UNIVERSITAT DE BARCELONA

Cardinal Arithmetic: From Silver's Theorem to Shelah's PCF Theory

MASTER'S THESIS

Curial Gallart Rodríguez

Advisor: Prof. Joan Bagaria Pigrau

Master's Degree in Pure and Applied Logic

July, 2020

Our minds are finite, and yet even in these circumstances of finitude we are surrounded by possibilities that are infinite, and the purpose of life is to grasp as much as we can out of that infinitude.

Alfred North Whitehead

Acknowledgements

I am very proud to have been a student of Professor Joan Bagaria. His passion for set theory can only be compared to his extensive knowledge of the field, and this was reflected in his classes. He has been an inspiration for me and for many of my classmates. Thank you for your advice and your guidance.

I would also like to thank all the professors of the Master in Pure and Applied Logic of the University of Barcelona, specially Ramon Jansana. Thank you for keeping the master alive, and allowing students like me to learn the beautiful field of logic.

Many thanks to all my classmates in the master, those who left and those who stayed. Thank you for sharing the pain, but also the reliefs. Your laughs have made it easy to go through these two years.

I am deeply thankful to my always supportive parents. Thank you for encouraging me to be whatever I wanted to be, for coming along with me through this arduous path, which has just started. I will always be in debt with you.

I would also like to extend my deepest gratitude to my family, and to those who have behaved as such. Some of you have been inspiring me through the years and still do.

Last but not least, I would like to thank all my friends, those who are near and those who are far away. You have all contributed to this, some of you struggled as much as I did. This was possible because you were part of it.

Contents

Introduction

During the last quarter of the 19th century, mathematicians started an enterprise to formalise some of the most fundamental concepts in mathematics, which at that time were defined rather vaguely and based on mere intuitions. Georg Cantor in 1872 defined a *set* as a collection of objects that share some property, and introduced the notion of a transfinite ordinal number, in his own attempt to formalize the notion of *infinity*. His work, which wasn't exempt of criticism and mistrust, soon came into vogue, making set theory an independent area of mathematics. But in 1902 his imprecise definition of a set led to an inconsistency, found by the mathematician and philosopher Bertrand Russel, and with the disclosure of that antinomy came a foundational crisis in mathematics. The way to overcome this obstacle was to adopt an axiomatic system, begun by Zermelo and completed by Fraenkel and Skolem, known as ZFC. Up to this day this system of axioms hasn't seen any contradiction and has become the standard axiomatization for set theory, and therefore for the whole of mathematics.

Cantor had proposed as early as 1978 his famous *Continuum Hypothesis* (CH), namely $2^{\aleph_0} = \aleph_1$, when his investigations were centered on the study of certain definable subsets of the reals. The apparent difficulty to define a subset of $\mathbb{R}$ of cardinality strictly between $\aleph_0$ and $\mathfrak{c} = 2^{\aleph_0}$, led him to conjecture that there wasn't in fact such a set. Cantor spent many years on a failed attempt to solve his hypothesis, except for some particular cases. It is worth mentioning his result that every closed set in $\mathbb{R}$ is either countable, i.e., has cardinality $\aleph_0$, or contains a perfect subset, and has, therefore, size $\mathfrak{c}$. At this point, the development of set theory found its two main guidelines in the study of the definable sets of the reals, an active area of research to this day that we know as *descriptive set theory*, and cardinal arithmetic, which is the central topic of this thesis.

Cardinal arithmetic deals with the rules that govern the behaviour of the different operations that can be performed between infinite cardinals. The operations of addition and multiplication are a natural generalization of such operations on integers, and it turns out that they are completely determined in ZFC: when at least one of the two cardinals κ and λ is infinite

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

However, cardinal exponentiation was a completely different story. In the simplest non-trivial case, when κ is any cardinal, 2^κ represents the cardinality of the power set $P(\kappa)$ (Adopting the usual convention of set theory that the number κ is identified with the set of all ordinals smaller than κ.). Being the indeterminacy of the power-set operation one of the main reasons for the independence results, the intimate bond between this operation and cardinal exponentiation made its understanding much slower, and usually dependant on the assumption of additional hypotheses such as the CH or the large cardinal axioms.

Julius König presented at the third international congress of mathematicians in Heidelberg, in 1904, what he thought was a solution to Cantor's most famous problem, the Continuum Hypothesis. König had allegedly refuted it by proving that the continuum could not equal any of the alephs in Cantor's list of infinite cardinals. The proof relied on a theorem from Felix Bernstein's dissertation, that it turned out to be false. Indeed, Bernstein's lemma was false for exactly the type of cardinals required in König's use of it. In his words, they were described as *singularities* at which certain inductive arguments broke down. König hadn't refuted the continuum hypothesis, what he had proved was that the continuum could not have countable cofinality. This was a particular case of his famous inequality

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i,$$

if $\kappa_i < \lambda_i$, for every $i \in I$, one of the few rules that govern the behaviour of the regular cardinals, an extraordinary result overshadowed by his erroneous use of it.

The name *singular* cardinal was adopted by Hausdorff to describe the cardinals whose cofinality was smaller than themselves, after the introduction of the notion of cofinality in his paper of 1906 on order-types. The division of infinite cardinals in "regular" and "singular" suggested that regular cardinals were the ones that deserved the most attention. Singular cardinals were just regarded as a curiosity, as an obstacle to Cantor's conjecture, and many years had to pass by until set theorists gave them the importance they deserved.

After the first period of developement of cardinal arithmetic, from 1870 to 1930, the first great advance, which started new lines of research in set theory, was due to Kurt Gödel. In his paper of 1938, "The consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis", Gödel takes for the first time axiomatic set theory as a proper mathematical discipline and object of study, and introduces the inner model L, the class of constructible sets. In ZF Gödel showed that $L \models \mathrm{ZFC} + \mathrm{CH}$, and hence that there is no counterexample to the Continuum Hypothesis in ZFC, unless ZF is already inconsistent. But even with this result, it wasn't clear that the Continuum Hypothesis could be proved.

The *forcing* technique, invented by Paul Cohen in 1963, was the next great advance in set theory. This method, which emerged unexpectedly since it didn't follow the lines of research of the time, was a very intuitive and flexible result that allowed Cohen to show that the negation of the Axiom of Choice is consistent with ZF and that the negation of the Continuum Hypothesis is consistent with ZFC, assuming that ZF is consistent. Forcing has been of immeasurable utility for proving consistency results, and its trascendence can only be compared to Gödel's work. One of the most surprising applications of forcing was due to Easton, who showed in his thesis that the behaviour of the exponentiation is almost arbitrary in the context of regular cardinals. Formally, if E is a function on the class of regular cardinals and it satisfies that for all κ, λ regular,

- $\kappa \leq \lambda$ implies $E(\kappa) \leq E(\lambda)$, and
- $\kappa < cf(E(\kappa))$,

then it is consistent with ZFC that for every regular cardinal κ, $E(\kappa) = 2^\kappa$. In particular, it is consistent with the ZFC axioms that, for instance, $2^{\aleph_0} = \aleph_{762}$, or that $2^{\aleph_0} = \aleph_{\omega_1}$.

Set theorists in the early 1970s thought that the limitation of Easton's result to regular cardinals was due to some wakness in the proof, and that eventually it would end up being generalized to singular cardinals. However, Silver showed in 1974 that the *Generalized Continuum Hypothesis* (GCH), namely $2^\kappa = \kappa^+$ for every infinite cardinal κ, can't fail for the first time at a singular cardinal of uncountable cofinality. This result changed the situation dramatically, as it showed that there were in fact non-trivial theorems about cardinal arithmetic which applied only to singular cardinals.

Silver's Theorem triggered a new wave of results revolving around the understanding of the arithmetic of singular cardinals, which came in many different flavours. F. Galvin and A. Hajnal bounded 2^κ for every strong limit singular cardinal κ with uncountable cofinality (Galvin-Hajnal Theorem). Ronald Jensen proved his Covering Theorem for L, which was one of the most striking results in his newly developed Inner Model Theory, where he retook the study of Gödel's constructible universe.

The understanding of the consistent behaviour of the exponentaition of singular cardinals is known as the *Singular Cardinals Problem*, and the main driving force of this challenge is the *Singular Cardinal Hypothesis* (SCH), which states that for every singular cardinal κ

$$2^{cf(\kappa)} < \kappa \text{ implies } \kappa^{cf(\kappa)} = \kappa^+.$$

The results in Inner Model Theory, specially Jensen's Covering Theorem, showed that large cardinal assumptions were necessary in the construction of models of SCH. More precisely, the Covering Theorem implies the SCH, and thus the negation of the SCH implies the existence of some large cardinal. Another absolute result that was dependant on the existence of large cardinals was obtained in 1974 by Robert Solovay, who showed that the SCH is eventually true. In particular, he showed that the SCH holds above the first strongly compact cardinal, which implies, for instance, that in the presence of large cardinals, the GCH holds for a proper class of cardinals. Large cardinals with consistency strength of at least that of a measurable cardinal are required, together with fairly sophisticated forcing techniques, to build models of ZFC in which the SCH is false. Menachem Magidor in two papers that appeared almost simultaneously in 1977 showed that modulo some large cardinal, $\aleph_\omega$ could be a counterexample to the SCH and, in fact, be the first counterexample to the GCH.

S. Shelah in 1980 obtained an analog of the Galvin-Hajnal Theorem for singular cardinals of countable cofinality. That is, he bounded 2^κ for κ a strong limit singular cardinal of countable cofinality. The ideas used in the proof of this result were developed further into his famously known pcf (possible cofinalities) theory. In his book of 1994 "Cardinal Arithmetic" he exposes all the advances in his newly founded theory, whose major application in cardinal arithmetic is the following theorem:

$$\text{If } \delta \text{ is a limit ordinal such that } |\delta|^{cf(\delta)} < \aleph_\delta, \text{ then } \aleph_\delta^{cf(\delta)} < \aleph_{|\delta|^{+4}}.$$

In particular, if $\aleph_\omega$ is a strong limit, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Apart from this extraordinary result, pcf theory has been used in many other mathematical contexts such as model theory, infinitary logics, algebra, or general topology. In all of these areas Shelah's pcf theory has been revolutionary, but in the understanding of the behaviour of singular cardinals it has been unparalleled. It has exposed cardinal arithmetic, showing its inner workings from a completely different point of view, obtained by focusing in concepts far more fundamental for cardinal arithmetic than cardinal exponentiation.

The main goal of this master's thesis is to give a detailed description of the major ZFC advances in cardinal arithmetic from Silver's Theorem to Shelah's pcf theory and his bound on

$2^{\aleph_\omega}$. In our attempt to make this thesis as self-contained as possible, we have devoted the first chapter to review the most elementary concepts of set theory, which include all the classical results from the first period of developement of cardinal arithmetic, from 1870 to 1930, due to Cantor, Hausdorff, König, and Tarski.

In the second chapter we introduce the technique of ultrapowers, a method for constructing models of set theory that has its origins in model theory. This technique is fundamental in the proof of Silver's theorem, which is included in full detail, following Silver's original argument. The chapter finishes with Solovay's theorem asserting that the SCH holds above the first strongly compact cardinal, again by means of the ultrapower construction.

The third chapter serves as a bridge between the techniques of the last chapter and Shelah's pcf theory, that will be developed in the next one. We will develop a general theory of reduced products of sets of ordinals, while introducing the notions of true cofinality and scale, which are fundamental in pcf theory, among many others. Then we will start a search for sufficient conditions for the existence of exact upper bounds for sequences of ordinal functions, that will finish with two formidable results due to Shelah. Using all the machinery developed in this chapter, we will discuss a generalization of Silver's theorem and an improvement on Solovay's theorem from the last chapter.

The fourth chapter is devoted to pcf theory. We will give a detailed exposition of all the mathematical architecture developed by Shelah to obtain his famous bound $2^{\aleph_\omega} < \aleph_{\omega_4}$, assuming that $\aleph_\omega$ is a strong limit. The chapter starts with the definition of the pcf operator and its most elemental preperties, and is followed by the study of the ideals $J_{<\lambda}$ and their generators. At this point we will be ready to find a connection between cardinal arithmetic and pcf theory, through an exhaustive study of a certain kind of chains of elementary substructures. A first bound for $2^{\aleph_\omega}$ will follow from this analysis, which will be improved in the later sections to get the famous bound that we have mentioned above. We will close the chapter with a brief description of some of the most important applications of pcf theory.

In the last chapter we will discuss some of the most important open problems in cardinal arithmetic, and we will describe some of the major recent findings that have occurred in the last 20 years.

None of the results nor the ideas presented in this thesis are my own, and except for some details in the proofs and some minor remarks, I do not claim the authorship of any of them.

Chapter 1

Preliminaries

This preliminar chapter serves three main purposes: fix the notation, define precisely the framework in which this thesis will be developed, and contextualize historically the fundamental results about singular cardinals, and cardinal arithmetic in general. Our aim is to make this thesis as self-contained as possible, and hence we are forced to review very basic topics, which are completely dispensable for the reader that has a basic knowledge of first-order logic and set theory at the level of an undergraduate course, including ordinals and cardinals, basic combinatorial set theory, as well as some familiarity with the ZFC axioms. All the results presented in this chapter can be found in any basic set theory textbook. For a more detailed exposition, containing all the proofs that have been omitted, we refer the reader to [36] or [21].

1.1 Framework

We shall work in Zermelo Fraenkel set theory with the Axiom of Choice (AC), abbreviated ZFC. This first-order axiomatic system, proposed by Ernst Zermelo and Abraham Fraenkel, has been the standard form of axiomatic set theory since the early twentieth century, when it was presented. A list of the ZFC axioms can be found in Appendix A.

Since ZFC is a theory with a recursive set of axioms in which elementary arithmetic is interpretable, by Gödel's second incompleteness theorem if ZFC is consistent, one cannot prove this in ZFC itself. Therefore, we shall assume that the axiom system ZFC is consistent, and thus by Gödel's completeness theorem for first-order logic, that it has a model.

A *model* of (a fragment of) ZFC is a pair $\mathcal{M} = \langle M, E\rangle$, where M is a non-empty set or proper class, called the *universe* of the model, and E is a binary relation on M, such that $\mathcal{M}$ satisfies the (fragment of) ZFC axioms. A model $\langle M, E\rangle$ is called *standard* if $E = \in \cap (M \times M)$, where $\in$ is the membership relation between sets.

If $\mathcal{M}$ and $\mathcal{N}$ are models (whose universes may be proper classes) of the same language $\mathcal{L}$, a function $j : M \to N$ is an *elementary embedding* if for every formula $\varphi(x_1, \dots, x_n)$ of language $\mathcal{L}$ and every $a_1, \dots, a_n \in M$,

$$\mathcal{M} \models \varphi(a_1, \dots, a_n) \text{ if and only if } \mathcal{N} \models \varphi(j(a_1), \dots, j(a_n)).$$

If j is the identity, then we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ (or that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$), and we write $\mathcal{M} \preceq \mathcal{N}$.

Theorem 1.1.1 (Löwenheim-Skolem). *Let $\mathcal{M}$ be a model of language $\mathcal{L}$.*

Downward: If $A \subseteq M$, then there is some elementary substructure $\mathcal{N} \preceq \mathcal{M}$ with $A \subseteq N$ and $|N| \leq |A| + |\mathcal{L}| + \omega$.

Upward: If M is infinite, for every cardinal $\kappa \geq |M| + |\mathcal{L}|$, there is some elementary extension $\mathcal{N} \succeq \mathcal{M}$ such that $|N| = \kappa$.

Proof. For a proof see [11] or [32]. ∎

A model $\mathcal{M} = \langle M, E\rangle$ is *transitive* if the relation E is transitive, i.e., if aEb and bEc implies that aEc, for $a, b, c \in M$. The model $\mathcal{M}$ is *well-founded* if

(1) E is *well-founded*, i.e., there is no infinite descending E-chain

$$\ldots a_{n+1}Ea_n \ldots a_2Ea_1Ea_0,$$

of elements of M, and

(2) E is *set-like*, i.e., for every $a \in M$, the class $\{b \in M : bEa\}$ is a set.

Theorem 1.1.2 (Mostowski Collapse). *If $\langle M, E\rangle$ is a well-founded model of Extensionality, then there is a unique transitive standard model $\langle N, \in\rangle$, called the transitive, or Mostowski, collapse of $\langle M, E\rangle$, and a unique isomorphism $\pi : \langle M, E\rangle \to \langle N, \in\rangle$.*

Proof. For $x \in M$, let $\pi(x) = \{\pi(z) : z \in M \wedge zEx\}$. The existence of π is guaranteed by transfinite recursion on well-founded relations. The uniqueness comes from the fact that transitive isomorphic models are in fact equal. ∎

The Axiom of Choice is required throughout this exposition because we are constantly working with products of sets, and we need them to be non-empty to avoid trivialities. To be more precise, recall that the Axiom of Choice is the following statement:

(AC) *Every set has a choice function.*

A *choice function* for a familiy $\langle a_i : i \in I\rangle$ of non-empty sets is a function f with domain I such that $f(i) \in a_i$ for each $i \in I$. The *generalized cartesian product* (or simply, the *product*) of a family of sets $\langle a_i : i \in I\rangle$ is the set of all choice functions for it, in symbols

$$\prod_{i\in I} a_i = \{f : dom(f) = I \text{ and } \forall i \in I(f(i) \in a_i)\}.$$

The Axiom of Choice ensures that these products, which are central in the study of the behaviour of the *power-set function* $\aleph_\alpha \mapsto 2^{\aleph_\alpha}$ and in this thesis, are non-empty.

1.2 Ordinals and Cardinals

Definition 1.2.1. Let P be a non-empty set.

(1) A *strict partial ordering* is an irreflexive and transitive binary relation on P, usually denoted by $<$.

(2) A *quasi-ordering* is a reflexive and transitive binary relation on P.

(3) A *reflexive partial ordering* is an antisymmetric quasi-ordering on P, usually denoted by $\leq$.

In general, if $<$ is a strict partial ordering and $\leq$ a reflexive partial ordering, we will call both pairs $(P, <)$ and $(P, \leq)$ a *partial ordering*, but it's clear what we mean in each case.

Let R be any of the above relations on P. We say that R is a *linear ordering* on P, if for any two different $p, q \in P$, either pRq, or qRp. If in addition, every non-empty subset of A has a least element with respect to R (i.e., for every $X \subseteq A$ there is some $p_0 \in X$ such that $\forall q \in X(p_0 R q)$), then we say that R is a *well-ordering*.

A set or a proper class A is called *transitive* if it contains all elements of its elements, i.e., if $\in$ is transitive on A.

Definition 1.2.2 (Ordinal). An *ordinal number* (or simply, an *ordinal*) is a transitive set well-ordered by $\in$.

We will use lowercase Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ as variables for ordinals.

Usually, when talking about ordinals, the set $\mathbb{N}$ of natural numbers is represented by the Greek letter ω, and each natural number $n \in \omega$ is identified with the set of its predecessors, i.e.,

$$\begin{aligned} &0 = \emptyset, \\ &n = \{0, 1, \ldots, n-1\}, \text{ for } n > 0. \end{aligned}$$

It's easy to see that each natural number n and ω are ordinals. This is no surprise, since the definition of the ordinal numbers, due to Von Neumann, came as a generalization of the natural numbers, inspired by their representation as sets of their predecessors.

If α and β are ordinal numbers, then $\alpha \in \beta$ if and only if $\alpha \subset \beta$. Thus, $\alpha \in \beta$ if and only if α is a proper $\in$-initial segment of β. It follows that every ordinal α is precisely the set of all its $\in$-predecessors, which are themselves ordinals. We usually write $\alpha < \beta$ for $\alpha \subset \beta$, and $\alpha \leq \beta$ for $\alpha \subseteq \beta$.

Remark 1.2.3. *If α is an ordinal, then so is $\beta = \alpha \cup \{\alpha\}$, called the* successor ordinal *of α, and usually denoted as $\alpha + 1$. On the contrary, if for a given ordinal β, for every $\alpha < \beta$, there is $\gamma < \beta$ such that $\alpha < \gamma$, we will say that β is a* limit ordinal.

If X is a set of ordinals, then $\cup X$ (denoted $\sup X$) is also an ordinal. Thus, the ordinals form a proper class, denoted by OR, which is well ordered by $\leq$. If we let $\alpha > 0$ be a limit ordinal, and we let $\langle \beta_\gamma : \gamma < \alpha \rangle$ be a non-decreasing sequence of ordinals (i.e., $\gamma < \delta$ implies $\beta_\gamma \leq \beta_\delta$), then we define the *limit* of the sequence by

$$\lim_{\gamma \to \alpha} \beta_\gamma = \sup\{\beta_\gamma : \gamma < \alpha\}.$$

One of Cantor's major conjectures was the assertion that every set could be well-ordered. It turned out that this claim was, modulo ZF, equivalent to the Axiom of Choice, which wasn't surprising since Zermelo introduced this axiom for this exact purpose. There are many equivalent statements to the Axiom of Choice in set theory and in other branches of mathematics such as order theory, abstract algebra, functional analysis, or general topology. Here below we briefly discuss Zermelo's result, together with another statement equivalent to the AC, modulo ZF, which are the two most important of this kind.

(1) Zermelo's *Well-Ordering Principle* says that every set is well-orderable. Since every well-ordered set X is order isomorphic to a unique ordinal, they are the canonical representatives of the orders of the well-ordered sets. The unique ordinal order isomorphic to X is denoted by $ot(X)$, and called the *order-type* of X.

(2) *Zorn's Lemma* states that every non-empty partial order in which every chain has an upper bound has a maximal element.

As we have mentioned above, the ordinals are a generalization of the natural numbers. One of the main features of natural numbers are the induction process and the definitions by recursion. The following are generalizations of these processes over the class OR of all ordinals.

Theorem 1.2.4 (Transfinite Induction). *Given a formula $\varphi(x)$ in the language of set theory, if*

(1) $\varphi(0)$,

(2) *for very ordinal α, if $\varphi(\alpha)$, then $\varphi(\alpha+1)$, and*

(3) *for every limit ordinal α, if $(\forall\beta<\alpha)\varphi(\beta)$, then $\varphi(\alpha)$,*

then $\forall\alpha\varphi(\alpha)$.

Theorem 1.2.5 (Transfinite Recursion). *If G is a set-theoretic operation, there exists a unique set-theoretic operation F, such that for every ordinal α,*

$$F(\alpha) = G(F \restriction \alpha).$$

A straightforward application of the transfinite recursion is the definition of ordinal arithmetic. In a very similar way as in the case of natural numbers, addition, multiplication, and exponentiation can be defined for arbitrary ordinal numbers by transfinite recursion as follows:

Definition 1.2.6 (Ordinal Addition).

(1) $\alpha + 0 = \alpha$,

(2) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all $\beta \in OR$,

(3) $\alpha + \beta = \sup\{\alpha + \gamma : \gamma < \beta\}$, for every limit $\beta \in OR$.

Definition 1.2.7 (Ordinal Multiplication).

(1) $\alpha \cdot 0 = 0$,

(2) $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$, for all $\beta \in OR$,

(3) $\alpha \cdot \beta = \sup\{\alpha \cdot \gamma : \gamma < \beta\}$, for every limit $\beta \in OR$.

Definition 1.2.8 (Ordinal Exponentiation).

(1) $\alpha^0 = 1$,

(2) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$, for all $\beta \in OR$,

(3) $\alpha^\beta = \sup\{\alpha^\gamma : \gamma < \beta\}$, for every limit $\beta \in OR$.

Also by transfinite recursion we can define the Von Neumann universe V (also known as the *cumulative hierarchy* of sets, or the *universe of all sets*).

Definition 1.2.9. We define V as follows:

$$V_0 = \emptyset$$
$$V_{\alpha+1} = P(V_\alpha)$$
$$V_\delta = \bigcup_{\alpha<\delta} V_\alpha, \text{ if } \delta \text{ is a limit ordinal.}$$

Then, $V = \bigcup_{\alpha \in OR} V_\alpha$.

V is a proper class, which is a model of ZFC, and by virtue of the axiom of foundation, every set belongs to V.

Definition 1.2.10 (Cardinal). A *cardinal number* (or simply, a *cardinal*) is an ordinal that is not bijectable with any smaller ordinal.

We will use lowercase Greek letters $\kappa, \lambda, \mu, \eta, \nu, \ldots$ to denote infinite cardinals.

Every infinite cardinal is a limit ordinal. Given an infinite cardinal κ, the set of all ordinals bijectable with some $\lambda \leq \kappa$ is the least cardinal greater than κ, it is called the *successor cardinal* of κ, and it is denoted by κ^+. In contrast, a non-zero non-successor cardinal is called a *limit cardinal.* Moreover, if X is a set of cardinals, then $\bigcup X = \text{sup}X$ is also a cardinal. Hence, the cardinals form a proper class contained in OR, that we denote by $CARD$.

Example 1.2.11.

(1) The natural numbers $n \in \omega$ are cardinals.

(2) The ordinal ω is a cardinal.

(3) The ordinal $\omega + 1$ is not a cardinal. Indeed, we can define the bijection $f : \omega \to \omega + 1$, that sends 0 to ω, and $n + 1$ to n, for every natural number n.

The Well-Ordering Principle implies that every set X has a *cardinality*, i.e., that it is bijectable with a unique cardinal. The cardinality of X is denoted by $|X|$. It's easy to check that "having the same cardinality" is an equivalence "relation" (in quotation marks because the domain of this relation is the class of all sets), which tells us that there exists a bijection between equivalent sets.

Given two cardinals κ and λ, we say that κ is less than or equal λ, and denote it by $\kappa \leq \lambda$, if there is an injective function from κ into λ (or, if there is a function from λ onto κ). Similarly, we say that κ is less than λ, and denote it by $\kappa < \lambda$, if there is an injective function from κ into λ, and no bijection between them.

Theorem 1.2.12 (Cantor-Bernstein). *Let κ and λ be cardinals. If $\kappa \leq \lambda$ and $\lambda \leq \kappa$, then $\kappa = \lambda$.*

The Cantor-Bernstein Theorem ensures that the relation $\leq$ is a reflexive partial ordering on the class $CARD$. Moreover, assuming the Axiom of Choice, since every set has a cardinality, every cardinal is an ordinal, and OR is a linearly-ordered class, we can conclude that the partial ordering $\leq$ on the cardinals is a linear ordering.

The Transfinite Recursion Theorem allows us to define the set-theoretical operation $\aleph$ (*aleph*), which enumerates the infinite cardinals in increasing order.

Definition 1.2.13. We define the function $\aleph$, writing '$\aleph_\alpha$' instead of '$\aleph(\alpha)$', for every ordinal α, as

$$\begin{aligned} &\aleph_0 = \omega \\ &\aleph_{\alpha+1} = \aleph_\alpha^+ \\ &\aleph_\alpha = \sup\{\aleph_\beta : \beta < \alpha\}, \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

We also let $\omega_\alpha = \aleph_\alpha$. Since every cardinal is an ordinal, in order to prevent any kind of ambiguity, we will use the notation ω_α when talking about ordinals, and $\aleph_\alpha$ when talking about cardinals. We will call an infinite set *countable* if it has cardinality $\aleph_0$, and *uncountable* if it has cardinality $> \aleph_0$.

We can also define cardinal addition and multiplication:

Definition 1.2.14. Let κ, λ and μ be cardinals.

(+) $\kappa + \lambda = \mu$ if and only if there are two disjoint sets A and B such that $|A| = \kappa$, $|B| = \lambda$, and $|A \cup B| = \mu$.

(·) $\kappa \cdot \lambda = \mu$ if and only if there are sets A and B such that $|A| = \kappa$, $|B| = \lambda$, and $|A \times B| = \mu$.

These definitions make sense because if two sets A and B have cardinalities κ and λ, respectively, we can make them disjoint by taking $A \times \{0\}$ instead of A, and $B \times \{1\}$ instead of B. Hence, addition and multiplication are defined for every pair of cardinals.

In fact, we could have defined cardinal addition and multiplication as follows:

(+) $\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$.

(·) $\kappa \cdot \lambda = |\kappa \times \lambda|$.

In general, if $\{\kappa_i : i \in I\}$ is an indexed set of cardinals, we define

$$\sum_{i \in I} \kappa_i = |\bigcup_{i \in I} A_i|,$$

$$\prod_{i \in I} \kappa_i = |\prod_{i \in I} A_i|,$$

where $\{A_i : i \in I\}$ is a disjoint family of sets such that $|A_i| = \kappa_i$ for each $i \in I$.

Next proposition tells us that cardinal addition and multiplication are in fact trivial.

Proposition 1.2.15. *If $\aleph_0 \leq \kappa$, then for every cardinal λ:*

(1) $\kappa + \lambda = max\{\kappa, \lambda\}$.

(2) *If $\lambda \neq 0$, then* $\kappa \cdot \lambda = \max\{\kappa, \lambda\}$.

1.3 Cofinality

The concept of cofinality, adopted by Hausdorff in 1906, was defined for a linearly ordered set as the smallest well-order-type of an unbounded subset. This notion was introduced to describe the "singularities" in which König's erroneous solution of the Continuum Hypothesis failed.

Definition 1.3.1. Let α be a limit ordinal.

(1) A subset X of α is *unbounded* in α if and only if sup$X = \alpha$, i.e., if for all $\beta < \alpha$, there is $\gamma \in X$ such that $\beta < \gamma$.

(2) An ordinal β is *cofinal* in α if and only if there is a strictly increasing function $f : \beta \to \alpha$ whose range is unbounded α.

(3) The *cofinality* of α, in symbols $cf(\alpha)$, is the least ordinal which is cofinal in α.

Lemma 1.3.2. *Let α, β and γ be limit ordinals. If α is cofinal in β and β is cofinal in γ, then α is cofinal in γ.*

Proof. If $f : \alpha \to \beta$ and $g : \beta \to \gamma$ are the functions that witness that α is cofinal in β and β is cofinal in γ, respectively, then $g \circ f$ witnesses that α is cofinal in γ. ■

Lemma 1.3.3. *If α is a limit ordinal, $cf(\alpha)$ is the least ordinal such that there is a function $f : cf(\alpha) \to \alpha$ (strictly increasing or not) whose range is unbounded in α.*

Theorem 1.3.4. *For every limit ordinal α, $cf(\alpha)$ is a cardinal.*

Proof. Let $f : cf(\alpha) \to \alpha$ witness that $cf(\alpha)$ is cofinal in α and let g be a bijection between $|cf(\alpha)|$ and $cf(\alpha)$. The composition $f \circ g : |cf(\alpha)| \to \alpha$ is a function with range unbounded in α, and thus by lemma 1.3.3, $cf(\alpha) \leq |cf(\alpha)|$. Hence $|cf(\alpha)| = cf(\alpha)$, i.e., $cf(\alpha)$ is a cardinal. ■

Theorem 1.3.5. *Let α and β be limit ordinals. If α is cofinal in β, then $cf(\alpha) = cf(\beta)$.*

Proof. Suppose α is cofinal in β. Since $cf(\alpha)$ is cofinal in α, $cf(\alpha)$ is also cofianl in β, by lemma 1.3.2. Hence, $cf(\beta) \leq cf(\alpha)$. Now, let $f : cf(\beta) \to \beta$ and $g : \alpha \to \beta$ witness that $cf(\beta)$ and α, respectively, are cofinal in β. Let $h : cf(\beta) \to \alpha$ be the function such that, for every $\xi < cf(\beta)$,

$$h(\xi) = \min\{\gamma < \alpha : f(\xi) < g(\gamma)\}.$$

The range of h is unbounded in α, hence by lemma 1.3.3 $cf(\alpha) \leq cf(\beta)$. Therefore, $cf(\alpha) = cf(\beta)$. ■

Definition 1.3.6. A limit ordinal α is *regular* if and only if $cf(\alpha) = \alpha$, and it is *singular* if and only if $cf(\alpha) < \alpha$.

The name "singular" was adopted by Hausdorff, as we have mentioned above, to refer to those "singularities" of König. But even then, Hausdorff's division of cardinals to "regular" and "singular" suggested that regular cardinals were the objects that deserved serious atention, and that the singulars were less important. This view was sustained, surprisingly, until the 1970s when Easton's and Silver's results pointed out that, in fact, it was the singular cardinals the ones that had to be looked upon.

Proposition 1.3.7. *The following conditions are equivalent for an infinite cardinal κ.*

(1) *κ is regular.*

(2) *If $\alpha < \kappa$, every function $f : \alpha \to \kappa$ is bounded below κ, i.e., the range of f is not unbounded in κ.*

(3) *Every subset of κ of cardinality less than κ is bounded in κ.*

Proof. By lemma 1.3.3. ∎

Theorem 1.3.8. *If α is a limit ordinal, $cf(\alpha)$ is a regular cardinal.*

Proof. Let α be a limit ordinal and let $\kappa = cf(\alpha)$. Since $cf(\kappa)$ is cofinal in κ and κ is cofinal in α, $cf(\kappa)$ is cofinal in α, by lemma 1.3.2. Thus, $\kappa = cf(\alpha) \leq cf(\kappa)$. But $cf(\kappa) \leq \kappa$, so $cf(\kappa) = \kappa$. ∎

Theorem 1.3.9. *If κ is an infinite cardinal, $cf(\kappa)$ is the least cardinal such that κ is the union of a familiy of $cf(\kappa)$-many sets, all of them of cardinality less than κ.*

A very useful characterization of regular cardinals, which follows directly form the last theorem, is the following one:

Corollary 1.3.10. *An infinite cardinal κ is regular if an only if the union of every family of less than κ sets each of cardinality less than κ is a set of cardinality less than κ.*

Corollary 1.3.11. *Every infinite successor cardinal is regular.*

1.4 Cardinal Arithmetic

We will begin by defining cardinal exponentiation. We saw in proposition 1.2.15 that cardinal addition and multiplication are trivial. However, the exponentiation is, in contrast, highly non-trivial. Indeed, even the value of $2^{\aleph_0}$ cannot be decided in ZFC. That's why, when talking about cardinal arithmetic, we refer uniquely to cardinal exponentiation.

Definition 1.4.1. Let κ, λ and μ be cardinals. $\kappa^\lambda = \mu$ if and only if there are sets A and B with $|B| = \kappa$, $|A| = \lambda$, and $|B^A| = \mu$, where recall

$$B^A := \{f : f \text{ is a function with } dom(f) = A \text{ and } ran(f) \subseteq B\}.$$

Note that if $|A| = |C|$ and $|B| = |D|$, then $|B^A| = |D^C|$. Hence, we could have defined cardinal exponentiation simply as:

$$\kappa^\lambda = |\{f : f \text{ is a function with } dom(f) = \lambda \text{ and } ran(f) \subseteq \kappa\}|.$$

Or, equivalently, as

$$\kappa^\lambda = |\prod_{\alpha<\lambda} \kappa|.$$

Definition 1.4.2. Let κ and λ be infinite cardinals. We define

$$\kappa^{<\lambda} := \sup\{\kappa^\mu : \mu \text{ is a cardinal and } \mu < \lambda\}.$$

If A is a set with $|A| \geq \lambda$, we let

$$\begin{aligned} [A]^{<\lambda} &:= \{X \subseteq A : |X| < \lambda\} \\ [A]^{\leq\lambda} &:= \{X \subseteq A : |X| \leq \lambda\} \\ [A]^{\lambda} &:= \{X \subseteq A : |X| = \lambda\} \end{aligned}$$

We also denote $[A]^{<\lambda}$ by $P_\lambda(A)$. It's fairly easy to see, using some of the results that we will introduce hereunder, that $|[A]^{<\lambda}| = |A|^{<\lambda}$ and $|[A]^\lambda| = |A|^\lambda$.

Some of the basic properties of cardinal exponentiation, that follow from the definition, are the following:

Proposition 1.4.3. *Let κ, λ and μ be cardinals.*

(1) $\kappa^0 = 1$.

(2) *If $\kappa \neq 0$, then $0^\kappa = 0$.*

(3) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

(4) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

(5) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.

(6) *If $\kappa \neq 0$ and $\lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$.*

(7) *If $\kappa \leq \lambda$, then $\kappa^\mu \leq \lambda^\mu$.*

By the end of the 19th century, Cantor showed, by means of his diagonalization argument, the following fundamental result.

Theorem 1.4.4 (Cantor). *For every set A, it holds that $|A| < |P(A)|$.*

Cantor's Theorem had immediate important consequences for the philosophy of mathematics. For instance, if we take any infinite set and we apply the power set operation iteratively, by Cantor's Theorem we get an strictly increasing sequence of infinite cardinals. Consequently, the theorem implies that there is no largest cardinal number, in the same sense that there is no largest natural number, because we can always add 1 to any given natural.

We usually refer to the function $\aleph_\alpha \mapsto 2^{\aleph_\alpha}$, defined on the class of all cardinals, as the *power-set function.* The next theorem explains why it is named like this.

Theorem 1.4.5. *For every set A, $|P(A)| = 2^{|A|}$.*

Corollary 1.4.6. *For every cardinal κ, $\kappa < 2^\kappa$. Thus, for every ordinal α, $\aleph_{\alpha+1}$ is the least possible value that the power-set function can take, i.e., $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$.*

The conjecture that $2^{\aleph_0}$ has the smallest possible value, namely $\aleph_1$, is known as the *Continuum Hypothesis* (CH). First formulated in 1874 by Cantor, it was the first of Hilbert's problems, presented at the Paris conference of the International Congress of Mathematicians in 1900. After countless unsuccessful attempts by some of the greatest mathematicians of the time, such as Hausdorff, Hilbert, König and Cantor himself, it was proven to be independent of the ZFC axioms, after K. Gödel proved its consitency in 1931, and after P. J. Cohen proved the consistency of its negation in 1963.

The CH can be extended to every infinite cardinal, resulting in the *Generalized Continuum Hypothesis* (GCH), which is the statement $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, for every $\alpha \in OR$, and it is also independent of ZFC.

One of the most notorious attempts at solving the Continuum Hypothesis was due to König, which, as we have mentioned in the introduction, contained a flaw in the proof. But even though König's argument was wrong, hidden inside the proof there was one of the most important theorems of the first period of development of cardinal arithmetic.

Theorem 1.4.7 (König). *If I is an index set, and $\kappa_i < \lambda_i$ for each $i \in I$, then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

König's theorem, and in particular the next corollary, was later discovered to be one of the few rules that govern the behaviour of cardinal exponentiation at regular cardinals, but at the time, overshadowed by his mistake, it didn't recive the attention that it deserved.

Corollary 1.4.8. *If κ is an infinite cardinal and $2 \leq \lambda$, then $\kappa < cf(\lambda^\kappa)$. In particular, every infinite cardinal κ satisfies $\kappa < cf(2^\kappa)$.*

Corollary 1.4.9. *For every infinite cardinal κ, it holds that $\kappa < \kappa^{cf(\kappa)}$.*

Theorem 1.4.10 (Hausdorff's Formula). *If κ and λ are infinite cardinals, then $(\kappa^+)^\lambda = \kappa^+ \cdot \kappa^\lambda$.*

Proof. If $\kappa^+ \leq \lambda$, then $\kappa^\lambda = (\kappa^+)^\lambda$. But $\kappa^+ \leq \kappa^\lambda$. Thus, $(\kappa^+)^\lambda = \max\{\kappa^+, \kappa^\lambda\} = \kappa^+ \cdot \kappa^\lambda$.

On the other hand, if $\lambda \leq \kappa$, since κ^+ is regular, by proposition 1.3.7, every function from λ to κ^+ is bounded below κ^+. Hence,

$$(\kappa^+)^\lambda = \left| \bigcup_{\alpha<\kappa^+} \alpha^\lambda \right| \leq \kappa^+ \cdot \kappa^\lambda.$$

But $\kappa^+ \cdot \kappa^\lambda \leq (\kappa^+)^\lambda$. Thus, $(\kappa^+)^\lambda = \kappa^+ \cdot \kappa^\lambda$. ■

Next theorem, due to Cantor and Hessenberg, shows that exponentiations in which the base is not larger than the exponent, can be reduced to exponentiations with base 2.

Theorem 1.4.11. *If λ is an infinite cardinal and $2 \leq \kappa \leq \lambda$, then*

$$2^\lambda = \kappa^\lambda = \lambda^\lambda.$$

Proof. $2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda$. ■

Theorem 1.4.12. *Let λ be an infinite cardinal. Then for all infinite cardinals κ, the value of κ^λ is computed as follows, by induction on κ:*

(1) *If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.*

(2) *If there exists some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*

(3) *If $\kappa > \lambda$ and if $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then:*

   (i) *if $cf(\kappa) > \lambda$, then $\kappa^\lambda = \kappa$,*

   (ii) *if $cf(\kappa) \leq \lambda$, then $\kappa^\lambda = \kappa^{cf(\kappa)}$.*

Proof. For a proof see [36]. ■

Last theorem not only shows how to compute κ^λ, for any two infinite cardinals κ and λ, but also that the computation of the value of κ^λ is reducible to the *gimmel function*, given by $\kappa \mapsto \kappa^{cf(\kappa)}$. It was Bukovský who proved the next result, which follows from last theorem:

Corollary 1.4.13. *For every κ and λ, the value of κ^λ is either 2^λ, or κ, or $\mu^{cf(\mu)}$ for some μ such that $cf(\mu) \leq \lambda < \mu$.*

Proof. If $\kappa^\lambda > 2^\lambda \cdot \kappa$, let μ be the least cardinal such that $\mu^\lambda = \kappa^\lambda$, and by Theorem 1.4.12 (for μ and λ), $\mu^\lambda = \mu^{cf(\mu)}$. ■

Thus the key to cardinal arithmetic is the function $\kappa^{cf(\kappa)}$. Note that if κ is regular then $\kappa^{cf(\kappa)} = \kappa^\kappa = 2^\kappa$, but if κ is singular then $\kappa^{cf(\kappa)} \leq \kappa^\kappa = 2^\kappa$. By Corollary 1.4.9 of König's Theorem, $\kappa^{cf(\kappa)} > \kappa$, and so $\kappa^{cf(\kappa)} \geq \kappa^+$. Of course if $2^{cf(\kappa)} \geq \kappa$, then $\kappa^{cf(\kappa)} = 2^{cf(\kappa)}$. Therefore, the question lies in what is the behaviour of $\kappa^{cf(\kappa)}$ for those κ for which $2^{cf(\kappa)} < \kappa$. The simplest possibility, known as the *Singular Cardinal Hypothesis* (SCH), is when

$$2^{cf(\kappa)} < \kappa \text{ implies } \kappa^{cf(\kappa)} = \kappa^+.$$

Under the assumption of the SCH, cardinal exponentiation is completely determined by the power-set function on regular cardinals, but the SCH is independent of the ZFC axioms. It's consistency follows from the fact that the GCH implies it, and its negation can be proven to be consistent as well, if one assumes the existence of a certain large cardinal number.

This analysis of cardinal arithmetic shows that the fundamental question related to the singular cardinals problem is whether the SCH can fail. We will see in subsequent chapters various results that revolve around the satisfaction of the SCH. For instance, in Chapter 2 we will see that the SCH holds above a certain large cardinal, and in Chapter 3 we will see that if the SCH holds for all singular cardinals of countable cofinality then it holds for all singular cardinals.

Since the singular cardinals problem is so closely related to the large cardinal axioms, as they are needed to prove that the SCH can consitently fail (see [15], [49], or [25]), we will introduce the simplest of the large cardinal notions, the inaccessible cardinals, and its main properties.

Definition 1.4.14 (Strong Limit). A cardinal κ is a *strong limit* cardinal if $2^\lambda < \kappa$ for every $\lambda < \kappa$.

Every strong limit cardinal is a limit cardinal, and the converse holds under the GCH. It is worth mentioning that if κ is a strong limit cardinal, then $2^\kappa = \kappa^{cf(\kappa)}$.

Definition 1.4.15 (Inaccessible Cardinal). A cardinal κ is (*strongly*) *inaccessible* if it is uncountable, regular, and a strong limit.

It can be shown that if κ is an inaccessible cardinal, then V_κ is a model of ZFC. Therefore, by Gödel's Second Incompleteness Theorem, it cannot be proven in ZFC that an inaccessible cardinal exists.

In this thesis we won't focus too much on the development of the theory of large cardinals. We will just introduce a couple of them when needed, together with their most relevant properties for our purposes.

The *transitive closure* of a set A, denoted $tc(A)$, is the smallest transitive set that contains A, and it can be recursively defined as

$$tc(A) = A \cup tc\Big(\bigcup A\Big).$$

Definition 1.4.16. Let κ be an infinite cardinal. A set A is *hereditarily of cardinality less than* κ if its transitive closure has size strictly less than κ.

Definition 1.4.17. For every infinite cardinal κ, we let H_κ be the set of all sets hereditarily of cardinality less than κ, i.e.,

$$H_\kappa = \{A : |tc(A)| < \kappa\}.$$

The H_κ's form a very interesting hierarchy, whose main properties are the following ones.

Proposition 1.4.18.

(1) *For every infinite cardinal κ, H_κ is transitive.*

(2) $H_{\aleph_0} = V_\omega$.

(3) *For every infinite cardinal κ, $H_\kappa \subseteq V_\kappa$.*

(4) *The H_κ's form a cummulative hierarchy, i.e., if $\lambda \leq \kappa$ are infinite cardinals then $H_\lambda \subseteq H_\kappa$, and if κ is a limit cardinal then $H_\kappa = \bigcup_{\lambda<\kappa} H_\lambda$.*

(5) $V = \bigcup_{\kappa \in CARD} H_\kappa$.

Note however that $H_{\aleph_1} \neq V_{\omega_1}$, since $P(\omega) \in V_{\omega+2} \setminus H_{\omega_1}$, so H_κ and V_κ may differ at some points. The following result tells us that they coincide for very specific κ's.

Theorem 1.4.19. *If κ is a regular infinite cardinal, then $H_\kappa = V_\kappa$ if and only if $\kappa = \omega$, or else κ is inaccessible.*

Therefore, when κ is an inaccesible cardinal H_κ is a model of ZFC. If we remove the hypothesis of κ being a strong limit, i.e., if we assume that κ is a regular uncountable cardinal, then H_κ is a model of ZFC minus the Power Set axiom.

It is also worth mentioning that even though H_κ is transitive, an elementary substructure $M \preceq H_\kappa$ need not be transitive. For example, if we let $\kappa > \omega_1$ and we let M be a countable elementary substructure of H_κ, M is not transitive. Indeed, note that $\omega \in M$, and that ω_1 can be defined as the least ordinal in H_κ for which there is no onto function from ω to that ordinal. Hence, by elementarity M also has this ordinal, which is necessarily ω_1, but since M is countable $\omega_1 \not\subseteq M$.

1.5 Filters and Ideals

When studying subsets of a given set A, one might be interested in the subsets of A that are large (small) enough to satisfy a certain criterion. Intuitively, a filter (ideal) on a set A is a structure defined on the power set $P(A)$, that makes this precise by giving a notion of bigness (smallness) for the subsets of A.

Definition 1.5.1 (Filters and Ideals). Let A be a non-empty set. A *filter* on A is a set F of subsets of A such that:

(1) $A \in F$ and $\emptyset \notin F$.

(2) If $X, Y \in F$, then $X \cap Y \in F$.

(3) If $X \in F$ and $X \subseteq Y \subseteq A$, then $Y \in F$.

An *ideal* on A is a set I of subsets of A such that:

(1) $\emptyset \in I$.

(2) If $X, Y \in I$, then $X \cup Y \in I$.

(3) If $X \in I$ and $Y \subseteq X$, then $Y \in I$.

If $A \notin I$ we say that the ideal I is proper, but we don't require it. We will see in chapter 4 that the ideal $J_{<\lambda}[A]$ need not be proper.

If F is a filter on A, then the set $F^* = \{A \setminus X : X \in F\}$ is an ideal on A. Conversely, if I is an ideal on A, then the set $I^* = \{A \setminus X : X \in I\}$ is a filter on A. We call F^* and I^* *dual ideal* and *dual filter*, respectively.

If I is an ideal on A, then I^+ denotes the collection of sets of *positive I-measure*, namely

$$I^+ = \{X \subseteq A : X \notin I\}.$$

Example 1.5.2.

(1) If we let X be a non-empty subset of A, then the set $F = \{Y \subseteq A : X \subseteq Y\}$ is a filter, and it's called a *principal* filter on A. Note that every filter on a finite set is principal.

(2) An example of a filter which is non-principal is the *Fréchet filter*, which is the set of all co-finite subsets of ω. More generally, if κ is an infinite cardinal, we define the *Fréchet filter on κ* as the filter
$$F = \{X \subseteq \kappa : |\kappa \setminus X| < \kappa\}.$$

We say that a family F of subsets of A has the *finite intersection property* if every finite collection $\{X_1, \ldots, X_n\} \subseteq F$ has non-empty intersection $X_1 \cap \cdots \cap X_n \neq \emptyset$. Clearly, every filter has the finite intersection property.

Lemma 1.5.3. *If $F \subseteq P(A)$ is non-empty and has the finite intersection property, then F can be extended to a filter on A.*

Proof. Let G be the set of all $X \subseteq A$ such that there is a finite $\{X_1, \ldots, X_n\} \subseteq F$ with $X_1 \cap \cdots \cap X_n \subseteq X$. Then, G is a filter and $F \subseteq G$. ■

Definition 1.5.4 (Ultrafilter). A filter U on a set A is called an *ultrafilter* if for every $X \in U$, either $X \in U$ or $A \setminus X \in U$.

Remark 1.5.5. *Let U be an ultrafilter on a set A.*

(1) *The ultrafilter U is principal if and only if it is such that $U = \{X \subseteq A : a \in X\}$, for some $a \in A$. In this case we call a the* principal element *of U.*

(2) *If $X, Y \subseteq A$ are such that $X \cup Y \in U$, then either $X \in U$, or $Y \in U$.*

Definition 1.5.6 (Maximal filter). A filter F on A is called *maximal* if there is no filter G on A such that $F \subseteq G$ and $F \neq G$.

Proposition 1.5.7. *A filter F on A is maximal if and only if it is an ultrafilter.*

Theorem 1.5.8 (A. Tarski). *Every filter can be extended to an utlrafilter.*

Proof. Let F be a filter on some set A. Let $\mathbb{P}$ be the set of all filters on A that contain F, ordered by $\subseteq$. Then $\mathbb{P}$ is a partial ordering. If C is a chain in $\mathbb{P}$, then $\bigcup C$ is also a filter on A, and therefore an upper bound of C in $\mathbb{P}$. Hence by Zorn's Lemma $\mathbb{P}$ has a maximal element which, by the proposition above, is an ultrafilter. ■

Therefore, any familiy F of subsets of A that has the finite intersection property can be extended to an ultrafilter on A.

If F is a filter (ultrafilter) on a set A, then we say that $X \subseteq A$ is F*-positive* if it doesn't belong to the dual ideal F^*.

If $X \subseteq A$ is an F-positive subset of A, then we define the *projection* of F to X as the set

$$F_X := \{X \cap Y : Y \in F\}.$$

One can easily verify that F_X is a filter (ultrafilter) on X.

Definition 1.5.9 (κ-complete filter). Let F be a filter on a set A, and κ an infinite cardinal. F is said to be κ*-complete* if it holds that for all $\lambda < \kappa$, if $\{X_\alpha : \alpha < \lambda\}$ is a family of elements of F, then $\bigcap_{\alpha<\lambda} X_\alpha \in F$. ω_1-complete filters are also called σ*-complete*.

Dually, an ideal I on A is κ*-complete*, if for every $\lambda < \kappa$ and every family $\{X_\alpha : \alpha < \lambda\}$ of elements of I, it holds that $\bigcup_{\alpha<\lambda} X_\alpha \in I$.

Example 1.5.10. The Fréchet filter on κ, $F = \{X \subseteq \kappa : |\kappa \setminus X| < \kappa\}$, is κ-complete. The filter of subsets of $[0, 1]$ of Lebesgue measure 1, and the filter of co-meager subsets of $[0, 1]$ are examples of σ-complete filters.

Definition 1.5.11 (Uniform filter). A filter F on an infinite cardinal κ is *uniform* if $|X| = \kappa$ for all $X \in F$.

An uncountable cardinal κ is called *measurable* if there exists a κ-complete non-principal ultrafilter on κ. Measurable cardinals are inaccessible, thus one cannot prove in ZFC that measurable cardinals exist.

Proposition 1.5.12. *Every κ-complete non-principal ultrafilter on κ is uniform.*

Proof. Let U be a κ-complete non-principal ultrafilter on κ and assume, on the contrary, that $X \in U$ has cardinality $\lambda < \kappa$. Since U is non-principal, for every $\alpha \in X$, there exists $X_\alpha \in U$ such that $\alpha \notin X_\alpha$. Hence, by κ-completeness, $Y := \bigcap_{\alpha\in X} X_\alpha \in U$. But then $X \cap Y = \emptyset$, which is impossible. ■

1.6 The Club Filter and Stationary Sets

If α is an infinite limit ordinal, a set $C \subseteq \alpha$ is said to be *unbounded* if for every $\beta < \alpha$ there is $\gamma \in C$ such that $\beta < \gamma$. We say that C is *closed* if the supremum of every increasing sequence of elements of C belongs to C, provided this supremum is $< \alpha$, i.e., if for every limit ordinal $\beta < \alpha$ such that $\beta = sup(C \cap \beta)$, it holds that $\beta \in C$. We say that C is a *club* subset of α if it is closed and unbounded.

If α is an infinite limit ordinal, tipical examples of club subsets of α include the *tail sets* $C_\beta := \{\gamma < \alpha : \beta < \gamma\}$, for any $\beta < \alpha$, or the set of limit ordinals smaller than α.

Proposition 1.6.1. *If α is a limit ordinal of uncountable cofinality and C and D are club subsets of α, then $C \cap D$ is club.*

Proof. That $C \cap D$ is closed is immediate. To show that $C \cap D$ is unbounded, let $\beta < \alpha$. Since C is unbounded, there exists $\gamma_1 > \beta$ such that $\gamma_1 \in C$, and since D is unbounded, there is $\gamma_2 > \gamma_1$ such that $\gamma_2 \in D$. Iterate this process to get an increasing sequence

$$\beta < \gamma_1 < \gamma_2 < \cdots < \gamma_{2n} < \gamma_{2n+1} < \ldots$$

so that $\{\gamma_{2n} : n < \omega\} \subseteq C$ and $\{\gamma_{2n+1} : n < \omega\} \subseteq D$. Then, since C and D are closed

$$\gamma := sup\{\gamma_{2n} : n < \omega\} = sup\{\gamma_{2n+1} : n < \omega\} \in C \cap D,$$

and $\gamma < \alpha$, because α has uncountable cofinality. This shows that $C \cap D$ is unbounded. ■

Therefore, if α is a limit ordinal of uncountable cofinality, the set of all club subsets of α has the finite intersection property. So we can consider the filter on α generated by the club subsets of α, that consists of all $X \subset \alpha$ for which there exists some club $C \subseteq \alpha$ such that $C \subseteq X$. We call this filter the *club filter* on α, and we denote it by $Club(\alpha)$.

Theorem 1.6.2. *If κ is a regular uncountable cardinal, then $Club(\kappa)$ is κ-complete.*

Proof. Let $\lambda < \kappa$, and $\langle C_\alpha : \alpha < \lambda \rangle$ a sequence of club subsets of κ. We will prove that $\bigcap_{\alpha<\lambda} C_\alpha$ is club by induction on λ.

The base and successor cases follow directly from proposition 1.6.1. So let λ be a limit and assume that $\bigcap_{\alpha<\mu} C_\alpha$ is club, for every $\mu < \lambda$.

By taking $\bigcap_{\beta\leq\alpha}$ instead of C_α, we may assume that the sequence of C_α's is decreasing, i.e.,

$$C_0 \supseteq C_1 \supseteq \cdots \supset C_\alpha \supseteq \ldots$$

Let $C = \bigcap_{\alpha<\lambda} C_\alpha$. Clearly C is closed, since so are all the C_α's. To show that it is unbounded, fix some $\beta < \kappa$, and define a sequence $\langle \beta_\alpha : \alpha < \lambda \rangle$ as follows:

- $\beta_0 = \beta$.
- Having obtained β_α, let $\beta_{\alpha+1}$ be the least ordinal in $C_{\alpha+1}$ greater than β_α (this is possible because $C_{\alpha+1}$ is unbounded).
- If α is limit, and we have obtained β_γ, for every $\gamma < \alpha$, then let β_α be the least ordinal in C_α greater than $sup\{\beta_\gamma : \gamma < \alpha\}$ (this is possible because C_α is unbounded and κ is regular).

Then $sup\{\beta_\alpha : \alpha < \lambda\} \in C$, because $\{\beta_\gamma : \alpha < \gamma\} \subseteq C_\alpha$, for every $\alpha < \lambda$. ■

The dual of the club filter on an ordinal α of uncountable cofinality is the ideal NS_α of non-stationary sets. A subset S of α is called *stationary* if it is positive with respect to the club filter, i.e., if it intersects all club subsets of κ. An informal analogy that helps getting some intuition on what is the idea behind club and stationary sets is the following one: In a measure space of measure 1, club sets would be analogous to sets of measure 1, while stationary sets would be analogous to sets of positive measure.

Since the intersection of two club subsets of α is club, every club subset of α is stationary, but the converse is not necessarily true. Moreover, it is immediate to check that if S is stationary and C is club, then $S \cap C$ is stationary. By duality, it follows from theorem 1.6.2 that if κ is regular and uncountable, then NS_κ is κ-complete, that is, the union of less than κ-many non-stationary sets is non-stationary.

Probably the most relevant example of a stationary set, specially in this thesis, is the following one:

Proposition 1.6.3. *If κ has uncountable cofinality and $\lambda < cf(\kappa)$ is an infinite regular cardinal, then the set*

$$E^{\kappa}_{\lambda} := \{\alpha < \kappa : cf(\alpha) = \lambda\}$$

is stationary.

Proof. Let C be a club subset of κ. Since $\lambda < cf(\kappa)$, the λ-th element α of C is less than κ, and since λ is regular, α has cofinality λ. ■

If A is a non-empty set, a function f in OR^A, i.e., a function with domain A and such that every image is an ordinal, is called an *ordinal function*. An ordinal function f on a set A is *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in A$, $\alpha > 0$.

The next theorem, which is refered to as Fodor's Theorem or the Pressing-Down Lemma, is a tremendously useful tool that can be applied in a vast number of arguments.

Theorem 1.6.4 (Fodor). *Let κ be a regular uncountable cardinal, and let $S \subseteq \kappa$ be a stationary set. If $f : S \to \kappa$ is regressive, then there is a stationary $T \subseteq S$ on which f is constant, i.e., there exists some $\beta < \kappa$ such that $f(\alpha) = \beta$ for all $\alpha \in T$.*

1.7 Lévy Collapse

In this thesis we will avoid using forcing as much as possible. However, the Lévy Collapse is a notion of forcing that cannot be circumvented. It will be used in the original proof of Silver's Theorem, and it is worth saying a few words about it. For a detailed exposition see [36] or [47].

Let $(P, \leq)$ be a partial ordering. We say that $p, q \in P$ are *compatible* if there exists $r \in P$ such that $r \leq p$ and $r \leq q$, otherwise we call them *incompatible*. A subset $A \subseteq P$ is an *antichain* of P if for every $p, q \in A$, if $p \neq q$, then p and q are incompatible.

Definition 1.7.1. If κ is a cardinal, we say that a partial ordering $(P, \leq)$ satisfies the *κ-chain condition* (or that it is *κ-cc*) if every antichain of P has cardinality less than κ. If $\kappa = \aleph_1$, we say that P has the *countable chain condition* (or that it is *ccc*).

Lemma 1.7.2. *If $(P, \leq)$ is a κ-cc partial ordering, for κ a cardinal, then P does not collapse any cofinalities greater than or equal to κ, i.e., every cardinal greater than or equal to κ has the same cofinality in the generic extension. Hence, all cardinals greater than or equal to κ are preserved in the generic extension.*

Theorem 1.7.3 (Lévy Collapse). *Let κ be a regular cardinal and let $\lambda > \kappa$ be a cardinal. Let $(P, \leq)$ be the partial ordering such that:*

(1) *P is the set of all functions p such that*

  (i) *$dom(p) \subseteq \kappa$ and $|dom(p)| < \kappa$,*

  (ii) *$ran(p) \subseteq \lambda$.*

(2) *The order $\leq$ is defined for all $p, q \in P$ as*

$$p \leq q \text{ if and only if } p \supseteq q.$$

Then $(P, \leq)$ is a notion of forcing, denoted by $Col(\kappa, \lambda)$, that collapses λ onto κ, i.e., λ has cardinality κ in the generic extension. Moreover, $Col(\kappa, \lambda)$ is λ^+-cc, and thus all cardinals $\geq \lambda^+$ are preserved.

Chapter 2

Ultrapowers and Elementary Embeddings

In this chapter we will present one of the main techniques in model theory for constructing models, the ultrapower method. This technique originated with Skolem in the 1930's, and has been used extensively since the work of Łoś in 1955, time in which set theory had a decisive advance thanks to the infusion of model-theoretic methods.

We will start with a first section devoted to the definitions and most fundamental results, which are relevant to the ultrapower construction. In the next section we will describe in full detail how Silver used the ultrapower technique to prove his famous theorem, which revitalized the field of cardinal arithmetic. In the last section we will introduce the notion of a strongly compact cardinal, and give all the details of the proof of Solovay's theorem asserting that the Singular Cardinal Hypothesis eventually holds if we assume the existence of such cardinals.

2.1 Definitions

All the definitions and results in this section can be found in [36].

Definition 2.1.1 (Reduced products). Let S be a nonempty set and $\{\mathcal{M}_x : x \in S\}$ a system of models for language $\mathcal{L}$. For every $x \in S$, we denote by M_x the universe of the model $\mathcal{M}_x$. Let F be a filter on S, and consider the set

$$M := \prod_{x \in S} M_x / =_F,$$

where $=_F$ is the binary relation on $\prod_{x\in S} M_x$ defined by

$$f =_F g \text{ if and only if } \{x \in S : f(x) = g(x)\} \in F.$$

It follows easily that $=_F$ is an equivalence relation. We denote the equivalence classes by $[f]_F$, or simply $[f]$ if there is no conflict with any other filter.

We define the *reduced product* $\mathcal{M}$ of $\{\mathcal{M}_x : x \in S\}$ by F as the model of language $\mathcal{L}$ with universe M and with interpretations:

(1) If c is a constant symbol in $\mathcal{L}$,

$$c^{\mathcal{M}} = [f], \text{ where } f(x) = c^{\mathcal{M}_x}, \ \forall x \in S.$$

(2) If G is an n-ary function symbol in $\mathcal{L}$,

$$G^{\mathcal{M}}([f_1],\ldots,[f_n]) = [f], \text{ where } f(x) = G^{\mathcal{M}_x}(f_1(x),\ldots,f_n(x)), \ \forall x \in S.$$

(3) If R is an n-ary relation symbol in $\mathcal{L}$,

$$R^{\mathcal{M}}([f_1],\ldots,[f_n]) \text{ if and only if } \{x \in S : R^{\mathcal{M}_x}(f_1(x),\ldots,f_n(x))\} \in F.$$

If all the models $\mathcal{M}_x$ coincide, we call $\mathcal{M}$ a *reduced power*.

Reduced products are particularly important when the filter is an ultrafilter. If U is an ultrafilter on S then the reduced product $\mathcal{M}$ is called the *ultraproduct* of $\{\mathcal{M}_x : x \in S\}$ by U, and we denote it by

$$\mathcal{M} = \mathrm{Ult}_U(\{\mathcal{M}_x : x \in S\}).$$

Again, if all the models $\mathcal{M}_x$ coincide, the ultraproduct $\mathcal{M}$ is called an *ultrapower*, and it is denoted by $\mathrm{Ult}_U(\mathcal{M}_x)$.

We say that the equivalence class $[f]_U$ is *represented* in the ultraproduct by the function f on S. In the case of ultrapowers the equivalence classes represented by the *constant functions* are of great relevance (as we will see below). For each $a \in M_x$, the constant function with value a is the function $c_a \in \prod_{x\in S} M_x$, defined by $c_a(x) = a$, for every $x \in S$.

The importance of ultraproducts is due mainly to the following fundamental property, which reduces satisfaction in the ultrapower to satisfaction on a large set of coordinates, large in the sense of the ultrafilter U.

Theorem 2.1.2 (Łoś). *Let U be an ultrafilter on S and let $\mathcal{M}$ be the ultraproduct of the collection $\{\mathcal{M}_x : x \in S\}$ by U.*

(1) *If φ is a formula, then for every $f_1,\ldots,f_n \in \prod_{x\in S} M_x$,*

$$\mathcal{M} \models \varphi([f_1],\ldots,[f_n]) \textit{ if and only if } \{x \in S : \mathcal{M}_x \models \varphi(f_1(x),\ldots,f_n(x))\} \in U.$$

(2) *If σ is a sentence,*

$$\mathcal{M} \models \sigma \textit{ if and only if } \{x \in S : \mathcal{M}_x \models \sigma\} \in U.$$

The proof is by induction on the complexity of φ using the filter properties of U, the ultrafilter property for the negation step, and the Axiom of Choice for the existential quantifier step.

It is also convenient to adopt some terminology when working with ultraproducts. We say that $\mathcal{M}_x$ satisfies $\varphi(f_1(x),\ldots,f_n(x))$ for *almost all* x, or that $\mathcal{M}_x \models \varphi(f_1(x),\ldots,f_n(x))$ holds *almost everywhere*, if

$$\{x \in S : \mathcal{M}_x \models \varphi(f_1(x),\ldots,f_n(x))\} \in U.$$

Corollary 2.1.3. *If U is an ultrafilter on a nonempty set S, $\mathcal{N}$ is a model of language $\mathcal{L}$ with universe N, and $Ult_U(\mathcal{N})$ is the ultrapower of $\mathcal{N}$ by U, then:*

(1) *$\mathcal{N}$ is elementary equivalent to $Ult_U(\mathcal{N})$, i.e., if σ is a sentence in the language $\mathcal{L}$, then*

$$\mathcal{N} \models \sigma \textit{ if and only if } \mathrm{Ult}_U(\mathcal{N}) \models \sigma.$$

(2) *The canonical embedding* $j_U : \mathcal{N} \to \mathrm{Ult}_U(\mathcal{N})$*, defined by* $j_U(a) = [c_a]_U$*, for every* $a \in N$*, is an elementary embedding, i.e., for every formula* $\varphi(x_1, \ldots, x_n)$ *of language* $\mathcal{L}$*, and every* $a_1, \ldots, a_n \in N$*,*

$$\mathcal{N} \models \varphi(a_1, \ldots, a_n) \text{ if and only if } \mathrm{Ult}_U(\mathcal{N}) \models \varphi(j_U(a_1), \ldots, j_U(a_n)).$$

Of course, our interests revolve around the construction of ultrapowers of models $\langle \mathcal{N}, \in \rangle$ of set theory. Hence, if U is an ultrafilter on a nonempty set S, restating the above definitions for this particular case, $\mathrm{Ult}_U(\mathcal{N})$ is made into a model of set theory through the following interpretation of the $\in$ relation on N:

$$[f]_U \in^{\mathrm{Ult}_U(\mathcal{N})} [g]_U \text{ if and only if } \{x \in S : \mathcal{N} \models f(x) \in g(x)\} \in U.$$

Then we consider the model $\langle \mathrm{Ult}_U(\mathcal{N}), \in^{\mathrm{Ult}_U(\mathcal{N})} \rangle$, that we denote by $\mathrm{Ult}_U(\mathcal{N})$. If there is no conflict with other ultrapowers, we will write $\in^{\mathrm{Ult}_U}$, or even $\in^{\mathrm{Ult}}$, instead of $\in^{\mathrm{Ult}_U(\mathcal{N})}$.

Of great importance is the construction of ultrapowers in which $\langle \mathrm{Ult}_U(\mathcal{N}), \in^{\mathrm{Ult}} \rangle$ is *well-founded*. Recall that a model is well-founded when

(1) there is no infinite descending $\in^{\mathrm{Ult}}$-chain

$$\ldots [f_{n+1}] \in^{\mathrm{Ult}} [f_n] \ldots [f_2] \in^{\mathrm{Ult}} [f_1] \in^{\mathrm{Ult}} [f_0],$$

and

(2) $\in^{\mathrm{Ult}}$ is *set-like*, i.e., $ext([f]) := \{[g] \in \mathrm{Ult}_U(\mathcal{N}) : [g] \in^{\mathrm{Ult}} [f]\}$ is a set for every $[f] \in \mathrm{Ult}_U(\mathcal{N})$.

The second condition is clearly satisfied for any ultrafilter U, but for the first condition we need an extra requirement on the ultrafilter:

Lemma 2.1.4. *If* U *is a* σ*-complete ultrafilter, then* $\langle Ult_U(\mathcal{N}), \in^{Ult} \rangle$ *is a well-founded model.*

When the ultrapower is well-founded, by the Mostowski Collapse Theorem 1.1.2, there exists an isomorphism π from $\mathrm{Ult}_U(\mathcal{N})$ into its transitive collapse. In this situation we identify the ultrapower with its transitive collapse, and hence the canonical elementary embedding $j_U : \mathcal{N} \to Ult_U(\mathcal{N})$ is defined by $j_U(a) = \pi([c_a]_U)$. It's common in this context not to make a distinction between the ultrapower and its transitive collapse, and denote both of them by $\mathrm{Ult}_U(\mathcal{N})$.

The use of ultraproducts in set theory is central in the systematic study of large cardinals, where the above ideas are generalized to construct ultrapowers of proper classes, and in particular of the universe V. For this purpose, to make things work properly, we have to replace the equivalence classes $[f]_U$ by sets

$$\{g \in [f]_U : g \text{ has minimal rank}\},$$

known as Scott's trick (due to Dana Scott). To be more precise:

Definition 2.1.5 ($\mathbf{Ult}_U(V)$). Let U be an ultrafilter on a set S and consider the class of all functions with domain S. We define the following relations:

$$f =^* g \text{ if and only if } \{x \in S : f(x) = g(x)\} \in U,$$

$$f \in^* g \text{ if and only if } \{x \in S : f(x) \in g(x)\} \in U.$$

For each f, we denote $[f]_U$ the equivalence class of f in $=^*$:

$$[f]_U := \{g : f =^* g \text{ and } \forall h(h =^* f \to \text{rank}(g) \leq \text{rank}(h))\}.$$

Let $\text{Ult}_U(V)$ be the class of all $[f]_U$, where f is a function on S, then we consider the model $\langle \text{Ult}_U(V), \in^{\text{Ult}_U} \rangle$, where $\in^{\text{Ult}_U}$ (or simply $\in^{\text{Ult}}$) is the relation defined by

$$[f]_U \in^{\text{Ult}_U} [g]_U \text{ if and only if } f \in^* g.$$

Thanks to the Scott's trick, all the properties seen above for ultrapowers still hold for $\text{Ult}_U(V)$. Namely,

(1) Łoś's Theorem 2.1.2 holds for $\text{Ult}_U(V)$. That is, if $\varphi(x_1, \ldots, x_n)$ is a formula in the language of set theory, then

$$\text{Ult}_U(V) \models \varphi([f_1], \ldots, [f_n]) \text{ if and only if } \{x \in S : \varphi(f_1(x), \ldots, f_n(x))\} \in U.$$

If σ is a sentence in the language of set theory, then

$$\text{Ult}_U(V) \models \sigma \text{ if and only if } \sigma \text{ holds},$$

that is, $\text{Ult}_U(V)$ is elementary equivalent to the universe V.

(2) The map $j_U : V \to \text{Ult}_U(V)$ defined by $j_U(a) = [c_a]_U$, for every set a, is an elementary embedding.

Definition 2.1.6 (Critical point). Let $j : \mathcal{N} \to \mathcal{M}$ be an elementary embedding, where $\mathcal{N}$ and $\mathcal{M}$ are transitive calsses and j is definable in $\mathcal{N}$ by a formula of set theory with parameters in $\mathcal{N}$. If $j(\alpha) = \alpha$ for all $\alpha < \kappa$ and $\kappa < j(\kappa)$, then κ is said to be the *critical point* of j.

We have mentioned above that the use of ultraproducts was central in the development of the theory of large cardinals, just to name one application, Dana Scott took the ultrapower of V by a κ-complete ultrafilter U, where κ is a measurable cardinal, to get an elementary embedding $j : V \to \text{Ult}_U(V)$, for which κ is the critical point of j. The converse also holds, namely if $j : V \to \mathcal{M}$ is an elementary embedding for some transitive class $\mathcal{M}$ with critical point κ, then κ is a measurable cardinal (see [36]). This result provided a new characterization of measurable cardinals, and it was the seed of new characterizations and definitions of many large cardinal properties in terms of elementary embeddings of the universe.

Finally, some observations concerning the behaviour of the canonical elementary embedding at the ordinals for a well-founded $\text{Ult}_U(V)$:

Proposition 2.1.7. *Let U be a σ-complete ultrafilter on a set S and let $j_U : V \to \text{Ult}_U(V)$ be the canonical elementary embedding.*

(1) *If α is an ordinal, then $j_U(\alpha)$ is an ordinal.*

(2) *If $\alpha < \beta$, then $j_U(\alpha) < j_U(\beta)$.*

(3) *$\alpha \leq j_U(\alpha)$, for every ordinal α.*

(4) *$j_U(\alpha + 1) = j_U(\alpha) + 1$, for every ordinal α.*

(5) *$j_U(n) = n$, for all natural numbers n, and thus $j_U(\omega) = \omega$.*

(6) *If* $[f]_U < \omega$, *then* $f(x) < \omega$ *for almost all* $x \in S$. *Moreover, by* σ*-completeness, there is some* $n < \omega$ *such that* $f(x) = n$ *for almost all* $x \in S$.

(7) *If* U *is* κ*-complete, then* $j_U(\alpha) = \alpha$ *for every ordinal* $\alpha < \kappa$.

In this presentation of the ultrapower method it was mandatory to adopt a precise notation for the model-theoretic notions. From now on, this degree of precision is no longer needed, hence we won't make any distinction between a model and its universe. That is, we will use M (or N in some cases) both for refering to a model and its universe.

2.2 Silver's Theorem

Easton's result [17] had given set theorists in the early 70's the wrong intuition that, as happens with regular cardinals, there were no deep theorems in ZFC about the arithmetic of singular cardinals. Easton had shown that the behaviour of the power-set function at regular cardinals was almost arbitrary, and set theorists of that time thought that the result could be generalized to singular cardinals. This position changed drastically when Silver published his result.

Silver's Theorem (1974), that originally appeared in [71], is the statement that the Generalized Continuum Hypothesis (GCH) can't fail for the first time at a singular cardinal of uncountable cofinality. The beginning of modern cardinal arithmetic can certainly be dated to the publication of Silver's result, which revealed that there was actually a theory of singular cardinals to be discovered.

It is worth noting that Silver's proof was inspired by a result of Magidor, who in 1974 was very interested in ultrapowers of the universe of sets, and was trying to establish that non-regular ultrafilters on ω_1 (an ultrafilter U is (μ, κ)-regular if there is a family of κ members of U such that the intersection of any μ-many members of the family is empty) could not exist. Rather than disproving the existence of such ultrafilters, what he proved was that if there is a regular, nonuniform ultrafilter on ω_1 and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$. By modifying Magidor's proof, Silver was able to get in ZFC his famous theorem.

Lemma 2.2.1. *Let* $(P, <)$ *be a linear ordering and let* κ *be a cardinal. If every initial segment of* P *has cardinality* $< \kappa$, *then* $|P| \leq \kappa$.

Proof. Denote by $P_q := \{p \in P : p < q\}$, for every $q \in P$, all the initial segments of P, and suppose that they have cardinality $< \kappa$. Since P is linearly ordered, we can assume that $\langle P_q : q \in P\rangle$ is a $\subsetneq$-increasing sequence. Hence, $|P| = sup_{q\in P}|P_q| \leq \kappa$. ■

Theorem 2.2.2 (Silver). *If* κ *is a singular cardinal of uncountable cofinality and the set* $\{\nu < \kappa : 2^\nu = \nu^+\}$ *is a stationary subset of* κ, *then* $2^\kappa = \kappa^+$.

The principal idea behind Silver's original proof is to work inside an extension $M[G]$, obtained by the Lévy Collapse, $\mathrm{Col}(\omega, 2^{cf(\kappa)})$, where we construct an ultraproduct $\mathrm{Ult}_D(M)$, in which $P(\kappa)^{\mathrm{Ult}_D(M)}$ is small. Since $\mathrm{Col}(\omega, 2^{cf(\kappa)})$ preserves κ, 2^κ is still small in the ground model.

Proof. Let $T := \{\nu < \kappa : 2^\nu = \nu^+\}$ and let h be a continuous (i.e., $h(\alpha) = sup_{\beta<\alpha}h(\beta)$ for every limit ordinal α), strictly-increasing (i.e., if $\alpha < \beta$, then $h(\alpha) < h(\beta)$) map from $cf(\kappa)$ onto a cofinal subset of κ. It can easily be seen that the set $X = \{\alpha < cf(\kappa) : h(\alpha) \in T\}$

is stationary. Note that for any $C \subseteq cf(\kappa)$ club, $h[C] = \{h(\alpha) : \alpha \in C\}$ is club, as it's clearly unbounded, and the continuity of h makes it closed. Therefore, $h[cf(\kappa)]$ is club, and thus, $h[X] = T \cap h[cf(\kappa)] = \{h(\alpha) : \alpha \in X\}$ is stationary. Hence, for every $C \subseteq cf(\kappa)$ club, $h[X] \cap h[C] \neq \emptyset$, and therefore, $X \cap C \neq \emptyset$, because h is one-to-one.

Now, denote by M the ground model and consider the generic extension $M[G]$ given by the Lévy Collapse $\mathrm{Col}(\omega, 2^{cf(\kappa)})$, in which $2^{cf(\kappa)}$ is countable and all the cardinals above $2^{cf(\kappa)}$ are preserved. From now on we will work inside the generic extension $M[G]$, where we will construct an ultraproduct of the ground model M in which 2^κ is small.

First note that $U = cf(\kappa)^{cf(\kappa)} \cap M$ is countable in the generic extension, so we can consider an enumeration $\{f_i : i < \omega\}$ of the regressive functions of U. Now, we construct inductively a sequence $X_0 \supseteq X_1 \supset X_2 \supseteq \dots$ of stationary subsets of $cf(\kappa)$, for which f_{i-1} is constant on X_i, for every natural $i > 0$. Let $X_0 = X$. Assume that we have built X_i for some $i < \omega$. Since f_i is regressive and X_i is stationary, by Fodor's Lemma there is some $X_{i+1} \subseteq X_i$ stationary, on which f_i is constant. Let

$$D = \{B \in M : B \subseteq cf(\kappa) \text{ and } X_i \subseteq B, \text{ some } i < \omega\}.$$

Claim: D is an utlrafilter on $P(cf(\kappa))^M$.

Proof: Checking that D is a filter is almost straightforward. Let $B_1, B_2 \in D$. Then, there are $i, j < \omega$ such that $X_i \subseteq B_1$ and $X_j \subseteq B_2$, and since $\langle X_i : i < \omega\rangle$ is a $\subseteq$-descending sequence, either $X_i \subseteq B_1 \cap B_2$ or $X_j \subseteq B_1 \cap B_2$. Now, let $B_1 \in D$ and suppose that $B_1 \subseteq B_2$. Then, there is $X_i \subseteq B_1 \subseteq B_2$.

To show that it is an ultrafilter, consider $B \in P(cf(\kappa))^M$ such that $B \notin D$. Then, $X_i \nsubseteq B$, for every $i < \omega$. Define the function g on $cf(\kappa)$ by

$$g(\alpha) = \begin{cases} f_i(\alpha), & \text{if } \alpha \in X_0 \cap B, \\ 0, & \text{otherwise.} \end{cases}$$

Note that g is clearly regressive, so there is some $j < \omega$ such that $g = f_j$. By definition f_j is constant on X_{j+1}, so there is some $\beta < cf(\kappa)$ for which $f_j(\alpha) = \beta$, for every $\alpha \in X_{j+1}$. Hence, two possible cases arise:

(i) If $\beta = 0$, then for every $\alpha \in X_{j+1}$, $g(\alpha) = f_j(\alpha) = 0$, so by the definition of g, we get that $\alpha \notin X_0 \cap B$. Hence, as $X_j \subseteq X_0$, this means that $X_j \cap B = \emptyset$, and therefore, that $X_j \subseteq B^c$. So, the complement of B is in D.

(ii) If $\beta \neq 0$, then for every $\alpha \in X_{j+1}$, we have that $g(\alpha) = f_j(\alpha) \neq 0$, so $\alpha \in X_0 \cap B$, and thus $X_{j+1} \subseteq X_0 \cap B$, but this contradicts the fact that no X_i is totally contained in B. Hence making this case impossible.

□

Now that we have proven that D is an ultrafilter, we are ready to define the ultraproduct $\mathrm{Ult}_D(M)$, with $j : M \to \mathrm{Ult}_D(M)$ its canonical elementary embedding.

Our objective is proving that $2^\kappa = \kappa^+$ holds inside the ultraproduct. But D is not σ-complete, so $\mathrm{Ult}_D(M)$ need not be well-founded. Therefore, we have to find an element in $\mathrm{Ult}_D(M)$ that has cardinality κ, so that we have an object that acts just as κ inside the ultraproduct. This object will be $j(h)([d])$, where d is the diagonal function on $cf(\kappa)$, defined by $d(\alpha) = \alpha$. Our goal is proving that $j(h)([d])$ has exactly κ-many $\in^{\mathrm{Ult}}$-predecessors, so that $\kappa^{\mathrm{Ult}_D(M)} = j(h)([d])$.

The first step towards this purpose will be showing that $\{j(\alpha) : \alpha < cf(\kappa)\}$ is exactly the set of $\in^{\mathrm{Ult}}$-predecessors of $[d]$, denoted by $ext([d])$. Łoś's Theorem tells us that $[f] \in \mathrm{Ult}_D(M)$ is an $\in^{\mathrm{Ult}}$-predecessor of $[d]$ if and only if

$$\{\alpha < cf(\kappa) : f(\alpha) \in \alpha\} \in D.$$

That is, $[f]$ is an $\in^{\mathrm{Ult}}$-predecessor of $[d]$ if and only if f is regressive at D-many places. Changing every $\in^{\mathrm{Ult}}$-predecessor of $[d]$ on a null set, we get that the set of $\in^{\mathrm{Ult}}$-predecessors of $[d]$ is exactly the set of classes of $\mathrm{Ult}_D(M)$ represented by some regressive function, i.e.,

$$ext([d]) = \{[f] \in \mathrm{Ult}_D(M) : f \text{ regressive}\}$$

<u>Claim:</u> $\{[f] \in \mathrm{Ult}_D(M) : f \text{ regressive}\} = \{j(\alpha) : \alpha < cf(\kappa)\}$.

<u>Proof:</u> An easy remark that will be useful is that for every $\beta < cf(\kappa)$,

$$(\beta, cf(\kappa)) := \{\alpha < cf(\kappa) : \beta < \alpha\} \in D.$$

Indeed, since every X_i is stationary, it is unbounded in $cf(\kappa)$, and thus, $X_i \nsubseteq (\beta, cf(\kappa))^c$, for every $i < \omega$. Now we are ready to prove the equality:

$\supseteq$) Let $\beta < cf(\kappa)$. The above result tells us that

$$(\beta, cf(\kappa)) = \{\alpha < cf(\kappa) : c_\beta(\alpha) \in \alpha\} \in D.$$

Therefore, c_β is regressive at D-many places, so there is some f regressive such that $j(\beta) = [c_\beta] = [f]$.

$\subseteq$) Let f be a regressive function on $cf(\kappa)$. Then, $f = f_i$ for some $i < \omega$. So, by definition, there is some $\beta < cf(\kappa)$ such that $f(\alpha) = \beta$, for every $\alpha \in X_{i+1}$. Hence, as $X_{i+1} \in D$,

$$X_{i+1} \subseteq \{\alpha < cf(\kappa) : f(\alpha) = \beta\} \in D.$$

Hence, $[f] = [c_\beta] = j(\beta)$.

$\square$

This proves that $ext([d])$, the set of $\in^{\mathrm{Ult}}$-predecessors of $[d]$, is exactly $\{j(\alpha) : \alpha < cf(\kappa)\}$.

Recall that we have defined h as a continuous strictly increasing function from $cf(\kappa)$ onto a cofinal subset of κ. So by elementarity, we have that $j(h)$ is a continuous strictly increasing function in $\mathrm{Ult}_D(M)$, for which for every $\alpha < cf(\kappa)$,

$$\mathrm{Ult}_D(M) \models j(h)(j(\alpha)) = j(h(\alpha)) < j(\kappa).$$

<u>Claim:</u> $[f] \in \mathrm{Ult}_D(M)$ is a $\in^{\mathrm{Ult}}$-predecessor of $j(h)([d])$ if and only if it is a $\in^{\mathrm{Ult}}$-predecessor of $j(h)(j(\alpha))$, for some $\alpha < cf(\kappa)$.

<u>Proof:</u> An immediate consequence of Łoś's Theorem is that $[d]$ is an ordinal in the ultraproduct. To check that it is a limit ordinal, assume towards a contradiction that

$$succ(cf(\kappa)) := \{\alpha < cf(\kappa) : \alpha \text{ successor}\} \in D.$$

Hence, $X_i \subseteq succ(cf(\kappa))$ for some $i < \omega$, i.e., all the ordinals in X_i are successors. But this is contradictory with the fact that X_i is stationary, as it has to intersect every club subset of $cf(\kappa)$, in particular, the set of all limit ordinals of $cf(\kappa)$.

Now, since $\{j(\alpha) : \alpha < cf(\kappa)\}$ is the set of $\in^{\text{Ult}}$-predecessors of $[d]$, a limit ordinal, by the continuity of $j(h)$ we have that

$$\text{Ult}_D(M) \models j(h)([d]) = sup\{j(h)(j(\alpha)) : \alpha < cf(\kappa)\}.$$

Therefore, $ext(j(h)([d])) = \bigcup_{\alpha<cf(\kappa)} ext(j(h(\alpha)))$.

□

Now we are ready to prove that $\kappa^{\text{Ult}_D(M)} = j(h)([d])$, as we have anticipated above. From now on, we might indistinctly use both the notations κ_α and $h(\alpha)$, for every $\alpha < cf(\kappa)$.

Claim: $j(h)([d])$ has exactly κ-many $\in^{\text{Ult}}$-predecessors.

Proof: First note that having an unbounded set of cardinals that satisfy the GCH below κ makes it a strong limit in M: For every $\lambda < \kappa$, there is some $\nu \in T$ such that $\nu \geq \lambda$, and thus $2^\lambda \leq 2^\nu = \nu^+ < \kappa$. As a consequence, for every $\alpha < cf(\kappa)$,

(i) $cf(\kappa) < \kappa_\alpha$ implies $\kappa_\alpha^{cf(\kappa)} \leq \kappa_\alpha^{\kappa_\alpha} = 2^{\kappa_\alpha} < \kappa$,

(ii) $\kappa_\alpha \leq cf(\kappa)$ implies $\kappa_\alpha^{cf(\kappa)} = 2^{cf(\kappa)} < \kappa$.

Hence, for every $\alpha < cf(\kappa)$,

$$\kappa_\alpha^{cf(\kappa)} < \kappa. \tag{2.1}$$

On the other hand, note that $[f] \in \text{Ult}_D(M)$ is a $\in^{\text{Ult}}$-predecessor of $j(h(\alpha))$ if and only if

$$\{\beta < cf(\kappa) : f(\beta) < \kappa_\alpha\} \in D.$$

So we can assume that $[f]$ is a $\in^{\text{Ult}}$-predecessor of $j(h(\alpha))$ if and only if $f \in {}^{cf(\kappa)}\kappa_\alpha$ (by changing f on a null set if necessary). Therefore, by (2.1), $j(h(\alpha))$ has less than κ-many $\in^{\text{Ult}}$-predecessors, for every $\alpha < cf(\kappa)$.

Furthermore, note that for a given $\alpha < cf(\kappa)$, every $j(\beta)$ is a $\in^{\text{Ult}}$-predecessor of $j(h(\alpha))$, $\beta < \kappa_\alpha$, because

$$\{\gamma < cf(\kappa) : c_\beta(\gamma) = \beta < \kappa_\alpha = c_{\kappa_\alpha}(\gamma)\} \in D.$$

So for any $\alpha < cf(\kappa)$, we have that $\kappa_\alpha \leq |ext(j(h(\alpha)))| < \kappa$. Hence, it follows from this inequality, the last claim, and the fact that $\langle \kappa_\alpha : \alpha < cf(\kappa) \rangle$ is a cofinal sequence on κ, that

$$|ext(j(h)([d]))| = \kappa.$$

□

Now that we have proven that $j(h)([d]) = \kappa^{\text{Ult}_D(M)}$, we are ready to enter the last stage of the proof, divided into three claims, where we show that $P(\kappa)$ is small in the ultraproduct, that is, $\text{Ult}_D(M) \models |P(\kappa)| \leq \kappa^+$. We denote $P(\kappa)^{\text{Ult}_D(M)} = P(j(h)([d]))$ by

$$Q := \{Z : \text{Ult}_D(M) \models Z \subseteq j(h)([d])\}.$$

First we claim that there is a 1-1 map from Q into $ext\big(2^{j(h)([d])}\big)$. Since $ext\big(2^{j(h)([d])}\big)$ is the set of functions from $j(h)([d])$ into 2, we can define the map that sends every $Z \in Q$ to its characteristic function (that belongs to $2^{j(h)([d])}$), which is clearly 1-1.

Next, we claim that $\mathrm{Ult}_D(M) \models ext\big(2^{j(h)([d])}\big) = ext\big(j(h)([d])^+\big)$. But since

$$X = \{\alpha < cf(\kappa) : 2^{h(\alpha)} = h(\alpha)^+\} \in D,$$

by elementarity and Łoś's Theorem,

$$\mathrm{Ult}_D(M) \models 2^{j(h)([d])} = j(h)([d])^+,$$

which implies the claim.

Finally, we claim that there is a 1-1 map from $ext\big(j(h)([d])^+\big)$ into κ^+. The first thing to note is that

$$ext\big(j(h)([d])^+\big) = ext(j(h)([d])) \cup \{j(h)([d])\}.$$

We proved that $j(h)([d])$ had exactly κ-many predecessors, so every element of $ext\big(j(h)([d])^+\big)$ has at most κ-many predecessors. By lemma 2.2.1, once we show that $\big(ext\big(j(h)([d])^+\big), \in^{\mathrm{Ult}}\big)$ is a linear ordering, we will get that $\mathrm{Ult}_D(M) \models \big|ext\big(j(h)([d])^+\big)\big| \leq \kappa^+$ for free. To do so, consider $[f], [g] \in \big(ext(j(h)([d])^+\big)$ such that $[f] \neq [g]$. Then, two possible cases arise:

(i) $[f]$ and $[g]$ are predecessors of $j(h)([d])$. Then $[f]$ and $[g]$ are predecessors of $j(h(\alpha))$ and $j(h(\beta))$, respectively, for some $\alpha, \beta < cf(\kappa)$. So by Łoś's Theorem,

$$\{\gamma < cf(\kappa) : f(\gamma) < \kappa_\alpha \text{ and } g(\gamma) < \kappa_\beta\} \in D.$$

We can assume that f and g are ordinal functions, by changing them on a null set if necessary, and thus for every $\gamma < cf(\kappa)$, one of the following three holds: $f(\gamma) < g(\gamma)$, $f(\gamma) = g(\gamma)$, or $g(\gamma) < f(\gamma)$. But as $[f]$ and $[g]$ were assumed to be different, and D is an ultrafilter, either

$$\{\gamma < cf(\kappa) : f(\gamma) < g(\gamma)\} \in D,$$

or

$$\{\gamma < cf(\kappa) : g(\gamma) < f(\gamma)\} \in D.$$

Therefore, either $[f] \in^{\mathrm{Ult}} [g]$, or $[g] \in^{\mathrm{Ult}} [f]$.

(ii) $[f]$ is a predecessor of $j(h)([d])$ and $[g] = j(h)([d])$. Then it's clear that $[f] \in^{\mathrm{Ult}} [g]$.

Therefore, $\big(ext\big(j(h)([d])^+\big), \in^{\mathrm{Ult}}\big)$ is a linear ordering as we wanted.

As a summary of the last three claims that we have just proven, we have obtained the following results:

- $\mathrm{Ult}_D(M) \models |P(\kappa)| = |Q| \leq |ext\big(2^{j(h)([d])}\big)|$
- $\mathrm{Ult}_D(M) \models ext\big(2^{j(h)([d])}\big) = ext\big(j(h)([d])^+\big)$.
- $\mathrm{Ult}_D(M) \models |ext\big(j(h)([d])^+\big)| \leq \kappa^+$

Therefore, $\mathrm{Ult}_D(M) \models |P(\kappa)| \leq \kappa^+$.

Now, we finish the argument by showing that if $2^\kappa = \kappa^+$ fails in M, then it holds that $\mathrm{Ult}_D(M) \models \kappa^{++} \leq |P(\kappa)|$, contradicting the above result. We saw in theorem 1.7.3 that the Lévy Collapse $\mathrm{Col}(\omega, 2^{cf(\kappa)})$ is $(2^{cf(\kappa)})^+$-cc, so all the cardinals above $2^{cf(\kappa)}$, and in particular

κ^{++}, are preserved. Hence, if $2^\kappa = \kappa^+$ fails in M, it holds that $\kappa^{++} \leq |P(\kappa)|$, so there exists a 1-1 map from κ^{++} into $P(\kappa)$. Let $\langle C_\alpha : \alpha < \kappa^{++}\rangle$ be the 1-1 sequence of subsets of κ given by this map.

Define in the ultrapower $\text{Ult}_D(M)$ the map F from κ^{++} into Q, by $F(\alpha) = B_\alpha$, where

$$\text{Ult}_D(M) \models B_\alpha = j(h)([d]) \cap j(C_\alpha).$$

Let $\alpha, \beta \in \kappa^{++}$ be different ordinals. Then $C_\alpha \neq C_\beta$, and thus either $C_\alpha \setminus C_\beta \neq \emptyset$, or $C_\beta \setminus C_\alpha \neq \emptyset$. Assume, without loss of generality, that the first case holds (the argument is the same for both cases), and let $\gamma \in C_\alpha \setminus C_\beta$. Hence, by elementarity,

$$\text{Ult}_D(M) \models j(\gamma) \in^{\text{Ult}} j(C_\alpha) \wedge j(\gamma) \notin^{\text{Ult}} j(C_\beta).$$

Recall that $\langle \kappa_\alpha : \alpha < cf(\kappa)\rangle$ was a cofinal sequence in κ, so as $\gamma \in C_\alpha \subseteq \kappa$, there is some $\delta < cf(\kappa)$ for which $\gamma < \kappa_\delta = h(\delta)$. So, again by elementarity,

$$\text{Ult}_D(M) \models j(\gamma) \in^{\text{Ult}} j(h(\delta)).$$

Therefore, $j(\gamma)$ is a $\in^{\text{Ult}}$-predecessor of $j(h)([d])$, and thus

$$\text{Ult}_D(M) \models j(\gamma) \in^{\text{Ult}} \big(j(C_\alpha) \cap j(h)([d])\big) \wedge j(\gamma) \notin^{\text{Ult}} \big(j(C_\beta) \cap j(h)([d])\big).$$

Or, equivalently, $j(\gamma) \in^{\text{Ult}} B_\alpha \setminus B_\beta$, which implies that $B_\alpha \neq B_\beta$. Hence, F is a 1-1 map from κ^{++} into Q. But we showed that there was a 1-1 map from Q into κ^+, and as κ^+ and κ^{++} are preserved in the ultrapower, we get a contradiction.

■

The following corollary is a very useful consequence of Silver's Theorem.

Corollary 2.2.3. *If the SCH holds for all singular cardinals of countable cofinality, then it holds for all singular cardinals.*

2.3 Solovay's Theorem

Solovay's Theorem, that originally appeared in [72] around the same time as Silver's Theorem, is the statement that above the first strongly compact cardinal, the SCH holds for any singular cardinal.

We begin by defining strongly compact cardinals and stating the most basic properties of this large cardinal notion. Introduced by Keisler and Tarski in [40], strong compactness has its origins in infinitary logics, appearing as a natural generalization of the compactness of first order logic.

Let $\kappa \geq \lambda$ be any cardinals. $L_{\kappa,\lambda}$ is the language consisting of κ-many variables, and whose formulas are constructed from the atomic formulas by taking conjunctions and disjunctions of $< \kappa$-many formulas, or by quantifying (universally or existentially) $< \lambda$-many variables, allowing in each case $< \lambda$-many free variables.

Keisler and Tarski defined a *strongly compact* cardinal as a κ for which $L_{\kappa,\omega}$ satisfies κ-compactness: If for every set of formulas Σ, every $\Gamma \subseteq [\Sigma]^{<\kappa}$ is satisfiable, then Σ is satisfiable as well. However, we are more interested in the following equivalent formulation of strong compactness:

Definition 2.3.1 (Strongly compact cardinal). An uncountable regular cardinal κ is *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

If κ is an infinite cardinal and A a set with $|A| \geq \kappa$, recall that we denote by $P_\kappa(A)$ the set of all subsets of A of cardinality $< \kappa$.

Definition 2.3.2 (Fine measure). Let κ be a cardinal and A a set such that $|A| \geq \kappa$. For every $x \in P_\kappa(A)$, let

$$\hat{x} := \{y \in P_\kappa(A) : x \subseteq y\}$$

and consider the filter on $P_\kappa(A)$ generated by the sets $\hat{x}$, for all $x \in P_\kappa(A)$. That is, the filter

$$F := \{X \subseteq P_\kappa(A) : \hat{x} \subseteq X, \text{ some } x \in P_\kappa(A)\}.$$

If κ is a regular cardinal, then F is κ-complete. We call U a *fine measure* on $P_\kappa(A)$ if U is a κ-complete ultrafilter on $P_\kappa(A)$ that extends the filter F; i.e, $\hat{x} \in U$, for all $x \in P_\kappa(A)$.

Lemma 2.3.3. *The following are equivalent for any regular cardinal κ:*

(1) *κ is a strongly compact cardinal.*

(2) *For any set A such that $|A| \geq \kappa$, there exists a fine measure on $P_\kappa(A)$.*

Proposition 2.3.4. *If κ is a strongly compact cardinal, κ is measurable.*

The idea behind the proof of Solovay's Theorem, splitted into the next three lemmas, is to show that for every regular λ above a strongly compact cardinal κ, $\lambda^{<\kappa} = \lambda$. Then, with a very simple argument, and using the corollary 2.2.3, we get the desired result. For this purpose we construct an ultrapower $\mathrm{Ult}_D(V)$ of the universe, with respect to an ultrafilter D on λ, and we use it to build a collection $\{M_\alpha : \alpha < \lambda\}$ of sets in $P_\kappa(\lambda)$, whose power sets cover the whole $P_\kappa(\lambda)$.

Lemma 2.3.5. *If κ is a strongly compact cardinal and $\lambda > \kappa$ is a regular cardinal, then there exists a κ-complete nonprincipal uniform ultrafilter D on λ with the property that almost all* $(\text{mod } D)$ *ordinals $\alpha < \lambda$ have cofinality less than κ.*

Proof. As κ is strongly compact, by lemma 2.3.3 there exists a fine measure U on $P_\kappa(\lambda)$, which satisfies that every $\gamma < \lambda$ belongs to almost all (mod U) $x \in P_\kappa(\lambda)$. Indeed, it's clear that $\{\gamma\} \in P_\kappa(\lambda)$ for every $\gamma < \lambda$, so

$$\hat{\{\gamma\}} = \{x \in P_\kappa(\lambda) : \{\gamma\} \subseteq x\} = \{x \in P_\kappa(\lambda) : \gamma \in x\} \in U.$$

Let us consider the ultrapower $\mathrm{Ult}_U(V)$, let $j_U : V \to \mathrm{Ult}_U(V)$ be the corresponding ultrapower embedding, and let f be the least ordinal function in $\mathrm{Ult}_U(V)$ such that $j_U(\gamma) = [c_\gamma]_U < [f]_U$, for all $\gamma < \lambda$. Then,

$$[f]_U = \lim_{\gamma \to \lambda} j_U(\gamma).$$

<u>Claim:</u> $f(x) < \lambda$ for almost all $x \in P_\kappa(\lambda)$.

<u>Proof:</u> Define the function

$$\begin{aligned} g : P_\kappa(\lambda) &\longrightarrow \lambda \\ x &\longmapsto \sup(x) \end{aligned}$$

We will prove that $[f]_U \leq [g]_U < j_U(\lambda)$, which by Łoś's Theorem is equivalent to the statement of the claim. First note that given $\gamma < \lambda$, for any $x \in P_\kappa(\lambda)$, if $\gamma \in x$, then $\gamma \leq \sup(x) = g(x)$. Therefore,

$$\hat{\{\gamma\}} = \{x \in P_\kappa(\lambda) : \gamma \in x\} \subseteq \{x \in P_\kappa(\lambda) : \gamma \leq g(x)\},$$

and as $\hat{\{\gamma\}} \in U$, we have that $\{x \in P_\kappa(\lambda) : \gamma \leq g(x)\} \in U$. Hence, for all $\gamma < \lambda$, $j_U(\gamma) \leq [g]_U$, and thus $[f]_U \leq [g]_U$.

Now note that for every $x \in P_\kappa(\lambda)$, as λ is regular, $g(x) < \lambda$. Therefore

$$\{x \in P_\kappa(\lambda) : g(x) < \lambda\} = P_\kappa(\lambda) \in U,$$

and thus $[g]_U < j_U(\lambda)$.

□

Now we are ready to define the ultrafilter D on λ by:

$$X \in D \text{ if and only if } f^{-1}(X) \in U.$$

It is straightforward to check that it is an ultrafilter. To see that it is κ-complete, let $\mu < \kappa$ and let $\langle X_\alpha : \alpha < \mu \rangle$ be a family of elements of D. By definition of D we have that for every $\alpha < \mu$, $\{x \in P_\kappa(\lambda) : f(x) \in X_\alpha\} \in U$, and as U is κ-complete (because κ is regular),

$$\bigcap_{\alpha<\mu} \{x \in P_\kappa(\lambda) : f(x) \in X_\alpha\} = \{x \in P_\kappa(\lambda) : f(x) \in \bigcap_{\alpha<\mu} X_\alpha\} \in U.$$

Therefore $f^{-1}(\bigcap_{\alpha<\mu} X_\alpha) \in U$, and hence, $\bigcap_{\alpha<\mu} X_\alpha \in D$.

To prove that D is nonprincipal suppose, aiming for a contradiction, that there is some $\alpha_0 < \lambda$ for which $D = \{X \subseteq \lambda : \alpha_0 \in X\}$. Then $\{\alpha_0\} \in D$, so $\{x \in P_\kappa(\lambda) : f(x) = \alpha_0\} \in U$. But f is greater than c_{α_0} in $\mathrm{Ult}_U(V)$ by definition, hence

$$\{x \in P_\kappa(\lambda) : \alpha_0 < f(x)\} \in U,$$

a contradiction.

Claim: D is uniform.

Proof: First note that the diagonal function d on λ, defined by $d(\alpha) = \alpha$, dominates the constant functions on λ at almost every $\alpha < \lambda$ (mod D): By definition of f we have that $[c_\gamma]_U < [f]_U$ for every $\gamma < \lambda$, that is, $\{x \in P_\kappa(\lambda) : \gamma < f(x)\} \in U$, but note that

$$\{x \in P_\kappa(\lambda) : \gamma < f(x)\} = \{x \in P_\kappa(\lambda) : f(x) \in \{\alpha < \lambda : \gamma < \alpha\}\},$$

so by definition of D, $\{\alpha < \lambda : c_\gamma(\alpha) = \gamma < \alpha = d(\alpha)\} \in D$. Consequently, for every $\gamma < \lambda$, $\{\alpha < \lambda : \alpha \leq \gamma\} \notin D$, that is, the bounded sets are not in D. Therefore, as λ is regular, any subset of λ with cardinality $< \lambda$ is bounded, and thus, any set in D must have cardinality λ. So D is uniform.

□

Now we just need to check that almost all (mod D) ordinals $\alpha < \lambda$ have cofinality less than κ, i.e., that $\{\alpha < \lambda : cf(\alpha) < \kappa\} \in D$. By definition of the ultrafilter D, this is equivalent to show that

$$\{x \in P_\kappa(\lambda) : f(x) \in \{\alpha < \lambda : cf(\alpha) < \kappa\}\} = \{x \in P_\kappa(\lambda) : cf(f(x)) < \kappa\} \in U.$$

Which will follow immediately once we show that for almost all x (mod U), it holds that $f(x) = \sup\{\alpha \in x : \alpha < f(x)\}$, i.e., that $f(x)$ has a cofinal sequence of length $< \kappa$, for almost all (mod U) $x \in P_\kappa(\lambda)$. Let's check this fact:

$\geq$) Clear for every $x \in P_\kappa(\lambda)$.

$\leq$) Let $h(x) := \sup\{\alpha \in x : \alpha < f(x)\}$. Recall that we showed in the beginning that every $\gamma < \lambda$ belongs to almost all (mod U) $x \in P_\kappa(\lambda)$, and that f is defined as the least ordinal function greater than all the constant functions (mod U), hence

$$\{x \in P_\kappa(\lambda) : \gamma \in x\} \cap \{x \in P_\kappa(\lambda) : \gamma < f(x)\} \in U.$$

Note that given $x \in P_\kappa(\lambda)$, if $\gamma \in x$ and $\gamma < f(x)$, then $\gamma \in \{\alpha \in x : \alpha < f(x)\}$, and thus, $\gamma \leq h(x)$. Therefore, $\{x \in P_\kappa(\lambda) : \gamma \leq h(x)\} \in U$, which by Łoś's Theorem 2.1.2 is equivalent to $j_U(\gamma) \leq [h]_U$, for every $\gamma < \lambda$. So by definition of f,

$$[f]_U = lim_{\gamma \to \lambda} j_U(\gamma) \leq [h]_U,$$

and thus $f(x) \leq h(x)$ for almost all (mod U) $x \in P_\kappa(\lambda)$.

■

Lemma 2.3.6. *If κ is a strongly compact cardinal and $\lambda > \kappa$ is a regular cardinal, then there exists a κ-complete nonprincipal ultrafilter D on λ and a collection $\{M_\alpha : \alpha < \lambda\}$ such that*

(1) *$|M_\alpha| < \kappa$ for all $\alpha < \lambda$,*

(2) *for every $\gamma < \lambda$, γ belongs to M_α for almost all α (mod D).*

Proof. Consider the ultrafilter D on λ defined in the proof of the last lemma. Recall that U was a fine measure on $P_\kappa(\lambda)$, and if f is the least ordinal function in $\mathrm{Ult}_U(V)$ greater than all the constant functions, then D was defined as

$$X \in D \text{ if and only if } f^{-1}(X) \in U.$$

Let us consider the ultrapower $\mathrm{Ult}_D(V)$, and let $j_D : V \to \mathrm{Ult}_D(V)$ be the canonical elementary embedding.

It follows immediately from the fact that the diagonal function d is greater in $\mathrm{Ult}_D(V)$ than all the constant functions (shown in the proof of the last lemma), that $[d]_D = \lim_{\gamma \to \lambda} j_D(\gamma)$. Moreover, since almost all (mod D) ordinals $\alpha < \lambda$ have cofinality less than κ, there exist $A_\alpha \subseteq \alpha$ cofinal in α, with $|A_\alpha| < \kappa$, for almost all $\alpha < \lambda$ (mod D). Now, consider the function $\langle A_\alpha : \alpha < \lambda \rangle$, where $A_\alpha = \emptyset$, if $cf(\alpha) \geq \kappa$, and let A be the set of ordinals represented in Ult_D by this function. It follows immediately from the definition of $\langle A_\alpha : \alpha < \lambda \rangle$ and from Łoś's Theorem that in the ultrapower A is cofinal in $[d]_D$. More precisely, $\mathrm{Ult}_D(V)$ satisfies that for every ordinal $[f]_D$ smaller than $[d]_D$, there exists $[g]_D \in A$ such that $[f]_D \leq [g]_D$.

We claim that from this fact follows that for every $\alpha < \lambda$, there is $\beta > \alpha$ in λ such that the interval

$$\{[f]_D : j_D(\alpha) \leq [f]_D < j_D(\beta)\},$$

which we denote by $[\alpha, \beta)_D$, has nonempty intersection with A. This amounts to show that for every $\alpha < \lambda$, there is $\beta > \alpha$ in λ and $[g]_D \in \mathrm{Ult}_D(V)$ such that

$$\{\gamma < \lambda : g(\gamma) \in A_\gamma \text{ and } \alpha \leq g(\gamma) < \beta\} \in D.$$

Since A is cofinal in $[d]_D$ and for every $\alpha < \lambda$, $j_D(\alpha) < [d]_D$, there exists $[g]_D \in A$ such that $j_D(\alpha) \leq [g]_D$, which by Łoś's Theorem translates to

$$\{\gamma < \lambda : g(\gamma) \in A_\gamma \text{ and } \alpha \leq g(\gamma)\} \in D. \tag{2.2}$$

Recall that $A_\gamma \subseteq \gamma$ for almost all $\gamma < \lambda$ (mod D), so as $[g]_D \in A$, we have that

$$\{\gamma < \lambda : g(\gamma) < \gamma\} \in D,$$

and thus, that $[g]_D < [d]_D$. Therefore, as $[d]_D = \lim_{\gamma \to \lambda} j_D(\gamma)$, there is some $\beta < \lambda$ such that $[g]_D < j_D(\beta)$, so $\{\gamma < \lambda : g(\gamma) < \beta\} \in D$. If we intersect this last set with (2.2), we get that

$$\{\gamma < \lambda : g(\gamma) \in A_\gamma \text{ and } \alpha \leq g(\gamma) < \beta\} \in D.$$

Hence, as we claimed above, for every $\alpha < \lambda$, there is $\beta > \alpha$ in λ such that $A \cap [\alpha, \beta)_D \neq \emptyset$.

Now we build a sequence $\langle \alpha_\delta : \delta < \lambda \rangle$ of ordinals smaller than λ inductively as follows:

- Let $\alpha_0 = 0$.
- Having built α_δ for $\delta < \lambda$, we know from the argument above that $A \cap \ [\alpha_\delta, \beta)_D \neq \emptyset$ for some $\beta > \alpha_\delta$ smaller than λ. Let $\alpha_{\delta+1} := \beta$.
- If $\delta < \lambda$ is a limit ordinal let $\alpha_\delta = \sup_{\beta < \delta} \alpha_\beta$.

If we denote by I_δ the interval $\{\gamma < \lambda : \alpha_\delta \leq \gamma < \alpha_{\delta+1}\}$, it's clear from the way we defined $\alpha_{\delta+1}$ from α_δ that for any $\delta < \lambda$

$$\{\gamma < \lambda : A_\gamma \cap I_\delta \neq \emptyset\} \in D. \tag{2.3}$$

Now for every $\alpha < \lambda$ let

$$M_\alpha := \{\delta < \lambda : I_\delta \cap A_\alpha \neq \emptyset\}.$$

We claim that the collection $\{M_\alpha : \alpha < \lambda\}$ satisfies (1) and (2) of the statement:

(1) It's clear by the way we defined $\langle A_\alpha : \alpha < \lambda \rangle$, that $|A_\alpha| < \kappa$ for every $\alpha < \lambda$. This implies that for every $\alpha < \lambda$, A_α intersects $< \kappa$-many intervals of the form I_δ, because these intervals are mutually disjoint. Therefore, $|M_\alpha| < \kappa$, for every $\alpha < \lambda$.

(2) We have to check that for every $\gamma < \lambda$, $\{\alpha < \lambda : \gamma \in M_\alpha\} \in D$. This set is exactly

$$\{\alpha < \lambda : A_\alpha \cap I_\gamma \neq \emptyset\},$$

which was proven to belong to D in (2.3).

■

Lemma 2.3.7. *If κ is strongly compact and $\lambda > \kappa$ is a regular cardinal, then there exists a collection $\{M_\alpha : \alpha < \lambda\} \subseteq P_\kappa(\lambda)$ such that*

$$P_\kappa(\lambda) = \bigcup_{\alpha < \lambda} P(M_\alpha).$$

Consequently, $\lambda^{<\kappa} = \lambda$.

Proof. Consider the collection $\{M_\alpha : \alpha < \lambda\}$ given in the proof of the last lemma, and let D be the ultrafilter on λ that we defined in the proof of lemma 2.3.5.

To prove the left-to-right inclusion let $x \in P_\kappa(\lambda)$. We claim that $x \subseteq M_\alpha$ for almost all $\alpha < \lambda$ (mod D). For every $\gamma \in x$, by property (2) of the last lemma, $\{\alpha < \lambda : \gamma \in M_\alpha\} \in D$. Hence, as $|x| < k$ and D is κ-complete we have that

$$\{\alpha < \lambda : x \subseteq M_\alpha\} = \bigcap_{\gamma \in x} \{\alpha < \lambda : \gamma \in M_\alpha\} \in D.$$

Therefore, it follows that for every $x \in P_\kappa(\lambda)$, there is some $\alpha < \lambda$ such that $x \subseteq M_\alpha$. Thus,

$$P_\kappa(\lambda) \subseteq \bigcup_{\alpha<\lambda} P(M_\alpha).$$

For the other inclusion, let $x \in P(M_\alpha)$ for any $\alpha < \lambda$. By (1) of the last lemma, $|M_\alpha| < \kappa$, so $|x| < \kappa$, and thus, as $M_\alpha \subseteq \lambda$, we get that $x \in P_\kappa(\lambda)$.

Now we have to check that $\lambda^{<\kappa} = \lambda$. It's clear that $|P_\kappa(\lambda)| = \lambda^{<\kappa}$ and that $\lambda \leq \lambda^{<\kappa}$, which imply that $|\bigcup_{\alpha<\lambda} P(M_\alpha)| \geq \lambda$. This last fact combined with the fact that $\bigcup_{\alpha<\lambda} P(M_\alpha) \subseteq \lambda$ implies that $|\bigcup_{\alpha<\lambda} P(M_\alpha)| = \lambda$, and hence, that

$$\lambda^{<\kappa} = \lambda.$$

■

Theorem 2.3.8 (Solovay). *If κ is a strongly compact cardinal, then the SCH holds above κ. That is, if $\lambda > \kappa$ is a singular cardinal such that $2^{cf(\lambda)} < \lambda$, then $\lambda^{cf(\lambda)} = \lambda^+$.*

Proof. Let $\lambda > \kappa$ be any cardinal. Since λ^+ is always regular, by the last lemma,

$$\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+.$$

In particular, $\lambda^{\aleph_0} \leq \lambda^+$. Hence, for every singular cardinal $\lambda > \kappa$ with $cf(\lambda) = \omega$,

$$\lambda^{cf(\lambda)} = \lambda^{\aleph_0} = \lambda^+.$$

Therefore, by corollary 2.2.3, since the Singular Cardinal Hypothesis holds above κ for all singular cardinals of countable cofinality, then it holds for all singular cardinals above κ.

■

Chapter 3

Ordinal Functions, Scales, and Exact Upper Bounds

In this chapter, based on the presentation of Abraham and Magidor in [1], we develop a general theory of ordinal functions that live in products of sets of ordinals modulo an ideal. That is, we will study reduced products of sets of ordinals by an ideal I, together with the relations $<_I$, $\leq_I$ and $=_I$, which as we will see here below, they are defined akin to the relation $=_F$ seen in the last chapter. In particular, if A is an infinite set of ordinals and I is an ideal over A, we are interested in certain conditions that ensure the existence of exact upper bounds for sequences of ordinal functions on A, increasing in $<_I$.

The approach carried out in this chapter allows us to see the inner workings of cardinal arithmetic, and acts as a bridge between the techniques developed in the second chapter and Shelah's pcf theory, that will be presented in the next chapter. The connection to the last chapter will be made clear with a slight generalization of Silver's Theorem 2.2.2, with a completely different approach than that of the original proof of Silver, and an improvement of Solovay's Theorem 2.3.8, thanks to the introduction of the notion of λ-strongly compact cardinals.

All the theorems and results of the first two sections of this chapter are due to Shelah, unless stated otherwise. They originally appeared in [68].

3.1 Definitions

Let A be an infinite set and let I be an ideal on A. We define relations on OR^A

$$f =_I g \text{ if and only if } \{a \in A : f(a) \neq g(a)\} \in I,$$
$$f \leq_I g \text{ if and only if } \{a \in A : f(a) > g(a)\} \in I,$$
$$f <_I g \text{ if and only if } \{a \in A : f(a) \geq g(a)\} \in I.$$

We might as well use the notations $X \subseteq_I Y$ and $X =_I Y$ for subsets $X, Y \subseteq A$, with the obvious meaning.

If F is a filter on A, then $f <_F g$ means $f <_I g$, where I is the dual ideal of F, and similarly for $f =_F g$ and $f \leq_F g$.

Remark 3.1.1. *The relation $<_I$ is a partial ordering (different from $\leq_I$ unless I is a prime ideal), and if I is σ-complete, then $<_I$ is well-founded.*

Proof. It is straightforward to check that $<_I$ is a partial ordering. So assume that I is a σ-complete ideal on A and suppose towards a contradiction, that $f_{n+1} <_I f_n$ is a $<_I$-descending ω-sequence of ordinal functions on A. For every $n \in \omega$

$$\{a \in A : f_n(a) \leq f_{n+1}(a)\} \in I,$$

and since I is σ-complete,

$$\bigcup_{n\in\omega} \{a \in A : f_n(a) \leq f_{n+1}(a)\} \in I.$$

So, by duality,

$$\bigcap_{n\in\omega} \{a \in A : f_{n+1}(a) < f_n(a)\} \neq \emptyset.$$

Let $a \in A$ be such that $f_{n+1}(a) < f_n(a)$ for every $n \in \omega$. Then, $\langle f_n(a) : n \in \omega\rangle$ is a strictly descending infinite sequence of ordinals, which contradicts the well-foundedness of $\in$.

■

Definition 3.1.2 (Products of sets of ordinals). Let A be a non-empty set. If we let $S = \langle S(a) : a \in A\rangle$ be a sequence of non-empty sets of ordinals, we define the *product* of S as

$$\prod_{a\in A} S(a) = \{f : dom(f) = A \text{ and } \forall a \in A(f(a) \in S(a))\},$$

that we also denote as $\prod S$.

If A is a set of limit ordinals, we define the product $\prod_{a\in A} a$ as the set of functions f with domain A and such that $f(a) \in a$ for all $a \in A$. This product is also denoted $\prod A$.

More generally, if h is an ordinal function defined on A, then $\prod_{a\in A} h(a)$ denotes the set of ordinal functions f with domain A and such that $f(a) \in h(a)$ for every $a \in A$. We also use the notation $\prod h$ for this product. Note that if we consider the identity function on A, id_A, then $\prod id_A = \prod A$. In most cases, we will be interested in the product $\prod h$, in which $h(a) > 0$ is a limit ordinal for every $a \in A$.

Definition 3.1.3 (Reduced products of sets of ordinals). Let A be a non-empty set and let $S = \langle S(a) : a \in A\rangle$ be a sequence of non-empty sets of ordinals. If F is a filter on A, we can define, in an analogous way as we did in chapter 2, the reduced product $\prod S/F$ consisting of all $=_F$ equivalence classes.

In this chapter we are more interested in the reduced product $\prod S/I$, where I is an ideal on A, defined in a similar way as $\prod S/F$. Even though the relations $<_I$ and $\leq_I$ are defined on $\prod S$, our general setting will be the triple $(\prod S/I, <_I, \leq_I)$, where we will identify equivalent functions, and thus we will work with single functions instead of equivalence classes, so we might write things like $f \in \prod S/I$, instead of $[f] \in \prod S/I$.

To simplify some arguments, it will be common to change functions on null sets. For instance, if we have two functions $f, g \in \prod S$ such that $f <_I g$, note that f is I-equivalent to a function f' that is everywhere dominated by g, and since we identify equivalent functions, we can assume that $f(a) < g(a)$ for every $a \in A$. These are the kind of simplifications we will make.

Remark 3.1.4. *Let A be a non-empty set, let I be an ideal on A, and let $S = \langle S(a) : a \in A \rangle$ be a sequence of non-empty sets of ordinals. Then:*

(1) $(\prod S, <_I)$ *is a strict partial order.*

(2) $(\prod S, \leq_I)$ *is a quasi-ordering.*

Moreover, for every $f, g, h \in \prod S$ the following holds:

(1) $f <_I g$ *or* $f =_I g$ *implies that* $f \leq_I g$*, but the converse is not true in general.*

(2) *If* $f <_I g \leq_I h$ *or* $f \leq_I g <_I h$*, then* $f <_I h$.

(3) *If* $h = S$ *is an ordinal function such that* $h(a) > 0$ *is a limit ordinal for every* $a \in A$*, then there is no* $<_I$*-maximal element of* $\prod h$.

The following definitions are presented in the setting $(\prod S, <_I, \leq_I)$, which is the one that we are intereseted in for the development of pcf theory in the next chapter, but they make perfect sense in any triple $(P, <, \leq)$, where $<$ is a strict linear ordering and $\leq$ a quasi-ordering on P.

Definition 3.1.5. Let F be a set of ordinal functions on a non-empty set A, let I be an ideal on A, and let g be an ordinal function on A.

(1) g is an *upper bound* of F if $f \leq_I g$ for all $f \in F$.

(2) g is a *least upper bound* of F if it is an upper bound, and $g \leq_I h$ for every upper bound h of F.

(3) If g is an upper bound of F, we say that it is a *minimal upper bound* if for every upper bound h of F, $h \not\leq_I g$ and $g \leq_I h$.

(4) If g is an upper bound of F, F is said to be *bounded below* g if it has an upper bound $h <_I g$.

Definition 3.1.6 (Cofinality). Let I be an ideal on a non-empty set A and $S = \langle S(a) : a \in A \rangle$ a sequence of non-empty sets of ordinals. A set $F \subseteq \prod S$ is *cofinal* if for every $g \in \prod S$, there is $f \in F$ such that $g \leq_I f$. The *cofinality* of $(\prod S, \leq_I)$, written $cf(\prod S, \leq_I)$, or even $cf(\prod S)$, is the smallest size of a cofinal set (it need not be a regular cardinal if the ordering is not linear).

We will say that a set of ordinal functions F on A is *cofinal* in $h \in OR^A$, if it is cofinal in $\prod h$.

Example 3.1.7. An example of a partial order whose cofinality is a singular cardinal is $(\prod_{n<\omega} \aleph_n, <)$, where $<$ is the everywhere dominance order. That is, for every $f, g \in \prod_{n<\omega} \aleph_n$,

$$f < g \text{ if and only if } f(n) < g(n) \text{ for every } n < \omega.$$

The following notion is crucial for the development of pcf theory, as we will see in the next chapter:

Definition 3.1.8 (True cofinality). Let I be an ideal on a non-empty set A and consider a sequence of non-empty sets of ordinals $S = \langle S(a) : a \in A \rangle$. We say that $\prod S$ has *true cofinality* if there exists $F \subseteq \prod S$ linearly ordered by $<_I$ and cofinal in $\prod S$. We denote by $tcf(\prod S, <_I)$, or $tcf(\prod S)$ for short, the least cardinality of such a subset F of $\prod S$, which is always a regular cardinal (if it exists).

Example 3.1.9. A similar construction as the one above in example 3.1.7 is a particular case of a partial order that doesn't have true cofinality. Consider the partial order $(\omega \times \omega_1, <)$, where $<$ is the order defined by

$$(n, \alpha) < (m, \beta) \text{ if and only if } n < m \text{ and } \alpha < \beta.$$

Note that if B is a subset of $\omega \times \omega_1$ there are two possibilities: either B has cardinality $\aleph_0$, in which case it is clearly not cofinal in $\omega \times \omega_1$, or B has cardinality $\aleph_1$ and so there are ω_1-many pairs in B that have the same first component, making them incomparable. Therefore, there is no cofinal subset of $\omega \times \omega_1$ linearly ordered by $<$.

Definition 3.1.10 (λ-directed product). If I is an ideal on a non-empty set A and we let $S = \langle S(a) : a \in A \rangle$ be a sequence of non-empty sets of ordinals, we say that $(\prod S, <_I)$ is λ-*directed*, for any infinite cardinal λ, if any subset $F \subseteq \prod S$ of size $< \lambda$ has an upper bound in $(\prod S, <_I)$.

Remark 3.1.11. *For any infinite cardinal λ, $tcf(\prod S, <_I) = \lambda$ if and only if*

(1) *$(\prod S, <_I)$ has a cofinal subset of size λ.*

(2) *$(\prod S, <_I)$ is λ-directed.*

It follows that if $tcf(\prod S, <_I) = \lambda$ and $G \subseteq \prod S$ is a cofinal subset, then $tcf(G, <_I) = \lambda$ as well.

Due to the importance of the concept of true cofinality of a reduced product, we give a name to the λ-sequences in $\prod S/I$ that witness that $\prod S/I$ has true cofinality λ:

Definition 3.1.12 (Scale). Let I be an ideal on an infinite set A, and let $S = \langle S(a) : a \in A \rangle$ be a sequence of non-empty sets of ordinals. A *scale* of length λ (or λ-*scale*) in $(\prod S, <_I)$ is a $<_I$-increasing transfinite sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in $\prod S$ that is cofinal in $(\prod S, <_I)$.

Hence, $tcf(\prod S, <_I) = \lambda$ if and only if λ is a regular cardinal and $(\prod S, <_I)$ has a λ-scale.

In particular, if μ is a singular cardinal and I an ideal on $cf(\mu)$, a sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of ordinal functions on $OR^{cf(\mu)}$ is said to be a λ-*scale on* μ, if it is a λ-scale in $\prod_{\beta < cf(\mu)} \mu_\beta$, where $\langle \mu_\beta : \beta < cf(\mu) \rangle$ is an increasing sequence of regular cardinals with limit μ.

Definition 3.1.13 (Exact upper bound). If F is a set of ordinal functions on A, then a least upper bound g of F is said to be an *exact upper bound* if F is cofinal in $\{h : h <_I g\}$, i.e., if for every ordinal function h on A with $h <_I g$, there exists $f \in F$ such that $h \leq_I f$.

A trivial example is the ordinal function $h = \langle h(a) : a \in A \rangle$, with $h(a) > 0$ a limit ordinal for every $a \in A$, which is an exact upper bound of $(\prod h, <_I)$.

There exists a close bound between exact upper bounds and true cofinality: If the sequence $\vec{f} = \langle f_\gamma : \gamma < \lambda \rangle$ is $<_I$-increasing and has an exact upper bound g, then $\vec{f}$ is a scale in $\prod g/I$, and hence $tcf(\prod g/I) = \lambda$. This makes them so important for pcf theory, that our next objective will be finding sufficient conditions for the existence of exact upper bounds of short sequences of ordinal functions. The word "short" is important here, because as the next lemma shows, long sequences of ordinal functions always have exact upper bounds.

Lemma 3.1.14. *Let I be an ideal on a non-empty set A. If $\lambda > 2^{|A|}$ is a regular cardinal, then every $<_I$-increasing sequence of ordinal functions on A of length λ has an exact upper bound.*

Proof. Let $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be a $<_I$-increasing sequence of ordinal functions on A. By Löwenheim-Skolem, let $\mathcal{M} \preceq H_\kappa$, with κ large enough so that $I \in M$, $\vec{f} \in M$, $|M| = 2^{|A|}$ and $M^{|A|} \subseteq M$. For every $\alpha < \lambda$ define the function $g_\alpha : A \to (M \cap OR)$ by

$$g_\alpha(a) = \text{the least } \beta \in (M \cap OR) \text{ such that } \beta \geq f_\alpha(a),$$

for every $a \in A$. Since $M^{|A|} \subseteq M$, we have that $g_\alpha \in M$, and since $|M| = 2^{|A|} < \lambda$, there are λ-many g_α's that coincide. Let g^* denote all of these functions. Note that $g^* \geq f_\alpha$ for λ-many α's. Thus, since $\langle f_\alpha : \alpha < \lambda \rangle$ is a $<_I$-increasing sequence of functions and λ is regular, we have that $f_\alpha \leq_I g^*$ for all $\alpha < \lambda$. Indeed, if $\beta < \lambda$, there is some $\alpha < \lambda$ such that $\beta \leq \alpha$ and $f_\beta \leq_I f_\alpha \leq_I g^*$. Therefore g^* is an upper bound of $\vec{f}$ and it's the least one, by the way we defined the g_α's.

Now we have to check that $\vec{f}$ is cofinal in $\{h \in OR^A : h <_I f\}$. It's enough to check it for every function in M, so let $h \in M$ be an ordinal function on A such that $h <_I g^*$. Let $\alpha < \lambda$ be any ordinal such that $g^* = g_\alpha$. Then, as $h <_I g_\alpha$,

$$\{a \in A : h(a) \geq g_\alpha(a)\} \in I. \tag{3.1}$$

Since $h \in M$ it's clear be the definition of g_α that $h(a) < f_\alpha(a)$, for every $a \in A$ such that $h(a) < g_\alpha(a)$. Therefore, it follows from (3.1) that $h <_I f_\alpha$.

■

3.2 Finding Exact Upper Bounds

As we have mentioned above, we want to find sufficient conditions for the existence of exact upper bounds of short sequences of ordinal functions. That is, we want to drop the hypothesis of the last lemma that a sequence of ordinal functions must have length $\lambda > 2^{|A|}$ to have an exact upper bound.

Here below we will define these sufficient conditions, which arise naturally from the proof of Shelah's Trichotomy Theorem 3.2.13, a wonderful result that will conclude this section, before we deal with the improvements of Silver's and Solovay's Theorems.

Definition 3.2.1 (Projection). If A is a non-empty set and $S = \langle S(a) : a \in A \rangle$ a sequence of non-empty sets of ordinals, we define the function sup_of_S on A by $sup\_of\_S(a) = sup(S(a))$, for every $a \in A$.

If an ordinal function $f \in OR^A$ is bounded by sup_of_S, i.e., $f(a) < sup(S(a))$ for every $a \in A$, we define the *projection* of f onto $\prod S$, denoted $proj(f, S)$, as the least function in $\prod S$ that bounds f. To be more precise, $proj(f, S)$ is the function $f^+ \in \prod S$ defined by

$$f^+(a) = min\big(S(a) \setminus f(a)\big),$$

for every $a \in A$. Clearly $f \leq g$ implies $f^+ \leq g^+$.

More generally, even if $f \in OR^A$ is not bounded by sup_of_S, we can define the projection of f onto $\prod S$ as the function $f^+ \in \prod S$ defined by

$$f^+(a) = \begin{cases} min\big(S(a) \setminus f(a)\big), & \text{if } f(a) < sup(S(a)), \\ 0, & \text{otherwise,} \end{cases}$$

for every $a \in A$. So, if I is an ideal on A and $f \in OR^A$ is such that $f <_I sup\_of\_S$, then f^+ is the $\leq_I$-least function in $\prod S$ that $\leq_I$-bounds f, up to I-equivalence. It's clear that if $f \leq_I g$, then $f^+ \leq_I g^+$, by changing the functions on an I-set.

Definition 3.2.2 (Strongly increasing). Let I be an ideal on A, let B be a set of ordinals and let $\vec{f} = \langle f_\alpha : \alpha \in B \rangle$ be a $<_I$-increasing sequence of ordinal functions on A. We say that $\vec{f}$ is *strongly increasing* if there is a sequence of sets $\langle X_\alpha : \alpha \in B \rangle$ in the ideal I, such that for every $\alpha, \beta \in B$, if $\alpha < \beta$, then

$$a \in A \setminus (X_\alpha \cup X_\beta) \implies f_\alpha(a) < f_\beta(a).$$

Definition 3.2.3 (Property $(*)_\kappa$). Let I be an ideal on a non-empty set A, let λ be a regular cardinal, and let $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be a $<_I$-increasing sequence of ordinal functions on A. For any regular cardinal $\kappa \leq \lambda$ we define the property $(*)_\kappa$ of $\vec{f}$ as:

> $\forall X \subseteq \lambda$ unbounded, $\exists X_0 \subseteq X$ with $ot(X_0) = \kappa$, such that $\langle f_\alpha : \alpha \in X_0 \rangle$ is strongly increasing.

It's clear that $(*)_\kappa$ implies $(*)_\mu$ for every regular $\mu < \kappa$.

Definition 3.2.4 (Bounding Projection Property). Let I be an ideal on a non-empty set A, λ a regular cardinal, and $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ a $<_I$-increasing sequence of functions in OR^A. We say that $\vec{f}$ has the *bounding projection property* for κ if for every sequence $S = \langle S(a) : a \in A \rangle$ of non-empty sets of ordinals such that

(1) $|S(a)| < \kappa$, and

(2) sup_of_S bounds $\vec{f}$ in the order $<_I$,

there exists $\beta < \lambda$ such that $f_\beta^+ = proj(f_\beta, S)$ is an upper bound of $\vec{f}$ in the $<_I$ order.

Clearly, if $\vec{f}$ has the bounding projection property for κ, then it has it for every $\mu > \kappa$.

Property $(*)_\kappa$ defined above, will be proven to be a sufficient condition for a $<_I$-increasing sequence of ordinal functions of length $\lambda > |A|^+$ to have an exact upper bound. The following two results establish a connection between $(*)_\kappa$ and the bounding projection property, that results in the aforementioned theorem. A detailed proof can be found in [1].

Lemma 3.2.5 (The Bounding Projection Lemma). *Let I be an ideal on A, $\lambda > |A|$ a regular cardinal, and $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ a $<_I$-increasing sequence of ordinal functions on A satisfying $(*)_\kappa$, where κ is a regular cardinal such that $|A| < \kappa \leq \lambda$. Then, $\vec{f}$ satisfies the bounding projection property for κ.*

Theorem 3.2.6 (Exact Upper Bounds). *Let I be an ideal on A, $\lambda > |A|^+$ a regular cardinal, and $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ a $<_I$-increasing sequence of ordinal functions on A satisfying the bounding projection property for $|A|^+$. Then $\vec{f}$ has an exact upper bound.*

As we have anticipated, the combination of these two results implies the existence of an exact upper bound for sequences of ordinal functions that satisfy $(*)_\kappa$. But in fact, next theorem says even more.

Theorem 3.2.7. *Let I be an ideal on A, $\lambda > |A|^+$ a regular cardinal, and $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ a $<_I$-increasing sequence of ordinal functions on A. For every regular cardinal κ such that $|A|^+ \leq \kappa \leq \lambda$, the following are equivalent:*

(1) $(*)_\kappa$ *holds for* $\vec{f}$.

(2) $\vec{f}$ *satisfies the bounding projection property for* κ.

(3) $\vec{f}$ *has an exact upper bound* g *for which*

$$\{a \in A : cf(g(a)) < \kappa\} \in I.$$

Proof. For our purposes, we are just interested in the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). The complete proof can be found in [1].

The existence of an exact upper bound g of $\vec{f}$ is a consequence of lemma 3.2.5 and theorem 3.2.6. So the only thing that needs to be checked is that

$$\{a \in A : cf(g(a)) < \kappa\} \in I.$$

Since $\vec{f}$ is a $<_I$-increasing sequence cofinal in $\{h \in OR^A : h <_I g\}$, we may assume that $g(a) > 0$ is a limit ordinal, for all $a \in A$.

Assume, aiming for a contradiction, that $B = \{a \in A : cf(g(a)) < \kappa\} \in I^+$. Define the sequence $S = \langle S(a) : a \in A\rangle$ of sets of ordinals as

$$S(a) = \begin{cases} \{g(a)\}, & \text{if } a \notin B, \\ S^*(a), & \text{otherwise,} \end{cases}$$

where $S^*(a)$ is a cofinal subset of $g(a)$ such that $ot(S^*(a)) < \kappa$. Observe that $sup(S(a)) = g(a)$ for every $a \in A$, so $\vec{f}$ is $<_I$-bounded by sup_of_S. Therefore, as $\vec{f}$ has the bounding projection property for κ (by lemma 3.2.5), there is $\beta < \lambda$ for which $f^+_\beta = proj(f_\beta, S)$ is an upper bound of $\vec{f}$ in the order $<_I$.

It follows from the definition of S that for all $a \in B$,

$$f^+_\beta(a) = min\big(S(a) \setminus f_\beta(a)\big) < g(a),$$

i.e., $g \not\leq_I f^+_\beta$. But this is in contradiction with g being a least upper bound of $\vec{f}$.

■

Now that we have proven that $(*)_\kappa$ is a sufficient condition for the existence of exact upper bounds, the natural step to take now is to show under which circumstances we can produce sequences that satisfy $(*)_\kappa$. For this purpose, we present a combinatorial principle introduced in [68] by Shelah, which is a weakening of Jensen's diamond principle ($\diamondsuit$).

Definition 3.2.8 (Club Guessing Sequence). A *club guessing sequence* is a sequence $\langle C_\delta : \delta \in S\rangle$, where $S \subseteq \lambda$ is a stationary set, every C_δ is a club subset of δ, and such that for every club $D \subseteq \lambda$, there exists some $\delta \in S$ with $C_\delta \subseteq D$.

Club guessing sequences exist in ZFC, unlike $\diamondsuit$, a feature that will prove to be useful in the next chapter about pcf theory, in particular it will be used in the proof of Shelah's bound on $2^{\aleph_\omega}$. The following is a well-known result of Shelah.

Theorem 3.2.9 (Club Guessing). *For every regular cardinal* κ, *if* λ *is a cardinal such that* $cf(\lambda) \geq \kappa^{++}$, *then any stationary set* $S \subseteq E^\lambda_\kappa$ *has a club guessing sequence* $\langle C_\delta : \delta \in S\rangle$, *such that* $ot(C_\delta) = \kappa$ *for every* $\delta \in S$ *(Recall that* E^λ_κ *is the set of ordinals in* λ *that have cofinality* κ*).*

Club guessing can be used to prove the following lemma, which produces sequences that satisfy $(*)_\kappa$. A detailed proof can be found in [1].

Lemma 3.2.10. *Let I be a proper ideal on A, κ and λ regular cardinals such that $\kappa^{++} < \lambda$, and $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ a $<_I$-increasing sequence of functions in OR^A that satisfies the following requirement: for every $\delta \in E^\lambda_{\kappa^{++}}$, there exists a club set $E_\delta \subseteq \delta$ such that for some $\gamma \geq \delta$ in λ,*

$$sup\{f_\alpha : \alpha \in E_\delta\} <_I f_\gamma.$$

Then $(*)_\kappa$ holds for $\vec{f}$.

Theorem 3.2.11. *Let I be a proper ideal on a set of regular cardinals A, λ a regular cardinal such that $\prod A/I$ is λ-directed, and $\vec{g} = \langle g_\alpha : \alpha < \lambda \rangle$ a sequence in $\prod A$. Then there exists a $<_I$-increasing sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ in $\prod A/I$ such that*

(1) *$g_\alpha < f_{\alpha+1}$, for every $\alpha < \lambda$, and*

(2) *$(*)_\kappa$ holds for $\vec{f}$ for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\kappa^{++} <_I id_A$, i.e., $\{a \in A : a \leq \kappa^{++}\} \in I$.*

Proof. We shall define a $<_I$-increasing sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ in $\prod A/I$ inductively as follows:

- Having built f_α, let $f_{\alpha+1}$ be any function in $\prod A$ satisfying $f_\alpha, g_\alpha < f_{\alpha+1}$. This can be done because A is a set of regular cardinals.
- Let $\alpha < \lambda$ be a limit, and suppose that $\langle f_\beta : \beta < \alpha \rangle$ has been constructed. There are two possible situations:

  (1) $cf(\alpha) = \kappa^{++}$, where κ is a regular cardinal such that

  $$\{a \in A : a \leq \kappa^{++}\} \in I.$$

  Fix some club set $E_\alpha \subseteq \alpha$ of order-type $cf(\alpha)$, and define

  $$f_\alpha = sup\{f_\beta : \beta \in E_\alpha\}.$$

  Then if $a > \kappa^{++}$, $f_\alpha(a) < a$. Otherwise, there would be some $\beta \in E_\alpha$ such that for every $a > \kappa^{++}$, $f_\beta(a) \geq a$, and hence $id_A \leq_I f_\beta(a)$, which would imply that $f_\beta \notin \prod A/I$.

  Therefore, $f_\alpha <_I id_A$, and thus $f_\alpha \in \prod A/I$.

  (2) If case (1) doesn't hold, let $f_\alpha \in \prod A$ be any $\leq_I$-upper bound of $\langle f_\beta : \beta < \alpha \rangle$, given by the λ-directedness of $\prod A/I$.

Lemma 3.2.10 ensures that $(*)_\kappa$ holds for $\vec{f}$.

■

Combining this last theorem with theorem 3.2.7, we have a method for producing sequences of functions that have property $(*)_\kappa$, and thus an exact upper bound.

A very interesting application of this result is the following theorem, which gives a way to represent successors of singular cardinals of uncountable cofinality by the true cofinality of a product of regular cardinals. This is the underlying basis of Silver's Theorem.

Theorem 3.2.12 (Representation Theorem). *Let μ be a singular cardinal with uncountable cofinality. Then there exists a club set $C \subseteq \mu$ such that*

$$\mu^+ = tcf(\prod C^{(+)}/J^{bd}),$$

where $C^{(+)} = \{\kappa^+ : \kappa \in C\}$ denotes the set of successors of cardinals in C, and J^{bd} is the ideal of bounded subsets of $C^{(+)}$.

Proof. Let $C_0 \subseteq \mu$ be a club set such that $|C_0| = cf(\mu)$, and consisting of singular cardinals above $cf(\mu)$.

First note that the product $\prod C_0^{(+)}/J^{bd}$ is μ^+-directed. Indeed, if $F \subseteq \prod C_0^{(+)}$ has cardinality $\eta < \mu$, the function $h \in \prod C_0^{(+)}$ defined by

$$h(\kappa^+) = \begin{cases} sup\{f(\kappa^+) : f \in F\}, & \text{if } \kappa^+ > \eta, \\ \text{arbitrary}, & \text{if } \kappa^+ \leq \eta, \end{cases}$$

is an upper bound of F in the $\leq_{J^{bd}}$ relation. On the other hand, if $F \subseteq \prod C_0^{(+)}$ has cardinality μ, let $\langle F_\alpha : \alpha < cf(\mu)\rangle$ be a cofinal sequence on F with respect to $\subseteq$, such that $|F_\alpha| < \mu$ for every $\alpha < cf(\mu)$. By the μ-directedness proved above, there is an upper bound h_α of F_α for every $\alpha < cf(\mu)$, and by the same reason an upper bound h of $\{h_\alpha : \alpha < cf(\mu)\}$, which is an upper bound of F.

Now we can apply theorem 3.2.11 and theorem 3.2.7 to obtain a $<_{J^{bd}}$-increasing sequence $\vec{f} = \langle f_\alpha : \alpha < \mu^+\rangle$ of functions in $\prod C_0^{(+)}/J^{bd}$, and an exact upper bound h of $\vec{f}$ for which

$$\{\kappa^+ \in C_0^{(+)} : cf(h(\kappa^+)) < \lambda\} \in J^{bd}, \tag{3.2}$$

for every regular cardinal $\lambda < \mu$. Since the identity function on $C_0^{(+)}$ is an upper bound of $\vec{f}$, we can assume, changing h in a bounded set if necessary, that $h(\kappa^+) \leq \kappa^+$ for every $\kappa^+ \in C_0^{(+)}$.

Claim: There is a club set $C^{(+)} \subseteq \{\kappa^+ \in C_0^{(+)} : h(\kappa^+) = \kappa^+\}$.

Proof: Assume towards a contradiction that there is no such set. This means that all club subsets of $C_0^{(+)}$ have unbounded-many κ^+'s for which $h(\kappa^+) < \kappa^+$. Hence, the set $S \subseteq C_0^{(+)}$ of all $\kappa^+ \in C_0^{(+)}$ that belong to some club subset of $C_0^{(+)}$ and such that $h(\kappa^+) < \kappa^+$, is stationary.

Note that $cf(h(\kappa^+)) < \kappa$ for every $\kappa^+ \in S$, otherwise there would be some $\kappa^+ \in S$ such that $h(\kappa^+) = \kappa$, and thus that $cf(\kappa) = cf(h(\kappa^+)) = \kappa$, contradicting the fact that every $\kappa \in C_0$ was chosen to be singular.

So by Fodor's Theorem 1.6.4 there is some $T \subseteq S$ stationary and some $\lambda < \mu$, for which $cf(h(\kappa^+)) = \lambda$, for every $\kappa^+ \in T$. So, for every $\kappa^+ \in T$ we have that $cf(h(\kappa^+)) < \lambda^+$, and since T is unbounded we get a contradiction with (3.2). □

Therefore, there is a club set $C^{(+)} \subseteq \mu^+$ such that for every $\kappa^+ \in C^{(+)}$, $h(\kappa^+) = \kappa^+$, hence $h \restriction C^{(+)}$ is the identity function, and thus

$$\prod C^{(+)} = \prod h \restriction C^{(+)}.$$

Since $\vec{f}$ was a scale on $\prod h/J^{bd}$ (because h is an exact upper bound of $\vec{f}$), the sequence $\langle f_\alpha \restriction C^{(+)} : \alpha < \mu^+\rangle$ is a scale on $\prod h \restriction C^{(+)}/J^{bd}$, and thus

$$tcf(\prod C^{(+)}/J^{bd}) = tcf(\prod h \restriction C^{(+)}/J^{bd}) = \mu^+$$

■

An analogous representation theorem for successors of singular cardinals of countable cofinality can be found in [36]. The statement is very similar than that of theorem 3.2.12 above, but we consider the ideal of finite sets instead of the ideal of bounded sets.

Regular cardinals are typically divided into three classes: successors of regular cardinals, limit regular cardinals, and successors of singular cardinals. Representation theorems reveal a glimpse of the peculiar nature of the last of these classes (see [18]), but the ulterior meaning of these theorems is that for regular uncountable products, true cofinality exists and is equal to the least possible value.

Next theorem is a fundamental result concerning exact upper bounds. It appeared for the first time as "Claim 1.2" on page 41 in [68], a presentation which does little justice to its formidable contents.

Theorem 3.2.13 (The Trichotomy Theorem). *Let I be an ideal on A, $\lambda > |A|^+$ regular, and $\vec{f} = \langle f_\alpha : \alpha < \lambda\rangle$ a $<_I$-increasing sequence of ordinal functions on A. Then for every regular cardinal κ with $|A| < \kappa \leq \lambda$, one of the following holds:*

(Good)$_\kappa$ *$\vec{f}$ has an exact upper bound g such that $cf(g(a)) \geq \kappa$, for all $a \in A$.*

(Bad)$_\kappa$ *There are sets of ordinals $S(a)$ for $a \in A$ such that*

(i) $|S(a)| < \kappa$

(ii) *sup_of_S $<_I$-dominates $\vec{f}$, i.e., for every $\alpha < \lambda$,*

$$\{a \in A : f_\alpha(a) \geq sup(S(a))\} \in I.$$

And there is an ultrafilter D over A extending the dual filter of I such that for every $\alpha < \lambda$, there exists some $\beta < \lambda$ for which $f_\alpha^+ <_D f_\beta$.

(Ugly) *There exists $g \in OR^A$ such that, forming $t_\alpha := \{a \in A : g(a) < f_\alpha(a)\}$, the sequence $\langle t_\alpha : \alpha < \lambda\rangle$ (which is $\subseteq_I$-increasing, because $\vec{f}$ is $<_I$-increasing) doesn't stabilize modulo I. That is, for every $\alpha < \lambda$ there exists $\beta > \alpha$ in λ such that $t_\beta \setminus t_\alpha \in I^+$.*

Proof. First of all we prove that **(Good)**$_\kappa$ can be weakened in the theorem to the existence of a least upper bound instead of an exact upper bound:

Claim: If $\vec{f}$ is not **(Ugly)**, then every least upper bound h of $\vec{f}$ is an exact upper bound.

Proof: Assume, aiming for a contradiction, that h is a least upper bound of $\vec{f}$, but not an exact upper bound. Then there is an ordinal function g on A such that $g <_I h$, and $g \not<_I f_\alpha$ for every $\alpha < \lambda$. Let $\langle t_\alpha : \alpha < \lambda\rangle$ be the sequence of sets in I^+ defined by $t_\alpha = \{a \in A : g(a) < f_\alpha(a)\}$, as in **(Ugly)**. Since we are under the assumption that $\vec{f}$ is not **(Ugly)**, the sequence $\langle t_\alpha : \alpha < \lambda\rangle$ stabilizes modulo I. That is, there is some $\alpha < \lambda$ such that for every $\beta > \alpha$ in λ, $t_\beta \setminus t_\alpha \in I$, or equivalently, $t_\beta \subseteq_I t_\alpha$. Therefore, $t_\alpha =_I t_\beta$ because $\langle t_\alpha : \alpha < \lambda\rangle$ is a $\subseteq_I$-increasing sequence.

Note that if $A \setminus t_\alpha \in I$, then $\{a \in A : g(a) \geq f_\alpha(a)\} \in I$, but this is in contradiction with $g \not<_I f_\alpha$. Hence, $A \setminus t_\alpha \in I^+$.

Let f' be the function defined as follows:

$$f'(a) = \begin{cases} h(a), & \text{if } a \in t_\alpha, \\ g(a), & \text{if } a \in A \setminus t_\alpha. \end{cases}$$

Fix some $\beta > \alpha$ in λ. Then the set $\{a \in A : f_\beta(a) > f'(a)\}$ can be broken into two pieces

$$\{a \in A \setminus t_\alpha : f_\beta(a) > g(a)\} \cup \{a \in t_\alpha : f_\beta(a) > h(a)\}.$$

On the one hand, since h is an upper bound of $\vec{f}$, $f_\beta \leq_I h$, and thus

$$\{a \in t_\alpha : f_\beta(a) > h(a)\} \in I.$$

On the other hand, $\{a \in A \setminus t_\alpha : f_\beta(a) > g(a)\} = t_\beta \setminus t_\alpha \in I$. Therefore,

$$\{a \in A : f_\beta(a) > f'(a)\} \in I,$$

i.e., $f_\beta \leq_I f'$, and thus f' is an upper bound of $\vec{f}$.

We will finish the proof by showing that $h \not\leq_I f'$, and since h is a least upper bound of $\vec{f}$ we will get a contradiction. For this purpose, assume, on the contrary, that $h \leq_I f'$. Then,

$$\{a \in A : h(a) > f'(a)\} = \{a \in A \setminus t_\alpha : h(a) > g(a)\} \in I.$$

But recall that $g <_I h$, so $\{a \in A \setminus t_\alpha : h(a) \leq g(a)\} \in I$, which implies that $A \setminus t_\alpha \in I$. But this is impossible because we saw in the beginning that $A \setminus t_\alpha \in I^+$.

□

Assume that $\vec{f}$ is not **(Ugly)** and let κ be a regular cardinal with $|A| < \kappa \leq \lambda$. We will either prove that there is a least upper bound g of $\vec{f}$ (which is an exact upper bound by the last claim) such that $cf(g(a)) \geq \kappa$, for every $a \in A$, or find sets of ordinals $S(a)$ and an ultrafilter D disjoint from I, for which **(Bad)**$_\kappa$ holds.

Define by induction on $\beta < |A|^+$ upper bounds g_β of $\vec{f}$ and sequences of sets of ordinals $S_\beta = \langle S_\beta(a) : a \in A\rangle$ as follows:

- Let $g_0 := sup\{f_\alpha(a) + 1 : \alpha < \lambda\}$ for every $a \in A$. Let $S_0 = \emptyset$.
- Assume g_β and S_β have been constructed for some $\beta < |A|^+$. If there is no upper bound of $\vec{f}$ smaller than g_β in the $<_I$ relation, then g_β is a least upper bound of $\vec{f}$, and by the last claim, since we are under the assumption that $\vec{f}$ is not **(Ugly)**, g_β is an exact upper bound of $\vec{f}$.

  Otherwise, let $g_{\beta+1}$ be an upper bound of $\vec{f}$ such that $g_{\beta+1} <_I g_\beta$, and let $S_{\beta+1} = \langle S_{\beta+1}(a) : a \in A\rangle$ be defined as

  $$S_{\beta+1}(a) := \{g_\gamma(a) : \gamma < \beta + 1\}.$$

- Suppose $\beta < \lambda$ is limit and assume that g_γ and S_γ have been constructed for all $\gamma < \beta$. As in the successor step, in the case it exists, we let g_β be any upper bound of $\vec{f}$ such that $g_\beta <_I g_\gamma$ for every $\gamma < \beta$. We define S_β in the exact same way as we did above.

Observe that for every $\beta < \lambda$, the sequence $\langle proj(f_\alpha, S_\beta) : \alpha < \lambda\rangle$ is $\leq_I$-increasing in $\prod_{a\in A} S_\beta(a)$, because $\vec{f}$ is $<_I$-increasing. Clearly, the construction results in two possible cases:

Case 1) There is some $\beta < |A|^+$ for which g_β is an exact upper bound of $\vec{f}$. We will prove that **(Good)**$_\kappa$ holds by showing that $\vec{f}$ has the bounding projection property for $|A|^+$, and thus for all $\kappa \geq |A|^+$, which by theorem 3.2.7 implies that there is an exact upper bound g of $\vec{f}$ such that

$$\{a \in A : cf(g(a)) < \kappa\} \in I.$$

As usual, g will be changed on a null set so that $cf(g(a)) \geq \kappa$ for all $a \in A$, if necessary.

Pick any $\gamma < \beta$. Since g_γ is an upper bound of $\vec{f}$ and $g_\gamma <_I g_\delta$ for every $\delta < \gamma$, we have that for every $\xi < \lambda$, $proj(f_\xi, S_{\gamma+1}) =_I g_\gamma$. Fix some $\xi < \lambda$, then for any $\alpha < \lambda$

$$f_\alpha \leq_I g_\beta <_I g_\gamma =_I proj(f_\xi, S_{\gamma+1}).$$

Hence, as $|S_{\gamma+1}(a)| = |\gamma+1| \leq |\beta| < |A|^+$ and $sup\_of\_S_{\gamma+1} =_I g_0$ (so $\vec{f}$ is $<_I$-bounded by $sup\_of\_S_{\gamma+1}$), the sequence $\vec{f}$ has the projection property with respect to $S_{\gamma+1}$.

Case 2) There is no $\beta < |A|^+$ for which g_β is the least upper bound of $\vec{f}$. We will show that $\vec{f}$ satisfies $\mathbf{(Bad)}_{|A|^+}$, which clearly implies $\mathbf{(Bad)}_\kappa$ for every $\kappa \geq |A|^+$.

Fix some $\beta < |A|^+$, and define for all $\alpha, \gamma < \lambda$ the sets

$$t^\alpha_\gamma := \{a \in A : proj(f_\gamma, S_\beta)(a) < f_\alpha(a)\}.$$

Note that $\vec{f}$ doesn't have the bounding projection property, otherwise by theorem 3.2.7 $\vec{f}$ would have an exact upper bound, but this is not the case by the way we have constructed the g_β's. Therefore, for every $\gamma < \lambda$ there exists some $\alpha_\gamma < \lambda$ such that

$$f_{\alpha_\gamma} \not<_I proj(f_\gamma, S_\beta),$$

and thus, since $\vec{f}$ is $<_I$-increasing, for every $\alpha > \alpha_\gamma$, $f_\alpha \not\leq_I proj(f_\gamma, S_\beta)$, i.e, $t^\alpha_\gamma \in I^+$. Since $\vec{f}$ is not **(Ugly)**, for every $\gamma < \lambda$ the sequence $\langle t^\alpha_\gamma : \alpha < \lambda\rangle$ stabilizes mod I. Hence, for every $\gamma < \lambda$ there is some $\alpha(\gamma) < \lambda$ big enough, such that for every $\alpha \geq \alpha(\gamma)$,

(a) $t^\alpha_\gamma \in I^+$, and

(b) $t^{\alpha(\gamma)}_\gamma =_I t^\alpha_\gamma$.

If we fix some $\alpha < \lambda$, it follows from the fact that $\langle proj(f_\alpha, S_\beta) : \alpha < \lambda\rangle$ is $\leq_I$-increasing, that the sequence $\langle t^\alpha_\gamma : \gamma < \lambda\rangle$ is $\subseteq_I$-decreasing. Therefore, for every $\gamma < \lambda$, if we let $\alpha > max\{\alpha(\gamma), \alpha(\gamma+1)\}$,

$$t^{\alpha(\gamma+1)}_{\gamma+1} =_I t^\alpha_{\gamma+1} \subseteq_I t^\alpha_\gamma =_I t^{\alpha(\gamma)}_\gamma.$$

This proves that the sequence $\langle t^{\alpha(\gamma)}_\gamma : \gamma < \lambda\rangle$ is $\subseteq_I$-decreasing, and thus that it has the finite intersection property.

We claim that the set $I^* \cup \{t^{\alpha(\gamma)}_\gamma : \gamma < \lambda\}$ has the finite intersection property. Since I^* already has the property because it is a filter, we only need to check that for every $\gamma < \lambda$, and every $X \in I^*$, $t^{\alpha(\gamma)}_\gamma \cap X \neq \emptyset$. Assume, on the contrary, that $t^{\alpha(\gamma)}_\gamma \cap X = \emptyset$, then $t^{\alpha(\gamma)}_\gamma \subseteq A \setminus X$, but since $X \in I^*$, $A \setminus X \in I$, and thus $t^{\alpha(\gamma)}_\gamma \in I$, contradicting $t^{\alpha(\gamma)}_\gamma \in I^+$.

Hence, by lemma 1.5.3 and theorem 1.5.8, there is an ultrafilter D on A extending $I^* \cup \{t^{\alpha(\gamma)}_\gamma : \gamma < \lambda\}$, which clearly witnesses $\mathbf{(Bad)}_{|A|^+}$, because for every $\gamma < \lambda$, it holds that $t^{\alpha(\gamma)}_\gamma \in D$, which is equivalent to

$$proj(f_\gamma, S_\beta) <_D f_{\alpha(\gamma)}.$$

■

An immediate corollary of the Trichotomy Theorem is that if the ideal I is maximal or if $\lambda > 2^{|A|}$, neither of the latter two alternatives can hold, and thus the sequence $\vec{f}$ has an exact upper bound, thus giving an alternative proof of lemma 3.1.14. But probably the most important application of the Trychotomy Theorem, that appears as "Theorem 1.5" in chapter II of [68], is the proof of the existence of a μ^+-scale for every singular cardinal μ, i.e., the existence of a sequence $\langle f_\alpha : \alpha < \mu^+ \rangle$ of ordinal functions on $OR^{cf(\mu)}$, which is a μ^+-scale on $\prod_{\beta < cf(\mu)} \mu_\beta$, where $\langle \mu_\beta : \beta < cf(\mu) \rangle$ is an increasing sequence of regular cardinals with limit μ.

3.3 Silver's Theorem revisited

As we have mentioned above, theorem 3.2.12 is in the core of the proof of Silver's Theorem. A slightly more general form of the theorem is presented below, where we avoid using ultraproducts of systems of models as in the original proof, and all the machinery involving ordinal functions and scales developed in this chapter is put into work.

Theorem 3.3.1 (Silver). *Let κ be a singular cardinal of uncountable cofinality. Suppose that there exists a stationary set $S \subseteq \kappa$ such that, for every $\lambda \in S$, $\lambda^{cf(\kappa)} = \lambda^+$. Then*

$$\kappa^{cf(\kappa)} = \kappa^+.$$

Proof. Assume that the stationary set S has order-type $cf(\kappa)$. By theorem 3.2.12, there exists a club set $C \subseteq \kappa$ such that

$$\kappa^+ = tcf(\prod C^{(+)}/J^{bd}).$$

If we let $\vec{g} = \langle g_\alpha : \alpha < \kappa^+ \rangle$ be a scale on $\prod C^{(+)}/J^{bd}$, since $(S \cap C)^{(+)}$ is unbounded in $C^{(+)}$ (as $S \cap C$ is stationary), the sequence $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$, where $f_\alpha = g_\alpha \restriction (S \cap C)^{(+)}$ for every $\alpha < \kappa^+$, is a scale on $\prod (S \cap C)^{(+)}/J^{bd}$. Therefore,

$$tcf(\prod (S \cap C)^{(+)}/J^{bd}) = \kappa^+.$$

From now on, to simplify the notation, we will denote $S \cap C$ by S.

Since $\lambda^{cf(\kappa)} = \lambda^+$ for all $\lambda \in S$, there is an encoding of all sets $X \in [\lambda]^{\leq cf(\kappa)}$ by ordinals in λ^+. That is, there exists a bijection t_λ from $[\lambda]^{\leq cf(\kappa)}$ into λ^+, for every $\lambda \in S$. Therefore, we can encode every $X \in [\kappa]^{cf(\kappa)}$ by a function $h_X \in \prod S^{(+)}$, where

$$h_X(\lambda^+) = t_\lambda(X \cap \lambda),$$

for every $\lambda \in S$. One thing to note about the codes h_X, is that they are pairwise eventually disjoint. To see this, let $X, Y \in [\kappa]^{cf(\kappa)}$ be different sets, and suppose towards a contradiction that $h_X =_{J^{bd}} h_Y$. This is equivalent to

$$\{\lambda \in S : t_\lambda(X \cap \lambda) \neq t_\lambda(Y \cap \lambda)\} \in J^{bd},$$

hence there is some $\lambda \in S$ for which $t_\nu(X \cap \nu) = t_\nu(Y \cap \nu)$, for every $\nu > \lambda$. But this implies that $X \cap \nu = Y \cap \nu$ for every $\nu > \lambda$, and thus that $X = Y$, which contradicts our assumption.

<u>Claim:</u> For every $g \in \prod S^{(+)}$, the collection

$$G_g = \{X \in [\kappa]^{cf(\kappa)} : h_X <_{J^{bd}} g\}$$

has cardinality $\leq \kappa$.

Proof: Suppose, aiming for a contradiction that $|G_g| \geq \kappa^+$. For every $X \in G_g$ and every $\lambda \in S$, as $h_X(\lambda^+) = t_\lambda(X \cap \lambda) \in \lambda^+$, h_X is regressive on S. Therefore, by Fodor's Theorem 1.6.4 there is a stationary set $S_X \subseteq S$ and some $\lambda_X < \lambda^+$, such that $h_X(\lambda^+) = \lambda_X$, for every $\lambda \in S_X$.

Recall that we chose S to have order-type $cf(\kappa)$, so if we pick any $\lambda \in S$, we get that

$$|P(S)| \leq 2^{cf(\kappa)} \leq \lambda^{cf(\kappa)} = \lambda^+ < \kappa.$$

Therefore, there exists $G_g^* \subseteq G_g$ with $|G_g^*| = \kappa^+$, a fixed stationary set $S_0 \subseteq S$, and a fixed cardinal $\lambda_0 < \lambda$, such that $S_X = S_0$ and $\lambda_X = \lambda_0$, for every $X \in G_g^*$. Thus if we let $X, Y \in G_0$ be different sets, for every $\lambda \in S_0$,

$$h_X(\lambda^+) = t_\lambda(X \cap \lambda) = \lambda_0 = t_\lambda(Y \cap \lambda) = h_Y(\lambda^+).$$

But since S_0 is stationary, and thus unbounded, this is in contradiction with h_X and h_Y being eventually disjoint.

□

Observe that

$$[\kappa]^{cf(\kappa)} = \bigcup_{g \in \prod S^{(+)}} G_g.$$

The right to left inclusion is clear, and the left to right is easy to check: let $X \in [\kappa]^{cf(\kappa)}$, since $\vec{f}$ is cofinal in $\prod S^{(+)}$ there is some $\alpha < \kappa^+$ such that $h_X <_{J^{bd}} f_\alpha$.

Hence, it follows from the claim above that

$$\kappa^{cf(\kappa)} = \left| \bigcup_{g \in \prod S^{(+)}} G_g \right| \leq \kappa^+.$$

■

A very interesting consequence is the following result, that we already used in the proof of Solovay's Theorem 2.3.8. We restate it and prove it from the last theorem.

Corollary 3.3.2 (Silver). *If the SCH holds for all singular cardinals of countable cofinality, then it holds for all singular cardinals.*

Proof. Assume, on the contrary, that κ is the first singular cardinal that does't satisfy the SCH, i.e., such that $2^{cf(\kappa)} < \kappa$ and $\kappa^+ < \kappa^{cf(\kappa)}$, and has uncountable cofinality. Since every tail set is club, the set

$$S := \{\lambda < \kappa : cf(\lambda) = \omega \text{ and } 2^{cf(\kappa)} < \lambda\}$$

resulting of the intersection of E^κ_ω with the tail subset of κ of ordinals bigger than $2^{cf(\kappa)}$, is stationary. Note that since κ has uncountable cofinality, for every $\lambda \in S$, it holds that $2^{\aleph_0} \leq 2^{cf(\kappa)} < \lambda$, and as the SCH holds for every cardinal with countable cofinality, $\lambda^{\aleph_0} = \lambda^+$.

Claim: $\lambda^{cf(\kappa)} < \kappa$ for every $\lambda < \kappa$.

Proof: We prove it by induction on λ. The base case follows from the fact that κ has uncountable cofinality,

$$\aleph_0^{cf(\kappa)} = 2^{cf(\kappa)} < \kappa.$$

For the successor case assume that $\lambda^{cf(\kappa)} < \kappa$. It follows from the Hausdorff formula 1.4.10 that

$$(\lambda^+)^{cf(\kappa)} = \lambda^+ \cdot \lambda^{cf(\kappa)} < \kappa.$$

Now, suppose that $\lambda < \kappa$ is a limit cardinal and that $\mu^{cf(\kappa)} < \kappa$ for every $\mu < \lambda$. Then there are three possible situations:

(i) If $2^{cf(\kappa)} < \lambda$ and $cf(\kappa) < cf(\lambda)$, then $\lambda^{cf(\kappa)} = \lambda < \kappa$, by theorem 1.4.12.

(ii) If $2^{cf(\kappa)} < \lambda$ and $cf(\kappa) \geq cf(\lambda)$, then $2^{cf(\lambda)} \leq 2^{cf(\kappa)} < \lambda$, and by theorem 1.4.12, $\lambda^{cf(\kappa)} = \lambda^{cf(\lambda)}$. Now, two possibilities arise:

   (a) If λ is regular, then $\lambda^{cf(\kappa)} = \lambda^{cf(\lambda)} = 2^{cf(\lambda)} \leq 2^{cf(\kappa)} < \lambda < \kappa$.

   (b) If λ is singular it satisfies the SCH (because κ was chosen to be the first cardinal that doesn't satisfy the SCH), and since $2^{cf(\lambda)} < \lambda$, we have that $\lambda^{cf(\kappa)} = \lambda^{cf(\lambda)} = \lambda^+ < \kappa$.

(iii) If $2^{cf(\kappa)} \geq \lambda$, then $\lambda^{cf(\kappa)} \leq (2^{cf(\kappa)})^{cf(\kappa)} = 2^{cf(\kappa)} < \kappa$.

□

Note that for every $\lambda \in S$, it holds that λ is a limit, $\mu^{cf(\kappa)} < \kappa$ for every $\mu < \lambda$, because of the claim above, $2^{cf(\kappa)} < \lambda$, and $\aleph_0 = cf(\lambda) < cf(\kappa)$, so we are in the situation (ii) of the last induction, and hence

$$\lambda^{cf(\kappa)} = \lambda^{cf(\lambda)} = \lambda^{\aleph_0} = \lambda^+.$$

So by theorem 3.3.1,

$$\kappa^{cf(\kappa)} = \kappa^+.$$

■

3.4 On ω_1-Strongly Compact Cardinals

In this section we present an improvement of Solovay's Theorem 2.3.8 that appeared in [7], due to J. Bagaria and M. Magidor. In the article, they show that the SCH holds above the first ω_1-strongly compact cardinal, a large cardinal property slightly weaker than that of a strongly compact cardinal that we define here below.

Definition 3.4.1 (λ-strongly compact cardinal). If $\lambda < \kappa$ are uncountable cardinals, which may be singular, we say that κ is λ-*strongly compact* if for every set A, every κ-complete filter on A can be extended to a λ-complete ultrafilter on A.

The following characterization of λ-strongly compact cardinals appears in [6]:

Lemma 3.4.2. *Let $\lambda < \kappa$ be uncountable cardinals. The following are equivalent:*

(1) *κ is λ-strongly compact.*

(2) *For every set A, there exists a λ-complete fine measure on $P_\kappa(A)$.*

(3) *For every $\alpha \geq \kappa$, there exists an elementary embedding $j : V \to M$, with M transitive, and critical point $\geq \lambda$, such that j is definable in V, and there exists $D \in M$ such that $j''\alpha := \{j(\beta) : \beta < \alpha\} \subseteq D$ and $M \models |D| < j(\kappa)$.*

Let $\langle \lambda_n : n < \omega \rangle$ be an increasing sequence of regular cardinals with limit λ, a singular cardinal of countable cofinality. Our setting for the rest of the section will be the reduced product $\prod_{n<\omega} \lambda_n / J^{bd}$, where recall, J^{bd} is the ideal of bounded sets. Even though we restrict ourselves to such a specific set-up, all of the definitions stated here below have equivalent fomulations for singular cardinals of any cofinality, and reduced products modulo any ideal (see [18]).

Definition 3.4.3 (Good points). Let $\vec{f} = \langle f_\alpha : \alpha < \delta \rangle$ be a $<_{J^{bd}}$-increasing sequence of functions in $\prod_{n<\omega} \lambda_n$. An ordinal $\alpha < \delta$ with uncountable cofinality is said to be *good for* $\vec{f}$ if there exists $D \subseteq \alpha$ cofinal in α, and $n < \omega$ such that

$$f_\beta(m) < f_\gamma(m)$$

for all $\beta < \gamma$ in D, and all $m > n$. The sequence $\vec{f}$ is said to be *good* if every limit $\alpha < \delta$ is good for $\vec{f}$.

It is important not to confuse a scale that satisfies $(good)_\kappa$, as it appears in the statement of the Trychotomy Theorem 3.2.13, with the concept of goodness presented in this section. However, if $\vec{f}$ is a scale that satisfies $(good)_\kappa$, then κ is a good point for $\vec{f}$ (see [18]), so they are related, but not exactly the same thing.

Definition 3.4.4 (Better scale). A scale $\vec{f} = \langle f_\alpha : \alpha < \delta \rangle$ in $\prod_{n<\omega} \lambda_n / J^{bd}$ is said to be *better* if for every limit $\alpha < \delta$ with uncountable cofinality, there exists a club set $C \subseteq \alpha$ with order-type $cf(\alpha)$, and $n < \omega$ such that

$$f_\beta(m) < f_\gamma(m)$$

for all $\beta < \gamma$ in C, and all $m > n$. Of course, any better scale is good.

Next theorem, due to Shelah, tells us that if the SCH fails at λ, then there is a good λ^+-scale for λ. For a proof see [68] or [18].

Theorem 3.4.5. *If μ is a strong limit singular cardinal and $2^\mu > \mu^+$, then there is a better scale for μ.*

Theorem 3.4.6. *Suppose κ is an ω_1-strongly compact cardinal. Then for every $\lambda > \kappa$ with $cf(\lambda) = \omega$ there is no good λ^+-scale for λ.*

Proof. Let $\lambda > \kappa$ be a singular cardinal with countable cofinality, and let $\langle \lambda_n : n < \omega \rangle$ be a sequence of cardinals with limit λ. Suppose towards a contradiction that $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ is a good λ^+-scale for λ, with respect to $\langle \lambda_n : n < \omega \rangle$.

Since κ is ω_1-strongly compact, there exists an ω_1-complete fine measure U in $P_\kappa(\lambda^+)$. Consider the ultrapower $Ult_U(V)$, which is well-founded by lemma 2.1.4, and let

$$j : V \to Ult_U(V) \cong M$$

be the canonical elementary embedding, where M is the transitive collapse of the ultrapower.

It can be shown, by means of the identity function on $P_\kappa(\lambda^+)$, that $\lambda^+ < j(\lambda^+)$. Moreover, if we let $\beta := sup(j''\lambda^+)$, it can be easily checked that $\beta < j(\lambda^+)$.

By elementarity we have that in M, $j(\vec{f}) := \langle f^*_\alpha : \alpha < \lambda^+ \rangle$ is a good $j(\lambda^+)$-scale in $\prod_{n<\omega} j(\lambda_n)$. Since $\beta < j(\lambda^+)$ and $cf(\beta) = \lambda^+$, there exists $D \subseteq \beta$ cofinal in β, and $n < \omega$, such that for all $\gamma < \gamma'$ in D and every $m > n$,

$$f^*_\gamma(m) < f^*_{\gamma'}(m).$$

We define an increasing sequence of ordinals $D^* = \{\gamma_\delta : \delta < \lambda^+\} \subseteq D$ by induction on $\delta < \lambda^+$ as follows:

- Let γ_0 be the first ordinal in D.
- Suppose we have obtained γ_δ. Let α_δ be the least ordinal such that $\gamma_\delta < j(\alpha_\delta)$. Then, let $\gamma_{\delta+1} \in D$ be such that $j(\alpha_\delta) < \gamma_{\delta+1}$.
- Suppose that δ is a limit and we have built $\langle \gamma_\xi : \xi < \delta \rangle$. Let γ_δ be the least ordinal in D greater than all γ_ξ, $\xi < \delta$.

Note that for all $\delta < \lambda^+$, $\gamma_\delta \in D \subseteq \beta$, so $\gamma_\delta < j(\alpha)$, for some $\alpha < \lambda^+$. Since α_δ is the least ordinal such that $\gamma_\delta < j(\alpha_\delta)$, we have that $j(\alpha_\delta) \leq j(\alpha) < j(\lambda^+)$, and thus by elementarity $\alpha_\delta < \lambda^+$. Therefore,

$$f^*_{\gamma_0} <_{J^{bd}} f^*_{j(\alpha_0)} <_{J^{bd}} f^*_{\gamma_1} <_{J^{bd}} f^*_{j(\alpha_1)} <_{J^{bd}} \dots$$

For every $\delta < \lambda^+$, it's clear that for every $m > n$, $f^*_{\gamma_\delta}(m) < f^*_{\gamma_{\delta+1}}(m)$, because $\gamma_\delta, \gamma_{\delta+1} \in D$. For every $\delta < \lambda^+$, let $n_\delta > n$ be such that for every $m > n_\delta$,

$$f^*_{\gamma_\delta}(m) < f^*_{j(\alpha_\delta)}(m) < f^*_{\gamma_{\delta+1}}(m).$$

Of course, since $n_\delta < \omega$ for every $\delta < \lambda^+$, there is $E \subseteq \lambda^+$ of cardinality λ^+ and $k < \omega$, such that for all $\delta \in E$, $n_\delta = k$. Hence, for every $\delta, \delta' \in E$ such that $\gamma_\delta < \gamma_{\delta'}$,

$$f^*_{\gamma_\delta}(k) < f^*_{j(\alpha_\delta)}(k) < f^*_{\gamma_{\delta+1}}(k) \leq f^*_{\gamma_{\delta'}}(k) < f^*_{j(\alpha_{\delta'})}(k) < f^*_{\gamma_{\delta'+1}}(k).$$

Therefore, $\langle f^*_{\gamma_\delta}(k) : \delta \in E \rangle$ and $\langle f^*_{j(\alpha_\delta)}(k) : \delta \in E \rangle$ are two strictly increasing λ^+-sequences.

Note that since $\vec{f}(\alpha_\delta) = f_{\alpha_\delta}$, for every $\delta < \lambda^+$, by elementarity

$$M \models f^*_{j(\alpha_\delta)} = j(\vec{f})(j(\alpha_\delta)) = j(\vec{f}(\alpha_\delta)) = j(f_{\alpha_\delta}),$$

and thus

$$f^*_{j(\alpha_\delta)}(k) = j(f_{\alpha_\delta}(k)).$$

Since $f_{\alpha_\delta} \in \prod_{n<\omega} \lambda_n$ for every $\delta \in E$, $f_{\alpha_\delta}(k) \in \lambda_k$, and thus $f^*_{j(\alpha_\delta)}(k) = j(f_{\alpha_\delta}(k)) \in j''\lambda_k$. But this is impossible because $j''\lambda_k$ has order-type $\lambda_k < \lambda^+$, and $\langle f^*_{j(\alpha_\delta)}(k) : \delta \in E \rangle$ has order-type λ^+.

■

Corollary 3.4.7. *If κ is an ω_1-strongly compact cardinal, then the SCH holds above κ.*

Proof. By theorem 3.4.6, for every $\lambda > \kappa$ with countable cofinality there is no good λ^+-scale for λ. By the contraposition of theorem 3.4.5, this implies that the SCH holds at every $\lambda > \kappa$ with countable cofinality. Hence, by theorem 3.3.2, the SCH holds at every singular cardinal above κ.

■

Chapter 4

PCF Theory

After blurring the perspective that there were no non-trivial theorems in ZFC concerning singular cardinals, many results inspired by Silver's Theorem triggered the start of a new era of developement of cardinal arithmetic. One of the main directions taken by set theorists was the search for upper bounds on the size of $2^{\aleph_\alpha}$, for strong limit singular cardinals $\aleph_\alpha$. Galvin and Hajnal [24] proved in 1975 that if $\aleph_\alpha$ is a strong limit singular cardinal of uncountable cofinality, then $2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$. Using additional assumptions, Jech and Prikry [37] extended such bounds to fixed points of the aleph function, and Magidor [48] obtained the bound $\aleph_{\omega_1}^{\aleph_1} < \aleph_{\omega_2}$. However, the question whether an upper bound existed for $2^{\aleph_\alpha}$, where $\aleph_\alpha$ is of countable cofinality, remained open.

It was Shelah who, in 1980, proved that if $\aleph_\omega$ is a strong limit, then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$, a result that appeared in his book *Proper Forcing* [63]. In full generality, if δ is a limit ordinal, then $\aleph_\delta^{cf(\delta)} < \aleph_{(|\delta|^{cf(\delta)})^+}$. The proof used, in a very strong sense, ideas that he had introduced before in the context of Jónsson algebras (an algebra with countably-many finitary operations, which has no proper subalgebra of the same cardinality) [61], regarding the cofinalities of ultraproducts of sets of cardinals. This led Shelah to a systematic study of the possible cardinals that can be realized as the cofinalities of theses ultraproducts.

By the end of the 1980s, in a sequence of papers [62], [65] and [66], that led to his famous book *Cardinal Arithmetic* [68], Shelah developed a beautiful theory that brought a number of unexpected results and yielded deep applications to cardinal arithmetic. The central concept of this theory, known as pcf theory (pcf stands for "possible cofinalities"), is the pcf operator, defined on sets A of regular uncountable cardinals as

$$\text{pcf}(A) := \{\lambda \in CARD : \lambda = cf(\prod A/U), \text{ for some ultrafilter } U \text{ on } A\}.$$

This simple definition leads to many surprising results, the most notorious being the following one, that we shall discuss in section 4.6 of this chapter: If $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} < \aleph_{\omega_4}$. This result is based on the fact that the size of the pcf operator can be bounded in certain situations, and in fact, much of the work on pcf theory goes into finding these bounds on $|\text{pcf}(A)|$. Since there are $2^{2^{|A|}}$-many ultrafilters on A, pcf(A) could be quite large a priori, but as we will see, there are various uniformities that lead to a useful structure theory for pcf.

In a few words, what this theory does is give an algebraic description of the inner workings of cardinal arithmetic, and the key point is the turn of focus to the cofinalities, rather than to the cardinal exponentiation. In Shelah's words, "the exponentiation function can be misleading when used to measure the number of subsets of a given singular cardinal". The correct way to study the size of, for example, $\aleph_\omega^{\aleph_0}$ is by studying $cf([\aleph_\omega]^{\aleph_0}, \subseteq)$, rather than $\aleph_\omega^{\aleph_0}$ itself, and this will be made clear once we show in section 4.4 that, unlike $2^{\aleph_0}$, $cf([\aleph_\omega]^{\aleph_0}, \subseteq)$ is bounded in ZFC.

In this chapter we will try to cover the main features of the pcf function through the study of the sequence of ideals $J_{<\lambda}[A]$ on A, for $\lambda \in \text{pcf}(A)$. This sequence of ideals has the particular property that each one of them is generated by the previous one together with a single set $B_\lambda \subseteq A$. The sets B_λ, called the generators for the ideals $J_{<\lambda}$, will give us an alternative way of representing every $\lambda \in \text{pcf}(A)$ as the true cofinality of the product of B_λ over the ideal $J_\lambda[A]$. This is a key property of the pcf function, that together with a deep study of a certain chain of elementary substructures of H_θ, allows us to shape the generators in a way that can be put to use to bound $cf([\aleph_\omega]^{\aleph_0}, \subseteq)$. In the later sections, there is a finer analysis of the generators B_λ, and the bound $2^{\aleph_\omega} < \aleph_{\omega_4}$, when $\aleph_\omega$ is a strong limit, is established. We will finish this chapter by reviewing some applications of pcf theory in cardinal arithmetic, and in other areas of mathematics like general topology or model theory.

All the details of the results in this chapter can be found in [68], but we will follow the exposition of Abraham and Magidor in [1], as well as [10], in some cases. Unless stated otherwise, all theorems and results in this chapter are due to Shelah.

4.1 Basic Properties of the pcf Function

Initially pcf theory was developed under the assumption that $2^{|A|} < \min A$, where A is a set of regular uncountable cardinals. At the time it was reasonable to make this assumption in the context of the applications, but Shelah was well aware of the necessity of separating pcf theory from cardinal exponentiation. In [68] Shelah replaced this assumption by $|A| < \min A$, an assumption so vital in pcf theory, which avoids so many pathologies, that we shall give it a name. We say that a set A of regular uncountable cardinals is *progressive* if $|A| < \min A$. Our canonical progressive set of regular cardinals, which we will use as an example to illustrate some of the concepts, will be $A_0 = \{\aleph_n : 1 < n < \omega\}$.

The central notion of pcf theory is the cofinality of an ultraproduct of a set of regular cardinals. An interesting feature of the ultraproduct $\prod A/U$, where A is a set of regular cardinals and U is an ultrafilter on A, is that it is linearly ordered by the relation $<_U$ (hence cofinality and true cofinality of the ultraproduct coincide, and thus $cf(\prod A/U)$ is always a regular cardinal). Moreover, it is a linear order with an upper bound in OR^A, namely the identity function on A, $id_A(a) = a$ for all $a \in A$. What we are interested in is the values that $cf(\prod A/U, <_U)$ can take under different ultrafilters. For this purpose, and due to its importance, we restate the definition of the pcf operator, and give an alternative equivalent definition that will be useful in some situations.

Definition 4.1.1. For any set A of regular uncountable cardinals we define the *pcf function* (or *pcf operator*) as

$$\begin{aligned} \text{pcf}(A) &= \{cf(\prod A/U) : U \text{ is an ultrafilter on } A\} \\ &= \{tcf(\prod A/I) : I \text{ is an ideal on } A\}. \end{aligned}$$

The equivalence of both definitions follows from the fact that every ideal I on A can be extended to a dual of an ultrafilter. This combined with the fact any scale $\vec{f}$ in $\prod A/I$ is also a scale in $\prod A/I'$, for every ideal $I' \supseteq I$, gives us the equivalence.

An immediate observation that can be made about the pcf function, that follows directly from the fact that A is a set of regular cardinals, is that $\min(A) \leq \min(\text{pcf}(A))$. Other easily verifiable properties are the following ones:

Remark 4.1.2. *Let A and B be sets of regular uncountable cardinals.*

(1) $A \subseteq \text{pcf}(A)$.

(2) *If $A \subseteq B$, then* $\text{pcf}(A) \subseteq \text{pcf}(B)$.

(3) $\text{pcf}(A \cup B) = \text{pcf}(A) \cup \text{pcf}(B)$.

Proof.

(1) Let $b \in A$, and consider the principal ultrafilter on A generated by b,

$$U_b = \{X \subseteq A : b \in X\}.$$

Note that for every $f, g \in \prod A$, $f <_{U_b} g$ if and only if $f(b) < g(b)$. Let $\vec{f} = \langle f_\alpha : \alpha < a \rangle$ be a sequence of functions in $\prod A$, for which $\langle f_\alpha(b) : \alpha < a \rangle$ is increasing and cofinal in b. Then $\vec{f}$ is a b-scale in $\prod A/U_b$, and thus $b = cf(\prod A/U_b)$, so $b \in \text{pcf}(A)$.

(2) Let $\lambda \in \text{pcf}(A)$, let U be an ultrafilter on A such that $\lambda = cf(\prod A/U)$, and let $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be a λ-scale in $\prod A/U$. Extend U to an ultrafilter D on B. Then, the sequence $\vec{f'} = \langle f'_\alpha : \alpha < \lambda \rangle$, obtained by extending arbitrarily every function f_α in $\vec{f}$ to f'_α so that it belongs to $\prod B$, is a λ-scale in $\prod B/D$. Hence, $\lambda = cf(\prod B/D)$, and thus $\lambda \in \text{pcf}(B)$.

(3) Let $\lambda \in \text{pcf}(A \cup B)$, and let U be an ultrafilter on $A \cup B$ such that $cf(\prod(A \cup B)/U) = \lambda$. Note that since $A \cup B \in U$, by remark 1.5.5, either $A \in U$, or $B \in U$. Assume, without loss of generality, that $A \in U$. Then $\lambda = cf(\prod A/U_A)$, where recall, U_A is the projection of U to A, defined as $U_A = \{A \cap X : X \in U\}$. Therefore, $\lambda \in \text{pcf}(A)$.

For the other direction we can use the last item: Since $A, B \subseteq A \cup B$, both $\text{pcf}(A)$ and $\text{pcf}(B)$ are subsets of $\text{pcf}(A \cup B)$, and thus $\text{pcf}(A) \cup \text{pcf}(B) \subseteq \text{pcf}(A \cup B)$.

■

An immediate consequence of item (1) of this remark is that $\min(A) = \min(\text{pcf}(A))$. Also note that that pcf is almost a closure operator, the only property missing being idempotency, i.e., $\text{pcf}(\text{pcf}(A)) = \text{pcf}(A)$ for every set of regular cardinals A. We will see in the next section a couple of conditions that are sufficient for a progressive set A to satisfy $\text{pcf}(\text{pcf}(A)) = \text{pcf}(A)$.

Definition 4.1.3 (Interval of Regular Cardinals). We say that A is an *interval of regular cardinals* if for some cardinals $\kappa < \lambda$,

$$A = \{\mu \in CARD : \mu \text{ is regular and } \kappa \leq \mu < \lambda\}.$$

Since there is no possibile confusion, we will use the following notation for intervals of regular cardinals

$$[\kappa, \lambda) := \{\mu \in CARD : \mu \text{ is regular and } \kappa \leq \mu < \lambda\}$$
$$(\kappa, \lambda) := \{\mu \in CARD : \mu \text{ is regular and } \kappa < \mu < \lambda\}$$

Note that $\text{pcf}(A)$ is not necessarily an interval of regular cardinals, since for instance, $A = \{\aleph_{2n} : 1 < n < \omega\}$ doesn't witness that $\aleph_{2n+1}$ belongs to $\text{pcf}(A)$. Our canonical set $A_0 = \{\aleph_n : 1 < n < \omega\}$ is an example of an interval of regular cardinals.

It is worth mentioning that progressive intervals of regular cardinals don't have a maximal element, and thus they are infinite and unbounded. Indeed, suppose that A has a maximal element, say μ. Then $\mu = \sup(A)$, and thus $cf(\mu) \leq |A| < \min(A) < \mu$, so μ is singular. But this is impossible because A is an interval of regular cardinals.

To close the section we make an observation that shows how fundamental is pcf theory with respect to cardinal arithmetic. Let us consider $A_0 = \{\aleph_n : 1 < n < \omega\}$. As we have seen in the proof of (1) of the remark 4.1.2, every $\aleph_n \in A_0$ is a possible cofinality of $\prod A_0$ by a principal ultrafilter. If U is a non-principal ultrafilter on A_0, then clearly $cf(\prod A_0/U) \geq \aleph_{\omega+1}$, and since $|\prod A_0| = \aleph_\omega^{\aleph_0}$, we have $\aleph_{\omega+1} \leq cf(\prod A_0/U) \leq \aleph_\omega^{\aleph_0}$. So unless $\aleph_{\omega+1} < \aleph_\omega^{\aleph_0}$, the set $\text{pcf}(A_0)$ has only one element outside A_0, and so a meaningful theory of possible cofinalities requires the negation of the Singular Cardinal Hypothesis.

4.2 The Ideal $J_{<\lambda}$

Definition 4.2.1. Let A be a set of regular cardinals. For any cardinal λ define

$$J_{<\lambda}[A] = \{X \subseteq A : \text{pcf}(X) \subseteq \lambda\}.$$

Equivalently, $X \in J_{<\lambda}[A]$ if and only if for every ultrafilter U on A, if $X \in U$, then $cf(\prod A/U) < \lambda$. It's easy to check that $J_{<\lambda}[A]$ is an ideal on A using remark 4.1.2, but it is not necessarily a proper ideal. Note that $A \in J_{<\lambda}[A]$ is possible, unless $\lambda \in \text{pcf}(A)$ (if $A \in J_{<\lambda}[A]$, then $\text{pcf}(A) \subseteq \lambda$, but this is impossible if $\lambda \in \text{pcf}(A)$).

If $X \in J_{<\lambda}[A]$ we say that X *forces* $\prod A$ to have cofinality less than λ. When the set A is clear from the context, we write $J_{<\lambda}$ instead of $J_{<\lambda}[A]$.

The idea behind the ideals $J_{<\lambda}[A]$ is to identify the subsets of A that are too small to obtain a regular λ as the cofinality of an ultraproduct of A. The $J_{<\lambda}$ are essential to the study of pcf theory, as we will see, they have a very nice structure, and will give us a characterization of the pcf operator through the study of their generators. Some of their most elementary properties, which can be easily checked, are the following ones:

Proposition 4.2.2. *Let A and B be sets of regular cardinals, and let λ and μ be cardinals.*

(1) $J_{<\lambda}[A] \subseteq P(A \cap \lambda)$.

(2) *If μ is singular, then* $J_{<\mu^+}[A] = J_{<\mu}[A]$.

(3) *If* $\lambda \leq \min(\text{pcf}(A))$, *then* $J_{<\lambda}[A] = \{\emptyset\}$.

(4) *If* $\lambda > \sup(\text{pcf}(A))$, *then* $J_{<\lambda}[A] = P(A)$.

(5) *If* $A \subseteq B$, *then* $J_{<\lambda}[A] = J_{<\lambda}][B] \cap P(A)$.

(6) *If* $\lambda < \mu$, *then* $J_{<\lambda}[A] \subseteq J_\mu[A]$.

The next theorem is a central result in pcf theory. A great amount of consequences will be derived from this theorem, which thanks to the results of the last chapter on exact upper bounds, uncover the good behaviour of the pcf operator.

Theorem 4.2.3 (λ-directedness). *Let A be a progressive set of regular cardinals. Then for every cardinal λ, the reduced product $\prod A/J_{<\lambda}[A]$ is λ-directed, i.e., any subset $F \subseteq \prod A$ of cardinality $< \lambda$ is bounded in $\prod A/J_{<\lambda}[A]$.*

Proof. We can assume that $A \notin J_{<\lambda}[A]$, otherwise $J_{<\lambda}[A] = P(A)$, so $|\prod A/J_{<\lambda}| = 1$, and the theorem holds trivially.

We show by induction on $\mu < \lambda$ that $\prod A/J_{<\lambda}$ is μ^+-directed. Assume that $\prod A/J_{<\lambda}$ is μ-directed, and let $F = \{f_\alpha : \alpha < \mu\} \subseteq \prod A$ be such that $|F| = \mu$. By reordering F if necessary, we assume that it is increasing in $<_{J_{<\lambda}}$. We show that F has an upper bound in $\prod A/J_{<\lambda}$:

Case 1) μ singular.
Let $\langle \beta_\gamma : \gamma < cf(\mu)\rangle$ be an increasing and cofinal sequence on μ. For every $\gamma < cf(\mu)$, since $cf(\mu) < \mu$, we can use the μ-directedness of $\prod A/J_{<\lambda}$ to get an upper bound g_γ of $\{f_\alpha : \alpha < \beta_\gamma\}$. Therefore, a successive application of the induction hypothesis gives us an upper bound h of $\{g_\gamma : \gamma < cf(\mu)\}$, which is in turn an upper bound of F.

Case 2) μ is regular and $\mu \leq a_n$, where $a_0 = \min(A)$ and a_n is the $(n+1)$-th element of A, for any $n < \omega$.
Note that for every $a > \mu$ in A, since a is regular, $\sup\{f_\alpha(a) : \alpha < \mu\}$ is bounded by a. Let g be defined as $g(a) = \sup\{f_\alpha(a) : \alpha < \mu\}$ for all $a > \mu$ in A, and let $g(a) = 0$ for all $a \leq \mu$ in A. Then the function g belongs to $\prod A$, and since $|A \cap (\mu + 1)|$ is finite, it is clearly an upper bound of F with respect to $<_{J_{<\lambda}}$.

Case 3) μ is regular and $\mu > a_n$, for every $n < \omega$.
We assumed in the beginning that $J_\lambda[A]$ is proper. Since $\prod A/J_{<\lambda}$ is μ-directed by induction hypothesis, we can apply theorem 3.2.11 to get a $<_{J_{<\lambda}}$-increasing sequence $\vec{f'} = \langle f'_\alpha : \alpha < \mu\rangle$ that satisfies $(*)_{|A|^+}$, and is such that $f_\alpha < f'_{\alpha+1}$ for every $\alpha < \mu$. Since $\vec{f'}$ satisfies $(*)_{|A|^+}$, by theorem 3.2.7 it has an exact upper bound g with respect to $<_{J_{<\lambda}}$, which is in turn an upper bound of F.

Since the identity function on A, id_A, taking every $a \in A$ to a, is an upper bound of $\vec{f'}$, and g is a least upper bound of $\vec{f'}$ (so $g \leq_{J_{<\lambda}} id_A$), by changing g on a null set if necessary, we can assume that $g(a) \leq a$ for all $a \in A$.
Define the set

$$B = \{a \in A : g(a) = a\}.$$

Note that if $B \in J_{<\lambda}[A]$, we can redefine g as $g(a) = 0$ for every $a \in B$ so that $g \in \prod A$, and since B is a null set, g remains an upper bound of F, but now in $\prod A/J_{<\lambda}$. Therefore, the next claim finishes the proof.

Claim: $B \in J_{<\lambda}[A]$.

Proof: Suppose, aiming for a contradiction, that $B \notin J_{<\lambda}[A]$. Then, by definition of $J_{<\lambda}[A]$, there is an ultrafilter U on A, disjoint from $J_{<\lambda}[A]$, for which $B \in U$ and $cf(\prod A/U) \geq \lambda$. Since $|\vec{f'}| = \mu$ and $\mu < \lambda \leq cf(\prod A/U)$, the sequence $\vec{f'}$ is bounded in $\prod A/U$. Let h be such a bound, so that $h(a) < a$ for every $a \in A$, by changing it on a null set if necessary. Then, $h(a) < a = g(a)$, for all $a \in B$, i.e., $h \restriction B <_{J_{<\lambda}} g \restriction B$. Hence, by definition of exact upper bound, there is some $\alpha_0 < \mu$ such that $h \restriction B <_{J_{<\lambda}} f'_{\alpha_0} \restriction B$, and thus $h(a) < f'_{\alpha_0}(a)$ for every $a \in B$. Since $B \in U$, this implies that

$$B = \{a \in B : h(a) < f'_{\alpha_0}(a)\} \subseteq \{a \in A : h(a) < f'_{\alpha_0}(a)\} \in U.$$

But this would mean that $h <_U f_{\alpha_0}$, which is in contradiction with h being an upper bound of $\vec{f}'$. □

■

We list now several corollaries of the λ-directedness Theorem. The first one being one of the most useful applications of the ideals $J_{<\lambda}$, which is a characterization of pcf in terms of the $J_{<\lambda}$'s.

Corollary 4.2.4. *Let A be a progressive set of regular cardinals. Then for every ultrafilter U on A,*

$$cf(\prod A/U) < \lambda \text{ if and only if } J_{<\lambda}[A] \cap U \neq \emptyset.$$

Proof.

$\Longleftarrow$) By definition of $J_{<\lambda}[A]$, if $X \in J_{<\lambda} \cap U$, then $cf(\prod A/U) < \lambda$.

$\Longrightarrow$) Assume that $J_{<\lambda} \cap U = \emptyset$. This means that U extends the dual filter of $J_{<\lambda}[A]$, and since $\prod A/J_{<\lambda}$ is λ-directed by theorem 4.2.3, then $\prod A/U$ is λ-directed as well (any set $F \subseteq \prod A$ bounded by $g \in \prod A$ with respect to $<_{J_{<\lambda}}$, is also bounded by g with respect to $<_{J^*_{<\lambda}}$, and thus with respect to $<_U$). Therefore, $cf(\prod A/U) < \lambda$ is impossible in this case.

■

The next characterization of the pcf operator in terms of $J_{<\lambda}$ follows directly from the corollary:

$$\begin{aligned} cf(\prod A/U) = \lambda &\iff J_{<\lambda^+}[A] \cap U \neq \emptyset \text{ and } J_{<\lambda}[A] \cap U = \emptyset \\ &\iff \lambda^+ \text{ is the first cardinal such that } J_{<\lambda^+}[A] \cap U \neq \emptyset. \end{aligned}$$

An easily verifiable consequence of this characterization is the following remark:

Remark 4.2.5. *If $\lambda \in \mathrm{pcf}(A)$, then $J_{<\lambda^+} \setminus J_{<\lambda} \neq \emptyset$, and thus $J_{<\lambda} \subset J_{<\lambda^+}$.*

Note that since every ultrafilter on a set A of regular cardinals is a subset of $P(A)$, the number of ultrafilters on A is at most $2^{2^{|A|}}$ (in fact it is exactly this number). Hence $|\mathrm{pcf}(A)| \leq 2^{2^{|A|}}$. The next corollary gives us the first non-trivial bound on pcf.

Corollary 4.2.6. *If A is a progressive set of regular cardinals, then*

$$|pcf(A)| \leq 2^{|A|}.$$

Proof. Define a map from $\mathrm{pcf}(A)$ to $P(A)$ that sends every λ to X_λ, where X_λ is any set in $J_{<\lambda^+} \setminus J_{<\lambda}$, whose existence is granted by the remark 4.2.5. It can be easily checked that this map is one-to-one.

■

One of the most notorious conjectures in pcf theory, known as the *pcf conjecture*, was the assertion that $|\mathrm{pcf}(A)| \leq |A|$ for any progressive set of regular cardinals A. The conjecture would imply the sharp bound $2^{\aleph_\omega} < \aleph_{\omega_1}$, if $\aleph_\omega$ is a strong limit cardinal. However, Gitik proved in [30] that assuming the existence of a certain large cardinal, it is consistent that the pcf conjecture fails.

In the next section we will see that $J_{<\lambda}$ together with one of the sets $X_\lambda \in J_{<\lambda^+} \setminus J_{<\lambda}$, that appear in the proof of the last corollary, generate the next ideal $J_{<\lambda^+}$, so they play a crucial role in the structure of the $J_{<\lambda}$'s.

Corollary 4.2.7. *If λ is a limit cardinal then*

$$J_{<\lambda}[A] = \bigcup_{\mu<\lambda} J_{<\mu}[A].$$

Proof. Define $I := \bigcup_{\mu<\lambda} J_{<\mu}$ for some limit cardinal λ. Recall from remark 4.2.2 that if $\mu_1 < \mu_2$, then $J_{<\mu_1} \subseteq J_{<\mu_2}$, and hence, as I is a union of a chain of ideals, it is also an ideal. The inclusion $I \subseteq J_{<\lambda}$ is immediate by the definition of $J_{<\lambda}$.

For the other inclusion, suppose towards a contradiction that there exists $X \in J_{<\lambda}$ such that $X \notin I$. Let D be an ultrafilter extending the dual filter of I, then $X \in D$ and $D \cap I = \emptyset$. Since $D \cap J_{<\mu} = \emptyset$ for every $\mu < \lambda$, by corollary 4.2.4 we have $cf(\prod A/D) \geq \mu$. Thus $cf(\prod A/D) \geq \lambda$, and again by corollary 4.2.4, $D \cap J_{<\lambda} = \emptyset$, but this is impossible because $X \in J_{<\lambda}$ by assumption. ■

Another way of stating the above corollary is that for every cardinal λ (not necessarily limit)

$$J_{<\lambda}[A] = \bigcup_{\mu<\lambda} J_{<\mu^+}[A].$$

Corollary 4.2.8. *For every $X \subseteq A$ there is a unique regular λ such that $X \in J_{<\lambda^+} \setminus J_{<\lambda}$.*

Proof. Since $cf(\prod A/U) < |\prod A|^+$ for all ultrafilters U on A, for every $X \subseteq A$ it holds that $X \in J_{<|\prod A|^+}$, and therefore, there is a minimal μ, for which $X \in J_{<\mu}[A]$. By the previous corollary, such a μ can't be a limit cardinal, so there is a cardinal λ such that $\mu = \lambda^+$, and thus $X \in J_{<\lambda^+} \setminus J_{<\lambda}$. The cardinal λ must be regular, for otherwise by remark 4.2.2 we would have $J_{<\lambda^+} = J_{<\lambda}$.

■

Corollary 4.2.9 (max pcf). *If A is a progressive set of regular cardinals, then the set* $\mathrm{pcf}(A)$ *contains a maximal cardinal.*

Proof. Let λ be the unique regular cardinal for which $A \in J_{<\lambda^+} \setminus J_{<\lambda}$, given by the last corollary. Then, $cf(\prod A/U) < \lambda^+$ for all ultrafilters U on A, and thus $\lambda \geq \sup(\mathrm{pcf}(A))$.

Since $A \notin J_{<\lambda}$, there exists an ultrafilter D on A extending the dual filter of $J_{<\lambda}$, and thus $D \cap J_{<\lambda} = \emptyset$. Therefore, by corollary 4.2.4, $cf(\prod A/D) \geq \lambda$, and by the previous paragraph $cf(\prod A/D) = \lambda$, which means $\lambda \in \mathrm{pcf}(A)$. Thus $\lambda = \max(\mathrm{pcf}(A))$.

■

Lemma 4.2.10. *Let A be a progressive interval of regular cardinals, and λ a regular cardinal with* $\sup(A) < \lambda$. *Let I be a proper ideal on A such that $\prod A/I$ is λ-directed. Then* $\lambda \in \mathrm{pcf}(A)$.

Proof. For a proof see [1]. ■

Theorem 4.2.11 (No-holes). *If A is a progressive interval of regular cardinals, then* $\mathrm{pcf}(A)$ *is again an interval of regular cardinals.*

Proof. If A is finite, then every ultrafilter on A is principal with principal element some $a \in A$, and therefore $A = \mathrm{pcf}(A)$.

Assume that A is infinite. Let $\lambda_0 = \max(\mathrm{pcf}(A))$. Since $A \subseteq \mathrm{pcf}(A)$, and A doesn't have a maximal element (because it is a progressive interval of regular cardinals, and so $\sup(A)$ is a singular cardinal), it is enough to show that every regular cardinal μ such that $\sup(A) < \mu \leq \lambda_0$ belongs to $\mathrm{pcf}(A)$.

Let μ be a regular cardinal such that $\sup(A) < \mu \leq \lambda_0$. Then $J_{<\mu}[A]$ is a proper ideal, otherwise $\mathrm{pcf}(A) \subseteq \max(\mathrm{pcf}(A))$, which is impossible. By theorem 4.2.3, $\prod A/J_{<\mu}$ is μ-directed, hence we find ourselves under the hypotheses of lemma 4.2.10, and thus $\mu \in \mathrm{pcf}(A)$. ■

The No-Holes Theorem is a fundamental result in pcf theory that will give us the next non-trivial bound on the size of pcf. But most importantly, this theorem is the basis of the connection between cardinal arithmetic and pcf theory.

We mentioned earlier that the pcf operator was missing idempotency to be a closure operator. To finish the section we state two theorems that give sufficient conditions for this to happen.

Theorem 4.2.12. *Let A be a progressive set of regular cardinals, and let $B \subseteq$* $\mathrm{pcf}(A)$ *be progressive. Then*

$$\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A).$$

Hence if $\mathrm{pcf}(A)$ *is progressive, then* $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.

Proof. The proof can be found in [1]. ■

Theorem 4.2.13. *Let A be a progressive set of regular cardinals. If there is no inaccessible cardinal in the set of accumulation points of* $\mathrm{pcf}(A)$*, namely*

$$\mathrm{pcf}(A)' := \{\lambda \in \mathrm{pcf}(A) : \lambda = \sup(\mathrm{pcf}(A) \cap \lambda)\},$$

then $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.

Proof. For a proof see [41]. ■

Therefore, in a model of set theory without inaccessible cardinals, such as V_κ where κ is the first inaccessible cardinal, the pcf function is a closure operator.

4.3 Generators for $J_{<\lambda}$

This section is devoted to the investigation of the consequences of remark 4.2.5, a result that could seem insignificant a priori, but it will prove to be fundamental in the uncovering of the skeleton of the ideals $J_{<\lambda}$. In the last section, after the proof of corollary 4.2.6, we mentioned that there are sets $X_\lambda \in J_{<\lambda^+} \setminus J_{<\lambda}$ that play a very important part in the study of the structure of the $J_{<\lambda}$'s. In this section a relation between $J_{<\lambda^+}$ and $J_{<\lambda}$ is established by means of these sets, through a fine study of a certain kind of sequences of ordinal functions.

In particular, we will see that for every cardinal $\lambda \in \text{pcf}(A)$ there is a set $B_\lambda[A] \in J_{<\lambda^+} \setminus J_{<\lambda}$ that satisfies, for every $X \subseteq A$, the following property called *normality*:

$$\begin{aligned} X \in J_{<\lambda^+}[A] &\iff X \setminus B_\lambda[A] \in J_{<\lambda}[A] \\ &\iff X \subseteq_{J_{<\lambda}} B_\lambda[A]. \end{aligned}$$

This property, that we represent by

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B_\lambda[A],$$

tells us that the ideal $J_{<\lambda^+}[A]$ is generated from $J_{<\lambda}[A]$ by the addition of a single set $B_\lambda[A]$, called a *generator* for $J_{<\lambda^+}[A]$. If there is no possible confusion we write B_λ instead of $B_\lambda[A]$.

The importance of the generators B_λ lies not only in the study of the structure of the ideals $J_{<\lambda}$, but in the whole structure of pcf. Among many other important features, the generators B_λ will give us an alternative characterization of pcf, a compactness theorem for which every subset of A can be covered by finitely many generators, and the most important of them all, a way to represent every $\lambda \in \text{pcf}(A)$ as a true cofinality.

But first, as we have mentioned above, we have to focus on a special kind of sequences of ordinal functions:

Definition 4.3.1 (Universally Cofinal Sequence). Let A be a set of regular cardinals and let $\lambda \in \text{pcf}(A)$. A sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ of functions in $\prod A$ is a *universal* sequence for λ if and only if

(1) $\vec{f}$ is increasing in $<_{J_{<\lambda}}$,

(2) $\vec{f}$ is cofinal in $\prod A/D$, for every ultrafilter D on A such that $\lambda = cf(\prod A/D)$.

Theorem 4.3.2 (Universally Cofinal Sequences). *If A is a progressive set of regular cardinals, then every $\lambda \in \text{pcf}(A)$ has a universal sequence.*

Remark 4.3.3. *Note that if $\lambda \in \text{pcf}(A)$ and D is an ultrafilter on A such that $cf(\prod A/D) = \lambda$, then $A \cap (\lambda + 1) \in D$. Indeed, if $\lambda > \sup(A)$ then it's clear that*

$$\{a \in A : a > \lambda\} = \emptyset \notin D.$$

On the other hand, if $\lambda \leq \sup(A)$ and we fix a sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ cofinal in $\prod A/D$, then for every $a > \lambda$ in A, the sequence $\langle f_\alpha(a) : \alpha < \lambda \rangle$ is bounded by some $b_a \in a$. Let $g \in \prod A$ be $g(a) = b_a$ for every $a > \lambda$, and arbitrary for every $a \leq \lambda$. Then, for every $\alpha < \lambda$

$$\{a \in A : a > \lambda\} \subseteq \{a \in A : f_\alpha(a) < g(a)\}.$$

Hence, $\{a \in A : a > \lambda\} \in D$ would imply that $f_\alpha <_D g$ for every $\alpha < \lambda$, which is impossible because $\vec{f}$ has been chosen to be cofinal in $\prod A/D$.

As a consequence, if $\langle f_\alpha : \alpha < \lambda \rangle$ is a universal sequence for λ, we may assume that $f_\alpha(a) = \alpha$ for all $a \in A \setminus \lambda$, since $A\setminus$ is a null set.

Theorem 4.3.4. *For every progressive set A of regular cardinals, if $<$ is the everywhere dominance order on $\prod A$, namely $f < g$ if and only if $f(a) < g(a)$ for every $a \in A$, then*

$$cf(\prod A, <) = \max(\text{pcf}(A)).$$

Hence $cf(\prod A, <)$ is a regular cardinal.

Proof. We only give a sketch of the proof. Let $\lambda = \max(\mathrm{pcf}(A))$, and let D be an ultrafilter on A such that $\lambda = cf(\prod A/D)$. Note that $<_D$ extends $<$ on $\prod A$ because for any $f, g \in \prod A$, if $f < g$, then

$$\{a \in A : f(a) < g(a)\} = A \in D,$$

and thus $f <_D g$. Therefore, any cofinal set in $(\prod A, <)$ is also cofinal in $(\prod A, <_D)$, and hence

$$\lambda = cf(\prod A, <_D) \leq cf(\prod A, <).$$

Now we have to show that $cf(\prod A, <) \leq \lambda$ by exhibiting a cofinal subset of $(\prod A, <)$ of cardinality λ.

For every $\mu \in \mathrm{pcf}(A)$, let $\vec{f^\mu} = \langle f^\mu_\alpha : \alpha < \mu \rangle$ be a universal sequence for μ. For all $\mu_1, \ldots, \mu_n \in \mathrm{pcf}(A)$ and all $\alpha_i < \mu_i$, for any $n < \omega$ and $1 \leq i \leq n$, let $\sup\{f^{\mu_1}_{\alpha_1}, \ldots, f^{\mu_n}_{\alpha_n}\}$ be the function in $\prod A$ defined as

$$\sup\{f^{\mu_1}_{\alpha_1}, \ldots, f^{\mu_n}_{\alpha_n}\}(a) = \max\{f^{\mu_1}_{\alpha_1}(a), \ldots, f^{\mu_n}_{\alpha_n}(a)\},$$

for every $a \in A$. If we let F be the set of all functions of the form $\sup\{f^{\mu_1}_{\alpha_1}, \ldots, f^{\mu_n}_{\alpha_n}\}$, then clearly $|F| = \lambda$, and it can be checked that F is cofinal in $(\prod A, <)$.

■

Lemma 4.3.5. *Let A be a progressive set of regular cardinals and let $\lambda \in \mathrm{pcf}(A)$. If γ is the least ordinal such that $A \cap \gamma \notin J_{<\lambda}[A]$, then there is a universal sequence for λ that satisfies $(*)_\kappa$ with respect to $J_{<\lambda}[A]$ for every regular cardinal $\kappa < \gamma$, and in particular for $\kappa = |A|^+$.*

Proof. First note that $\gamma \leq \lambda + 1$. Indeed, if we assume the contrary, then $A \cap (\lambda+1) \in J_{<\lambda}[A]$. Let D be an ultrafilter on A such that $\lambda = cf(\prod A/D)$. Then, by remark 4.3.3 $A \cap (\lambda+1) \in D$, so we can consider the projection $D_{A\cap(\lambda+1)}$ of D to $A \cap (\lambda+1)$, and thus

$$\lambda = cf(\prod A/D) = cf(\prod A \cap (\lambda+1)/D_{A\cap(\lambda+1)}).$$

But this implies that $\lambda \in \mathrm{pcf}(A \cap (\lambda+1))$, which is impossible because $A \cap (\lambda+1) \in J_{<\lambda}[A]$, so $\mathrm{pcf}(A \cap (\lambda+1)) \subseteq \lambda$.

Also note that $\gamma \neq \lambda$, because $|A \cap \lambda| \leq |A| < \min(A) \leq \lambda$, and since λ is regular, $A \cap \lambda$ is bounded by some $\lambda_0 < \lambda$, so $A \cap \lambda = A \cap \lambda_0$, and hence λ can't be the least ordinal such that $A \cap \lambda \notin J_{<\lambda}[A]$.

If $\gamma = \lambda + 1$, then $A \cap \lambda \in J_{<\lambda}[A]$, and thus $J_{<\lambda}[A] = P(A \cap \lambda)$. In this situation any universal sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ such that $f_\alpha(a) = \alpha$ for all $a \in A \setminus \lambda$, as in remark 4.3.3, is as required. Note that any sequence of this kind clearly satisfies $(*)_\lambda$ by letting $X_\alpha = A \cap \lambda$ for every $\alpha < \lambda$, where X_α are the sets that witness that $\vec{f}$ is strongly increasing, as in definition 3.2.2.

Assume now that $\gamma < \lambda$, and thus that $A \cap \gamma$ is unbounded in γ. Let $\vec{f'} = \langle f'_\alpha : \alpha < \lambda \rangle$ be any universal sequence for λ. Since $J_{<\lambda}[A]$ is a λ-directed proper ideal, theorem 3.2.11 can be applied to the sequence $\vec{f'}$. This gives a $<_{J_{<\lambda}}$-increasing sequence $\vec{g} = \langle g_\alpha : \alpha < \lambda \rangle$ in $\prod A$ such that $f'_\alpha < g_{\alpha+1}$ for every $\alpha < \lambda$, and satisfies $(*)_\kappa$ for every regular cardinal κ such that $\kappa^{++} < \lambda$. Since $\vec{f'}$ is a universal sequence and $\vec{g}$ dominates $\vec{f'}$, then $\vec{g}$ is also universal and $(*)_\kappa$ holds for every regular cardinal $\kappa < \gamma$.

■

As a consequence of the above lemma, if $\lambda \in \mathrm{pcf}(A)$, there is a universal sequence for λ, which by theorem 3.2.7 has an exact upper bound g with respect to $<_{J_{<\lambda}}$, and is such that

$$\{a \in A : cf(g(a)) \leq |A|\} \in J_{<\lambda}[A].$$

Lemma 4.3.6. *Let A be a progressive set of regular cardinals and $B \subseteq A$. Then*

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B$$

if and only if

(a) $B \in J_{<\lambda^+}[A]$, *and*

(b) $B \in D$ *for every ultrafilter D on A with $cf(\prod A/D) = \lambda$.*

Proof.

$\Longrightarrow$) Assume that $J_{<\lambda^+}[A] = J_{<\lambda}[A] + B$. Property (a) holds trivially. To show (b) let D be any ultrafilter on A with $cf(\prod A/D)$. By corollary 4.2.4, $D \cap J_{<\lambda^+} \neq \emptyset$ and $D \cap J_{<\lambda} = \emptyset$. If we let $X \in D \cap J_{<\lambda^+}$, then $X \setminus B \in J_{<\lambda}$. Suppose that $A \setminus B \in D$. Since $X \in D$, it holds that $(A \setminus B) \cap X = X \setminus B \in D$, and thus $X \setminus B \in D \cap J_{<\lambda}$, but this is impossible because $D \cap J_{<\lambda} = \emptyset$. Therefore, $B \in D$.

$\Longleftarrow$) From (a) and (b) we prove that $J_{<\lambda^+}[A] = J_{<\lambda}[A] + B$.

Let $X \in J_{<\lambda}[A] + B$, namely $X \subseteq A$ and $X \setminus B \in J_{<\lambda}$. Since $J_{<\lambda} \subseteq J_{<\lambda^+}$, X belongs to $J_{<\lambda^+}$, and as $B \in J_{<\lambda^+}$ by (a),

$$(X \setminus B) \cup B \in J_{<\lambda^+}.$$

Since $X \subseteq (X \setminus B) \cup B$, this proves that $X \in J_{<\lambda^+}$, and thus that $J_{<\lambda^+}[A] \supseteq J_{<\lambda}[A] + B$.

To prove the other inclusion we let $X \in J_{<\lambda^+}$, and we show that $X \setminus B \in J_{<\lambda}$. Let D be any ultrafilter on A such that $X \setminus B \in D$. Then $X \in D \cap J_{<\lambda^+}$, and by corollary 4.2.4, $cf(\prod A/D) < \lambda^+$. Note that since $X \setminus B \in D$, then $A \setminus B \in D$. Therefore, $cf(\prod A/D) < \lambda$, because otherwise $cf(\prod A/D) = \lambda$, and by (b) we get that $B \in D$, which is impossible.

■

Now we are ready to prove, as we have pointed out at the beginning of the section, the Normality Theorem, which asserts the existence of generators for $J_{<\lambda^+}$, for every $\lambda \in \mathrm{pcf}(A)$. This theorem is sometimes called the *pcf theorem*, specially in expositions (see for example [36]) where the existence of the B_λ's is proven first (In this situation, the sets B_λ are defined as the ones that satisfy (a) and (b) of the last lemma.), and $J_{<\lambda}$ is defined afterwards as the the ideal of subsets of A generated by the sets B_μ, for $\mu \in \mathrm{pcf}(A) \cap \lambda$.

Theorem 4.3.7 (Normality). *Let A be a progressive set of regular cardinals, then for every cardinal $\lambda \in \mathrm{pcf}(A)$ there exists a set $B_\lambda \subseteq A$ such that*

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B_\lambda.$$

Proof. Let $\lambda \in \mathrm{pcf}(A)$. As a consequence of lemma 4.3.5, we mentioned that there is a universal sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ for λ, which has an exact upper bound h in $OR^A/J_{<\lambda}$. Since the identity function id_A on A is an upper bound of $\vec{f}$ and h is a least upper bound, $h \leq_{J_{<\lambda}} id_A$, and therefore we can assume that $h(a) \leq a$ for every $a \in A$. Define the set

$$B_\lambda = \{a \in A : h(a) = a\}.$$

We will show that B_λ is a generator for $J_{<\lambda^+}[A]$ by showing that it satisfies (a) and (b) of the last lemma:

(a) We have to show that $B_\lambda \in J_{<\lambda^+}$. Let D be any ultrafilter on A such that $B_\lambda \in D$. We have to check that $cf(\prod A/D) < \lambda^+$.

Case 1) If $D \cap J_{<\lambda} \neq \emptyset$, then $cf(\prod A/D) < \lambda$ by corollary 4.2.4.

Case 2) If $D \cap J_{<\lambda} = \emptyset$, then D is an extension of the dual filter of $J_{<\lambda}$, and hence $<_{J_{<\lambda}} \subseteq <_D$. Therefore, $\vec{f}$ is $<_D$-increasing and h remains an exact upper bound of $\vec{f}$ with respect to $<_D$. Since $B_\lambda \in D$, it holds that $h =_D id_A$, and thus

$$cf(\prod A/D) = cf(\prod h/D).$$

Moreover, $\vec{f}$ is a λ-scale on $\prod h/D$, as h is an exact upper bound of $\vec{f}$, and therefore $cf(\prod A/D) = \lambda$.

(b) Let D be an ultrafilter on A with $cf(\prod A/D) = \lambda$. We have to show that $B_\lambda \in D$. Assume towards a contradiction that $B_\lambda \notin D$. Then

$$A \setminus B_\lambda = \{a \in A : h(a) < a\} \in D,$$

and thus $h \in \prod A/D$. Note that since $cf(\prod A/D) = \lambda$, by corollary 4.2.4, $D \cap J_{<\lambda} = \emptyset$, so D is an extension of the dual filter of $J_{<\lambda}[A]$, and $<_{J_{<\lambda}} \subseteq <_D$ as before. But then $f_\alpha <_D h$ for every $\alpha < \lambda$, and since $h \in \prod A/D$, this implies that $\vec{f}$ is bounded in $\prod A/D$, which is in contradiction with $\vec{f}$ being a universal sequence for λ.

■

Note that the generators B_λ are not uniquely determined, since they depend on the choice of the exact upper bound h, but if B_λ and B'_λ are two generators, then $B_\lambda =_{J_{<\lambda}} B'_\lambda$, or equivalently, the symmetric difference $B_\lambda \triangle B'_\lambda = (B_\lambda \setminus B'_\lambda) \cup (B'_\lambda \setminus B_\lambda)$ is in $J_{<\lambda}$. This is why any set $B \subseteq A$ that satisfies

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B,$$

is called a "$B_\lambda[A]$ set".

If we let A be a progressive set of regular cardinals, a sequence $\langle B_\lambda[A] : \lambda \leq \max(\text{pcf}(A))\rangle$, where $B_\lambda = \emptyset$ if $\lambda \notin \text{pcf}(A)$, is called a *generating sequence* for A.

Remark 4.3.8. *Let A be a progressive set of regular cardinals and let $\lambda = \max(\text{pcf}(A))$. Note that $A \in J_{<\lambda^+}[A]$, as $\text{pcf}(A) \subseteq \max(\text{pcf}(A))^+$, and therefore $A \setminus B_\lambda \in J_{<\lambda}$. Hence*

$$A =_{J_{<\lambda}} B_{\max(\text{pcf}(A))}.$$

The Compactness Theorem, which we will state next, tells us that any set $X \in J_{<\lambda}$ is covered by a finite collection of generators. From this we will conclude that the ideal $J_{<\lambda}[A]$ is finitely generated by the sets B_μ, for $\mu \in \text{pcf}(A) \cap \lambda$.

Theorem 4.3.9 (Compactness). *Let A be a progressive set of regular cardinals and let $\langle B_\lambda[A] : \lambda \leq \max(\mathrm{pcf}(A))\rangle$ be a generating sequence for A. Then for any $X \subseteq A$, there exists a finite set $\{\lambda_1, \dots, \lambda_n\} \subseteq \mathrm{pcf}(X)$ such that*

$$X \subseteq B_{\lambda_1} \cup B_{\lambda_2} \cup \cdots \cup B_{\lambda_n}.$$

Proof. For every $X \subseteq A$, we prove by induction on $\lambda = \max(\mathrm{pcf}(X))$ that there exist $\lambda_1, \dots, \lambda_n \in \mathrm{pcf}(X)$ such that $X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n}$.

Let $X \subseteq A$ be such that $\lambda^+ = \max(\mathrm{pcf}(X))$, and assume that the statement holds for every $Y \subseteq A$ with $\max(\mathrm{pcf}(Y)) \leq \lambda$. Note that $\mathrm{pcf}(X) \subseteq \lambda^{++}$, so $X \in J_{<\lambda^{++}}[A]$, and by definition of generator of $J_{<\lambda^{++}}[A]$ we have that

$$X \setminus B_{\lambda^+} \in J_{<\lambda^+}[A].$$

Therefore $\mathrm{pcf}(X \setminus B_{\lambda^+}) \subseteq \lambda^+$, and thus $\max(\mathrm{pcf}(X \setminus B_{\lambda^+})) \leq \lambda$. So by induction hypothesis there are $\lambda_1, \dots, \lambda_n \in \mathrm{pcf}(X \setminus B_{\lambda^+})$ such that

$$X \setminus B_{\lambda^+} \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n},$$

and hence

$$X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n} \cup B_{\lambda^+}.$$

■

Corollary 4.3.10. *Let A be a progressive set of regular cardinals. Then for every cardinal λ and every set $X \subseteq A$,*

$$X \in J_{<\lambda}[A] \iff X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n}, \quad \text{for some finite } \{\lambda_1, \dots, \lambda_n\} \subseteq \lambda.$$

Lemma 4.3.11. *Let A be a progressive set of regular cardinals. For every cardinal $\lambda \in \mathrm{pcf}(A)$ and every $B \in J_{<\lambda^+}[A]$,*

$$B \text{ is a generator for } J_{<\lambda^+}[A] \iff \lambda \notin \mathrm{pcf}(A \setminus B).$$

Proof.

$\Longrightarrow$) Assume that B is a generator for $J_{<\lambda^+}[A]$. Let D' be an ultrafilter on $A \setminus B$ and extend it to an ultrafilter D on A. Since the map that sends every $f \in \prod A$ to it's restriction $f \restriction (A \setminus B)$ is an isomorphism between $\prod A/D$ and $\prod(A \setminus B)/D'$, it's enough to check that $\lambda \neq cf(\prod A/D)$.

Assume, aiming for a contradiction, that $cf(\prod A/D) = \lambda$. Then by (b) of lemma 4.3.6, $B \in D$. But this is impossible because D is an extension of the ultrafilter D' on $A \setminus B$, and therefore $A \setminus B \in D$.

$\Longleftarrow$) Assume that $\lambda \notin \mathrm{pcf}(A \setminus B)$. Let D be an utlrafilter on A with $cf(\prod A/D) = \lambda$. Since $B \in J_{<\lambda^+}$ by assumption, to show that B is a generator for $J_{<\lambda^+}[A]$ we only need to check (b) of lemma 4.3.6, that is, that $B \in D$.

Suppose towards a contradiction that $A \setminus B \in D$, and consider the projection $D_{(A\setminus B)}$ of the ultrafilter D to $A \setminus B$. Since

$$\prod A/D \cong \prod(A \setminus B)/D_{(A\setminus B)}$$

we have that $cf(\prod(A \setminus B)/D_{(A\setminus B)}) = \lambda$, but this contradicts $\lambda \notin \mathrm{pcf}(A \setminus B)$.

■

Theorem 4.3.12. *Let A be a progressive set of regular cardinals and let $\lambda \in \text{pcf}(A)$. Then, any universal sequence for λ is cofinal in $\prod B_\lambda / J_{<\lambda}$, and moreover*

$$tcf(\prod B_\lambda / J_{<\lambda}) = \lambda.$$

Proof. We saw in the proof of theorem 4.3.7 that if $\lambda \in \text{pcf}(A)$ and $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is a universal sequence for λ, then there is an exact upper bound h of $\vec{f}$ such that $B_\lambda = \{a \in A : h(a) = a\}$. Therefore,

$$\prod B_\lambda = \prod_{a \in B_\lambda} a = \prod_{a \in B_\lambda} h(a),$$

and hence, $\vec{f}$ is cofinal on $\prod B_\lambda / J_{<\lambda}$. This shows that the sequence $\vec{f} \restriction B_\lambda := \langle f_\alpha \restriction B_\lambda : \alpha < \lambda \rangle$ is cofinal on $\prod B_\lambda / J_{<\lambda}$, and moreover, since $\vec{f} \restriction B_\lambda$ is $<_{J_{<\lambda}}$-increasing, it is a λ-scale on $\prod B_\lambda / J_{<\lambda}$, therefore

$$tcf(\prod B_\lambda / J_{<\lambda}) = \lambda.$$

■

In particular, the last theorem shows that $\lambda \in \text{pcf}(B_\lambda)$, and as $B_\lambda \in J_{<\lambda^+}$, we have the following corollary:

Corollary 4.3.13. *If A is a progressive set of regular cardinals and $\lambda \in \text{pcf}(A)$, then*

$$\lambda = \max(\text{pcf}(B_\lambda)).$$

Now we are ready to prove the alternative characterization of pcf that we have mentioned at the beginning of the section. First we give a characterization of pcf in terms of the generators, and then we combine it with corollary 4.2.4 to get a second characterization by means of the ideal $J_{<\lambda}$.

Theorem 4.3.14. *Let A be a progressive set of regular cardinals and let D be an ideal on A. Then for every cardinal λ,*

$$\begin{aligned} cf(\prod A/D) = \lambda &\iff \lambda \text{ is the least cardinal such that } B_\lambda \in D \\ &\iff B_\lambda \in D \text{ and } D \cap J_{<\lambda} = \emptyset. \end{aligned}$$

Proof. Assume that $cf(\prod A/D) = \lambda$. Suppose towards a contradiction that there is some $\mu < \lambda$ such that $B_\mu \in D$. Since $J_{<\mu^+} \subseteq J_{<\lambda}$, and $B_\mu \in J_{<\mu^+}$ by (a) of lemma 4.3.6, $B_\mu \in D \cap J_{<\lambda}$, but this contradicts $D \cap J_{<\lambda} = \emptyset$, given by corollary 4.2.4.

Assume now that λ is the least cardinal such that $B_\lambda \in D$. Suppose, aiming for a contradiction that $D \cap J_{<\lambda} \neq \emptyset$, and let $X \in D \cap J_{<\lambda}$. Then, by corollary 4.3.10, there are $\lambda_1, \ldots, \lambda_n \in \lambda$, some $n < \omega$, for which

$$X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n},$$

and since $X \in D$, the union $B_{\lambda_1} \cup \cdots \cup B_{\lambda_n}$ also belongs to D. By remark 1.5.5 there is some $\lambda_i \in \{\lambda_1, \ldots, \lambda_n\}$ such that $B_{\lambda_i} \in D$, but this is impossible because by assumption λ is the least cardinal such that $B_\lambda \in D$ and λ_i is smaller than λ.

For the last implication assume that $B_\lambda \in D$ and that $D \cap J_{<\lambda} = \emptyset$. By lemma 4.3.6

$B_\lambda \in J_{<\lambda^+}$, so $B_\lambda \in D \cap J_{<\lambda^+}$, and hence the intersection $D \cap J_{<\lambda^+}$ is nonempty. So by corollary 4.2.4

$$cf(\prod A/D) = \lambda.$$

■

Up until now we have seen many different ways to characterize pcf(A) for a progressive set of regular cardinals A. Hereafter we review all of them, which essentially come from the last theorem and corollary 4.2.4:

$$\begin{aligned} \mathrm{pcf}(A) &= \{\min\{\lambda : J_{<\lambda^+} \cap D \neq \emptyset\} : D \text{ ultrafilter on } A\} \\ &= \{\min\{\lambda : B_\lambda \in D\} : D \text{ ultrafilter on } A\} \\ &= \{\lambda : B_\lambda \in D \text{ and } D \cap J_{<\lambda} = \emptyset, \text{ any ultrafilter } D \text{ on } A\} \end{aligned}$$

Lemma 4.3.15. *Let A be a progressive set of regular cardinals. If $X \subseteq A$ and $\lambda \in \mathrm{pcf}(A)$, then*

$$B_\lambda[X] =_{J_{<\lambda}[X]} X \cap B_\lambda[A].$$

Proof. It's enough to check that (a) and (b) of lemma 4.3.6 hold for $X \cap B_\lambda[A]$ with respect to $J_{<\lambda}[X]$.

■

Lemma 4.3.16. *Let A be a progressive set of regular cardinals. The following are equivalent for every filter F on A and every cardinal λ:*

(1) $tcf(\prod A/F) = \lambda$.

(2) $B_\lambda \in F$*, and F contains the dual filter of* $J_{<\lambda}[A]$.

(3) $cf(\prod A/D) = \lambda$ *for every ultrafilter D that extends F.*

Proof.

(1) $\Longrightarrow$ (3) Clear.

(3) $\Longrightarrow$ (2) If we assume (3), by lemma 4.3.6, $B_\lambda \in D$ for every ultrafilter D extending F. Hence $B_\lambda \in F$, because

$$F = \bigcap\{D : D \text{ ultrafilter extending } F\}.$$

Suppose that F doesn't contain the dual filter of $J_{<\lambda}[A]$, then there is some ultrafilter D extending F such that $D \cap J_{<\lambda} \neq \emptyset$. But this is impossible, because then by corollary 4.2.4, $cf(\prod A/D) < \lambda$.

(2) $\Longrightarrow$ (1) Assume that $B_\lambda \in F$, and that F contains the dual filter of $J_{<\lambda}[A]$. Since $B_\lambda \in F$, by the same reason as in the proof of lemma 4.3.11, $\prod A/F$ and $\prod B_\lambda/F$ are isomorphic. Therefore, by lemma 4.3.12

$$\begin{aligned} tcf(\prod A/F) &= tcf(\prod B_\lambda/F) \\ &= tcf(\prod B_\lambda/J_{<\lambda}) = \lambda. \end{aligned}$$

■

In theorem 3.2.12 we proved that successors of singular cardinals with uncountable cofinality can be represented by the true cofinality of a product of cardinals modulo the ideal of bounded sets. We finish this section with a stronger assertion, that will be used in the proof of Shelah's bound $2^{\aleph_\omega} < \aleph_{\omega_4}$, if $\aleph_\omega$ is a strong limit.

Theorem 4.3.17 (Representation Theorem). *Let μ be a singular cardinal with uncountable cofinality. Then there exists a club set $C \subseteq \mu$ such that*

$$\mu^+ = tcf(\prod C^{(+)}/J_{<\mu}[C^{(+)}]).$$

Therefore, $\mu^+ = \max(\mathrm{pcf}(C^{(+)}))$.

Proof. By theorem 3.2.12 there exists a club set $C_0 \subseteq \mu$ such that

$$\mu^+ = tcf(\prod C_0^{(+)}/J^{bd}),$$

where recall, J^{bd} is the ideal of bounded sets in $C_0^{(+)}$. We can assume, as we did in the proof of that theorem, that C_0 is a club set of limit cardinals above $cf(\mu)$ and such that $|C_0| = cf(\mu)$, thus making C_0 progressive. Therefore, since $\mu^+ \in \mathrm{pcf}(C_0^{(+)})$, by normality there exists $B_{\mu^+}[C_0^{(+)}] \subseteq C_0^{(+)}$ such that

$$J_{<\mu^{++}}[C_0^{(+)}] = J_{<\mu^+}[C_0^{(+)}] + B_{\mu^+}[C_0^{(+)}].$$

We claim that $C_0^{(+)} \setminus B_{\mu^+}[C_0^{(+)}]$ is bounded in $C_0^{(+)}$. To show this, it's enough to check that $B_{\mu^+}[C_0^{(+)}]$ belongs to the dual filter $(J^{bd})^*$. Note that since $<_{J^{bd}} = <_{(J^{bd})^*}$,

$$\mu^+ = tcf(\prod C_0^{(+)}/J^{bd}) = tcf(\prod C_0^{(+)}/(J^{bd})^*),$$

and thus by lemma 4.3.16, $B_{\mu^+}[C_0^{(+)}] \in (J^{bd})^*$. Therefore, $\sup(C_0^{(+)} \setminus B_{\mu^+}[C_0^{(+)}]) \in C_0^{(+)}$.

Define $C := C_0 \setminus \sup\{\alpha \in C_0 : \alpha^+ \in C_0^{(+)} \setminus B_{\mu^+}\}$. Note that since $C_0^{(+)} \setminus C^{(+)} \in J^{bd}$, we have that $C_0^{(+)} =_{J^{bd}} C^{(+)}$, and thus

$$\mu^+ = tcf(\prod C^{(+)}/J^{bd}).$$

So $\mu^+ \in \mathrm{pcf}(C^{(+)})$, and we can apply theorem 4.3.7 to get a generator $B_{\mu^+}[C^{(+)}]$, which by theorem 4.3.12 satisfies

$$\mu^+ = tcf(\prod B_{\mu^+}[C^{(+)}]/J_{<\mu^+}[C^{(+)}]).$$

Hence, it's enough to check that $C^{(+)} =_{J_{<\mu^+}[C^{(+)}]} B_{\mu^+}[C^{(+)}]$ to finish the proof. But this an immediate consequence of lemma 4.3.15, which asserts that

$$B_{\mu^+}[C^{(+)}] =_{J_{<\mu^+}[C^{(+)}]} C^{(+)} \cap B_{\mu^+}[C_0^{(+)}] = C^{(+)}.$$

■

4.4 The Cofinality of $[\mu]^\kappa$ and the First Bound

In this section we present Shelah's first bound on the power-set function at strong limit singular points, which appeared in [63]. In full generality, Shelah's result states that if δ is a limit ordinal such that $\delta < \aleph_\delta$ (i.e., it is not a fixed point of the aleph function), then

$$\aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}.$$

In particular, if $\aleph_\omega$ is a strong limit, then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$. For this purpose, we will focus our attention on the partial ordering $([\mu]^\kappa, \subseteq)$, and more precisely on its cofinality $cf([\mu]^\kappa, \subseteq)$. As we have mentioned in the introduction of this chapter, Shelah's most insightful contribution to cardinal arithmetic was the change of focus from the exponentiation to the cofinality. The next theorem is one of the reasons why this is the case:

Theorem 4.4.1. *Let κ and μ be cardinals such that $\kappa \leq \mu$. Then,*

$$|[\mu]^\kappa| = cf([\mu]^\kappa, \subseteq) \cdot 2^\kappa.$$

Proof. The inequality $|[\mu]^\kappa| \geq cf([\mu]^\kappa, \subseteq) \cdot 2^\kappa$ is clear. For the other direction suppose that $cf([\mu]^\kappa, \subseteq) = \lambda$ and let $Y = \{Y_\alpha \in [\mu]^\kappa : \alpha < \lambda\}$ be cofinal in $([\mu]^\kappa, \subseteq)$. Define a map from $[\mu]^\kappa$ to $Y \times P(\kappa)$ by sending every $E \in [\mu]^\kappa$ to the pair $\langle Y_\alpha, S\rangle$, where Y_α is such that $E \subseteq Y_\alpha$ (it exists because Y is cofinal in $([\mu]^\kappa, \subseteq)$), and S is the subset of κ isomorphic to E (Y_α and κ are bijectable).

This map is one-to-one: Let $E_1, E_2 \in [\mu]^\kappa$ be different sets, and suppose that both E_1 and E_2 are sent to $\langle Y_\alpha, S\rangle$. Then E_1 and E_2 are isomorphic, as they are both isomorphic to S. Since $E_1, E_2 \subseteq Y_\alpha$, and the isomorphism is inherited from the bijection between Y_α and κ, this makes E_1 and E_2 equal, which is impossible by assumption. Hence, the map is one-to-one.

■

Another reason why it is interesting to study $cf([\mu]^\kappa, \subseteq)$ instead of μ^κ is that cofinalities are more resistant to forcing than powers of cardinals. For instance, adding any number of Cohen reals has absolutely no effect on $cf(\aleph_\omega^{\aleph_0}, \subseteq)$. But the main reason will be presented soon, when we make the connection between pcf theory and cardinal arithmetic.

Lemma 4.4.2. *Let $\kappa \leq \mu$ be any cardinals, and let E be a cofinal subset of $[\mu]^\kappa$. Then there exists a cofinal subset in $([\mu^+]^\kappa, \subseteq)$ of cardinality $|E| \cdot \mu^+$.*

Proof. For every $\mu \leq \gamma < \mu^+$ let $f_\gamma : \gamma \to \mu$ be a bijection. We claim that the set

$$F = \{f_\gamma^{-1}(X) : X \in E,\ \mu \leq \gamma < \mu^+\},$$

which clearly has cardinality $|E| \cdot \mu^+$, is cofinal in $([\mu^+]^\kappa, \subseteq)$.

Let $Y \in [\mu^+]^\kappa$. Since $|Y| = \kappa \leq \mu$ and μ^+ is regular, Y is bounded below μ^+, and thus there is some $\mu \leq \gamma < \mu^+$ such that $\sup(Y) < \gamma$. Let $X \in E$ be any set such that $f_\gamma(Y) \subseteq X$, then $Y \subseteq f_\gamma^{-1}(X)$, and thus F is cofinal in $([\mu^+]^\kappa, \subseteq)$.

■

Now observe the following. Let μ be a singular cardinal, and let $\kappa < \mu$ be an infinite cardinal such that the interval $A = (\kappa, \mu)$ of regular caridnals has size $\leq \kappa$, and thus it is progressive. Suppose that $cf([\mu]^\kappa, \subseteq) = \lambda$, and let $\{X_\alpha : \alpha < \lambda\}$ be cofinal in $([\mu]^\kappa, \subseteq)$. For every $\alpha < \lambda$ let h_α be the function defined by $h_\alpha(a) = \sup(X_\alpha \cap a)$, for every $a \in A$. Note that $\{h_\alpha : \alpha < \lambda\}$ is cofinal in $(\prod A, <)$. Indeed, if we let $f \in \prod A$, then for every $a \in A$ we have that $f(a) < a < \mu$, and that $ran(f) \in [\mu]^{\leq\kappa}$, as $|A| \leq \kappa$. Hence, there is some $\alpha < \lambda$ such that $ran(f) \subseteq X_\alpha$, and therefore $f(a) \in X_\alpha \cap a$, for every $a \in A$, which implies that $f(a) < \sup(X_\alpha \cap a) = h_\alpha(a)$. Consequently, $cf(\prod A, <) \leq \lambda = cf([\mu]^\kappa, \subseteq)$. But recall from theorem 4.3.4 that $cf(\prod A, <) = \max(\text{pcf}(A))$, hence

$$\max(\text{pcf}(A)) \leq cf([\mu]^\kappa, \subseteq). \tag{4.1}$$

The inequality $\geq$ holds as well, and the rest of the chapter will be devoted to prove it, and thus show the equality $cf([\mu]^\kappa, \subseteq) = \max(\text{pcf}(A))$. This is the main reason why we decide to focus on the cofinality rather than to the exponentiation, because if we combine this equality with theorem 4.4.1, we get the following connection between pcf theory and cardinal arithmetic:

$$\mu^\kappa = \max(\text{pcf}(A)) \cdot 2^\kappa.$$

Once this has been proved, getting the bound $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$, assuming that $\aleph_\omega$ is a strong limit, will follow almost immediately, but the proof of the above inequality requires a bit of work. We plan to exhibit a cofinal subset of $[\mu]^\kappa$ of cardinality $\leq \max(\text{pcf}(A))$, and to get this set we will investigate a certain kind of universal sequences called *minimally obedient* and their relationship with characteristic functions of elementary substructures.

Recall from the proof of theorem 4.3.7 how the generator $B_\lambda[A]$ was obtained for a progressive set of regular cardinals A: We defined a universal sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ for λ with an exact upper bound h, and then B_λ was defined as the set $\{a \in A : h(a) = a\}$. The use of minimally obedient universal sequences will give us a greater flexibility to tune-up B_λ by means of a certain type of elementary substructures of H_θ that will be presented here below.

Recall that H_θ is defined as the collection of all sets having transitive closure of cardinality $< \theta$. We will usually use the expressions "big enough" and "sufficiently large" to refer to the cardinal θ. This expressions essentially mean that θ is regular and that it is sufficiently large to include all the sets that we are intereseted in, such as a progressive set of regular cardinals A, a sequence of functions $\vec{f}$, or even a certain cardinal κ. Instead of considering H_θ as a structure with a single binary relation symbol $\in$, we will treat it as a triple $(H_\theta, \in, <^*)$, where $<^*$ is a well-ordering of its universe.

Definition 4.4.3. A *chain of elementary substructures* of H_θ of length λ is a sequence of structures $\langle M_\alpha : \alpha < \lambda \rangle$ such that

(1) $M_\alpha \preceq H_\theta$, for every $\alpha < \lambda$, and

(2) $\alpha < \beta < \lambda$ implies $M_\alpha \subseteq M_\beta$.

If moreover M_δ is the structure with universe $\bigcup_{\alpha<\delta} M_\alpha$ and interpretation of the $\in$ relation $\in^{M_\delta} = \bigcup_{\alpha<\delta} \in^{M_\alpha}$, for every limit ordial $\delta < \lambda$, then then we say that the chain $\langle M_\alpha : \alpha < \lambda \rangle$ is *continuous*.

Definition 4.4.4 (κ-presentable elementary substructure). Let κ be any cardinal. An elementary substructure $M \preceq H_\theta$ is *κ-presentable* if and only if $M = \bigcup_{\alpha<\kappa} M_\alpha$, where $\langle M_\alpha : \alpha < \kappa \rangle$ is a continuous chain of elementary substructures of H_θ of length κ such that

(1) $|M| = \kappa$ and $\kappa + 1 \subseteq M$ (i.e., M contains all ordinals $\leq \kappa$).

(2) $\alpha < \beta < \kappa$ implies $M_\alpha \in M_\beta$.

In order to define a κ-presentable elementary substructure of H_θ for a sufficiently large θ, use Löwenheim-Skolem to get an elementary substructure $M_0 \preceq H_\theta$ such that $\kappa+1 \subseteq M_0$ and $|M_0| \leq \kappa$. By recursive applications of Löwenheim-Skolem we can obtain $M_{\alpha+1} \preceq H_\theta$ so that $M_\alpha \cup \{M_\alpha\} \subseteq M_{\alpha+1}$ and $|M_{\alpha+1}| \leq \kappa$, for every $\alpha < \kappa$. At limit points $\delta < \kappa$ let M_δ be the structure with universe $\bigcup_{\alpha<\delta} M_\alpha$ and interpretation of the $\in$ relation $\in^{M_\delta} = \bigcup_{\alpha<\delta} \in^{M_\alpha}$.

Let $M = \bigcup_{\alpha<\kappa} M_\alpha$ be a κ-presentable elementary substructure. We define the *ordinal closure* of $M_\alpha \cap OR$, denoted by $\overline{M}_\alpha$, as the set of ordinals that belong to M_α and the limits of these ordinals. Note that since $M_\alpha \in M_{\alpha+1}$ and $M_\alpha \subseteq M_{\alpha+1}$, we have that $\overline{M}_\alpha \in M_{\alpha+1}$ and $\overline{M}_\alpha \subseteq M_{\alpha+1}$.

Definition 4.4.5 (Characteristic function). For any structure M we define the *characteristic function* of M, denoted χ_M, as

$$\chi_M(\mu) = \sup(M \cap \mu),$$

for every regular cardinal $\mu > |M|$. It follows from the definition that $\chi_M(\mu) \in \overline{M}$, and since μ is regular and $|M \cap \mu| \leq |M| < \mu$, then $\chi_M(\mu) \in \mu$. Therefore, if A is a set of regular cardinals such that $|M| < \min(A)$, then $\chi_M \restriction A \in \prod A$.

One of the key features of the characteristic function is that if M is a κ-presentable elementary substructure with $A \in M$, for a progressive set of regular cardinals A such that $\kappa = |M| < \min(A)$, then $\chi_M \restriction A$ determines $M \cap \mu$, for any $\mu > |M|$, i.e., $\chi_M \restriction A = \chi_N \restriction A$ implies $M \cap \mu = N \cap \mu$. Here below we prove a certain form of this fact.

Lemma 4.4.6. *Let $M, N \preceq H_\theta$, let κ be a regular uncountable cardinal, and let $\mu > \kappa$ be any cardinal. If $M \cap \kappa \subseteq N \cap \kappa$, and for every successor cardinal $\lambda^+ \in M \cap (\mu+1)$, it holds that* $\sup(M \cap \lambda^+) = \sup(M \cap N \cap \lambda^+)$, *then*

$$M \cap \mu \subseteq N \cap \mu.$$

Proof. For a proof see [1]. ■

Theorem 4.4.7. *Let κ be a regular uncountable cardinal, and let $\mu > \kappa$ be any cardinal. If M and N are two κ-presentable substructures of H_θ and for every successor cardinal $\lambda^+ \in \mu+1$ it holds that* $\sup(M \cap \lambda^+) = \sup(N \cap \lambda^+)$, *then*

$$M \cap \mu = N \cap \mu.$$

Proof. Let $M = \bigcup_{\alpha<\kappa} M_\alpha$ and $N = \bigcup_{\alpha<\kappa} N_\alpha$ be κ-presentations for M and N. We show by induction on $\lambda \leq \mu$, that $M \cap \lambda = N \cap \lambda$.

The case $\lambda \leq \kappa$ is clear, since M and N are κ-presentable and thus $\kappa+1 \subseteq M, N$. If λ is a limit cardinal, then $M \cap \lambda = N \cap \lambda$ follows by induction hypothesis. Therefore, let $\lambda \in (\kappa, \mu]$ and assume that $M \cap \lambda = N \cap \lambda$. We have to check that $M \cap \lambda^+ = N \cap \lambda^+$.

Let $\gamma = \sup(M \cap \lambda^+) = \sup(N \cap \lambda^+)$. Note that the set $\{\sup(M_\alpha \cap \lambda^+) : \alpha < \kappa\}$ is clearly closed. And if we observe that

$$\begin{aligned}\gamma = \sup(M \cap \lambda^+) &= \sup((\bigcup_{\alpha<\kappa} M_\alpha) \cap \lambda^+) \\ &= \sup(\bigcup_{\alpha<\kappa} (M_\alpha \cap \lambda^+)) \\ &= \bigcup_{\alpha<\kappa} (\sup(M_\alpha \cap \lambda^+)),\end{aligned}$$

it's also clear that it is unbounded. The same argument can be used to show that the set $\{\sup(N_\alpha \cap \lambda^+) : \alpha < \kappa\}$ is also a club subset of γ, and hence the intersection

$$C := \{\sup(M_\alpha \cap \lambda^+) : \alpha < \kappa\} \cap \{\sup(N_\alpha \cap \lambda^+) : \alpha < \kappa\}$$

is a club subset of γ that belongs to $M \cap N$. Note that C satisfies that

$$\sup(C) = \gamma = \sup(M \cap \lambda^+) = \sup(N \cap \lambda^+),$$

and since $M \cap N \cap \gamma = M \cap N \cap \lambda^+$, it also holds that $\sup(C) = \sup(M \cap N \cap \lambda^+)$.

Therefore, $\sup(M \cap \lambda^+) = \sup(N \cap \lambda^+) = \sup(M \cap N \cap \lambda^+)$ for every successor cardinal $\lambda^+ \in \mu + 1$, and by lemma 4.4.6, $M \cap \mu = N \cap \mu$.

■

Now let's rewind a little bit. Recall that our objective is to prove that

$$cf([\mu]^\kappa, \subseteq) \leq \max(\text{pcf}(A)),$$

where μ is a singular cardinal, and $\kappa < \lambda$ an infinite cardinal such that A is an interval (κ, μ) of regular cardinals of size $\leq \kappa$. To do so we have to present a cofinal subset of $[\mu]^\kappa$ of cardinality $\leq \max(\text{pcf}(A))$. Define the set

$$F = \{M \cap \mu : M \ \kappa\text{-presentable and } A \in M\}.$$

Claim: F is cofinal in $[\mu]^\kappa$.

Proof: Let $X \in [\mu]^\kappa$. We show that there exists a κ-presentable M with $A \in M$ such that $X \subseteq M \cap \mu$. By Löwenheim-Skolem there exists an elementary substructure $M_0 \preceq H_\theta$ such that $X \cup \{A\} \cup (\kappa + 1) \subseteq M_0$ and $|M_0| \leq \kappa$. We can construct, starting from M_0, a κ-presentable elementary substructure M of H_θ such that $A \in M$ and $X \subseteq M$, by a recursive application of Löwenheim-Skolem. If we combine this with the fact that $X \in [\mu]^\kappa$, we get that $X \subseteq M \cap \mu$.

□

Also note that theorem 4.4.7 implies that for any two κ-presentable elementary substructures M and N with $A \in M$ and $A \in N$, if $M \cap \mu \neq N \cap \mu$, then $\chi_M \restriction A \neq \chi_N \restriction A$, and hence

$$|F| \leq |\{\chi_M \restriction A : M \ \kappa\text{-presentable and } A \in M\}|.$$

Therefore, it suffices to show that

$$|\{\chi_M \restriction A : M \ \kappa\text{-presentable and } A \in M\}| \leq \max(\text{pcf}(A)) \tag{4.2}$$

to prove that F is a cofinal subset of $[\mu]^\kappa$ of cardinality $\leq \max(\text{pcf}(A))$, and hence that $cf([\mu]^\kappa, \subseteq) \leq \max(\text{pcf}(A))$. To achieve it we will study, as we have anticipated in the beginning, minimally obedient universal sequences and how they are related to characteristic functions of elementary substructures.

Definition 4.4.8 (Supremum along a club). Let A be a progressive set of regular cardinals, δ a limit ordinal such that $|A| < \delta < \min(A)$, and $\vec{f} = \langle f_\alpha : \alpha < \delta \rangle$ a sequence of functions in $\prod A$. For every club set $E \subseteq \delta$ of order-type $cf(\delta)$, let h_E be the function defined for every $a \in A$ as

$$h_E(a) = \sup\{f_\alpha(a) : \alpha \in E\}.$$

We say that h_E is the *supremum algong* E of the sequence $\vec{f}$. Note that since $cf(\delta) < \min(A)$, the function h_E belongs to $\prod A$, and also that if $E_1 \subseteq E_2$ then $h_{E_1} \leq h_{E_2}$.

It is interesting to note that there exists a $\leq$-minimal supremum along the club subsets of δ of order-type $cf(\delta)$. Namely, there exists a club set $C \subseteq \delta$ of order-type $cf(\delta)$ such that

$$h_C(a) \leq h_E(a)$$

for every $a \in A$ and every club set $E \subseteq \delta$ of order-type $cf(\delta)$.

To see this assume towards a contradiction that there is no such club set $C \subseteq \delta$. Hence, for every club $E \subseteq \delta$ with $ot(E) = cf(\delta)$ there is some club $E' \subseteq \delta$ with $ot(E') = cf(\delta)$ such that $h_E \not\leq h_{E'}$, i.e., there is some $a \in A$ such that $h_E(a) > h_{E'}(a)$. We construct inductively, using this idea, a sequence $\langle E_\alpha : \alpha < |A|^+ \rangle$ of club subsets of δ with order-type $cf(\delta)$, so that for every $\alpha < |A|^+$, $a_\alpha \in A$ is such that $h_{E_\alpha}(a_\alpha) > h_{E_{\alpha+1}}(a_\alpha)$. Observe that there are $|A|^+$-many a_α's that coincide, and denote them by a. Then, there is an unbounded set $B \subseteq |A|^+$ of indices for which $\langle h_{E_\alpha}(a) : \alpha \in B \rangle$ is a strictly decreasing sequence of ordinals of length $|A|^+$, which contradicts the well-foundedness of $\in$.

If J is an ideal on A, and the sequence $\vec{f} = \langle f_\alpha : \alpha < \delta \rangle$ is $<_J$-increasing, the minimal supremum h_C along the club subsets of δ of order-type $cf(\delta)$ is an upper bound of the subsequence $\langle f_\alpha : \alpha \in C \rangle$ with respect to $\leq$, and therefore all the functions f_α, for $\alpha < \delta$, are $\leq_J$-bounded by h_C. This function h_C is called *minimal club-obedient bound* of $\vec{f}$.

Definition 4.4.9 (Minimally obedient universal sequence). Let A be a progressive set of regular cardinals, $\lambda \in \mathrm{pcf}(A)$, and $\vec{f} = \langle f_\alpha : \alpha < \lambda \rangle$ a universal sequence for λ. Let κ be a fixed regular cardinal such that $|A| < \kappa < \min(A)$. We say that $\vec{f}$ is *minimally obedient* (*at cofinality* κ) if for every $\delta \in E^\lambda_\kappa$, f_δ is the minimal club-obedient bound of $\vec{f}$ (with respect to the ideal $J_{<\lambda}$).

The universal sequence $\vec{f}$ is said to be *minimally obedient* if $|A|^+ < \min(A)$ and it is minimally obedient at cofinality κ, for every regular cardinal κ such that $|A| < \kappa < \min(A)$.

Suppose that A is a progressive set of regular cardinals, and let $\lambda \in \mathrm{pcf}(A)$. We can build a minimally obedient universal sequence $\langle f_\alpha : \alpha < \lambda \rangle$ for λ, from a universal sequence $\langle f^0_\alpha : \alpha < \lambda \rangle$ by induction on $\alpha < \lambda$:

Base) $f_0 = f^0_0$.

Successor) Having built f_α, we let $f_{\alpha+1}$ be any function such that

$$\max\{f_\alpha, f^0_\alpha\} < f_{\alpha+1}.$$

Limit) If $\delta < \lambda$ is a limit ordinal with $cf(\delta) = \kappa$ and such that $|A| < \kappa < \min(A)$, let f_δ be the minimal club-obedient bound of $\langle f_\alpha : \alpha < \delta \rangle$.

If $\delta < \lambda$ is a limit ordinal not of the form above, we can use the λ-directedness of $\prod A/J_{<\lambda}$, given by theorem 4.2.3, to define f_δ as a $<_{J_{<\lambda}}$-bound of $\langle f_\alpha : \alpha < \delta \rangle$.

Definition 4.4.10 (Persistently cofinal sequence). Let A be a progressive set of regular cardinals, and let $\langle B_\lambda[A] : \lambda \leq \max(\mathrm{pcf}(A)) \rangle$ be a generating sequence for A. We say that a sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in $\prod A$ is *persistently cofinal* for λ if for every $h \in \prod A$ there is some $\alpha_0 < \lambda$ such that for every $\alpha \geq \alpha_0$ in λ,

$$h \restriction B_\lambda <_{J_{<\lambda}} f_\alpha \restriction B_\lambda.$$

By making use of theorems 4.3.12 and 4.3.16, it can be easily seen that a λ-sequence $\langle f_\alpha : \alpha < \lambda\rangle$ of functions in $\prod A$ is universal for λ if and only if it is $<_{J_{<\lambda}}$-increasing and persistently cofinal.

The next lemma is crucial in the proof of (4.2), and will also be used in the next section.

Lemma 4.4.11. *Let A be a progressive set of regular cardinals, let κ be a regular cardinal such that $|A| < \kappa < \min(A)$, and let $\lambda \in \mathrm{pcf}(A)$. Let $\vec{f} = \langle f_\alpha : \alpha < \lambda\rangle$ be a sequence of functions in $\prod A$. Let N be a κ-presentable elementary substructure of H_θ (for sufficiently large θ) such that $\lambda, A, \vec{f} \in N$, and let $\gamma = \chi_N(\lambda)$.*

(1) *If $\vec{f}$ is persistently cofinal for λ, then*

$$\{a \in A : \chi_N(a) \leq f_\gamma(a)\}$$

is a $B_\lambda[A]$ set.

(2) *If $\vec{f}$ is a minimally obedient universal sequence for λ at cofinality κ, then there is a club set $C \subseteq N \cap \gamma$ of order-type κ such that*

$$f_\gamma(a) = \sup\{f_\alpha(a) : \alpha \in C\} \in \overline{N} \cap a,$$

for every $a \in A$, and moreover

(a) $f_\gamma \leq \chi_N$.

(b) *For every $h \in N \cap \prod A$ there exists some $d \in N \cap \prod A$ such that*

$$h \restriction B_\lambda <_{J_{<\lambda}} d \restriction B_\lambda,$$

and $d \leq f_\gamma$.

Proof. For a proof see [1]. ■

Lemma 4.4.12. *Let A be a progressive set of regular cardinals and κ a regular cardinal such that $|A| < \kappa < \min(A)$. Let $\lambda \in \mathrm{pcf}(A)$, and let $\vec{f^\lambda} = \langle f^\lambda_\alpha : \alpha < \lambda\rangle$ be a sequence of functions in $\prod A$. Let N be a κ-presentable elementary substructure of H_θ (for θ big enough) so that $A, \lambda, \vec{f^\lambda} \in N$. Let $\gamma = \chi_N(\lambda)$. Suppose that $\vec{f^\lambda}$ satisfies (1) and (2) of the last lemma. Then*

$$b_\lambda = \{a \in A : \chi_N(a) = f^\lambda_\gamma(a)\}$$

is a $B_\lambda[A]$ set, and there exists a set $b'_\lambda \subseteq b_\lambda$ that belongs to N and is such that $b'_\lambda =_{J_{<\lambda}} b_\lambda$. Hence, b'_λ is also a $B_\lambda[A]$ set.

Proof. Note that since $f_\gamma \leq \chi_N$ by (2) of the last lemma,

$$b_\lambda = \{a \in A : \chi_N(a) = f^\lambda_\gamma(a)\}$$

is a $B_\lambda[A]$ set by (1).

For the second part, note that since $\vec{f^\lambda}, \gamma \in N$, if $b_\lambda \in N$, we could recover N from b_λ, and thus get $N \in N$, which is impossible. We shall find an approximation b'_λ of b_λ that lies in N.

Since $N = \bigcup_{\alpha<\kappa} N_\alpha$, if $a \in A$ and $f^\lambda_\gamma(a) < \chi_N(a)$, there is some $\alpha < \kappa$ for which $f^\lambda_\gamma(a) < \chi_{N_\alpha}(a)$. As $\kappa > |A|$ is a regular cardinal, $\sup_{a\in A} f^\lambda_\gamma(a) < \kappa$, and hence there is some $\alpha < \kappa$ such that

$$f^\lambda_\gamma(a) < \chi_N(a) \text{ if and only if } f^\lambda_\gamma(a) < \chi_{N_\alpha}(a),$$

for every $a \in A$. If we negate both sides we get that

$$\chi_N(a) \leq f^\lambda_\gamma(a) \text{ if and only if } \chi_{N_\alpha}(a) \leq f^\lambda_\gamma(a),$$

and since $f^\lambda_\gamma \leq \chi_N$ by (2) of the last lemma, we get that

$$a \in b_\lambda \text{ if and only if } \chi_{N_\alpha}(a) \leq f^\lambda_\gamma(a), \tag{4.3}$$

for every $a \in A$.

Since $\vec{f^\lambda}$ satisfies (2) of the last lemma and $\chi_{N_\alpha} \in N \cap \prod A$, there exists $d \in N \cap \prod A$ such that $\chi_{N_\alpha} \restriction B_\lambda <_{J_{<\lambda}} d \restriction B_\lambda$, and $d \leq f^\lambda_\gamma$. Therefore, $\{a \in B_\lambda : \chi_{N_\alpha}(a) \geq d(a)\} \in J_{<\lambda}$. If we define

$$b'_\lambda = \{a \in A : \chi_{N_\alpha}(a) \leq d(a)\},$$

all the parameters in the definition are in N, it clearly holds that $b'_\lambda \subseteq b_\lambda$, and moreover $\{a \in B_\lambda : a \notin b'_\lambda\} \in J_{<\lambda}$, that is, $B_\lambda \setminus b'_\lambda \in J_{<\lambda}$, or equivalently $B_\lambda \subseteq_{J_{<\lambda}} b'_\lambda$. Therefore, since b_λ is a $B_\lambda[A]$ set, $b_\lambda =_{J_{<\lambda}} B_\lambda \subseteq_{J_{<\lambda}} b'_\lambda$, which combined with $b'_\lambda \subseteq b_\lambda$ implies that $b_\lambda =_{J_{<\lambda}} b'_\lambda$.

■

Now, we can finally bound the number of characteristic functions $\chi_M \restriction A$ by $\max(\mathrm{pcf}(A))$.

Corollary 4.4.13. *Let A be a progressive set of regular cardinals, let κ be a regular cardinal such that $|A| < \kappa < \min(A)$, and let N be a κ-presentable elementary substructure of H_θ (for θ large enough) containing A and a sequence $\vec{f^\lambda} = \langle f^\lambda_\alpha : \alpha < \lambda\rangle$, for every $\lambda \in N \cap \mathrm{pcf}(A)$, satisfying* (1) *and* (2) *of lemma 4.4.11. Then there are cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ in $N \cap \mathrm{pcf}(A)$ such that*

$$\chi_N \restriction A = \sup\{f^{\lambda_0}_{\gamma_0}, \ldots, f^{\lambda_n}_{\gamma_n}\},$$

where $\gamma_i = \chi_N(\lambda_i)$, for all $1 \leq i \leq n$.

Proof. We use lemma 4.4.12 to get $B_\lambda[A]$ sets $b'_\lambda \in N$, for every $\lambda \in \mathrm{pcf}(A) \cap N$, so that

$$b'_\lambda \subseteq \{a \in A : \chi_N(a) = f_{\chi_N(\lambda)}(a)\} = b_\lambda. \tag{4.4}$$

Now, let $\lambda_0 = \max(\mathrm{pcf}(A))$ and $A = A_0$, and define by induction on $n < \omega$ sets $A_{n+1} = A \setminus (b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_n})$, and, if $A_{n+1} \neq \emptyset$, cardinals $\lambda_{n+1} = \max(\mathrm{pcf}(A_{n+1}))$. Note that since $b'_{\lambda_0}, \ldots, b'_{\lambda_n} \in N$, A_{n+1} and λ_{n+1} are also in N, and since $\lambda_0 > \cdots > \lambda_n$ forms a strictly decreasing sequence of cardinals, by lemmas 4.3.15 and 4.3.14, there must be some $n < \omega$ for which $A_{n+1} = \emptyset$. Hence,

$$A = b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_n}. \tag{4.5}$$

Since (2) of lemma 4.4.11 ensures that $f^\lambda_{\chi_N(\lambda)} \leq \chi_N$, for every $\lambda \in N \cap \mathrm{pcf}(A)$, we have that

$$f^{\lambda_0}_{\chi_N(\lambda_0)}, \ldots, f^{\lambda_n}_{\chi_N(\lambda_n)} \leq \chi_N. \tag{4.6}$$

By (4.5), for every $a \in A$, there is some $i \leq n$ for which $a \in b'_{\lambda_i}$, and thus by (4.4)

$$\chi_N(a) = f^{\lambda_i}_{\chi_N(\lambda_i)}(a).$$

Therefore, for every $a \in A$

$$\chi_N(a) \leq \sup\{f^{\lambda_0}_{\chi_N(\lambda_0)}(a), \ldots, f^{\lambda_n}_{\chi_N(\lambda_n)}(a)\},$$

which combined with (4.6) implies that

$$\chi_N \restriction = \sup\{f^{\lambda_0}_{\chi_N(\lambda_0)}, \dots, f^{\lambda_n}_{\chi_N(\lambda_n)}\}.$$

■

Note that this corollary tells us that

$$\begin{aligned} |\{\chi_N \restriction A : N \ \kappa\text{-presentable and } A \in N\}| &\leq \left| [\{\vec{f^\lambda} : \lambda \in N \cap \mathrm{pcf}(A)\}]^{<\omega} \right| \\ &= |\{\vec{f^\lambda} : \lambda \in N \cap \mathrm{pcf}(A)\}| \\ &= |\mathrm{pcf}(A)| \leq \max(\mathrm{pcf}(A)), \end{aligned}$$

which is exactly (4.2). Therefore, we have proven the following crucial theorem, anticipated in the beginning, which connects pcf theory and cardinal arithmetic.

Theorem 4.4.14. *Let μ be a singular cardinal, and let $\kappa < \mu$ be an infinite cardinal such that the interval $A = (\kappa, \mu)$ of regular cardinals has size $\leq \kappa$, and thus it is progressive. Then*

$$cf([\mu]^\kappa, \subseteq) = \max(\mathrm{pcf}(A)).$$

Therefore, by theorem 4.4.1

$$\mu^\kappa = \max(\mathrm{pcf}(A)) \cdot 2^\kappa. \tag{4.7}$$

This has important consequences for cardinal arithmetic as it implies the following bound, which was found by Shelah in 1980.

Corollary 4.4.15. *Let δ be a limit ordinal such that $\delta < \aleph_\delta$ (i.e., it is not a fixed point of the aleph function), then*

$$cf([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{(2^{|\delta|})^+},$$

and therefore,

$$\aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}.$$

Proof. Consider the progressive interval of regular cardinals $A = (|\delta|, \aleph_\delta)$. We know from the last theorem that

$$cf([\aleph_\delta]^{|\delta|}, \subseteq) = \max(\mathrm{pcf}(A)),$$

and hence,

$$\aleph_\delta^{|\delta|} = \max(\mathrm{pcf}(A)) \cdot 2^{|\delta|}.$$

By the No-holes Theorem 4.2.11, since A is an interval of regular cardinals, $\mathrm{pcf}(A)$ is also an interval of regular cardinals. If we let $\max(\mathrm{pcf}(A)) = \aleph_\gamma$, then $|\mathrm{pcf}(A)| \leq |\gamma|$, and as $A = (|\delta|, \aleph_\delta) \subseteq \mathrm{pcf}(A)$, we have that $\mathrm{pcf}(A) = [|\delta|^+, \aleph_\gamma]$. But note that $\aleph_\gamma$ is a regular cardinal, so if we assume that $|\mathrm{pcf}(A)| < \gamma$, then there exists a sequence of cardinals $\langle \kappa_\alpha : \alpha < \lambda \rangle$, with $\lambda < |\gamma|$, such that $\lim_{\alpha \to \lambda} \kappa_\alpha = \aleph_\gamma$, which is impossible. Therefore,

$$|\mathrm{pcf}(A)| = |\gamma|.$$

Since $|A| \leq |\delta|$, by theorem 4.2.6,

$$|\text{pcf}(A)| = |\gamma| \leq |P(A)| \leq 2^{|\delta|} < (2^{|\delta|})^+,$$

and therefore,

$$cf([\aleph_\delta]^{|\delta|}, \subseteq) = \max(\text{pcf}(A)) = \aleph_\gamma < \aleph_{(2^{|\delta|})^+}.$$

Finally, as $2^{|\delta|} < (2^{|\delta|})^+ \leq \aleph_{(2^{|\delta|})^+}$ and $\gamma < (2^{|\delta|})^+$, we can conclude that

$$\aleph_\delta^{|\delta|} = \max(\text{pcf}(A)) \cdot 2^{|\delta|} = \aleph_\gamma \cdot 2^{|\delta|} < \aleph_{(2^{|\delta|})^+}.$$

■

In particular, if we consider our canonical interval $A_0 = \{\aleph_n : 1 < n < \omega\}$ of regular cardinals, we can immediately infer that

$$cf([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{(2^{\aleph_0})^+},$$

and therefore, that

$$\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}.$$

So if we assume that $\aleph_\omega$ is a strong limit, then $\aleph_\omega^{\aleph_0} = 2^{\aleph_\omega}$, and hence

$$2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}.$$

4.5 Transitivity and Localization

Minimally obedient persistently cofinal sequences allowed us to get, in lemma 4.4.12, special generating sequences for progressive sets of regular cardinals A,

$$b_\lambda = \{a \in A : \chi_N(a) = f_\gamma^\lambda(a)\},$$

where recall, N κ-presentable structure. Minimally obedient sequences can be tuned further to get *elevated* sequences of ordinal functions and, from them, even more specialized generating sequences. We won't describe in full the detail how to get elevated sequences from minimally obedient ones, this can be found in [1]. We will briefly outline the hypothesis needed to get them, and what are their main properties.

Fix a progressive set of regular cardinals A, and let κ be a regular cardinal such that $|A| < \kappa < \min(A)$. Let N be a κ-presentable elementary substructure of H_θ, for θ large enough, such that $A \in N$. For every $\lambda \in N \cap \text{pcf}(A)$ let $\vec{f^\lambda} = \langle f_\alpha^\lambda : \alpha < \lambda \rangle$ be a minimally obedient universal sequence for λ (at cofinality κ) contained in N. The *elevation* of the array $\langle \vec{f^\lambda} : \lambda \in \text{pcf}(A) \rangle$ is another array $\langle \vec{F^\lambda} : \lambda \in \text{pcf}(A) \rangle$ of persistently cofinal sequences (but not club-obedient), which satisfy properties (1) and (2) of lemma 4.4.11, and belong to N.

Definition 4.5.1 (Transitive generator). Let A be a progressive set of regular cardinals. Let Z be such that $A \subseteq Z \subseteq \text{pcf}(A)$, and let $\vec{B} = \langle B_\lambda : \lambda \in Z \rangle$ be a generating sequence for A. $\vec{B}$ is said to be *transitive* (or *smooth*) if for every $\lambda \in Z$, if $\mu \in B_\lambda$, then $B_\mu \subseteq B_\lambda$.

Transitive generators, and in particular the *localization* property of the pcf operator that can be obtained from them, are the underlying basis of the next bound on the size of $\mathrm{pcf}(A)$, and consequently of Shelah's famous bound $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$, if $2^{\aleph_0} < \aleph_\omega$. The existence of a transitive generating sequence is granted thanks to the elevated sequences. A proof of the following theorem can be found either in [1] or in [10].

Theorem 4.5.2 (Transitive Generators). *Let A be a progressive set of regular cardinals and κ a regular cardinal such that $|A| < \kappa < \min(A)$. Let N be a κ-presentable elementary substructure of H_θ, for θ large enough, such that $A \in N$. Let $\langle \vec{f}^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ be an array of minimally obedient universal sequences at cofinality κ contained in N, and let $\langle \vec{F}^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ be the elevated array. For every $\lambda \in \mathrm{pcf}(A) \cap N$, if we let $\gamma = \chi_N(\lambda)$, then*

$$b_\lambda = \{a \in A : \chi_N(a) = F^\lambda_\gamma(a)\}$$

is a $B_\lambda[A]$ set, and there exists a set $b'_\lambda \subseteq b_\lambda$ that belongs to N and is such that $b'_\lambda =_{J_{<\lambda}} b_\lambda$. Hence, b'_λ is also a $B_\lambda[A]$ set. Moreover, the collection $\langle b_\lambda : \lambda \in \mathrm{pcf}(A) \cap N\rangle$ is transitive, i.e., if $\mu \in b_\lambda$, then $b_\mu \subseteq b_\lambda$.

We state below the *localization* property of the pcf function for the sake of completeness, but we won't require it in full generality:

> Let A be a progressive set of regular cardinals. If $B \subseteq \mathrm{pcf}(A)$ is also progressive, then for every $\lambda \in \mathrm{pcf}(B)$ there exists $B_0 \subseteq B$ with $|B_0| \leq |A|$ and such that $\lambda \in \mathrm{pcf}(B_0)$.

Localization is a structural principle asserting that $\mathrm{pcf}(A)$ is "small", "localised". Observe that in the case $B = \mathrm{pcf}(A)$, if $\mathrm{pcf}(A)$ is progressive, $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$ by theorem 4.2.12, and then for every $\lambda \in \mathrm{pcf}(A)$ there exists $A_0 \subseteq \mathrm{pcf}(A)$ with $|A_0| \leq |A|$ and such that $\lambda \in A_0$.

We prove here below a simpler case of localization, which is enough to get the upper bound on the size of $pcf(A)$ that will be obtained in the next section.

Theorem 4.5.3. *Let A be a progressive set of regular cardinals. There is no set $B \subseteq \mathrm{pcf}(A)$ such that $|B| = |A|^+$, and, for every $b \in B$, $b > \max(\mathrm{pcf}(B \cap b))$.*

Proof. Assume, towards a contradiction, that there is $B \subseteq \mathrm{pcf}(A)$ such that $|B| = |A|^+$, and $b > \max(\mathrm{pcf}(B \cap b))$ for every $b \in B$.

Since A is progressive $|A| < \min(A)$, and if $|A|^+ \in A$, we may remove the first cardinal of A and assume that $|A|^+ < \min(A)$.

The set $E = A \cup B$ has cardinality $|A|^+$ and satisfies $|E| < \min(E)$. Indeed, as $B \subseteq \mathrm{pcf}(A)$, it holds that $\min(A) \leq \min(B)$, and thus

$$|E| = |A|^+ < \min(A) = \min(A \cup B) = \min(E).$$

Hence, as E is progressive the Transitive Generator Theorem 4.5.2 can be applied to E.

Let $\kappa = |E|$, and let N be a κ-presentable substructure of H_θ that contains A and B. Let $\langle b_\lambda : \lambda \in \mathrm{pcf}(E) \cap N\rangle$ be a transitive generating sequence for E given by theorem 4.5.2, and let $b'_\lambda \in N$ be the sets such that $b'_\lambda \subseteq b_\lambda$ and $b'_\lambda =_{J_{<\lambda}} b_\lambda$.

Since $|A| < |B| = |A|^+$, there is an initial segment $B_0 \subseteq B$ with $|B_0| = |A|$ such that for every $a \in A$, if there exists $\beta \in B$ for which $a \in b_\beta$, then a is already in some $b_{\beta'}$ with $\beta' \in B_0$. Let $\beta_0 = \min(B \setminus B_0)$. Then $B_0 = B \cap \beta_0$ and $B_0 \in N$.

Claim: There exists a finite descending sequence of cardinals $\lambda_0 > \cdots > \lambda_n$ in $N \cap \mathrm{pcf}(B_0)$ such that

$$B_0 \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}.$$

Proof: Let $\lambda_0 = \max(\mathrm{pcf}(B_0))$. Since $B_0 \in N$ and $N \preceq H_\theta$, $\lambda_0 \in N$, and hence $b'_{\lambda_0} \in N$. Let $B_1 = B_0 \setminus b'_{\lambda_0}$. Since $B_0 \in N$ and $b'_{\lambda_0} \in N$, then $B_1 \in N$, and if we let $\lambda_1 = \max(\mathrm{pcf}(B_1))$, then clearly $\lambda_1 \in N$. Now, define $B_2 = B_1 \setminus b'_{\lambda_1}$, and keep iterating this process for $i < \omega$. Note that for all $i < \omega$, $B_{i+1} \subsetneq B_i$, hence $\lambda_0 > \lambda_1 > \lambda_2 > \dots$ is a strictly decreasing sequence of cardinals. Therefore, there must be some $n < \omega$ such that $B_{n+1} = B_n \setminus b'_{\lambda_n} = \emptyset$. Note that

$$\begin{aligned}
B_1 &= B_0 \setminus b'_{\lambda_0} \\
B_2 &= B_1 \setminus b'_{\lambda_1} = (B_0 \setminus b'_{\lambda_0}) \setminus b'_{\lambda_1} \\
B_3 &= B_2 \setminus b'_{\lambda_2} = ((B_0 \setminus b'_{\lambda_0}) \setminus b'_{\lambda_1}) \setminus b'_{\lambda_2} \\
&\vdots
\end{aligned}$$

hence $B_0 \subseteq b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_n} \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}$.

□

We will finish the proof by getting a contradiction after showing that $b_{\beta_0} \cap A \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}$. Let $a \in b_{\beta_0} \cap A$. Then, $a \in b_\beta$ for some $\beta \in B_0$. Since $\beta \in B_0 \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}$, β belongs to some b_{λ_i}, and by transitivity $b_\beta \subseteq b_{\lambda_i}$. Hence, $a \in b_{\lambda_i}$ as we wanted.

This shows that $\max(\mathrm{pcf}(b_{\beta_0} \cap A)) < \beta_0$, as

$$\mathrm{pcf}(b_{\beta_0} \cap A) \subseteq \mathrm{pcf}(b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}) = \mathrm{pcf}(b_{\lambda_0}) \cup \cdots \cup \mathrm{pcf}(b_{\lambda_n}),$$

and hence by corollary 4.3.13,

$$\begin{aligned}
\max(\mathrm{pcf}(b_{\beta_0} \cap A)) &\leq \max(\mathrm{pcf}(b_{\lambda_0})) + \cdots + \max(\mathrm{pcf}(b_{\lambda_n})) \\
&= \max\{\lambda_0, \dots, \lambda_n\} = \lambda_0 < \beta_0.
\end{aligned}$$

Now, recall that $\beta_0 = \min(B \setminus B_0)$, hence by our initial assumption,

$$\beta_0 > \max(\mathrm{pcf}(B \cap \beta_0)) = \max(\mathrm{pcf}(B_0)) = \lambda_0 > \lambda_1 > \cdots > \lambda_n.$$

Note that since $\beta_0 \in \mathrm{pcf}(A)$, it holds that $\beta_0 \in \mathrm{pcf}(b_{\beta_0} \cap A)$, otherwise $\beta_0 \in \mathrm{pcf}(A \setminus b_{\beta_0})$, there would be an ultrafilter D on $A \setminus b_{\beta_0}$ witnessing $\prod(A \setminus b_{\beta_0})$ having cofinality β_0, and thus, by theorem 4.3.14, β_0 would be the least cardinal such that $b_{\beta_0} \in D$, but this is obviously false because D is an ultrafilter on $A \setminus b_{\beta_0}$.

Therefore, we have proven that $\max(\mathrm{pcf}(b_{\beta_0} \cap A)) < \beta_0$ and $\beta_0 \in \mathrm{pcf}(b_{\beta_0} \cap A)$, but note that this contradictory since they are clearly mutually exclusive.

■

4.6 Size Limitation on pcf and The Bound

This rather short section will be devoted to the proof of Shelah's famous bound $2^{\aleph_\omega} < \aleph_{\omega_4}$, assuming $\aleph_\omega$ is a strong limit. Theorem 4.2.6 asserts that for a progressive set of regular cardinals the size of $\mathrm{pcf}(A)$ doesn't exceed $2^{|A|}$. The following theorem is an improvement to that result, and is the best bound on the size of $\mathrm{pcf}(A)$ found so far.

Theorem 4.6.1. *Let A be a progressive interval of regular cardinals. Then*

$$|\mathrm{pcf}(A)| < |A|^{+4}.$$

Proof. Suppose, aiming for a contradiction that $|\mathrm{pcf}(A)| \geq |A|^{+4}$. We will obtain a sequence B of length $|A|^+$ of cardinals in $\mathrm{pcf}(A)$ such that for every $b \in B$, $b > \max(\mathrm{pcf}(B \cap b))$, contradicting theorem 4.5.3.

Let $S = E^{|A|^{+3}}_{|A|^+} = \{\kappa < |A|^{+3} : cf(\kappa) = |A|^+\}$, and consider a club guessing sequence $\langle C_\alpha : \alpha \in S\rangle$ such that $ot(C_\alpha) = |A|^+$, given by theorem 3.2.9. Then, for every club set $D \subseteq |A|^{+3}$, there exists some $\alpha \in S$ such that $C_\alpha \subseteq D$.

Suppose that $\aleph_\sigma = \sup(A)$. Note that as $\mathrm{pcf}(A)$ is an interval of regular cardinals by the No-holes Theorem 4.2.11, and $|\mathrm{pcf}(A)| \geq |A|^{+4}$,

$$\{\aleph_{\sigma+\beta} : \beta < |A|^{+4}\} \subseteq \mathrm{pcf}(A). \tag{4.8}$$

We will define by induction on $i < |A|^{+3}$ ordinals γ_i for which $D = \{\gamma_i : i < |A|^{+3}\}$ is a club subset of $|A|^{+4}$ of order-type $|A|^{+3}$.

- $\gamma_0 = 0$.
- If $i < |A|^{+3}$ is a limit ordinal, we let

$$\gamma_i = \sup\{\gamma_j : j < i\}.$$

- Let $i < |A|^{+3}$ and suppose that $\{\gamma_j : j \leq i\}$ has been built. For every $\alpha \in S$ define the set of cardinals

$$e_\alpha = \{\aleph_{\sigma+\gamma_j} : j \in C_\alpha \cap (i+1)\}.$$

Then the set of successors of e_α, $e_\alpha^{(+)}$, is a set of regular cardinals such that $e_\alpha^{(+)} \subseteq \mathrm{pcf}(A)$, because $\gamma_j \in D$, so $\gamma_j < |A|^{+4}$, and because of (4.8). Now, for every $\alpha \in S$, ask whether $\max(\mathrm{pcf}(e_\alpha^{(+)})) < \aleph_{\sigma+|A|^{+4}}$ or not. Since $|S| = |A|^{+3}$ there are $|A|^{+3}$ such questions, and hence we can define $\gamma_{i+1} < |A|^{+4}$ so that

(1) $\gamma_i < \gamma_{i+1}$, and

(2) $\forall \alpha \in S$, if $\max(\mathrm{pcf}(e_\alpha^{(+)})) < \aleph_{\sigma+|A|^{+4}}$ (i.e., the answer to the question is "yes"), then

$$\max(\mathrm{pcf}(e_\alpha^{(+)})) < \aleph_{\sigma+\gamma_{i+1}}.$$

Let $\delta = \sup(D)$. Then $\aleph_{\sigma+\delta}$ is a singular cardianl with $cf(\aleph_{\sigma+\delta}) = |D| = |A|^{+3}$. Hence, by the Representation Theorem 4.3.17 there exists a club $C \subseteq D$ such that

$$\aleph^+_{\sigma+\delta} = \max(\mathrm{pcf}(\{\aleph^+_{\sigma+\alpha} : \alpha \in C\})). \tag{4.9}$$

Note that there is an isomorphism between D and $|A|^{+3}$, hence, under this isomorphism, we can transform C into a club subset of $|A|^{+3}$

$$E = \{i < |A|^{+3} : \gamma_i \in C\}.$$

By the club-guessing property there exists $\alpha \in S$ such that $C_\alpha \subseteq E$. If C'_α denotes the non-accumulation points of C_α, i.e.,

$$C'_\alpha = \{\xi \in C_\alpha : C_\alpha \cap \xi \text{ bounded in } \xi\},$$

we claim that $B = \{\aleph^+_{\sigma+\gamma_j} : j \in C'_\alpha\}$ is the seqeunce of elements of pcf(A) that contradicts theorem 4.5.3.

First note that since $ot(C_\alpha) = |A|^+$, then $ot(C'_\alpha) = |A|^+$. Hence, it suffices to prove that for every $i \in C_\alpha$

$$\max(\text{pcf}(\{\aleph^+_{\sigma+\gamma_j} : j \in C_\alpha \cap (i+1)\})) < \aleph_{\sigma+\gamma_{i+1}}. \tag{4.10}$$

Recall how we defined γ_{i+1}. Since $e^{(+)}_\alpha \subseteq \{\aleph^+_{\sigma+\alpha} : \alpha \in C\}$, (4.9) implies that

$$\max(\text{pcf}(e^{(+)}_\alpha)) \leq \max(\text{pcf}(\{\aleph^+_{\sigma+\alpha} : \alpha \in C\})) = \aleph^+_{\sigma+\delta}.$$

So the answer to the question for e_α was "yes", and thus it implies (4.10).

∎

This leads to the following improvement of theorem 4.4.15.

Theorem 4.6.2. *Let $\aleph_\delta$ be a singular cardinal such that $\delta < \aleph_\delta$, then*

$$cf([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{(|\delta|^{+4})}.$$

Proof. Let A be the progressive interval of regular cardinals $(|\delta|, \aleph_\delta)$. We can apply theorem 4.4.14 to get

$$cf([\aleph_\delta]^{|\delta|}, \subseteq) = \max(\text{pcf}(A)).$$

But since A is an interval of regular cardinals, by theorem 4.6.1, $|\text{pcf}(A)| < |A|^{+4}$, and hence $\max(\text{pcf}(A)) < \aleph_{(\delta+|A|^{+4})} \leq \aleph_{(|\delta|^{+4})}$. Therefore

$$cf([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{(|\delta|^{+4})}.$$

∎

Corollary 4.6.3. *Let δ be a limit ordinal such that $|\delta|^{cf(\delta)} < \aleph_\delta$, then*

$$\aleph_\delta^{cf(\delta)} < \aleph_{(|\delta|^{+4})}.$$

Proof. Recall that theorem 4.4.1 asserts that

$$\aleph_\delta^{cf(\delta)} = cf([\aleph_\delta]^{cf(\delta)}, \subseteq) \cdot 2^{cf(\delta)}.$$

Hence, if we combine this fact with the previous theorem we get that

$$\begin{aligned}\aleph_\delta^{cf(\delta)} &\leq cf([\aleph_\delta]^{cf(\delta)}, \subseteq) \cdot |\delta|^{cf(\delta)} \\ &< \aleph_{(|\delta|^{+4})} \cdot |\delta|^{cf(\delta)} < \aleph_{(|\delta|^{+4})} \cdot \aleph_\delta = \aleph_{(|\delta|^{+4})}.\end{aligned}$$

∎

In particular, if we consider the singular cardinal $\aleph_\omega$, we get that

$$cf([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{\omega_4},$$

and therefore

$$\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}.$$

If we assume that $\aleph_\omega$ is a strong limit, then $\aleph_\omega = 2^{\aleph_\omega}$, and thus

$$2^{\aleph_\omega} < \aleph_{\omega_4}.$$

4.7 Applications of pcf Theory

Shelah's pcf theory most fruitful application has undoubtedly been in cardinal arithemtic. The bounds on the powers of singular cardinals were unprecedented, and have been unparalleled since their discovery. However, pcf theory has seen many other applications, in cardinal arithmetic, and in many other areas of mathematics. Many of the applications are generalizations of results that were known to hold for a long time for regular cardinals, but were completely unattainable for singular cardinals.

An example of this is a result of Shelah in infinitary logic that says that if λ is a singular cardinal of cofinality greater than $\aleph_1$ then there are two $L_{\infty,\lambda}$-equivalent non-isomorphic models of cardinality λ. This result was knwon to hold for λ regular, but it wasn't until the developement of pcf theory that an analogous result was found for λ singular.

Another example is Stevo Todorčević's result on infinite Ramsey theory that asserts that if μ is a singular cardinal and there exists a cardinal $\kappa < \mu$ such that for every $\lambda \in [\kappa, \mu)$, $\lambda \not\to (\lambda)^2_\lambda$, then $\mu^+ \not\to (\mu^+)^2_{\mu^+}$. Recall that $\lambda \to (\kappa)^n_\mu$ means that for every n-coloring on λ with μ colors there is a homogeneous set of size κ, i.e., for every function $c : [\lambda]^n \to \mu$ there exists $X \subseteq \lambda$ of cardinality κ such that c is constant on $[X]^n$.

Here below we will briefly discuss in a bit more detail other applications of pcf theory.

4.7.1 Jónsson Algebras

The exact place where pcf theory was born was in Shelah's construction of a Jónsson algebra on $\aleph_{\omega+1}$ [61] in 1978. A *Jónsson algebra* is a first-order structure $\langle M, \{f_n : n < \omega\}\rangle$ with countably-many functions f_n, which has no proper substructure of the same cardinality.

Example 4.7.1. For every $m < \omega$ let f_m be the constant function on ω defined by $f_m(n) = m$, for every $n < \omega$. Then $\langle \omega, \{f_m : m < \omega\}\rangle$ is a Jónsson algebra of cardinality $\aleph_0$.

Many results ensured the existence of Jónsson algebras, in ZFC and under the assumption of additional axioms:

(1) If there exists a Jónsson algebra of cardinality κ then there exists a Jónsson algebra of cardinality κ^+.

(2) For every regular cardinal κ there exists a Jónsson algebra of cardinality κ^+.

(3) If the GCH holds then there exists a Jónsson algebra of cardinality κ^+ for every infinite cardinal κ.

(4) If we assume $V = L$ then there is a Jónsson algebra of cardinality κ for every infinite cardinal κ.

However, it was unknown whether a Jónsson algebra of size μ^+ for μ singular existed in ZFC. The study of cofinalities of ultraproducts of sets of regular cardinals led Shelah to deduce from the existence of a $\aleph_{\omega+1}$-scale in $\prod B_{\aleph_{\omega+1}}/J_{<\aleph_{\omega+1}}$, given by theorem 4.3.12 with respect to the progressive set of regular cardinals $A_0 = \{\aleph_n : 1 < n < \omega\}$, the following theorem.

Theorem 4.7.2. *Let μ be a singular cardinal. If there exists a cardinal $\kappa < \mu$ such that for every regular $\lambda \in (\kappa, \mu)$ there exists a Jónsson algebra of cardinality λ, then there exists a Jónsson algebra of cardinality μ^+.*

The existence of a Jónsson algebra of cardinality $\aleph_0$, combined with (1) of the enumeration above guarantees the existence of Jónsson algebras of size $\aleph_n$ for all $n < \omega$, through a straightforward induction. Hence, theorem 4.7.2 ensures the existence of a Jónsson algebra of cardinality $\aleph_{\omega+1}$, by setting $\kappa = \aleph_0$ and $\mu = \aleph_\omega$.

To this day, it is still an open problem whether a Jónsson algebra of size $\aleph_\omega$ exists.

4.7.2 On a Conjecture of Tarski

In 1925 Tarski showed that for every limit ordinal β it holds that $\prod_{\alpha<\beta} \aleph_\alpha = \aleph_\beta^{|\beta|}$, and conjectured that

$$\prod_{\alpha<\beta} \aleph_{\delta_\alpha} = \aleph_\delta^{|\beta|}, \tag{4.11}$$

for every ordinal β and every increasing sequence $\langle \delta_\alpha : \alpha < \beta \rangle$ such that $\lim_{\alpha\to\beta} \delta_\alpha = \delta$. The inequality $\prod_{\alpha<\beta} \aleph_{\delta_\alpha} \leq \aleph_\delta^{|\beta|}$ always holds, and if β has $|\beta|$-many disjoint cofinal subsets the other inequality holds as well. Indeed, if we let $\{A_i : i < |\beta|\}$ be a collection of disjoint cofinal subsets of β, then

$$\prod_{\alpha<\beta} \aleph_{\delta_\alpha} \geq \prod_{i<|\beta|} \prod_{\alpha\in A_i} \aleph_{\delta_\alpha} \geq \prod_{i<|\beta|} \aleph_\delta = \aleph_\delta^{|\beta|}.$$

Note that if $\aleph_\gamma$ is a singular cardinal of cofinality $\aleph_1$ that satisfies

$$\text{(A)} \qquad \aleph_\gamma > \aleph_{\omega_1}^{\aleph_1} \text{ and } \aleph_\gamma^{\aleph_1} > \aleph_{\gamma+\omega}^{\aleph_0},$$

then the sequence $\{\aleph_\alpha : \alpha < \omega_1\} \cup \{\aleph_{\gamma+n} : n < \omega\}$ of lenght $\omega_1 + \omega$ is a counterexample to (4.11):

$$\prod_{\alpha<\omega_1} \aleph_\alpha \cdot \prod_{n<\omega} \aleph_{\gamma+n} = \aleph_{\omega_1}^{\aleph_1} \cdot \aleph_{\gamma+\omega}^{\aleph_0} < \aleph_{\gamma+\omega}^{|\omega_1+\omega|}.$$

Such a cardinal exists in one of Magidor's models [49], and thus Tarski's conjecture can be proved to consistently fail in a model in which, for example, $\aleph_\gamma = \aleph_{\omega_1+\omega_1}$ is a strong limit, $\aleph_{\omega_1+\omega_1}^{\aleph_1} = \aleph_{\omega_1+\omega_1+\omega+2}$ and $\aleph_{\omega_1+\omega_1+\omega}^{\aleph_0} = \aleph_{\omega_1+\omega_1+\omega+1}$.

In 1991 Jech and Shelah [38] used pcf theory to show that if Tarski's conjecture fails, then it fails in the exact way that we have just mentioned, that is, if there is a counterexample to Tarski's conjecture, then there is one of length $\omega_1 + \omega$. The main result of the paper is the assertion that a necessary and sufficient condition for Tarski's conjecture to fail is the existence of a singular cardinal $\aleph_\gamma$ of cofinality $\aleph_1$ that satisfies condition (A) above.

4.7.3 Dowker Spaces

The last application that we will discuss is the construction of a Dowker space of size $\aleph_{\omega+1}$ by M. Kojman and S. Shelah [46].

Definition 4.7.3 (Normal Space). A topological space (X, τ) is said to be *normal* if for every two disjoint closed sets F_1 and F_2 of X there exist two disjoint open sets $U_1, U_2 \in \tau$ such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

Definition 4.7.4 (Paracompact Space). A topological space (X, τ) is said to be *paracompact* if every open cover has an *open refinement* that is *locally finite*. That is, for every open cover $U = \{U_i : i \in I\}$ of X there exists another open cover $V = \{V_j : j \in J\}$ of X such that for every $j \in J$ there is some $i \in I$ for which $V_j \subseteq U_i$ (*open refinement*), and every $x \in X$ has a neighborhood O such that the set $\{j \in J : V_j \cap O \neq \emptyset\}$ is finite (*locally finite*).

(X, τ) is *countably paracompact* if every countable open cover of X has a locally finite refinement.

Definition 4.7.5 (Dowker Space). A *Dowker space* is a normal topological space which is not countably paracompact.

The problem of the existence of a Dowker space in ZFC was raised by C. H. Dowker in 1951, and was answered positively 20 years later by M. E. Rudin, who constructed in [59] a Dowker space of size $\aleph_\omega^{\aleph_0}$. In the 1 page long paper, Rudin topologizes the set X of functions in $\prod_{n<\omega}(\omega_n + 2)$ for which there exists some $k < \omega$ such that $\omega < cf(f(n)) < \omega_k$, for every $n < \omega$, by using the collection of all sets of the form

$$(f, g] := \{h \in X : \forall n < \omega(f(n) < h(n) \leq g(n))\},$$

for $f, g \in \prod_{n<\omega}(\omega_n + 2)$, as a basis for the topology.

In 1996 Z. T. Balogh constructed in [8] another Dowker space in ZFC whose cardinality is $2^{\aleph_0}$. While both are constructed in ZFC, the problem with both Rudin's and Balogh's spaces is that their size is not decided in ZFC, that's why after proving the existence of such a space, the search for a "small" Dowker space (i.e., one whose size didn't depend on the exponentiation) continued. This problem was known as the "small Dowker space problem"

An answer didn't came until 1998, when Kojman and Shelah proved in [46], using pcf theory, the existence of a Dowker space of cardinality $\aleph_{\omega+1}$, by exhibiting a Dowker subspace of Rudin's space. The proof of the existence of such a space uses again the existence of an $\aleph_{\omega+1}$-scale in $\prod B_{\aleph_{\omega+1}}/J_{<\aleph_{\omega+1}}$, which is given by theorem 4.3.12, when we consider the progressive set of regular cardinals $A_0 = \{\aleph_n : 1 < n < \omega\}$.

It still is an open problem whether $\aleph_{\omega+1}$ is the least cardinal in which one can prove the existence of a Dowker space in ZFC.

Chapter 5

Open Problems and Recent Findings

The results presented in this thesis are between 30 and 50 years old, and the theory in which they are sustained is astounding. The progress in cardinal arithmetic has been tied to the emergence of numerous new techniques, which have evolved very fast and have become increasingly difficult and complex. In this chapter we will review some of these new techniques that have thrown light on the understanding of singular cardinals, and the problems which are still out of their reach.

Throughout this thesis we have already mentioned some problems that are still open to this day. In the last chapter we have pointed out that it is still unknown whether a Jónsson algebra of size $\aleph_\omega$ exists, or if $\aleph_{\omega+1}$ is the least cardinal in which there exists a Dowker space in ZFC. We have also mentioned that one of the most notorious open problems in pcf theory was the pcf conjecture, which asserted that $|\text{pcf}(A)| \leq |A|$ for any progressive set of regular cardinals A. Moti Gitik [30] proved in 2013 that assuming the consistency of infinitely many strong cardinals (a large cardinal whose consitency strength is smaller than that of a strongly compact cardinal), one can force a countable set of successor cardinals A such that $|\text{pcf}(A)| = \aleph_1$, thus proving that the pcf conjecture can consistently fail.

A positive anwser to the pcf conjecture would have implied the sharp bound $2^{\aleph_\omega} < \aleph_{\omega_1}$, assuming $\aleph_\omega$ is a strong limit. To this day, whether Shelah's bound can be improved and how is still one of the most important open problems in pcf theory, and in cardinal arithmetic in general. Just improving it to $\aleph_{\omega_3}$ would account for an extraordinary result. On the opposite side there is another major problem: can $2^{\aleph_\omega}$ be strictly greater than $\aleph_{\omega_1}$, while $\aleph_\omega$ being a strong limit? The consistency strength of a result of this kind must be enormous, and the analysis provided by the pcf theory suggests that an entire new approach is needed.

Shelah has proposed countless questions in pcf theory, and in many other areas of mathematics. In [69] there is a good batch of these questions. Just to mention one, he asks whether it is possible that if A is a progressive set of odd (even) regular cardinals, then every $\lambda \in \text{pcf}(A)$ is odd (even). Where $\aleph_{2\alpha}$ is an even cardinal, and $\aleph_{2\alpha+1}$ is an odd cardinal. More work is being done by Shelah and others on pcf theory without the Axiom of Choice. Applications of pcf theory to topology, algebra and measure theory continue to flow. Probably the two most important driving forces in this matter are Shelah's Strong and Weak Hypotheses, but before stating them we need to introduce the concept of *pseudopower*.

Definition 5.0.1 (Pseudopower). For $cf(\lambda) \leq \kappa < \lambda$ we define the *pseudopower* $\text{pp}_\kappa(\lambda)$ as the supremum of the cofinalities $cf(\prod A/D)$, where A is a set of at most κ regular cardinals below λ, and D is an ultrafilter on A containing no bounded sets bounded below λ.

We write $\mathrm{pp}(\lambda)$ instead of $\mathrm{pp}_{cf(\lambda)}(\lambda)$.

In terms of the pseudopower, theorem 4.6.3 can be reformulated as $\mathrm{pp}(\aleph_\delta) < \aleph_{|\delta|^{+4}}$, where δ is a limit ordinal such that $\delta < \aleph_\delta$, and in particular, $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$.

The Strong Hypothesis. For all singular cardinals $\mathrm{pp}(\lambda) = \lambda^+$

This is a replacement for the GCH, implied by it, and hard to change by forcing, but it is consistent with large cardinals. Theorem 2.3.8 clearly implies that the Strong Hypothesis holds above any strongly compact cardinal.

The Weak Hypothesis. For any singular cardinal λ, there are at most countably many singular cardinals $\mu < \lambda$ with $\mathrm{pp}(\mu) \geq \lambda$.

Lemma 5.0.2. *The Weak Hypothesis implies:*

- $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_1}$*, and more generally if* $\delta < \aleph_\delta$ *then* $\mathrm{pp}(\aleph_\delta) < \aleph_{|\delta|^+}$.
- *For every progressive set of regular cardinals* A*,* $\mathrm{pcf}(A)$ *has cardinality at most* $|A|$.
- $\mathrm{pp}(\lambda)$ *has cofinality at least* λ^+ *for* λ *singular.*

In 2018 M. Gitik introduced the extender based Prikry-Magidor forcing with overlapping extenders [28] and used it to construct a model in which Shelah's Weak Hypothesis for uncountable cofinality fails. More precisely, he showed that for a fixed λ, for every $\eta < \lambda$ and every regular cardinal μ, there is a generic extension in which there is an increasing sequence $\langle \kappa_\alpha : \alpha < \eta \rangle$ of cardinals of cofinality μ such that $\mathrm{pp}(\kappa_\alpha) \geq \lambda$ for all $\alpha < \eta$.

For example if we let $\lambda = \aleph_9$, there is a generic extension in which there is an increasing sequence $\langle \kappa_\alpha : \alpha < \aleph_8 \rangle$ of cardinals of cofinality $\mu = \aleph_1$ (so uncountably-many of them are singular) such that $\mathrm{pp}(\kappa_\alpha) \geq \aleph_9$ for all $\alpha < \aleph_8$. Therefore, the set $\{\kappa_\alpha : \alpha < \aleph_8\}$ witnesses the failure of the Weak Hypothesis.

The forcing introduced by Gitik was in fact a refinement of the forcing that he developed together with Magidor in [31], to show that $\aleph_\omega$ can be the first cardinal at which the GCH fails, which was in turn a specialisation of Magidor's forcing [49]. All these forcing notions are part of a group of forcing notions known in the community as Prikry-type forcings [27].

In 1970 Karel Prikry constructed a model in which he introduced an unbounded ω-sequence to a measurable cardinal κ, while preserving all cardinals $\geq \kappa$ and without introducing bounded subsets of κ. In short, Prikry's forcing does virtually nothing else other than changing the cofinality of κ to ω. Prikry observed himself that modifying his forcing he could provide a counterexample to the SCH if he could arrange it in a way that κ violated the GCH. Silver managed to start with a supercompact cardinal (a large cardinal of consistency strength bigger than that of a strongly compact) and by forcing turn it into a measurable κ satisfying $2^\kappa > \kappa^+$.

The next major developement was Magidor's forcing, which combined a system of Lévy collapses with Prikry's method of adding an ω-sequence to a large cardinal, and obtained in 1977 a model in which $\aleph_\omega$ was a strong limit and $2^{\aleph_\omega} = \aleph_{\omega+\omega+1}$. Thus the first singular cardinal was shown to consistently violate the SCH. In a subsequent paper [50] Magidor managed to construct a model in which $\aleph_\omega$ was the first cardinal violating the GCH. More precisely, he obtained a model in which $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$, and $2^{\aleph_\omega} = \aleph_{\omega+2}$.

After the introductions of Radin's forcing [58] in 1980, with which a club set is added to a large cardinal, Foreman and Woodin [22] were able to improve Magidor's forcing to obtain a model in which the GCH fails at every cardinal. Cummings [13] used Radin Forcing to obtain a model in which the GCH holds at regular cardinals and fails at singular cardinals.

In the same year, Woodin presented a proof using large cardinals that $\aleph_\omega^{\aleph_0}$ could be arbitrarily large, while $\aleph_\omega$ being a strong limit. This result, which reminds Easton's result on powers of regular cardinals, turned out to be false, as Shelah proved that same year his bound $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$.

It was Gitik in 2005 who proved for the first fixed point of the aleph function what Woodin tried to prove for $\aleph_\omega$. Indeed, in [26] Gitik uses the full machinery of large cardinal forcing developed by Magidor to show that it is impossible to find a bound for the power of the first fixed point of the aleph function. However, it is still unknown for which cardinals κ, fixed points of the aleph function, there are bounds for the exponentiation.

The methods developed in [31] by Gitik and Magidor can be used to get for each $\alpha < \omega_1$ a model in which κ is the first fixed point, the GCH holds below κ and 2^κ has α-many fixed points below it. However, it is not known how to get a similar result for $\alpha = \omega_1$.

Finally, let us say a few words about a couple recent results in cardinal arithmetic.

Definition 5.0.3 (Dense set). Let $(\mathbb{P}, \leq)$ be a partial ordering. $D \subseteq \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$, there exists $q \in D$ such that $q \leq p$.

The following generalization of stationarity is due to Jech.

Definition 5.0.4 (Stationary). Let κ be any cardinal and let X be a set such that $|X| \geq \kappa$. $S \subseteq [X]^\kappa$ is *stationary* if and only if it intersects every club subset of $[X]^\kappa$, where a club subset of $[X]^\kappa$ is a set unbounded under $\subseteq$ and closed under union of chains of length at most κ.

Definition 5.0.5 (Proper forcing notion). A notion of forcing $(\mathbb{P}, \leq)$ is *proper* if for every uncountable cardinal λ, every stationary subset of $[\lambda]^\omega$ remains stationary in the generic extension.

Definition 5.0.6 (PFA). The *Proper Forcing Axiom* (PFA) asserts that if $(\mathbb{P}, \leq)$ is proper and $\langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ such that $D_\alpha \cap G \neq \emptyset$ for every $\alpha < \omega_1$.

The Proper Forcing Axiom is a powerful extension of the Baire Category Theorem which has proved highly effective in settling mathematical statements which are independent of ZFC. For instance, it is a result of Todorčević and Veličković that PFA implies that $2^{\aleph_0} = \aleph_2$. Matteo Viale proved in 2006 the following result, which built on early work of Moore [55]:

Theorem 5.0.7 (Viale [74]). *PFA implies $2^\mu = \mu^+$ whenever μ is a singular strong limit cardinal.*

The other result that we wanted to mention is Itay Neeman's proof [56] of the consistency of the failure of SCH at a singular cardinal of countable cofinality μ together with the tree property at μ^+.

Definition 5.0.8 (Tree). A *tree* is a partially ordered set $(T, <)$ such that for each $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by the relation $<$.

For every $t \in T$, the order-type of $\{s \in T : s < t\}$ is called the *height* of t. The height of T is the least ordinal greater than the height of each element of T. For each ordinal α, the

α-th *level* of T is the set of all elements of T of height α. A *branch* of T is a maximal chain in $(T, <)$.

Definition 5.0.9 (The Tree Property). A cardinal κ has the *tree property* if every tree of height κ and whose levels have size less than κ, has a branch of length κ.

Appendix A

The Axioms of ZFC

Extensionality: If two sets a and b have the same elements, then they are equal.

$$\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b$$

Pairing: Given any sets a and b, there exists a set containing a and b as elements.

$$\exists x(a \in x \wedge b \in x)$$

Union: For every set a, there is a set containing all elements of the elements of a.

$$\exists x \forall y \in a \forall z \in y(z \in x)$$

PowerSet: For every set a there is a set that contains all subsets of a.

$$\exists x \forall y(\forall z \in y(z \in a) \rightarrow y \in x)$$

Infinity: There exists an infinite set.

$$\exists x(\exists y(y \in x) \wedge \forall y \in x \exists z \in x(y \in z))$$

Foundation: Every non-empty set a contains an $\in$-minimal element.

$$\exists y(y \in a) \rightarrow \exists y \in a \forall z \in a(z \notin y)$$

Separation: For every set a and every property, there is a set containing the elements of a that have this property.

$$\exists x \forall y(y \in x \leftrightarrow y \in a \wedge \varphi(y))$$

for all formulas φ of the language of set theory in which x does not occur free.

Replacement: For every definable function on a set a, there is a set containing all the values of the function.

$$\forall x \in a \exists! y \varphi(x, y) \rightarrow \exists z \forall x \in a \exists y \in z \varphi(x, y)$$

for all formulas φ of the language of set theory in which z does not occur free. The quantifier $\exists! y$ is read as "there is a unique y".

Choice: For every set a of pairwise disjoint non-empty sets, there exists a set that contains exactly one element from each set in a.

$$\exists x(\forall z \in x \exists y \in a(z \in y) \wedge \forall y \in a \exists! z \in y(z \in x \wedge (\exists t \in y(t \neq z) \rightarrow t \notin x)))$$

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