Vacuum decay in the presence of gravity

Author: Guim Planella Planas
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Jaume Garriga Torres

Abstract: In this paper, two mechanisms for vacuum decay in field theory are compared. First, using the Coleman-de Lucia method, the transition probability between two non-degenerate vacua is computed. Then this calculation is repeated according to the newly proposed flyover method. For this purpose, a numerical simulation is used which solves Einstein's equation for a spherically symmetric metric and with a scalar field as a source. The obtained result shows the flyover decay is dominant for a certain parametric range and it has a larger probability of upward transitions relative to CdL.

I. INTRODUCTION

Consider a field theory with a potential with two non-degenerate minima (see Fig. 1). Classically, both of them are stable as long as they are not overly perturbed. However, quantum effects allow for tunneling between them, making the one with the highest energy unstable and allowing for the process known as vacuum decay. Actually, in the presence of gravity, where energy is not conserved, both vacua are unstable and both downward and upward transitions are allowed.

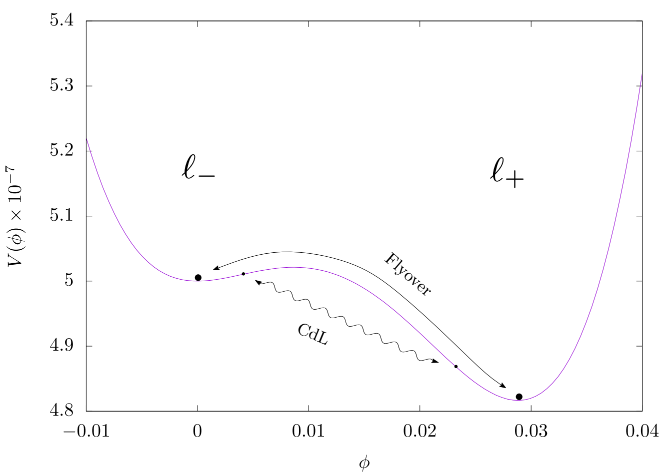

FIG. 1: Potential with two minima corresponding to (35). All values are in natural units of $G = c = \hbar = 1$. The CdL and flyover mechanisms are indicated schematically for transitions between the two vacua (with curvature length-scales ℓ_- and ℓ_+).

The standard method for calculating vacuum decay probabilities uses the WKB approximation, which is generalized from the usual theory of a particle tunneling through a barrier in one dimension to the tunneling of a quantum field. This is what Coleman did in [1, 2] developing a semiclassical theory of vacuum decay in the absence of gravity. Coleman's methods were extended in [3] to account for a non-flat initial or final vacuum. This extension forms the basis of our current understanding of these processes. An interesting fact about the result noticed by Lee and Weinberg [4] is that the probabilities of upward and downward transitions are related through the entropy of the respective de Sitter space-times. This relation is sometimes referred to as detailed balance condition and fits well with the idea of quantum de Sitter space being a thermal state.

Recently, there have been indications of another mechanism of vacuum decay [5] giving different results than the CdL approach. This new method appears to be dominant for some range of parameters of the field theory potential and it seems to put into question the validity of detailed balance. It is based on considering an initial quantum fluctuation of the field that subsequently evolves classically over the barrier and onto the true vacuum.

In this work, both methods are studied and compared to more accurately establish which one dominates vacuum decay for a given potential and to determine with more precision whether the suspected violation of detailed balance holds true.

All formulas and numerical values are expressed in natural units of $G = c = \hbar = 1$.

II. COLEMAN-DE LUCCIA VACUUM DECAY

In this section, the Coleman-de Luccia formalism is presented and an expression for the detailed balance condition is obtained. The first step is to consider a one-dimensional particle in a metastable minimum that tunnels through the barrier to reach a lower energy state. We will take the barrier to be given by the potential $V(x)$ being $[\sigma_+, \sigma_-]$ the classically forbidden region at zero energy. Then, the probability of the particle passing through is given by

$$P = Ae^{-B}, \tag{1}$$

where

$$B = 2\int_{\sigma_+}^{\sigma_-} \sqrt{2mV(x)}\mathrm{d}x, \tag{2}$$

and the prefactor A can be determined from the potential. Here, we will ignore it as we are only interested in the leading order in the WKB approximation.

On the other hand, consider a classical particle moving under the same potential. Its Lagrangian will be

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x). \tag{3}$$

Taking time to be imaginary $t = i\tau$, the kinetic term changes sign and the Lagrangian becomes

$$\mathcal{L}_E = \frac{1}{2}m\dot{x}^2 + V(x). \tag{4}$$

An overall sign has been dropped to make it positive. This is the same as if the particle evolved in real time and the potential changed sign. Now, the allowed region where the particle can move is exactly that which was classically forbidden in the quantum system. Moreover, we can relate it to the transition probability of tunneling through the barrier. For this reason, we calculate the action associated with this particle moving from one end of the classically forbidden region to the other and back. This is called a bounce and we refer to this action in imaginary time as the Euclidean action.

$$\begin{aligned}\mathcal{S}_E &= \int_{-\infty}^{\infty} \frac{1}{2}m\dot{x}^2 + V(x)\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} 2V(x)\mathrm{d}\tau \\ &= 2\int_{\sigma_+}^{\sigma_-} \sqrt{2mV(x)}\mathrm{d}x = B\end{aligned} \tag{5}$$

where we have used that the energy is zero and the equations of motion obtained from (4). Hence, we can find the probability of tunneling by calculating the Euclidean action of the bounce. Generalizing this result to any type of quantum tunneling is the standard tool for calculating the probability of a vacuum decay in the presence of gravity.

In the case we are interested in, the potential corresponds to a given self-interacting scalar field which we will suppose is the only relevant matter component of our universe. The value of its potential at the minimum will correspond to a positive vacuum energy density and thus we are initially in a de Sitter space-time. Tunneling to another minimum will imply that the vacuum energy will change and so will the Hubble scale ℓ. Therefore we have to consider the metric on a de Sitter space-time which is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \ell^2\cosh^2\frac{t}{\ell}\left(\mathrm{d}\chi^2 + \sin^2\chi\mathrm{d}S_2\right), \tag{6}$$

where $\mathrm{d}S_n$ is the volume element of the unit n-sphere. By Euclideanizing with $t = i\tilde{\tau}$, we get

$$\mathrm{d}s^2 = \mathrm{d}\tilde{\tau}^2 + \ell^2\cos^2\frac{\tilde{\tau}}{\ell}\mathrm{d}S_3. \tag{7}$$

Finally, taking $\tau - \frac{\pi}{2} = \frac{\tilde{\tau}}{\ell}$ we obtain

$$\mathrm{d}s^2 = \ell^2\left(\mathrm{d}\tau^2 + \sin^2\tau\mathrm{d}S_3\right) = \ell^2\mathrm{d}S_4, \tag{8}$$

that corresponds with the metric of a 4-sphere. In an exact calculation, we should solve Einstein's equations in this Euclideanized space-time under the condition that both the field and the metric are in the initial vacuum at infinity (in our case the bounce is compact meaning the field never reaches the minima and we should start from slightly different positions as indicated in Fig 1). The obtained solution would be our bounce. Let us assume the bounce has $O(4)$ symmetry and the transition between vacua happens in a thin wall. This means our bounce is a spherical bubble of new vacuum surrounded by a thin wall with surface tension σ separating it from the initial vacuum in the rest of space. In this approximation, Einstein's equations reduce to the conditions [6]

(i) The restrictions of the metrics of both vacuum solutions coincide on the wall.

(ii) The extrinsic curvature of the wall fulfills

$$K_{+ab} - K_{-ab} = -8\pi\left(S_{ab} - \frac{1}{2}h_{ab}S\right), \tag{9}$$

being $K_{\pm ab}$ the extrinsic curvatures of the surface from the interior and exterior respectively, S_{ab} the stress-energy tensor of the wall and h_{ab} the induced metric.

In our case, the interior and exterior metrics will be $\mathrm{d}s_\pm^2 = \ell_\pm^2\mathrm{d}S_{4\pm}$ respectively. We are taking ℓ_+ as the length scale of the vacuum in the lower minimum and ℓ_- as the one in the higher. The separating surface Σ will be at $\tau_\pm = \tau_\pm^0$ where $\tau_\pm^0$ is the value of τ in the wall according to the inside and outside metrics respectively. Then, condition (i) implies that

$$\ell_+\sin\tau_+^0 = \ell_-\sin\tau_-^0. \tag{10}$$

For the second condition we need to calculate the extrinsic curvature, so we first need the normal vector to the surface. This will be

$$\hat{n}_\pm = \frac{1}{\ell_\pm}\frac{\partial}{\partial\tau_\pm}. \tag{11}$$

Then, according to its definition

$$K_{\pm ab} = \nabla_a n_{\pm b} = \frac{\cot\tau_\pm^0}{\ell_\pm}h_{ab}, \tag{12}$$

where we used $\Gamma^\tau_{ab} = -\frac{\cot\tau}{\ell^2}h_{ab}$ and the normal vector being constant. Using the fact that $S_{ab} = -\sigma h_{ab}$, condition (ii) reduces to

$$\frac{\cot\tau_+}{\ell_+}h_{ab} - \frac{\cot\tau_-}{\ell_-}h_{ab} = -8\pi(-\sigma h_{ab} + \frac{3}{2}\sigma h_{ab}). \tag{13}$$

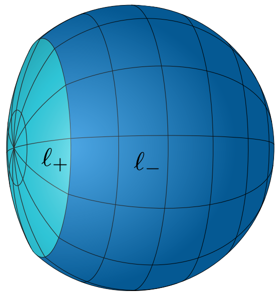

FIG. 2: Representation of the geometry of the Euclidean bounce interpolating between the two vacua. Both spheres must be understood to be four-dimensional and are joined at the locus of the domain wall's worldvolume.

That is

$$\frac{\cot\tau_-^0}{\ell_-} - \frac{\cot\tau_+^0}{\ell_+} = 4\pi\sigma. \tag{14}$$

Equations (10) and (14) completely determine τ_+^0 and τ_-^0, thus they fix the metric on the whole space. This bounce is represented in Fig. 2. We can observe both equations stay the same when we exchange the two minima, meaning there is only one way to patch the two vacua and so, the bounce for upward or downward transitions is exactly the same. This is a general fact and would hold even without the thin-wall approximation.

This allows us to calculate the Euclidean action. However, we still need to subtract the contribution corresponding to the background. Then we would have

$$B = \mathcal{S}_{bounce} - \mathcal{S}_{background}. \tag{15}$$

Using the Hilbert-Einstein action, these can be expressed in terms of easily computable integrals. However, we will not do that as we are only interested in the detailed balance condition which only requires the Euclidean action of the background. For this purpose we calculate

$$\Delta S = \ln\frac{P_\uparrow}{P_\downarrow} = B_\downarrow - B_\uparrow = \mathcal{S}_{bg\uparrow} - \mathcal{S}_{bg\downarrow}, \tag{16}$$

as the identical contributions of the upward and downward bounces cancel out. In terms of the Hubble length scales, the Euclidean action of the background becomes

$$\begin{aligned}\mathcal{S}_{bg} &= \int\left(\frac{-\mathcal{R}}{16\pi} + \mathcal{L}_E\right)\sqrt{g}\mathrm{d}^4x = -\frac{3}{8\pi}\int\frac{1}{\ell^2}\sqrt{g}\mathrm{d}^4x \\ &= -\frac{3\pi\ell^2}{4}\int_0^\pi \sin^3\tau\mathrm{d}\tau = -\pi\ell^2,\end{aligned} \tag{17}$$

Where $\mathcal{R} = 12/\ell^2$ is the Ricci scalar in the convention where the curvature of the sphere is positive and $\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2 + V(\phi)$ in analogy to equation (4). This gives

$$\Delta S = \pi\left(\ell_+^2 - \ell_-^2\right). \tag{18}$$

As indicated before, this corresponds to the entropy difference between the two spaces, as the quantity $\pi\ell^2$ is identified with the entropy of a de Sitter space-time. Thus, we have established the detailed balance condition.

III. FLYOVER VACUUM DECAY

In this section, we consider a different channel, the flyover vacuum decay. The basic idea is that the scalar field momentum develops an exceptionally large quantum fluctuation, that then evolves classically over the barrier and on to the true vacuum. Therefore, the probability of decay will be the probability of observing the minimal fluctuation whose length scale and magnitude are large enough to evolve over the barrier and not recollapse. The first step is to calculate the probability for a given velocity fluctuation of magnitude $\dot\phi_0$ and length scale $\bar\ell$. The form of the perturbation will be given by

$$\dot\phi(r, t=0) = \dot\phi_0 \exp\left(-\frac{r^2}{2\bar\ell^2}\right) \tag{19}$$

A calculation of this probability requires considering a real scalar quantum field in a background space that is not Minkowsky but de Sitter. This is beyond the scope of this work and we will simply use the result detailed in [5] where they find that

$$P \propto e^{-\frac{\dot\phi_0^2}{2\langle\dot\phi^2\rangle_{\bar\ell}}}, \tag{20}$$

with the variance given by

$$\left\langle\dot\phi^2\right\rangle_{\bar\ell} = \frac{H^4\eta^2}{4\pi^2}\int_0^\infty k^2\left|\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{(-\eta)^{\frac{3}{2}}e^{if(-k\eta)}}{(k^2\eta^2+\nu^2)^{\frac{1}{4}}}\right|^2 e^{-H^2\eta^2k^2\bar\ell^2}\mathrm{d}k, \tag{21}$$

being $f(x) = \sqrt{x^2+\nu^2} - \nu\ln\left(\frac{\nu+\sqrt{x^2+\nu^2}}{x}\right)$ and $\nu = m/H$.The variable η is the conformal time coordinate in the flat chart which will cancel out in the final result and thus can be set to any value when evaluating this integral numerically. The expansion rate H and the mass m can be calculated from the potential by

$$\ell^{-1} = H = \sqrt{\frac{8\pi V(\phi_{min})}{3}}, \qquad m = \sqrt{V''(\phi_{min})}. \tag{22}$$

As we are only interested in the leading exponential term, we just need to consider the values which give the maximum probability, as the rest will correspond to higher order corrections. To find these maximum values we will need to maximize the exponent of (20) so we will use a simulation of Einstein's equations coupled to a massive scalar field to determine which values produce a decay. The stress-energy tensor we must use is given by

$$\begin{aligned}T_{\mu\nu} &= -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L} \\ &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - g_{\mu\nu}V(\phi),\end{aligned} \tag{23}$$

where we have used the Lagrangian $\mathcal{L} = -\frac{1}{2}\left(\partial\phi\right)^2 - V(\phi)$. We will assume spherical symmetry for our space-time, meaning the metric will be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + B^2\mathrm{d}r^2 + R^2\mathrm{d}S_2, \tag{24}$$

where B and R are arbitrary functions of r and t. This is the main difference with the CdL method, where we assumed a more restrictive Euclidean $O(4)$ symmetry. The evolution for the coupled system of space-time and scalar field gives rise to the following equations, which can be easily obtained by calculating the Einstein tensor from (24) and equating it to (23) (we have used a symbolic calculation software [9]):

$$\dot{K} = -\left(K - \frac{2U}{R}\right)^2 - 2\frac{U^2}{R^2} - 8\pi\left(\dot{\phi}^2 - V(\phi)\right), \tag{25}$$

$$\dot{U} = -\frac{1-\Gamma^2+U^2}{2R} - 4\pi R\left(\frac{\dot{\phi}^2}{2} + \frac{\phi'^2}{2B^2} - V(\phi)\right), \tag{26}$$

$$\dot{\Gamma} = -\frac{4\pi R}{B}\dot{\phi}\phi', \tag{27}$$

$$\dot{B} = B\left(K - \frac{2U}{R}\right), \tag{28}$$

$$\ddot{\phi} = -K\dot{\phi} + \frac{1}{BR^2}\left(\frac{R^2}{B}\phi'\right)' - \partial_\phi V. \tag{29}$$

We have defined, following [7],

$$U = \dot{R}, \qquad \Gamma = \frac{R'}{B}, \qquad K = \frac{\dot{B}}{B} + \frac{2\dot{R}}{R}. \tag{30}$$

The initial conditions for the field momentum are detailed in (19) and we will keep the field fixed at the initial minimum. As for the variables related to the metric tensor, they can be obtained taking into account that at $t = 0$, $B = 1$ and $R = r$ and using the Misner-Sharp mass defined as [7]

$$M = \frac{R}{2}\left(1 - \Gamma^2 + U^2\right). \tag{31}$$

The choice of initial conditions for B and R implies we are using a flat slicing for the background de Sitter spacetime. From both the 00 and the 01 components of the Einstein tensor, M can be seen to fulfill at time $t = 0$

$$M'(r, t=0) = 4\pi T_{00} r^2. \tag{32}$$

Finally, integrating (32), gives $M(r)$ and then, from (31) and G_{01}, we can find

$$U(r, t=0) = \sqrt{\frac{2M}{r}}, \tag{33}$$

$$K(r, t=0) = 2\pi T_{00} r^{\frac{3}{2}}\sqrt{\frac{2}{M}} + 3r^{-\frac{3}{2}}\sqrt{\frac{M}{2}}. \tag{34}$$

To solve this system of coupled equations we use a Fortran program that implements a simple finite difference algorithm [8]. Spatial derivatives are calculated using a three-point formula and the system is evolved in time using a fourth order Runge-Kutta scheme. At each time step, the mesh is refined to compensate the exponentially increasing separation between points due to the background de Sitter expansion. A code for the simulation is available at `https://github.com/guim278/Vacuum-decay`. We will use a potential given by

$$V(\phi) = m_0^4\left(\left(\frac{\phi}{m_0}\right)^2 - \left(\frac{\phi}{m_0}\right)^3 + \frac{1}{5}\left(\frac{\phi}{m_0}\right)^4\right) + 50m_0^4, \tag{35}$$

with $m_0 = 10^{-2}$ in Planck units. This is represented in Fig. 1.

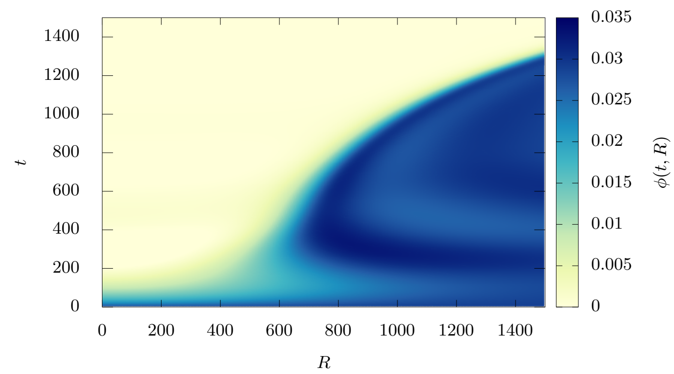

FIG. 3: Expansion of a bubble created by a perturbation with $\bar{\ell} = 520.5$ and $\dot{\phi}_0 = -3.4 \cdot 10^{-4}$ using the potential (35) which gives $\ell_+ \approx 498$ and $\ell_- \approx 489$. All quantities are in Planck units.

Starting from the lower minimum at $\phi \approx 0.0288$, we can see the results of various perturbations in figures 3 and 4. In the first case, an expanding bubble with $\phi = 0$ forms with a clearly visible separation wall. On the other hand, the second case gives place to a bubble that recollapses after some time, not leading to a transition. The separation wall is also distinguishable and we can see there is a strong perturbation in the field when the bubble recollapses. This actually poses a problem for the convergence of the simulation which diverges at the origin shortly after the recollapse. For downward transitions, the result is qualitatively the same even though the disturbances caused by a recollapse are much less important.

IV. DETAILED BALANCE VIOLATION

In this final section, we will look at whether the quantity in equation (18) is the same when calculating the decay probability using the flyover method or whether the detailed balance condition is violated, as suggested in [5].

We fix the value of $\bar{\ell}$ and, using a bisection method, we

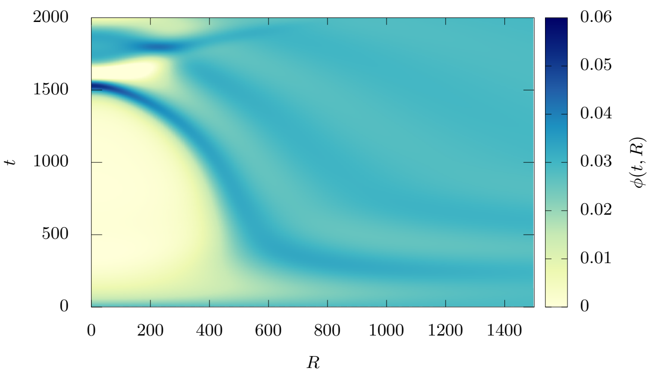

FIG. 4: Recollapse of a bubble created by a perturbation with $\bar{\ell} = 520.5$ and $\dot{\phi}_0 = -3.2 \cdot 10^{-4}$ using the potential (35) which gives $\ell_+ \approx 498$ and $\ell_- \approx 489$.

determine the value of $\dot{\phi}_0$ at which the shift between the bubble collapsing or expanding indefinitely takes place. Repeating this process over different values of $\bar{\ell}$ we find a critical curve where the velocity fluctuation is the smallest possible such that it causes a transition. Then, we numerically evaluate expression (20) over this critical curve and find at which $\bar{\ell}$ does this probability become maximal. For the upward transition the result is $\ell_{max} \approx 285.8$ and $B_\uparrow \approx 10808$. Moreover, for the downward transition we obtain $\ell_{max} \approx 102.5$ and $B_\downarrow \approx 400$, giving

$$\ln \frac{P_\downarrow}{P_\uparrow} = \Delta B = B_\uparrow - B_\downarrow \approx 1.041 \cdot 10^4. \qquad (36)$$

On the other hand, in a CdL vacuum decay, we would have

$$\Delta S = \frac{3}{8}\left(\frac{1}{V(\phi_+)} - \frac{1}{V(\phi_-)}\right) \approx 2.856 \cdot 10^4, \qquad (37)$$

meaning $e^{\Delta B} \ll e^{\Delta S}$ and thus the detailed balance condition is strongly violated. Furthermore, a numerical calculation ignoring the gravitational effects of the transition using the CdL method gives $B_{CdL\downarrow} \approx 408$. Thus, while for downward transitions both methods give comparable results, upward transitions are much more likely in the flyover method. We must note this fact depends on the potential we choose and we expect CdL to dominate for other potentials. In fact, this potential was chosen with this in mind based on estimates in [5].

V. CONCLUSIONS

To conclude, we have presented and compared two different mechanisms for vacuum decay in quantum field theory in the presence of gravity. Starting from a massive scalar field with a potential in the form of figure 1, we have considered transitions between the two non-degenerate minima.

We first use the Coleman-de Luccia method, which is the most widely known and the standard approach to vacuum decay. It assumes $O(4)$ symmetry for the semiclassical solution interpolating between the two minima. This channel satisfies the detailed balance condition for transitions between the two de Sitter vacua.

The second method is the recently proposed flyover method, which assumes an initial quantum fluctuation of the scalar field classically evolving over the barrier. In this case, we have used a numerical simulation involving Einstein's equations with a scalar field as a source and assuming an $O(3)$ symmetric space-time. For this purpose, we have introduced the Misner-Sharp formalism, commonly used in numerical simulations involving spherical symmetry.

Using these results we have confirmed that, for the potential we have studied, the latter dominates upward transitions and it does not fulfill the detailed balance condition.

Acknowledgments

This work was made possible thanks to the help of my advisor Jaume Garriga and his support despite the difficult situation due to the pandemic.

[1] S. Coleman, "Fate of the false vacuum: Semiclassical theory," *Physical Review D*, vol. 15, no. 10, p. 2929, 1977.

[2] C. G. Callan Jr and S. Coleman, "Fate of the false vacuum. ii. first quantum corrections," *Physical Review D*, vol. 16, no. 6, p. 1762, 1977.

[3] S. Coleman and F. De Luccia, "Gravitational effects on and of vacuum decay," *Physical Review D*, vol. 21, no. 12, p. 3305, 1980.

[4] K. Lee and E. J. Weinberg, "Decay of the true vacuum in curved space-time," *Physical Review D*, vol. 36, no. 4, p. 1088, 1987.

[5] J. J. Blanco-Pillado, H. Deng, and A. Vilenkin, "Flyover vacuum decay," *Journal of Cosmology and Astroparticle Physics*, vol. 2019, no. 12, p. 001, 2019.

[6] E. Poisson, "An advanced course in general relativity," *lecture notes at University of Guelph*, 2002.

[7] C. W. Misner and D. H. Sharp, "Relativistic equations for adiabatic, spherically symmetric gravitational collapse," *Physical Review*, vol. 136, no. 2B, p. B571, 1964.

[8] T. W. Baumgarte and S. L. Shapiro, *Numerical relativity: solving Einstein's equations on the computer*. Cambridge University Press, 2010.

[9] The package Einsteinpy for symbolic calculation in gravitational physics using Python.