Self-dual solutions to claims problems with ex-ante conditions

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Abstract

A claims problem involves the distribution of a divisible and scarce resource within a group of individuals with a specific right over this endowment. In the classic model, the final allocation of the estate among the agents is based on a single criterion: their own claims. However, what happens when the initial conditions of the individuals are not equal?. To answer this, we provide an extension of two of the most well-known rules: the proportional rule and the Talmudic rule. These extensions include new relevant information that characterizes the claimants and impacts the final distribution. We provide a formal definition for these generalized rules, check the properties that each of these satisfies, and give a respective characterization to each of them. Finally, we propose an empirical approach for the analysis of this kind of problems, and the perceptions from the agents among them, with and without ex-ante conditions. From this, we found a significant preference among the proportional rule in situations with ex-ante conditions. While for the cases with ex-ate conditions the perceptions tend to vary depending on the role played by the agent.

Keywords: claims problem, proportional rule, Talmudic rule, Equal losses, Equal awards, Experimental design.

JEL codes: C79, D63, D74

Contents

1 Introduction

A claims problem can be defined as a distribution problem in which a group of individuals has various entitlements to a divisible and limited resource, whose amount is not enough to jointly fulfill the shares of each individual. This type of problem has been studied under different names such as bankruptcy, taxation, rights, and rationing problems, all of which tend to converge around the same question: how should a scarce resource be divided among the claimants? – and to present parallel elements: an estate, that is, an available endowment to be divided, and the claims, which characterize the equally valid right or demand of each agent to the resource. Real-life cases of this are reflected in situations like defining an efficient and fair way of distributing the liquidation value of a firm among its creditors, or paying out the inheritance assigned to each heir when a person dies and the estate is insufficient, or, at a national level, assigning taxes or subsidies, or distributing scarce resources like food, water, medical supplies, and services among states and provinces.

An extensive literature has been devoted to the analysis and estimation of several solutions to problems of this kind, beginning with the fundamental paper by O'Neill (1982), in which the author proposed a mathematical description to the allocation problem of overlapping claims, through the study of examples from antiquity and medieval times, and analyzed the respective resolutions proposed for each of them (from the Talmud to the Proportional solution). From this point, the model proposed by O'Neill (1982) has been extended in a variety of ways, including the findings of Aumann and Maschler (1985), who reported a game-theoretic analysis of the classic bankruptcy problem and presented an explicit characterization of the nucleolus of the coalitional game that is naturally associated with such problems, using the cases proposed in the Talmud.

Another important contribution was made by Young (1988), who incorporated a characterization of possible solutions (including the Talmudic and Proportional rules) to such problems in terms of their appealing properties, especially their consistency. A similar analysis was offered by Chun (1988), who also used an axiomatic approach to answer the question as to why the proportional solution is the most widely used. By analyzing a set of axioms related to an attractive solution, he proved that only the proportional rule satisfies them. Additionally, in this same line, Thomson (1983) focused on axiomatic studies of the bargaining problems related to the dependence of solutions on the disagreement point and on the number of agents. The same author also presents an extensive literature, including a survey (Thomson, 2003, 2015), associated with this topic.

The foregoing studies usually start with the mathematical evaluation of the claims problems and, more specifically, with the possible solutions proposed in the literature, commonly known as allocation or division rules. This concept can be equally conceived as rationing schemes implemented to assess the appropriate partition of the limited endowment between a group of agents with conflicting claims (Herrero and Villar, 2001). All division rules described in the literature are characterized by satisfying natural lower and upper bound requirements, which in other words implies that no matter what rule is implemented or the situation considered, an agent can never receive a negative part or a higher share than her initial claim. Moreover, these rules are designed taking into account some requirements of how the division should be carried out (Thomson and Chun, 1992) according to the characteristics of each situation, and they tend to represent in a certain way the application of certain ethical principles and operational conditions in the determination of the proper allocation of the total resource.

Another aspect to keep in mind in the examination of the rules proposed as solutions for the general claims problem is the group of structural properties that can be applied to them. These reflect different value judgments (O'Neill, 1982) and are used in the axiomatic characterization of these alternative solutions. In general, each rule can be characterized by distinctive sets of independent axioms, relying on the group of intuitive properties that this rule satisfies, and each characterization states an intuition regarding the problems that can be solved with that specific rule (Herrero and Villar, 2001). Furthermore, the axiomatic approach is considered as a key instrument in the design of division rules, starting with the specification of the problem and involving the preferences of the agents over the alternatives, which rely on the claims of each individual in such problems, and usually results in characterization theorems used to identify a particular solution or a set of solutions for problems of this type (Thomson, 2001).

This study focuses on two specific and well-known rules: the Proportional rule and the Talmudic rule, and uses as reference and comparison two additional and common methods employed for solving these problems: the Constrained Equal Awards (CEA) and the Constrained Equal Losses (CEL) rules. As mentioned above, all these rules indicate some ethical principles, and these four rules are no exception, considering that they share an 'egalitarian intuition' in the sense that each of them seeks to divide the available resource into equal parts among the agents based on different elements. In the first case, the CEA provides an equal non-negative retribution for each agent bounded above by her claim. In the second case, the Proportional rule distributes awards proportional to claims and considers the principle that each unit related to individuals' claims should be treated equally (Aumman and Maschler, 1985). In the third case, the CEL rule imposes equal losses for all the agents subject to no-one receiving a negative amount. Finally, the Talmudic rule is a combination of two previous rules and works in the following way: in a first step, everyone receives her half-claim, in a second step, if the sum of the half-claims of all agent is less than the endowment, we use the CEA formula, and if this sum is higher, then the CEL formula is applied.

In the classic model of the resolution of claim problems, agents are represented under a single criterion: their own claims and, therefore, they are compensated according to these. In essence, the claims are considered as the only relevant information that should be included in the computation of the final allocation of the limited resource. However, what happens when we include exogenous features that also describe important characteristics of the agents? More specifically, what happens when the initial conditions of the individuals are not equal? How should these commonly used rules be reformulated to incorporate these differences in the final distribution? In addressing these questions, Timoner and Izquierdo (2016) considered the cases of the CEA and CEL rules with the inclusion of exogenous information other than claims, which reflects the initial stock or endowment (named ex-ante conditions) for each agent of the corresponding resource, and proposed a generalized version of these rules and the axiomatic characterization of them.

To illustrate the relevance of introducing these differences between agents in the specification and resolution of claim problems, we use the following example: consider the situation in which every year the different specialties of a hospital claim financing from a limited and common fund but in the last year one of these areas received an extra donation from a pharmaceutical firm. It can be assumed, therefore, that this additional resource should influence the present distribution; in other words, a positive initial stock is fixed for this specialty for the present and maybe for future allocations. Another case to analyze is the one presented by Timoner and Izquierdo (2016) regarding the distribution of irrigation water among a group of farmers during a drought. In this case, the authors described a situation in which climate change affects the initial stock of water available to each farmer. Then, although the area each farmer has under crop is equal, the distribution of water should take into account the inequalities between the farmers' water reserves.

In line with the previous studies and the implications associated with the introduction of ex-ante conditions, we aim to include these initial differences between agents in the allocation of the resource for the Proportional and the Talmudic rule. These rules are both characterized to please the self-duality property. This implies that they behave symmetrically on the solution, treating equally awards and losses from a problem. We want to check how this equal treatment of losses and gains can be influenced through the inclusion of new information about the agents. To do this, we define the extensions of these rules, then provide a revision of the properties that these rules satisfy or not, and give a specific characterization of them, which allows us to understand under which situations they can be implemented.

Moreover, the proportional rule is commonly considered as the definition of fairness for claims problems (Thomson, 2003), and so tends to be the first choice in the adjudication of conflicting claims. Given this, we also test how this fairness perception can be affected by the inclusion of new information. To do this, we propose an empirical approach, based on a non-formal experiment that allows us to determine if there exist some differences in the perception of the subjects in the application of the CEA rule, CEL rule, the proportional rule, and the Talmudic rule, with and without ex-ante conditions. This experiment consists of a questionnaire with eight multiple-choice real-life cases, with four different possible solutions, each of them associated with a well-known rule.

This paper is structured as follows. In Section 2, we start by formally introducing the basic framework of the analysis of claims problems, with a description of the classic model, how it works with ex-ante conditions and a delineation of the most common properties used in the axiomatic characterization of this rules. In section 3, we present the generalized version for the proportional rule, with its respective definition, properties' revision, and characterization. In Section 4, we repeat the same analysis for the generalized Talmudic rule, including an examination for the two-claims situation. In Section 5, we describe and analyze a possible empirical approach for such problems, based on a small experiment that considers different real-life situations. Finally, the last section includes some general conclusions for the proposed rules. The questionnaire used in the experiment and some proofs are provided in the appendix.

2 The problem

2.1 Classic model

In order to generate a solid base for the application of claim problems, we introduce the mathematical description of the classic model as proposed by O'Neill (1982). Consider, therefore, a finite set of agents $N \equiv \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$, and each agent $i \in N$ has a **claim** represented as $c_i \geq 0$ over a limited and divisible **estate** of value $E \in \mathbb{R}_+$.. Altogether, a **claims problem** is defined as a pair $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ where $c = (c_1, c_2, \ldots, c_n)$ is the **claims vector** and $0 \leq E \leq \sum_{i \in N} c_i$, which reflects insufficiency of endowment to satisfy simultaneously everyone's demand. Using Thomson's (2019) notation, let $\mathcal{C}^N$ denote the family of all these problems. Moreover, let C denote the aggregate claim and L the aggregate loss, which are computed as:

$$C = \sum_{i \in N} c_i \text{ and } L = C - E.$$

A problem of this kind can be solved using of a division **rule**, defined as a function that indicates how the estate should be divided among the agents, based only on the economically relevant variables, which are the individuals' claims and the amount of the limited resource (Gachter and Riedl, 2006). Formally, a rule connects each $(E, c) \in \mathcal{C}^N$ with a unique point $x \in \mathbb{R}^n$, defined as **awards vector**, that satisfies (i) the **non-negativity** and **claims boundedness** conditions $0 \leq x \leq c$ and (ii) the **balance** requisite $\sum_{i \in N} c_i = E$ (Thomson, 2015). This vector is interpreted as a recommendation for the problem, allocating the estate between the claimants, taking into account that (i) each of them cannot receive a negative retribution or a higher payment than their own claim, and also the requirement respect to (ii) the total distribution of the state between the group of agents N.

As mentioned above, this paper examines four well-known and commonly used rules for claim problems: the proportional rule, the constraint equal awards rule, the constraint equal losses rule, and the Talmud rule. These rules are of particular interest given the idea of equality that underlies each of them, focusing on different factors answering the question about what exactly should be equated (losses, ratios, awards, or a combination of them), especially when claimants are not identical. Another aspect to recall about these rules is their fulfillment of some specific properties, including equal treatment of equals, scale invariance, composition, path-independence, and consistency (described in the following sections), which are simultaneously satisfied by the first three rules (Herrero and Villar, 2001).

First, we describe the proportional rule, considered as the best known and most widely practiced rule. This rule allocates the resource proportional to claims, reaching an equalization of the ratios between claims and retributions for all the agents. Formally,

Definition 1 *(Proportional rule - P). For any problem $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, the proportional rule is*

$$P_i(E, c) = \lambda c_i \text{ for all } i \in N,$$

where λ is computed to solve $\lambda C = E$.

The second rule is the constrained equal-awards rule, which aims to equalize as possible the awards between claimants. This rule is grounded in the notion that all agents should get the same amount, ignoring differences in claims, and subject to no one receiving more than her claim (Thomson, 2016). It is formally defined as:

Definition 2 *(Constrained equal-awards rule - CEA). For any problem $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, the CEA rule is*

$$CEA_i(E, c) = \min\{c_i, \lambda\} \text{ for all } i \in N,$$

where λ is computed to solve $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

The next rule, the constrained equal-losses rule, remains close to the CEA rule in the sense that it intends to generate an equal impact for all agents but through the equalization of the losses among them. In other words, this rule allocates the endowment such that from each agent is removed an equal share (loss) from their claims until the sum of residuals is equal to the total estate, subject to the condition that no claimant ends up with a negative award. Formally,

Definition 3 *(Constrained equal-losses rule - CEL). For any problem $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, the CEL rule is*

$$CEL_i(E, c) = \max\{0, c_i - \lambda\} \text{ for all } i \in N,$$

where λ is computed to solve $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

Finally, the last rule treated in this research is the Talmudic rule, derived from some numerical examples given in The Talmud and formally defined by Aumann and Maschler (1985), can be described as a "hybrid" of the two previous rules, such that: in a first step everyone receives her half-claim, in a second step, if the sum of the half-claims of all agent is less than the endowment, we use the CEA formula, if this sum is higher, then the CEL formula is applied. The following is a compact definition:

Definition 4 *(Talmudic rule - T). For any problem $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, and for all $i \in N$ the Talmudic rule is,*

$$T_i(E, c) = \begin{cases} \min\left\{\frac{1}{2}c_i, \lambda\right\} & if\ E \leq \frac{1}{2}C \\ c_i - \min\left\{\frac{1}{2}c_i, \lambda\right\} & if\ E \geq \frac{1}{2}C, \end{cases}$$

where λ is computed to solve $\sum_{i \in N} T_i(E, c) = E$.

In order to illustrate the application of these four rules, let us evaluate the following example:

Example 2.1 *Consider the following three-person standard claims problem (E, c):*

$$E = 21 \text{ and } (c_1, c_2, c_3) = (5, 10, 15).$$

*If we apply the **CEA** rule to this problem we obtain the allocation $(x_1, x_2, x_3) = (5, 8, 8)$, that can be obtained by assigning progressively equal additional units of payoff: first, assign 5 units to each agent (then agent 1 is fully satisfied and leaves the game); second, the assignment process follows by additionally assigning 3 units (resulting for the part that has not divided yet, equal to 6) to agents 1 and 3. If we apply the **CEL** rule to this problem we obtain the allocation $(x_1, x_2, x_3) = (2, 7, 12)$, that can be obtained by subtracting equal units of loss between agents: first, estimate the loss of the game as the subtraction between the sum of the claims and the resource: $L = C - E = 9$ and divided it between the number of agents (in this case 3); second, the assignment process follows by subtracting 3 units to each agents claim $(5-3, 10-3, 15-3)$. If we apply the **proportional** rule to this problem we obtain the allocation $(x_1, x_2, x_3) = (3.5, 7, 10.5)$, that can be obtained by dividing equally proportional units of the resource between agents: first, estimate an equal proportion to split as the ratio between the total resource divided by the sum of the claims: $\lambda = \dfrac{E}{C} = \dfrac{7}{10}$; second, the assignment process follows by applying this proportion to each agents claim $(5 * \frac{7}{10}, 10 * \frac{7}{10}, 15 * \frac{7}{10})$. Finally, If we apply the **Talmudic** rule to this problem we obtain the allocation $(x_1, x_2, x_3) = (2.5, 6.75, 11.75)$, that can be obtained by the following process: first, assign the respective half-claim to each agent $\dfrac{c_i}{2} = (2.5, 5, 7.5)$, then since the sum of the half-claims of all agent is lower than the endowment $\frac{C}{2} = 15 < 21 = E$, we add also the solution from the CEL formula applied over the part that has not been divided yet, equal to 6, and taking into account as claims the half of them, therefore*

$$\begin{aligned} (x_1, x_2, x_3) &= \frac{c_i}{2} + CEL\big(6,\ (2.5, 6.75, 11.75)\big) \\ &= (2.5, 5, 7.5) + (0, 1.75, 4.25) \\ &= (2.5, 6.75, 11.75). \end{aligned}$$

A final, but no less important, element in the analysis of the classic model is the Path Awards of a rule, which can be defined as follows: For a given claim vector, c is the locus of the awards vector resulting from a rule as the amount to divide (the estate) varies from 0 to the sum of the claims. The relevance of this representation for the claim problems arises from the fact that it allows us to describe geometrically the behavior that characterizes each rule and to visualize some of their important properties. To elaborate on this concept, in Figure 1, we present an illustration of the path awards for the rules under examination for the two-claimant case.

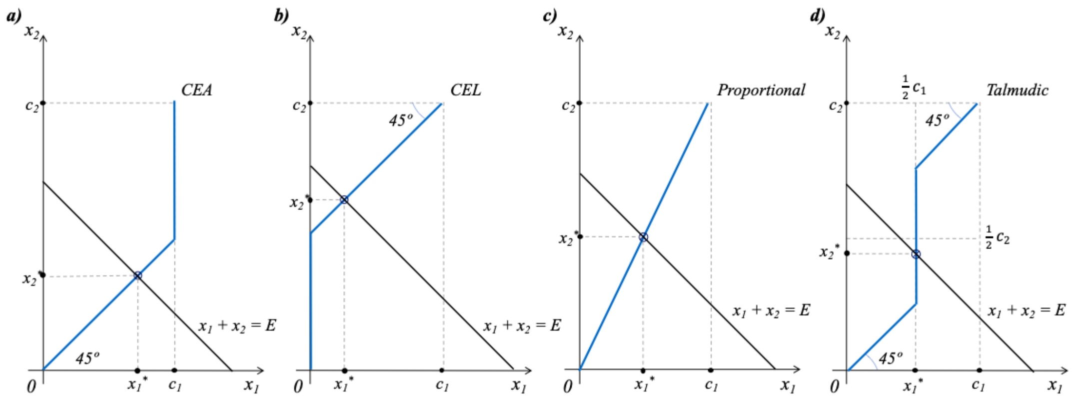

Figure 1: Paths of awards of (a) Constrained equal awards rule. (b) Constrained equal losses rule. (c) Proportional rule and (d) Talmudic rule, for two-claimant case.

As Figures 1 illustrates, the CEA rule and the CEL rule are characterized by following a 45° line from opposite points of origin, in the first case, this line is followed until it awards the whole claim to the lowest claimant (in this case c_1) and, thereafter, it continues with a vertical line until the total resource is distributed. In the second case, the CEL rule starts with a vertical line until the claimant with the maximum claim (in this case c_2) loses a quantity equal to the lowest claim, and continue from this as a line of slope 1 until claims vector. On the other hand, the proportional rule is represented through a straight line between the origin and the claims vector point. Lastly, we can observe clearly with this how the Talmudic rule works as a combination con the CEA rule and CEL rule depending on the size of the claims respect to the total estate.

2.2 Model with ex-ante conditions

Based on the classic model for conflicting claims, several authors have considered possible extensions of this framework with the incorporation of some features that could also characterize claimants, and therefore should impact the final allocation. For instance, Hougaard et al. (2013a and 2013b) worked on more complex division situations where not only claims, but also individual rights, references, or entitlements, can be used as relevant information in the distribution process. Moreover, Pulido et al. (2002 and 2008) dealt with claims problems taking into account exogenous reference points that adjust the needs or claims of each individual and therefore influence the division of the limited endowment among agents. Finally, and more close to this research, we can find the work developed by Timoner and Izquierdo (2016), in which the authors presented the axiomatic characterization for the generalized version from the CEA rule and the CEL rule with the inclusion of ex-ante conditions associated with previous individual stock or endowment of the corresponding resource.

For this research, we use the same notion given by Timoner and Izquierdo (2016) over the definition of ex-ante conditions. They are introduced in the classic model to reveal inequalities between claimants, which imply some compensations for some agents and some detriments for others, that are reflected in the final distribution of the insufficient resource. Specifically, the ex-ante conditions in our study are interpreted as an initial stock or endowment related to each agent. These are in the same units as the estate. Given this, a positive value of this variable can be understood as a better position of an agent, implying less urgency over the limited good. On the other hand, a negative value represents a greater necessity of a claimant over the endowment, which can be associated with an exogenous and negative context for the agent.

With this in mind, we can adapt the mathematical description of the classic model with the presence of ex-ante conditions as follows:

Definition 5 *Given a finite set of agents* $N \equiv \{1, 2, \dots, n\}$*, we define a* ***claims problem with ex-ante conditions*** *as a triple* (E, c, δ)*, where* $E \in \mathbb{R}_+$ *represents a limited and divisible estate,* $c \in \mathbb{R}^n_+$ *is the claims vector, and* $\delta \in \mathbb{R}^n$ *is the* ***vector of ex-ante conditions****. The problem is characterized by* $0 \leq E \leq \sum_{i \in N} c_i$*, which reflects insufficiency of endowment to honor everyone's claims.*

Using Timoner and Izquierdo (2016) notation, let $\mathcal{R}^N$ denote family of all those problems. Within this line, the definition of a allocation rule with ex-ante condition is adjusted to:

Definition 6 *A* ***generalized rule*** *is a function* F *that connects each* $(E, c, \delta) \in \mathcal{R}^N$ *with a unique point* $x = F(E, c, \delta)$*, that satisfies (i) the non-negativity and claims boundedness conditions* $0 \leq x \leq c$ *and (ii) the balance requisite* $\sum_{i \in N} c_i = E$*.*

With this two well delineated concepts, Timoner and Izquierdo (2016) proceed to define the generalized form for the CEA rule and the CEL rule as[1]:

Definition 7 *(Generalized equal-wards Rule - GEA). For any problem* $(E, c, \delta) \in \mathcal{R}^N$*, the GEA rule is defined as*

$$GEA_i(E, c, \delta) = \min\{c_i, (\lambda - \delta_i)_+\} \text{ for all } i \in N,$$

where λ *is computed to solve* $\sum_{i \in N} GEA_i(E, c, \delta) = E$*.*

Definition 8 *(Generalized equal-losses rule - GEL). For any problem* $(E, c, \delta) \in \mathcal{R}^N$*, the GEL rule is defined as*

$$GEL_i(E, c, \delta) = \max\{0, c_i - (\lambda + \delta_i)_+\} \text{ for all } i \in N,$$

where λ *is computed to solve* $\sum_{i \in N} GEL_i(E, c, \delta) = E$*.*

We illustrate how these rules can be applied, using the initial parameters from the previous example and adding some values for the ex-ante conditions.

Example 2.2 *Consider the following three-person standard claims problem* (E, c, δ) *with ex-ante conditions:*

$$E = 21 \text{ and } (c_1, c_2, c_3) = (5, 10, 15) \text{ and } (\delta_1, \delta_2, \delta_3) = (-1, 0, 2).$$

[1] From now we will use the following notation: for all $a \in \mathbb{R}$, $(a)_+ = \max\{0, a\}$

If we apply ***the GEA rule*** *to this problem we obtain the allocation* $(x_1, x_2, x_3) = (5, 9, 7)$*, that can be obtained by applying the following rule:* $GEA_i(E, c, \delta) = min\{c_i, (\lambda - \delta_i)_+\}$ *where* λ *satisfies that* $\sum_{i=1}^{3} GEA_i(E, c, \delta) = E$*. If we apply* ***the GEL rule*** *to this problem we obtain the allocation* $(x_1, x_2, x_3) = (\frac{10}{3}, \frac{22}{3}, \frac{31}{3})$*, that can be obtained by applying the following rule:* $GEL_i(E, c, \delta) = max\{0, c_i - (\lambda + \delta_i)_+\}$ *where* λ *satisfies that* $\sum_{i=1}^{3} GEL_i(E, c, \delta) = E$*. A graphic comparison between this example and the previous one is represented in Figure 2 which is based on the hydraulic representation of rationing rules given by Kaminski (2000).*

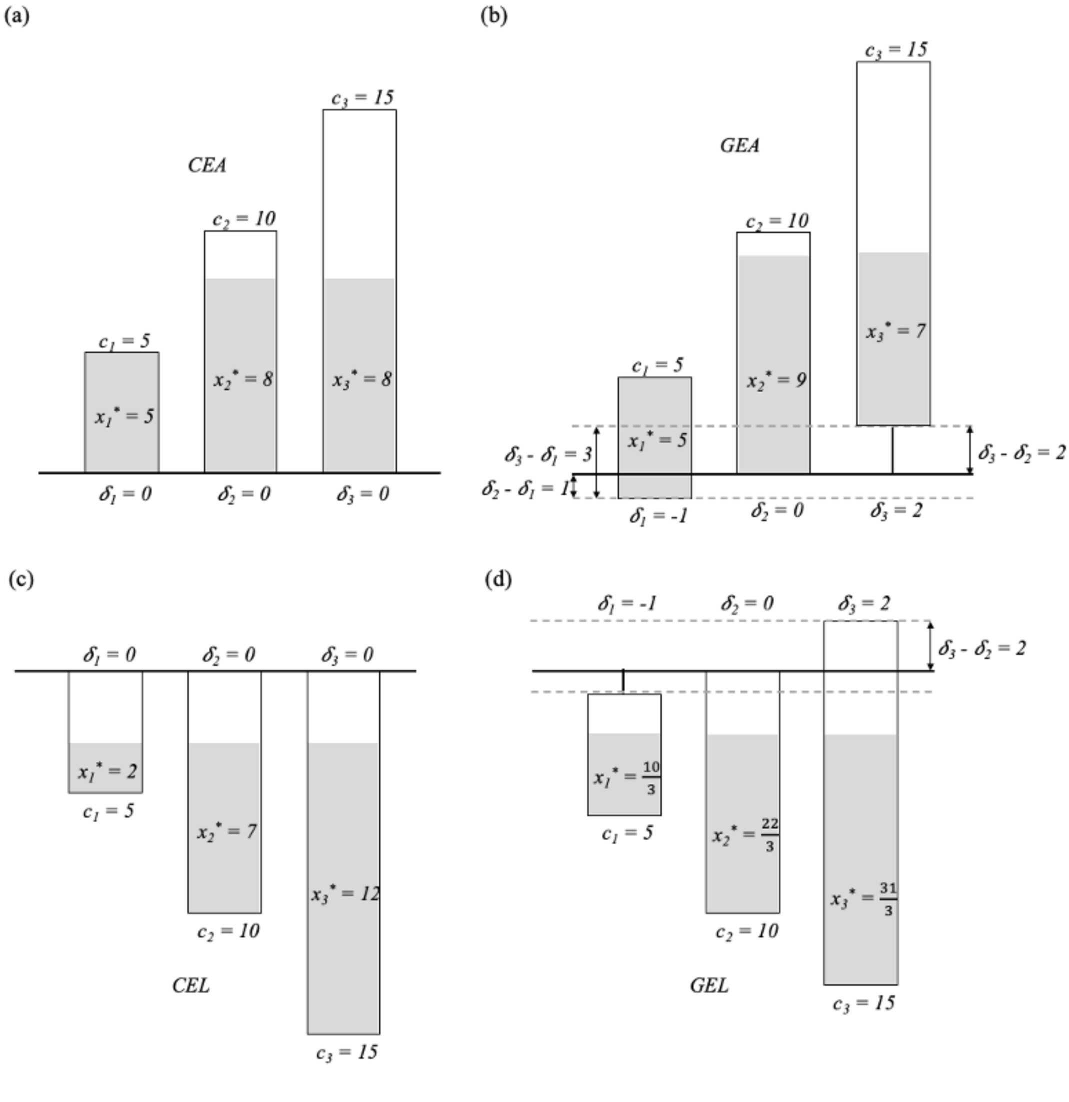

Figure 2: Equalizing (a) awards without ex-ante conditions (CEA). (b) awards with ex-ante conditions (GEA). (c) losses without ex-ante conditions (CEL) and (d) losses with ex-ante conditions (GEL)

According to Kaminski (2000), a claims problem can be visualized through a hydraulic metaphor. Each panel from Figure 2 illustrates the application of and specific rule on the resolution of a standard problem, based on a hydraulic system. Each system consists of a set of vessels that are connected through a central system of pipes. The amount of water that can be pumped through these pipes represents the total quantity of the endowment, while the vessels are related to each agent and the size of his/her claim. Once we let the water flows down through the system, it starts to fill the vessels until the water runs out. Moreover, given the connection between the vessels, it should reach an equal level among all of them, except in the case when the size of the claim is lower than the equal share (as in the case of Panel a).

For our example, we can observe in Panels (b) and (c), how the inclusion of ex-ante conditions to the problems translates into a vertical movement of the vessels. This shift captures the differences among the agents driven by this new information about them. Therefore, in the case of the GEA rule, the agents with a positive ex-ante condition present a delay to start filling up, given the benefits of their initial conditions, while the agents with a negative ex-ante condition, move down their vessels, to obtain a higher quantity of the estate before the other agents start to receive some of this. The contrary effects can be applied in the case of the GEL rule.

2.3 Properties

Once we have set down the essential elements of the model describing the adjudication of conflicting claims, with and without ex-ante conditions, we proceed to explore in greater depth the main component from the axiomatic method, concern to the set of operational properties applied to each appealing solution proposed to this type of problems. The evaluation of the properties that each alternative satisfy, allows us to characterize them, which in turn provides an insight into the kind of situations for which a rule is suitable. This idea rests on the fact that each property is intuitive and denotes a single and clear ethical or operational principle (Hougaard et al., 2013), that permits, on one hand, easy comparison between the rules and the cases where they should be applied, on the basis of the properties that they share, and on the other hand, a mean of distinguishing among them through a set of distinctive properties associated with an individual solution.

For our research, we delineate first the properties that most of our interested rules do satisfy (as equal treatment of equals, scale invariance, consistency, and resource and claims monotonicity), and then we continue with distinctive properties implemented in the literature for the axiomatic characterization of these rules. To do this, we take into account that for a given problem (E, c) we denote by $x(E, c)$ a solution for the claims problem, and for any problem (E, c, δ) we indicate by $x(E, c, \delta)$ a generalized solution for the claims problem.

- Equal treatment of equals

We start with a principle of equal distribution among agents with the same claims. This property is also known as "Symmetry" among agents and is based on an ethical matter of impartiality. In this kind of problem, the single relevant information about the agents is their claims and therefore is the only aspect that is taken into account to compute the allocation between them. In this way, agents with similar characteristics (claims) should be treated equally (same payment) without any priority and excluding not relevant information about the agents as names, gender, religion, political ideas, etc.

Now, in the case of claims problems with ex-ante conditions, we reformulated this property keeping the same principle of impartiality but introducing more information within the characterization of the agents. This implies that two agents are considered equal if and only if they share the same claims and the same ex-ante conditions. Formally,

Property 1 *(Equal treatment of equals for generalized rules). For any problem $(E, c, \delta) \in \mathcal{R}^N$, and each $i, j \in N$, we have $x_i(E, c, \delta) = x_j(E, c, \delta)$, whenever $c_i = c_j$ and $\delta_i = \delta_j$.*

- Scale invariance

The second property states that the size and units of the estate and the claims should not affect the solution of the problem. To clarify this idea, consider an arbitrary claims problem that is initially expressed in euros but now we want to know the solution in dollars. This axiom specifies that the new solution would correspond to the initial distribution but expressed now in dollars, according to the same exchange rate applied over the estate and claims. In the case of the size, the principle also implies that it should be the same divide one dollar to one million dollars.

As in the case of the first property, we adjusted this one maintaining the principle of homogeneity of degree one and now in (E, c, δ), which allows to keep the essence of the axiom respect to the size and units of the estate, claims and ex-ante conditions.

Property 2 *(Scale invariance for generalized rules). For any problem* $(E, c, \delta) \in \mathcal{R}^N$*, and all* $\alpha > 0$*, we have* $x(\alpha E, \alpha c, \alpha\delta) = \alpha x(E, c, \delta)$.

- Consistency

The third property is consistency, which was first noticed by Aumann and Maschler (1985) and reflects a powerful principle, based on the coherence between the solution to a problem applied for a group and the solutions to the problems implemented by sub-groups. In other words, this property states that the performance of a rule should be the same for a whole set and their corresponding subsets. Besides, this axiom allows us to avoid possible coalitions between agents, since if it is satisfied no group of agents has incentives to re-apply the rule in a reduced set. This applies also in the other way, if the solution for a two-person problem can be consistently extended to any number of them, then that extension is unique, or in other words, what is good for the smallest group is good for larger ones (Herrero and Villar, 2001).

Notice that for the extension version, the ex-ante conditions are introduced under the same treatment of the claims, taking into account only those belonging to the evaluated subgroup, as follows[2]:

Property 3 *(Consistency for generalized rules). For any problem* $(E, c, \delta) \in \mathcal{R}^N$ *and for all* $S \subseteq N$*, we have* $x(E, c, \delta)_{|S} = x\left(E - \sum_{i \in N \setminus S} x_i(E, c, \delta),\ c_{|S},\ \delta_{|S}\right)$.

- Resource and claims monotonicity

We now refer to monotonicity requirements, related to elementary notions as efficiency and fairness in distribution. In this study, we focus on both of them: resource monotonicity and claims monotonicity. On one hand, the resource monotonicity axiom states that if the endowment destined to be divided increase, by the principle of efficiency, each claimant would receive at least as much as with the initial state. This is reflected in different situations where due to external causes the resource can vary (positive o negative) and in such cases, all agents should suffer the consequences in a similar way.

[2]We use the following notation: for all $a \in \mathbb{R}^N$ and $S \subseteq N$, $a_{|S} = (a_i)_i \in S$.

On the other hand, claims monotonicity specifies that if an individual's claim rises, she should receive at least as much as with the initial claim. This principle is based on the fact that the main element that characterizes the agents in this kind of problem is their claims, therefore a higher claim implies a greater position over the limited resource. Formally, they are defined as:

Property 4 *(Resource monotonicity for generalized rules). For any problem* $(E, c, \delta) \in \mathcal{R}^N$ *and all* $E' > E$ *such that* $C \geq E'$*, we have* $x(E', c, \delta) \geq x(E, c, \delta)$.

Property 5 *(Claims monotonicity for generalized rules). For any problem* $(E, c, \delta) \in \mathcal{R}^N$ *and all* $c'_i > c_i$*, we have* $x\Big(E,\ (c'_i, c_{N\setminus[i]}),\ \delta\Big) \geq x\Big(E,\ (c_i, c_{N\setminus[i]}),\ \delta\Big)$.

- Composition up and composition down

To describe these properties, we follow the description made by Thomson (2003), through the following situations: Consider the case in which after having allocated the liquidation value of a firm between its creditors, its assets are re-evaluated and found to be worth less than initially thought, which lead us to rethink the problem in two alternatives: (i) we cancel the initial problem and apply the rule to the adjusted problem; or (ii) we consider the initial awards as claims and apply the rule with the lower estate. Given this, our first property, composition down states that these two ways to implement a rule should lead to the same result. Formally,

Property 6 *(Composition down for generalized rules). For any problem* $(E, c, \delta) \in \mathcal{R}^N$*, and all* $E' > E$*, we have* $x(E, c, \delta) = x(E, x(E', c, \delta), \delta)$.

On the contrary, suppose we are in the case where after having divided the liquidation value of a firm among its creditors, the amount to divide result to be higher than initially estimated. We have again two parallel options: (i) we cancel the initial problem and apply the rule to the adjusted problem; or (ii) we let agents keep their initial awards, adjust claims down by these amounts,and reapply the rule to divide the additional amount. For this, our second property, composition up, specifies that these two ways to implement a rule should lead to the same result. Formally,

Property 7 *(Composition up for generalized rules). For any problem* $(E, c, \delta) \in \mathcal{R}^N$*, and all* $E' > E$ *such that* $C \geq E'$*, we have* $x(E', c, \delta) = x(E, c, \delta) + x(E' - E, c - x(E, c, \delta), \delta)$.

- Self-duality

Finally, the last and most important property of our study is self-duality, which states that the same principle can be applied to the resolution of a claim problem, trying to distribute awards or to allocate losses among agents. This property implies a certain symmetry in the behavior of the solution, in the sense that it treats equally both parts of the problem. Moreover, this notion has its origin in the Talmud, which determines that dividing 'what is there' and dividing 'what is not there' should be treated symmetrically (Herrero and Villar, 2001).

This is a strong axiom, that just a few solutions satisfy, including the Proportional rule and the Talmudic rule, while the CEA and the CEL rule do not hold this property, on the contrary, they represent the dual version of each other. The relevance of the self-duality notion arises from the different perceptions that can be applied over the resolution of a claims problem. There are certain situations in which it is natural to think in terms of dividing the award (as in the case of a heritage distribution), while there are other cases where it makes sense to think in terms of dividing the loss (as a bankruptcy problem). However, there are other cases in which it is equally attractive to think in terms of losses or of gains, and is in these cases where self-dual solutions are called for (Aumann and Mashler, 1985).

For the generalized rules, we apply the same method proposed by Timoner and Izquierdo (2015), adapting this axiom taking into account that positive ex-ante conditions become negative (and vice versa) when passing from the initial problem (E, c, δ) to the dual problem $(L, c, -\delta)$. Therefore, the definition of this axiom for this case is

Property 8 *(Self-duality for generalized rules). For any problem $(E, c, \delta) \in \mathcal{R}^N$, we have $x(E, c, \delta) = c - x(E, c, \delta)$.*

3 The generalized proportional rule

3.1 Definition

We extend the proportional rule, keeping the notion of equalizing payments through a unique ratio applied over each claim, but in the presence of ex-ante conditions. To achieve this, we propose the following definition:

Definition 9 *Let $(E, c, \delta) \in \mathcal{R}^N$ be a rationing problem with ex-ante conditions. Then, the generalized proportional rule, GP is defined as*

$$GP(E, c, \delta) = \min\{(\lambda c_i - \delta_i)_+,\ c_i\},$$

where λ is computed to solve $\sum_{i \in N} GP_i(E, c, \delta) = E$.

In order to explain the application of this definition, let us consider the following example:

Example 3.1 *Consider the following two-person standard claims problem (E, c):*

$$E = 18 \text{ and } (c_1, c_2) = (10, 20).$$

Applying the proportional rule to this problem we get the allocation $(x_1, x_2) = (6, 12)$, which is computed through the division of the resource into equally proportional units among agents: first, we estimate an equal proportion to split, as the ratio between the total resource divided by the sum of the claims: $\lambda = E/\sum_{i \in 2} c_i = \frac{3}{5}$; second, the assignment process follows by applying this proportion to each agents claim $(10 * \frac{3}{5}, 20 * \frac{3}{5})$. Let now add to the problem the following vector of ex-ante conditions:

$$(\delta_1, \delta_2) = (-5, 5).$$

To solve this, we start by matching the initial conditions between the set of agents, taking into account their initial ratios, $\frac{\delta_i}{c_i}$. In this case the initial ratios for claimant 1 and 2 are equal to $\frac{\delta_1}{c_1} = -\frac{1}{2}$ and $\frac{\delta_2}{c_2} = \frac{1}{4}$. Then, we *estimate the differences between these two ratios, as* $\frac{1}{4} - \left(-\frac{1}{2}\right) = \frac{3}{4}$*, and multiply this for the claim associated to the smallest initial ratio, in this case* c_1*, obtaining as result* 7.5 *units. This amount is assigned to* c_1*, 5 units that correspond to the compensation for the initial negative condition and 2.5 units are to reach the initial ratio of* c_2*,* $\frac{2.5}{10} = \frac{1}{4} = \frac{\delta_2}{c_2}$*. The remaining 10.5 units from the estate, are allocated proportionally between agents, therefore we estimate an equal proportion for both claimants as* $\lambda' = \frac{10{,}5}{10+20}$*, and apply this over their respective claims, resulting in* $x_1' = \frac{10{,}5}{10+20} \cdot 10 = 3.5$ *and* $x_2' = \frac{10{,}5}{10+20} \cdot 20 = 7$*. But, if we sum both parts of the solution from* c_1 *we obtain a total payment for her equal to 11 units, which is not possible given the upper boundary of the problem of this kind* ($x_i \leq c_i$)*, then for* c_1 *we can only assign 10 units, and the remain one is submitted to* c_2*, obtaining as final solution* $x(E, c, \delta) = (10, 8)$*. A graphic representation of this procedure can be found in Figure 3 which reflects the path awards of both parts of this example.*

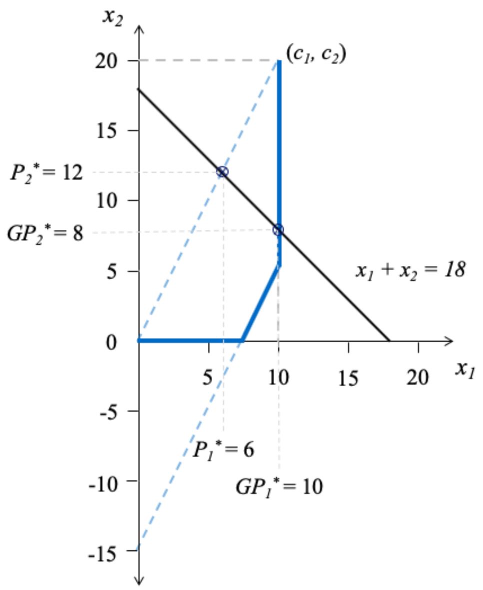

Figure 3: Paths of awards of Proportional rule (dotted line) and Generalized proportional rule, for two-claimant case.

As Figure 3 shows, the proportional rule and its generalized version start the awards' distribution from the same initial point $(0, 0)$. However, if we add some differences among agents, through the inclusion of ex-ante conditions to the problem, the generalized proportional rule tries to balance these initial differences, in proportional terms of what each agent initially have and what they demand. In example 3.1, we can detail how this rule distributes as many units as necessary so that the two claimants are at the same level (in this case, 7.5 units for c_1). From this point $(7.5, 0)$ the generalized proportional rule presents the same pattern from the original proportional rule, keeping the boundary conditions of the problem.

In the case of a problem with three or more agents, $n \geq 3$, we work with the same methodology applied as follows: we organize the claimants in ascending order according to their respective initial ratios, then we assign part of the endowment to the agent with the smallest initial ratio, until it reaches the initial level of the second-lowest, and we apply the same procedure now with respect to the third-lowest, and so on. Once the agents are at the same level, we assign the rest of the endowment with the same principles of the proportional rule, taking as claims, the part that has not yet been fulfilled of each agent.

Remark that if $\delta = (0, 0..., 0) \in R^N$, then $GP(E, c, \delta) = P(E, c)$. This means that the generalized proportional rule coincides with the proportional rule for standard claims problems, in the absence of ex-ante conditions. This is because in this scenario the agents start with equal conditions, so there is no need for an initial leveling in which one agent has priority over another. All start to receive at the same time and in the same proportion to their claims.

3.2 Properties

Proposition 1 *The generalized proportional rule satisfies scale invariance.*

Proof: Let $(E, c, \delta) \in \mathcal{R}^N$ be an arbitrary claims problem with ex-ante conditions and $\alpha > 0$. Then for all $i \in N$, we have:

$$\begin{aligned} GP(\alpha \cdot E,\, \alpha \cdot c,\, \alpha \cdot \delta) \quad &= \min\{(\lambda(\alpha \cdot c_i) - \alpha \cdot \delta_i)_+,\, \alpha \cdot c_i\} \\ &= \min\{\alpha \cdot (\,\lambda c_i - \delta_i)_+,\, \alpha \cdot c_i\} \\ &= \alpha \cdot \min\{(\,\lambda c_i - \delta_i)_+,\, c_i\} = \alpha \cdot GP(E,\, c,\, \delta). \end{aligned}$$

□

Proposition 2 *The generalized proportional rule does not satisfy composition up.*

Proof: Let us consider the following counterexample:

$$E = 20,\ (c_1, c_2) = (18, 32),\ (\delta_1, \delta_2) = (5, 10) \quad and \quad E' = 40.$$

With this we want to check if the following equality holds:

$$GP_i(E',\, c,\, \delta) = GP_i(E,\, c,\, \delta) + GP_i(E' - E,\, c - GP_i(E,\, c,\, \delta),\, \delta).$$

By definition we have:

$$GP_1\big(20, (18, 32), (5, 10)\big) = \min\{(\lambda 18 - 5)_+, 18\} \quad \text{and} \quad GP_2\big(20, (18, 32), (5, 10)\big) = \min\{(\lambda 32 - 10)_+, 32\}.$$

$$GP_1\big(40, (18, 32), (5, 10)\big) = \min\{(\lambda' 18 - 5)_+, 18\} \quad \text{and} \quad GP_2\big(40, (18, 32), (5, 10)\big) = \min\{(\lambda' 32 - 10)_+, 32\}.$$

Solving these equations we get that: $GP(E,\, c,\, \delta) = (7.60,\, 12.40)$ and $GP(E',\, c,\, \delta) = (14.80,\, 25.20)$. Subtracting the solution of $GP(E,\, c,\, \delta)$ of the claims on the initial problem, we get that:

$$GP_1\big(40 - 20, (18 - 7.60,\, 32 - 12.40), (5, 10)\big) = \min\{(\lambda \cdot (10.40) - 5)_+, (10.40)\}.$$

$$GP_2\big(40 - 20, (18 - 7.60,\, 32 - 12.40), (5, 10)\big) = \min\{(\lambda \cdot (19.60) - 10)_+, (19.60)\}.$$

From we get a solution of $GP(E - E',\, c - GP(E,\, c,\, \delta),\, \delta) = (7.13,\, 12.87)$, which sums with the solution of the $GP_i(E,\, c,\, \delta)$ problem is equal to $(14.73, 25.27)$, that differs from the solution with the original vector claims, $GP(E',\, c,\, \delta) = (14.80,\, 25.20)$.

Proposition 3 *The generalized proportional rule does not satisfy composition down.*

Proof: Let us consider the following counterexample:

$$E = 10,\ (c_1, c_2) = (10, 15),\ (\delta_1, \delta_2) = (5, 0) \quad and \quad E' = 15.$$

With this we want to check if the following equality holds:

$$GP_i(E,\, c,\, \delta) = GP_i(E,\, GP_i(E',\, c,\, \delta),\, \delta).$$

By definition we have:

$$GP_1\big(10, (10, 15), (5, 0)\big) = \min\{(\lambda 10 - 5)_+, 10\} \quad \text{and} \quad GP_2\big(10, (10, 15), (5, 0)\big) = \min\{(\lambda 15 - 0)_+, 15\}.$$

$$GP_1\big(15, (10, 15), (5, 0)\big) = \min\{(\lambda' 10 - 5)_+, 10\} \quad \text{and} \quad GP_2\big(15, (10, 15), (5, 0)\big) = \min\{(\lambda' 15 - 0)_+, 15\}.$$

Solving these equations we get that: $GP(E,\, c,\, \delta) = (1, 9)$ and $GP(E',\, c,\, \delta) = (3, 12)$. Replacing the solution of $GP(E',\, c,\, \delta)$ as claims on the initial problem, we get that:

$$GP_1(10, 3, 5) = \min\{(\lambda 3 - 5)_+, 3\} \quad \text{and} \quad GP_2(10, 12, 0) = \min\{(\lambda 12 - 0)_+, 12\}.$$

From we get a solution of $GP(E,\, GP(E',\, c,\, \delta),\, \delta) = (0, 10)$, which differs from the solution with the original vector claims, $GP(E,\, c,\, \delta) = (1, 9)$.

Proposition 4 *The generalized proportional rule satisfies consistency.*

Proof: In Theorem 1, presented in Section 3.3, we give a characterization for the generalized proportional rule by making payoff bilateral comparisons. Then, it is straightforward that if we reevaluate the problem for subgroups of claimants, then the solution would satisfy the same bilateral comparisons and thus the solution does not change (Timoner and Izquierdo, 2014). □

Proposition 5 *The generalized proportional rule satisfies resource monotonicty.*

Proof: Let $(E, c, \delta) \in \mathcal{R}^N$ be an arbitrary claims problem with ex-ante conditions and E' such that $C \geq E' > E$. Then, by definition we have:

$$GP_i(E, c, \delta) = \min\{(\lambda c_i - \delta_i)_+, c_i\}.$$

$$GP_i(E', c, \delta) = \min\{(\lambda' c_i - \delta_i)_+, c_i\}.$$

We must prove that $\lambda' \geq \lambda$. Suppose on the contrary that $\lambda' < \lambda$. Then

$$\min\{\max\{\lambda' c_i - \delta_i\}, c_i\} \leq \min\{\max\{\lambda c_i - \delta_i\}, c_i\},$$

which implies that

$$E' = \sum_{i \in N} \min\{(\lambda' c_i - \delta_i)_+, c_i\} \leq \sum_{i \in N} \min\{(\lambda c_i - \delta_i)_+, c_i\} = E,$$

and contradicts the assumption $E' > E$. Hence $\lambda' > \lambda$ and this $GP_i(E', c\delta) \geq GP_i(E, c, \delta)$. □

Proposition 6 *The generalized proportional rule satisfies self-duality.*

Proof: Let $(E, c, \delta) \in \mathcal{R}^N$ be an arbitrary claims problem with ex-ante conditions. By definition we have:

$$GP(L, c, -\delta) = \min\{(\lambda^* c + \delta)_+, c\} \quad \text{where} \quad \sum_{i \in N} GP_i(L, c, -\delta) = L.$$

Given the definition of self-duality, for all $i \in N$, we have that

$$\begin{aligned}
GP_i(E, c, \delta) \quad &= c_i - GP_i(L, c, -\delta) \\
&= c_i - \min\{(\lambda^* c_i + \delta_i)_+,\, c_i\} \\
&= c_i - \min\{\max\{\lambda^* c_i + \delta_i,\, 0\},\, c_i\} \\
&= c_i + \max\{-\max\{\lambda^* c_i + \delta_i,\, 0\},\, -c_i\} \\
&= c_i + \max\{\min\{-(\lambda^* c_i + \delta_i),\, 0\},\, -c_i\} \\
&= \max\{c_i + \min\{-(\lambda^* c_i + \delta_i),\, 0\},\, c_i - c_i\} \\
&= \max\{\min\{c_i - (\lambda^* c_i + \delta_i),\, c_i\},\, 0\} \\
&= \max\{\min\{(1 - \lambda^*) c_i - \delta_i),\, c_i\},\, 0\}
\end{aligned}$$

where $\lambda = 1 - \lambda^*$, then

$$\begin{aligned}
&= \max\{\min\{(\lambda c_i - \delta_i),\, c_i\},\, 0\} \\
&= \min\{(\lambda c_i - \delta_i)_+,\, c_i\} = GP_i(E, c, \delta).
\end{aligned}$$

In the last line, we know by definition that $c_i \geq 0$, then the minimization inside the maximization will be always higher or equal to zero if and only if $(\lambda c_i - \delta_i) \geq 0$ which is equal to say $(\lambda c_i - \delta_i)_+$. □

3.3 A characterization for the Generalized proportional rule

In this section, we provide a characterization for our proposed generalized proportional rule, through a description of it by means of comparing the ex-ante conditions for any pair of agents, as follows,

Theorem 1 *Let (E, c, δ) be a claims problem with ex-ante conditions and let $x \in \mathbb{R} : \sum_{i \in N} x_i = E$ and $0 \leq x_i \leq c_i$ for all $i \in N$. Then*

$$x = GP(E, c, \delta) \Leftrightarrow \begin{array}{c} \textit{for all } i, j \in N, \textit{ and } i \neq j, \textit{ such that } \dfrac{1}{c_i} \cdot (x_i + \delta_i) < \dfrac{1}{c_j} \cdot (x_j + \delta_j), \\ \textit{it holds that either } x_i = c_i \textit{ or } x_j = 0. \end{array}$$

Proof. Let $x = GP(E, c, \delta)$ but suppose there exists $i, j \in N$ such that

$$\frac{1}{c_i} \cdot (x_i + \delta_i) < \frac{1}{c_j} \cdot (x_j + \delta_j), \text{but } x_i < c_i \text{ and } x_j > 0.$$

Since $x_i < c_i$ it follows that

$$x_i = \max\{\lambda c_i - \delta_i, 0\}. \tag{1}$$

Moreover, $x_j > 0$ implies that

$$\lambda c_j - \delta_j > 0. \tag{2}$$

Then,

$$\frac{1}{c_i}\cdot(x_i+\delta_i) = \frac{\max\{\lambda c_i-\delta_i,0\}+\delta_i}{c_i} = \frac{\max\{\lambda c_i,\delta_i\}}{c_i} = \max\left\{\lambda,\frac{\delta_i}{c_i}\right\}$$

$$\geq \lambda = \lambda\frac{c_j}{c_j} \geq \min\left\{\frac{\lambda c_j}{c_j},\frac{c_j+\delta_j}{c_j}\right\} = \min\left\{\frac{\lambda c_j}{c_j},\frac{c_j+\delta_j}{c_j}\right\} - \frac{\delta_j}{c_j}+\frac{\delta_j}{c_j}$$

$$= \min\left\{\frac{\lambda c_j-\delta_j}{c_j},\frac{c_j}{c_j}\right\}+\frac{\delta_j}{c_j} = \frac{1}{c_j}\left(\min\{\lambda c_j-\delta_j,c_j\}\right)+\frac{\delta_j}{c_j}$$

$$= \frac{1}{c_j}\left[\min\{(\lambda c_j-\delta_j)_+,c_j\}+\delta_j\right] = \frac{x_j+\delta_j}{c_j},$$

where the first inequality follows from (1) and the penultimate one form (2). Hence, this contradicts the hypothesis $\frac{1}{c_i}\cdot(x_i+\delta_i) < \frac{1}{c_j}\cdot(x_j+\delta_j)$.

Now, Let us suppose that for all $i,j\in N$ such that

$$\frac{1}{c_i}\cdot(x_i+\delta_i) < \frac{1}{c_j}\cdot(x_j+\delta_j), \tag{3}$$

and it holds that either $x_i=c_i$ or $x_j=0$, but $x\neq GP(E,c,\delta)$. Since both x and GP are efficient vectors, there is $i,j\in N$ such that

$$x_i > GP_i(r,c,\delta) \text{ and } x_j < GP_j(r,c,\delta). \tag{4}$$

By (4) we can deduce that

$$x_i > \min\{(\lambda c_i-\delta_i)_+,c_i\} \Rightarrow x_i = (\lambda c_i-\delta_i)_+ \text{ and}$$

$$x_j < \min\{(\lambda c_j-\delta_j)_+,c_j\} \Rightarrow x_j < c_j \text{ and } x_j < \lambda\omega_j-\delta_j$$

Hence, it follows that

$$\frac{x_j+\delta_j}{c_j} < \frac{\lambda c_j-\delta_j+\delta_j}{c_j} = \lambda = \frac{\lambda c_i}{c_i} \leq \max\left\{\frac{\lambda c_i}{c_i},\frac{\delta_i}{c_i}\right\}+\frac{\delta_i}{c_i}-\frac{\delta_i}{c_i}$$

$$= \max\left\{\frac{\lambda c_i-\delta_i}{c_i},0\right\}+\frac{\delta_i}{c_i} = \frac{x_i+\delta_i}{c_i}.$$

Then, since x satisfies property (3) we reach a contradiction. □

This type of characterization of a rule allows us to understand in a deeper way how the rule can work for any pair of agents involved in the problems of this type, making it easier to identify the situations in which it can be applied.

4 The generalized Talmudic rule

4.1 Definition and properties

Once we have provided a characterization for the generalized proportional rule, we pass now to the study of an extended version of the Talmudic rule with the inclusion of ex-ante conditions. We denote this version as the generalized Talmudic rule, which keeps the principles of its initial version, in the sense that it also works as a combination of two well-known rules: the GEA rule and the GEL rule. Formally,

Definition 10 *Let (E, c, δ) be a claims problem with ex-ante conditions. The generalized Talmudic rule GT is defined as*

$$GT(E,c,\delta) = \begin{cases} GEA\Big(E, \dfrac{c}{2}, \delta\Big) & \text{if } E \leq \dfrac{C}{2} \\ \\ \dfrac{c}{2} + GEL\Big(E - \dfrac{C}{2}, \dfrac{c}{2}, \delta\Big) & \text{if } E > \dfrac{C}{2}. \end{cases}$$

As we can detail in the previous definition, the generalized Talmudic rule consists of two conditioned parts: a first section that evaluates the case where the estate is lower or equal to the half of the sum of the claims. This situation implies that the resource is not large enough to cover even half of the claims, then it allocates the endowment equally among the agents, taking into account as a final claims, only half of the agents' initial. On the other hand, when we situate in the inverse case, it works similarly with respect to the original rule. It starts assigning to each agent half of their claims, and the remaining part is distributed under the principle of equalizing losses among them, taking into account as claims the part of their demand that has not been satisfied.

One important aspect to highlight at this point is the fact that the ex-ante conditions are introduced in both parts of the formula in their total amount. This is because they represent the initial differences between the participants and therefore should be considered in their entirety, regardless of the size of the claims on the total resource.

In order to explain the application of this extension and the differences with respect to the original rule, let us consider the following example:

Example 4.1 *Consider the following standard claims problem (E, c) for $n = 3$:*

$$E = 110 \text{ and } (c_1, c_2, c_3) = (40, 60, 70).$$

Applying the Talmudic rule to this problem we get the allocation $(x_1, x_2, x_3) = (20, 40, 50)$, which is computed through the following process: given that the total state is higher than the half of the sum of the claims, $\frac{C}{2} = 85 < 110 = E$, we are placed in the second part of the formula. Therefore, we assign the respective half-claim to each agent, $\frac{c}{2} = (20, 30, 35)$, and we add to this the solution of the CEL formula applied over the remaining share

$E' = E - \frac{C}{2} = 25$, with the half of each claim as the variable of decision $c' = (20, 30, 35)$. Then,

$$\begin{aligned}(T_1, T_2, T_3) &= \frac{c}{2} + CEL\big(25,\ (20, 30, 35)\big) \\ &= (20, 30, 35) + (0, 10, 15) \\ &= (20, 40, 50).\end{aligned}$$

Let now add to the problem the following vector of ex-ante conditions:

$$(\delta_1, \delta_2, \delta_3) = (10, 0, 20).$$

To solve this, we applied the same process, taking into account that the problem is situated again in the second part of the equation, and therefore we use in this case the GEL rule in order to include the impact of the ex-ante conditions in the solution:

$$\begin{aligned}(GT_1, GT_2, GT_3) &= \frac{c}{2} + GEL\big(25,\ (20, 30, 35), (10, 0, 20)\big) \\ &= (20, 30, 35) + (0, 20, 5) \\ &= (20, 50, 40).\end{aligned}$$

With a $\lambda = 10$ for the GEL solution, we get the following distribution: $(GT_1, GT_2, GT_3) = (20, 50, 40)$. A hydraulic representation (Kaminski, 2000) is represented in Figure 4, and illustrates the principle coincidences and differences between these two rules.

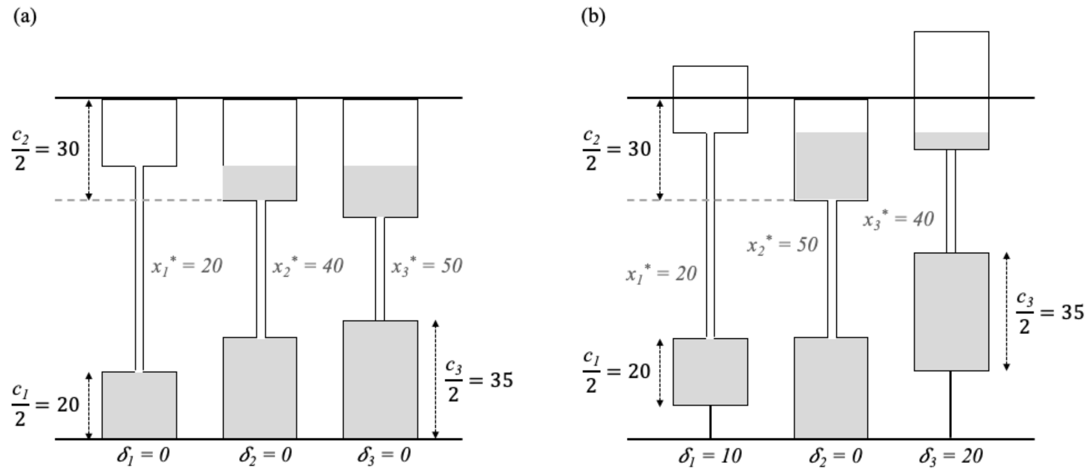

Figure 4: Hydraulic representation of the (a) Talmudic rule. and the (b) generalized Talmudic rule.

As Figure 4 illustrates, the hydraulic systems associated with the Talmudic rule and the generalized Talmudic rule reflects the behavior of them, illustrating the fact that both results from a combination of the CEA rule and the CEL rule, and their generalized versions for the second case. As mentioned before, the inclusion of ex-ante conditions to the problem generates vertical movements for the vessels in order to introduce the initial differences among the agents in the final allocation.

After we defined a formula for our extended rule and its operation, we proceed to test it according to the properties that characterize it. For this, we analyze the following axioms: consistency, resource monotonicity, and self-duality.

Proposition 7 *The generalized Talmudic rule satisfies consistency and resource monotonicity.*

Proof: To test this, we take into account the structure of the generalized Talmudic rule, which is composed of the GEA rule and the GEL rule. As mentioned before, these were proposed and characterized by Timoner and Izquierdo (2015), who found that they please both resource monotonicity and consistency. Given this, we can prove that the generalized Talmudic rule satisfies both properties, based on the premise that it is constructed over two rules that fulfill these two axioms. □

Now, in the case of the self-duality condition, we have that,

Proposition 8 *The generalized Talmudic rule satisfies self-duality.*

Proof: Let $(E, c, \delta) \in \mathcal{R}^N$ be an arbitrary claims problem with ex-ante. We must prove that for all $i \in N$:

$$GT_i(E, c, \delta) = c_i - GT_i(L, c, -\delta).$$

By definition,

$$GT(L, c, -\delta) = \begin{cases} GEA\Big(L, \dfrac{c}{2}, -\delta\Big) & \text{if } L \leq \dfrac{C}{2}. \\ \\ \dfrac{c}{2} + GEL\Big(L - \dfrac{C}{2}, \dfrac{c}{2}, -\delta\Big) & \text{if } L > \dfrac{C}{2} \end{cases}$$

In this case, due to the division of the formula given by the conditional, we can distinguish two cases:

(a) Case 1, when $E \leq \frac{C}{2}$. In this case, we place in the first part of the formula, then for all $i \in N$, we have that:

$$\begin{aligned} GT_i(E, c, \delta) &= GEA_i(E, \tfrac{c}{2}, \delta) \\ &= \min\left\{\tfrac{c_i}{2}, (\lambda - \delta_i)_+\right\} \\ &= \tfrac{c_i}{2} + \min\left\{0, (\lambda - \delta_i)_+ - \tfrac{c_i}{2}\right\} \\ &= \tfrac{c_i}{2} - \max\left\{0, \tfrac{c_i}{2} - (\lambda - \delta_i)_+\right\}. \end{aligned}$$

For the dual part, we have to take into account that, by definition $L = C - E$, then if $E \leq \frac{C}{2}$ this implies that $L > \frac{C}{2}$. Thus $GT_i(L, c, -\delta) = \frac{c_i}{2} + GEL_i\big(L - \frac{C}{2}, \frac{c}{2}, -\delta\big) = \frac{c_i}{2} + \max\left\{0, \frac{c_i}{2} - (\lambda - \delta_i)_+\right\}$. Replacing $GT_i(E, c, \delta)$ and $GT_i(L, c, -\delta)$ in the self-duality equation,

$$\begin{aligned} GT_i(E, c, \delta) &= c_i - GT_i(L, c, -\delta) \\ &= c_i - \tfrac{c_i}{2} - \max\left\{0, \tfrac{c_i}{2} - (\lambda - \delta_i)_+\right\} \\ &= \tfrac{c_i}{2} - \max\left\{0, \tfrac{c_i}{2} - (\lambda - \delta_i)_+\right\}. \end{aligned}$$

(b) Case 2, when $E > \frac{C}{2}$. Now, we are in the second part of the formula, then for all $i \in N$, we have that:

$$\begin{aligned} GT_i(E,c,\delta) &= \tfrac{c_i}{2} + GEL_i\big(E - \tfrac{C}{2}, \tfrac{c}{2}, \delta\big) \\ &= \tfrac{c_i}{2} + \max\big\{0, \tfrac{c_i}{2} - (\lambda + \delta_i)_+\big\} \\ &= \tfrac{c_i}{2} + \max\big\{-\tfrac{c_i}{2}, \tfrac{c_i}{2} - (\lambda + \delta_i)_+ - \tfrac{c_i}{2}\big\} + \tfrac{c_i}{2} \\ &= c_i + \max\big\{-\tfrac{c_i}{2}, -(\lambda + \delta_i)_+\big\} \\ &= c_i - \min\big\{\tfrac{c_i}{2}, (\lambda + \delta_i)_+\big\}. \end{aligned}$$

And for the dual part, now $L \leq \frac{C}{2}$ then $GT_i(L,c,-\delta) = GEA_i(L, \frac{c}{2}, -\delta) = \min\big\{\frac{c_i}{2}, (\lambda + \delta_i)_+\big\}$. Replacing $GT_i(E,c,\delta)$ and $GT_i(L,c,-\delta)$ in the self-duality equation,

$$\begin{aligned} GT_i(E,c,\delta) &= c_i - GT_i(L,c,-\delta) \\ &= c_i - \min\big\{\tfrac{c_i}{2}, (\lambda + \delta_i)_+\big\}. \end{aligned}$$

□

Other elements that characterize the Talmudic rule are its particularities when we analyze it for the two agents' case. We will elaborate on this in the following section, given the introduction of ex-ante conditions to the equation.

4.2 The case for two-claimants (n = 2)

The Talmudic rule comes up as the consistent extension of the contested garment rule, which is a defined solution for a claims problem with two agents. This extension was proposed by Aumann and Maschler (1985), following the solutions given in the Talmud to several practical distribution problems. Formally, the contested garment rule is defined as,

Definition 11 *Let $(E,(c_1,c_2))$.The contested garment rule, $CG(E,c)$, is*

$$\begin{aligned} CG_1(E,c) &= m_1(E,c) + \tfrac{1}{2}(E - m_1(E,c) - m_2(E,c)), \\ CG_2(E,c) &= m_2(E,C) + \tfrac{1}{2}(E - m_1(E,c) - m_2(E,c)), \end{aligned}$$

where $m_1(E,c) = max\{E - c_2, 0\}$ and $m_2(E,c) = max\{E - c_1, 0\}$.

Taking into account the different characterizations that have been made on the contested garment rule throughout the literature, it can be considered as a method for the compromise between the axioms of composition from minimal rights and independence of claims truncation (Herrero and Villar, 2001), which traduces in an agreement between the CEA rule and the CEL rule. To understand this, we connect these rules with another one known as equal losses rule from truncated claims. We start defining truncated claims as: for a given claims problem (E,c) we define the truncated claim of agent $i \in N$ as $tc_i(E) = \min\{E, c_i\}$. Then,

Definition 12 *Let $(E,(c_1,c_2))$. The equal losses rule from truncated claims, $EL(E,c)$, is*

$$\begin{aligned} EL_1(E,c) &= tc_1(E) + \tfrac{1}{2}(E - tc_1(E) - tc_2(E)), \\ EL_2(E,c) &= tc_2(E) + \tfrac{1}{2}(E - tc_1(E) - tc_2(E)). \end{aligned}$$

Similar to the case of the contested garment rule, the equal losses rule from truncated claims parts from the notion of a minimal right of each agent over the endowment, in the first case correspond to the part of the estate that is not demanded for other agent, while in the case of the equal losses rules this minimal right is based on the premise that an agent can receive as maximum the total of the estate, even if its claim is higher than it. This similarity leads to the following proposition:

Proposition 9 *For any, two-person claims problem (E,c), the contested garment rule coincides with the equal loss rule from truncated claims.*

Proof: Let $(E,c) \in \mathcal{R}^2$ be an arbitrary claims problem. We must prove that:

$$CG_1(E,c) = EL_1(E,c) \quad \text{and} \quad CG_2(E,c) = EL_2(E,c).$$

Starting from the contested garment rule, we have

$$\begin{aligned} CG_1(E,c) &= max\{E - c_2,\, 0\} + \tfrac{1}{2}(E - max\{E - c_2,\, 0\} - max\{E - c_1,\, 0\}) \\ &= max\{E - c_2,\, 0\} + \tfrac{1}{2}(E + min\{c_2 - E,\, 0\} + min\{c_1 - E,\, 0\}) \\ &= max\{E - c_2,\, 0\} + \tfrac{1}{2}(E + (min\{c_2 - E + E,\, E\} + min\{c_1 - E + E,\, E\} - 2 \cdot E)) \\ &= max\{E - c_2,\, 0\} + \tfrac{1}{2}(min\{c_2,\, E\} + min\{c_1,\, E\} - E) \\ &= max\{E - c_2,\, 0\} - \tfrac{1}{2}(E - min\{c_2,\, E\} - min\{c_1,\, E\}) \\ &= -\big[\, min\{c_2 - E,\, 0\} + \tfrac{1}{2}(E - min\{c_2,\, E\} - min\{c_1,\, E\})\big] \\ &= -\big[\, min\{c_2 - E + E,\, E\} - E + \tfrac{1}{2}(E - min\{c_2,\, E\} - min\{c_1,\, E\})\big] \\ &= -\big(\, min\{c_2,\, E\} - \tfrac{1}{2}E - \tfrac{1}{2}min\{c_2,\, E\} - \tfrac{1}{2}min\{c_1,\, E\}\big) \\ &= -\big(\, \tfrac{1}{2}min\{c_2,\, E\} - \tfrac{1}{2}E - \tfrac{1}{2}min\{c_1,\, E\}\big) \\ &= \tfrac{1}{2}(E - min\{c_2,\, E\} + min\{c_1,\, E\}) \\ &= \tfrac{1}{2}(E - min\{c_2,\, E\} + min\{c_1,\, E\} + min\{c_1,\, E\} - min\{c_1,\, E\}) \\ &= min\{c_1,\, E\} + \tfrac{1}{2}(E - min\{c_2,\, E\} - min\{c_1,\, E\}) = EL_1(E,c). \end{aligned}$$

It is straightforward that if we apply the same process for agent 2, we can get the same result. □

Now, in the case of a two-person claims problem with ex-ante conditions (E,c,δ), we define the relative ex-ante condition of agent i with respect to agent j as

$$\Delta\delta_j^i = (\delta_i - \delta_j)_+.$$

Moreover we define the relative net ex-ante condition of agent i with respect to agent j from half of claims as

$$\Delta\delta_j^i|\frac{c}{2} = \Big[(\delta_i - \frac{c_i}{2}) - (\delta_j - \frac{c_j}{2})\Big]_+ .$$

Finally, we define the respective adjusted truncated claims as

$$\overline{tc}_1(E,c,\delta) = \begin{cases} \min\left\{\left(E - \left(\Delta\delta_2^1 \wedge \frac{c_2}{2}\right)\right)_+ ; c_1 - \left(\Delta\delta_1^2 \wedge \frac{c_1}{2}\right)\right\} & \text{if } E \leq \frac{c_1+c_2}{2} \\ \\ \frac{c_1}{2} + \min\left\{\left(E - \left(\frac{c_1+c_2}{2}\right) - \left(\Delta\delta_2^1|\frac{c}{2} \wedge \frac{c_2}{2}\right)\right)_+ ; c_1 - \left(\Delta\delta_1^2|\frac{c}{2} \wedge \frac{c_1}{2}\right)\right\} & \text{if } E \geq \frac{c_1+c_2}{2}. \end{cases}$$

and

$$\overline{tc}_2(E,c,\delta) = \begin{cases} \min\left\{\left(E - \left(\Delta\delta_1^2 \wedge \frac{c_1}{2}\right)\right)_+ ; c_2 - \left(\Delta\delta_2^1 \wedge \frac{c_2}{2}\right)\right\} & \text{if } E \leq \frac{c_1+c_2}{2} \\ \\ \frac{c_2}{2} + \min\left\{\left(E - \left(\frac{c_1+c_2}{2}\right) - \left(\Delta\delta_1^2|\frac{c}{2} \wedge \frac{c_1}{2}\right)\right)_+ ; c_2 - \left(\Delta\delta_2^1|\frac{c}{2} \wedge \frac{c_2}{2}\right)\right\} & \text{if } E \geq \frac{c_1+c_2}{2}. \end{cases}$$

Definition 13 *Let (E, c, δ) be a two-person claims problem, Then the equal losses rule from adjusted truncated claims, $\overline{EL}(E, c, \delta)$, is*

$$\overline{EL}_1(E,c,\delta) \quad = \overline{tc}_1(E,c,\delta) + \frac{1}{2}\left[E - \overline{tc}_1(E,c,\delta) - \overline{tc}_2(E,c,\delta)\right],$$

$$\overline{EL}_2(E,c,\delta) \quad = \overline{tc}_2(E,c,\delta) + \frac{1}{2}\left[E - \overline{tc}_1(E,c,\delta) - \overline{tc}_2(E,c,\delta)\right].$$

Proposition 10 *For $n = 2$, the generalized Talmudic rule coincides with equal losses solution from adjusted truncated claims.*

We illustrate how these rules once applied lead to the same result, using the following example,

Example 4.2 *Consider the following two-person standard claims problem (E, c, δ):*

$$E = 16 \text{ and } (c_1, c_2) = (10, 18) \text{ and } (\delta_1, \delta_2) = (6, 3).$$

We can solve this problem, applying these two rules and obtain the same result:

1. *Generalized Talmudic rule Applying the Generalized Talmudic rule to this problem we get the allocation $(x_1, x_2) = (5, 11)$, which is computed through the following process: given that the total state is higher than*

the half of the sum of the claims, $\frac{C}{2} = 14 < 16 = E$*, we are placed in the second part of the formula. Therefore,*

$$\begin{aligned}(GT_1, GT_2) &= \frac{c}{2} + GEL\big(16-14,\ (5,9),(6,3)\big)\\ &= (5,9)+(0,2)\\ &= (5,11).\end{aligned}$$

2. *Equal losses solution from truncated claims Taking into account that the estate is higher than the sum of the half of the claims, and therfore we are placed in the second part of the equation,we start by estimating the relative net ex-ante condition of agent 1 with respect to agent 2, and in the opposite direction, as follows*

$$\Delta\delta_2^1|\frac{c}{2} = \left[(6-\frac{10}{2})-(3-\frac{18}{2})\right]_+ = 7$$

$$\Delta\delta_1^2|\frac{c}{2} = \left[(3-\frac{18}{2})-(6-\frac{10}{2})\right]_+ = 0$$

Then we compute the adjusted truncated claim for each agent, as follows

$$\begin{aligned}\overline{tc}_1(E,c,\delta) &= \frac{10}{2} + \min\left\{\left(16-\left(\frac{10+18}{2}\right)-(\min\{7,9\})\right)_+;\ 10-\min\left\{0,\frac{10}{2}\right\}\right\}\\ &= 5+\min\{0;\ 10\} = 5.\end{aligned}$$

$$\begin{aligned}\overline{tc}_2(E,c,\delta) &= \frac{18}{2} + \min\left\{\left(16-\left(\frac{10+18}{2}\right)-(\min\{0,5\})\right)_+;\ 18-\min\left\{7,\frac{18}{2}\right\}\right\}\\ &= 9+\min\{2;\ 11\} = 11.\end{aligned}$$

And we finish calculating the equal losses rule from adjusted truncated as

$$\overline{EL}_1(E,c,\delta) = 5 + \frac{1}{2}\left[16-5-11\right] = 5,$$

$$\overline{EL}_2(E,c,\delta) = 11 + \frac{1}{2}\left[16-5-11\right] = 11,$$

Figure 5 reflects the path awards for the application of both rules to this example.

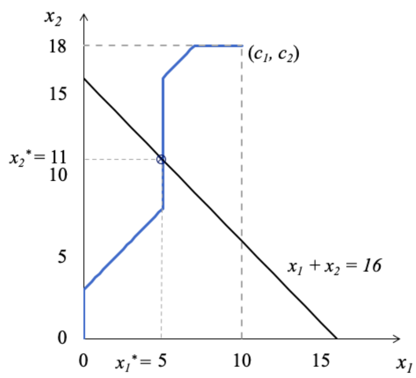

Figure 5: Equal losses from truncated claims and generalized Talmudic rule.

4.3 Axiomatic characterization

Finally, the next theorem characterizes the generalized Talmudic rule by means of the properties we have verified in Sections 4.1. and 4.2. Formally,

Theorem 2 *The generalized Talmudic rule is the unique rule that satisfies consistency, resource monotonicity, and for $n = 2$ coincides with the equal losses solution from adjusted truncated claims*

Proof: By Proposition 7, 10 the generalized Talmudic rule satisfy these properties.

Now, suppose there is another rule F that satisfies these properties, not equal to GT. This means there is a n-person problem (E, c, δ), $n \geq 3$, such that

$$x = F(E, c, \delta) \neq GT(E, c, \delta) = y.$$

Since $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = E$, there exists i and j, $i \neq j$ such that

$$x_i > y_i \text{ and } x_j < y_j.$$

Without lost of generality we can suppose $x_i + y_i \geq y_i + y_j$. We claim $x_i + y_i > y_i + y_j$. Indeed, suppose on the contrary that $x_i + y_i = y_i + y_j$. By consistency of both F and GT we know that

$$(x_i, x_j) = F(x_i + x_j, (c_i, c_j), (\delta_i, \delta_j)).$$

$$(y_i, y_j) = GT(y_i + y_j, (c_i, c_j), (\delta_i, \delta_j)).$$

However, since both F and GT coincides with the equal losses solution from adjusted truncated claims and $x_i + x_j = y_i + y_j$ we would have $x_i = y_i$ and $x_j = y_j$ reaching a contradiction. Hence, we conclude $x_i + x_j > y_i + y_j$. Finally

$$\begin{aligned}
(y_i, y_j) &= GT(y_i + y_j, (c_j, c_j), (\delta_i, \delta_j)) \\
&\leq GT(x_i + x_j, (c_j, c_j), (\delta_i, \delta_j)) \\
&= \overline{EL}(x_i + x_j, (c_j, c_j), (\delta_i, \delta_j)) \\
&= F(x_i + x_j, (c_j, c_j), (\delta_i, \delta_j)) = (x_i, x_j),
\end{aligned}$$

where the first equality follows from the consistency property of the GT rule, the inequality follows from resource monotonicity of the GT rule, the second and thirds equality since the GT and the F rule is equal to the $\overline{EL}$ rule for $n = 2$, and the last equality is due to the consistency of F. In particular, we get that $y_i \leq x_i$. However this contradicts our hypothesis that $y_i < x_i$. We conclude $F = GT$. □

As mentioned before, this type of analysis provides a deeper understanding of the behavior of a rule, and therefore the cases where it can be implemented.

5 Suggested empirical approach

5.1 Experimental design

In this section, we describe a preliminary empirical approach to the analysis of the standard claims problem. This approach is based on a non-formal experiment, which corresponds to a questionnaire, intended for undergraduate and master students between 18 and 35 years of age. The questionnaire consists of eight multiple-choice questions and describes different real-life situations with three agents and a limited resource or payment to divide among them. All problems have four possible answers, each of them associated with our interesting solutions: the CEA rule, the proportional rule, the CEL rule, and the Talmudic rule, and their respective generalized versions.

We follow the hypothesis worked by Herrero, Moreno-Ternero, and Ponti (2009) concerning to: "Is there any particular rule that is salient in subjects' perception of the optimal solution to a problem of adjudicating conflicting claims?". Our contribution to this idea is to analyze this question from a different methodology, proving if there is any difference in these subjects' perception of the optimal solution when we include additional information about the characteristics of agents (ex-ante conditions). To achieve this, we divided the questionnaire into two parts: in the first part, the only information that the respondents know about the agents are their claims. In the second part, we use similar situations from the first part, adding in these cases some exogenous characteristics that, combine with their claims, affect the estimation of the final distribution among the agents. With this in mind, we aim to check if there is any coincidence with the most voted answer in the first part, respect to the second one. The following scheme illustrate the structure of the experiment.

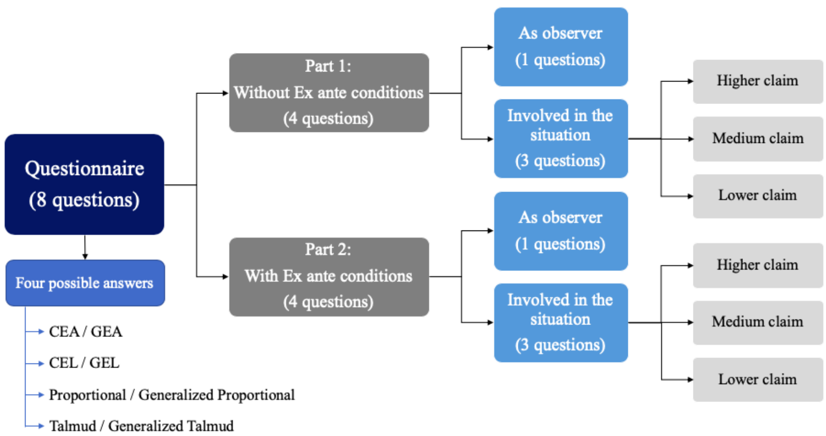

Figure 6: Structure of the questionnaire

As Figure 6 illustrates, we also implemented another element developed by Herrero, Moreno-Ternero, and Ponti (2009), concerning to the application of different perspectives for the respondent. For our experiment, we proposed the following alternatives: first, the subject has to solve the problem as an impartial observer. Second, the respondent is part of the problem, and therefore a claim is assigned for him/her. This last alternative is evaluated for three different cases, varying the size of the claim assigned to the subject (lowest, middle, or highest claim). Finally, to guarantee the randomness in the answers, we let that some respondents first answer questions in part 1, and others first answer questions in part 2.

5.2 Results

The questionnaire was answered by 52 students through the Google Forms application. We start analyzing the results, in an aggregate level, comparing first the responses obtained in each part of the experiment:

Result 1 *In the absence of ex-conditions, the proportional rules is the most attractive rule.*

As Figure 7 shows, for the first part of the experiment, which involves situations without ex-ante conditions, we can detail a clear preference for the proportional rule over the other, independent of the situation and the role played by the respondents. This result is consistent with the findings from Gächter and Riedl (2006), and is supported by two facts reflected in the figure, which are: a significantly high share for this rule, with values between 61.5 percent and 46.2 percent, and the high variability for the second most voted alternative in each case. Given this, we can assume that not only the proportional rule is the most intuitively selected alternative, but also the other rules do not present a selection pattern after this one.

Furthermore, this result was also tested taking into account the share of votes associated with the order in which the participants answered each part of the questionnaire. We find a higher preference for the proportional rule among the subjects that answered first Part I of the experiment, than in the set of respondents who answered first Part II. Although, this second group present lower percentages for the proportional rule, they also present an attractiveness for the proportional rule among the other rules. This behavior can be related to the fact that the participants of the first group had at first sight the proportional option, so that they could continue looking for it throughout the questionnaire, while those of the second group, not having this option at the beginning, may not have been able to detect it intuitively at the end.

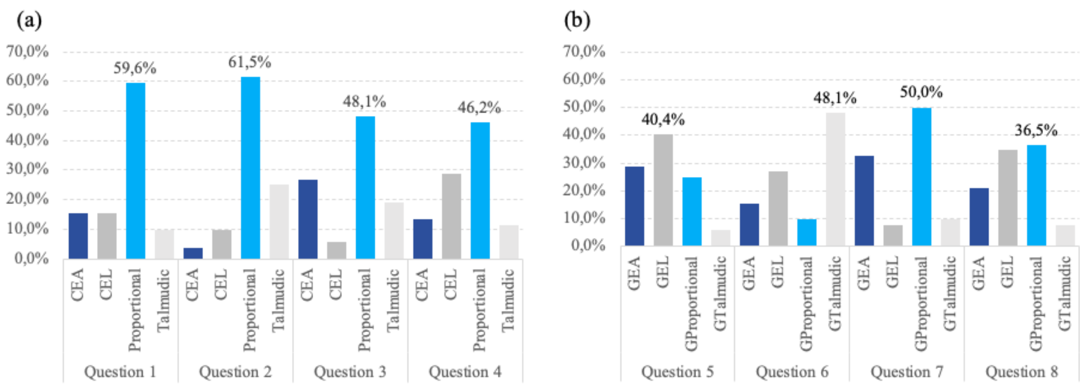

Figure 7: Share of responses by rule for (a) Part I and (b) Part II of the experiment.

Result 2 *With the presence of ex-ante conditions, there is no an unique preferred solution.*

In contrast to our first result, the second part from Figure 7 provides some evidence for the perspective of the respondents over the ex-ante conditions, and how their initial notion of fair distribution can change with the inclusion of this new information. In this case, we cannot perceive any type of preference of some rule over the others. In fact, the share of respondents for each rule tends to change significantly depending on the situation and/or the role played by the participant. This finding answers our research question in the sense that it shows how participants' initial perceptions of fairness tend to change with the inclusion of this new information. To deepen this, we now analyze each pair questions that are similar and belong to each part, in order to evaluate which factors can determine this change in perception.

Result 3 *The preferences over the proposed rules tend to be similar in the cases where the respondent answer as an impartial observer or as the agent of the problem with the lower claim.*

As we can observe in Figure 8, the pair of questions 3 and 7 presents a similar pattern in the selection of the rules, with a predominance of the proportional rule, followed by the CEA rule, even with the inclusion of ex-ante condition. In this case, both questions share a similar numerical problem and the respondent acts as an impartial observer. Then, we can presume that the inclusion of ex-ante conditions has a lower impact on the initial perceptions of the subject when he/she acts as an impartial observant of the problem.

The same result can be detailed in the case of the pair of questions 4 and 8, which also share a similar numerical problem, and the participant in both situations is part of the problem and the lowest claim is assigned to her/him. For this scenario, we can also detail a non-significant variation on the share of responses with an inclination, in both cases, over the proportional rule followed by the CEL rule, implying that with a lower claim the agent is not interested in changing his/her initial perception of the problem.

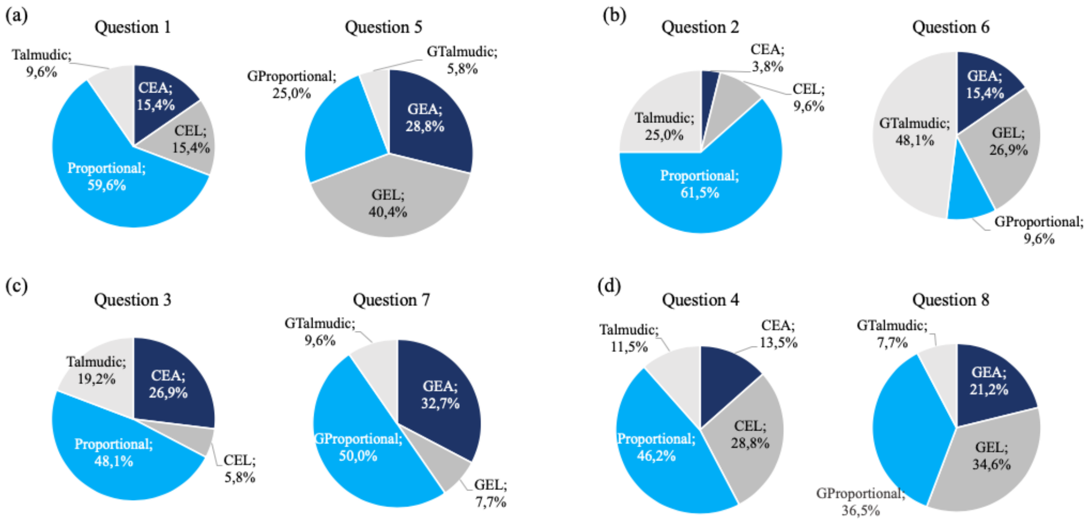

Figure 8: Comparison of share responses by pair of questions. (a) Middle claim, (b) highest claim, (c) impartial observant, and (d) lowest claim

Result 4 *The preferences over the proposed rules tend to vary significantly in the cases where the respondent answer as an agent of the problem with middle and higher claims.*

In the case of the pair of questions 1 and 5, they show important changes in the selection of the rules, with a predominance of the proportional rule in the case without ex-ante conditions, and an attractiveness of the GEL, with the inclusion of ex-ante condition. As mentioned before, each pair of questions share a similar numerical problem, but in this case, the participant is part of the problem and the medium claim is assigned to her/him. Then, we conclude that the inclusion of ex-ante conditions tend to affect the selection of a rule over the others when he/she acts as part of the problem with a medium claim.

The same result can be detailed in the case of the pair of questions 2 and 6, which also share a similar numerical problem, and the participant in both situations is part of the problem and the highest claim is assigned to her/him. For this scenario, we can also detail significant variation on the share of responses with an inclination, in the first case, over the proportional rule and in the second part for the Talmudic rule, implying that with a higher claim the agent is influenced to change his/her initial perception of the problem.

6 Conclusions

We have provided an extension of the classic model for claims problems. This extension aims to introduce new information about the agents, as ex-ante inequalities among them that should be included in the resolution process, in order to try to compensate these initial differences. Two of the most well-known rules for problems of this kind, as the proportional rule and the Talmudic, have been generalized and characterized within this new framework. For the first rule, we characterize it taking into account the properties that can be verified over this (as self-duality) and through a description of it by means of comparing the ex-ante conditions for any pair of agents. On the other hand, for the generalized Talmudic rule, we support our axiomatic characterization on the means of the properties that this rule satisfy.

Moreover, once we analyze these generalized versions from a theoretical point of view, we proceed to combine this with results presented by Timoner and Izquierdo (2016), to propose an experiment that allows us to understand the perceptions of different agents among this rules, and how these can be affected with the introduction of ex-ante conditions. From this, we found a significant preference among the proportional rule in situations with ex-ante conditions, independent of the role assigned to the respondent in each situation. While for the cases with ex-ate conditions the perceptions tend to vary depending on the role played by the agent, in the cases where the respondent was an impartial observant or an agent of the problem, but with the lower claim, the perceptions over the initial rules and their generalized versions do not change significantly. However, in situations where the respondent was associated with the problem with a medium or higher claim, the initial perception of the fairest rule change with the inclusion of ex-ante conditions.

Given these results, as future research work, we propose to elaborate a more formal experiment, that permits to evaluate different frames among the agents in order to determine the principal causes of the changes in their perceptions of a fair rule produced by the inclusion of ex-ante conditions.

7 Appendices

7.1 Appendix A: Proof from Proposition 10

Proposition 10: *The generalized Talmudic rule coincides with equal losses solution from adjusted truncated claims.*

Proof. Let us first analyzed the case when $E \leq \dfrac{c_1 + c_2}{2}$. Without loss of generality we can suppose that $\delta_1 \geq \delta_2$. Then, we can distinguish four cases: (a) $\delta_1 - \delta_2 < \dfrac{c_2}{2}$ and $\delta_1 - \delta_2 < E$; (b) $\delta_1 - \delta_2 < \dfrac{c_2}{2}$ and $\delta_1 - \delta_2 \geq E$; (c) $\delta_1 - \delta_2 \geq \dfrac{c_2}{2}$ and $\delta_1 - \delta_2 < E$; (d) $\delta_1 - \delta_2 \geq \dfrac{c_2}{2}$ and $\delta_1 - \delta_2 \geq E$.

(a) In this case, the adjusted truncated claims are

$$\overline{tc}_1(E,c,\delta) = \min\left\{E - \Delta\delta_2^1;\ c_1\right\},$$

$$\overline{tc}_2(E,c,\delta) = \min\left\{E;\ c_2 - \Delta\delta_2^1\right\}.$$

Therefore, we can analyze four subcases:

(a1) $\overline{tc}_1(E,c,\delta) = E - \Delta\delta_2^1$ and $\overline{tc}_2(E,c,\delta) = E$. In this case the equal losses rule is

$$\overline{EL}_1 = \frac{1}{2}[E - \Delta\delta_2^1] \text{ and } \overline{EL}_2 = \frac{1}{2}[E + \Delta\delta_2^1].$$

To check that this is the generalized talmudic rule, let us first remark that, by the conditions of this case, $0 \leq \overline{EL}_1 \leq \frac{c_1}{2}$ and $0 \leq \overline{EL}_2 \leq \frac{c_2}{2}$. Then, we must check that there exists $\lambda \in \mathbb{R}$ such that

$\overline{EL}_1 = \frac{1}{2}[E - \Delta\delta_2^1] = GEA_1(E, \frac{c}{2}, \delta) = min\{(\lambda - \delta_1)_+,\ \frac{c_1}{2}\}$ and

$\overline{EL}_2 = \frac{1}{2}[E + \Delta\delta_2^1] = GEA_2(E, \frac{c}{2}, \delta) = min\{(\lambda - \delta_2)_+,\ \frac{c_2}{2}\}$.

To make true the first equality, we must take $\lambda = \frac{1}{2}[E - \Delta\delta_2^1] + \delta_1$.

But then if we replace this λ in the expression $min\{(\lambda - \delta_2)_+,\ c_2\}$ we get

$$\begin{aligned}\min\{(\tfrac{1}{2}[E - \Delta\delta_2^1] + \delta_1 - \delta_2)_+,\ \tfrac{c_2}{2}\} &= \min\{(\tfrac{1}{2}[E + \Delta\delta_2^1])_+,\ \tfrac{c_2}{2}\} \\ &= \min\{\overline{EL}_2,\ \tfrac{c_2}{2}\} = \overline{EL}_2.\end{aligned}$$

.

(a2) $\overline{tc}_1 = E - \Delta\delta_2^1$ and $\overline{tc}_2(E,c,\delta) = c_2 - \Delta\delta_2^1$.

In this case the equal losses rule is

$$\overline{EL}_1 = E - \frac{1}{2}c_2 \text{ and } \overline{EL}_2 = \frac{1}{2}c_2.$$

It is clear that $0 \leq \overline{EL}_2 \leq \frac{1}{2}c_2$ and $0 \leq \overline{EL}_1 \leq \frac{1}{2}c_1$, since $E \leq \frac{c_1+c_2}{2}$. Then, we must check that there exists $\lambda \in \mathbb{R}$ such that

$\overline{EL}_1 = E - \frac{1}{2}c_2 = GEA_1(E, \frac{c}{2}, \delta) = min\{(\lambda - \delta_1)_+,\ \frac{c_1}{2}\}$ and

$\overline{EL}_2 = \frac{1}{2}c_2 = GEA_2(E, \frac{c}{2}, \delta) = min\{(\lambda - \delta_2)_+,\ \frac{c_2}{2}\}$.

To make true the first equality, we must take $\lambda = E - \frac{1}{2}c_2 + \delta_1$.

But then, if we replace this λ in the expression $min\{(\lambda - \delta_2)_+,\ \frac{c_2}{2}\}$ we get

$$\min\{(E - \tfrac{1}{2}c_2 + \delta_1 - \delta_2)_+,\ \tfrac{c_2}{2}\} = \tfrac{c_2}{2},$$

where the last inequality follows since in case (a) we are supposing $\delta_1 - \delta_2 < E \leq E\cdot$

(a3) $\overline{tc}_1(E,c,\delta) = c_1$ and $\overline{tc}_2(E,c,\delta) = E$

In this case the equal losses rule is

$$\overline{EL}_1 = \frac{1}{2}c_1 \text{ and } \overline{EL}_2 = E - \frac{1}{2}c_1.$$

It is clear that $0 \leq \overline{EL}_2 \leq \frac{1}{2}c_2$ and $0 \leq \overline{EL}_1 \leq \frac{1}{2}c_1$, since $E \leq \frac{c_1+c_2}{2}$. Then, we must check that there exists $\lambda \in \mathbb{R}$ such that

$\overline{EL}_1 = \frac{1}{2}c_1 = GEA_1(E, \frac{c}{2}, \delta) = min\{(\lambda - \delta_1)_+,\ \frac{c_1}{2}\}$ and

$\overline{EL}_2 = E - \frac{1}{2}c_1 = GEA_2(E, \frac{c}{2}, \delta) = min\{(\lambda - \delta_2)_+,\ \frac{c_2}{2}\}$.

To make true the second equality, we must take $\lambda = E - \frac{1}{2}c_1 + \delta_2$.

But then, if we replace this λ in the expression $min\{(\lambda - \delta_1)_+,\ \frac{c_1}{2}\}$ we get

$$\min\{(E - \tfrac{1}{2}c_1 + \delta_2 - \delta_1)_+,\ \tfrac{c_1}{2}\} \quad = \tfrac{c_1}{2} \quad ,$$

where the last inequality follows since in case (a3) we are supposing $c_1 < E - (\delta_1 - \delta_2)\cdot$

(a4) $\overline{tc}_1(E,c,\delta) = c_1$ and $\overline{tc}_2(E,c,\delta) = c_2 - \Delta\delta_2^1$.

This case reduces to the case $E = \frac{c_1+c_2}{2}$. Indeed, by the hypothesis of the case we have $c_1 \leq E - \Delta\delta_2^1$ and $c_2 - \Delta\delta_2^1 \leq E$. Adding these two expressions we get $\frac{c_1+c_2}{2} \geq E$, and thus

$$\frac{c_1 + c_2}{2} = E \tag{5}$$

(recall that we are analyzing the case $\frac{c_1+c_2}{2} \geq E$.)

Furthermore, since $E - (\delta_1 - \delta_2) \geq c_1$ and by (5) we get that $\frac{c_1+c_2}{2} \geq \delta_1 - \delta_2$. On the other hand, since $E \geq c_2 - (\delta_1 - \delta_2)$ we get $\delta_1 - \delta_2 \geq \frac{c_2-c_1}{2}$. Hence, we conclude

$$\delta_1 - \delta_2 = \frac{c_2 - c_1}{2}. \tag{6}$$

In this case, by (5) and (6), the equal losses rule is

$$\overline{EL}_1 = \frac{1}{2}c_1 \text{ and } \overline{EL}_2 = \frac{1}{2}c_2.$$

We end this case since, by definition of the GEA rule that

$$GEA(\frac{c_1 + c_2}{2}, \frac{c}{2}, \delta) = (\frac{c_1}{2}, \frac{c_2}{2}).$$

(b) In this case, the adjusted truncated claims are

$$\overline{tc}_1(E,c,\delta) = \min\{0;\, c_1\} = 0$$

$$\overline{tc}_2(E,c,\delta) = \min\left\{E;\, c_2 - \Delta\delta_2^1\right\}.$$

However, since $\delta_1 - \delta_2 < \dfrac{c_2}{2}$ we have $\delta_1 - \delta_2 < c_2 - (\delta_1 - \delta_2)$ Furthermore, since $E \leq \delta_1 - \delta_2 nd$, we have that $E < c_2 - (\delta_1 - \delta_2)$ and thus

$$\overline{tc}_2(E,c,\delta) = \min\left\{E;\, c_2 - \Delta\delta_2^1\right\} = E.$$

Therefore, $\overline{tc}_2(E,c,\delta) = E$ and

$$\overline{EL}_1 = 0 \text{ and } \overline{EL}_2 = E.$$

This, solution coincides with the GEA rule. Indeed, we must check that there exists $\lambda \in \mathbb{R}$ such that

$$\overline{EL}_1 = 0 = GEA_1(E, \tfrac{c}{2}, \delta) = min\{(\lambda - \delta_1)_+,\ \tfrac{c_1}{2}\} \text{ and}$$

$$\overline{EL}_2 = E = GEA_2(E, \tfrac{c}{2}, \delta) = min\{(\lambda - \delta_2)_+,\ \tfrac{c_2}{2}\}.$$

To make true the equality in $\overline{EL}_2$, we must take $\lambda = E + \delta_2$. But then, it is easy to check that

$$min\{(E + \delta_2 - \delta_1)_+,\ \frac{c_1}{2}\} = 0,$$

since, by hypothesis, $\delta_1 - \delta_2 \geq E$.

(c) In this case, the adjusted truncated claims are

$$\overline{tc}_1(E,c,\delta) = \min\left\{E - \tfrac{c_2}{2};\ c_1\right\} = E - \tfrac{c_2}{2},$$

$$\overline{tc}_2(E,c,\delta) = \min\left\{E;\ \tfrac{c_2}{2}\right\} = \tfrac{c_2}{2}.$$

since, by hypothesis of this case,

$$E > \delta_1 - \delta_2 \geq \frac{c_2}{2},$$

$$E - \frac{c_2}{2} \leq \frac{c_1 + c_2}{2} - \frac{c_2}{2} = \frac{c_1}{2} < c_1$$

Therefore,

$$\overline{EL}_1 = E - \frac{c_2}{2} \text{ and } \overline{EL}_2 = \frac{c_2}{2}.$$

This solution coincides with the GEA rule. Indeed, we must check that there exists $\lambda \in \mathbb{R}$ such that

$$\overline{EL}_1 = E - \tfrac{c_2}{2} = GEA_1(E, \tfrac{c}{2}, \delta) = min\{(\lambda - \delta_1)_+,\ \tfrac{c_1}{2}\} \text{ and}$$

$$\overline{EL}_2 = \tfrac{c_2}{2} = GEA_2(E, \tfrac{c}{2}, \delta) = min\{(\lambda - \delta_2)_+,\ \tfrac{c_2}{2}\}.$$

To make true the equality in $\overline{EL}_1$, we must take $\lambda = E - \frac{c_2}{2} + \delta_1$. But then, it is easy to check that

$$min\{(E - \frac{c_2}{2} + \delta_1 - \delta_2)_+,\ \frac{c_2}{2}\} = \frac{c_2}{2},$$

since, by hypothesis, $E - \frac{c_2}{2} + \delta_1 - \delta_2 \geq E - (\delta_1 - \delta_2) + \delta_1 - \delta_2 = E \geq \frac{c_2}{2}$.

(d) In this case, the adjusted truncated claims are

$$\overline{tc}_1(E,c,\delta) = \min\left\{\left(E - \tfrac{c_2}{2}\right)_+ ;\, c_1\right\},$$

$$\overline{tc}_2(E,c,\delta) = \min\left\{E;\, \tfrac{c_2}{2}\right\}.$$

We can distinguish two subcases:

(d1) If $E \leq \frac{c_2}{2}$, then $\overline{tc}_1(E,c,\delta) = 0$ and $\overline{tc}_2(E,c,\delta) = E$, and thus

$$\overline{EL}_1 = 0 \text{ and } \overline{EL}_2 = E.$$

This solution coincides with the GEA rule. The proof is analagously to case (b).

(d2) If $E > \frac{c_2}{2}$, then $c_1 < E - \frac{c_2}{2}$; otherwise, $c_1 < E - \frac{c_2}{2} \leq \frac{c_1+c_2}{2} - \frac{c_2}{2} = \frac{c_1}{2} < c_1$ that leads to a contradiction. Then, $\overline{tc}_1(E,c,\delta) = E - \frac{c-2}{2}$ and $\overline{tc}_2(E,c,\delta) = \frac{c_2}{2}$, and thus

$$\overline{EL}_1 = E - \frac{c_2}{2} \text{ and } \overline{EL}_2 = \frac{c_2}{2}.$$

This solution coincides with the GEA rule. The proof is analagously to case (c).

To see the case when $E \geq \dfrac{c_1+c_2}{2}$, recall that the expression for the Talmudic rule is

$$T(E,c,\delta) = \frac{c}{2} + GEL(E - \frac{C}{2}, \frac{c}{2}, \delta).$$

However, notice that, for $i = 1, 2$,

$$\begin{aligned}
\frac{c_i}{2} + GEL_i(E - \frac{C}{2}, \frac{c}{2}, \delta) &= \frac{c_i}{2} + \max\left\{0, \frac{c_i}{2} - \max\{\lambda + \delta_i, 0\right\} \\
&= \frac{c_i}{2} + \max\left\{0, \frac{c_i}{2} + \min\{-\lambda - \delta_i, 0\}\right\} \\
&= \frac{c_i}{2} + \max\left\{0, \min\{-\lambda - \delta_i + \frac{c_i}{2}, \frac{c_i}{2}\}\right\} \\
&= \frac{c_i}{2} + \max\left\{0, \min\left\{-\lambda - \left[\delta_i - \frac{c_i}{2}\right], \frac{c_i}{2}\right\}\right\} \\
&= \frac{c_i}{2} + \min\left\{-\lambda - \left[\delta_i - \frac{c_i}{2}\right]_+, \frac{c_i}{2}\right\} \\
&= \frac{c_i}{2} + \min\left\{\lambda^* - \left[\delta_i - \frac{c_i}{2}\right]_+, \frac{c_i}{2}\right\}.
\end{aligned}$$

where $\lambda^* = -\lambda$ is such that

$$\min\left\{\lambda^* - \left[\delta_1 - \frac{c_1}{2}\right]_+, \frac{c_1}{2}\right\} + \min\left\{\lambda^* - \left[\delta_2 - \frac{c_2}{2}\right]_+, \frac{c_2}{2}\right\} = E - \frac{C}{2}.$$

Notice that $\min\left\{\lambda^* - \left[\delta_i - \frac{c_i}{2}\right]_+, \frac{c_i}{2}\right\}$ is the formula of $GEA_i(E - \frac{C}{2}, \frac{c}{2}, \delta_{\frac{c}{2}})$. Hence, following the same four cases than for the case $E \leq \frac{c_1 + c_2}{2}$, but now considering $\Delta\delta_j^i|_{\frac{c}{2}}$ instead of $\Delta\delta_j^i$, it is easy to prove the equality

$$T(E, c, \delta) = \frac{c}{2} + GEL(E - \frac{C}{2}, \frac{c}{2}, \delta) = \overline{EL}(E, c, \delta).$$

□

7.2 Appendix B: Questionnaire

This questionnaire has eight multiple-choice questions. Each of them involves the allocation of a scarce good among three agents, with 4 possible answers. We ask you to select the one that you intuitively consider the most appropriate distribution between the agents.

PART I

1. Question 1: Suppose that three faculties (A1, A2, A3), apply for a university fund, and you are the director of the first one. Each faculty has demanded the following amounts (30, 15, 60) for its operation. The fund is equal to 81 monetary units, which is insufficient to meet all the requests. What do you consider would be an adequate allocation for your faculty and the other ones?
   (a) A1 = 30, A2 = 15, A3 = 36.
   (b) A1 = 22, A2 = 7, A3 = 52.
   (c) A1 = 23, A2 = 12, A3 = 46.
   (d) A1 = 22, A2 = 7, A3 = 52.
2. Question 2: You and two friends (A1, A2, A3), have created a web page that won an award of 120 monetary units. Each of you invested the following hours on the project (70, 30, 60), and each hour has a price of one monetary unit. What do you think would be the most appropriate award allocation for you and your friends?
   (a) A1 = 57.5, A2 = 15, A3 = 47.5.
   (b) A1 = 45, A2 = 30, A3 = 45.
   (c) A1 = 56.67, A2 = 16.67, A3 = 46.67.
   (d) A1 = 52.5, A2 = 22.5, A3 = 45.
3. Question 3: Consider that a firm goes bankrupt and is liquidated for an amount of 70,000 monetary units. This amount is destined to settle the following debts (10,000, 50,000, 50,000) contracted with three different banks (A1, A2, A3). What do you consider is the fairest distribution of payments to each entity?
   (a) A1 = 0, A2 = 35,000, A3 = 35,000.
   (b) A1 = 6,000, A2 = 32,000, A3 = 32,000.

(c) A1 = 5,000, A2 = 32,500, A3 = 32,500.

(d) A1 = 10,000, A2 = 30,000, A3 = 30,000.

4. Question 4: Imagine that you and your two roommates, (A1, A2, A3), decided to move to a new place, and so you ask the owner to give back the deposit. In the beginning, each one paid for this concept the following amounts (300, 360, 420) euros. But, given certain damages within the property, the owner decides to return only 600 euros. What do you consider is the fairest distribution between you and your roommates?

   (a) A1 = 150, A2 = 195, A3 = 255.

   (b) A1 = 200, A2 = 200, A3 = 200.

   (c) A1 = 140, A2 = 200, A3 = 260.

   (d) A1 = 170, A2 = 200, A3 = 230.

PART II

5. Question 5: Suppose a public fund destined for three infrastructure projects, (A1, A2, A3), where you are responsible for the first one. Each project presents the following monetary needs for completion (10, 5, 20). The fund is equal to 27 units and the projects have received the following contributions from other funds (15, 10, 20). Which do you consider would be an adequate allocation for your project and the other ones?

   (a) A1 = 10, A2 = 5, A3 = 12.

   (b) A1 = 8.5, A2 = 5, A3 = 13.5.

   (c) A1 = 6, A2 = 1, A3 = 20.

   (d) A1 = 7, A2 = 0, A3 = 20.

6. Question 6: Suppose that Sony wants to distribute a fund for the investment of its three main areas (A1: games, A2: TVs, A3: Smartphones). You are in charge of the games area. The total amount to be divided is 600,000 euros and each area has the following investment needs (350,000, 150,000, 300,000). Additionally, last year's profits by area were as follows (200,000, 100,000, 0). How do you think should be the distribution of the fund among your area and the remain two?

   (a) A1 = 350,000, A2 = 100,000, A3 = 150,000.

   (b) A1 = 150,000, A2 = 150,000, A3 = 300,000.

   (c) A1 = 200,000, A2 = 100,000, A3 = 300,000.

   (d) A1 = 220,000, A2 = 80,000, A3 = 300,000.

7. Question 7: Consider the situation in which a man dies and leaves a fortune of 1,750 monetary units, to be distributed among his three heirs (A1, A2, A3). But, during his lifetime the man promised the following amounts to each of them (250, 1,250, 1,250). Besides this, before the man dies, he gave to his second heir assets that sum a total of 500 monetary units and nothing to his other heirs. What do you consider to be the fairest distribution of the fortune among the heirs?

   (a) A1 = 83, A2 = 583, A3 = 1,083.

   (b) A1 = 205, A2 = 523, A3 = 1,022.

   (c) A1 = 0, A2 = 1,125, A3 = 625.

   (d) A1 = 250, A2 = 500, A3 = 1,000.

8. Question 8: Imagine that you and two neighbors (A1, A2, A3), share 36,000-liter water well. Under this measure, each one uses the following quantities of water from the well (10,000, 12,000, 14,000) liters. But, due to weather conditions of the last period the well only filled up to 20,000 liters. Besides this, each of you has a private water reserve with the following sizes (5,000, 15,000, 10,000). What do you consider is the fairest distribution of water between you and your neighbors?

   (a) A1 = 0, A2 = 11,500, A3 = 8,500.

   (b) A1 = 10,000, A2 = 2,500, A3 = 7,500.

   (c) A1 = 9,670, A2 = 1,670, A3 = 8,670.

   (d) A1 = 9,000, A2 = 2,000, A3 = 9,000.

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