbf{A}}(F)]$ of the lattice $\mathrm{Con}\mathbf{A}$.

The map $\Omega^{\mathbf{A}} : \mathcal{F}i_{\vdash}(\mathbf{A}) \to \mathrm{Con}\,\mathbf{A}$ that sends every $\vdash$-filter F of $\mathbf{A}$ to its Leibniz congruence $\Omega^{\mathbf{A}}(F)$ is called the *Leibniz operator*.

Lemma 1.21 *Let* $\langle \mathbf{A}, F\rangle$ *and* $\langle \mathbf{B}, G\rangle$ *be matrix models of a logic* $\vdash$*, and let* $h : \mathbf{A} \to \mathbf{B}$ *be a homomorphism of algebras. Then*

1.- $h^{-1}[G]$ *is a* $\vdash$*-filter of* $\mathbf{A}$.

2.- *If* h *is surjective and* $\ker h$ *is compatible with* F*, then* $h[F]$ *is a* $\vdash$*-filter of* $\mathbf{B}$.

The *reduction* of a matrix $\langle \mathbf{A}, F\rangle$ is the matrix $\langle \mathbf{A}, F\rangle^* := \langle \mathbf{A}/\Omega^{\mathbf{A}}(F), F/\Omega^{\mathbf{A}}(F)\rangle$, where $F/\Omega^{\mathbf{A}}(F) := \{a/\Omega^{\mathbf{A}}(F) : a \in F\}$. A matrix $\langle \mathbf{A}, F\rangle$ is *reduced* when $\Omega^{\mathbf{A}}(F) = \mathrm{id}_{\mathbf{A}}$.

For a given logic $\vdash$ of type L we associate with $\vdash$ the class of algebras $\mathsf{Alg}^* \vdash$ defined by

$$\mathsf{Alg}^* \vdash := \{\mathbf{A} : \text{there is } F \in \mathcal{F}i_{\vdash}(\mathbf{A}) \text{ such that } \Omega^{\mathbf{A}}(F) = \mathrm{id}_{\mathbf{A}}\}.$$

Definition 1.22 *A logic $\vdash$ is said to be* protoalgebraic *if there exists a set of formulas $\Lambda \subseteq \mathrm{Fm}(2)$ such that*

1.- $\vdash \Lambda(v_1, v_1)$.

2.- $\Lambda(v_1, v_2), v_1 \vdash v_2$.

An important characterization of this kind of logics is the following result.

Theorem 1.23 *A logic $\vdash$ is protoalgebraic if and only if whenever F and G are $\vdash$-filters of any algebra $\mathbf{A}$ and θ is a congruence of $\mathbf{A}$, if $F \subseteq G$ and θ is compatible with F, then θ is compatible with G.*

Proof. Assume that whenever F and G are $\vdash$-filters of an algebra $\mathbf{A}$ and θ is a congruence of $\mathbf{A}$, if $F \subseteq G$ and θ is compatible with F, then θ is compatible with G. Consider the formula algebra $\mathbf{Fm}$ and the set $\Sigma = \{\varphi \in \mathrm{Fm} : \emptyset \vdash h(\varphi)\}$, where h is the substitution defined by $h(v_2) = v_1$ and $h(v) = v$ for any other variable $v \neq v_2$. Thus, $\Sigma = \{\varphi(v_1, v_2, \bar{v}) \in \mathrm{Fm} : \emptyset \vdash \varphi(v_1, v_1, \bar{v})\}$. Observe that $\Sigma = h^{-1}[\mathrm{Theo}_{\vdash}]$ where $\mathrm{Theo}_{\vdash}$ is the set of theorems of $\vdash$, so Σ is a $\vdash$-filter of $\mathbf{Fm}$. Also, by definition of Σ, it holds that $\vdash h[\Sigma]$. Now, we show that $\Sigma, v_1 \vdash v_2$. Indeed, let us consider the congruence $\Omega^{\mathbf{Fm}}(\Sigma)$. It is well known that $\langle v_1, v_2\rangle \in \Omega^{\mathbf{Fm}}(\Sigma)$. On the other hand we have that $\Sigma \subseteq \mathrm{Cn}_{\vdash}(\Sigma \cup \{v_1\})$. Hence, by assumption, $\Omega^{\mathbf{Fm}}(\Sigma)$ is compatible with $\mathrm{Cn}_{\vdash}(\Sigma \cup \{v_1\})$. Therefore, we conclude that $v_2 \in \mathrm{Cn}_{\vdash}(\Sigma \cup \{v_1\})$. Let h^* be a substitution given by $h^*(v_1) = v_1$ and $h^*(v) = v_2$ for the remaining variables v different from v_1. Take $\Lambda(v_1, v_2) = h^*[\Sigma]$. Notice that $\vdash \Lambda(v_1, v_1)$ is a consequence of $\vdash h[\Sigma]$ by substitution invariance and $\Lambda(v_1, v_2), v_1 \vdash v_2$ follows from $\Sigma, v_1 \vdash v_2$ again by substitution invariance. So, we have a set $\Lambda(v_1, v_2)$ that satisfies the conditions of Definition 1.19. Consequently, $\vdash$ is protoalgebraic.

Conversely, assume that $\vdash$ is protoalgebraic. Let $\mathbf{A}$ be an algebra and F, G be $\vdash$-filters of $\mathbf{A}$ with $F \subseteq G$. Take $\theta \in \mathrm{Con}\,\mathbf{A}$ and suppose θ is compatible with F. Let $\langle a, b\rangle \in \theta$ and assume $a \in G$. As $\vdash$ is protoalgebraic, we have a set $\Lambda(v_1, v_2) \subseteq \mathrm{Fm}(2)$ that holds the Definition 1.19. Hence, for any formula $\lambda(v_1, v_2) \in \Lambda(v_1, v_2)$, $\langle \lambda^{\mathbf{A}}(a, a), \lambda^{\mathbf{A}}(a, b)\rangle \in \theta$ and $\lambda^{\mathbf{A}}(a, a) \in F$ (because $\lambda(v_1, v_1)$ is a theorem). Thus, $\lambda^{\mathbf{A}}(a, b) \in F$ because θ is compatible with F. From this it follows that $\Lambda^{\mathbf{A}}(a, b) \subseteq F \subseteq G$. Therefore, as $a \in G$ and $\Lambda(v_1, v_2), v_1 \vdash v_2$, we conclude $b \in G$. ■

Corollary 1.24 *A logic $\vdash$ is protoalgebraic if and only if for every algebra $\mathbf{A}$, $\Omega^{\mathbf{A}}$ is monotone on the set of $\vdash$-filters, that is, for all $\vdash$-filters F and G of $\mathbf{A}$ such that $F \subseteq G$, we have $\Omega^{\mathbf{A}}(F) \subseteq \Omega^{\mathbf{A}}(G)$.*

A useful result that characterizes the process of filter generation in algebras for the protoalgebraic case is the following.

Lemma 1.25 *Let $\vdash$ be a protoalgebraic logic, and let* $\mathbf{A}$ *be an algebra with* $Y \cup \{a\} \subseteq A$. *Then,* $a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash} Y$ *iff there exists* $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}$ *and a homomorphism* $h : \mathbf{Fm} \to \mathbf{A}$ *such that* $\Gamma \vdash \alpha$ *and* $h[\Gamma] \subseteq Y \cup \mathrm{Fg}^{\mathbf{A}}_{\vdash}\emptyset$ *and* $h(\alpha) = a$.

Proof. Consider the following set

$$X = \{a \in A : \text{there is } \Gamma \cup \{\alpha\} \subseteq \mathrm{Fm} \text{ and a homomorphism } h : \mathbf{Fm} \to \mathbf{A} \text{ such that } \Gamma \vdash \alpha \text{ and } h[\Gamma] \subseteq Y \cup \mathrm{Fg}^{\mathbf{A}}_{\vdash}\emptyset \text{ and } h(\alpha) = a\}.$$

In order to show the Lemma we prove that X is a $\vdash$-filter of $\mathbf{A}$.

Indeed, let $\Delta \cup \{\lambda\} \subseteq \mathrm{Fm}$ such that $\Delta \vdash \lambda$ and take any homomorphism h from $\mathbf{Fm}$ to $\mathbf{A}$. Suppose $h[\Delta] \subseteq X$. Since we are working with finitary logics, we can take Δ as a finite set. So, $\Delta = \{\delta_1, ..., \delta_n\}$. Thus, for every $i \in \{1, ..., n\}$ there exists $\Gamma_i \cup \{\alpha_i\} \subseteq \mathrm{Fm}$ and some homomorphism h_i such that $\Gamma_i \vdash \alpha_i$, $h_i(\alpha_i) = h(\delta_i)$ and $h_i[\Gamma_i] \subseteq Y \cup \mathrm{Fg}^{\mathbf{A}}_{\vdash}\emptyset$. We may assume that the sets of variables occurring in $\Gamma_i \cup \{\alpha_i\}$, in $\Gamma_j \cup \{\alpha_j\}$ and in $\Delta \cup \{\lambda\}$ are mutually disjoint for all distinct $i, j \leq n$. On the other hand, since $\vdash$ is protoalgebraic, there is a set of formulas $\Lambda(v_1, v_2)$ that satisfies the conditions in Definition 1.19. Hence, we obtain

$$\Gamma_i \cup \Lambda(\alpha_i, \delta_i) \vdash \delta_i,$$

for all $i \leq n$ and so it follows that

$$\bigcup_{i \leq n}(\Gamma_i \cup \Lambda(\alpha_i, \delta_i)) \vdash \lambda,$$

because $\Delta = \{\delta_1, ..., \delta_n\} \vdash \lambda$. Observe that, as $h(\delta_i) = h_i(\alpha_i)$ and $\vdash \Lambda(v_1, v_1)$ we get that $h[\Lambda(\alpha_i, \delta_i)] \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash}\emptyset$. Now, consider the homomorphism h^* such that for each $i \leq n$ it acts on the variables occurring in $\Gamma_i \cup \{\alpha_i\}$ in the same way as h_i and for the remaining variables it acts as h. Therefore,

$$h^*[\bigcup_{i \leq n} \Gamma_i \cup \Lambda(\alpha_i, \delta_i)] \subseteq Y \cup \mathrm{Fg}^{\mathbf{A}}_{\vdash}\emptyset \quad \text{and} \quad h^*(\lambda) = h(\lambda).$$

Consequently, $h(\lambda) \in X$ and hence X is a $\vdash$-filter of $\mathbf{A}$.

Since $Y \subseteq X$ and $X \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash} Y$, we conclude that $\mathrm{Fg}^{\mathbf{A}}_{\vdash} Y = X$, showing the Lemma. ■

We define some relations between equational consequences and logics. Given two sets of equations Π and Π', $\Pi \vDash \Pi'$ means that for every equation $\varphi \approx \psi \in \Pi'$, $\Pi \vDash \varphi \approx \psi$.

Given a set of formulas $\Lambda(x, y)$ and a pair of formulas $\langle \varphi, \psi \rangle$

$$\Lambda(\varphi, \psi) := \{\lambda(\varphi, \psi) : \lambda \in \Lambda(x, y)\}.$$

Let Π be a set of equations. We define

$$\Lambda(\Pi) := \bigcup\{\Lambda(\varphi, \psi) : \varphi \approx \psi \in \Pi\}.$$

If $\mathcal{E}(x)$ is a set of equations on at most the variable x and $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, we define

$$\mathcal{E}(\varphi) = \{\delta(x/\varphi) \approx \gamma(x/\varphi) : \delta \approx \gamma \in \mathcal{E}(x)\},$$

and

$$\mathcal{E}(\Gamma) = \bigcup_{\varphi\in\Gamma} \mathcal{E}(\varphi).$$

Given an equational consequence $\vDash$ and a logic $\vdash$ we say that the equational consequence is *interpretable* in the logic $\vdash$ by means of a set of formulas $\Lambda(x, y)$ if for every set of equations Π and every equation $\varphi \approx \psi$ it holds that

$$\Pi \vDash \varphi \approx \psi \quad \text{iff} \quad \Lambda(\Pi) \vdash \Lambda(\varphi, \psi).$$

Analogously, given a logic $\vdash$ and an equational consequence $\vDash$ we say that the logic is *interpretable* into the equational consequence by means of a set of equations $\mathcal{E}(x)$ in at most one variable x if for any set of formulas Γ and any formula φ the following holds

$$\Gamma \vdash \varphi \quad \text{iff} \quad \mathcal{E}(\Gamma) \vDash \mathcal{E}(\varphi).$$

Definition 1.26 *A finitary logic* $\vdash$ *is* algebraizable *if there exists a class of algebras* K*, a set* $\mathcal{E}(x)$ *of equations and a set* $\Lambda(x, y)$ *of formulas such that the following two conditions hold:*

1.- $\vdash$ *is interpretable in* $\vDash_{\mathsf{K}}$ *by means of* $\mathcal{E}(x)$,

2.- $x \approx y \vDash_{\mathsf{K}} \mathcal{E}(\Lambda(x, y))$ *and* $\mathcal{E}(\Lambda(x, y)) \vDash_{\mathsf{K}} x \approx y$.

or equivalently, the following conditions hold:

3.- $\vDash_{\mathsf{K}}$ *is interpretable in* $\vdash$ *by means of* Λ,

4.- $x \dashv\vdash \Lambda(\mathcal{E}(x))$.

If the above conditions hold for a quasivariety K, $\vdash$ and $\vDash_{\mathsf{K}}$, we say that the logic $\vdash$ is *algebraized by* K or equivalently, that K is its *equivalent algebraic semantics.* If K is a quasivariety we say that $\vdash$ is *(elementarily) algebraizable*, and if K is a variety, then $\vdash$ is said to be *strongly algebraizable.*

The following result is a characterization of algebraizable logics by means of an isomorphism between lattices. Before stating it we need some definitions.

Given a homomorphism $h : \mathbf{A} \to \mathbf{B}$ and any subset X of $B \times B$, let

$$h^{-1}[X] := \{\langle a, b\rangle \in A \times A : \langle h(a), h(b)\rangle \in X\}.$$

and for every $Y \subseteq A \times A$, let

$$h[Y] := \{\langle h(a), h(b)\rangle : \langle a, b\rangle \in Y\}.$$

For each algebra $\mathbf{A}$ we say that the Leibniz operator $\Omega^{\mathbf{A}} : \mathcal{F}i_{\vdash}(\mathbf{A}) \to \mathrm{Con}_{\vdash}\,\mathbf{A}$ *commutes with inverse homomorphisms* if for every $F \in \mathcal{F}i_{\vdash}(\mathbf{A})$ and any homomorphism h it holds that

$$h^{-1}[\Omega^{\mathbf{A}}(F)] = \Omega^{\mathbf{A}}(h^{-1}[F]).$$

Theorem 1.27 *For any logic* $\vdash$ *the following are equivalent:*

1.- $\vdash$ *is algebraizable.*

2.- *There exists a class of algebras* K *such that for any algebra* $\mathbf{A} \in \mathsf{K}$, $\Omega^{\mathbf{A}}$ *is an isomorphism between* $\langle \mathcal{F}i_{\vdash}(\mathbf{A}), \subseteq\rangle$ *and* $\langle \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}, \subseteq\rangle$ *that commutes with inverse homomorphisms.*

Chapter 2

The inconsistency lemma

In this chapter we discuss an abstract form of the inconsistency lemma that we have formulated in the Introduction. Our goal is to show an algebraic counterpart of this formulation for (finitary) deductive systems $\vdash$. The content of this chapter covers much of the results presented in [9, 29]. In Section 2.1 we see that, when a logic $\vdash$ is algebraized by a quasivariety K, then $\vdash$ has an inconsistency lemma (in the abstract sense) if and only if every algebra in K has a *dually pseudocomplemented join semilattice* of compact *relative* congruences (Theorem 2.11). Besides, if $\vdash$ has a *classical* inconsistency lemma (which is an abstract formulation of the inconsistency lemma for classical propositional logic), then the compact relative congruences of any algebra $\mathbf{A}$ in K form a *Boolean sublattice* of the relative congruence lattice of $\mathbf{A}$ and vice-versa, as we prove in Section 2.3 (Theorem 2.26). This is an important characterization of the abstract form of the classical inconsistency lemma that provide us an algebraic meaning of this statement. We see that a version for protoalgebraic logics of the latter result is also true (Theorem 2.14).

In Section 2.3 we study the algebraic meaning of the *deduction-detachment theorem*, namely the property of *equationally definable principal relative congruences* (EDPRC), in order to prove that a deductive system $\vdash$ which is algebraized by a quasivariety K has a classical inconsistency lemma if and only if K is *relatively semisimple* with EDPRC (Theorem 2.26). In this context arises another important algebraic property for a quasivariety K, the property of having *equationally definable principal relative congruence meets* (EDPRM). We characterize, as in [11], the quasivarieties K with EDPRM as all those whose class of *relative finitely sudirectly irreducible* (RFSI) members form a universal class and for every algebra in K the set of its relative congruences has the structure of a distributive lattice (Theorem 2.22). This characterization and many other results of this section appear in [4, 11].

Within the Section 2.3 we presented a remarkable example, following [13, 14], of a quasivariety whose RFSI members are not all finitely subdirectly irreducible (FSI) which shows us that in a quasivariety the relative congruences and the congruences of any of its members do not always have the same properties.

At the end of this chapter we analyze some important logics and its equivalent algebraic semantics. We see how are their corresponding inconsistency lemmas and find that some notable results (as Corollary 2.10) are not reversible.

2.1 The inconsistency lemmas

We say that a set Ξ of formulas of $\vdash$ is *inconsistent* in $\vdash$ if $\Xi \vdash \alpha$ for all $\alpha \in \mathrm{Fm}$. Observe that:

(E) If Ξ is *finite* and inconsistent in $\vdash$, then $\Xi \dashv\vdash h[\Xi]$ for all substitutions h.

Indeed, let $\beta = \beta(x_1, ..., x_n) \in \Xi$ and take an arbitrary substitution h. Consider the formula $\beta(v_1, ..., v_n)$ where v_i is a new variable that is not among the variables of Ξ for all $i \in \{1, ..., n\}$. Then, since Ξ is inconsistent, $\Xi \vdash \beta(v_1, ..., v_n)$. Now, as Ξ is finite, we can define another substitution h' such that it agrees with h on the variables of Ξ and $h'(v_i) = x_i$ for $i \in \{1, ..., n\}$ and it sends to x_1 any other variable. In this way we have $h'[\Xi] \vdash h'(\beta(v_1, ..., v_n))$ and hence $h[\Xi] \vdash \beta(x_1, ..., x_n)$. Therefore, $h[\Xi] \vdash \Xi$.

Thus, using (E) we can prove the equivalence of the following conditions.

Proposition 2.1 *The following statements are equivalent*:

(i) $\vdash$ *has a greatest compact theory.*

(ii) Fm *is a compact* $\vdash$*-theory.*

(iii) *Some finite set of formulas is inconsistent in* $\vdash$.

(iv) *Some finite subset of* $\mathrm{Fm}(1)$ *is inconsistent in* $\vdash$.

(v) *There is a finite* $\Xi \subseteq \mathrm{Fm}(1)$ *such that* $A = \mathrm{Fg}^{\mathbf{A}}_{\vdash}\, \Xi^{\mathbf{A}}(a)$ *for every algebra* $\mathbf{A}$ *and all* $a \in A$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\vdash$ has a greatest compact theory, call it Γ. Note that we have $\mathrm{Fm} = \bigcup\{\mathrm{Cn}_{\vdash}(\Delta) : \Delta \subseteq \mathrm{Fm}$ and Δ is finite$\}$. By assumption, there must be some finite set Δ_0 of formulas such that $\mathrm{Cn}_{\vdash}(\Delta_0) = \Gamma$ and $\mathrm{Cn}_{\vdash}(\Delta) \subseteq \Gamma$ for every $\Delta \subseteq \mathrm{Fm}$ finite. Therefore, $\mathrm{Fm} = \mathrm{Cn}_{\vdash}(\Delta_0)$, which means that Fm is a compact $\vdash$-theory.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv). Suppose Ξ is a finite set of formulas inconsistent in $\vdash$. Consider the substitution h that sends each variable to v_1. Thus, $h[\Xi]$ is a finite subset of $\mathrm{Fm}(1)$ and, by (E), we have that it is also inconsistent.

(iv) $\Rightarrow$ (v). Let Ξ be a finite inconsistent subset of $\mathrm{Fm}(1)$. Consider an arbitrary algebra $\mathbf{A}$ and take $a \in A$. We have that $A = \bigcup\{\mathrm{Fg}^{\mathbf{A}}_{\vdash} X : X \subseteq A$ and X is finite$\}$. On the other hand, for every finite set $X \subseteq A$, $\mathrm{Fg}^{\mathbf{A}}_{\vdash} X \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash}\, \Xi^{\mathbf{A}}(a)$, because Ξ is inconsistent. Therefore, it must be the case that $A = \mathrm{Fg}^{\mathbf{A}}_{\vdash}\, \Xi^{\mathbf{A}}(a)$.

(v) $\Rightarrow$ (i). Just take $\mathbf{A} = \mathbf{Fm}$ in (v). ■

In the case of classical and intuitionistic propositional logic, the theory Fm is compact. Among the finite sets that generate Fm we find $\Xi = \{v_1, \neg v_1\}$ and also $\Xi = \{\bot\}$.

Definition 2.2 *Let* $\Psi_n \subseteq \mathrm{Fm}(n)$ *for all* $n \in \mathbb{N}$. *We call* $\{\Psi_n : n \in \mathbb{N}\}$ *an* IL-sequence *for* $\vdash$ *if whenever* $\Gamma \cup \{\alpha_1, ..., \alpha_n\} \subseteq \mathrm{Fm}$, *then*

$$\Gamma \cup \{\alpha_1, ..., \alpha_n\} \textit{ is inconsistent in } \vdash \textit{ iff } \Gamma \vdash \Psi_n(\alpha_1, ..., \alpha_n).$$

As a first consequence, we have the following facts.

Remark 2.3 *Suppose* $\{\Psi_n : n \in \mathbb{N}\}$ *is an IL-sequence for* $\vdash$. *Then*

(i) $\Psi_n(\alpha_1, ..., \alpha_n) \cup \{\alpha_1, ..., \alpha_n\}$ *is inconsistent in* $\vdash$ *for all* $n \in \mathbb{N}$ *and* $\alpha_1, ..., \alpha_n \in \mathrm{Fm}$.

(ii) $\Psi_n(\alpha_1, ..., \alpha_n) \dashv\vdash \Psi_n(\alpha_{f1}, ..., \alpha_{fn})$ *for any permutation f of 1,...,n.*

(iii) *If* $\{\Phi_n : n \in \mathbb{N}\}$ *is another IL-sequence for* $\vdash$, *then* $\Psi_n \dashv\vdash \Phi_n$ *for all* n.

Proof. (i) This is a consequence of $\Psi_n(\alpha_1, ..., \alpha_n) \vdash \Psi_n(\alpha_1, ..., \alpha_n)$.

(ii) Since the elements of $\{\alpha_1, ..., \alpha_n\}$ are not ordered, using the previous item and Definition 2.2 we have this statement.

(iii) Let $n \in \mathbb{N}$. Observe that $\Psi_n(v_1, ..., v_n) \cup \{v_1, ..., v_n\}$ and $\Phi_n(v_1, ..., v_n) \cup \{v_1, ..., v_n\}$ are inconsistent. Thus, as both are IL-sequences, $\Psi_n \dashv\vdash \Phi_n$. ■

Remark 2.4 *The following statements are equivalent.*

(i) $\emptyset$ *occurs in some IL-sequence for* $\vdash$.

(ii) $v_1 \vdash v_2$.

(iii) $\{\emptyset : n \in \mathbb{N}\}$ *is an IL-sequence for* $\vdash$.

Proof. (i) $\Rightarrow$ (ii). Let $\{\Psi_n : n \in \mathbb{N}\}$ be an IL-sequence such that $\Psi_m = \emptyset$ for some $m \in \mathbb{N}$. Then, by Remark 2.3(i), $\emptyset(v_1, ..., v_m) \cup \{v_1, ..., v_m\}$ is inconsistent and hence $\{v_1, .., v_m\}$ too. Now, consider the substitution h that sends all the variables to v_1. Thus, by condition (E), $h[\{v_1, ..., v_m\}] = \{v_1\}$ is inconsistent. Therefore, $v_1 \vdash v_2$.

(ii) $\Rightarrow$ (iii). Let $\Gamma \cup \{\alpha_1, ..., \alpha_n\} \subseteq \mathrm{Fm}$. If $\Gamma \cup \{\alpha_1, ..., \alpha_n\}$ is inconsistent in $\vdash$, then, using (ii), we obtain $\Gamma \vdash \emptyset(\alpha_1, ..., \alpha_n)$. Conversely, if $\Gamma \vdash \emptyset(\alpha_1, ..., \alpha_n)$, then, by (ii), $\Gamma \vdash \{\alpha_1, ..., \alpha_n\}$. Also by (ii) we have $\alpha_1 \vdash \beta$ for any $\beta \in \mathrm{Fm}$. Therefore, $\Gamma \cup \{\alpha_1, ..., \alpha_n\}$ is inconsistent. Hence, $\{\emptyset : n \in \mathbb{N}\}$ is an IL-sequence for $\vdash$.

(iii) $\Rightarrow$ (i) is clear. ■

Definition 2.5 *An IL-sequence* $\{\Psi_n : n \in \mathbb{N}\}$ *for* $\vdash$ *is* elementary *if* Ψ_n *is a finite set for every n. We say that* $\vdash$ *has an* inconsistency lemma (*briefly an IL) if it has an elementary IL-sequence.*

This definition is a generalization of the familiar inconsistency lemmas of intuitionistic and classical propositional logic. For the intuitionistic case, the lemma is of the form

$$\Gamma \cup \{\alpha_1, ..., \alpha_n\} \text{ is inconsistent iff } \Gamma \vdash \neg(\alpha_1 \wedge ... \wedge \alpha_n)$$

for all $n \in \mathbb{N}$.

In this situation our convention that $0 \notin \mathbb{N}$ becomes important. Without it, the constant-free formulation of classical logic would have no inconsistency lemma, because if we take $0 \in \mathbb{N}$, the set Ψ_0 cannot exist since in this formulation we do not have any constant.

Remark 2.6 *Let $\{\Psi_n : n \in \mathbb{N}\}$ be an IL-sequence for $\vdash$. Then, $\vdash$ has an inconsistency lemma iff it has a greatest compact theory.*

Proof. Suppose that $\vdash$ has an IL. By Remark 2.3(i), $\Psi_1 \cup \{v_1\}$ is inconsistent. It follows that $\Psi_1 \cup \{v_1\} \dashv\vdash \mathrm{Fm}$. Thus, $\mathrm{Fm} = \mathrm{Cn}_{\vdash}(\Psi_1 \cup \{v_1\})$. Hence, Fm is compact because Ψ_1 is finite.

Conversely, if $\vdash$ has a greatest compact theory, by Proposition 2.1, there is some finite set $\Xi \subseteq \mathrm{Fm}(1)$ inconsistent in $\vdash$. Then, as $\Psi_n \cup \{v_1, ..., v_n\}$ is inconsistent, we obtain $\Psi_n \cup \{v_1, ..., v_n\} \vdash \Xi$. Now, since $\vdash$ is finitary and Ξ is finite, there is a finite $\Psi'_n \subseteq \Psi_n$ such that $\Psi'_n \cup \{v_1, ..., v_n\} \vdash \Xi$. This shows that each $\Psi'_n \cup \{v_1, ..., v_n\}$ is inconsistent. Whence, by means of condition (E), $\{\Psi'_n : n \in \mathbb{N}\}$ is an elementary IL-sequence for $\vdash$. ■

Notice that if $\{\Psi_n : n \in \mathbb{N}\}$ is an elementary IL-sequence for $\vdash$, then it is the case that for any $\vdash$-filter F of the algebra of formulas **Fm** and any $\alpha_1, ..., \alpha_n \in \mathrm{Fm}$, we have

$$\mathrm{Fm} = F + \mathrm{Cn}_{\vdash}(\{\alpha_1, ..., \alpha_n\}) \text{ iff } \Psi_n(\alpha_1, ..., \alpha_n) \subseteq F.$$

This holds because $\mathrm{Fm} = F + \mathrm{Cn}_{\vdash}(\{\alpha_1, ..., \alpha_n\})$ iff $F \cup \{\alpha_1, ..., \alpha_n\}$ is inconsistent iff $F \vdash \Psi_n(\alpha_1, ..., \alpha_n)$ iff $\Psi_n(\alpha_1, ..., \alpha_n) \subseteq F$.

In the next theorem we extend this equivalence from **Fm** to *all* algebras provided that $\vdash$ is protoalgebraic.

Theorem 2.7 *Let $\{\Psi_n : n \in \mathbb{N}\}$ be an elementary IL-sequence for a protoalgebraic deductive system $\vdash$. Let F be a $\vdash$-filter of an algebra $\mathbf{A}$, and let $a_1, ..., a_n \in A$, where $n \in \mathbb{N}$. Then*

$$A = F +^{\mathbf{A}} \mathrm{Fg}^{\mathbf{A}}_{\vdash}\{a_1, ..., a_n\} \quad \textit{iff} \quad \Psi^{\mathbf{A}}_n(a_1, ..., a_n) \subseteq F.$$

Proof. Observe that, by Remark 2.6 and Proposition 2.1, there is some finite set $\Xi \subseteq \mathrm{Fm}(1)$ such that $A = \mathrm{Fg}^{\mathbf{A}}_{\vdash} \Xi^{\mathbf{A}}(a)$ for all $a \in A$.

Now, assume $\Psi^{\mathbf{A}}_n(a_1, ..., a_n) \subseteq F$. Using Remark 2.3(i) we have that $\Psi_n \cup \{v_1, ..., v_n\}$ is inconsistent. Thus, $\Psi_n \cup \{v_1, ..., v_n\} \vdash \Xi(v_{n+1})$. Whence, for all $a \in A$, $\Xi^{\mathbf{A}}(a) \subseteq \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\Psi^{\mathbf{A}}_n(a_1, ..., a_n) \cup \{a_1, ..., a_n\})$. Then, by our assumption, $\mathrm{Fg}^{\mathbf{A}}_{\vdash} \Xi^{\mathbf{A}}(a) \subseteq F +^{\mathbf{A}} \mathrm{Fg}^{\mathbf{A}}_{\vdash}\{a_1, ..., a_n\}$. Therefore, $A = F +^{\mathbf{A}} \mathrm{Fg}^{\mathbf{A}}_{\vdash}\{a_1, ..., a_n\}$.

Conversely, let $a \in A$ and suppose $A = F +^{\mathbf{A}} \mathrm{Fg}^{\mathbf{A}}_{\vdash}\{a_1, ..., a_n\}$. Hence, $\Xi^{\mathbf{A}}(a) \subseteq F + \mathrm{Fg}^{\mathbf{A}}_{\vdash}\{a_1, ..., a_n\}$. Since our system $\vdash$ is protoalgebraic, for each $\xi \in \Xi$, by Lemma 1.25, there exists $\Gamma_\xi \cup \{\beta_\xi\} \subseteq \mathrm{Fm}$ and a homomorphism $h_\xi : \mathbf{Fm} \to \mathbf{A}$ such that:

(i) $\Gamma_\xi \vdash \beta_\xi$

(ii) $h_\xi[\Gamma_\xi] \subseteq F \cup \{a_1, ..., a_n\}$

(iii) $h_\xi(\beta_\xi) = \xi^{\mathbf{A}}(a)$.

As $\vdash$ is finitary we may assume that each Γ_ξ is finite. Also, by substitution invariance, we can arrange that $\Gamma_\xi \cup \{\beta_\xi\}$ and $\Gamma_{\xi'} \cup \{\beta_{\xi'}\}$ involve disjoint sets of variables whenever $\xi, \xi' \in \Xi$ are distinct formulas. Due to this, we can choose a single homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ that agrees with h_ξ on the variables of $\Gamma_\xi \cup \{\beta_\xi\}$ for each $\xi \in \Xi$ and $h(v_i) = a_i$ for $i = 1, ..., n$ and $h(v_{n+1}) = a$, where v_i is a variable not in $\Gamma_\xi \cup \{\beta_\xi\}$ for all $\xi \in \Xi$. So, we have

$$\textstyle\bigcup_{\xi\in\Xi} \Gamma_\xi \vdash \{\beta_\xi : \xi \in \Xi\} \quad \text{ and } \quad h[\bigcup_{\xi\in\Xi} \Gamma_\xi] \subseteq F \cup \{a_1, ..., a_n\}.$$

In order to prove our theorem, we are looking for a set of formulas, say Δ, such that $h[\Delta] \subseteq F$ and $\Delta \cup \{v_1, ..., v_n\}$ is inconsistent.

So, first choose a finite set $\Lambda \subseteq \mathrm{Fm}(2)$ as in Definition 1.22 of protoalgebraic logic. For every $\xi \in \Xi$, it follows that

$$\Lambda(\beta_\xi, \xi(v_{n+1})), \beta_\xi \vdash \xi(v_{n+1}) \tag{2.1}$$

and for all $\lambda \in \Lambda$, as F is a $\vdash$-filter and $\vdash \Lambda(v_1, v_1)$,

$$h(\lambda(\beta_\xi, \xi(v_{n+1}))) = \lambda^{\mathbf{A}}(h(\beta_\xi), h(\xi(v_{n+1}))) = \lambda^{\mathbf{A}}(\xi^{\mathbf{A}}(a), \xi^{\mathbf{A}}(a)) \in F. \tag{2.2}$$

Hence, if we take

$$\Gamma := (\textstyle\bigcup_{\xi\in\Xi} \Gamma_\xi) \cup (\bigcup_{\xi\in\Xi} \Lambda(\beta_\xi, \xi(v_{n+1}))) \cup \{v_1, ..., v_n\},$$

then, by (2.1) and (i), $\Gamma \vdash \Xi(v_{n+1})$ and, by (ii), (iii) and (2.2), $h[\Gamma] \subseteq F \cup \{a_1, ..., a_n\}$.

Now, this set has the form

$$\Gamma := \Gamma' \cup \Pi_1 \cup ... \cup \Pi_n$$

where $h[\Gamma'] \subseteq F$ and $v_i \in \Pi_i$ and $h[\Pi_i] = \{a_i\}$ for $i = 1, ..., n$.

For each i and each $\alpha \in \Pi_i$, using the properties of Λ, we have

$$\Lambda(v_i, \alpha), v_i \vdash \alpha. \tag{2.3}$$

Also, it follows that $h(v_i) = h(\alpha)$. Thus, we may assume without loss of generality that $\Pi_i = \{v_i\}$. In fact, $h[\Gamma'] \subseteq F$ remains true if we add every

$$\lambda(v_i, \alpha) \qquad (\lambda \in \Lambda, v_i \neq \alpha \in \Pi_i, 1 \leq i \leq n)$$

to Γ', whereupon $\Gamma \vdash \Xi(v_{n+1})$ is still true if we delete every element other than v_i from each Π_i (because of (2.3)).

In other words, we can arrange that $\Gamma = \{\gamma_1, ..., \gamma_r, v_1, ..., v_n\}$, where h sends $\gamma_1, ..., \gamma_r$ into F. Besides, Γ is inconsistent (because $\Gamma \vdash \Xi(v_{n+1})$) and then $\{\gamma_1, ..., \gamma_r\} \vdash \Psi_n$, by the IL. Consequently, since $h[\{\gamma_1, ..., \gamma_r\}] \subseteq F$, we get $\Psi_n^{\mathbf{A}}(a_1, ..., a_n) = h[\Psi_n] \subseteq F$, as required. ∎

A deductive system $\vdash$ has a *deduction-detachment theorem* (DDT) if there is a set of binary formulas $\Sigma(v_1, v_2)$ such that

$$\Gamma \cup \{\alpha\} \vdash \beta \qquad \text{iff} \qquad \Gamma \vdash \Sigma(\alpha, \beta)$$

for all $\Gamma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}$.

Every deductive system with the DDT is protoalgebraic and in our case, since $\vdash$ is finitary, Σ can be chosen finite.

In classical and intuitionistic propositional logic the inconsistency lemma is a consequence of the standard DDT for these systems, namely $\Gamma \cup \{\alpha\} \vdash \beta$ iff $\Gamma \vdash \alpha \to \beta$. Indeed, for the case of intuitionistic logic we have $\Gamma \cup \{\alpha\}$ is inconsistent iff $\Gamma \cup \{\alpha\} \vdash \bot$, which is equivalent, by the DDT, to $\Gamma \vdash \alpha \to \bot$ and this is the same as $\Gamma \vdash \neg\alpha$. And for classical propositional logic we have $\Gamma \cup \{\neg\alpha\}$ is inconsistent iff $\Gamma \cup \{\neg\alpha\} \vdash \alpha$ iff $\Gamma \vdash \neg\alpha \to \alpha$ iff $\Gamma \vdash \neg\neg\alpha \vee \alpha$ iff $\Gamma \vdash \alpha$.

Systems with an IL may lack a conjunction-like connective $\wedge$. For example:

$$\{\{v_1 \to (v_2 \to ... \to (v_n \to \bot))\} : n \in \mathbb{N}\}$$

is an IL-sequence for the $\{\to, \bot\}$-fragment of intuitionistic logic. This is because in this system we also have a standard deduction theorem. So, for any $\Gamma \cup \{\alpha_1, ..., \alpha_n\} \subseteq \mathrm{Fm}$, $\Gamma \cup \{\alpha_1, ..., \alpha_n\}$ is inconsistent iff $\Gamma \cup \{\alpha_1, ..., \alpha_n\} \vdash \bot$ which is equivalent to $\Gamma \vdash \alpha_1 \to (\alpha_2 \to ... \to (\alpha_n \to \bot))$.

We can now characterize the inconsistency lemma by semantic means thanks to Theorem 2.7. Before that let us recall some important concepts.

An idempotent commutative semigroup $\langle S; + \rangle$ is called a *join semilattice with* 0 if it has a least element with respect to the order defined by

$$x \leq y \qquad \text{iff} \qquad x + y = y.$$

It is said to be *dually pseudo-complemented* if it has a greatest element 1 and, for each $a \in S$, there exists a smallest $b \in S$ such that $a + b = 1$. This element b is denoted as a^*. Note that $a^{**} \leq a$ for all $a \in S$.

Clearly, the compact $\vdash$-filters of an algebra $\mathbf{A}$ form a join semilattice with 0 under the operation $+^{\mathbf{A}}$ and taking the semilattice order $\leq$ as $\subseteq$ where the least element is $\mathrm{Fg}^{\mathbf{A}}_{\vdash} \emptyset$.

Theorem 2.8 *Let $\vdash$ be a protoalgebraic deductive system. Then the following conditions are equivalent.*

(i) *$\vdash$ has an inconsistency lemma.*

(ii) *For every algebra $\mathbf{A}$, the compact $\vdash$-filters of $\mathbf{A}$ form a dually pseudo-complemented similattice with respect to $+^{\mathbf{A}}$.*

(iii) *The join semilattice of compact $\vdash$-theories is dually pseudo-complemented.*

Proof. We may assume that Fm is a compact $\vdash$-theory, because this is a consequence of all three conditions, by Proposition 2.1. In other words, there exists some finite $\Xi \subseteq \mathrm{Fm}(1)$ inconsistent in $\vdash$.

(i) $\Rightarrow$ (ii). Let $\{\Psi_n : n \in \mathbb{N}\}$ be an elementary IL-sequence for $\vdash$. Take arbitrary elements $a_1, ..., a_n$ of an algebra $\mathbf{A}$, where $n \in \mathbb{N}$, and let $H = \mathrm{Fg}^{\mathbf{A}}_{\vdash} \{a_1, ..., a_n\}$. By Theorem 2.7, for any $\vdash$-filter F of $\mathbf{A}$ we have

$$A = F +^{\mathbf{A}} H \quad \text{iff} \quad \Psi_n^{\mathbf{A}}(a_1, ..., a_n) \subseteq F.$$

As Ψ_n is finite, $\mathrm{Fg}_{\vdash}^{\mathbf{A}}\, \Psi_n^{\mathbf{A}}(a_1, ..., a_n)$ is compact and equal to H^* in the semilattice of compact $\vdash$-filters of $\mathbf{A}$, since in view of the previous equivalence, $A = H +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}} \Psi_n^{\mathbf{A}}(a_1, ..., a_n)$. Finally, if $\emptyset$ is a $\vdash$-filter of $\mathbf{A}$, then $\emptyset^* = A$, which is also compact by Proposition 2.1, because $\vdash$ has a greatest compact theory.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). Let $n \in \mathbb{N}$ and choose a finite set $\Psi_n' \subseteq \mathrm{Fm}$ so that

$$\mathrm{Cn}_{\vdash} \Psi_n' = (\mathrm{Cn}_{\vdash}\{v_1, ..., v_n\})^*$$

in the semilattice of compact $\vdash$-theories. Let

$$\Psi_n = g[\Psi_n'],$$

where g is a substitution that fixes $v_1, ..., v_n$ and sends all other variables to v_1. So, Ψ_n is a finite subset of $\mathrm{Fm}(n)$.

For any $\vdash$-theory Γ (compact or not), we have

$$\mathrm{Fm} = \Gamma + \mathrm{Cn}_{\vdash}\{v_1, ..., v_n\} \text{ iff } \mathrm{Cn}_{\vdash}\, \Psi_n' \subseteq \Gamma \text{ (i.e., } \Psi_n' \subseteq \Gamma). \tag{2.4}$$

This is due to the compactness of Fm and the fact that Γ is a join of compact elements of the lattice of all $\vdash$-theories, because this lattice is algebraic.

Given $\Gamma \cup \{\alpha_1, ..., \alpha_n\} \subseteq \mathrm{Fm}$, let h be a surjective substitution that sends v_i to α_i for $i = 1, ..., n$ and that sends to v_1 all other variables occurring in Ψ_n'. This substitution exists because Ψ_n' is finite. Then $h[\Psi_n'] = \Psi_n(\alpha_1, ..., \alpha_n)$ and $\Gamma = h[h^{-1}[\Gamma]]$. It suffices to show that $\Gamma \cup \{\alpha_1, ..., \alpha_n\}$ is inconsistent in $\vdash$ iff $\Gamma \vdash \Psi_n(\alpha_1, ..., \alpha_n)$.

By Lemma 1.21, $h^{-1}[\mathrm{Cn}_{\vdash}\, \Gamma]$ and $h^{-1}[\mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, ..., \alpha_n\})]$ are $\vdash$-theories and ker h is compatible with $h^{-1}[\mathrm{Cn}_{\vdash}\, \Gamma]$. Now, by protoalgebraicity and Theorem 1.23, ker h is also compatible with the larger theory

$$Y := h^{-1}[\mathrm{Cn}_{\vdash}\, \Gamma] + \mathrm{Cn}_{\vdash}\, \{v_1, ..., v_n\} = \mathrm{Cn}_{\vdash}(h^{-1}[\mathrm{Cn}_{\vdash}\, \Gamma] \cup \{v_1, ..., v_n\}).$$

Therefore, $h[Y]$ is a theory, by Lemma 1.21. It follows that $h[Y]$ extends $\mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, ..., \alpha_n\})$, because it contains

$$h[h^{-1}[\Gamma] \cup \{v_1, ..., v_n\}] = \Gamma \cup \{\alpha_1, ..., \alpha_n\}.$$

On the other hand, it is clear that $Y \subseteq h^{-1}[\mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, ..., \alpha_n\})]$, hence

$$h[Y] = \mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, ..., \alpha_n\}). \tag{2.5}$$

Consequently,

$\Gamma \cup \{\alpha_1, ..., \alpha_n\}$ is inconsistent in $\vdash$ iff $\Gamma, \alpha_1, ..., \alpha_n \vdash \Xi$

iff $\Gamma, \alpha_1, ..., \alpha_n \vdash h[\Xi]$ (as $\Xi \dashv\vdash h[\Xi]$)

iff $h[\Xi] \subseteq h[Y]$ (by (2.5))

iff $\Xi \subseteq Y$ (as ker h is compatible with Y)

iff $\mathrm{Fm} = h^{-1}[\mathrm{Cn}_{\vdash}\, \Gamma] + \mathrm{Cn}_{\vdash}\{v_1, ..., v_n\}$

iff $\Psi'_n \subseteq h^{-1}[\mathrm{Cn}_{\vdash}\,\Gamma]$ (by (2.4))

iff $h[\Psi'_n] \subseteq \mathrm{Cn}_{\vdash}\,\Gamma$

iff $\Gamma \vdash h[\Psi'_n]$, i.e., $\Gamma \vdash \Psi_n(\alpha_1, ..., \alpha_n)$.

Thereby, $\vdash$ has an inconsistency lemma. ∎

Some immediate consequences of this result are the following facts.

Corollary 2.9 *If a protoalgebraic deductive system* $\vdash$ *has an inconsistency lemma, then every finite algebra has a dually pseudo-complemented lattice of* $\vdash$*-filters.*

Proof. Let $\mathbf{A}$ be a finite algebra. As it is finite, its compact $\vdash$-filters and its $\vdash$-filters are the same. Therefore, by the previous theorem and taking into account that the $\vdash$-filters of $\mathbf{A}$ form a lattice, it follows that $\mathbf{A}$ has a dually-pseudo complemented lattice of $\vdash$-filters. ∎

Recall that a join semilattice $\langle S; +\rangle$ with 0 is *dually Brouwerian* if, for any $a, b \in S$, there is a smallest $c \in S$ such that $a \leq b + c$.

In this setting it is worth mentioning that a deductive system has a DDT iff it is protoalgebraic and has a dually Brouwerian join semilattice of compact theories. A dually Brouwerian join semilattice with 0 is dually pseudo-complemented if it has a greatest element. So, the next result also follows from Theorem 2.8.

Corollary 2.10 *If a deductive system with a greatest compact theory has a deduction-detachment theorem, then it has an inconsistency lemma.*

Proof. Consider a deductive system $\vdash$ with a DDT and assume it has a greatest compact theory, say Γ. Let Δ be an arbitrary compact $\vdash$-theory. As $\vdash$ has a DDT, it is protoalgebraic and has a dually Brouwerian join semilattice of compact $\vdash$-theories. Hence, there must be a smallest compact $\vdash$-theory Θ such that $\Gamma = \Delta +^{\mathbf{A}} \Theta$. So, the join semilattice of compact $\vdash$-theories is dually pseudo-complemented. Therefore, by Theorem 2.8, it must be the case that $\vdash$ has an inconsistency lemma. ∎

A deductive system with an inconsistency lemma need not be protoalgebraic. Indeed, the usual IL of intuitionistic propositional logic (**IPL**) is also a IL for the implication-less fragment of **IPL** (i.e., the $\{\wedge, \vee, \neg, \bot, \top\}$-fragment). However, this system is not protoalgebraic.

Recall that for every quasivariety K and an algebra $\mathbf{A}$ of the same type, the K-congruence lattices of $\mathbf{A}$ and $\mathbf{A}/\theta_{\mathsf{K}}(\emptyset)$ are isomorphic, where $\theta_{\mathsf{K}}(\emptyset)$ denotes the K-congruence generated by $\emptyset$. As it is true that $\mathbf{A}/\theta_{\mathsf{K}}(\emptyset) \in \mathsf{K}$, the following conclusion is a consequence of Theorem 2.8 and of the fact that every algebraizable deductive system is protoalgebraic.

Theorem 2.11 *Let* K *be a quasivariety that algebraizes a deductive system* $\vdash$*. Then the following conditions are equivalent.*

(i) $\vdash$ *has an inconsistency lemma.*

(ii) *For every algebra* $\mathbf{A}$*, the join semilattice of compact* K*-congruences of* $\mathbf{A}$ *is dually pseudo-complemented.*

(iii) *For every* $\mathbf{A} \in \mathsf{K}$*, the join semilattice of compact* K*-congruences of* $\mathbf{A}$ *is dually pseudo-complemented.*

If K *is a variety, then these conditions are equivalent to*

(iv) *For every* $\mathbf{A} \in \mathsf{K}$*, the join semilattice of compact congruences of* $\mathbf{A}$ *is dually pseudo-complemented.*

Proof. As K algebraizes $\vdash$, we get that $\vdash$ is protoalgebraic.

(i) $\Rightarrow$ (ii). Assuming (i), by Theorem 2.8, it follows that for every algebra $\mathbf{A}$, its compact $\vdash$-filters form a dually pseudo-complemented semilattice. Since K algebraizes $\vdash$, as we saw in the Preliminaries in Theorem 1.27, there is an isomorphism from the $\vdash$-filters of $\mathbf{A}$ onto the lattice of K-congruences of $\mathbf{A}$. Thus, the join semilattice of compact K-congruences of $\mathbf{A}$ is dually pseudo-complemented.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). Since the lattice of K-congruences of $\mathbf{Fm}$ is isomorphic to $\mathbf{Fm}/\theta_{\mathsf{K}}(\emptyset)$ and $\mathbf{Fm}/\theta_{\mathsf{K}}(\emptyset) \in \mathsf{K}$, by (iii), as K algebraizes $\vdash$, the join semilattice of compact $\vdash$-theories is dually pseudo-complemented. Therefore, by Theorem 2.8, $\vdash$ has an inconsistency lemma.

Now, if K is a variety, the K-congruences and the congruences of any algebra $\mathbf{A}$ in K are the same. Thus, (iv) is equivalent to (iii). ■

2.2 Classical inconsistency lemmas

If $\langle S, + \rangle$ is a dually pseudo-complemented join semilattice with 0, we know, by a result of Glivenko, that the sub-poset $\{a^* : a \in S\}$ of $\langle S, \leq \rangle$ is a complemented distributive lattice, that is, a Boolean lattice. The join operation of this lattice is $+$ and the meet operation $\cdot$ is defined as $a \cdot b = (a^* + b^*)^*$, for every $a, b \in S$. Besides, if $\langle S, + \rangle$ satisfies $x^{**} = x$, then it is a Boolean lattice with respect to the join semilattice order. The converse is also true, because in a Boolean lattice the (dual) pseudo-complements coincide with complements and these are unique.

Notice that, even when the compact $\vdash$-filters of an infinite algebra form a Boolean lattice, it is not always the case that they form a sublattice of the lattice of all $\vdash$-filters, because the intersection of two compact $\vdash$-filters need not be compact. This observation will be used later on.

In classical propositional logic, the inconsistency lemma can be formulated, taking $n \in \mathbb{N}$, as fallows

$$\Gamma \cup \{\neg(\alpha_1 \wedge ... \wedge \alpha_n)\} \text{ is inconsistent} \quad \text{iff} \quad \Gamma \vdash \{\alpha_1, ..., \alpha_n\}$$

Nevertheless, this variant of the inconsistency lemma is false in the context of intuitionistic logic, where $x \vdash \neg\neg x$ is derivable but $\neg\neg x \vdash x$ is not.

Because of this, it is sensible to introduce the following strengthening of an inconsistency lemma for arbitrary deductive systems.

Definition 2.12 *An IL-sequence* $\{\Psi_n : n \in \mathbb{N}\}$ *for* $\vdash$ *will be called* classical *provided that, whenever* $n \in \mathbb{N}$ *and* $\Gamma \cup \{\alpha_1, ..., \alpha_n\} \subseteq \mathrm{Fm}$,

$$\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n) \textit{ is inconsistent in } \vdash \quad \textit{iff} \quad \Gamma \vdash \{\alpha_1, ..., \alpha_n\}.$$

We say that a deductive system $\vdash$ has a *classical inconsistency lemma* if it has a classical elementary IL-sequence.

Observe that if $\{\Psi_n : n \in \mathbb{N}\}$ is a classical IL-sequence for $\vdash$ and $\{\Phi_n : n \in \mathbb{N}\}$ is an IL-sequence for $\vdash$, then it is also classical. Indeed, $\Gamma \cup \Phi_n(\alpha_1, ..., \alpha_n)$ is inconsistent iff $\Gamma \cup \Phi_n(\alpha_1, ..., \alpha_n) \vdash \Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n)$ iff (by Remark 3.3(iii)) $\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n) \vdash \Gamma \cup \Phi_n(\alpha_1, ..., \alpha_n)$ iff $\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n)$ is inconsistent, which is equivalent to $\Gamma \vdash \{\alpha_1, ..., \alpha_n\}$. Therefore, when $\vdash$ has a classical IL-sequence, then every IL-sequence for $\vdash$ is classical.

We say that a deductive system $\vdash^*$ is an *axiomatic extension* of $\vdash$ in the same language if there is a set of formulas Δ closed under substitutions such that, whenever $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, it holds

$$\Gamma \vdash^* \varphi \quad \text{iff} \quad \Gamma \cup \Delta \vdash \varphi.$$

It is also true that a classical IL-sequence $\{\Psi_n : n \in \mathbb{N}\}$ for a system $\vdash$ remains a classical IL-sequence for any axiomatic extension $\vdash^*$ of $\vdash$ in the same language. This is because, taking Δ as the set of formulas that exists in view of the definition of axiomatic extension, we have, whenever $n \in \mathbb{N}$ and $\Gamma \cup \{\alpha_1, ..., \alpha_n\} \subseteq \mathrm{Fm}$, that $\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n)$ is inconsistent in $\vdash^*$ iff $\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n) \vdash^* \beta$ for all $\beta \in \mathrm{Fm}$ iff $\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n) \cup \Delta \vdash \beta$ for any $\beta \in \mathrm{Fm}$ iff $\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n) \cup \Delta$ is inconsistent in $\vdash$ iff $\Gamma \cup \Delta \vdash \{\alpha_1, ..., \alpha_n\}$ iff $\Gamma \vdash^* \{\alpha_1, ..., \alpha_n\}$.

It is well known that for the classical propositional logic we have $x \dashv\vdash \neg\neg x$. Consequently, $\alpha_1, ..., \alpha_n \dashv\vdash \neg\neg(\alpha_1 \wedge ... \wedge \alpha_n)$ for all $\alpha_1, ..., \alpha_n \in \mathrm{Fm}$. We can obtain an abstract form of this property that characterizes the classical elementary IL-sequences for any deductive system $\vdash$. In order to do that, let us define before some important sets.

Suppose $\{\Psi_n : n \in \mathbb{N}\}$ is an elementary IL-sequence for $\vdash$, where no Ψ_n is empty. For each $n \in \mathbb{N}$, put $\#n = |\Psi_n|$. So, $\Psi_n = \{\psi_n^1, ..., \psi_n^{\#n}\}$. Let

$$\Psi_{\#n}\Psi_n := \Psi_{\#n}(\psi_n^1, ..., \psi_n^{\#n})$$

Notice that, by Remark 2.3(ii), each $\Psi_{\#n}\Psi_n$ is a well defined finite non-empty subset of $\mathrm{Fm}(n)$. For any $\alpha_1, ..., \alpha_n \in \mathrm{Fm}$, as $\{\alpha_1, ..., \alpha_n\} \cup \Psi_n(\alpha_1, ..., \alpha_n)$ is inconsistent in $\vdash$, whence

$$\alpha_1, ..., \alpha_n \vdash \Psi_{\#n}\Psi_n(\alpha_1, ..., \alpha_n). \tag{2.6}$$

Writing $\bar{\alpha}$ for $\alpha_1, ..., \alpha_n$ we deduce from $\Psi_{\#n}\Psi_n(\bar{\alpha}) \vdash \Psi_{\#n}\Psi_n(\bar{\alpha})$ that

$$\Psi_{\#n}\Psi_n(\bar{\alpha}) \cup \Psi_n(\bar{\alpha}) \text{ is inconsistent in } \vdash . \tag{2.7}$$

Thus, we have the following equivalence.

Lemma 2.13 *Let* $\{\Psi_n : n \in \mathbb{N}\}$ *be an elementary IL-sequence for* $\vdash$*, with* $\Psi_n \neq \emptyset$ *for all* n*. This sequence is classical iff, for any formulas* $\alpha_1, ..., \alpha_n$ $(n \in \mathbb{N})$*, we have*

$$\alpha_1, ..., \alpha_n \dashv\vdash \Psi_{\#n}\Psi_n(\alpha_1, ..., \alpha_n). \tag{2.8}$$

In that case, whenever $a_1, ..., a_n$ *are elements of an algebra* $\mathbf{A}$*, then*

$$\mathrm{Fg}^{\mathbf{A}}_{\vdash}\{a_1, ..., a_n\} = \mathrm{Fg}^{\mathbf{A}}_{\vdash}\Psi^{\mathbf{A}}_{\#n}\Psi^{\mathbf{A}}_n(a_1, ..., a_n). \tag{2.9}$$

Proof. Assume $\{\Psi_n : n \in \mathbb{N}\}$ is classical. So, $\Psi_{\#n}\Psi_n(\bar{\alpha}) \vdash \{\alpha_1, ..., \alpha_n\}$, because (2.7). On the other hand, by (2.6), we also have $\{\alpha_1, ..., \alpha_n\} \vdash \Psi_{\#n}\Psi_n(\bar{\alpha})$. Hence, $\alpha_1, ..., \alpha_n \dashv\vdash \Psi_{\#n}\Psi_n(\alpha_1, ..., \alpha_n)$.

Conversely, suppose $\alpha_1, ..., \alpha_n \dashv\vdash \Psi_{\#n}\Psi_n(\alpha_1, ..., \alpha_n)$. Then, since $\{\Psi_n : n \in \mathbb{N}\}$ is an IL-sequence, $\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n)$ is inconsistent in $\vdash$ if and only if $\Gamma \vdash \Psi_{\#n}\Psi_n(\alpha_1, ..., \alpha_n)$. Therefore, by assumption, $\Gamma \cup \Psi_n(\alpha_1, ..., \alpha_n)$ is inconsistent in $\vdash$ if and only if $\Gamma \vdash \{\alpha_1, ..., \alpha_n\}$. Hence, $\{\Psi_n : n \in \mathbb{N}\}$ is classical. ■

As we mentioned before, this result is an abstract form of the property of elimination of double negation in the classical case.

We can now give an important semantic characterization of protoalgebraic deductive system having a classical inconsistency lemma.

Theorem 2.14 *Let* $\vdash$ *be a protoalgebraic deductive system. Then the following conditions are equivalent.*

(i) $\vdash$ *has a classical inconsistency lemma.*

(ii) *For every algebra* $\mathbf{A}$*, the compact* $\vdash$*-filters of* $\mathbf{A}$ *form a Boolean sublattice of the lattice of all* $\vdash$*-filters of* $\mathbf{A}$.

(iii) *The join semilattice of compact* $\vdash$*-theories is a Boolean lattice.*

In this case, every finite algebra has a Boolean lattice of $\vdash$*-filters.*

Proof. If $v_1 \vdash v_2$, then $\{\emptyset : n \in \mathbb{N}\}$ is a classical elementary IL-sequence for $\vdash$ and all three conditions hold (by Remark 2.4). Thereby, we may assume that $v_1 \nvdash v_2$, that is, $\emptyset$ does not occur in any IL-sequence for $\vdash$. On the other hand, the set Λ given by the definition of protoalgebraic logic cannot be empty, since $\Lambda(v_1, v_2), v_1 \vdash v_2$. So, as $\vdash \Lambda(v_1, v_1)$, it follows that $\emptyset$ is not a filter of any algebra.

(i) $\Rightarrow$ (ii). Let $\{\Psi_n : n \in \mathbb{N}\}$ be a classical elementary IL-sequence for $\vdash$ and let H be a compact $\vdash$-filter of an algebra $\mathbf{A}$. Since $H \neq \emptyset$, it has the form $\mathrm{Fg}^{\mathbf{A}}_{\vdash}\{a_1, ..., a_n\}$, where $n \in \mathbb{N}$, for some $a_1, ..., a_n \in A$. By the proof of Theorem 2.8, in the join semilattice of compact $\vdash$-filters of $\mathbf{A}$, we have $H^* = \mathrm{Fg}^{\mathbf{A}}_{\vdash}\Psi^{\mathbf{A}}_n(a_1, ..., a_n)$. Because $\Psi_n \neq \emptyset$, the same argument shows that

$$H^{**} = \mathrm{Fg}^{\mathbf{A}}_{\vdash}\Psi^{\mathbf{A}}_{\#n}\Psi^{\mathbf{A}}_n(a_1, ..., a_n).$$

Hence, by (2.9), $H^{**} = H$. Thus, the compact $\vdash$-filters of $\mathbf{A}$ form a Boolean lattice with respect to $\subseteq$, by the result of Glivenko mentioned at the beginning of this section.

It remains to show that the intersection of any two compact $\vdash$-filters of $\mathbf{A}$ is compact.

Let $\bar{a} = a_1, ..., a_n \in A$ and $\bar{b} = b_1, ..., b_m \in A$, where $n, m \in \mathbb{N}$ (so $\Psi_n, \Psi_m \neq \emptyset$). We claim that

$$\mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{a}\} \cap \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{b}\} = \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_{\#n+\#m}^{\mathbf{A}}(\Psi_n^{\mathbf{A}}(\bar{a}), \Psi_m^{\mathbf{A}}(\bar{b})), \tag{2.10}$$

where the right hand side abbreviates

$$\mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_{\#n+\#m}^{\mathbf{A}}(\psi_n^{1\,\mathbf{A}}(\bar{a}), ..., \psi_n^{\#n\,\mathbf{A}}(\bar{a}), \psi_m^{1\,\mathbf{A}}(\bar{b}), ..., \psi_m^{\#m\,\mathbf{A}}(\bar{b})).$$

Indeed, by Theorem 2.7, we have

$$\begin{aligned} A &= \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{a}\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_n^{\mathbf{A}}(\bar{a}) \\ &= \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{b}\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_m^{\mathbf{A}}(\bar{b}), \end{aligned}$$

whence

$$\begin{aligned} A &= \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{a}\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}(\Psi_n^{\mathbf{A}}(\bar{a}) \cup \Psi_m^{\mathbf{A}}(\bar{b})) \\ &= \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{b}\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,(\Psi_n^{\mathbf{A}}(\bar{a}) \cup \Psi_m^{\mathbf{A}}(\bar{b})), \end{aligned}$$

and so $\mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_{\#n+\#m}^{\mathbf{A}}(\Psi_n^{\mathbf{A}}(\bar{a}), \Psi_m^{\mathbf{A}}(\bar{b})) \subseteq \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{a}\} \cap \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{b}\}$, also by Theorem 2.7.

Conversely, let $c \in \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{a}\} \cap \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{b}\}$. Then, by Theorem 2.7, $A = \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{c\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_1^{\mathbf{A}}(c)$. Whence

$$A = \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{c\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_1^{\mathbf{A}}(c) \subseteq \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{a}\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_1^{\mathbf{A}}(c)$$

and similarly for $\bar{b}$, so

$$A = \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{a}\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_1^{\mathbf{A}}(c) = \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{\bar{b}\} +^{\mathbf{A}} \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_1^{\mathbf{A}}(c).$$

Then, again by Theorem 2.7, $\Psi_n^{\mathbf{A}}(\bar{a}) \cup \Psi_m^{\mathbf{A}}(\bar{b}) \subseteq \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_1^{\mathbf{A}}(c)$. Now, Boolean lattices satisfy $x \leq y^* \Rightarrow y \leq x^*$, hence

$$\mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\{c\} \subseteq \mathrm{Fg}_{\vdash}^{\mathbf{A}}\,\Psi_{\#n+\#m}^{\mathbf{A}}(\Psi_n^{\mathbf{A}}(\bar{a}), \Psi_m^{\mathbf{A}}(\bar{b})),$$

completing the proof of (2.10). As the right hand of (2.10) is compact, we have proved (ii).

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). By (iii), the semilattice of compact $\vdash$-theories is dually pseudo-complemented, hence by Theorem 2.8, $\vdash$ has an elementary IL-sequence $\{\Psi_n : n \in \mathbb{N}\}$ and $(\mathrm{Cn}_{\vdash}\,\{\alpha_1, ..., \alpha_n\})^* = \mathrm{Cn}_{\vdash}\,\Psi_n(\alpha_1, ..., \alpha_n)$ whenever $\alpha_1, ..., \alpha_n \in \mathrm{Fm}$. The identity $x^{**} = x$ shows that (2.8) holds, so it follows that the IL-sequence is classical, by Lemma 2.13. ■

Corollary 2.15 *If a protoalgebraic deductive system has a classical inconsistency lemma, then it has a deduction-detachment theorem.*

Proof. Let $\vdash$ be a protoalgebraic deductive system with a classical inconsistency lemma. By Theorem 2.14, the join semilattice of compact $\vdash$-theories is a Boolean lattice. Hence, as the $+$-reduct of a Boolean lattice is a dually Brouwerian join semilattice with 0, in which $a \cdot b^*$ is always the least c such that $a \leq b + c$, we get that $\vdash$ has a dually Brouwerian join semilattice of compact $\vdash$-theories. Therefore, $\vdash$ has a DDT. ■

We can verify that, whenever $\{\Psi_n : n \in \mathbb{N}\}$ is a classical elementary IL-sequence for $\vdash$ and $\Gamma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}$, then

$$\Gamma \cup \{\alpha\} \vdash \beta \quad \text{iff} \quad \Gamma \vdash \Psi_{1+\#1}(\alpha, \psi_1^1(\beta), ..., \psi_1^{\#1}(\beta)).$$

Indeed,
$\Gamma \cup \{\alpha\} \vdash \beta$ iff $\Gamma \cup \{\alpha\} \cup \Psi_1(\beta)$ is inconsistent iff $\Gamma \cup \{\alpha, \psi_1^1(\beta), ..., \psi_1^{\#1}(\beta)\}$ is inconsistent iff $\Gamma \vdash \Psi_{1+\#1}(\alpha, \psi_1^1(\beta), ..., \psi_1^{\#1}(\beta))$.

Another important consequence of Theorem 2.14 is the following.

Corollary 2.16 *If a protoalgebraic deductive system* $\vdash$ *has a classical inconsistency lemma, then the* $\vdash$*-filters of any algebra* $\mathbf{A}$ *form a pseudo-complemented distributive lattice, i.e., a Heyting algebra.*

Proof. By Corollary 2.15, $\vdash$ has a DDT and DDT always entails filter distributivity. As the $\vdash$-filter lattice of $\mathbf{A}$ is algebraic, it is isomorphic to the ideal lattice of the join semilattice $\mathbf{S}$ of compact filters of $\mathbf{A}$ and, by Theorem 2.14, we know that $\mathbf{S}$ is a Boolean lattice. But, by another result of Glivenko, the ideal lattice of a distributive lattice with 0 is always pseudo-complemented. Therefore, the filter lattice of $\mathbf{A}$ is pseudo-complemented and distributive. ■

2.3 Semisimplicity, EDPRC and filtrality

Let us examine now the algebraic counterpart of an algebraizable deductive system $\vdash$. We will see that, when $\vdash$ is algebraized by a quasivariety K, $\vdash$ has a classical inconsistency lemma if and only if for every $\mathbf{A} \in \mathsf{K}$ the compact K-congruences of $\mathbf{A}$ form a Boolean sublattice of the K-congruence lattice of $\mathbf{A}$.

A quasivariety K is said to be *relatively congruence distributive* if, for every member $\mathbf{A}$ of K, the lattice $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ of all K-congruences on $\mathbf{A}$ is distributive. K has the *relative congruence extension property* if every K-congruence on a subalgebra of a member $\mathbf{A}$ of K is the restriction of some K-congruence on $\mathbf{A}$.

Recall that in a complete lattice $\langle L, \wedge, + \rangle$ an element $a \in L$ is said to be *meet irreducible* if it is not the greatest element of L and, whenever $b, c \in L$, if $a = b \wedge c$, then $a = b$ or $a = c$. Besides, $a \in L$ is *completely meet-irreducible* if for every $X \subseteq L$, if $a = \bigwedge X$, then $a \in X$. We say that $a \in L$ is a *co-atom*, if $a < 1$ and there is no $b \in L$ such that $a < b < 1$, where 1 is the greatest element in L. It is an *atom* if $0 < a$ and there is no $b \in L$ such that $0 < b < a$, where 0 is the least element of L.

An algebra $\mathbf{A}$ in a quasivariety K is K-*subdirectly irreducible* or *finitely* K-*subdirectly irreducible* or K-*simple* if, in the lattice $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$, the identity relation $\mathrm{id}_{\mathbf{A}} = \{\langle a, a \rangle : a \in A\}$ is completely meet-irreducible or meet irreducible or a co-atom, respectively. Notice that the K-subdirectly irreducible and the

K-simple algebras cannot be trivial, while the finitely K-subdirectly irreducible algebras can be.

As we saw in Proposition 1.15, each member of a quasivariety K is isomorphic to a subdirect product of K-subdirectly irreducible algebras in K.

If every K-subdirectly irreducible algebra in a quasivariety K is K-simple, then we say that K is *relatively semisimple.* A quasivariety K is said to have *equationally definable principal relative congruences* (EDPRC) if there exists a finite set $\Phi \subseteq \mathrm{Fm}(4)\times\mathrm{Fm}(4)$ such that, whenever $\mathbf{A}$ in K and $a, b, c, d \in A$, then

$$\langle c, d\rangle \in \theta_{\mathsf{K}}(a, b) \quad \text{iff} \quad (\varphi^{\mathbf{A}}(a, b, c, d) = \eta^{\mathbf{A}}(a, b, c, d) \quad \text{for all } \langle\varphi, \eta\rangle \in \Phi).$$

Varieties with EDPRC are studied in [4, 23].

We say that K has *equationally definable principal relative congruence meets* (EDPRM) if there exists a finite set $\Delta \subseteq \mathrm{Fm}(4)\times\mathrm{Fm}(4)$ such that

$$\theta_{\mathsf{K}}(a, b) \cap \theta_{\mathsf{K}}(c, d) = \bigvee\nolimits_{\langle\varphi,\eta\rangle\in\Delta} \theta_{\mathsf{K}}(\varphi^{\mathbf{A}}(a, b, c, d), \eta^{\mathbf{A}}(a, b, c, d))$$

for all $\mathbf{A}$ in K and $a, b, c, d \in A$, where $\bigvee$ is the join formed in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. In this situation, the set Δ is called a *system of principal intersection formulas.*

Quasivarieties with EDPRM are studied comprehensively in [11]. Most of the results presented in this section appear there.

Recall that if K is a variety the congruences and K-congruences of algebras in K are the same, so the prefixes 'K-' and 'relatively' can be dropped.

A quasivariety K is *filtral* if for every K-congruence θ on a subdirect product $\mathbf{A}$ of K-subdirectly irreducible algebras in K, there exists a filter X_θ over the index set I of the product such that

$$\theta = \{\langle a, b\rangle \in A \times A : \{i \in I : a(i) = b(i)\} \in X_\theta\}.$$

The filtral quasivarieties turn out to be just the relative semisimple quasivarieties with EDPRC [9, 18, 19]. For further informations about this kind of quasivarieties we refer to [24, 25, 26].

We denote by $\mathsf{K}_{\mathrm{RFSI}}$ the class of all finitely K-subdirectly irreducible algebras in a quasivariety K. The lattice meet of $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ as well as of $\mathrm{Con}\,\mathbf{A}$ coincides with the set-theoretical intersection and it will be denoted by $\wedge$ while to denote the lattice join of $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ we shall use the symbol $+^{\mathsf{K}}$.

Lemma 2.17 *For a quasivariety* K *of algebras the following conditions are equivalent*:

(i) K *is relatively congruence distributive.*

(ii) *For every* $\mathbf{A} \in \mathsf{K}$ *and* $\theta_0, \theta_1, \psi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$, *if* ψ *is finitely meet irreducible in* $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ *and* $\theta_0 \wedge \theta_1 \leq \psi$, *then* $\theta_0 \leq \psi$ *or* $\theta_1 \leq \psi$.

(iii) *For every* $\mathbf{A} \in \mathsf{K}$, $\theta_0, \theta_1 \in \mathrm{Con}\,\mathbf{A}$ *and* $\psi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$, *if* ψ *is finitely meet irreducible in* $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ *and* $\theta_0 \wedge \theta_1 \leq \psi$, *then* $\theta_0 \leq \psi$ *or* $\theta_1 \leq \psi$.

(iv) *For every* $\mathbf{A} \in \mathsf{K}$, $a, b, c, d \in A$ *and* $\psi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$, *if* ψ *is finitely meet irreducible in* $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ *and* $\theta_{\mathsf{K}}(a, b) \wedge \theta_{\mathsf{K}}(c, d) \leq \psi$ *then* $\langle a, b\rangle \in \psi$ *or* $\langle c, d\rangle \in \psi$.

Proof. (i) $\Rightarrow$ (ii). Assume (i). Since $\theta_0 \wedge \theta_1 \leq \psi$, then $(\theta_0 \wedge \theta_1) +^{\mathsf{K}} \psi = \psi$. Thus, by distributivity, $(\theta_0 +^{\mathsf{K}} \psi) \wedge (\theta_1 +^{\mathsf{K}} \psi) = \psi$. As ψ is finitely meet irreducible, $\theta_0 +^{\mathsf{K}} \psi = \psi$ or $\theta_1 +^{\mathsf{K}} \psi = \psi$. Therefore, $\theta_0 \leq \psi$ or $\theta_1 \leq \psi$.

(ii) $\Rightarrow$ (iii). First we show that (ii) implies the following property:

(M) For every $\mathbf{A}$ in K, $\theta_0, \theta_1 \in \mathrm{Con}\, \mathbf{A}$ and $\psi \in \mathrm{Con}_{\mathsf{K}} \mathbf{A}$, if ψ is finitely meet irreducible in $\mathrm{Con}_{\mathsf{K}} \mathbf{A}$, $\{\theta_0, \theta_1\} \cap \mathrm{Con}_{\mathsf{K}} \mathbf{A} \neq \emptyset$ and $\theta_0 \wedge \theta_1 \leq \psi$, then $\theta_0 \leq \psi$ or $\theta_1 \leq \psi$.

We proceed by contraposition in order to show that (M) follows from (ii). So, suppose that on a certain algebra $\mathbf{A} \in \mathsf{K}$ we have congruence relations θ_0, θ_1 and ψ satisfying:

1.- ψ is a finitely meet irreducible element of $\mathrm{Con}_{\mathsf{K}} \mathbf{A}$.

2.- $\{\theta_0, \theta_1\} \cap \mathrm{Con}_{\mathsf{K}} \mathbf{A} \neq \emptyset$.

3.- $\theta_0 \wedge \theta_1 \leq \psi$.

4.- Neither $\theta_0 \leq \psi$ nor $\theta_1 \leq \psi$.

Assume $\theta_0 \in \mathrm{Con}_{\mathsf{K}} \mathbf{A}$; in the case $\theta_1 \in \mathrm{Con}_{\mathsf{K}} \mathbf{A}$ we proceed similarly. Let $B = \{\langle a, b\rangle \in A \times A : \langle a, b\rangle \in \theta_1\}$. As $\mathbf{B}$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$ whose projections $\pi_1, \pi_2 : \mathbf{B} \to \mathbf{A}$ fulfill $\pi_1(B) = \pi_2(B) = A$, we have $\pi_1^{-1}(\theta_1) = \pi_2^{-1}(\theta_1)$, where

$$\pi_1^{-1}(\theta_1) = \{\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle \in B \times B : \langle a_1, b_1\rangle \in \theta_1\}$$

and

$$\pi_2^{-1}(\theta_1) = \{\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle \in B \times B : \langle a_2, b_2\rangle \in \theta_1\}.$$

From this it follows

5.- $\pi_1^{-1}(\theta_0) \wedge \pi_2^{-1}(\mathrm{id}_{\mathbf{A}}) \leq \pi_1^{-1}(\psi)$.

Indeed, as $\pi_2^{-1}(\mathrm{id}_{\mathbf{A}}) \leq \pi_2^{-1}(\theta_1)$, it follows that $\pi_1^{-1}(\theta_0) \wedge \pi_2^{-1}(\mathrm{id}_{\mathbf{A}}) \leq \pi_1^{-1}(\theta_0) \wedge \pi_2^{-1}(\theta_1) = \pi_1^{-1}(\theta_0) \wedge \pi_1^{-1}(\theta_1) = \pi_1^{-1}(\theta_0 \wedge \theta_1) \leq \pi_1^{-1}(\psi)$ (by (3)).

Thus, since $\theta_0 \not\leq \psi$ and $\pi_1(B) = A$, we have

6.- $\pi_1^{-1}(\theta_0) \not\leq \pi_1^{-1}(\psi)$.

Now, take $\langle a, b\rangle \in (\theta_1 \setminus \psi)$ (by (4) such pair exists). As $\langle a, b\rangle, \langle b, b\rangle \in B$, we obtain $\langle\langle a, b\rangle, \langle b, b\rangle\rangle \in \pi_2^{-1}(\mathrm{id}_{\mathbf{A}})$ and $\langle\langle a, b\rangle, \langle b, b\rangle\rangle \notin \pi_1^{-1}(\psi)$. Therefore, we get

7.- $\pi_2^{-1}(\mathrm{id}_{\mathbf{A}}) \not\leq \pi_1^{-1}(\psi)$.

On the other hand, we claim that $\pi_1^{-1}(\theta_0) \in \mathrm{Con}_{\mathsf{K}} \mathbf{B}$. Indeed, as $\mathbf{B}$ and $\mathbf{A}/\theta_0$ are elements of K and the composition $\pi_{\theta_0} \circ \pi_1 : \mathbf{B} \to \mathbf{A}/\theta_0$ is an onto homomorphism (where $\pi_{\theta_0} : \mathbf{A} \to \mathbf{A}/\theta_0$ is the canonical homomorphism defined by $\pi_{\theta_0}(a) = a/\theta_0$ for every $a \in A$), it follows that $\mathbf{B}/\ker(\pi_{\theta_0} \circ \pi_1) \cong \mathbf{A}/\theta_0$, by the First Isomorphism Theorem. But

$$\begin{aligned}\ker(\pi_{\theta_0} \circ \pi_1) &= \{\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle \in B \times B : \pi_{\theta_0}(a_1) = \pi_{\theta_0}(b_1)\}\\ &= \{\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle \in B \times B : \langle a_1, b_1\rangle \in \theta_0\} = \pi_1^{-1}(\theta_0).\end{aligned}$$

Hence, $\mathbf{B}/\pi_1^{-1}(\theta_0) \cong \mathbf{A}/\theta_0$ and therefore $\pi_1^{-1}(\theta_0) \in \mathrm{Con}_{\mathsf{K}} \mathbf{B}$.

Similarly, we have $\pi_2^{-1}(\mathrm{id}_{\mathbf{A}}), \pi_1^{-1}(\psi) \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{B}$. Moreover, using the Correspondence Theorem we obtain that $\pi_1^{-1}(\psi)$ is finitely meet irreducible in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{B}$ because ψ is finitely meet irreducible in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ and π_1 is an onto homomorphism.

Consequently, by (5), (6) and (7), the condition (ii) is not satisfied, showing that (ii) implies (M).

Applying the same kind of arguments we obtain that (M) yields (iii). That is, proceeding again by contraposition, assume that on a certain algebra $\mathbf{A} \in \mathsf{K}$ we have congruence relations θ_0, θ_1, ψ satisfying:

1.- ψ is a finitely meet irreducible element of $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$.

2.- $\theta_0 \wedge \theta_1 \leq \psi$.

3.- Neither $\theta_0 \leq \psi$ nor $\theta_1 \leq \psi$.

So, in the same way as in (ii) implies (M), we obtain an algebra $\mathbf{B}$ and congruence relations $\pi_2^{-1}(\mathrm{id}_{\mathbf{A}}), \pi_1^{-1}(\theta_0), \pi_1^{-1}(\psi)$ such that:

4.- $\pi_1^{-1}(\psi)$ is finitely meet irreducible in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{B}$.

5.- $\pi_1^{-1}(\theta_0) \wedge \pi_2^{-1}(\mathrm{id}_{\mathbf{A}}) \leq \pi_1^{-1}(\psi)$.

6.- $\pi_1^{-1}(\theta_0) \not\leq \pi_1^{-1}(\psi)$ and $\pi_2^{-1}(\mathrm{id}_{\mathbf{A}}) \not\leq \pi_1^{-1}(\psi)$.

7.- $\pi_2^{-1}(\mathrm{id}_{\mathbf{A}}) \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{B}$.

Thus, the condition (M) is not satisfied, showing that (iii) follows from (M).

Consequently, (ii) implies (iii).

(iii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (i). Assume (iv) and let $\mathbf{A}$ in K. We claim that for all $a, b \in A$:

$$\theta_{\mathsf{K}}(a,b) \wedge \bigvee_{\langle c,d\rangle \in H} \theta_{\mathsf{K}}(c,d) = \bigvee_{\langle c,d\rangle \in H} \theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \tag{2.11}$$

where H is a finite subset of $A \times A$ and $\bigvee$ is the join formed in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$.

Indeed, as the lattice $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ is algebraic (i.e., every $\theta \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ coincides with the join of all $\theta_{\mathsf{K}}(a,b)$ where $\langle a,b\rangle \in \theta$), we have that each element of $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ is the meet of the finitely meet irreducible elements greater or equal than it. So, in order to show (2.11) it is enough to prove that for every finitely meet irreducible $\psi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$,

$$\theta_{\mathsf{K}}(a,b) \wedge \textstyle\bigvee_{\langle c,d\rangle \in H} \theta_{\mathsf{K}}(c,d) \leq \psi \quad \text{iff} \quad \textstyle\bigvee_{\langle c,d\rangle \in H} \theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \leq \psi.$$

Since for all $\langle c,d\rangle \in H$ it holds that

$$\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \leq \theta_{\mathsf{K}}(a,b) \wedge \textstyle\bigvee_{\langle c,d\rangle \in H} \theta_{\mathsf{K}}(c,d),$$

we obtain the implication from left to right. To prove the other implication, assume that $\bigvee_{\langle c,d\rangle \in H} \theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \leq \psi$. As for every $\langle c,d\rangle \in H$ we have $\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \leq \psi$, using the assumption (iv), it follows that $\theta_{\mathsf{K}}(a,b) \leq \psi$ or

$\theta_{\mathsf{K}}(c,d) \leq \psi$ for all $\langle c,d\rangle \in H$. If $\theta_{\mathsf{K}}(a,b) \leq \psi$, we are done. If $\theta_{\mathsf{K}}(c,d) \leq \psi$ for all $\langle c,d\rangle \in H$, then $\bigvee_{\langle c,d\rangle\in H} \theta_{\mathsf{K}}(c,d) \leq \psi$ and hence $\theta_{\mathsf{K}}(a,b) \wedge \bigvee_{\langle c,d\rangle\in H} \theta_{\mathsf{K}}(c,d) \leq \psi$.

Therefore, (2.11) is true.

Now, let $\theta_0, \theta_1, \theta_2 \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. Thus, since $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ is algebraic,

$$\begin{aligned}\theta_0 &= \bigvee\{\theta_{\mathsf{K}}(a_0,b_0) : \langle a_0,b_0\rangle \in \theta_0\},\\ \theta_1 &= \bigvee\{\theta_{\mathsf{K}}(a_1,b_1) : \langle a_1,b_1\rangle \in \theta_1\},\\ \theta_2 &= \bigvee\{\theta_{\mathsf{K}}(a_2,b_2) : \langle a_2,b_2\rangle \in \theta_2\}.\end{aligned}$$

Hence, using (2.11), we get that for all $\langle a_0,b_0\rangle \in \theta_0$, $\langle a_1,b_1\rangle \in \theta_1$ and $\langle a_2,b_2\rangle \in \theta_2$ it holds that $\theta_{\mathsf{K}}(a_0,b_0) \wedge (\theta_{\mathsf{K}}(a_1,b_1) +^{\mathsf{K}} \theta_{\mathsf{K}}(a_2,b_2)) = (\theta_{\mathsf{K}}(a_0,b_0) \wedge \theta_{\mathsf{K}}(a_1,b_1)) +^{\mathsf{K}} (\theta_{\mathsf{K}}(a_0,b_0) \wedge \theta_{\mathsf{K}}(a_2,b_2))$.

Consequently, $\theta_0 \wedge (\theta_1 +^{\mathsf{K}} \theta_2) = (\theta_0 \wedge \theta_1) +^{\mathsf{K}} (\theta_0 \wedge \theta_2)$.

Therefore, $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ is distributive. ∎

As a consequence of the above lemma we have:

Proposition 2.18 *Let* K *be a quasivariety with* EDPRM *and* Δ *a system of principal intersection terms. Then the following conditions are fulfilled*:

(i) K *is relatively congruence distributive.*

(ii) *For every* $\mathbf{A} \in \mathsf{K}$, $\mathbf{A} \in \mathsf{K}_{\mathrm{RFSI}}$ *iff* $\mathbf{A} \models \forall xyzw[(\bigwedge_{\langle\varphi,\eta\rangle\in\Delta} \varphi(x,y,z,w) = \eta(x,y,z,w)) \to (x = y \text{ or } z = w)]$.

Proof. (i). Let $\mathbf{A} \in \mathsf{K}$ and $a,b,c,d \in A$. Let ψ be a finitely meet irreducible element of $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ such that $\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \leq \psi$. Then, for all $\langle\varphi,\eta\rangle \in \Delta$, $\langle\varphi^{\mathbf{A}}(a,b,c,d), \eta^{\mathbf{A}}(a,b,c,d)\rangle \in \psi$. Hence, for each $\langle\varphi,\eta\rangle \in \Delta$, $\varphi^{\mathbf{A}}(a,b,c,d)/\psi = \eta^{\mathbf{A}}(a,b,c,d)/\psi$ and therefore $\langle\varphi^{\mathbf{A}}(a,b,c,d)/\psi, \eta^{\mathbf{A}}(a,b,c,d)/\psi\rangle \in \mathrm{id}_{\mathbf{A}/\psi}$. So, it follows that $\bigvee_{\langle\varphi,\eta\rangle\in\Delta} \theta_{\mathsf{K}}(\varphi^{\mathbf{A}}(a,b,c,d)/\psi, \eta^{\mathbf{A}}(a,b,c,d)/\psi) = \mathrm{id}_{\mathbf{A}/\psi}$. That is, $\theta_{\mathsf{K}}(a/\psi, b/\psi) \wedge \theta_{\mathsf{K}}(c/\psi, d/\psi) = \mathrm{id}_{\mathbf{A}/\psi}$. Consequently, $\langle a/\psi, b/\psi\rangle \in \mathrm{id}_{\mathbf{A}/\psi}$ or $\langle c/\psi, d/\psi\rangle \in \mathrm{id}_{\mathbf{A}/\psi}$, because $\mathbf{A}/\psi \in \mathsf{K}_{\mathrm{RFSI}}$. Therefore, $\langle a,b\rangle \in \psi$ or $\langle c,d\rangle \in \psi$.

Thus, by Lemma 2.17, K is congruence distributive.

(ii). Let $\mathbf{A} \in \mathsf{K}$. Suppose $\mathbf{A} \in \mathsf{K}_{\mathrm{RFSI}}$ and $\varphi^{\mathbf{A}}(a,b,c,d) = \eta^{\mathbf{A}}(a,b,c,d)$ for all $a,b,c,d \in A$ and $\langle\varphi,\eta\rangle \in \Delta$. Thus, $\langle\varphi^{\mathbf{A}}(a,b,c,d), \eta^{\mathbf{A}}(a,b,c,d)\rangle \in \mathrm{id}_{\mathbf{A}}$. So, $\theta_{\mathsf{K}}(a,b) \cap \theta_{\mathsf{K}}(c,d) = \mathrm{id}_{\mathbf{A}}$. Hence, $\langle a,b\rangle \in \mathrm{id}_{\mathbf{A}}$ or $\langle c,d\rangle \in \mathrm{id}_{\mathbf{A}}$.

Conversely, let $\theta_1, \theta_2 \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ and assume $\mathrm{id}_{\mathbf{A}} = \theta_1 \wedge \theta_2$. Since $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ is algebraic, $\theta_1 = \bigvee\{\theta_{\mathsf{K}}(a,b) : \langle a,b\rangle \in \theta_1\}$ and $\theta_2 = \bigvee\{\theta_{\mathsf{K}}(c,d) : \langle c,d\rangle \in \theta_2\}$. Thus, for all $\langle a,b\rangle \in \theta_1$ and $\langle c,d\rangle \in \theta_2$, it holds that $\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) = \mathrm{id}_{\mathbf{A}}$. So, for every $\langle\varphi,\eta\rangle \in \Delta$, $\langle\varphi^{\mathbf{A}}(a,b,c,d), \eta^{\mathbf{A}}(a,b,c,d)\rangle \in \mathrm{id}_{\mathbf{A}}$. Hence, $\varphi^{\mathbf{A}}(a,b,c,d) = \eta^{\mathbf{A}}(a,b,c,d)$ which implies that $a = b$ or $c = d$ (by assumption). Whence, the congruence $\mathrm{id}_{\mathbf{A}}$ is finitely meet irreducible in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. Thus, $\mathbf{A} \in \mathsf{K}_{\mathrm{RFSI}}$. ∎

The next consequence of Lemma 2.17 give us an important property of the quasivarieties that are relatively congruence distributive.

Proposition 2.19 *If a quasivariety* K *is relatively congruence distributive, then every relatively subdirectly irreducible member of* K *is subdirectly irreducible in the absolute sense, i.e.,* $\mathsf{K}_{\mathrm{RFSI}} = \mathbf{V}(\mathsf{K})_{\mathrm{FSI}} \cap \mathsf{K}$.

Proof. Let $\mathbf{A} \in \mathsf{K}_{\mathrm{RFSI}}$ and suppose $\theta_0 \wedge \theta_1 = \mathrm{id}_{\mathbf{A}}$ where $\theta_0, \theta_1 \in \mathrm{Con}\,\mathbf{A}$. As $\mathrm{id}_{\mathbf{A}}$ is finitely meet irreducible in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$, by Lemma 2.17, we get $\theta_0 = \mathrm{id}_{\mathbf{A}}$ or $\theta_1 = \mathrm{id}_{\mathbf{A}}$. Thus, $\mathbf{A} \in \mathbf{V}(\mathsf{K})_{\mathrm{FSI}}$. ■

For a class M of similar algebras, by $\mathbf{Q}(\mathsf{M})$ we denote the least quasivariety containing M, that is, the quasivariety *generated* by M. From the Preliminaries in Theorem 1.11 we have $\mathbf{Q}(\mathsf{M}) = \mathbf{ISPP}_U(\mathsf{M})$.

Lemma 2.20 *Let* M *be a class of similar algebras. Then, every nontrivial member of* $\mathbf{Q}(\mathsf{M})_{\mathrm{RFSI}}$ *belongs to* $\mathbf{ISP}_U(\mathsf{M})$.

Proof. Let $\mathbf{A} \in \mathbf{Q}(\mathsf{M})_{\mathrm{RFSI}}$ and assume $|A| > 1$. Then, $\mathbf{A}$ is a subalgebra of $\prod_{i\in I} \mathbf{C}_i$, where $\mathbf{C}_i \in \mathbf{IP}_U(\mathsf{M})$ for all $i \in I$.

In order to show that $\mathbf{A} \in \mathbf{ISP}_U(\mathsf{M})$ we want to find an ultrafilter $\mathcal{U}$ over I such that $\mathbf{A}$ is embeddable into the ultraproduct $\prod_{i\in I} \mathbf{C}_i/\mathcal{U}$.

So, for every $S \subseteq I$ define a congruence relation θ_S of $\prod_{i\in I} \mathbf{C}_i$ as follows:

$$\langle a, b\rangle \in \theta_S \quad \text{iff} \quad S \subseteq \{i \in I : a(i) = b(i)\}$$

for all $a, b \in \prod_{i\in I} \mathbf{C}_i$.

Let $\mathbb{F}$ be the set of all filters $\mathcal{F}$ over I satisfying $\theta_S \restriction \mathbf{A} = \mathrm{id}_{\mathbf{A}}$ for all $S \in \mathcal{F}$. Observe that $\mathbb{F} \neq \emptyset$ because $\{I\} \in \mathbb{F}$. As the poset $(\mathbb{F}, \subseteq)$ is *inductive* (i.e., every chain of $\mathbb{F}$ has an upper bound on it), then it has maximal elements (by Zorn's Lemma). Choose one of them and denoted it by $\mathcal{U}$. We claim that $\mathcal{U}$ is an ultrafilter over I.

Indeed, as $|A| > 1$, we have at least two elements $a, b \in A$ such that $a \neq b$. Thus, there must be a subset $H \subseteq I$ such that $a(i) \neq b(i)$ for all $i \in H$. So, $H \notin \mathcal{U}$. Hence, $\mathcal{U} \neq 2^I$. This means that $\mathcal{U}$ is proper. Now suppose, for a contradiction, that there is $S \subseteq I$ such that $S \notin \mathcal{U}$ and $I \setminus S \notin \mathcal{U}$. Then, for some $G \in \mathcal{U}$ it holds that $\theta_{S\cap G} \restriction \mathbf{A} \neq \mathrm{id}_{\mathbf{A}}$ and $\theta_{(I\setminus S)\cap G} \restriction \mathbf{A} \neq \mathrm{id}_{\mathbf{A}}$ (in particular this is true for I). But $(\theta_{S\cap G} \restriction \mathbf{A}) \wedge (\theta_{(I\setminus S)\cap G} \restriction \mathbf{A}) = \theta_G \restriction \mathbf{A}$. Therefore, as $\theta_G \restriction \mathbf{A} = \mathrm{id}_{\mathbf{A}}$ and $\mathbf{A} \in \mathrm{Q}(\mathsf{M})_{\mathrm{RFSI}}$, we get that $\theta_{S\cap G} \restriction \mathbf{A} = \mathrm{id}_{\mathbf{A}}$ or $\theta_{(I\setminus S)\cap G} \restriction \mathbf{A} = \mathrm{id}_{\mathbf{A}}$, a contradiction. Thus, for every $S \subseteq I$, $S \in \mathcal{U}$ or $I \setminus S \in \mathcal{U}$, showing the claim.

Since $\bigvee_{S\in\mathcal{U}}(\theta_S \restriction \mathbf{A}) = \mathrm{id}_{\mathbf{A}}$, we have $\mathbf{A}/\mathrm{id}_{\mathbf{A}} = \mathbf{A}/\bigvee_{S\in\mathcal{U}}(\theta_S \restriction \mathbf{A})$ and hence $\mathbf{A}/\mathrm{id}_{\mathbf{A}}$ is embeddable into $\prod_{i\in I} \mathbf{C}_i/\mathcal{U}$ by means of the map $a/\mathrm{id}_{\mathbf{A}} \mapsto a/\mathcal{U}$. For if $a/\mathrm{id}_{\mathbf{A}} \neq b/\mathrm{id}_{\mathbf{A}}$, then $a \neq b$. So, it follows that $\langle a, b\rangle \notin \theta_S \restriction \mathbf{A}$ for all $S \in \mathcal{U}$. Whence, $S \nsubseteq \{i \in I : a(i) = b(i)\}$ for every $S \in \mathcal{U}$. Consequently, $\{i \in I : a(i) = b(i)\} \notin \mathcal{U}$. Thus, $a/\mathcal{U} \neq b/\mathcal{U}$.

Therefore, $\mathbf{A}$ is embeddable into the ultraproduct of $\{\mathbf{C}_i : i \in I\}$ modulo $\mathcal{U}$. Hence, $\mathbf{A} \in \mathbf{ISP}_U(\mathsf{M})$. ■

It is worth mentioning that there exists quasivarieties whose RFSI members are not all FSI. Let us consider the following example.

A *Heyting algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice and for every $a, b, c \in A$, $a \wedge b \leq c$ iff $a \wedge b \to c$.

Let X be a poset. We can construct a Heyting algebra from X as follows [13,14].

A set $U \subseteq X$ is said to be an *upset* if for every $x, y \in X$, if $x \in U$ and $x \leq y$, then $y \in U$. We use $\mathrm{Up}(X)$ to denote the collection of all upsets of X.

We define the operation $\Rightarrow$ between upsets of X by:

$$U \Rightarrow V := X \setminus \downarrow (U \setminus V),$$

for all $U, V \in \mathrm{Up}(X)$.

It is known that $\mathbf{Up}(X) = \langle \mathrm{Up}(X), \cap, \cup, \Rightarrow, \emptyset, X \rangle$ is a Heyting algebra.

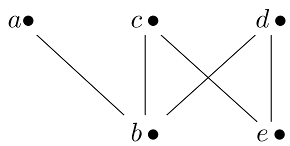
Let X be a finite poset. The following facts hold.

1.- $\mathbf{Up}(X)$ is FSI iff X is *rooted*, i.e., it has a minimum.

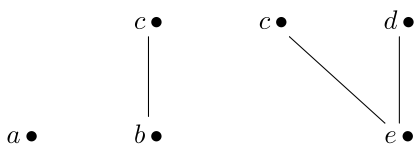
2.- The subalgebras of $\mathbf{Up}(X)$ are (up to isomorphisms) the algebras of the form $\mathbf{Up}(Y)$, where Y is a *p-morphic* image of X, i.e., there exists a surjective order-preserving map $f : X \to Y$ such that for all $x \in X$, $y \in Y$, if $f(x) \leq^Y y$, then there is $z \in X$ with $x \leq^X z$ and $f(z) = y$.

3.- The FSI subalgebras of $\mathbf{Up}(X)$ are (up to isomorphisms) the algebras of the form $\mathbf{Up}(Y)$ where Y is a rooted p-morphic image of X.

Consider the poset X depicted below



Its rooted p-morphic images are



A $\qquad$ B $\qquad$ C

under the following maps:

$$f_1 : X \to A \qquad x \mapsto a$$

$$f_2 : X \to B \qquad a, c, d, e \mapsto c \qquad b \mapsto b$$

$$f_3 : X \to C \qquad a, c \mapsto c \qquad b, e \mapsto e \qquad d \mapsto d$$

Let us consider the quasivariety $\mathbf{Q}(\mathbf{Up}(X))$. In view of Lemma 2.20, we have

$$\mathbf{Q}(\mathbf{Up}(X))_{\mathrm{RFSI}} \subseteq \mathbf{ISP}_U(\mathbf{Up}(X)).$$

Since X is finite, so is $\mathbf{Up}(X)$, whence $\mathbf{P}_U(\mathbf{Up}(X)) \subseteq \mathbf{I}(\mathbf{Up}(X))$. Therefore,

$$\mathbf{Q}(\mathrm{Up}(X))_{\mathrm{RFSI}} \subseteq \mathbf{IS}(\mathrm{Up}(X)).$$

Now, suppose that the FSI and RFSI members of $\mathbf{Q}(\mathrm{Up}(X))$ are the same.

Thus, $\mathbf{Q}(\mathrm{Up}(X))_{\mathrm{RFSI}} \subseteq \mathbf{IS}(\mathrm{Up}(X))_{\mathrm{FSI}} \subseteq \mathbf{I}(\{\mathrm{Up}(A), \mathrm{Up}(B), \mathrm{Up}(C)\})$.

As $\mathbf{Q}(\mathrm{Up}(X)) = \mathbf{P}_{\mathrm{SD}}\mathbf{Q}(\mathrm{Up}(X))_{\mathrm{RFSI}}$, we obtain that $\mathrm{Up}(X)$ is a subdirect product of $\mathrm{Up}(A)$, $\mathrm{Up}(B)$, $\mathrm{Up}(C)$.

In particular, every equation valid in $\mathrm{Up}(A)$, $\mathrm{Up}(B)$, $\mathrm{Up}(C)$ is also valid in $\mathrm{Up}(X)$.

But this is not the case, because the equation

$$(x_0 \to (x_1 \vee x_2)) \vee (x_1 \to (x_0 \vee x_2)) \vee (x_2 \to (x_0 \vee x_1)) \approx 1$$

does not hold in $\mathrm{Up}(X)$.

This equation clearly is true for the case of $\mathrm{Up}(A) = \{\emptyset, A\}$, $\mathrm{Up}(B) = \{\emptyset, \{c\}, B\}$ and $\mathrm{Up}(C) = \{\emptyset, \{c\}, \{d\}, \{c, d\}, C\}$.

However, it is not valid for $\mathrm{Up}(X)$. Take $\{a\}, \{c\}, \{d\} \in \mathrm{Up}(X)$ and set $x_0 = \{a\}$, $x_1 = \{c\}$ and $x_2 = \{d\}$. Then

$$\begin{aligned}(\{a\} \Rightarrow (\{c\} \cup \{d\})) \cup (\{c\} \Rightarrow (\{a\} \cup \{d\})) \cup (\{d\} \Rightarrow (\{a\} \cup \{c\})) &= \\ (X \setminus \downarrow \{a\}) \cup (X \setminus \downarrow \{c\}) \cup (X \setminus \downarrow \{d\}) &= \\ \{c, d, e\} \cup \{a, d\} \cup \{a, c\} &\neq X.\end{aligned}$$

This shows that the RFSI members of $\mathbf{Q}(\mathrm{Up}(X))$ are not all FSI.

Lemma 2.21 *For a quasivariety* K *of algebras,* $\mathbf{A}, \mathbf{B} \in \mathsf{K}$*,* $a, b \in A$*,* $\theta_0, \theta_1 \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ *and a surjective homomorphism* $h : \mathbf{A} \to \mathbf{B}$ *it holds*:

(i) $h(\theta_{\mathsf{K}}(a, b) +^{\mathsf{K}} \ker h) = \theta_{\mathsf{K}}(h(a), h(b))$.

(ii) *If* $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ *is distributive, then* $h(\theta_0 \wedge \theta_1 +^{\mathsf{K}} \ker h) = h(\theta_0 +^{\mathsf{K}} \ker h) \wedge h(\theta_1 +^{\mathsf{K}} \ker h)$.

Proof. (i). Observe that for for all $\theta \geq \ker h$ and all $\psi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{B}$ we have $\mathbf{A}/h^{-1}(\psi) \cong \mathbf{B}/\psi$ and $h^{-1}h(\theta) = \theta$. This is because h is surjective and therefore the composition $\pi_\psi \circ h : \mathbf{A} \to \mathbf{B}/\psi$ too. So, we can use the First Isomorphism Theorem to get that $\mathbf{A}/\ker(\pi_\psi \circ h) \cong \mathbf{B}/\psi$. But $\ker(\pi_\psi \circ h) = \{\langle a, b\rangle \in A \times A : h(a)/\psi = h(b)/\psi\} = \{\langle a, b\rangle \in A \times A : \langle h(a), h(b)\rangle \in \psi\} = h^{-1}(\psi)$. Thus, we obtain the desired isomorphism.

Consequently, $h(\theta_{\mathsf{K}}(a, b) +^{\mathsf{K}} \ker h) \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{B}$. Hence, $h(\theta_{\mathsf{K}}(a, b) +^{\mathsf{K}} \ker h) \geq \theta_{\mathsf{K}}(h(a), h(b))$. Besides, it holds that $\theta_{\mathsf{K}}(a, b) +^{\mathsf{K}} \ker h \leq h^{-1}(\theta_{\mathsf{K}}(h(a), h(b)))$. Therefore, $h(\theta_{\mathsf{K}}(a, b) +^{\mathsf{K}} \ker h) \leq \theta_{\mathsf{K}}(h(a), h(b))$. So, we have (i).

(ii). Since $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ is distributive, $h((\theta_0 \wedge \theta_1) +^{\mathsf{K}} \ker h) = h((\theta_0 +^{\mathsf{K}} \ker h) \cap (\theta_1 +^{\mathsf{K}} \ker h))$. So, we show

$$h((\theta_0 +^{\mathsf{K}} \ker h) \cap (\theta_1 +^{\mathsf{K}} \ker h)) = h(\theta_0 +^{\mathsf{K}} \ker h) \wedge h(\theta_1 +^{\mathsf{K}} \ker h).$$

Indeed, let $\langle a,b\rangle \in h(\theta_0 +^{\mathsf{K}} \ker h) \wedge h(\theta_1 +^{\mathsf{K}} \ker h)$. Hence, for some $\langle x,y\rangle \in \theta_0 +^{\mathsf{K}} \ker h$, $\langle a,b\rangle = \langle h(x), h(y)\rangle$. Then, $\langle x,y\rangle \in h^{-1}h(\theta_1 +^{\mathsf{K}} \ker h)$. So, $\langle x,y\rangle \in (\theta_0 +^{\mathsf{K}} \ker h) \wedge (\theta_1 +^{\mathsf{K}} \ker h)$. Thus, $\langle a,b\rangle \in h((\theta_0 +^{\mathsf{K}} \ker h) \cap (\theta_1 +^{\mathsf{K}} \ker h))$.

The converse contention is immediate ■

A first order formula Φ is a *universal formula* if it is in prenex form and all the quantifiers occurring in it are universal. A class of similar algebras M is a *universal class* if it can be axiomatized by universal formulas, i.e., $\mathsf{M} = \mathrm{Mod}(\Sigma) = \{\mathbf{A} : \mathbf{A} \vDash \Sigma\}$ for some set Σ of universal formulas. We know that universal classes are exactly those classes that are closed under subalgebras and ultraproducts.

By Proposition 2.18, we have that every quasivariety with EDPRM is relatively congruence distributive and its finitely subdirectly irreducible members form a universal class. It turns out that the converse implication is also true. This follows from the following result.

Theorem 2.22 *For a quasivariety* K *the following conditions are equivalent*:

(i) K *has* EDPRM.

(ii) *For every member* $\mathbf{A} \in \mathsf{K}$ *the lattice* $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ *is distributive and the set of its compact elements forms a sublattice.*

(iii) K *is relatively congruence distributive and* $\mathsf{K}_{\mathrm{RFSI}}$ *forms a universal class.*

(iv) *The lattice* $\mathrm{Con}_{\mathsf{K}}\, F_{\mathsf{K}}(4)$ *is distributive and the set of its compact elements forms a sublattice.*

(v) *There exists a finite set* $\Delta \subseteq \mathrm{Fm}(4) \times \mathrm{Fm}(4)$ *such that*

$$\mathsf{K}_{\mathrm{RFSI}} \vDash \forall xyzw[(\textstyle\bigwedge_{\langle\varphi,\eta\rangle\in\Delta} \varphi(x,y,z,w) = \eta(x,y,z,w)) \leftrightarrow (x = y \text{ or } z = w)].$$

Proof. By Proposition 2.18(i), we already have (i) implies (ii). The converse implication also holds. To show this, consider the free algebra $F_{\mathsf{K}}(4)$ whose free generators are x,y,z,w. Let $\Gamma = \{\langle p_\alpha, q_\alpha\rangle : \alpha < \beta\}$ be a set of generators of the congruence $\theta_{\mathsf{K}}(x,y) \wedge \theta_{\mathsf{K}}(z,w)$, i.e., Γ is a set of pairs of elements of $F_{\mathsf{K}}(4)$ such that the K-congruence generated by it is precisely $\theta_{\mathsf{K}}(x,y) \wedge \theta_{\mathsf{K}}(z,w)$. By (ii), the set of compact elements of $\mathrm{Con}_{\mathsf{K}}\, F_{\mathsf{K}}(4)$ forms a sublattice, consequently the congruence $\theta_{\mathsf{K}}(x,y) \wedge \theta_{\mathsf{K}}(z,w)$ is compact and so we can take Γ to be finite.

Let $\mathbf{A} \in \mathsf{K}$ and $a,b,c,d \in A$. Consider the homomorphism h from $F_{\mathsf{K}}(4)$ to the subalgebra generated by $\{a,b,c,d\}$ such that $h(x) = a$, $h(y) = b$, $h(z) = c$, $h(w) = d$. As $\mathrm{Con}_{\mathsf{K}}\, F_{\mathsf{K}}(4)$ is distributive by (ii), applying Lemma 2.21 we deduce that $h[\theta_{\mathsf{K}}(x,y) \wedge \theta_{\mathsf{K}}(z,w)] = h[\theta_{\mathsf{K}}(x,y)] \wedge h[\theta_{\mathsf{K}}(z,w)]$ and this congruence is generated by the set $\{\langle h(p_i(x,y,z,w)), h(q_i(x,y,z,w))\rangle : i < n\}$.

Whence,

$$\begin{aligned}
\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) &= h[\theta_{\mathsf{K}}(x,y)] \wedge h[\theta_{\mathsf{K}}(z,w)]\\
&= h[\theta_{\mathsf{K}}(x,y) \wedge \theta_{\mathsf{K}}(z,w)]\\
&= \textstyle\bigvee_{i<n} \theta_{\mathsf{K}}(p_i^{\mathbf{A}}(a,b,c,d), q_i^{\mathbf{A}}(a,b,c,d)).
\end{aligned}$$

Therefore, K has EDPRM. Thus, the conditions (i) and (ii) are equivalent.

(i) $\Rightarrow$ (iii). By Proposition 2.18.

(iii) $\Rightarrow$ (iv). Assume (iii). Let x, y, z, w be the free generators of $F_{\mathsf{K}}(4)$ and let $\Gamma = \{\langle p_\alpha, q_\alpha \rangle : \alpha < \beta\}$ be a set of generators of the congruence $\theta_{\mathsf{K}}(x,y) \wedge \theta_{\mathsf{K}}(z,w)$.

We claim

$$\mathsf{K}_{\mathrm{RFSI}} \vDash \forall xyzw[(\textstyle\bigwedge_{\alpha<\beta} p_\alpha(x,y,z,w) = q_\alpha(x,y,z,w)) \rightarrow (x = y \text{ or } z = w)].$$

Indeed, let $\mathbf{A} \in \mathsf{K}_{\mathrm{RFSI}}$ and let $h : F_{\mathsf{K}}(4) \rightarrow \mathbf{A}$ be a homomorphism such that $h(p_\alpha(x,y,z,w)) = h(q_\alpha(x,y,z,w))$ for all $\alpha < \beta$. Thus, $\langle p_\alpha, q_\alpha \rangle \in \ker h$ for every $\alpha < \beta$ and hence $\theta_{\mathsf{K}}(x,y) \wedge \theta_{\mathsf{K}}(z,w) \leq \ker h$. But $\ker h$ is finitely meet irreducible in $\mathrm{Con}_{\mathsf{K}} F_{\mathsf{K}}(4)$ because $F_{\mathsf{K}}(4)/\ker h$ is embeddable into $\mathbf{A}$ and, since $\mathsf{K}_{\mathrm{RFSI}}$ forms a universal class, $\mathbf{S}(\mathsf{K}_{\mathrm{RFSI}}) \subseteq \mathsf{K}_{\mathrm{RFSI}}$. Consequently, as $\mathrm{Con}_{\mathsf{K}} F_{\mathsf{K}}(4)$ is distributive by (iii) and using Lemma 2.17(ii), we get $\theta_{\mathsf{K}}(x,y) \leq \ker h$ or $\theta_{\mathsf{K}}(z,w) \leq \ker h$. Therefore, $h(x) = h(y)$ or $h(z) = h(w)$, proving the claim.

By the claim and the fact that $\mathsf{K}_{\mathrm{RFSI}}$ is closed under ultraproducts (which implies that $\vDash_{\mathsf{K}}$ is finitary), we conclude the existence of a finite subset of Γ, say $\{\langle p_i, q_i \rangle : i < n\}$ such that

$$\mathsf{K}_{\mathrm{RFSI}} \vDash \forall xyzw[(\textstyle\bigwedge_{i<n} p_i(x,y,z,w) = q_i(x,y,z,w)) \rightarrow (x = y \text{ or } z = w)].$$

On the other hand, we also have that $\mathsf{K}_{\mathrm{RFSI}} \vDash \forall xyzw[(x = y \text{ or } z = w) \rightarrow p_\alpha(x,y,z,w) = q_\alpha(x,y,z,w)]$ for all $i < n$. Indeed, let $\mathbf{A} \in \mathsf{K}_{\mathrm{RFSI}}$ and let $h : F_{\mathsf{K}}(4) \rightarrow \mathbf{A}$ be a homomorphism such that either $h(x) = h(y)$ or $h(z) = h(w)$. It follows that $\theta_{\mathsf{K}}(x,y) \leq \ker h$ or $\theta_{\mathsf{K}}(z,w) \leq \ker h$. Thus, $\theta_{\mathsf{K}}(x,y) \wedge \theta_{\mathsf{K}}(z,w) \leq \ker h$. Whence, $\langle p_\alpha, q_\alpha \rangle \in \ker h$ for all $i < n$. Therefore, $h(p_\alpha(x,y,z,w)) = h(q_\alpha(x,y,z,w))$ for each $i < n$.

Thus, we can conclude

$$\mathsf{K}_{\mathrm{RFSI}} \vDash \forall xyzw[(\bigwedge_{i<n} p_i(x,y,z,w) = q_i(x,y,z,w)) \leftrightarrow (x = y \text{ or } z = w)]. \tag{2.12}$$

Now, let $a, b, c, d \in F_{\mathsf{K}}(4)$. We show that

$$\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) = \textstyle\bigvee_{i<n} \theta_{\mathsf{K}}(p_i(a,b,c,d), q_i(a,b,c,d)),$$

which by distributivity of $\mathrm{Con}_{\mathsf{K}} F_{\mathsf{K}}(4)$ would imply that the set of compact elements of $\mathrm{Con}_{\mathsf{K}} F_{\mathsf{K}}(4)$ forms a sublattice, since we would have that this set is closed under arbitrary meets and joins.

Recall that if L is an algebraic lattice and $a, b \in L$, then $a \leq b$ if and only if for every finitely meet irreducible element m of L such that $b \leq m$, we have $a \leq m$.

So, let $\psi \in \mathrm{Con}_{\mathsf{K}} F_{\mathsf{K}}(4)$ be a finitely meet irreducible element and suppose $\bigvee_{i<n} \theta_{\mathsf{K}}(p_i(a,b,c,d), q_i(a,b,c,d)) \leq \psi$ for all $a, b, c, d \in F_{\mathsf{K}}(4)$. Then,

$$p_i(a/\psi, b/\psi, c/\psi, d/\psi) = q_i(a/\psi, b/\psi, c/\psi, d/\psi)$$

for all $i < n$. Hence, by $F_{\mathsf{K}}(4)/\psi \in \mathsf{K}_{\mathrm{RFSI}}$ and (2.12), we get $\theta_{\mathsf{K}}(a,b) \leq \psi$ or $\theta_{\mathsf{K}}(c,d) \leq \psi$. Thus, $\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \leq \psi$. Therefore, $\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \leq \bigvee_{i<n} \theta_{\mathsf{K}}(p_i(a,b,c,d), q_i(a,b,c,d))$.

For the converse direction we proceed by contraposition. Let $\langle e, f\rangle \notin \theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d)$. We can assume $\langle e, f\rangle \notin \theta_{\mathsf{K}}(a,b)$ (when $\langle e, f\rangle \notin \theta_{\mathsf{K}}(c,d)$ we proceed similarly). Thus, there exists a finitely meet irreducible element ψ of $\mathrm{Con}_{\mathsf{K}}\, F_{\mathsf{K}}(4)$ with $\langle e, f\rangle \notin \psi$ and $\theta_{\mathsf{K}}(a,b) \leq \psi$. As $a/\psi = b/\psi$ in $F_{\mathsf{K}}(4)/\psi$ and $F_{\mathsf{K}}(4)/\psi \in \mathsf{K}_{\mathrm{RFSI}}$, by (2.12) we obtain

$$p_i(a/\psi, b/\psi, c/\psi, d/\psi) = q_i(a/\psi, b/\psi, c/\psi, d/\psi)$$

for all $i < n$. Thus, $p_i(a,b,c,d)/\psi = q_i(a,b,c,d)/\psi$ for every $i < n$. That is, $\langle p_i(a,b,c,d), q_i(a,b,c,d)\rangle \in \psi$ for all $i < n$.

Hence, $\bigvee_{i<n} \theta_{\mathsf{K}}(p_i(a,b,c,d), q_i(a,b,c,d)) \leq \psi$.

Consequently, $\langle e, f\rangle \notin \bigvee_{i<n} \theta_{\mathsf{K}}(p_i(a,b,c,d), q_i(a,b,c,d))$. Whence, $\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) \geq \bigvee_{i<n} \theta_{\mathsf{K}}(p_i(a,b,c,d), q_i(a,b,c,d))$.

Therefore, the compact elements of $\mathrm{Con}_{\mathsf{K}}\, F_{\mathsf{K}}(4)$ form a sublattice.

(iv) $\Rightarrow$ (v). Assuming (iv) and proceeding as in the part (iii) implies (iv) (just taking $\Delta = \Gamma$), we get a set of formulas Δ satisfying (2.12). This set Δ can be chosen finite, because is a set of generators of the compact congruence $\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d)$ as we have seen.

(v) $\Rightarrow$ (i). Assume (v) and next proceeding as in the proof of (iii) implies (iv) to get that for all $\mathbf{A} \in \mathsf{K}$ and every $a,b,c,d \in A$, $\theta_{\mathsf{K}}(a,b) \wedge \theta_{\mathsf{K}}(c,d) = \bigvee_{i<n} \theta_{\mathsf{K}}(p_i(a,b,c,d), q_i(a,b,c,d))$. It follows that K has EDPRM. ■

Proposition 2.23 *Let* K *be a quasivariety with* EDPRC. *Then, there exists a universal formula* Ψ *such that for every* $\mathbf{A} \in \mathsf{K}$,

$$\mathbf{A} \textit{ is } \mathsf{K}\textit{-simple} \quad \textit{iff} \quad \mathbf{A} \vDash \Psi$$

Proof. Let $\mathbf{A} \in \mathsf{K}$. Since K has EDPRC, there is a finite set of formulas $\Phi \subseteq \mathrm{Fm}(4) \times \mathrm{Fm}(4)$ such that for all $a,b,c,d \in \mathbf{A}$, we have

$$\langle c,d\rangle \in \theta_{\mathsf{K}}(a,b) \quad \text{iff} \quad (\varphi^{\mathbf{A}}(a,b,c,d) = \eta^{\mathbf{A}}(a,b,c,d) \quad \text{for every } \langle \varphi, \eta\rangle \in \Phi).$$

We claim:

$\mathbf{A}$ is K-simple iff $\mathbf{A} \vDash \forall xyzw(\neg(x = y) \to \bigwedge_{\langle\varphi,\eta\rangle\in\Phi} \varphi(x,y,z,w) = \eta(x,y,z,w))$.

Indeed, assume $\mathbf{A}$ is K-simple. Let $a,b,c,d \in A$ and a suppose $a \neq b$. Thus, $\langle a,b\rangle \notin \mathrm{id}_{\mathbf{A}}$. That is, $\theta_{\mathsf{K}}(a,b) \neq \mathrm{id}_{\mathbf{A}}$. It follows that $\theta_{\mathsf{K}}(a,b) = A \times A$, because $\mathbf{A}$ is K-simple. So, $\langle c,d\rangle \in \theta_{\mathsf{K}}(a,b)$. Hence, $\varphi^{\mathbf{A}}(a,b,c,d) = \eta^{\mathbf{A}}(a,b,c,d)$ for all $\langle \varphi, \eta\rangle \in \Phi$.

Conversely, assume that for all $a,b,c,d \in A$, $\langle c,d\rangle \in \theta_{\mathsf{K}}(a,b)$ provided that $a \neq b$. Let $\phi \in \mathrm{Con}_{\mathsf{K}}\, \mathbf{A}$ and suppose $\phi \neq \mathrm{id}_{\mathbf{A}}$. Thus, there exists $\langle a,b\rangle \in \phi$ such that $a \neq b$. Consequently, by assumption, $\langle c,d\rangle \in \theta_{\mathsf{K}}(a,b)$ for all $c,d \in A$. Hence, $\theta_{\mathsf{K}}(a,b) = A \times A$. That is, $\phi = A \times A$. Therefore, $\mathbf{A}$ is K-simple. ■

An important consequence of the above results is the following fact.

Corollary 2.24 *If an algebra* $\mathbf{A}$ *belongs to a relatively semisimple quasivariety* K *with* EDPRC, *then the compact* K*-congruences of* $\mathbf{A}$ *form a sublattice of the* K*-congruence lattice of* $\mathbf{A}$.

Proof. Since K has EDPRC, it is relatively congruence distributive and the class of K-simple algebras in K is closed under nontrivial subalgebras and nontrivial ultraproducts as a direct consequence of Proposition 2.23, because this result shows that this class is in fact a universal class. On the other hand, by Lemma 2.20, whenever Q is the smallest quasivariety containing a class M of similar algebras, every algebra in $\mathsf{Q}_{\mathrm{RFSI}}$ can be embedded into an ultraproduct of members of M. So, since K is relatively semisimple, the nontrivial algebras in $\mathsf{K}_{\mathrm{RFSI}}$ are K-simple and therefore $\mathsf{K}_{\mathrm{RFSI}}$ is closed under subalgebras and ultraproducts. Hence, it is a universal class. Thus, by Theorem 2.22, the compact K-congruences of $\mathbf{A}$ form a sublattice of the K-congruence lattice of $\mathbf{A}$. ■

A *dual generalized Boolean lattice* is a distributive lattice $\langle L, \cdot, + \rangle$ with a least element 0 such that for any $a, b \in L$, there exists $c \in L$ with $a \cdot c = 0$ and $a + c = a + b$. Note that, in this case, $\langle L, \cdot, + \rangle$ is a Boolean lattice if and only if it has a greatest element.

Blok and Pigozzi proved that the join semilattice of compact congruences of an algebra $\mathbf{A}$ in a filtral variety M is always a dual generalized Boolean lattice [4, Cor.4.3]. This result remains true for the K-congruences of $\mathbf{A}$ when K is a relatively semisimple quasivariety with EDPRC.

Lemma 2.25 *Let* K *be a relatively semisimple quasivariety with* EDPRC *such that, for every* $\mathbf{A} \in \mathsf{K}$*, the total congruence* $A \times A$ *is compact in the lattice* $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$*. Then, the compact* K*-congruences of any algebra in* K *form a Boolean lattice with respect to* $\subseteq$.

Proof. Let $\mathbf{A} \in \mathsf{K}$. By Corollary 2.24, the compact K-congruences of $\mathbf{A}$ form a sublattice of the K-congruence lattice of $\mathbf{A}$. If we show that this lattice of compact K-congruences of $\mathbf{A}$ is a dual generalized Boolean lattice, as it has a greatest element, we would have that it is in fact a Boolean lattice. Let us see how to show that.

Since K has EDPRC, there is a finite set of formulas $\Phi \subseteq \mathrm{Fm}(4) \times \mathrm{Fm}(4)$ such that whenever $a, b, c, d \in \mathbf{A}$,

$$\langle c, d\rangle \in \theta_{\mathsf{K}}(a, b) \quad \text{iff} \quad (\varphi^{\mathbf{A}}(a, b, c, d) = \eta^{\mathbf{A}}(a, b, c, d) \quad \text{for all } \langle \varphi, \eta\rangle \in \Phi).$$

Observe that:

The lattice of compact K-congruences of $\mathbf{A}$ is a dual generalized Boolean lattice

if and only if

for all $a, b, c, d \in A$ there are $g, h \in A$ such that $\theta_{\mathsf{K}}(a, b) \wedge \theta_{\mathsf{K}}(g, h) = \mathrm{id}_{\mathbf{A}}$ and $\theta_{\mathsf{K}}(a, b) +^{\mathsf{K}} \theta_{\mathsf{K}}(g, h) = \theta_{\mathsf{K}}(a, b) +^{\mathsf{K}} \theta_{\mathsf{K}}(c, d)$,

which is equivalent to:

For every $a, b, c, d, e, f \in A$, there exists $g, h \in A$ such that

$$(\varphi^{\mathbf{A}}(a, b, e, f) = \eta^{\mathbf{A}}(a, b, e, f) \text{ and } \varphi^{\mathbf{A}}(g, h, e, f) = \eta^{\mathbf{A}}(g, h, e, f) \text{ for all } \langle \varphi, \eta\rangle \in \Phi$$
$$\Leftrightarrow \quad e = f)$$

and

$$(\varphi^{\mathbf{A}}(a, b, e, f) = \eta^{\mathbf{A}}(a, b, e, f) \text{ or } \varphi^{\mathbf{A}}(g, h, e, f) = \eta^{\mathbf{A}}(g, h, e, f)) \text{ for all } \langle \varphi, \eta\rangle \in \Phi$$

$$\Leftrightarrow$$

$\varphi^{\mathbf{A}}(a,b,e,f) = \eta^{\mathbf{A}}(a,b,e,f)$ or $\varphi^{\mathbf{A}}(c,d,e,f) = \eta^{\mathbf{A}}(c,d,e,f)$ for all $\langle \varphi, \eta \rangle \in \Phi$).

But these latter equivalences are valid in any relatively simple member $\mathbf{B}$ of K, because these members have just two compact K-congruences, namely, $\mathrm{id}_{\mathbf{B}}$ and $B \times B$. Thus, all the K-simple members of K have a dual generalized Boolean lattice of compact K-congruences.

Consequently, since every algebra $\mathbf{A} \in \mathsf{K}$ is isomorphic to a subdirect product of K-simple algebras in K (because K is relatively semisimple), we conclude that the compact K-congruences of any algebra in K form a Boolean lattice with respect to $\subseteq$. ■

Now we can prove the following important characterization.

Theorem 2.26 *Let K be a quasivariety that algebraizes a deductive system $\vdash$. Then, the following conditions are equivalent*:

(i) $\vdash$ *has a classical inconsistency lemma.*

(ii) $\vdash$ *has a greatest compact theory and* K *is relatively semisimple with* EDPRC.

(iii) *For every $\mathbf{A} \in \mathsf{K}$, the compact K-congruences of $\mathbf{A}$ form a Boolean sublattice of the lattice of all K-congruences of $\mathbf{A}$.*

Proof. Again, we may assume without loss of generality that $\vdash$ has a greatest compact theory. Consequently, for every $\mathbf{A} \in \mathsf{K}$, the set A is a compact $\vdash$-filter of $\mathbf{A}$, whence $A \times A$ is a compact K-congruence of $\mathbf{A}$, because K algebraizes $\vdash$.

(i) $\Rightarrow$ (ii). By (i) and Theorem 2.14, the join semilattices of compact K-congruences of all algebras in K are Boolean lattices. Let $\mathbf{A} \in \mathsf{K}$ and suppose it is K-subdirectly irreducible. We show that $\mathbf{A}$ is K-simple, that is, $\mathrm{id}_{\mathbf{A}}$ is a co-atom in the lattice $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$.

As $\mathbf{A}$ is K-subdirectly irreducible, $\mathrm{id}_{\mathbf{A}}$ is completely meet irreducible and therefore it has exactly one cover in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$, say μ. This cover is also compact because if $\mu \leq \bigvee X$ for some $X \subseteq \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$, then $\mathrm{id}_{\mathbf{A}} < \mu \leq \bigvee X$. Hence, as $\mathrm{id}_{\mathbf{A}}$ is compact, there must be some $Y \subseteq X$ finite such that $\mathrm{id}_{\mathbf{A}} \leq \bigvee Y$. Thus, since μ is the unique cover of $\mathrm{id}_{\mathbf{A}}$, $\mu \leq \bigvee Y$. So, μ is compact. Consequently, μ is an element of the Boolean lattice of compact K-congruences of $\mathbf{A}$. It follows that there is a compact K-congruence ϕ such that $\mu \wedge \phi = \mathrm{id}_{\mathbf{A}}$ and $\mu +^{\mathsf{K}} \phi = A \times A$. Whence, this congruence ϕ must be $\mathrm{id}_{\mathbf{A}}$ and so $\mu = A \times A$. This means that the unique cover in $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ of $\mathrm{id}_{\mathbf{A}}$ is $A \times A$. Therefore, $\mathrm{Con}_{\mathsf{K}}\,\mathbf{A} = \{\mathrm{id}_{\mathbf{A}}, A \times A\}$. Hence, $\mathbf{A}$ is K-simple.

On the other hand, by Corollary 2.15, $\vdash$ has a DDT. Thereby, K has EDPRC, because EDPRC is the algebraic counterpart of the DDT.

(ii) $\Rightarrow$ (iii) follows from Lemma 2.25 and Corollary 2.24.

(iii) $\Rightarrow$ (i). As for every algebra $\mathbf{A}$, the lattice of K-congruences of $\mathbf{A}$ is isomorphic to that of $\mathbf{A}/\theta_{\mathsf{K}}(\emptyset)$ and $\mathbf{A}/\theta_{\mathsf{K}}(\emptyset) \in \mathsf{K}$, hence, by (iii) and since K algebraizes $\vdash$, for every algebra $\mathbf{A}$ the compact $\vdash$-filters of $\mathbf{A}$ form a Boolean sublattice of the lattice of all $\vdash$-filters of $\mathbf{A}$. Therefore, by Theorem 2.14, $\vdash$ has a classical inconsistency lemma. ■

Some immediate consequences are the following.

Corollary 2.27 *Let $\vdash$ be a strongly algebraizable deductive system with a greatest compact theory. Then, $\vdash$ has a classical inconsistency lemma iff it is algebraized by a filtral variety.*

Proof. Let K the variety that algebraizes $\vdash$. We know, as we already mentioned, that the filtral varieties are exactly the semisimple varieties with EDPC. So, as $\vdash$ has a greatest compact theory, it follows, by Theorem 2.26, that $\vdash$ has a classical inconsistency lemma iff K is semisimple with EDPC, which is equivalent to be a filtral variety. ■

This latter result is also valid for every algebraizable logic, because the filtral quasivarieties are just the relatively semisimple quasivarieties with EDPRC. For the proof of this fact see [9].

Corollary 2.28 *Let $\vdash$ be a deductive system that is algebraized by some quasivariety K, where $\vdash$ has a classical inconsistency lemma. Then, $\vdash$ is finitely axiomatized iff the class of K-simple algebras in K is strictly elementary.*

Proof. By Theorem 2.26, K is relatively semisimple with EDPRC. So, by Corollary 2.24, K is relatively congruence distributive and the compact K-congruences of algebras in K are closed under finite intersections. A quasivariety Q with these two properties is finitely axiomatized iff $\mathsf{Q}_{\mathrm{FSI}}$ is strictly elementary [11, Theo. 3.4]. But, in our case, again by Corollary 2.24 and its proof, $\mathsf{K}_{\mathrm{FSI}}$ is the class of K-simple (or trivial) algebras in K. Therefore, since K algebraizes $\vdash$, this system $\vdash$ is finitely axiomatized iff the class of K-simple algebras in K is strictly elementary. ■

Corollary 2.29 *If two categorically equivalent varieties K and M algebraize deductive systems $\vdash$ and $\vdash^{'}$ respectively, then $\vdash$ has a (classical) inconsistency lemma iff $\vdash^{'}$ does.*

Proof. The equivalence functor between K and M preserves the isomorphism type of the congruence lattice of any algebra, and a lattice isomorphism between complete lattices give us an isomorphism between their semilattices of compact elements, so the result follows from Theorems 2.11 and 2.26. ■

2.4 Examples

Let us analyze briefly some notable deductive systems and their corresponding algebraic counterpart. We will see that some results we have presented cannot be characterizations of classical inconsistency lemmas, since their converse formulation is not always true.

Example 2.30 The filtral variety BA of all Boolean algebras is the only nontrivial semisimple variety of Heyting algebras. Hence, using Theorem 2.26, we get that no consistent axiomatic extension of intuitionistic propositional logic

(**IPL**) has a classical inconsistency lemma, except for classical propositional logic (**CPL**). Similarly, **CPL** is the only axiomatic consistent extension of the $\{\to, \bot\}$-fragment of **IPL** having a classical inconsistency lemma. Observe also that the converse of Corollaries 2.15 and 2.16 does not hold because we have that **IPL** has a DDT and the congruence lattices of Heyting algebras are distributive and pseudo-complemented but **IPL** does not have a classical inconsistency lemma.

Example 2.31 The *substructural logics* are logics associated with varieties of residuated lattices. We consider here the substructural logic $\mathbf{FL_{ew}}$ associated with the variety of commutative residuated lattices that are integral and zero bounded that we proceed to present.

A *commutative residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \odot, \to, \bot, \top\rangle$ such that:

(i) $\langle A, \wedge, \vee\rangle$ is a lattice.

(ii) $\langle A, \odot, \top\rangle$ is a commutative monoid.

(iii) $a \odot c \leq b$ iff $c \leq a \to b$ for all $a, b, c \in A$.

(iv) $\bot$ is an arbitrary element of A.

A commutative residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \odot, \to, \bot, \top\rangle$ is *integral* if $\top$ is the maximum of the lattice order, and it is *zero bounded* if $\bot \leq x$ for all $x \in A$.

The language of the logic $\mathbf{FL_{ew}}$ is $L = \{\to, \wedge, \vee, \odot, \top, \bot\}$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$. A Hilbert style axiomatization of $\mathbf{FL_{ew}}$ is given by the following axiom system:

(A1) $\alpha \to \alpha$
(A2) $(\alpha \to \beta) \to [(\delta \to \alpha) \to (\delta \to \beta)]$
(A3) $[\alpha \to (\beta \to \delta)] \to [\beta \to (\alpha \to \delta)]$
(A4) $[(\alpha \wedge \top) \odot (\beta \wedge \top)] \to (\alpha \wedge \beta)$
(A5) $(\alpha \wedge \beta) \to \alpha$
(A6) $(\alpha \wedge \beta) \to \beta$
(A7) $[(\alpha \to \beta) \wedge (\alpha \to \delta)] \to [\alpha \to (\beta \wedge \delta)]$
(A8) $\alpha \to (\alpha \vee \beta)$
(A9) $\beta \to (\alpha \vee \beta)$
(A10) $[(\alpha \to \delta) \wedge (\beta \to \delta)] \to [(\alpha \vee \beta) \to \delta]$
(A11) $\beta \to (\alpha \to (\alpha \odot \beta))$
(A12) $[\beta \to (\alpha \to \delta)] \to [(\alpha \odot \beta) \to \delta]$
(A13) $\top$
(A14) $\top \to (\alpha \to \alpha)$
(A15) $\bot \to \alpha$
(A16) $\alpha \to (\beta \to \alpha)$
(A17) $(\alpha \odot \beta) \to (\beta \odot \alpha)$,

whose rules of inference are:

(MP) $\alpha, \alpha \to \beta / \beta$ (Modus Ponens)
(Ad) $\alpha / \alpha \wedge \top$ (Adjunction unit)

This logic $\mathbf{FL_{ew}}$ is algebraized by the variety of commutative residuated lattices that are integral and zero bounded and it has a greatest compact theory generated by $\{\bot\}$. When a variety K algebraizes an axiomatic extension of $\mathbf{FL_{ew}}$, then K is filtral if and only if the theorems of the extension include $p \vee (p^k \to \bot)$ for some positive integer k [20, Chap.11]. By Theorem 2.26, these extensions are exactly the axiomatic extensions of $\mathbf{FL_{ew}}$ with a classical inconsistency lemma. The respective classical IL-sequences are

$$\{\{(v_1 \odot ... \odot v_n)^k \to \bot\} : n \in \mathbb{N}\},$$

where $\odot$ denotes fusion.

Example 2.32 The **BCK** logic is presented axiomatically in the language $\{\to\}$ of type $\langle 2 \rangle$ by the following formulas:

(B) $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$,
(C) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \psi))$,
(K) $\varphi \to (\psi \to \varphi)$.

These axioms together with *modus ponens* constitute the logic **BCK**.

This logic is algebraizable and its equivalent quasivariety semantics is precisely the quasivariety BCK of *BCK-algebras*, that is, algebras $\langle A, \to, 1 \rangle$ satisfying the following three equations and a quasiequation:

(1) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(2) $1 \to x = x$,
(3) $x \to 1 = 1$,
(4) $(x \to y = 1$ and $y \to x = 1) \Rightarrow (x = y)$.

The logic **BCK** is an example of a deductive system whose equivalent algebraic semantics is not a variety.

A *relative subvariety* of a quasivariety M is the intersection of M with a variety of the same type. The relative subvarieties K of the quasivariety BCK algebraize the axiomatic extensions of the $\{\to, \bot\}$-fragment $\mathbf{BCK}_\bot$ of $\mathbf{FL_{ew}}$.

An axiomatic extension of $\mathbf{BCK}_\bot$ has a classical inconsistency lemma iff its theorems include

$$(p \to^{k+1} q) \to (p \to^k q) \quad \text{and} \quad (p \to q) \to (((p \to^m \bot) \to q) \to q)$$

for some $k, m \in \mathbb{N}$. Here, $p \to^0 q := q$ and $p \to^{k+1} q := p \to (p \to^k q)$.

Among these extensions of $\mathbf{BCK}_\bot$ we have the *Łukasiewicz k-valued logic* ($k \in \mathbb{N}$). The classical IL-sequence for this system is

$$\{\{\neg((v_1 \odot ... \odot v_n)^k)\} : n \in \mathbb{N}\},$$

where $p \odot q$ and $\neg(p \to \neg q)$ are interchangeable.

Example 2.33 Consider the following operations defined on the real unit interval $[0, 1]$:

$a \wedge b := \min\{a, b\}$,
$a \vee b := \max\{a, b\}$,
$a \to b := \min\{1, 1 - a + b\}$,
$\neg a := 1 - a$.

Let **[0,1]** be the algebra $\langle [0, 1], \wedge, \vee, \neg, \to \rangle$ defined on $[0, 1]$ by these operations.

The *infinite-valued Łukasiewicz logic* $\mathbf{Ł}_{\infty f}$ is the finitary companion of the logic $\mathbf{Ł}_{\infty}$ defined semantically in the language $\langle \wedge, \vee, \to, \neg \rangle$ of type $\langle 2, 2, 2, 1 \rangle$, from the matrix $\langle \mathbf{[0,1]}, \{1\} \rangle$. The logic $\mathbf{Ł}_{\infty f}$ has a deduction-like theorem, the so called *local deduction theorem*, that is, $\Sigma, \vdash \psi$ iff $\Sigma \vdash \varphi^n \to \psi$ for some natural number $n > 1$.

An *MV-algebra* is an algebra $\langle A, \neg, \oplus, 0 \rangle$ such that:

(1) $\langle A, \oplus, 0 \rangle$ is a commutative monoid,

(2) $\neg\neg x = x$,

(3) $x \oplus \neg 0 = \neg 0$,

(4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

The logic $\mathbf{Ł}_{\infty f}$ is algebraizable, its equivalent algebraic semantics is the class MV of MV-algebras and it can be defined as the variety generated by the algebra $\mathbf{[0,1]}$. This variety lacks EDPC. Therefore, by Theorem 2.26, the logic $\mathbf{Ł}_{\infty f}$ has no classical inconsistency lemma. In fact, this logic has no inconsistency lemma of the ordinary kind. Indeed, if we had an IL-sequence $\{\Psi_n : n \in \mathbb{N}\}$ for this system, then it would follow, from the inconsistency of $\Psi_1(p) \cup \{p\}$ and the local deduction theorem, that $\Psi_1(p) \vdash \neg(p^n)$ for some $n \in \mathbb{N}$ [1]. But we know that $mp \wedge \neg p$ is inconsistent for each $m \in \mathbb{N}$. Hence, we would have that $\vdash \Psi_1(mp \wedge \neg p)$ and therefore $\vdash \neg((mp \wedge \neg p)^n)$. However, for each n we can choose an m and a value of p in the real interval $(0, 1)$ for which $\neg((mp \wedge \neg p)^n)$ does not take the value 1 in $\mathbf{[0,1]}$. Consequently, $\mathbf{Ł}_{\infty f}$ cannot have an inconsistency lemma.

Example 2.34 A *t-norm* is a binary operation $*$ on the real interval $[0, 1]$ satisfying the following conditions:

(i) $*$ is associative and commutative.

(ii) $*$ is non-decreasing in both arguments, i.e.,

$$x_1 \leq x_2 \text{ implies } x_1 * y \leq x_2 * y$$
$$y_1 \leq y_2 \text{ implies } x * y_1 \leq x * y_2.$$

(iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0, 1]$.

$*$ is a *continuous t-norm* if it is a t-norm and is a continuous mapping of $[0, 1]^2$ into $[0, 1]$.

Let $*$ be a continuous t-norm. Then there is a unique operation $x \Rightarrow y$ satisfying, for all $x, y, z \in [0, 1]$, the condition $(x * z) \leq y$ iff $z \leq x \Rightarrow y$, namely, $x \Rightarrow y := \max\{z : x * z \leq y\}$. This operation is called the *residuum* of the t-norm $*$. The residuum $\Rightarrow$ defines its corresponding unary operation of *precomplement* $\neg x := x \Rightarrow 0$, for all $x \in [0, 1]$.

An important example of continuous t-norm is the *Product t-norm*: $x * y := x \cdot y$ for all $x, y \in [0, 1]$ (here $\cdot$ is the usual product of reals) and its residuum is the so called *Goguen implication* $\Rightarrow$ defined as: $x \Rightarrow y = 1$ if $x \leq y$, and $x \Rightarrow y = x/y$ otherwise (here / is the usual division), which corresponding precomplement is the *Gödel negation* given by $\neg 0 = 1$ and $\neg x = 0$ for all $x > 0$.

[1] We define $p^1 := p, \quad p^{k+1} := p^k \odot p$ and $1p := p, \quad (k+1)p := (kp) \oplus p$.

Given a continuous t-norm $*$ we define the algebra $\mathbf{[0{,}1]}_* = \langle [0,1], *, \Rightarrow, \neg, \wedge, \vee \rangle$ where $\Rightarrow$ is the residuum of $*$, $\neg$ is the precomplement of $\Rightarrow$, and taking $\wedge$ and $\vee$ as the minimum and maximum in the same way as in Example 2.33, respectively. Hence, the *fuzzy logic* associated to $*$ is the finitary companion of the logic obtained semantically from the matrix $\langle \mathbf{[0{,}1]}_*, \{1\} \rangle$.

If we take the t-norm $*$ as the Product t-norm $\cdot$ defined above, we obtain the *product logic* Π.

A residuated lattice $\langle A, \wedge, \vee, \Rightarrow, *, \bot, \top \rangle$ is a *BL-algebra* if the following two identities hold for all $x, y \in A$:

(1) $x \wedge y = x * (x \Rightarrow y)$,
(2) $(x \Rightarrow y) \vee (y \Rightarrow x) = \top$.

The class of BL-algebras is a variety of algebras.

We define a Π-*algebra* (or *product algebra*) to be a BL-algebra satisfying, for all x, y, z in its universe, the following:

(1) $\neg\neg z \leq ((x * z \Rightarrow y * z) \Rightarrow (x \Rightarrow y))$,
(2) $x \wedge \neg x = 0$.

Also, the class of all product algebras is a variety and it is the equivalent algebraic semantics of the algebraizable product logic Π. This variety has no EDPC. Consequently, the product logic Π is algebraized by a variety without EDPC but it does have an inconsistency lemma of the ordinary kind, with

$$\{\{\neg(v_1 \odot ... \odot v_n)\} : n \in \mathbb{N}\}$$

as IL-sequence, where $\odot$ will be interpreted as the product norm.

This shows that a strongly algebraizable system with an inconsistency lemma (and hence a greatest compact theory) need not have a DDT. Therefore, the converse of Corollary 2.10 does not hold.

Example 2.35 The weakening axiom, namely $p \to (q \to p)$, is a theorem of $\mathbf{FL_{ew}}$. Among the substructural logics without this axiom are the uninorm-based fuzzy logic **IUML** and its consistent extensions. These systems have connectives $\to, \wedge, \neg$ and constants $\mathbf{t}$, $\bot$ such that $\neg\neg p \leftrightarrow p$ and $(p \to \neg q) \to (q \to \neg p)$ are theorems.

All consistent extensions of **IUML** have a DDT and a greatest compact theory generated by $\{\bot\}$. The form of the DDT is $\Gamma \cup \{\alpha\} \vdash \beta$ iff $\Gamma \vdash (\alpha \wedge \mathbf{t}) \to \beta$.

IUML is algebraized by the variety $\mathsf{OSM}_\bot$ of *bounded odd Sugihara monoids*. We know that $\mathsf{OSM}_\bot$ is categorically equivalent to the variety of *Gödel algebras* (a subvariety of Heyting algebras). So, by Corollary 2.29, no consistent extension of **IUML** has a classical inconsistency lemma, except for the largest such extension $\mathbf{IUML}_3$, whose algebraic counterpart is categorically equivalent to BA.

The variety that algebraizes $\mathbf{IUML}_3$ is generated by the algebra $\mathbf{Z}_3^\bot$ on the idempotent commutative ordered monoid $-1 < 0 < 1$, where 0 is the identity and $-1 \odot 1 = -1$. In this algebra $\wedge$ is the minimum operation and $\neg$ is the additive inversion, $\mathbf{t} = 0$, $\bot = -1$, and $x \to y := \neg(x \odot \neg y)$. In $\mathbf{IUML}_3$ the formulas $\neg p$ and $p \to \bot$ are not logically equivalent.

In this system, $\{\{v_1 \to (... \to (v_n \to \bot))\} : n \in \mathbb{N}\}$ is a classical IL-sequence and also for its $\{\to, \bot\}$-fragment.

The $\{\to\}$ and $\{\to, \bot\}$ fragments of $\mathbf{IUML}_3$ have a common DDT, namely, $\Gamma \cup \{\alpha\} \vdash \beta$ iff $\Gamma \vdash (\alpha \to (\beta \to \beta)) \to (\alpha \to \beta)$. The $\{\to, \bot\}$-fragment is algebraized by the smallest quasivariety Q containing the $\{\to \bot\}$-reduct of $\mathbf{Z}_3^{\bot}$. Hence, by Theorem 2.26, Q is relatively semisimple with EDPRC. However, it is not a variety. So, an algebraizable system with a classical inconsistency lemma need not be strongly algebraizable.

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