Dynamic phase transition in the mean-field Ising model

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Abstract: We study the dynamic behavior of the Ising model under an oscillating magnetic field in the mean-field approximation. A dynamic phase transition causing a spontaneous symmetry breaking of the hysteresis loop arises due to the competition between the timescales associated with the spin relaxation and the oscillating field. A dynamic order parameter measuring the asymmetry of the hysteresis loop is defined and its critical behavior is explored. The critical exponents are found to match those of the equilibrium transition once the thermodynamic variables are replaced with certain dynamic variables.

I. INTRODUCTION

Consider a magnet, which is a collective many-body system, placed in an external oscillating magnetic field. The thermodynamic response of the system, i.e. the magnetization, will also oscillate, and it will lag behind the applied field due to relaxational delay. This delay in the dynamic response gives rise to a nonzero area of the magnetization-field loop, a phenomenon known as hysteresis. When the time period of the magnetic field becomes much shorter than the typical relaxation time of the thermodynamic system, the hysteresis loop becomes asymmetric around the origin. This signals the appearance of a new thermodynamic phase, which arises spontaneously from dynamically broken symmetries due to the competing timescales within the system.

This dynamic phase transition (DPT) can be characterized by a dynamic order parameter, which can be shown to obey a universal power-law behavior in the vicinity of the critical point. Remarkably, the critical exponents are the same as those for the magnetization in the equilibrium transition if the temperature and the static external field are substituted by the period of the oscillating field and a small bias field, respectively.

Beyond magnetic systems, many other physical systems exhibit dynamic hysteresis. For instance, very recently Geng *et al.* [1] investigated a laser-driven cavity with noninstantaneous photon-photon interactions arising from a nonlinear optical response within it. They showed that the transmitted intensity of the light traversing the cavity displays hysteresis when scanning its length, depending on whether it is being opened or closed. They also reported a power law with scaling exponent -1 for the hysteresis area as a function of the scanning period in the limit of fast scans. Furthermore, they were able to explain the experimental results with a mean-field equation for the steady-state photon density.

This report is organized as follows. In Sec. II, a mean-field Ising spin model displaying hysteresis is obtained from simple physical considerations. In Sec. III we give a brief overview of our numerical program. In Sec. IV we qualitatively describe the hysteretic behavior and the dynamic phase transition exhibited by the system under consideration. In Sec. V we analyze, both analytically and numerically, the critical behavior of the dynamic order parameter. In Sec. VI we present simulated phase diagrams for different choices of the relevant parameters. To conclude, we summarize our work and give prospects of possible future extensions.

II. A HEURISTIC MODEL

In this section we show how a dynamic equation for the magnetization in a spin system exhibiting hysteresis can be obtained by means of simple considerations [2]. Consider N Ising spins of moment μ in an external field h, where n_1 point in the direction of h and $n_2 = N - n_1$ in the opposite one. The time evolution can be described by a master equation,

$$\begin{cases} \dot{n}_1 = w_{21} n_2 - w_{12} n_1, \\ \dot{n}_2 = w_{12} n_1 - w_{21} n_2, \end{cases} \tag{1}$$

where w_{12} and w_{21} are the transition rates from states 1 to 2 and vice-versa, that must obey the principle of detailed balance, i.e. $w_{12}/w_{21} = \exp(-2\mu h/kT)$. The magnetization is $M = \mu(n_1 - n_2)$, so that eq. (1) gives

$$\dot{M} = -(w_{12} + w_{21})\, M + (w_{21} - w_{12})\, \mu N. \tag{2}$$

Defining a relaxation time $\tau \equiv (w_{12} + w_{21})^{-1}$ yields

$$\tau \dot{M} = -M + \mu N \tanh\left(\frac{\mu h}{kT}\right). \tag{3}$$

Now we define $m = M/N$ and introduce a mean field by letting $h \to h' = h + \lambda m$, since the field experienced by individual spins is a linear superposition of both the external field and the one due to the other dipoles. Then,

$$\tau \dot{m} = -m + \mu \tanh\left(\frac{\mu h + \mu \lambda m}{kT}\right), \tag{4}$$

which in the stationary case, $\dot{m} = 0$, reduces to the mean-field Ising model equation for the magnetization. However, interesting dynamics appear when the magnetic field is a function of time, $h = h(t)$. Setting $\mu = 1$ in analogy to the Ising model and introducing a reduced field h/λ and a reduced temperature kT/λ, one obtains

$$\tau \frac{\mathrm{d}m}{\mathrm{d}t} = -m + \tanh\left[\frac{h(t)+m}{T}\right]. \tag{5}$$

Eq. (5) is the main equation we will solve in this article when a sinusoidal magnetic field $h(t) = h_0 \sin \omega t$ is applied. In these conditions, the model exhibits some interesting physics depending on the specific values of the variables h_0, ω and T. In particular, hysteresis loops are readily found to appear, in principle, for any set of values of these parameters.

It is also possible to introduce a reduced time t/τ, in which case it is seen that the hysteresis depends only on three dimensionless parameters. These parameters correspond to the temperature, the applied field strength, and an effective frequency $\omega\tau$. Thus, without any loss of generality, we will consider $\tau = 1$ everywhere.

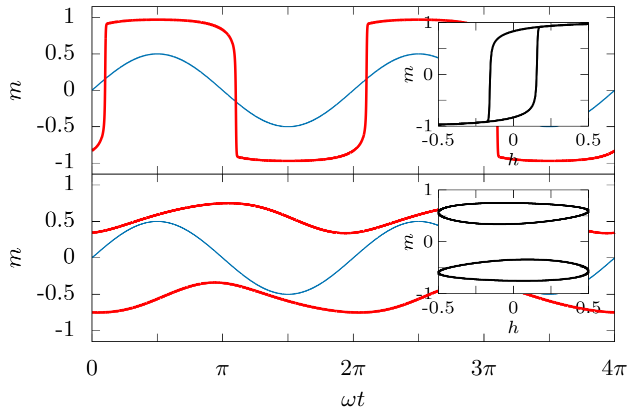

FIG. 1: Magnetization behavior during two cycles for $h_0 = 0.5$ and $T = 0.7$ and two different frequencies, (a) $\omega = 0.005$ and (b) $\omega = 2$. The blue lines show the external magnetic field. In (a) the magnetization can follow the oscillation of the field with a certain lag, and a symmetric hysteresis loop develops. In (b) the frequency is too high for the magnetization to keep up with the field, and the loop becomes asymmetric with two possible outcomes.

III. NUMERICAL METHOD

The model (5) has been solved with the 4th-order Runge-Kutta method after an initialization of $m = 1$. Time is measured in units of $\eta = \omega t$ and one period is discretized into $N = 2000$ timesteps (not to be confused with the number of spins). After each period of evolution i, the magnetization at each time site is compared with the value in the previous iteration $i-1$, and the program stops if $\max_{k\in[0,N]}[m_i(t_k) - m_{i-1}(t_k)] < \varepsilon$ (we have chosen the cutoff $\varepsilon = 10^{-10}$).

IV. DYNAMIC HYSTERESIS AND SYMMETRY BREAKING

When solving eq. (5) one observes that after an initial transient period, the average magnetization becomes a periodic function of time (although not necessarily sinusoidal) with the same period as the external field. Furthermore, the magnetization lags behind the field as a consequence of the relaxational delay associated with the characteristic spin-flip time τ. These behaviors are responsible for the appearance of a loop with nonzero area in the $m-h$ plane, a phenomenon known as dynamic hysteresis. The loop area is

$$A = -\oint m\,\mathrm{d}h \qquad \left(A_{\mathrm{num}} = -\frac{2\pi}{N} h_0 \sum_{i=1}^{N} m(\eta_i)\cos\eta_i\right), \tag{6}$$

and it is straightforward to see that it is equal to the energy dissipated by the system during one period of the oscillating field. The parenthesized expression is the numerical equivalent of A employed in our program.

For very low values of ω the area should approach zero, since the magnetization will be able to follow the external field for any value of τ. However, the hysteresis arising from solutions to eq. (5) does not quite vanish in this quasistatic limit, which is certainly an artifact of the mean field approach [3]. For higher values of ω, the area increases as the magnetization struggles to follow the field oscillation due to the time it takes the spins to invert their orientation. Eventually, ω exceeds a certain threshold value (dependent on h_0 and T) and the loop area starts decreasing. At the same time, the loop becomes asymmetric and is no longer centered at the origin of the $m-h$ plane, but can appear either in the upper half-plane or in the lower half-plane (in our study, it always appeared in the upper half-plane, since we always initialized the magnetization at $m = 1$). Therefore, there exists a symmetry breaking of dynamic origin, signaling a phase transition between a symmetric loop phase and an asymmetric one. These are sometimes called the dynamic paramagnetic and ferromagnetic phases, respectively. FIG. 1 illustrates the typical behavior of m below and above the transition frequency.

In order to quantitatively study this transition, one can define an associated order parameter Q as

$$Q = \frac{1}{P}\int_0^P m(t)\,\mathrm{d}t \qquad \left(Q_{\mathrm{num}} = \frac{1}{N}\sum_{i=1}^{N} m(\eta_i)\right), \tag{7}$$

where $P = 2\pi/\omega$ is the period of the applied field. When m oscillates around zero this order parameter vanishes, whereas it develops a nonzero value for high frequencies. In the limit when ω is so high that the magnetization remains constant at ± 1 one finds $|Q| = 1$. The order parameter can be shown to exhibit a discontinuous jump for low temperatures, whereas for higher temperatures the transition becomes continuous, implying the existence of a tricritical point along the phase boundary for fixed h_0 or ω (see Sec. VI).

V. CRITICAL BEHAVIOR

In this section we shall focus on the regime where the transition is continuous, i.e. for high enough temperatures. In this case, the order parameter Q of the DPT can be shown to exhibit a critical behavior similar to the one of the magnetization m in the case of the thermodynamic phase transition (TPT). In particular, a critical exponent $\beta = 1/2$ can be extracted from the evolution equation, such that $Q \sim (P_c - P)^\beta$ as $P \to P_c$.

It is clear that for $P < P_c$ the magnetization, despite being periodic, will oscillate around a certain offset so that $Q \neq 0$. This offset tends to zero as $P \to P_c$, so that m can be written as $m(\eta) = \varepsilon(P) + \xi(P, \eta)$, with $\varepsilon(P)$ a time-independent function satisfying $\varepsilon(P_c) = 0$ and $\xi(P, \eta)$ a periodic function with vanishing time average providing the time dependence. Then, it follows from the definition of the order parameter that ε must be Q itself, so that we may write

$$m(\eta) = Q + \xi(P, \eta). \tag{8}$$

Using this form for m, the RHS of the dynamic equation (5) can be expanded to third order in powers of Q. By defining the periodic function

$$\Xi(P, \eta) = \frac{h(t) + \xi(P, \eta)}{T} \tag{9}$$

and integrating both sides of the resulting expression, one arrives at

$$Q^2 = 3T^2 \left[\frac{I_1(P) - 2\pi T}{I_2(P)}\right]. \tag{10}$$

To obtain this relationship, we have introduced the following integrals:

$$I_1(P) = \int_0^{2\pi} \operatorname{sech}^2 [\Xi(P, \eta)]\, d\eta\,, \tag{11}$$

$$I_2(P) = \int_0^{2\pi} \{2 - \cosh [2\Xi(P, \eta)]\} \times \operatorname{sech}^4 [\Xi(P, \eta)]\, d\eta\,, \tag{12}$$

which are well behaved at $P = P_c$ in the sense that they do not show any singular behavior themselves. This can be understood from the fact that they only depend on the oscillating part of m, namely $\xi(P, \eta)$, which is not expected to exhibit abrupt changes at the critical point. In other words, the order parameter Q, which could introduce irregularities at $P = P_c$, has been effectively removed from the integrals according to eq. (8).

Eq. (10) reveals an analytical condition for the position of the critical point, namely

$$I_1(P_c) = 2\pi T, \tag{13}$$

since at $P = P_c$ it holds that $Q = 0$ and the RHS of (10) is expected to vanish.

It is worth mentioning that, even though $I_2(P)$ can change sign, $I_2(P_c)$ is positive. Furthermore, the first derivative of $I_1(P)$ is nonzero and negative [4], so that an expansion around P_c readily yields

$$Q^2 = \frac{3T^2}{I_2(P_c)} \left|\frac{\partial I_1(P)}{\partial P}\right|_{P=P_c} (P_c - P)\,, \tag{14}$$

and the exponent $\beta = 1/2$ is found. We have checked this result numerically, obtaining remarkable agreement, as shown in FIG. 2.

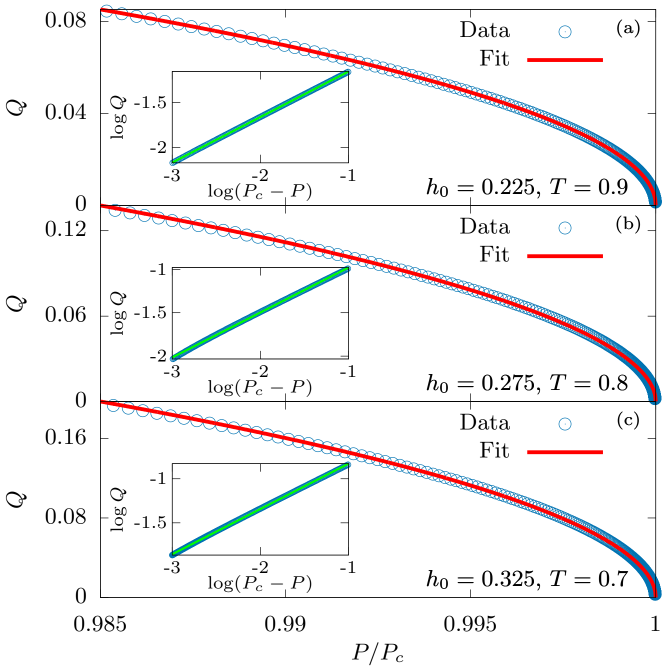

FIG. 2: Order parameter as a function of the period (normalized to P_c) for $h_b = 0$. The data has been fitted to the form $a(P_c - P)^\beta$ in order to extract P_c. The insets show the linear behavior of $\log Q$ vs. $\log(P_c - P)$ from which the critical exponents β have been obtained. The results are (a) $\beta = 0.5006 \pm 0.0012$, (b) $\beta = 0.5012 \pm 0.0013$, and (c) $\beta = 0.5042 \pm 0.0024$.

The analogy between the DPT and the TPT can be extended further by considering a bias field h_b accompanying the oscillating part, i.e. letting $h(t) = h_0 \sin \omega t + h_b$ [4]. In this case, an expansion to third order in Q and first order in h_b leads to an equation of state relating h_b and Q in the vicinity of the critical point, namely

$$h_b = \left[\frac{2\pi T - I_1(P)}{I_1(P)}\right] Q + \left[\frac{I_2(P)}{3T^2 I_1(P)}\right] Q^3. \tag{15}$$

At $P = P_c$, using the critical point condition given by eq. (13) yields

$$Q = T \left[\frac{6\pi}{I_2(P_c)}\right]^{1/3} h_b^{1/3}. \tag{16}$$

Thus, a critical exponent $\delta = 3$ is found, such that $Q \sim |h_b|^{1/\delta}$ at $P = P_c$ and $h_b \to 0$. We have checked this result numerically for the sets of values considered in FIG. 2 after extracting P_c, obtaining very good agreement (see FIG. 3). From this result, it can be interpreted that Q and h_b are conjugate variables in the same way m and h are conjugated in equilibrium.

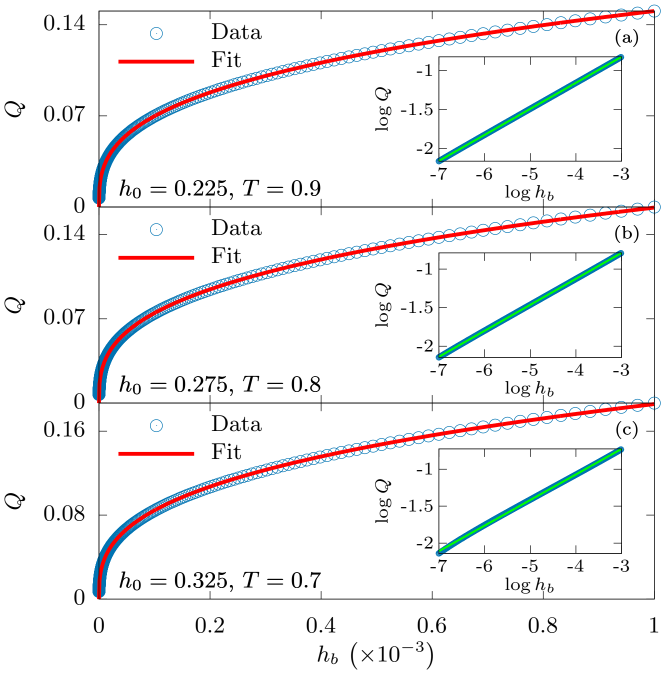

FIG. 3: Order parameter as a function of the bias field h_b. The data has been fitted to the form $ah_b^{1/\delta}$. The insets show the linear behavior of $\log Q$ vs. $\log h_b$ from which the critical exponents δ have been obtained. The results are (a) $\delta = 2.989 \pm 0.004$, (b) $\delta = 2.967 \pm 0.008$, and (c) $\delta = 2.891 \pm 0.032$.

Furthermore, after introducing the bias field h_b one can define a dynamic susceptibility as $\chi_Q = \partial Q/\partial h_b$. Differentiating eq. (15) with respect to Q, setting $Q = 0$ as corresponds to $P > P_c$, and expanding the remaining term around P_c yields

$$\chi_Q(P \gtrsim P_c) = \frac{1}{(P - P_c)} \frac{2\pi T}{|\partial_P I_1(P)|_{P=P_c}}. \tag{17}$$

Similarly, using eq. (14) for $P < P_c$ gives

$$\chi_Q(P \lesssim P_c) = \frac{1}{2} \frac{1}{(P_c - P)} \frac{2\pi T}{|\partial_P I_1(P)|_{P=P_c}}. \tag{18}$$

Thus, we find that χ_Q should diverge at the transition point just like the thermodynamic susceptibility $\partial m/\partial h$ diverges in the TPT, i.e. with a power law of the form $|P - P_c|^\gamma$ at both sides of P_c with $\gamma = -1$. Furthermore, the slopes, r, of the lines resulting from plotting $\chi_Q^{-1}(P)$ should satisfy $|r(P \lesssim P_c)/r(P \gtrsim P_c)| = 2$.

Remarkably, it is only necessary to assume $h(t+P/2) = -h(t)$ to obtain eq. (15). Thus, since all critical exponents follow from this expression, we may conclude that the critical behavior we have found is not unique to a harmonically oscillating field, but common to the whole set of fields satisfying this property. It is worth noting that this analogy with the TPT actually breaks down in systems with surfaces, but holds very well in bulk systems such as the one considered in this work [5].

VI. PHASE DIAGRAMS

One can obtain the shape of the phase boundaries in the $\omega - T$ and $h_0 - T$ planes by analyzing the evolution of the order parameter Q. It is found that for fixed h_0 the transition is discontinuous at low temperatures and becomes continuous at higher temperatures. Thus, the phase diagram in the ω–T plane (corresponding to slices of constant h_0) exhibits a tricritical point (TCP) for each h_0. Similarly, the boundary on the $h_0 - T$ plane is also comprised by a region of continuous transitions and another of discontinuous ones (for higher and lower temperatures respectively) joined together by a TCP. The global aspect of the phase diagram would thus be a 3-dimensional space ($h_0 - \omega - T$) separated by a surface of phase transitions, which in turn is split by a line of tricritical points. Even though it was initially believed that the discontinuous transition could be an artifact of the mean field approach itself, Monte Carlo studies of the dynamic Ising model have shown that Q still exhibits a discontinuous jump at low temperatures [3].

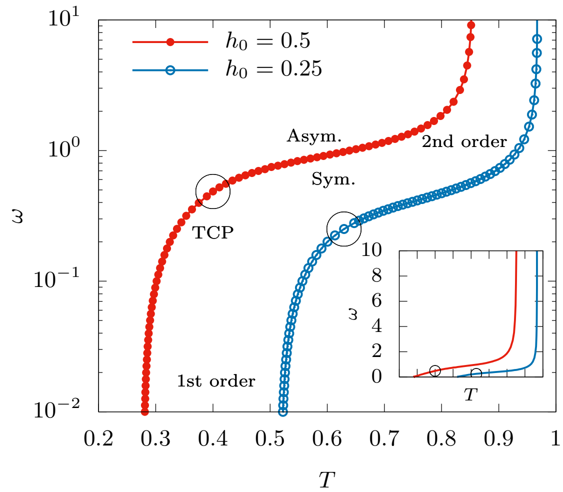

FIG. 4: Phase diagram in the $\omega - T$ plane for two fixed values of h_0, with approximate positions of the TCPs on the curves. The inset shows the shape of the boundary on a linear ω scale.

One can interpret that each value of h_0 has an associated value of the energy change due to the inversion of one spin simply given by $\pm\mu h_0$. Thus, the spins will tend to follow the field more easily when h_0 is higher, since the corresponding energy minimization is more significant. It can then be expected that given a certain temperature, a higher value of h_0 will in turn exhibit a higher value for the threshold frequency or critical frequency ω_c that drives the system into the dynamic ferromagnetic phase. This behavior can be seen qualitatively in FIG. 4.

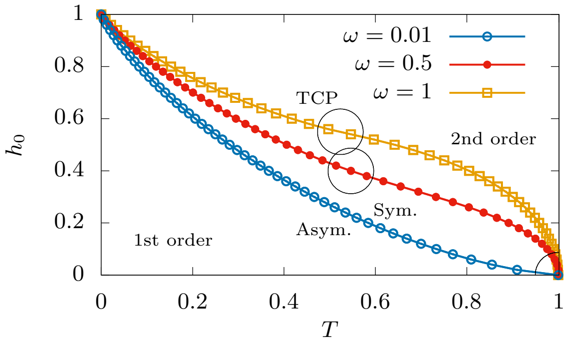

FIG. 5: Phase diagram in the $h_0 - T$ plane for three fixed values of ω, with approximate positions of the TCPs on the boundary curves. As ω decreases, the phase boundary becomes concave towards the origin and the diagram assumes the aspect known from the equilibrium transition.

The same argument allows to interpret the phase diagram in the $h_0 - T$ plane, shown in FIG. 5. It is interesting to notice how the phase boundary becomes convex towards the origin when ω decreases. This matches the result that one expects in the quasistatic limit, namely the equilibrium phase diagram where the transition takes place at $h = 0$ below the critical temperature $T_c = 1$.

The diagrams in the previous figures have been obtained using the numerical program described above. The discontinuous transition is easily found since Q just exhibits a relatively large jump. The critical points, on the other hand, have been computed with the transition condition (13). Furthermore, they have also been found through the position of the divergence of χ_Q, yielding the same results within error limits.

VII. CONCLUSIONS

In summary, we have shown how hysteretic behavior can be understood from a point of view of competing timescales in the dynamic mean-field Ising model. We have provided an overview of the dynamic transition that takes place when the time period of the oscillating field becomes much less than the typical spin relaxation time. Furthermore, we have analytically derived universal exponents relating the order parameter of the DPT to the relevant dynamic parameters of the system, and the predictions have been numerically verified.

The results for the mean-field critical exponents match those of the thermodynamic phase transition. However, this result is not restricted to a mean-field analysis. In Ref. [6] it was established that the DPT and the Ising model belong to the same universality class. Accordingly, in Ref. [7], critical exponents equal to those of the 2D Ising model were found by means of Monte Carlo simulations. Furthermore, Ref. [8] gathered evidence that the DPT exhibits universality with respect to the stochastic dynamics, which is remarkable since the dependency on the specific dynamics is still an open question in non-equilibrium systems. Additionally, experimental results for real systems exhibiting dynamic hysteresis are readily available. For instance, Ref. [9] demonstrates consistency with a mean-field Ising model.

Future extensions of this work could include the investigation of the tricritical exponents (in particular, $\beta = 1/4$ would be expected according to the analogy with mean-field equilibrium system). Furthermore, phase diagrams such as the ones above could be obtained through Monte Carlo simulations of the Ising system in order to compare them with the mean-field results. Finally, methods similar to those exposed in Ref. [9] could make it possible to obtain experimental validation of the results presented in this work.

Acknowledgments

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