A Study of the Effects of Automation on Labour

Miquel Oliver Vert

Supervisor: Marc Teignier

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Abstract

I develop a task-based model to study the effects of automation on labour demand. I differentiate two kinds of technological innovations: those that allow machines to replace labour in more tasks, and those that entail an improved version of a previous machine. Aggregate output is produced combining complementary tasks. Innovations improving previous machines are always beneficial for workers, since they just have a productivity effect. On contrast, innovations extending automation to more tasks, besides of a productivity effect, have a displacement effect. I find that if tasks are complementary enough then, no matter how productive the new machines are, an extension of automation to more tasks will always decrease wages. Then, motivated by the importance of this productivity effect, I incorporate a second final good to my baseline model. For the case of Leontief preferences, I find that if the second good is sufficiently capital intensive, then the effects of automation on labour are worsened.

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1 Introduction

The advances in automation technologies offer promising opportunities as they allow to expand the production possibilities frontier. At the same time, however, automation poses some threats because of its displacement effect, as it substitutes tasks previously performed by workers. This raises doubts on the capacity of future labor markets to create enough demand of labour, with the subsequent income distribution and social challenges.

The fear that machines could make human labour redundant is not a novel idea. Some historical examples go from the patent denial of the William Lee's knitting machine by Queen Elizabeth I in 1589 arguing that it would deprive her subjects of employment, to the British Luddite movement of the first half of the 19th century; or the predictions of some well-known economists such as Keynes, who predicted that our capacity to replace labour would outrun the pace at which we can find new uses for it (Keynes, 1933). Time has shown this fears to be false. However, some advocate that this time may be different. The decline in the labour share and the employment to population ratio in the US of last decades, as documented in Karabarbounis and Neiman (2014), and Oberfield and Raval (2014), supports this vision that the new automation wave may bring bad consequences for the future of work. Frey and Osborne (2013) assess the probability that occupations have of getting automated in the next decades, estimating that, with the current state of knowledge, around 47% of occupations are at high risk of getting computerised.

This paper focuses on the impact of automation on labour demand. Automation consists on the replacement of workers by machines that are more cost-effective, that is, that allow to produce more at the same cost. Therefore, automation by itself is not a bad thing. But it can be problematic if the profits of this automation process are not allocated properly among the population, concentrating the profits on too few winners and generating an unsustainable amount of losers which are worsened off due to losing their job and, thus, their major source of income. As explained above, labour, at least initially, is reduced by the adoption of these new machines; however, there are some countervailing forces which increase the demand for labour that can moderate, or even offset, the initial displacement effect (see also Acemoglu and Restrepo, 2018). On the one hand, as mentioned before, these new machines allow to increase the production, which rises the demand of labour in the tasks performed by labour that are needed to produce the final output. This is known as the productivity effect of automation. On the other hand, the extension of the tasks performed by machines (which, as I do in the model, they can be interpreted as capital) increases the demand for capital, rising its return, which, in turn, will incentive the accumulation of capital. This accumulation of capital will increase production and, as before, labour demand.

I use an intuitive task-based framework where capital and labour are perfect substitutes in tasks for which a machine has been invented. That way, capital will be used in a task if it's more cost-effective than labour. I differentiate two kinds of technological innovations: those that allow machines to replace labour in more tasks, and those that entail improving previous machines (or reducing their price). This framework allows us to study the effects of these two processes of automation on wages. As expected, improving current machines is always beneficial for labour, since it supposes an increase in the productivity of machines without displacing labour. In contrast, the effect on wages of extending automation to other tasks is not that clear and it will depend on whether the productivity effect or the displacement effect dominates. In general, we get that it is the adoption of not very productive machines (i.e. machines that are just a little bit preferable than labour or, as referred to in Acemoglu and Restrepo (2017), "so-so" technologies) the ones that lead to a decrease in the demand of labour (materialised by a drop in wages or an increase in unemployment). However, provided that tasks are complementary enough, I find that, no matter how productive these new machines are, the displacement effect will dominate over the productivity effect and wages will decrease.

The effects of automation can vary depending on the type of good whose production experiences the increase in automation. It may not be the same an increase in automation in a luxury good, where demand can increase exploiting the productivity effect of automation, than in a necessity good. The consequences may also vary depending on the labour intensity of the other goods. This motivates me to extend the model adding a second final good. To the best of my knowledge, this paper is the first attempt to study the effects of automation in an economy with more than one final good. This extension, for the case of Leontief preferences, shows how the reallocation of labour and capital between the two goods triggered by the increase in automation in one good can reduce the bad consequences of automation or intensify them, depending on the capital intensity of the other good.

The most related papers to this one are Hémous and Olsen (2017) and Acemoglu and Restrepo (2018, see also 2017 and 2019). Both present a task-based framework with two innovation processes: automation and creation of new tasks. In the former there are two types of workers (low- and high-skilled), machines can replace low-skilled ones, and innovation in new tasks takes the form of horizontal innovation. Their model features non-balanced growth, rising inequalities, in the form of increasing skilled premium, and a drop in the labor share. In contrast, the latter achieves balanced-growth as a consequence of assuming new tasks to be increasingly complex (the comparative advantage of labor is increasing in new tasks) and to replace tasks at the bottom performed by machines. This way, innovation in new tasks (labour-biased) can offset the displacement effect of automation (machine-biased) and the model can achieve balanced-growth. Zeira and Nakamura (2018) focus on automation unemployment, that is, the temporal unemployment due to the displaced workers by new automation that remain unemployed for one period as they look for another job. They also allow for the possibility of the creation of new tasks, though without giving any structure to this process (new tasks will appear and adopted if it is economically optimal to do so). With this, and assuming that labor share remains constant (following the Kaldor's facts), they find that automation unemployment converges to zero.

In all these models with some kind of expanding variety there is one crucial assumption which is that new tasks are more complex and difficult to be automated. We may agree that they may require a period of time in order to fully understand their production process, but after this relatively short time, it's not that clear that future tasks will be more difficult to automate. For this reason, and though undoubtedly, as we have already seen in the past, new technology will bring new tasks, I don't explicitly account for this. However, the effect of creating new tasks is equivalent to a reduction of the number of automated tasks. Also related to this paper is Peretto and Seater (2013), which presents a model with factor-eliminating technical change in which firms, through innovation that allows to increase the elasticity of output respect capital, reduce their need of labor.

The remaining of the paper is organized as follows. Section 2 introduces the baseline model. Section 3 studies the direct effects of extending and improving automation. Section 4 studies some countervailing forces. Section 5 presents two extensions: a simple way of endogenizing the two innovation processes, and another introducing a second final good in the model. Section 6 concludes and Section 7 contains the proves of the results not derived in the text.

2 Baseline Model

2.1 Consumers

I assume the preferences of the population can be identified as those of a representative household, with preferences u(c,l), where c is consumption of the single final good and l is the labour supply, as fraction of time. As usual, assume $u_c < 0$, $u_{cc} > 0$, $u_l < 0$, $u_{ll} < 0$. The optimization problem of the household is the following:

$$\max_{c_\tau, l_\tau} \quad U_t = \int_t^\infty e^{-\rho(\tau - t)} u(c_t, l_t) \; d\tau$$
$$\text{s.t.} \quad \dot{s}_t = s_t r_t + w_t l_t - c_t$$

Where ρ is the continuous discount rate, s are savings, r is the interest rate and w is the wage. Which leads to the optimality conditions:

$$w_t = -\frac{u_l}{u_c} \tag{1}$$

$$-\frac{u_{cc}\dot{c}_t}{u_c} = r_t - \rho \tag{2}$$

2.2 Producer of the final good

The production of the final good requires a continuum of N imperfectly complementary tasks (i.e. elasticity of substitution between tasks is $\sigma \in (0,1)$). We can find several examples supporting this complementarity, like the fatal disaster of the space shuttle *Challenger* caused by a single component, the O-rings (see Kremer, 1993). I assume a monopolist final good producer, with a mark-up μ (profits will be necessary to incentive innovation and this mark-up has to be at the final good producer level because, due to the complementarity of tasks, task producers would not have incentives to improve their productivity).

$$y = \left(\int_0^N y(x)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{3}$$

Where y(x) is the intermediate good produced by task x. Profit maximization leads to the following demand of tasks:

$$y(x) = yP^{\sigma}p(x)^{-\sigma} \tag{4}$$

Where P is the price of the final good and p(x) the (internal) price of the intermediate good of task x. Using (4) and that profits are defined by the mark-up on costs and then normalising P to 1:

$$0 = Py - (1+\mu)\int_0^N p(x)y(x)dx = Py - (1+\mu)P^{\sigma}y\int_0^N p(x)^{1-\sigma}dx$$

$$P = \left((1+\mu)\int_0^N p(x)^{1-\sigma}dx\right)^{\frac{1}{1-\sigma}} = 1 \tag{5}$$

2.3 Tasks Producers

Each task, indexed by x, can be produced by labour or capital (machines), conditional on the existence of a machine able to perform the task. Labour and capital are assumed to be perfect substitutes, so that the task production function is of the form:

$$y(x) = \gamma_L(x)l(x) + \gamma_K(x)k(x) \tag{6}$$

Task x will be produced by labour if the cost is lower using labour than using capital (if there is no machine for task x, then $\gamma_K(x) = 0$), that is, if $\frac{w}{\gamma_L(x)} < \frac{R}{\gamma_K(x)}$, where R is the rental price of capital.

I order tasks non-decreasingly in $\frac{\gamma_L(x)}{\gamma_K(x)}$ (i.e. from low to high comparative advantage of labour). That way, we have two intervals: one with the tasks for which there exists a machine that can make the task, $x \in [0, \theta^*]$, and the interval of tasks that have to be done forcefully by labour, $x \in (\theta^*, N]$. Further, let $[0, \theta]$ be the rang of tasks that are actually performed by machines. θ is defined as $\theta = min\{\bar{\theta}, \theta^*\}$, where $\bar{\theta}$ is the task with the highest index that satisfies $\frac{\gamma_L(\bar{\theta})}{\gamma_K(\bar{\theta})} \leq \frac{w}{R}$. That way, if $\theta = \bar{\theta}$, then there are tasks ($x \in [\bar{\theta}, \theta^*]$) for which there exists a machine but it is not used. On contrary, if $\theta = \theta^*$, then technology is binding.

Task producers work under perfect competition (i.e. p(x)=c(x)):

$$p(x) = \frac{R}{\gamma_K(x)}, \;\; x \in [0, \theta] \qquad\qquad p(x) = \frac{w}{\gamma_L(x)}, \;\; x \in (\theta, N] \tag{7}$$

2.4 Static Equilibrium

Using (7), we can write (4) as:

$$y(x) = y\left(\frac{R}{\gamma_K(x)}\right)^{-\sigma}, \;\; x \in [0, \theta] \qquad\qquad y(x) = y\left(\frac{w}{\gamma_L(x)}\right)^{-\sigma}, \;\; x \in (\theta, N] \tag{8}$$

Using this, factor demands are:

$$k(x) = yR^{-\sigma}\gamma_K(x)^{\sigma-1}, \;\; x \in [0, \theta] \qquad\qquad l(x) = yw^{-\sigma}\gamma_L(x)^{\sigma-1}, \;\; x \in (\theta, N] \tag{9}$$

Combining (5) with (7), leads to:

$$1 = (1+\mu)\left(\int_0^\theta R^{1-\sigma}\gamma_K(x)^{\sigma-1}dx + \int_\theta^N w^{1-\sigma}\gamma_L(x)^{\sigma-1}dx\right) \tag{10}$$

Market of factors clearing:

$$l = \int_\theta^N l(x)dx = yw^{-\sigma}\int_\theta^N \gamma_L(x)^{\sigma-1}dx$$

$$w = \left(\frac{y}{l}\int_\theta^N \gamma_L(x)^{\sigma-1}dx\right)^{\frac{1}{\sigma}} \tag{11}$$

$$k = \int_0^\theta k(x)dx = yR^{-\sigma}\int_0^\theta \gamma_K(x)^{\sigma-1}dx$$

$$R = \left(\frac{y}{K} \int_0^{\theta} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} \tag{12}$$

Plugging (11) and (12) into (10) we get the final good production function represented as a CES production function of K and l:

$$y = (1+\mu)^{-\frac{\sigma}{1-\sigma}} \left[K^{\frac{\sigma-1}{\sigma}} \left(\int_0^{\theta} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} + l^{\frac{\sigma-1}{\sigma}} \left(\int_{\theta}^{N} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{13}$$

For labour market to clear, both (1) and (11) have to be satisfied.

The law of motion of capital is:

$$\dot{K}_t = (R_t - \delta) K_t + w_t l_t - c_t \tag{14}$$

And asset market clearing imply $\mathrm{r}_t = R_t - \delta$.

2.5 Dynamic equilibrium

Given time paths for $\{\theta_t^*, \gamma_{K,t}(x)\}$ and an initial level of capital, K_0, the dynamic equilibrium is a time path for $\{\theta_t, K_t, c_t, l_t\}$ satisfying:

1. Labour-leisure equation (1) and Euler equation (2)

2. Factor prices (11) and (12)

3. $\theta = \begin{cases} \theta^*, & \text{if } \frac{w}{R} \geq \frac{\gamma_L(\theta^*)}{\gamma_K(\theta^*)} \\ \bar{\theta}, & \text{if } \frac{w}{R} < \frac{\gamma_L(\theta^*)}{\gamma_K(\theta^*)} \end{cases}$

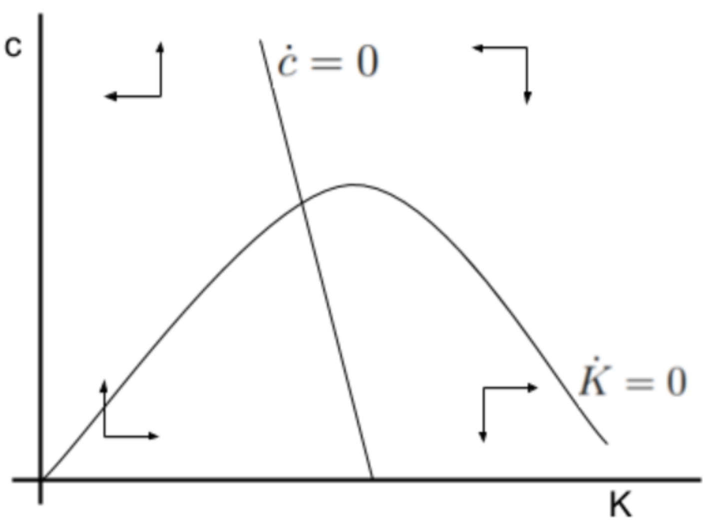
4. The law of motion of capital (14)

5. The transversality condition $\lim_{t\to\infty} K_t u_c e^{-\rho t} = 0$

Given θ_t and $\gamma_{K,t}(x)$, we can study the phase diagram on the plane c_t, K_t, and in particular the loci $\dot{c} = 0$ and $\dot{K} = 0$.

First, from the Euler equation, $\dot{c} = 0$ imply $R = \delta + \rho$. If labour were supplied inelastically, then there would be just one value of K satisfying it (and, thus, $\dot{c} = 0$ would be a vertical line in the phase diagram). But, if labour supply is elastic, then $\dot{c} = 0$ features a downward slope in K. The intuition is as follows. A decrease in c allows to increase K, which increases output using the same level of L and, so, wages increase (marginal product of labour increases). This will incentive to supply more labour (from $w = -\frac{u_L}{u_c}$, $u_L < 0$ and $u_{LL} < 0$) and this, in turn, will increase R, raising the level of capital that satisfies $R = \delta + \rho$.

Next, from the law of motion of capital, the locus $\dot{K} = 0$ is defined by the equation $c = y(l, K) - \delta K$. Then, the phase diagram takes the form of the following illustration. (See the Appendix for a proof of the Existence and Unicity of the steady state).



3 Extending Automation and Improving Machines

The model presented in the previous section allows us to differentiate between two processes of automation: the extension of automation to more tasks (i.e. increase θ) and the improvement of machines in already automated tasks (i.e. an increase of $\gamma_K(x)$ for at least some

$x \in [0, \theta]$). In this section I study how these two forms of automation affect the economy.

3.1 Change in $\gamma_K(x)$

First, using (13) we get that the marginal product of an improvement of the machine performing task x is positive but diminishing.

$$\frac{\partial y}{\partial \gamma_K(x)} = y^{\frac{1}{\sigma}} K^{\frac{\sigma-1}{\sigma}} \left(\int_0^\theta \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1-\sigma}{\sigma}} \gamma_K(x)^{\sigma-2} > 0$$

$$\frac{\partial^2 y}{\partial \gamma_K(x)^2} = \frac{\partial y}{\partial \gamma_K(x)} \left[(\sigma-1) \frac{\frac{1-\sigma}{\sigma} \gamma_K(x)^{\sigma-2}}{\int_0^\theta \gamma_K(x)^{\sigma-1} dx} + (\sigma-2)\gamma_K(x)^{\sigma-3} \right] < 0$$

In particular, if we assumed that machines have the same productivity in all tasks (and that increasing the productivity of machines in one task implied increasing the productivity of machines in all the other automated tasks in the same magnitude), that is, if $\gamma_K(x) = \gamma_K$; we would get:

$$\frac{\partial y}{\partial \gamma_K} = y^{\frac{1}{\sigma}} K^{\frac{\sigma-1}{\sigma}} \theta^{\frac{1}{\sigma}} \gamma_K^{-\frac{1}{\sigma}} > 0 \quad , \qquad \frac{\partial^2 y}{\partial \gamma_K(x)^2} = \frac{1}{\sigma} \frac{\partial y}{\partial \gamma_K} \left[\frac{\partial lny}{\partial \gamma_K} - \frac{1}{\gamma_K} \right] < 0$$

The second inequality follows from the fact that $y^{\frac{\sigma-1}{\sigma}} > k^{\frac{\sigma-1}{\sigma}} \theta^{\frac{1}{\sigma}} \gamma_K^{\frac{\sigma-1}{\sigma}}$ (which is straightforward from (13)).

From (11): $\dfrac{\partial w}{\partial \gamma_K(x)} = \dfrac{w}{\sigma} \dfrac{\partial lny}{\partial \gamma_K(x)} > 0$

From (12):

$$\begin{aligned} \frac{\partial R}{\partial \gamma_K(x)} &= \frac{R}{\sigma} \frac{\partial lny}{\partial \gamma_K(x)} + (\sigma-1) \frac{R}{\sigma} \frac{\gamma_K(x)^{\sigma-2}}{\int_0^\theta \gamma_K(x)^{\sigma-1} dx} \\ &= \frac{R}{\sigma} \frac{\gamma_K(x)^{\sigma-2}}{\int_0^\theta \gamma_K(x)^{\sigma-1} dx} \left[\frac{K^{\frac{\sigma-1}{\sigma}} \left(\int_0^\theta \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}}}{y^{\frac{\sigma-1}{\sigma}}} + (\sigma-1) \right] < R \frac{\gamma_K(x)^{\sigma-2}}{\int_0^\theta \gamma_K(x)^{\sigma-1} dx} \end{aligned}$$

Where the last inequality follows from (13), and constitutes an upper bound that correspond to the case where all tasks are performed by capital. On the one hand an improvement of machines unambiguously benefits wages. Note that this increase will be greater the higher the complementarity between labour and capital (i.e. lower σ), but the sign wouldn't be negative even if factors were substitutes. On the other hand, the sign of the effect on R is ambiguous. It will depend on the degree of complementarity between tasks and the amount of tasks performed by machines: the higher complementarity (lower σ) and the fewer tasks are performed by machines the more likely the effect will be negative. The intuition is clear: due to the complementarity of tasks in producing the final output, the increase in output due to the increase in productivity in machines is slowed down by the presence of labour tasks, the productivity of which is unchanged, driving down the demand of capital.

What we can say, is that the relative price of capital respect labour decreases.

$$\frac{\partial lnR}{\partial\gamma_K(x)} = \frac{1}{\sigma}\frac{\partial lny}{\partial\gamma_K(x)} + \frac{\sigma-1}{\sigma}\frac{\gamma_K(x)^{\sigma-2}}{\int_0^\theta \gamma_K(x)^{\sigma-1}dx} = \frac{\partial lnw}{\partial\gamma_K(x)} + \frac{\sigma-1}{\sigma}\frac{\gamma_K(x)^{\sigma-2}}{\int_0^\theta \gamma_K(x)^{\sigma-1}dx} < \frac{\partial lnw}{\partial\gamma_K(x)}$$

And so: $\dfrac{\partial\frac{R}{w}}{\partial\gamma_K(x)} < 0$

Next, we can ask how a change in $\gamma_K(x)$ affects $\dot{K}=0$ and $\dot{c}=0$ and, therefore, the steady state of the economy.

$\dot{c}=0$ implies $R=\delta+\rho$ so the locus will shift to the right (resp. left) if $\gamma_K(x)$ affects positively (resp. negatively) R, which depends on σ and θ, as discussed. If the complementarity between tasks is high enough, then $\dot{c}=0$ will shift to the left, which will tend to decrease both capital and consumption in the steady state.

$\dot{K}=0$ implies $c=y-\delta K$. So, since $\frac{\partial y}{\gamma_K}>0$, the locus will shift upwards. This shift tends to increase consumption in the steady state and reduce capital (due to the endogenous labour supply).

3.2 Change in θ

Now, I do the same analysis with the case of an increase in the number of tasks performed by machines.

$$\frac{\partial y}{\partial \theta} = \frac{y^{\frac{1}{\sigma}}}{1-\sigma}\left[l^{\frac{\sigma-1}{\sigma}}\left(\int_{\theta}^{N}\gamma_L(x)^{\sigma-1}dx\right)^{\frac{1-\sigma}{\sigma}}\gamma_L(\theta)^{\sigma-1} - K^{\frac{\sigma-1}{\sigma}}\left(\int_{0}^{\theta}\gamma_K(x)^{\sigma-1}dx\right)^{\frac{1-\sigma}{\sigma}}\gamma_K(\theta)^{\sigma-1}\right] \tag{15}$$

Note that, in order for the θ to increase, the machine implementation condition has to be satisfied (i.e. $\frac{\gamma_L(\theta)}{\gamma_K(\theta)} \leq \frac{w}{R}$). Since $\sigma \in (0,1)$, this condition is equivalent to $\gamma_L(\theta)^{\sigma-1} \geq \left(\gamma_K(\theta)\frac{w}{R}\right)^{\sigma-1}$. Using this, as well as (11) and (12), in (15) we get:

$$\frac{\partial y}{\partial \theta} \geq \frac{y^{\frac{1}{\sigma}}}{1-\sigma}\left[l^{\frac{\sigma-1}{\sigma}}\left(\int_{\theta}^{N}\gamma_L(x)^{\sigma-1}dx\right)^{\frac{1-\sigma}{\sigma}}\left(\gamma_K(\theta)\left(\frac{K}{l}\frac{\int_{\theta}^{N}\gamma_L(x)^{\sigma-1}dx}{\int_{0}^{\theta}\gamma_K(x)^{\sigma-1}dx}\right)^{\frac{1}{\sigma}}\right)^{\sigma-1}\right.$$
$$\left. -K^{\frac{\sigma-1}{\sigma}}\left(\int_{0}^{\theta}\gamma_K(x)^{\sigma-1}dx\right)^{\frac{1-\sigma}{\sigma}}\gamma_K(\theta)^{\sigma-1}\right] = 0$$

So, extending automation never harms production, conditional on the adoption of new technologies being based on efficiency criteria. The second derivative may become positive due to the productivity effect at a first moment, but we can see that it will decrease after some point, since:

$$\lim_{\theta\to\infty}\frac{\partial y}{\partial \theta} = \frac{y^{\frac{1}{\sigma}}}{1-\sigma}\left[-K^{\frac{\sigma-1}{\sigma}}\left(\int_{0}^{N}\gamma_K(x)^{\sigma-1}dx\right)^{\frac{1-\sigma}{\sigma}}\gamma_K(N)^{\sigma-1}\right] < 0$$

From (11):

$$\frac{\partial w}{\partial \theta} = \frac{w}{\sigma}\left[\frac{\partial lny}{\partial \theta} - \frac{\gamma_L(\theta)^{\sigma-1}}{\int_{\theta}^{N}\gamma_L(x)^{\sigma-1}dx}\right] \geq -\frac{w}{\sigma}\frac{\gamma_L(\theta)^{\sigma-1}}{\int_{\theta}^{N}\gamma_L(x)^{\sigma-1}dx} \tag{16}$$

This tells us that extending automation will lead to a decrease in wages if the producti-vity effect of automation is not enough to offset the displacement effect. In terminology of Acemoglu & Restrepo (2017), it is the "so-so" new machines (i.e. those machines that are just a bit more efficient than labour) the ones that are detrimental for workers. This leads us to the question: how many times (x^*) less costly has to be, at least, the newly invented machine in order to have a positive impact on wages. This is what x^* tells us if we define it as: $x^* = max\{x = \frac{\frac{w}{\gamma_L(\theta)}}{\frac{R}{\gamma_K(\theta)}} \quad | \quad \frac{\partial w}{\partial \theta} \leq 0\}$. This number, at the same time, informs us of what drop in wages would be necessary in order that keeping the task automated would necessarily require the productivity effect to offset the initial displacement effect. In other words, imagine a machine that displaces labour in task i but without any productivity effect. In this case, if automation is extended to more tasks decreasing wages, then, in order to keep task i automated, the productivity of machine i will have to increase, and this will offset a bit the initial displacement effect. Further extension of automation could drop wages even more, forcing further improvement of machine i, etc.

So, if we found that x^* tends to ∞, then wages would be always harmed by an increase of θ, no matter how productive the new machines are. Or with the wage interpretation. it would imply that even the productivity effect that would make the machine preferable than labour with $w = 0$ wouldn't be enough to compensate the initial displacement effect. From (16):

$\frac{\partial w}{\partial \theta} \leq 0 \iff \frac{\partial y}{\partial \theta} \leq \frac{\gamma_L(\theta)^{\sigma-1}}{\int_\theta^N \gamma_L(x)^{\sigma-1}dx}$, which, using (15), can be rewritten as:

$$\frac{y^{\frac{1-\sigma}{\sigma}}}{1-\sigma} l^{\frac{\sigma-1}{\sigma}} \frac{\gamma_L(\theta)^{\sigma-1}}{\left(\int_\theta^N \gamma_L(x)^{\sigma-1}dx\right)^{\frac{\sigma-1}{\sigma}}} \left[1 - \left(\frac{K}{l} \frac{\int_\theta^N \gamma_L(x)^{\sigma-1}dx}{\int_0^\theta \gamma_K(x)^{\sigma-1}dx}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\gamma_K(\theta)}{\gamma_L(\theta)}\right)^{\sigma-1}\right] \leq \frac{\gamma_L(\theta)^{\sigma-1}}{\int_\theta^N \gamma_L(x)^{\sigma-1}dx}$$

Note that, first using (11) and (12) and then the definition of x^*,

$\left(\frac{K}{l} \frac{\int_\theta^N \gamma_L(x)^{\sigma-1}dx}{\int_0^\theta \gamma_K(x)^{\sigma-1}dx}\right)^{\frac{\sigma-1}{\sigma}} = \left(\frac{w}{R}\right)^{\sigma-1} = \left(x^* \frac{\gamma_L(\theta)}{\gamma_K(\theta)}\right)^{\sigma-1}$. With which, we get:

$$\frac{1}{1-\sigma} \frac{l^{\frac{\sigma-1}{\sigma}} \left(\int_\theta^N \gamma_L(x)^{\sigma-1}dx\right)^{\frac{1}{\sigma}}}{y^{\frac{\sigma-1}{\sigma}}} \left[1 - (x^*)^{\sigma-1}\right] \leq 1$$

From (13), it is clear that $\frac{l^{\frac{\sigma-1}{\sigma}} \left(\int_\theta^N \gamma_L(x)^{\sigma-1}dx\right)^{\frac{1}{\sigma}}}{y^{\frac{\sigma-1}{\sigma}}} < 1$. Therefore, note that if $\sigma \to 0$, then even if $x^* \to \infty$, the inequality holds (i.e. even with the highest possible productivity effect, wages would decrease). That way, we have seen that, if tasks are complementary enough, then it is not only the increase of θ through "so-so" technologies what delivers $\frac{\partial w}{\partial \theta} < 0$. This is synthesised in the following theorem.

Theorem 1: If tasks are complementary enough (i.e. σ low enough), then an increase in θ always decreases wages, regardless of the efficiency of these new machines.

The intuition behind the previous result is that the higher the complementarity between tasks, the more limited the productivity effect will be, since the other (less productive) tasks will be more needed in order to complete the production process.

The previous reasoning of the necessary improvement of machines before extending automation to other tasks relies on wages falling enough. This increase in some $\gamma_K(x)$ won't be necessary if the wage doesn't descend below the level $\bar{w} = max\{w* > 0 \mid \frac{w^*}{R} = \frac{\gamma_L(x)}{\gamma_K(x)},\ x \in [0, \theta]\}$, under which, labour would be employed again in some previously automated tasks. This condition would be trivially achieved if there were a minimum wage above this threshold; in this case, extending automation through "so-so" machines would harm labour by throwing them out of the labour market without limit. Another sufficient condition to be able to increase θ through "so-so" machines without limit is to assume a sufficiently elastic labour supply. In such a circumstance, the reduction of the labour supply would slow down the decline of wages due to the increasing automation, preventing them to get below $\bar{w}$.

From (12): $\dfrac{\partial R}{\partial \theta} = \dfrac{R}{\sigma}\left[\dfrac{\partial lny}{\partial \theta} + \dfrac{\gamma_K(\theta)^{\sigma-1}}{\int_0^\theta \gamma_K(x)^{\sigma-1}dx}\right] > 0$

So, it is straightforward that: $\dfrac{\partial \frac{R}{w}}{\partial \theta} > 0$

Finally, as with γ_K, we can see how a change in θ affects $\dot{K} = 0$ and $\dot{c} = 0$. On the one hand, since an increase in θ increases R, $\dot{c} = 0$ locus will shift to the right, tending to increase capital. On the other hand, as with γ_K, an increase in θ increases output, so it also shifts $\dot{K} = 0$ locus upwards, tending to increase consumption.

4 Countervailing forces to the descend in wages

In this section, I want to see if wages can decrease with θ even after accounting for other forces that act in the opposite direction, such as capital accumulation towards the new steady state and the adjustment of the labour supply.

4.1 Capital Adjustment

Using (12) considering R constant (note that in the long run it will return to the steady state level, $R = \delta + \rho$), we can get how K changes after an increase of θ. And from (11), we get the variation of wages with a change in K. Therefore:

$$\frac{\partial w}{\partial K}\frac{\partial K}{\partial \theta} = \frac{w}{\sigma}\frac{\partial lny}{\partial K}K\left(\frac{\partial lny}{\partial \theta} + \frac{\gamma_K(\theta)^{\sigma-1}}{\int_\theta^N \gamma_K(x)^{\sigma-1}dx}\right) = \frac{w}{\sigma}R^{1-\sigma}\left(\frac{\partial lny}{\partial \theta}\int_\theta^N \gamma_K(x)^{\sigma-1}dx + \gamma_K(\theta)^{\sigma-1}\right) > 0 \tag{17}$$

Where in the second equality I have used again the definition of R from (12). To see that it may not be enough to offset the decline of wages observed in the previous section, assume the extreme case with no productivity effect (i.e. $\frac{\partial y}{\partial \theta} = 0$). That way, adding (16) and (17) leads to:

$$\frac{\partial w}{\partial \sigma}\left[-\frac{\gamma_L(\theta)^{\sigma-1}}{\int_\theta^N \gamma_L(x)^{\sigma-1}dx} + \left(\frac{\gamma_K(\theta)}{R}\right)^{\sigma-1}\right] = \frac{\partial w}{\partial \sigma}\gamma_L(\theta)^{\sigma-1}\left[-\frac{1}{\int_\theta^N \gamma_L(x)^{\sigma-1}dx} + w^{1-\sigma}\right]$$

Where I have used the assumption that there is no productivity effect (i.e. $\frac{\gamma_K(\theta)}{R} = \frac{\gamma_L(\theta)}{w}$). In order to conclude that the overall effect is still negative, we need the element in the square brackets to be negative, which is indeed the case since from (11) and (13):

$$w^{1-\sigma} = \frac{l^{\frac{\sigma-1}{\sigma}}\left(\int_\theta^N \gamma_L(x)^{\sigma-1}dx\right)^{\frac{1}{\sigma}}}{y^{\frac{\sigma-1}{\sigma}}}\frac{1}{\int_\theta^N \gamma_L(x)^{\sigma-1}dx} < \frac{1}{\int_\theta^N \gamma_L(x)^{\sigma-1}dx}$$

So, we conclude that this countervailing force may not be enough and, therefore, that in the new steady state we may have a lower wage.

4.2 Labour Supply Adjustment

As the demand for labour declines (i.e. for a same level of employment the firm will be paying a lower wage), labour supply adjusts declining too (if the reward to work is lower, then individuals will prefer to devote more time to leisure and less to work).

We can check that this labour supply adjustment acts on wages in the opposite direction that θ does:

$$\frac{\partial w}{\partial l}\frac{\partial l}{\partial \theta} = -\frac{w}{\sigma}\frac{1}{l}\frac{\partial l}{\partial \theta} > 0$$

The result comes from (1) and assuming that an θ decreases the demand for labour (i.e. that the displacement effect wins over the productivity and capital adjustment effects). If the increase in θ implied an increase in wages, then the labour supply effect would be negative on wages, smoothing the increase in wages.

Therefore, note that this countervailing force will never change the sign of the effect of θ on wages, it will just smooth the change.

5 Extensions

5.1 Endogenous Innovations

In line with the previous sections, there are two possible innovations: to increase γ_K and to increase θ. For simplicity, I am going to assume: i) $\gamma_L(x)$ is increasing in x; ii) $\gamma_K(x) = \gamma_K$ for all $x \in [0, \theta]$, that is, that machines have the same absolute productivity both in newly and in previously automated tasks (not comparative, since $\gamma_L(x)$ is increasing in x); and iii) $\frac{w}{\gamma_L(\theta)} > \frac{R}{\gamma_K}$, so that it is efficient to extend automation.

The profits of the firm (using (5)):

$$\pi = y - \int_0^N p(x)y(x) = y - y\int_0^N p(x)^{1-\sigma} = \frac{\mu}{1+\mu}y$$

And from previous results, we see that:

$$\frac{\partial \pi}{\partial \gamma_K} > 0 \ , \quad \frac{\partial^2 \pi}{\partial \gamma_K^2} < 0 \ , \quad \frac{\partial \pi}{\partial \theta} > 0 \ , \quad \frac{\partial \pi}{\partial \theta} \text{ decreases.}$$

How much will the firm spend in R&D? Let S be the spending in R&D and we can write the present value Hamiltonian: $H = e^{-\int_0^t r_s ds}(\pi_t - S_t) + \lambda_t S_t$

Which leads to the optimality condition: $r_t = \dfrac{\partial \pi_t}{\partial S}$

Which tells us that the firm will spend in R&D as long as the increase in profits is bigger than the financing cost. The innovation-market clearing condition (for the case it is optimal to allocate a positive amount of spending both in increasing current machines productivity, S_γ, and in increasing the amount of tasks they can perform, S_θ) implies: $r_t = \dfrac{\partial \pi_t}{\partial S_\gamma} = \dfrac{\partial \pi_t}{\partial S_\theta}$

In particular, assuming $\dot{\theta} = \eta_\theta S_\theta \ \ and \ \ \dot{\gamma}_K = \eta_\gamma S_\gamma$ we have:

$$\pi_\theta = \frac{\partial \pi}{\partial \theta} = \frac{\mu}{1+\mu}\frac{\partial y}{\partial \theta} = \frac{\mu}{1+\mu}\frac{y^{\frac{1}{\sigma}}}{1-\sigma}\left[l^{\frac{\sigma-1}{\sigma}} \left(\int_\theta^N \gamma_L(x)^{\sigma-1}dx\right)^{\frac{1-\sigma}{\sigma}} \gamma_L(\theta)^{\sigma-1} - \left(\frac{\theta}{K\gamma_K}\right)^{\frac{1-\sigma}{\sigma}} \right]$$

$$\pi_\gamma = \frac{\partial \pi}{\partial \gamma_K} = \frac{\mu}{1+\mu}\frac{\partial y}{\partial \gamma_K} = \frac{\mu}{1+\mu} y^{\frac{1}{\sigma}} K^{\frac{\sigma-1}{\sigma}} \theta^{\frac{1}{\sigma}} \gamma_K^{-\frac{1}{\sigma}}$$

Note that (using (11) and (12)) $\pi_\theta = 0$ if assumption iii) doesn't hold. This tells us, as it is logical, that the firm will not invest in innovation in θ if the new machines will not be implemented. That is, in this case, machines will have to improve before advancing in automating other tasks.

So, the innovation-market clearing condition becomes: $\eta_\theta \pi_\theta = \eta_\gamma \pi_\gamma$, which leads to:

$$\frac{\eta_\gamma}{\eta_\theta}(1-\sigma) = \left(\frac{K\int_\theta^N \gamma_L(x)^{\sigma-1}dx}{l}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\gamma_K}{\theta}\right)^{\frac{1}{\sigma}} \gamma_L(\theta)^{\sigma-1} - \frac{\gamma_K}{\theta} \tag{18}$$

The intuitive result that we would expect of this simple extension is to have a positive relationship between the relative facility to innovate and the relative state of technologies (i.e. that an increase in $\frac{\eta_\gamma}{\eta_\theta}$ implies an increase in $\frac{\gamma_K}{\theta}$). In order for this to hold, we need the derivative of the first term in the right hand side to dominate over the one of the second term. It is easy to see that this is satisfied if:

$$\frac{\gamma_K}{\theta} > \sigma^{\frac{\sigma}{1-\sigma}} \gamma_L(\theta)^\sigma \frac{l}{K \int_\theta^N \gamma_L(x)^{\sigma-1} dx}$$

Using (11) and (12) it can be rewritten as:

$$\sigma < \left(\frac{w}{\gamma_L(\theta)} \frac{\gamma_K}{R} \right)^{1-\sigma}$$

And this is certainly satisfied due to $\sigma \in (0, 1)$ and assumption iii).

5.2 Two Final Goods

In the previous analysis of the impact of automation on wages, the productivity effect played an important role. However, the possibility to produce more doesn't translate immediately in an increase of production. It was so in the previous model since we were assuming a single final good, but it wouldn't be the case in an economy with different goods if the demand doesn't support the increase in production of that good. Also, with a single good, all the labour and capital are used in the production of this good, while if there are different goods, we allow for movements of labour and capital across firms. With the inclusion of a second final good in the economy, I want to extend the previous model to take into account these issues. I divide the problem in two parts, first I find the equilibrium in the supply side and, then, I introduce the demand side.

Supply Side

The supply problem is analogous to the one developed in the baseline model but for two firms, indexed by $i = 1, 2$, each producing their single final good y_i.

$$\max_{y_i(x)} \quad P_i y_i - \int_0^{N_i} p_i(x) y_i(x) \ dx \quad \text{where:}$$

$$y_i = \left(\int_0^{N_i} y_i(x)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{19}$$

Which has the following optimality condition:

$$p_i(x) = P_i y_i^{\frac{1}{\sigma}} y_i(x)^{-\frac{1}{\sigma}} \tag{20}$$

The price of each good is derived from the zero profits condition (for this section, we don't care about endogenous innovation, so we can dispense with profits) and (20):

$$P_i = \left(\int_0^N p_i(x)^{1-\sigma} dx \right)^{\frac{1}{1-\sigma}}$$

Combining both:

$$P_i = \left(\int_0^{\theta_i} R_i^{1-\sigma} \gamma_K(x)^{\sigma-1} dx + \int_{\theta_i}^{N_i} w_i^{1-\sigma} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{1-\sigma}} \tag{21}$$

Defining l_i (resp. K_i) as the amount of labour (resp. capital) employed by firm i, and using the tasks production function (6), the demand of tasks (20) and $p_i(x) = c_i(x)$ with $c_i = \frac{R_i}{\gamma_K(x)}$ if $x \in [0, \theta_i)$ and $c_i = \frac{w_i}{\gamma_L(x)}$ if $x \in [\theta_i, N_i]$:

$$w_i = \left(\frac{y_i}{l_i} \int_{\theta_i}^{N_i} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} P_i \tag{22}$$

$$R_i = \left(\frac{y_i}{K_i} \int_0^{\theta_i} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} P_i \tag{23}$$

I normalize $P_1 = 1$. Then, note that w_1 and R_1 depend just on l_1 and K_1. Plugging (22) and (23) into (21) we get the final good production function represented as a CES production function of K_i and l_i:

$$y_i = \left[K_i^{\frac{\sigma-1}{\sigma}} \left(\int_0^{\theta_i} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} + l_i^{\frac{\sigma-1}{\sigma}} \left(\int_{\theta_i}^{N_i} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{24}$$

Perfect mobility of labour implies wage equalization ($w_i = w$). So, from (23):

$$\frac{y_1}{l_1} \int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx = P_2^{\sigma} \frac{y_2}{l_2} \int_{\theta_2}^{N_2} \gamma_L(x)^{\sigma-1} dx \tag{25}$$

Analogously, perfect capital mobility between firms implies capital rents equalization ($R_i = R$), which, using (24), leads to:

$$\frac{y_1}{K_1} \int_0^{\theta_1} \gamma_K(x)^{\sigma-1} dx = P_2^{\sigma} \frac{y_2}{K_2} \int_0^{\theta_2} \gamma_K(x)^{\sigma-1} dx \tag{26}$$

Note that in equilibrium (i.e. with $w_i = w$ and $R_i = R$), P_2 will be determined just by l_1 and K_1: $P_2(w_1(l_1, K_1), R_1(l_1, K_1))$. That way, we can characterise the supply side equilibrium by defining $F_1(l_1, K_1) = 0$ as:

$$\frac{y_1}{l_1} \int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx - \left(R_1^{1-\sigma} \int_0^{\theta_i} \gamma_K(x)^{\sigma-1} dx + w_1^{1-\sigma} \int_{\theta_i}^{N_i} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{\sigma}{1-\sigma}} \frac{y_2}{l_2} \int_{\theta_2}^{N_2} \gamma_L(x)^{\sigma-1} dx \tag{27}$$

I could have defined it from (26) instead of (25) without loss of generality, since $P_2(w_1(l_1, K_1), R_1(l_1, K_1))$ guarantees that if one is satisfied then the other will be satisfied too. The following lemmas characterise the locus in the $\{l_1, K_1\}$ plane.

Lemma 1: $\dfrac{\partial F_1}{\partial l_1} < 0$, $\dfrac{\partial F_1}{\partial K_1} < 0$ and, therefore $\left.\dfrac{dK_1}{dl_1}\right|_{F_1=0} > 0$.

Lemma 1 tells us that the supply locus is increasing. Next lemmas helps us to assess how does the supply locus moves after an increase in θ.

Lemma 2: $F_1(0, K) > 0 > F_1(L, 0)$.

Lemma 3: $\dfrac{\partial F_1}{\partial \theta_1} < 0$.

So, an increase of automation in good 1 shifts the equilibrium in the supply locus to the left (i.e. $F_1 = 0$ to the left).

Demand Side

I consider that the representative household's preferences are given by the nonhomothetic CES utility function as it is in Comín, Mestieri & Lashkari (2019).

$$1 = \sum_{i=1}^{2} \Omega_i^{\frac{1}{e}} \left(\frac{c_i}{g(U)^{\nu_i}} \right)^{\frac{e-1}{e}} \tag{28}$$

Which leads to the following demand function (see the Appendix for a derivation):

$$c_i = \Omega_i g(U)^{\nu_i(1-e)} \left(\frac{E}{P_i} \right)^e \tag{29}$$

And set the cardinalization $g(u) = \frac{E}{P}$ where P is the Price Index (for simplicity, we could use $P_1 = 1$ as Price Index).

Also to simplify the analysis, I will assume that the amount saved is a constant fraction of income, sY; so that expenditure is: $E = (1-s)Y$. In particular, $c_i = (1-s)y_i$. Then, (29) leads us to the following demand condition:

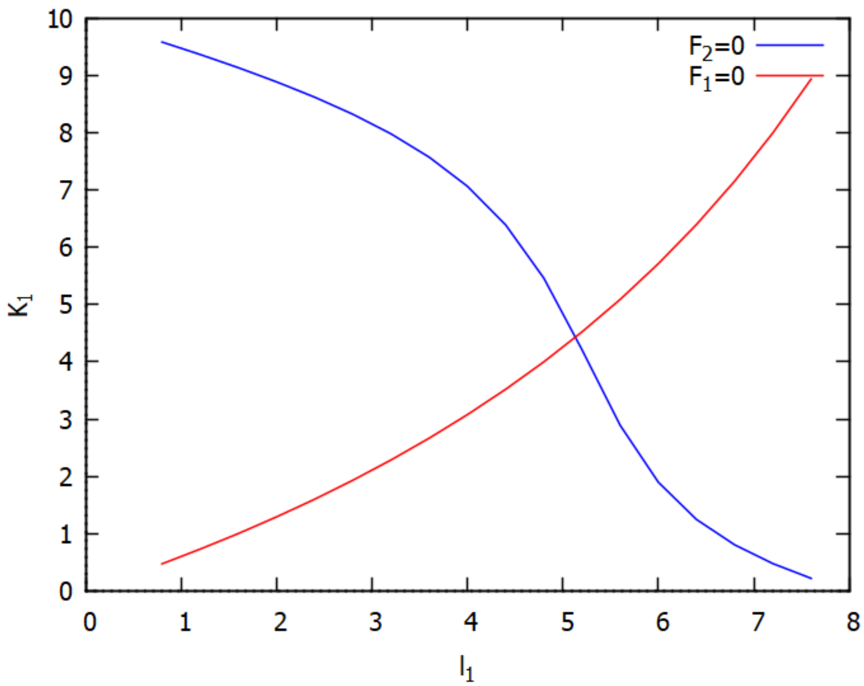
$$\frac{y_1}{y_2} = P_2^e \frac{\Omega_1}{\Omega_2} \left[(1-s)(P_2 y_2 + y_1) \right]^{(1-e)(\nu_1 - \nu_2)} \tag{30}$$

From this and using $P_2(w_1(l_1, K_1), R_1(l_1, K_1))$, as in the supply side, define $F_2(l_1, K_1) = 0$, which represents the demand locus. The following lemma is the analogous of Lemma 1 and characterises the shape of $F_2 = 0$ in the $\{l_1, K_1\}$ plane.

Lemma 4: Around the equilibrium, $\dfrac{\partial F_2}{\partial l_1} > 0, \dfrac{\partial F_2}{\partial K_1} > 0$ and, therefore $\left.\dfrac{dK_1}{dl_1}\right|_{F_2=0} < 0$.

Lemma 5: $F_2(0,0) < 0$ and $F_2(L,K) > 0$. $F_2(l_1, K_1) = 0$ tends to the corners (L,0) and (0,K), but without including these extreme points.

Therefore, the demand locus is decreasing and crosses exactly at one point with the supply locus (as illustrated in the following graph).



Lemma 6:

$$\frac{\partial F_2}{\partial \theta_1} = \frac{\frac{\partial y_1}{\partial \theta_1}}{y_2} - P_2^e \frac{\Omega_1}{\Omega_2} \left[(1-s)(P_2 y_2 + y_1)\right]^{(1-e)(\nu_1 - \nu_2)} \left[e \frac{\partial ln P_2}{\partial \theta_1} + (1-e)(\nu_1 - \nu_2) \frac{\partial ln(P_2 y_2 + y_1)}{\partial \theta_1} \right]$$

As we see with Lemma 6, the effect of an increase in automation in good 1 is not that clear as for the supply. The first term is clearly non-negative, but the second term will depend on the sign of the part in square brackets. The first term inside the square brackets corresponds to the price effect, while the second one to the income effect.

On the one hand, the price effect makes that as P_2 increases, the demand of good 1 increases (which implies increasing both l_1 and K_1). As it is known, this price effect is higher the more substitutes the two goods are.

On the other hand, the income effect is more diverse. We can distinguish two cases: if good 1 and 2 are substitutes or if they are complements. If they are substitutes (i.e. $e > 1$), then an increase in income will lead to an increase in relative consumption of the good with lower ν_i. In other words, if they are substitutes, the good with higher income elasticity is the one with lower ν_i. On the contrary, if they are complements, the good with higher income elasticity is the one with greater ν_i.

But we need to know how an increase in θ_1 affects both P_2 and income. We already know that $\frac{\partial y_1}{\partial \theta_1} \geq 0$, therefore it is just left to see $\frac{\partial lnP_2}{\partial \theta_1}$, which is the purpose of next lemma.

Lemma 7: i) A sufficient condition for $\dfrac{\partial lnP_2}{\partial \theta_1} > 0$ is: $\dfrac{l_1}{K_1}\dfrac{K_2}{l_2} > \left(\dfrac{w}{\gamma_L(\theta)}\dfrac{\gamma_K(\theta)}{R}\right)^{1-\sigma} \geq 1.$

ii) Also: $\dfrac{l_2}{K_2} < \dfrac{l_1}{K_1} < 2\dfrac{l_2}{K_2} \implies \dfrac{\partial lnP_2}{\partial \theta_1} > 0$

A complete study of the different possible forces in the model would require more time and, thus, I leave it to further research. However, and in order to gain intuition, I am going to study two particular cases: the Leontief and the perfect substitutes preferences.

5.2.1 Perfect Substitutes

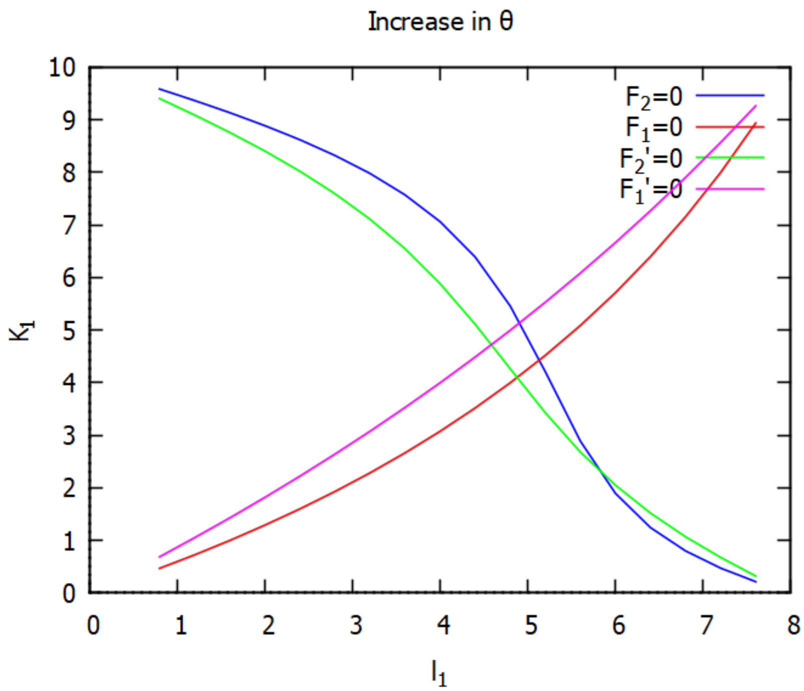
This case corresponds to setting $e = \infty$ and $\nu_1 = \nu_2$. In these preferences there is only consumption in the cheapest good (i.e. $y_2 = 0$ if $P_2 > 1$). Therefore, if $P_2 > 1$, the effect of an increase of θ_1 is equivalent to the one of the baseline model with a single final good; whereas if $P_2 < 1$, the effect is null.

5.2.2 Leontief preferences

These specific preferences correspond to the case where $e = 0$ and $\nu_1 = \nu_2$, so there is no price nor income effects. In such a case we can reformulate Lemma 6 as:

Lemma 6': $\dfrac{\partial F_2}{\partial \theta_1} = \dfrac{\frac{\partial y_1}{\partial \theta_1}}{y_2} \geq 0$.

Consequently, as θ_1 increases $F_2 = 0$ is curved down, as illustrated it in the following graph.



In the following I provide some intuition of the mechanism behind the graph. First, note that Lemma 3 tells us that w_1 falls more (or increases less) than w_2. This is because, since R_1 grows at a higher rate than w_1, then P_2 grows at a higher rate than w. For the same reason, R_1 increases more than R_2. Second, note that, at this first moment (without moving from the initial point), w_1 moves as w does in the single good model. Third, due to the perfect mobility of labour and capital, these imbalances of wages and rents of capital between the two goods will trigger a reallocation of factors, which will allow to increase both y_1 and y_2 maintaining the ratio marked by the Leontief preferences. In particular, labour will move from good 1 to good 2, pushing w_1 towards w_2; however, the direction of reallocation of capital is not that clear. On the one hand, without considering any productivity effect, as mentioned a few lines above, the shift in $F_1 = 0$ will tend to increase more R_1, tending to increase capital in good 1, which would also push w_1 towards w_2. On the other hand, if there is a productivity effect, the movement of $F_2 = 0$ will tend to decrease K_1 and l_1, which have opposite consequences for w_1. Note that the more steeper the $F_1 = 0$ is, the higher the descend in K_1 will be respect the descend in l_1 and, therefore, the more likely w_1 will decrease (and this could dominate over the initial increase due to exclusively the movement of $F_1 = 0$). Now, if we examine the explicit expressions of the slope of $F_1 = 0$ found in the Appendix in the Proof of Lemma 1 (see (33) or (34)), we can realise that it will be steeper the more capital intensive good 2 is. In order to get more intuition about the previous reasoning, suppose the extreme case where good 2 is fully automated and only uses capital. Note that graphically this would correspond to $F_1 = 0$ being a vertical line in $l_1 = L$ and $F_2 = 0$ would be just a point in that line. In this case, an increase of θ_1 will have the following effects. If the productivity effect is null, it will be as in the single good model, with no movement of factors. w_1 decreases while R_1 increases and, to preserve $R_1 = R_2$, we see that P_2 has to increase at the same rate (recall that a sufficient condition for $\frac{\partial P_2}{\partial \theta_1} > 0$ was that good 2 were more capital intensive, which is clearly satisfied). If the productivity effect is positive, since the demand won't accept a higher y_1 unless y_2 also increases, P_2 will increase more than R_1 to incentive capital to move from good 1 to good 2, so that the production of both goods is increased. Graphically, the only movement is from the point defined by $F_2 = 0$, which shifts down. Note that this movement of capital will depress even more w_1. The case where good 2 only uses labour is analogous. So, a conclusion that we can extract from the previous analysis is that the existence of a second good with Leontief preferences can smooth the bad consequences of automation on wages (or even make the overall effect positive) if good 2 is not too capital intensive, while if it is sufficiently capital intensive, it can intensify the bad consequences to labour.

6 Concluding Remarks

Automation poses one of the most important challenges of the future for our society and, especially, for the labour market. It is important that we get prepared before we are potentially hit and, for this reason, we need to increase our understanding of this phenomenon and its effects in order to be able to respond to it in the best possible way. The baseline model of this paper offers a study of the effects on labour that, to some extent, extends the previous literature, as it proves that, provided that the complementarity of tasks is high enough, an extension of automation will always damage labour regardless of the productivity of the new machines. Therefore, "so-so" machines may not be the only ones damaging labour.

I want to finish this thesis by mentioning how this research could be extended. The first clear line of research would be to complete the study of the Two Goods model. The mechanism behind this model is not trivial at all, and exploring the effects allowing income and price effects in the demand may bring some valuable insights. In this line, it would be interesting to see the different effects of automating a luxury from a necessity good, or (imperfect) complementary from (imperfect) substitute goods. A second interesting extension would be to relax the assumption of perfect labour mobility. In the reality, jobs are very diverse and so are their skill requirements. This can make the reallocation of labour more difficult and intensify the bad consequences of automation. A third extension could be to study the distributional implications of automation. As we have seen, automation is always good, in the sense that it allows to produce more, but it could be bad for social welfare if its benefits are retained by a minority.

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8 Appendix

8.1 Existence and Unicity of the Steady State

First, we can study a bit the shape of $\dot{K} = 0$. It contains the origin, since for $K = 0$, using (13), it is $y(L, 0) = 0$ (note that it is $K = 0$ to a negative exponent). Its slope is: $\dfrac{\partial c}{\partial K} = \dfrac{\partial y}{\partial K} + \dfrac{\partial y}{\partial L}\dfrac{\partial L}{\partial K} - \delta$. Note that when $K = 0$: $\dfrac{\partial y}{\partial K} \to \infty$, and since the adjustment of the labour supply is limited, also $\dfrac{\partial c}{\partial K} \to \infty$.

Next, I compute the marginal product of capital:

$$\frac{\partial y}{\partial K} = (1+\mu)^{-\frac{\sigma}{1-\sigma}} \left(\frac{y}{K} \int_0^\theta \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} = (1+\mu)^{-\frac{\sigma}{1-\sigma}} R \tag{31}$$

To ensure the existence and unicity of a steady state ($\dot{K} = 0$ and $\dot{c} = 0$) we need to see that $\frac{\partial y}{\partial K}$ is decreasing and that it decreases enough so that $R = \delta + \rho$ is satisfied for some K:

I first prove that R is decreasing in K:

$$\frac{\partial R}{\partial K} = \frac{1}{\sigma} \left(\frac{y}{K}\right)^{\frac{1-\sigma}{\sigma}} \left(\int_0^\theta \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} \left[\frac{\frac{\partial y}{\partial k}}{k} - \frac{y}{k^2} \right] < 0$$

It can be seen that the term in the square brackets is negative noting that:

$$\frac{\partial y}{\partial K} = \frac{y}{K} \frac{K^{\frac{\sigma-1}{\sigma}} \left(\int_0^{\theta} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}}}{K^{\frac{\sigma-1}{\sigma}} \left(\int_0^{\theta} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} + l^{\frac{\sigma-1}{\sigma}} \left(\int_{\theta}^{N} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}}} < \frac{y}{K} \tag{32}$$

Next, we see that it decreases enough. In particular, that $\lim_{K\to\infty} \frac{\partial y}{\partial K} = 0$

To see this, I use that, since $\sigma < 1$, when $K \to \infty$ then $K^{\frac{\sigma-1}{\sigma}} \to 0$ and so

$y \left(\int_{\theta}^{N} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma-1}} < \infty$. Using this in (31):

$$\lim_{K\to\infty} \frac{\partial y}{\partial K} = \lim_{K\to\infty} \left[\frac{(1+\mu)^{-\frac{\sigma}{1-\sigma}} l \left(\int_{\theta}^{N} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma-1}}}{K} \int_0^{\theta} \gamma_K(x)^{\sigma-1} dx \right]^{\frac{1}{\sigma}} = 0$$

8.2 Proof of Lemma 1:

$$\frac{\partial F_1}{\partial l_1} = \frac{y_1}{l_1} \int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx \left(\frac{\partial ln y_1}{\partial l_1} - \frac{1}{l_1} \right) - P_2^{\sigma} \frac{y_2}{l_2} \int_{\theta_2}^{N_2} \gamma_L(x)^{\sigma-1} dx \left(\frac{\partial ln y_2}{\partial l_1} + \frac{1}{l_2} + \sigma \frac{\partial ln P_2}{\partial l_1} \right)$$

Analogously to (32), it can be seen that $\frac{\partial y_i}{\partial l_i} - \frac{y_i}{l_i} < 0$. Now, I prove that $\frac{\partial P_2}{\partial l_1} = 0$:

$$\frac{\partial P_2}{\partial l_1} = P_2^{\sigma} \left[\int_0^{\theta_i} R^{1-\sigma} \gamma_K(x)^{\sigma-1} dx \frac{\partial ln R}{\partial l_1} + \int_{\theta_i}^{N_i} w^{1-\sigma} \gamma_L(x)^{\sigma-1} dx \frac{\partial ln w}{\partial l_1} \right]$$

Where, from logarithms of (11) and (12) and using that (by constant returns to scale in l and K) $y_1 = RK_1 + wl_1$:

$$\frac{\partial ln w}{\partial l_1} = \frac{1}{y_1 \sigma} \left(\frac{\partial y_1}{\partial l_1} - \frac{y_1}{l_1} \right) = \frac{w}{y_1 \sigma} \left(-\frac{RK_1}{wl_1} \right)$$

$$\frac{\partial ln R}{\partial l_1} = \frac{w}{y_1 \sigma}$$

So, in order to be 0, we need that: $\int_0^{\theta_2} R^{1-\sigma} \gamma_K(x)^{\sigma-1} dx = \int_{\theta_2}^{N_2} w^{1-\sigma} \gamma_L(x)^{\sigma-1} dx \frac{RK_1}{wl_1}$

Or, equivalently: $\left(\frac{w}{R} \right)^{\sigma} = \frac{K_1}{l_1} \frac{\int_{\theta_2}^{N_2} w^{1-\sigma} \gamma_L(x)^{\sigma-1} dx}{\int_0^{\theta_2} R^{1-\sigma} \gamma_K(x)^{\sigma-1} dx}$

Which is straightforward that it is satisfied by dividing (22) by (23).

So, using $\frac{\partial lny_2}{\partial l_1} = -\frac{\partial lny_2}{\partial l_2}$ and that $\frac{y_i}{l_i} > \frac{\partial y_i}{\partial l_i}$, we get that $\frac{\partial F_1}{\partial l_1} < 0$. $\frac{\partial F_1}{\partial K_1} > 0$ is proved similarly:

$$\frac{\partial F_1}{\partial K_1} = \frac{\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx}{l_1} \frac{\partial y_1}{\partial K_1} - P_2^{\sigma} \frac{y_2}{l_2} \int_{\theta_2}^{N_2} \gamma_L(x)^{\sigma-1} dx \left(\frac{\partial lny_2}{\partial K_1} + \sigma \frac{\partial lnP_2}{\partial K_1} \right)$$

Analogously as before, we can see that $\frac{\partial P_2}{\partial K_1} = 0$ and, using $\frac{\partial lny_2}{\partial K_1} = -\frac{\partial lny_2}{\partial K_2}$, we get that $\frac{\partial F_1}{\partial K_1} > 0$.

$\left. \frac{dK_1}{dl_1} \right|_{F_1=0} > 0$ is straightforward from applying the Implicit Function Theorem.

After some algebra, using the definitions (22) and (23), that $F_1 = 0$ implies both $w_i = w$ and $R_i = R$, and that $\frac{\partial lny_i}{\partial K_i} = \frac{R_i}{P_i y_i}$ and $P_i y_i = R_i K_i + w_i l_i$, we can simlify it to:

$$\left. \frac{dK_1}{dl_1} \right|_{F_1=0} = \frac{\frac{K_1}{y_1 l_1} + \frac{K_2}{P_2 y_2 l_2}}{\frac{1}{y_1} + \frac{1}{y_2}} > 0 \tag{33}$$

Or it can also be epressed as:

$$\left. \frac{dK_1}{dl_1} \right|_{F_1=0} = \left(\frac{w}{R} \right)^{\sigma} \frac{l_2 \int_0^{\theta_1} \gamma_K(x)^{\sigma-1} dx + P_2^{\sigma-1} l_1 \int_0^{\theta_2} \gamma_K(x)^{\sigma-1} dx}{l_2 \int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx + P_2^{\sigma-1} l_1 \int_{\theta_2}^{N_2} \gamma_L(x)^{\sigma-1} dx} > 0 \tag{34}$$

Note that if good 2 is very capital intensity, the slope will increase. In (33) it is easy to see since $\frac{K_2}{l_2}$ is in the numerator and in (34) note that in the extreme case of full automation, then $\theta_2 = N_2$ and $l_2 = 0$, so that the denominator becomes 0.

8.3 Proof of Lemma 2:

First, if $l_1 = 0$ and $K_1 = K$, then $y_1 = l_1 \left(\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma-1}} = 0$

Then $w_1 = \left(\frac{y_1}{l_1} \int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} = \left(\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma-1}}$ Plugging this into (21) with $P_1 = 1$, we get $R_1 = 0$. On the other hand, since $K_2 = 0$, then $y_2 = 0$. With the previous results, is easy to check that $F_1(0, K) > 0$.

Second, if $l_1 = L$, $K_1 = 0$, then $y_1 = K_1 \left(\int_0^{\theta_1} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma-1}} = 0$

Then $R_1 = \left(\frac{y_1}{K_1} \int_0^{\theta_1} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma}} = \left(\int_0^{\theta_1} \gamma_K(x)^{\sigma-1} dx \right)^{\frac{1}{\sigma-1}}$ Plugging this into (21) with $P_1 = 1$, we get $w_1 = 0$. On the other hand, since $l_2 = 0$, $y_2 = 0$. With the previous results, is easy to check that $F_1(0, K) < 0$.

8.4 Proof of Lemma 3:

$$\frac{\partial F_1}{\partial \theta_1} = \frac{y_1}{l_1} \int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx \left(\frac{\partial ln y_1}{\partial \theta_1} - \frac{\gamma_L(\theta_1)^{\sigma-1}}{\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx} \right) - P_2^{\sigma} \frac{y_2}{l_2} \int_{\theta_2}^{N_2} \gamma_L(x)^{\sigma-1} dx \cdot \sigma P_2^{\sigma-1} \frac{\partial P_2}{\partial \theta_1}$$

Using (22) and $\frac{\partial P_2}{\partial \theta_1}$ from (40):

$$\frac{\partial F_1}{\partial \theta_1} = \left(\frac{\partial ln y_1}{\partial \theta_1} - \frac{\gamma_L(\theta_1)^{\sigma-1}}{\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx} \right) \left[w^{\sigma} - P_2^{\sigma} \int_{\theta_2}^{N_2} \gamma_L(x)^{\sigma-1} dx \left(\frac{w}{P_2} \right) \right]$$
$$- \left(\frac{w}{P_2} \right)^{\sigma} P_2^{2\sigma-1} R^{1-\sigma} \int_0^{\theta_2} \gamma_K(x)^{\sigma-1} dx \left(\frac{\partial ln y_1}{\partial \theta_1} + \frac{\gamma_K(\theta_1)^{\sigma-1}}{\int_0^{\theta_1} \gamma_K(x)^{\sigma-1} dx} \right)$$

The term in the square brackets can be simplified using the definition of P_2 to $w^{\sigma} \frac{R^{1-\sigma} \int_0^{\theta_2} \gamma_K(x)^{\sigma-1} dx}{P_2^{1-\sigma}}$, so that we see that the terms containing $\frac{\partial ln y_1}{\partial \theta_1}$ cancel out, and we get:

$$\frac{\partial F_1}{\partial \theta_1} = w^{\sigma} \frac{R^{1-\sigma} \int_0^{\theta_1} \gamma_K(x)^{\sigma-1} dx}{P_2^{1-\sigma}} \left[-\frac{\gamma_L(\theta_1)^{\sigma-1}}{\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1} dx} - \frac{\gamma_K(\theta_1)^{\sigma-1}}{\int_0^{\theta_1} \gamma_K(x)^{\sigma-1} dx} \right] < 0 \qquad (35)$$

8.5 Derivation of the demand function:

From (28) and using the Implicit Function Theorem, we can get:

$$\frac{\partial U}{\partial c_i} = -\frac{\left(\Omega_i \frac{g(U)^{\nu_i}}{c_i} \right)^{\frac{1}{e}} \frac{1}{g(U)^{\nu_i}}}{\sum_{i=1}^{2} \Omega_i^{\frac{1}{e}} \left(\frac{c_i}{g(U)^{\nu_i}} \right)^{\frac{e-1}{e}} \frac{\nu_i}{g(U)}}$$ And call the denominator by ϕ.

From $\min_{c_i} \sum_{i=1}^{2} P_i c_i - \lambda(\bar{U} - U)$, we get:

$$c_i = \Omega_i g(U)^{\nu_i} \left(\frac{\lambda}{\phi P_i g(U)^{\nu_i}} \right)^e \tag{36}$$

$$1 = \sum_{i=1}^{2} \Omega_i \left(\frac{\lambda}{\phi P_i g(U)^{\nu_i}} \right)^{e-1} \tag{37}$$

Multiplying (36) by P_i and adding up, we get, using (37), that $E = \frac{\lambda}{\phi}$. And using this in (36), we get:

$$c_i = \Omega_i g(U)^{\nu_i(1-e)} \left(\frac{E}{P_i} \right)^e \tag{38}$$

8.6 Proof of Lemma 4:

$$\frac{\partial F_2}{\partial l_1} = \frac{\frac{\partial y_1}{\partial l_1}}{y_2} + \frac{y_1}{y_2^2}\frac{\partial y_2}{\partial l_2} - P_2^e \frac{\Omega_1}{\Omega_2} \left[(1-s)(P_2 y_2 + y_1)\right]^{(e-1)(\nu_1-\nu_2)} \frac{(e-1)(\nu_1-\nu_2)}{P_2 y_2 + y_1} \left(\frac{\partial y_1}{\partial l_1} - P_2 \frac{\partial y_2}{\partial l_2} \right)$$

Where I used that $\frac{\partial y_2}{\partial l_1} = -\frac{\partial y_2}{\partial l_2}$ and $\frac{\partial P_2}{\partial l_1} = 0$. Then, note that the third term is null since in equilibrium wages in both goods are equal (i.e. $w_1 = \frac{\partial y_1}{\partial l_1} = P_2 \frac{\partial y_2}{\partial l_2} = w_2$). So, around the equiibrium:

$$\frac{\partial F_2}{\partial l_1} = \frac{\frac{\partial y_1}{\partial l_1}}{y_2} + \frac{y_1}{y_2^2}\frac{\partial y_2}{\partial l_2} > 0$$

Similarly:

$$\frac{\partial F_2}{\partial K_1} = \frac{\frac{\partial y_1}{\partial K_1}}{y_2} + \frac{y_1}{y_2^2}\frac{\partial y_2}{\partial K_2} > 0$$

Therefore, applying again the Implicit Function Theorem, it is straightforward that $\left.\frac{dK_1}{dl_1}\right|_{F_2=0} < 0$.

8.7 Proof of Lemma 5:

I am going to examine the sign of $F_2(l_1, K_1)$ at the corners:

1. $l_1 = 0$, $K_1 = K$: Since $l_1 = 0$, then $y_1 = 0$. Analogously, since $K_2 = 0$, then $y_2 = 0$. So, $F_2(0, K)$ is not defined. However, it is easily seen that it cannot be satisfied neither for any point (0,z) with $z \in [0, K)$, since it would be $y_1 = 0$ while $y_2 > 0$, nor for any (z,K) with $z \in (0, L]$, since it would be $y_1 > 0$ while $y_2 = 0$. Therefore $F_2 = 0$ approaches the corner (0,K) but it is not defined at this point.

2. $l_1 = 0$, $K_1 = 0$: $F_2 < 0$ since $y_2 > 0$ while $y_1 = 0$.

3. $l_1 = L$, $K_1 = 0$: Analogously to case 1, we can guess that $F_2 = 0$ approaches the corner (L,0) but is not defined at it.

4. $l_1 = L$, $K_1 = K$: $F_2 > 0$ since $y_2 = 0$ while $y_1 > 0$.

8.8 Proof of Lemma 7:

Before going with the proof of Lemma 7, it may also be interesting to have some identity from the price normalization of P_1.

$$\frac{\partial P_1}{\partial \theta_1} = \frac{P_1^\sigma}{1-\sigma}\left[\left(\frac{R}{\gamma_K(\theta_1)}\right)^{1-\sigma} + \frac{1-\sigma}{\sigma}R^{1-\sigma}\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx\left(\frac{\partial lny_1}{\partial \theta_1} + \frac{\gamma_K(\theta_1)^{\sigma-1}}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx}\right)\right.$$
$$\left. - \left(\frac{w}{\gamma_L(\theta_1)}\right)^{1-\sigma} + \frac{1-\sigma}{\sigma}w^{1-\sigma}\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx\left(\frac{\partial lny_1}{\partial \theta_1} - \frac{\gamma_L(\theta_1)^{\sigma-1}}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}\right)\right]$$

Since P_1 is normalized to 1, $\frac{\partial P_1}{\partial \theta_1} = 0$. With this, grouping terms and using the definition of P_1 leads to:

$$0 = \left(\frac{R}{\gamma_K(\theta_1)}\right)^{1-\sigma} - \left(\frac{w}{\gamma_L(\theta_1)}\right)^{1-\sigma} + \frac{1-\sigma}{\sigma}\frac{\partial lny_1}{\partial \theta_1} + \frac{1-\sigma}{\sigma}\left[\left(\frac{R}{\gamma_K(\theta_1)}\right)^{1-\sigma} - \left(\frac{w}{\gamma_L(\theta_1)}\right)^{1-\sigma}\right]$$

So, we find that:

$$\frac{\partial lny_1}{\partial \theta_1} = \frac{1}{1-\sigma}\left[\left(\frac{R}{\gamma_K(\theta_1)}\right)^{1-\sigma} - \left(\frac{w}{\gamma_L(\theta_1)}\right)^{1-\sigma}\right] \tag{39}$$

Turning to the effect on P_2:

$$\frac{\partial P_2}{\partial \theta_1} = \frac{P_2^\sigma}{\sigma}\Bigg[R^{1-\sigma}\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dx\left(\frac{\partial lny_1}{\partial \theta_1} + \frac{\gamma_K(\theta_1)^{\sigma-1}}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx}\right) + w^{1-\sigma}\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx\left(\frac{\partial lny_1}{\partial \theta_1} - \frac{\gamma_L(\theta_1)^{\sigma-1}}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}\right)\Bigg] \tag{40}$$

From the previous equation it is straightforward that if the productivity effect is bigger than the displacement effect (i.e. if $\frac{\partial lny_1}{\partial \theta_1} > \frac{\gamma_L(\theta_1)^{\sigma-1}}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}$) then P_2 will increase.

First note that the previous equation can be written as:

$$\frac{\partial P_2}{\partial \theta_1} = \frac{P_2^\sigma}{\sigma}\left[\frac{\partial lny_1}{\partial \theta_1}P_2^{1-\sigma} + \frac{\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dx}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx}\left(\frac{R}{\gamma_K(\theta_1)}\right)^{1-\sigma} - \frac{\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}\left(\frac{w}{\gamma_L(\theta_1)}\right)^{1-\sigma}\right] \tag{41}$$

And using (15) together with (22) and (23), we can write

$$\frac{\partial lny_1}{\partial \theta_1} = \left(\frac{w}{\gamma_l(\theta_1)}\right)^{1-\sigma} - \left(\frac{R}{\gamma_K(\theta_1)}\right)^{1-\sigma}$$

And plugging this to (41) and due to the cost-efficiency condition of new machines (i.e. $\frac{w}{\gamma_l(\theta)} \geq \frac{R}{\gamma_K(\theta)}$), we get the following inequality:

$$\frac{\partial P_2}{\partial \theta_1} \geq \frac{P_2^\sigma}{\sigma}\left[\frac{\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dx}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx}\left(\frac{R}{\gamma_K(\theta_1)}\right)^{1-\sigma} - \frac{\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}\left(\frac{w}{\gamma_L(\theta_1)}\right)^{1-\sigma}\right]$$

Therefore, and using again (22) and (23), a sufficient condition for $\frac{\partial P_2}{\partial \theta_1} > 0$ is:

$$\frac{l_1}{K_1}\frac{K_2}{l_2} > \left(\frac{w}{\gamma_L(\theta)}\frac{\gamma_K(\theta)}{R}\right)^{1-\sigma} \geq 1 \tag{42}$$

This tells us that the effect on P_2 will be positive if good 1 is sufficiently more labour intensive than good 2. This is intuitive, since we have previously seen that an increase in θ increases $\frac{R}{w}$, and, therefore, it will tend to impact more on its price if the good requires more capital. In particular, this tells us that if the productivity effect is zero, then $\frac{\partial P_2}{\partial \theta_1} > 0$ (resp. < 0) if good 1 is more (resp. less) labour intensive than good 2.

Next, it may be interesting to see in which cases the productivity effect increases the effect of automation on P_2 (i.e. $\frac{\partial\left(\frac{\partial P_2}{\partial \theta_1}\right)}{\partial\left(\frac{\partial ln y_1}{\partial \theta_1}\right)} > 0$). Combining (36) with (38), we see that this is satisfied if: $P_2^{1-\sigma} > (1-\sigma)\left(\frac{\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dx}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx} - \frac{\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}\right)$

So, since $1 > 1-\sigma$, a sufficient condition is:

$$P_2^{1-\sigma} > \left(\frac{\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dx}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx} - \frac{\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}\right) = \frac{y_1}{y_2}\left(\frac{K_2}{K_1} - \frac{l_2}{l_1}\right)$$

Since $P_2 > 0$, this is clearly satisfied if the right-hand side is negative, which happens when good 2 is less capital intensive than good 1 ($\frac{l_1}{K_1} < \frac{l_2}{K_2}$). The intuition is: if good 2 requires less capital, then its price is less affected than the price of good 1 by the increase of the rental rate, but increases more than the one of good 1 as wages rise following the increase in the productivity effect. But this sufficient condition can be relaxed. Using the definition of P_2, the inequality becomes:

$$\begin{aligned} P_2^{1-\sigma} &= \int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dxR^{1-\sigma} + \int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dxw^{1-\sigma} \\ &> \frac{\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dx}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx}\frac{R^{1-\sigma}}{R^{1-\sigma}} - \frac{\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}\frac{w^{1-\sigma}}{w^{1-\sigma}} \end{aligned}$$

And grouping terms to one side:

$$\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dxR^{1-\sigma}\left(1 - \frac{R^{-(1-\sigma)}}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx}\right) + \int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dxw^{1-\sigma}\left(1 + \frac{w^{-(1-\sigma)}}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}\right) > 0$$

Using that $P_1^{1-\sigma} = 1 = R^{1-\sigma}\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx + w^{1-\sigma}\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx$, the term inside the first parenthesis becomes $-w^{1-\sigma}\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx$. So, we get:

$$w^{1-\sigma}\left[\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx - \int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx\frac{\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dx}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx}\right] + \frac{\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx} > 0$$

Multiplying by $\frac{w^{\sigma-1}}{\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx}$:

$$1 + \frac{w^{\sigma-1}}{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx} > \frac{\int_{\theta_1}^{N_1}\gamma_L(x)^{\sigma-1}dx}{\int_{\theta_2}^{N_2}\gamma_L(x)^{\sigma-1}dx}\frac{\int_0^{\theta_2}\gamma_K(x)^{\sigma-1}dx}{\int_0^{\theta_1}\gamma_K(x)^{\sigma-1}dx} = \frac{l_1}{K_1}\frac{K_2}{l_2}$$

Using first the definition of wage (22) and production function (24), we see that:

$$\frac{w^{\sigma-1}}{\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1}dx} = \frac{y_1^{\frac{\sigma-1}{\sigma}}}{w^{\frac{\sigma-1}{\sigma}} \left(\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1}dx\right)^{\frac{1}{\sigma}}} > 1$$

So, finally:

$$1 + \frac{w^{\sigma-1}}{\int_{\theta_1}^{N_1} \gamma_L(x)^{\sigma-1}dx} > 2 > \frac{l_1}{K_1}\frac{K_2}{l_2} \quad \rightarrow \quad 2\frac{l_2}{K_2} > \frac{l_1}{K_1} \tag{43}$$

We conclude, then, that a sufficient condition for $\frac{\partial\left(\frac{\partial P_2}{\partial \theta_1}\right)}{\partial\left(\frac{\partial lny_1}{\partial \theta_1}\right)} > 0$ is good 1 to be less than two times more labour intensive than good 2.

From the results above, we see that if good 1 is more labour intensive than good 2 but less than two times as good 2 (i.e. part ii of Lemma 7), then if the productivity effect is 0, $\frac{\partial P_2}{\partial \theta_1} > 0$ and if the productivity effect is positive, $\frac{\partial P_2}{\partial \theta_1}$ will be even bigger, which proves this part ii.