

# Anisotropic Gaussian random fields: Criteria for hitting probabilities and applications

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**Abstract:** We develop criteria for hitting probabilities of anisotropic Gaussian random fields with associated canonical pseudo-metric given by a class of gauge functions. This yields lower and upper bounds in terms of general notions of capacity and Hausdorff measure, respectively, therefore extending the classical estimates with the Bessel-Riesz capacity and the  $\gamma$ -dimensional Hausdorff measure. We apply the criteria to a system of linear stochastic partial differential equations driven by space-time noises that are fractional in time and either white or colored in space.

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## 1. Introduction

This paper is a contribution to the study of hitting probabilities for anisotropic Gaussian random fields. The motivation arises from applications of probabilistic potential theory to systems of linear stochastic partial differential equations (SPDEs) driven by a noise fractional in time and either white or colored in space.

Let  $X = \{X(x), x \in \mathbb{R}^d\}$  be a  $\mathbb{R}^D$ -valued Gaussian process with independent components. The canonical pseudo-distance corresponding to  $X$  is defined by  $\mathfrak{d}(x, y) = \|X(x) - X(y)\|_{L^2(\Omega)}$ . In this article, the process  $X$  is termed isotropic (respectively, anisotropic) if, up to non null multiplicative constants,  $\mathfrak{d}(x, y)$  is bounded below and above by an isotropic (respectively, anisotropic) function  $G$  of the variable  $x - y$ . We will write  $\mathfrak{d}(x, y) \asymp G(x - y)$ . The simplest example of  $G$  describing anisotropy is

$$G(x - y) = \sum_{j=1}^d |x_j - y_j|^{\alpha_j}, \quad x, y \in \mathbb{R}^d, \quad \alpha_j > 0, \quad (1.1)$$

where at least two of the  $\alpha_j$ 's are different. When  $\alpha_j = \alpha$  for all  $j$ ,  $G$  expresses isotropy. The fractional Brownian sheet and the random field solution to linear stochastic heat equations fall into this category of anisotropic processes, while the solution to linear wave equations is an example of isotropic process.

The study of hitting probabilities for  $X$  consists mainly in obtaining upper and lower bounds on the probabilities of random sets  $F_{I,A} := \{X^{-1}(A) \cap I \neq \emptyset\} = \{X(I) \cap A \neq \emptyset\}$ ,  $I \subset \mathbb{R}^d$ ,  $A \subset \mathbb{R}^D$ , in terms of the Hausdorff measure and/or the capacity of the set  $A$ . Such estimates provide the background to

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characterise polarity of sets for  $X$ , to deduce the Hausdorff dimension of  $F_{I,A}$  and, in general, to gain insight into analytic and geometric properties of the process.

For Gaussian processes with anisotropies described by (1.1), abstract results on hitting probabilities have been proved in [20, Theorem 7.6, p. 188] and [3, Theorem 2.1]. Extensions to non Gaussian processes are proved in [6, Theorems 2.1 and 2.4]. In these works, upper and lower bounds are given in terms of the usual  $\gamma$ -dimensional Hausdorff measure and the  $\beta$ -Bessel-Riesz capacity, respectively. There are several papers applying these results, or making use of similar approaches, to random fields that are solutions to SPDEs, either Gaussian or non Gaussian. The bounds are sharp when  $\gamma = \beta$ . We refer to [9] for a representative selection of references and for a survey on the state of the art.

Stochastic heat equations driven by fractional noises provide illustrations of anisotropic random fields with associated pseudo-metrics not fitting the above description. Among the many examples of such equations, we will focus here in the linear SPDE studied in [1] namely,

$$\frac{\partial v}{\partial t} = \Delta v + \dot{W}^{H,\alpha}, \quad (t, x) \in (0, T] \times \mathbb{R}^d; \quad v(0, x) = v_0(x), \quad (1.2)$$

where  $(\dot{W}^{H,\alpha})$  is a noise fractional in time (with Hurst parameter  $H \in (1/2, 1)$ ) and either white or colored in space (depending on the values of the parameter  $\alpha$ ). Details on the setting are given in Section 4 (see (4.3)).

When the initial condition  $v_0$  vanishes and the constraint  $4H - (d - \alpha) = 2$  holds, Tudor and Xiao prove that, for any  $t \in (0, T]$ ,  $\|v(t, x) - v(t, y)\|_{L^2(\Omega)} \asymp (\log(1/|x - y|))^{1/2} |x - y|$  ([20, Theorem 4]). This result suggests the use of more general notions of Hausdorff measures and capacities than the classical  $\gamma$ -dimensional Hausdorff measure and the  $\beta$ -Bessel-Riesz capacity respectively, to achieve sharp upper and lower bounds on the hitting probabilities for  $v$ .

Building on this fundamental idea, we develop our work into two steps. In the first one, we consider a class of increasing continuous real-valued functions  $q$  such that  $q(0) = 0$ , and Gaussian random fields  $X$  with canonical pseudo-metric satisfying  $\mathfrak{d}(x, y) \asymp q(|x - y|)$ . We term this type of processes *q-anisotropic*. In this setting, we establish abstract criteria for hitting probabilities. If  $q$  equals the function  $G$  in (1.1), we recover the results from [6], [3] and [20] mentioned above. The second step consists of an application of the new criteria to a system of random field solutions to (1.2).

After these introductory paragraphs, we describe with some detail the sections of the paper. In Section 2, we summarize the basic notions and notations used throughout the article. Section 3 is devoted to the discussion of abstract criteria on hitting probabilities for *q-anisotropic* Gaussian processes. First, we consider the case where anisotropy of the process is described by a single function  $q$  and then, we extend the analysis to the case where two (or more) different functions like  $q$  are needed in the description. In the first event, we denote the processes by  $M$  and call it *single q-anisotropic* while in the second, the process is denoted by  $U$  and is called *multiple q-anisotropic*. Although the first case could be deduced from the second one, for didactic reasons, we decided to take this path. However, in this introductory description, we will restrict to *single q-anisotropic* processes  $M$ .

The criteria for the upper bounds are proved using the strategy of [6, Section 2]. The main ingredient is Lemma 3.1, which takes the role of Lemma 2.5 in [6]. We remark that if the centred process  $M - E(M)$  satisfies  $\mathfrak{d}(x, y) \leq Cq(|x - y|)$  then the assumptions of Lemma 3.1 hold. Applying a re-scaling defined by means of  $q$ , with an approach close to the proof of [6, Theorem 2.6], we deduce upper bounds for hitting probabilities for small balls (see Theorem 3.1 and Lemma 3.2). In particular, Lemma 3.2 reveals that in our context, the  $g_q$ -Hausdorff measure  $\mathcal{H}_{g_q}(A)$  (see the definition in Section 2) with  $g_q(\tau) = \tau^D/(q^{-1}(\tau))$  is the suitable choice of geometric measure for upper bounds of the hitting probabilities. The classical

covering argument yields Theorem 3.2 (see also Theorem 3.3 for the *multiple  $q$ -anisotropic* case). We note by passing that our results hold for processes with continuous mean function, therefore removing the constraint of being centred in previous works.

Recall the definition of  $\mathfrak{g}$ -capacity given in Section 2 below. Assume that the process  $M$  satisfies  $\mathfrak{d}(x, y) \asymp q(|x - y|)$  ( $q$  is not necessarily the same function as in the preceding paragraph). In coherence with the classical anisotropic case, we expect the lower bounds on hitting probabilities to be given in terms of the  $(g_q)^{-1}$ -capacity. In fact, for  $\beta > 0$ , the  $\beta$ -Bessel-Riesz capacity is defined by the kernel  $\tau^{-\beta}$ . We prove that this is indeed the case if we restrict the class of functions  $q$  for which  $g_q(\tau) = \tau^D / (q^{-1}(\tau))$  satisfies a *rate growth* control at  $\tau = 0$ . More precisely, let  $\tau \mapsto v_q(\tau)$  be the radial integral of the function  $(q^D(|z|))^{-1}$  over the circular ring determined by the radii  $\tau$  and a constant  $c_0 > \tau$ . We require

$$[g_q(\tau)]^{-1} = O(v_q(\tau)), \quad \tau \downarrow 0 \tag{1.3}$$

(see (3.50)). When  $q(\tau) = \tau^\nu$  (the classical case) this imposes no restriction (see Section 5 for details). Then, adding to condition (1.3) the set of assumptions as in the classical anisotropic case (see Hypotheses  $(H_M)$  in Section 3.2) we establish in Theorem 3.4 criteria for lower bounds for hitting probabilities in terms of  $\text{Cap}_{(g_q)^{-1}}(A)$ . The proof combines the approach of [3, Theorem 2.1], [20, Theorem 7.6], based on weak approximations of measures, and the results of [6, Section 3].

In the case where the function  $q$  in the discussions on upper and lower bounds are equal we easily obtain that points are polar for  $M$  if and only if  $\lim_{\tau \downarrow 0} g_q(\tau) = 0$ .

We close Section 3 with a sample of generic and concrete examples where the above criteria apply and we recover known results on the linear stochastic heat, wave and Poisson equations.

In Section 4, we prove results on the random field solution to (1.2) that are required in the application of the abstract criteria of Section 3. These are on the covariance structure of the process and the identification of the associated anisotropic canonical pseudo-metric (see Lemmas 4.1 and 4.2, and Theorem 4.1, respectively). This leads eventually to Theorem 4.2 on sharp hitting probabilities for a system of SPDEs derived from (1.2). Finally, Section 5 gathers technical details on examples where the results can be applied.

Before this work was completed, we came across the arXiv document [14]. Both articles share the aim of establishing abstract criteria on anisotropic Gaussian processes beyond the classical case. Our setting is more general and there are many differences in the approaches. There is however similarity in the proof of the criterion for the lower bound. As was mentioned above, it relies on [3, Theorem 2.1], [20, Theorem 7.6].

There are several natural questions that are work in progress or in our research plans for the future. For example, the extension of the abstract criteria to non Gaussian  $q$ -anisotropic processes and the investigation of applications that could provide a strong motivation for this. For instance, SPDEs driven by multiplicative fractional type noises. Deepen in the understanding of polarity is also a challenging project. In particular, what means *critical dimension* in the setting of this article; and then, in connection with [8], how could polarity of points at critical dimension be characterized.

## 2. Preliminaries and notations

Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be monotone increasing and right-continuous. Assume that on a small non empty interval  $[0, \varepsilon_0]$ ,  $g$  is strictly increasing. The  $g$ -Hausdorff measure of a Borel set  $A \subset \mathbb{R}^D$  is defined by

$$\mathcal{H}_g(A) = \liminf_{\varepsilon \downarrow 0} \left\{ \sum_{i=1}^{\infty} g(2r_i) : A \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i), \sup_{i \geq 1} r_i \leq \varepsilon \right\}$$

(see e.g. [15]). In the particular case  $g(\tau) = \tau^\gamma$ , with  $\gamma > 0$ ,  $\mathcal{H}_g(A)$  is the  $\gamma$ -dimensional Hausdorff measure, usually denoted by  $\mathcal{H}_\gamma(A)$  (see e.g. [13]).

A function  $\mathfrak{g} : \mathbb{R}^D \rightarrow \mathbb{R}_+ \cup \{\infty\}$  is a *symmetric potential kernel* if: (a)  $\mathfrak{g}$  is symmetric; (b)  $\mathfrak{g}(z) > 0$ , for all  $z \neq 0$ ; (c)  $\mathfrak{g}(0) = \infty$ ; (d)  $\mathfrak{g}$  is continuous on  $\mathbb{R}^D \setminus \{0\}$ .

The energy of a measure  $\mu$  on  $\mathbb{R}^D$  relative to  $\mathfrak{g}$  is given by the expression

$$\mathcal{E}_{\mathfrak{g}}(\mu) = \int_{\mathbb{R}^D \times \mathbb{R}^D} \mathfrak{g}(y - \bar{y}) \mu(dy) \mu(d\bar{y}).$$

The  $\mathfrak{g}$ -capacity of a Borel set  $A \subset \mathbb{R}^D$  is defined by

$$\text{Cap}_{\mathfrak{g}}(A) = \left[ \inf_{\mu \in \mathbb{P}(A)} \mathcal{E}_{\mathfrak{g}}(\mu) \right]^{-1}, \tag{2.1}$$

where  $\mathbb{P}(A)$  denotes the set of probability measures on  $A$ . Since  $\mathfrak{g}$  is symmetric, this defines a Choquet capacity (see e.g. [12, Theorem 2.1.1, p. 533]).

When  $\mathfrak{g}$  is the Bessel-Riesz kernel of order  $\gamma \in \mathbb{R}$ , the  $\mathfrak{g}$ -capacity is the Bessel-Riesz capacity usually denoted by  $\text{Cap}_\gamma(A)$  (see e.g. [12, p. 376]).

Throughout this article, a *gauge function* means a strictly increasing continuous function  $q : [0, r) \subset \mathbb{R}_+ \mapsto \mathbb{R}_+$  satisfying  $q(0) = 0$ .

Whenever we consider the expression  $\log \frac{c}{\tau}$ ,  $\tau > 0$ , we assume that  $c$  is large enough to ensure  $\log \frac{c}{\tau} > 1$ .

Throughout the paper we will use the following notations. The Euclidean norm on  $\mathbb{R}^n$  is denoted by  $|\cdot|$ . For  $x \in \mathbb{R}^n$  and  $r \geq 0$ ,  $B_r(x)$  denotes the open Euclidean ball centred at  $x$  with radius  $r$ . Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , its Fourier transform is defined by the formula  $\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(x) dx$ , with  $x \cdot \xi$  denoting the scalar product. Let  $F$  be a set in a metric space  $(S, d)$ . For  $\rho > 0$ ,  $F^{(\rho)}$  denotes the set of points such that  $d(x, F) < \rho$ . Positive real constants are generically denoted by the letter  $C$ , or variants, like  $\tilde{C}$ ,  $\check{C}$ ,  $c$ , etc. If we want to make explicit the dependence on some parameters  $a_1, a_2, \dots$ , we write  $C(a_1, a_2, \dots)$  or  $C_{a_1, a_2, \dots}$ . The symbol  $\asymp$  between two mathematical expressions means equivalence up to multiplicative constants.

### 3. Criteria for hitting probabilities

We devote this section to investigate hitting probabilities of Gaussian random fields. The main results are Theorems 3.2, 3.3, 3.4 and 3.5, which yield upper and lower bounds in terms of the notions of  $g$ -Hausdorff measure and  $\mathfrak{g}$ -capacity, respectively.

#### 3.1. Upper bounds for hitting probabilities

The aim of this subsection is to prove extensions of Theorem 2.6 in [6] on sufficient conditions for upper bound estimates of hitting probabilities of Gaussian processes.

##### The single $q$ -anisotropic case

The results of the first part of the section concern a  $D$ -dimensional stochastic process denoted by

$$M = \{M(x) = (M_1(x), \dots, M_D(x)), x \in \mathbb{R}^d\}. \tag{3.1}$$

We start with a technical lemma which is a generalised version of [6, Lemma 2.5].

**Lemma 3.1.** *Assume that the process  $M$  has continuous sample paths a.s. Let  $q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a differentiable gauge function. Suppose that for all  $\varepsilon \in (0, 1)$  small enough and  $x \in \mathbb{R}^d$ ,*

$$E \left( \int_{B_\varepsilon(x)} dy \int_{B_\varepsilon(x)} d\bar{y} \exp \left( \frac{|M(y) - M(\bar{y})|}{q(|y - \bar{y}|)} \right) \right) \leq C\varepsilon^{2d}, \quad (3.2)$$

for some constant  $C$ . Set  $S_\varepsilon(x) = B_{\frac{q^{-1}(\varepsilon)}{2}}(x)$ . Then, the following statements hold.

1. For all  $p \geq 1$ , there exist constants  $C(p, d)$  and  $\tilde{C}(d)$  such that for  $\varepsilon$  small enough,

$$\begin{aligned} E \left( \sup_{y \in S_\varepsilon(x)} |M(y) - M(x)|^p \right) \\ \leq C(p, d) \varepsilon^{p-1} q^{-1}(\varepsilon) \int_0^1 \log^p \left( 1 + \frac{\tilde{C}(d)}{\tau^{2d}} \right) \dot{q}(q^{-1}(\varepsilon)\tau) d\tau, \end{aligned} \quad (3.3)$$

where  $\dot{q}$  denotes the derivative of  $q$ .

2. Assume that  $q$  is such that, for any  $r, \tau \in [0, c_0]$ , with  $c_0 > 0$  sufficiently small,

$$q(r\tau) \leq \varphi(\tau)q(r), \quad \dot{q}(r\tau) \leq \frac{1}{r}\psi(\tau)q(r\tau), \quad (3.4)$$

where  $\varphi$  and  $\psi$  are Borel functions such that, denoting  $\Phi(\tau) = \varphi(\tau)\psi(\tau)$ , we have

$$\int_0^1 \log^p \left( 1 + \frac{\tilde{C}(d)}{\tau^{2d}} \right) \Phi(\tau) d\tau < \infty. \quad (3.5)$$

Then, for all  $p \geq 1$ , there exists a constant  $C(p, d)$  such that for all  $\varepsilon$  small enough,

$$E \left( \sup_{y \in S_\varepsilon(x)} |M(y) - M(x)|^p \right) \leq C(p, d)\varepsilon^p. \quad (3.6)$$

*Proof.* 1. Let

$$\mathcal{C}_\varepsilon(\omega) = \int_{S_\varepsilon(x)} dy \int_{S_\varepsilon(x)} d\bar{y} \exp \left( \frac{|M(y, \omega) - M(\bar{y}, \omega)|}{q(|y - \bar{y}|)} \right). \quad (3.7)$$

From (3.2), we deduce  $\mathcal{C}_\varepsilon(\omega) < \infty$ , a.s. Notice that for almost all  $\omega$ ,  $\mathcal{C}_\varepsilon(\omega) \geq C_2 (q^{-1}(\varepsilon))^{2d}$ , for some constant  $C_2 > 0$ .

Applying [5, Proposition A.1, (A.3)] to  $S := S_\varepsilon(x)$  endowed with the Euclidean distance  $\rho$ ,  $\mu$  there the Lebesgue measure,  $\Psi(\tau) := e^\tau - 1$  and  $p(\tau) := q(\tau)$ , we deduce

$$\sup_{y \in S_\varepsilon(x)} |M(y) - M(x)| \leq 10 \int_0^{q^{-1}(\varepsilon)} \Psi^{-1} \left( \frac{C_1 \mathcal{C}_\varepsilon(\omega)}{\tau^{2d}} \right) \dot{q}(\tau) d\tau,$$

with  $C_1$  depending on  $d$ . Here, we have used that the volume of the  $d$ -dimensional Euclidean ball of radius  $r$  equals a multiple constant times  $r^d$ . Therefore, for any  $p \geq 1$ ,

$$\begin{aligned} E \left( \sup_{y \in S_\varepsilon(x)} |M(y) - M(x)|^p \right) &\leq 10^p E \left( \left| \int_0^{q^{-1}(\varepsilon)} \Psi^{-1} \left( \frac{C_1 C_\varepsilon(\omega)}{\tau^{2d}} \right) \dot{q}(\tau) d\tau \right|^p \right) \\ &\leq 10^p (q(q^{-1}(\varepsilon)))^{p-1} E \left( \int_0^{q^{-1}(\varepsilon)} \log^p \left( 1 + \frac{C_1 C_\varepsilon(\omega)}{\tau^{2d}} \right) \dot{q}(\tau) d\tau \right) \\ &= C(p) \varepsilon^{p-1} \int_0^{q^{-1}(\varepsilon)} E \left[ \log^p \left( 1 + \frac{C_1 C_\varepsilon(\omega)}{\tau^{2d}} \right) \right] \dot{q}(\tau) d\tau, \end{aligned} \quad (3.8)$$

where in the second inequality, we have applied Hölder's inequality with respect to the measure  $\dot{q}(\tau)d\tau$ . Observe that we may take  $C_1$  as large as we want.

The function  $x \mapsto \log^p(1+x)$  is concave on  $[e^{p-1}-1, \infty)$ . Hence, by taking  $C_1 \geq (e^{p-1}-1)C_2^{-1}$ , we can apply Jensen's inequality to estimate from above the term  $E \left[ \log^p \left( 1 + \frac{C_1 C_\varepsilon(\omega)}{\tau^{2d}} \right) \right]$  on the right-hand side of (3.8). By doing so, then using (3.7) and (3.2), and applying the change of variables  $\tau \mapsto (q^{-1}(\varepsilon))^{-1} \tau$ , we obtain,

$$\begin{aligned} E \left( \sup_{y \in S_\varepsilon(x)} |M(y) - M(x)|^p \right) &\leq C(p, d) \varepsilon^{p-1} q^{-1}(\varepsilon) \int_0^1 \log^p \left( 1 + \frac{\tilde{C}(d)}{\tau^{2d}} \right) \dot{q}(q^{-1}(\varepsilon)\tau) d\tau, \end{aligned} \quad (3.9)$$

with some constant  $\tilde{C}$  depending on  $d$ . This ends the proof of (3.3).

2. The conditions (3.4) imply  $r\dot{q}(r\tau) \leq \Phi(\tau)q(r)$ . For  $r := q^{-1}(\varepsilon)$  this yields

$$q^{-1}(\varepsilon)\dot{q}(q^{-1}(\varepsilon)\tau) \leq \Phi(\tau)\varepsilon.$$

Thus, up to the multiplicative constant  $C(p, d)$ , the right-hand side of (3.3) is equal to  $\varepsilon^p \int_0^1 \log^p \left( 1 + \frac{\tilde{C}(d)}{\tau^{2d}} \right) \Phi(\tau) d\tau$  and therefore, assuming (3.5), we obtain (3.6).  $\square$

**Examples 3.1.** We exhibit two examples of gauge functions  $q$  that satisfy the hypotheses of Lemma 3.1.

1.  $q(\tau) = \tau^\nu$ ,  $\tau > 0$ ,  $\nu > 0$ . The conditions (3.4) hold for any  $\tau, r > 0$ , with  $\varphi(\tau) = \tau^\nu$ ,  $\psi(\tau) = \frac{\nu}{\tau}$ . Since  $\int_0^1 \log^p \left( 1 + \frac{\tilde{C}(d)}{\tau^{2d}} \right) \tau^{\nu-1} d\tau < \infty$  for any  $p \geq 1$ , condition (3.5) holds.
2.  $q(\tau) = \tau^\gamma \left( \log \frac{c}{\tau} \right)^\delta$ ,  $\tau > 0$ , with  $\gamma > 0$ ,  $\delta \geq 0$ . Then,

$$\begin{aligned} q(r\tau) &= r^\gamma \tau^\gamma \left( \log \frac{c}{r\tau} \right)^\delta \leq r^\gamma \tau^\gamma \left( \log \frac{C}{r} + \log \frac{C}{\tau} \right)^\delta \\ &\leq C(\delta) r^\gamma \tau^\gamma \left( \log \frac{C}{r} \right)^\delta \left( 1 + \log \frac{C}{\tau} \right)^\delta, \end{aligned}$$

with  $C^2 \geq c$ . Hence,

$$q(r\tau) \leq \varphi(\tau)q(r), \quad \text{with } \varphi(\tau) = C(\delta)\tau^\gamma \left(1 + \log \frac{C}{\tau}\right)^\delta. \quad (3.10)$$

The derivative of  $q$  is  $\dot{q}(\tau) = \tau^{\gamma-1} \left(\log \frac{c}{\tau}\right)^{\delta-1} (\gamma \log \frac{c}{\tau} - \delta)$  and therefore, it is increasing on  $[0, ce^{-\frac{\delta}{\gamma}}]$ . In the sequel, we will restrict  $q$  to this interval, therefore  $\dot{q}(\tau) \leq \gamma\tau^{\gamma-1} \left(\log \frac{c}{\tau}\right)^\delta \leq \gamma\frac{1}{\tau}q(\tau)$ . Consequently,

$$\dot{q}(r\tau) \leq \frac{1}{r}\psi(\tau)q(r\tau), \quad \text{with } \psi(\tau) = \frac{\gamma}{\tau}. \quad (3.11)$$

Since  $\Phi(\tau) = C(\delta, \gamma)\tau^{\gamma-1} \left(1 + \log \frac{C}{\tau}\right)^\delta$ , we see that condition (3.5) holds.

**Remark 3.1.** Let  $q$  be a function as in Lemma 3.1. Assume that the process  $M$  in Lemma 3.1 is Gaussian, centred and such that, there exists a constant  $C$  and for any  $|y - \bar{y}| < 2\varepsilon$ ,

$$\|M(y) - M(\bar{y})\|_{L^2(\Omega)} \leq Cq(|y - \bar{y}|). \quad (3.12)$$

Then  $M$  satisfies the condition (3.2) for any  $x \in \mathbb{R}^d$ . Indeed, (3.12) implies

$$\exp\left(\frac{|M(y) - M(\bar{y})|}{q(|y - \bar{y}|)}\right) \leq \exp\left(\frac{1}{C} \frac{|M(y) - M(\bar{y})|}{\sqrt{\text{Var}(M(y) - M(\bar{y}))}}\right) = \exp(c|Z|),$$

where  $c = 1/C$  and  $Z$  is a standard Gaussian random variable. Since,  $E(\exp(c|Z|))$  is finite, (3.2) holds.

For any  $\varepsilon \in (0, 1)$ ,  $j \in \mathbb{Z}^d$ ,  $j = (j_1, \dots, j_d)$ , set

$$R_j^\varepsilon = \prod_{i=1}^d \left[ \frac{q^{-1}(\varepsilon)}{\sqrt{d}} j_i, \frac{q^{-1}(\varepsilon)}{\sqrt{d}} (j_i + 1) \right], \quad (3.13)$$

and for  $x \in R_j^\varepsilon$ , define  $x_j^\varepsilon := \left(\frac{q^{-1}(\varepsilon)}{\sqrt{d}} j_i\right)_{i=1, \dots, d}$ . Observe that  $\text{diam}(R_j^\varepsilon) = q^{-1}(\varepsilon)$  and  $R_j^\varepsilon \subset B_{\frac{q^{-1}(\varepsilon)}{2}}(\bar{x}_j^\varepsilon)$ , where  $\bar{x}_j^\varepsilon = \left(\frac{q^{-1}(\varepsilon)}{\sqrt{d}} (j_i + \frac{1}{2})\right)_{i=1, \dots, d}$ . Moreover, by the triangle inequality,

$$\sup_{x \in R_j^\varepsilon} (|M(x) - M(x_j^\varepsilon)|) \leq 2 \sup_{x \in B_{\frac{q^{-1}(\varepsilon)}{2}}(\bar{x}_j^\varepsilon)} (|M(x) - M(\bar{x}_j^\varepsilon)|). \quad (3.14)$$

The next statement provides an extension of [6, Theorem 2.6, (26)] to non necessarily centred processes.

**Theorem 3.1.** Fix  $\varepsilon \in (0, 1)$  small enough,  $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$ , and let  $R_j^\varepsilon$  be as in (3.13). Assume that the process  $M$  given in (3.1) is Gaussian, continuous, with i.i.d. components and such that  $\sigma^2 := \text{Var}(M_1(x_j^\varepsilon)) > 0$ .

1. Let  $f(x) = E(M(x))$  and  $\tilde{M}(x) = M(x) - f(x)$ . We assume that  $f : \mathbb{R}^d \rightarrow \mathbb{R}^D$  is a continuous map and furthermore, for some constant  $C(d, D)$ ,

$$E \left( \sup_{x \in R_j^\varepsilon} \left| \tilde{M}(x) - \tilde{M}(x_j^\varepsilon) \right|^2 \right) \leq C(d, D)\varepsilon^2. \quad (3.15)$$

Then there exists a constant  $C(\sigma, d, D)$  such that, for every  $z \in \mathbb{R}^D$ ,

$$P \left( M(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right) \leq C(\sigma, d, D)\varepsilon^D. \quad (3.16)$$

2. Suppose that for some constant  $\bar{C}(d, D)$

$$E \left( \sup_{x \in R_j^\varepsilon} \left| M(x) - M(x_j^\varepsilon) \right|^2 \right) \leq \bar{C}(d, D)\varepsilon^2. \quad (3.17)$$

Then there exists a constant  $\bar{C}(\sigma, d, D)$  such that, for every  $z \in \mathbb{R}^D$ ,

$$P \left( M(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right) \leq \bar{C}(\sigma, d, D)\varepsilon^D. \quad (3.18)$$

*Proof.* 1. We follow the approach of [6, Theorem 2.6] with some modifications due to the fact that the process  $M$  is not centred.

Because  $M$  is continuous, for any  $z \in \mathbb{R}^D$  we have

$$P \left( M(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right) = P \left( \inf_{x \in R_j^\varepsilon} |M(x) - z| \leq \varepsilon \right).$$

Assume we can prove that there exists a constant  $c(\sigma, d, D)$  such that for any  $z_1 \in \mathbb{R}$ ,

$$P \left( \inf_{x \in R_j^\varepsilon} |M_1(x) - z_1| \leq \varepsilon \right) \leq c(\sigma, d, D) \varepsilon, \quad (3.19)$$

where  $M_1(x)$  is the first component of the random vector  $M(x)$ . Then, because the components of  $M(x)$  are i.i.d, (3.19) yields (3.16) with  $C(\sigma, d, D) = [c(\sigma, d, D)]^D$ .

For the proof of (3.19), we fix  $x \in R_j^\varepsilon$  and compute the conditional expectation

$$E \left( M_1(x) | \tilde{M}_1(x_j^\varepsilon) \right) = f_1(x) + E \left( \tilde{M}_1(x) | \tilde{M}_1(x_j^\varepsilon) \right) = f_1(x) + c_j^\varepsilon(x) \tilde{M}_1(x_j^\varepsilon), \quad (3.20)$$

where

$$c_j^\varepsilon(x) = \frac{\text{Cov} \left( \tilde{M}_1(x), \tilde{M}_1(x_j^\varepsilon) \right)}{\text{Var} \left( \tilde{M}_1(x_j^\varepsilon) \right)}.$$

Define

$$Y_j^\varepsilon = \inf_{x \in R_j^\varepsilon} \left| E \left( M_1(x) | \tilde{M}_1(x_j^\varepsilon) \right) - z_1 \right|, \quad Z_j^\varepsilon = \sup_{x \in R_j^\varepsilon} \left| M_1(x) - E \left( M_1(x) | \tilde{M}_1(x_j^\varepsilon) \right) \right|.$$



These are independent random variables satisfying

$$P\left(\inf_{x \in R_j^\varepsilon} |M_1(x) - z_1| \leq \varepsilon\right) \leq P(Y_j^\varepsilon \leq \varepsilon + Z_j^\varepsilon). \quad (3.21)$$

We next prove that, for any  $r \geq 0$ ,

$$P(Y_j^\varepsilon \leq r) \leq C(\sigma, d, D)r. \quad (3.22)$$

As an auxiliary result for this, we first check that for all  $\varepsilon > 0$  small enough and  $x \in R_j^\varepsilon$ ,

$$|c_j^\varepsilon(x) - 1| \leq C(\sigma, d, D)\varepsilon, \quad (3.23)$$

implying that, for all  $\varepsilon > 0$  small enough, say  $\varepsilon \leq \varepsilon_0(\sigma, d, D)$ , and for all  $x \in R_j^\varepsilon$ , we have

$$c_j^\varepsilon(x) \geq \frac{1}{2}. \quad (3.24)$$

Indeed, because  $\text{Var}(\tilde{M}_1(x_j^\varepsilon)) = \text{Var}(M_1(x_j^\varepsilon)) = \sigma^2 > 0$ , using (3.15), similarly as in [6, (30), p.1356], we deduce

$$|c_j^\varepsilon(x) - 1| \leq \left( \frac{E \left[ \tilde{M}_1(x_j^\varepsilon) - \tilde{M}_1(x) \right]^2}{\text{Var}(\tilde{M}_1(x_j^\varepsilon))} \right)^{\frac{1}{2}} \leq C(\sigma, d, D)\varepsilon.$$

By (3.20),

$$\{Y_j^\varepsilon \leq r\} = \left\{ \inf_{x \in R_j^\varepsilon} |f_1(x) + E(\tilde{M}_1(x) | \tilde{M}_1(x_j^\varepsilon)) - z_1| \leq r \right\}$$

and the inequality  $|f_1(x) + E(\tilde{M}_1(x) | \tilde{M}_1(x_j^\varepsilon)) - z_1| \leq r$  is equivalent to

$$\frac{z_1 - f_1(x)}{c_j^\varepsilon(x)} - \frac{r}{c_j^\varepsilon(x)} \leq \tilde{M}_1(x_j^\varepsilon) \leq \frac{z_1 - f_1(x)}{c_j^\varepsilon(x)} + \frac{r}{c_j^\varepsilon(x)}.$$

Since by (3.24),  $\inf_{x \in R_j^\varepsilon} c_j^\varepsilon(x) \geq \frac{1}{2}$ , the above remarks yield

$$P(Y_j^\varepsilon \leq r) \leq \sup_{s \in \mathbb{R}} P\left(s - 2r \leq \tilde{M}_1(x_j^\varepsilon) \leq s + 2r\right) = \sup_{s \in \mathbb{R}} P\left(\tilde{M}_1(x_j^\varepsilon) \in B_{2r}(s)\right). \quad (3.25)$$

Because the density of  $\tilde{M}_1(x_j^\varepsilon)$  is bounded by  $(\text{Var}(M_1(x_j^\varepsilon))2\pi)^{-1/2} = \frac{1}{\sigma\sqrt{2\pi}}$ , we have

$$P(Y_j^\varepsilon \leq r) \leq \sup_{s \in \mathbb{R}} P\left(\tilde{M}_1(x_j^\varepsilon) \in B_{2r}(s)\right) \leq C(\sigma) r.$$

This proves (3.22).

We now address the last step in the proof of (3.19). From (3.21), and because  $Y_j^\varepsilon$  and  $Z_j^\varepsilon$  are independent, by using (3.22) we obtain,

$$\begin{aligned} P\left(\inf_{x \in R_j^\varepsilon} |M_1(x) - z| \leq \varepsilon\right) &\leq c(\sigma, d, D) E\left[(\varepsilon + Z_j^\varepsilon)\right] \\ &= c(\sigma, d, D) \left[\varepsilon + E\left((Z_j^\varepsilon)\right)\right]. \end{aligned} \quad (3.26)$$

Since  $M_1(x) - E\left(M_1(x)|\tilde{M}_1(x_j^\varepsilon)\right) = \tilde{M}_1(x) - c_j^\varepsilon(x)\tilde{M}_1(x_j^\varepsilon)$  (see (3.20)), by the triangle inequality we have  $Z_j^\varepsilon \leq Z_{j,1}^\varepsilon + Z_{j,2}^\varepsilon$ , with

$$Z_{j,1}^\varepsilon = \sup_{x \in R_j^\varepsilon} \left| \tilde{M}_1(x) - \tilde{M}_1(x_j^\varepsilon) \right|, \quad Z_{j,2}^\varepsilon = \sup_{x \in R_j^\varepsilon} \left| 1 - c_j^\varepsilon(x) \right| \left| \tilde{M}_1(x_j^\varepsilon) \right|.$$

Apply (3.15) to obtain  $E(Z_{j,1}^\varepsilon) \leq C(d, D)\varepsilon$ . Also, as a consequence of (3.15) and (3.23), we have  $E(|Z_j^\varepsilon|^2|D) \leq C(\sigma, d, D)\varepsilon$ . This yields  $E[(Z_j^\varepsilon)] \leq C(\sigma, d, D)\varepsilon$ . Along with (3.26), this implies (3.19) and, as was argued above, the proof of claim 1 is complete.

2. With minor changes in the previous proof, we can check that claim 2 holds. The details are left to the reader.  $\square$

**Remark 3.2.** *In the setting of Theorem 3.1, suppose in addition that the process  $M$  is continuous (which implies that  $\tilde{M}$  is continuous too). Assume that  $\tilde{M}$  satisfies the hypotheses of Lemma 3.1 with  $x := \bar{x}_j^\varepsilon$ . Then, applying (3.14) with  $M$  there replaced by  $\tilde{M}$ , we see that condition (3.15) holds for any  $p \geq 1$ . Similarly, if  $M$  satisfies the hypotheses of Lemma 3.1 with  $x := \bar{x}_j^\varepsilon$ . Then applying (3.14), we deduce that condition (3.17) holds for any  $p \geq 1$ .*

#### Hitting probabilities for small balls

Using a standard argument based on total probabilities, we can derive upper bounds for hitting probabilities of small balls in terms of the function  $q$ , as follows.

**Lemma 3.2.** *Let  $K \subset \mathbb{R}^d$  be a compact set of positive Lebesgue measure. Fix  $z \in \mathbb{R}^D$  and  $\varepsilon > 0$  (small enough). Let  $M$  be the process defined in (3.1) and assume that it is Gaussian and continuous, with i.i.d. components, and such that  $\sigma_K^2 := \inf_{x \in K^{(\eta)}} \text{Var}(M(x)) > 0$  (for some  $\eta$  sufficiently small). Let  $q$  be a function satisfying the conditions of Lemma 3.1, and assume that  $M$  satisfies (3.2) for any  $x \in K^{(\eta)}$ .*

*Then, there exists a constant  $C(K, \sigma_K, d, D)$  such that,*

$$P(M(K) \cap B_\varepsilon(z) \neq \emptyset) \leq C(K, \sigma_K, d, D) \frac{\varepsilon^D}{(q^{-1}(\varepsilon))^d}. \quad (3.27)$$

*Proof.* Since  $K$  is compact, there is a finite number of sets  $R_j^\varepsilon$  (defined in (3.13)) satisfying  $K \cap R_j^\varepsilon \neq \emptyset$ ; this number is a constant (depending on the dimension  $d$ ) multiple of  $\left(\frac{q^{-1}(\varepsilon)}{\sqrt{d}}\right)^{-d}$ . Moreover, by Lemma 3.1 and the inequality (3.14), we see that the condition (3.17) holds for any  $R_j^\varepsilon$  such that  $K \cap R_j^\varepsilon \neq \emptyset$  and this implies (3.18). Thus,

$$\begin{aligned} P(M(K) \cap B_\varepsilon(z) \neq \emptyset) &\leq \sum_{j \in \mathbb{Z}^d: K \cap R_j^\varepsilon \neq \emptyset} P(M(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset) \\ &\leq \tilde{C}(K, \sigma_K, d, D) \varepsilon^D \left(\frac{q^{-1}(\varepsilon)}{\sqrt{d}}\right)^{-d} = C(K, \sigma_K, d, D) \varepsilon^D (q^{-1}(\varepsilon))^{-d}. \end{aligned} \quad (3.28)$$

$\square$

For a gauge function  $q$ , define

$$g_q(\tau) = \frac{\tau^D}{(q^{-1}(\tau))^d}, \quad \tau \in \mathbb{R}_+. \quad (3.29)$$

From Lemma 3.2 we deduce conditions for points to be polar, as follows.

**Corollary 3.1.** *The hypotheses are as in Lemma 3.2. Assume further that*

$$\lim_{\tau \downarrow 0} g_q(\tau) = 0. \quad (3.30)$$

Then, for any  $z \in \mathbb{R}^D$ ,  $P(M(K) \cap \{z\} \neq \emptyset) = 0$ , that is  $\{z\}$  is polar for the process  $M$  restricted to  $K$ .

*Proof.* For any  $\varepsilon > 0$ , we have  $P(M(K) \cap \{z\} \neq \emptyset) \leq P(M(K) \cap B_\varepsilon(z) \neq \emptyset)$ . Applying (3.28) and using (3.30) yields the result.  $\square$

*Covering argument*

Assume that the hypotheses of Lemma 3.2 hold. Let  $g_q$  be the function defined in (3.29) and assume that on a sufficiently small interval  $(0, \rho_0)$ ,  $g_q$  is strictly increasing. Fix  $\varepsilon$  small enough. By the definition of the Hausdorff  $g_q$ -measure  $\mathcal{H}_{g_q}(A)$ , there exists a sequence of balls  $(B_i, i \geq 1)$  with radii  $r_i \in (0, \varepsilon)$ , such that  $B_i \cap A \neq \emptyset$ ,  $A \subset \cup_{i \geq 1} B_i$ , and  $\sum_{i \geq 1} g_q(2r_i) \leq \mathcal{H}_{g_q}(A) + \varepsilon$ . Then from (3.27), for any Borel set  $A \subset \mathbb{R}^D$  we deduce,

$$\begin{aligned} P(M(K) \cap A \neq \emptyset) &\leq \sum_{i \geq 1} P(M(K) \cap B_i \neq \emptyset) \\ &\leq C(K, \sigma_K, d, D) \sum_{i \geq 1} g_q(2r_i) \leq \mathcal{H}_{g_q}(A) + \varepsilon. \end{aligned}$$

Letting  $\varepsilon$  tend to zero, we obtain

$$P(M(K) \cap A \neq \emptyset) \leq C(K, \sigma_K, d, D) \mathcal{H}_{g_q}(A). \quad (3.31)$$

*Hitting probabilities in terms of  $g$ -Hausdorff measures*

We summarise the preceding discussion in the following statement.

**Theorem 3.2.** *Let  $K \subset \mathbb{R}^d$  be a compact set of positive Lebesgue measure. Consider a Gaussian continuous stochastic process  $M = \{M(x) = (M_1(x), \dots, M_D(x)), x \in \mathbb{R}^d\}$  with i.i.d. components and such that  $\sigma_K^2 := \inf_{x \in K^{(\eta)}} \text{Var}(M(x)) > 0$  (for some  $\eta > 0$  sufficiently small). Let  $q$  be a function satisfying the hypotheses of Lemma 3.1 and such that the function  $g_q$  given in (3.29) is strictly increasing on a small interval  $(0, \rho_0)$ . Assume also that the process  $M$  satisfies the condition (3.2) for any  $x \in K^{(\eta)}$ .*

*Then there exists a constant  $C(K, \sigma_K, d, D)$  such that for any Borel set  $A \subset \mathbb{R}^D$ ,*

$$P(M(K) \cap A \neq \emptyset) \leq C(K, \sigma_K, d, D) \mathcal{H}_{g_q}(A). \quad (3.32)$$

In Lemma 5.1 (sections 1. and 3.) examples of gauge functions  $q$  satisfying the assumptions of Theorem 3.2 and (3.30) are given.

### The multiple $q$ -anisotropic case

Through condition (3.2), the function  $q$  in Lemma 3.1 provides a control of the oscillations of the sample paths of the process  $M$ . However, as we will see in Section 4, there exist stochastic processes where two or a finite number of distinct functions  $q$  are needed for such control. In the second part of this section, we develop an extension of the previous results in a setting suitable for their application in Section 4.

Let

$$U = \{U(t, x) = (U_1(t, x), \dots, U_D(t, x)), (t, x) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\}, \quad (3.33)$$

be a stochastic process. In the examples in mind, the parameter  $t$  refers to time (therefore  $d_1 = 1$ ) while  $x$  refers to space.

**Lemma 3.3.** *Assume that the process  $U$  has continuous sample paths a.s. Let  $q_1, q_2$  be functions satisfying the properties of  $q$  in Lemma 3.1. Fix compact sets  $I \subset \mathbb{R}^{d_1}, J \subset \mathbb{R}^{d_2}$  of positive Lebesgue measure and assume that, for any  $\varepsilon$  small enough,*

$$\begin{aligned} E \left( \sup_{x \in J} \int_{B_\varepsilon(t)} ds \int_{B_\varepsilon(t)} d\bar{s} \exp \left( \frac{|U(s, x) - U(\bar{s}, x)|}{q_1(|s - \bar{s}|)} \right) \right) &\leq C\varepsilon^{2d_1}, \quad t \in I, \\ E \left( \sup_{t \in I} \int_{B_\varepsilon(x)} dy \int_{B_\varepsilon(x)} d\bar{y} \exp \left( \frac{|U(t, y) - U(t, \bar{y})|}{q_2(|y - \bar{y}|)} \right) \right) &\leq C\varepsilon^{2d_2}, \quad x \in J. \end{aligned} \quad (3.34)$$

Let  $S_\varepsilon^1(t) = B_{\frac{-1}{2}\varepsilon}(t)$ ,  $S_\varepsilon^2(x) = B_{\frac{-1}{2}\varepsilon}(x)$  and  $\tilde{S}_\varepsilon(t, x) = S_\varepsilon^1(t) \times S_\varepsilon^2(x)$ . Then, for all  $p \geq 1$ , there exists a constant  $C(p, d_1, d_2)$  such that, for all  $\varepsilon$  small enough and  $(t, x) \in I \times J$ ,

$$E \left( \sup_{(s, y) \in \tilde{S}_\varepsilon(t, x)} |U(s, y) - U(t, x)|^p \right) \leq C(p, d_1, d_2) \varepsilon^p. \quad (3.35)$$

*Proof.* We follow the steps of the proof of Lemma 3.1 considering first the processes  $M := U^{(x)} = \{U(t, x), t \in I\}$ , and then  $M := U^{(t)} = \{U(t, x), x \in J\}$ , obtained from  $U$  by fixing the indices  $x \in J$  and  $t \in I$ , respectively. The hypotheses (3.34) play the role of (3.2) in Lemma 3.1 for the proof of (3.6). In this way, we obtain,

$$\begin{aligned} E \left( \sup_{x \in J} \sup_{s \in S_\varepsilon^1(t)} |U(s, x) - U(t, x)|^p \right) &\leq C(p, d_1) \varepsilon^p, \\ E \left( \sup_{t \in I} \sup_{y \in S_\varepsilon^2(x)} |U(t, y) - U(t, x)|^p \right) &\leq C(p, d_2) \varepsilon^p, \end{aligned}$$

for some constants  $C(p, d_1), C(p, d_2)$ . Using the triangle inequality, we deduce (3.35).  $\square$

**Remark 3.3.** *Let  $q_1, q_2$  be functions as in Lemma 3.3. Assume that the process  $U$  in Lemma 3.3 is Gaussian, centred and such that for any  $|s - \bar{s}| < 2\varepsilon, |y - \bar{y}| < 2\varepsilon$ ,*

$$\begin{aligned} \sup_{x \in J} \|U(s, x) - U(\bar{s}, x)\|_{L^2(\Omega)} &\leq C_1 q_1(|s - \bar{s}|), \\ \sup_{t \in I} \|U(t, y) - U(t, \bar{y})\|_{L^2(\Omega)} &\leq C_2 q_2(|y - \bar{y}|), \end{aligned} \quad (3.36)$$

for some constants  $C_1, C_2$ . Then, arguing in a similar way as in Remark 3.1, we see that  $U$  satisfies (3.34).

For  $\varepsilon \in (0, 1)$ ,  $j = (j_1, \dots, j_{d_1}, j_{d_1+1}, \dots, j_{d_1+d_2}) \in \mathbb{Z}^{d_1+d_2}$ , define

$$\begin{aligned} R_j^{\varepsilon,1} &= \prod_{i=1}^{d_1} \left[ \frac{q_1^{-1}(\varepsilon)}{\sqrt{d_1}} j_i, \frac{q_1^{-1}(\varepsilon)}{\sqrt{d_1}} (j_i + 1) \right], \quad R_j^{\varepsilon,2} = \prod_{i=d_1+1}^{d_1+d_2} \left[ \frac{q_2^{-1}(\varepsilon)}{\sqrt{d_2}} j_i, \frac{q_2^{-1}(\varepsilon)}{\sqrt{d_2}} (j_i + 1) \right], \\ \tilde{R}_j^\varepsilon &= R_j^{\varepsilon,1} \times R_j^{\varepsilon,2}. \end{aligned} \quad (3.37)$$

For  $t \in R_j^{\varepsilon,1}$  let  $t_j^\varepsilon = \left( \frac{q_1^{-1}(\varepsilon)}{\sqrt{d_1}} j_i \right)_{i=1, \dots, d_1}$ , and for  $x \in R_j^{\varepsilon,2}$ , let  $x_j^\varepsilon = \left( \frac{q_2^{-1}(\varepsilon)}{\sqrt{d_2}} j_i \right)_{i=d_1+1, \dots, d_1+d_2}$ .

Given two gauge functions  $q_1, q_2$ , and denoting  $q = (q_1, q_2)$ , we define

$$\bar{g}_q(\tau) = \frac{\tau^D}{(q_1^{-1}(\tau))^{d_1} (q_2^{-1}(\tau))^{d_2}}, \quad \tau \in \mathbb{R}_+. \quad (3.38)$$

**Theorem 3.3.** *Let  $I$  and  $J$  be compact subsets of  $\mathbb{R}^{d_1}$  and  $\mathbb{R}^{d_2}$ , respectively, of positive Lebesgue measure. Assume that the stochastic process  $U$  defined in (3.33) is Gaussian, continuous, with i.i.d. components and such that  $\sigma_{I,J}^2 := \inf_{(t,x) \in I^{(\eta)} \times J^{(\eta)}} \text{Var } U(t,x) > 0$  (for  $\eta > 0$  small enough). Let  $q_1$  and  $q_2$  possess the same properties as the function  $q$  in Lemma 3.1 and the function  $\bar{g}_q$  given in (3.38) be strictly increasing on a small interval  $(0, \rho_0)$ . Assume also that the process  $U$  satisfies (3.34) on  $I^{(\eta)}$  and  $J^{(\eta)}$ , respectively.*

*Then there exists a constant  $C(I, J, \sigma_{I,J}, d_1, d_2, D)$  such that for any Borel set  $A \subset \mathbb{R}^D$ ,*

$$P(U(I \times J) \cap A \neq \emptyset) \leq C(I \times J, \sigma_{I,J}, D, d_1, d_2) \mathcal{H}_{\bar{g}_q}(A). \quad (3.39)$$

*Proof.* Let  $z \in \mathbb{R}^D$  and  $\varepsilon > 0$  (small enough). Since  $U$  satisfies the conditions of Lemma 3.3, and  $\tilde{R}_j^\varepsilon \subset \tilde{S}_\varepsilon(t, x)$ , for all  $p \geq 1$ , there exists a constant  $C(p, d_1, d_2)$  such that for all  $\varepsilon > 0$  small enough,

$$E \left( \sup_{(t,x) \in \tilde{R}_j^\varepsilon} |U(t,x) - U(t_j^\varepsilon, x_j^\varepsilon)|^p \right) \leq C(p, d_1, d_2) \varepsilon^p. \quad (3.40)$$

Applying Theorem 3.1 with  $M$  there replaced by  $U$ , we deduce

$$P \left( U(\tilde{R}_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right) \leq C(\sigma_{I,J}, d_1, d_2, D) \varepsilon^D,$$

for some constant  $C(\sigma_{I,J}, d_1, d_2, D)$ . Similarly as in the proof of Lemma 3.2, by an argument based on total probabilities (see (3.28)), we deduce

$$P(U(I \times J) \cap B_\varepsilon(z) \neq \emptyset) \leq C(I, J, \sigma_{I,J}, d_1, d_2, D) \frac{\varepsilon^D}{(q_1^{-1}(\varepsilon))^{d_1} (q_2^{-1}(\varepsilon))^{d_2}}. \quad (3.41)$$

We finish the proof by applying the covering argument laid out before the proof of Theorem 3.2.  $\square$

Similarly as we did in Corollary 3.1 and with the same arguments, from (3.41) we derive the following result on polarity of singletons.

**Corollary 3.2.** *The hypotheses are those of Theorem 3.3. In addition assume that*

$$\lim_{\tau \downarrow 0} \bar{g}_q(\tau) = 0. \quad (3.42)$$

Then  $P(U(I \times J) \cap \{z\} \neq \emptyset) = 0$ , that is, for the random field  $U$  restricted to  $I \times J$ , any set  $\{z\} \subset \mathbb{R}^D$  is polar.

In Lemma 5.1 (sections 1. and 2.) examples of gauge functions satisfying the conditions of Theorem 3.2 and (3.42) are given.

**Remark 3.4.** *The above discussion can be easily extended to the case where instead of  $q_1, q_2$ , we consider gauge functions  $q_1, \dots, q_{d_1+d_2}$  (repetitions are allowed). In this frame, setting  $q = (q_1, \dots, q_{d_1+d_2})$ , and defining*

$$\bar{g}_q(\tau) = \frac{\tau^D}{\prod_{j=1}^{d_1+d_2} q_j^{-1}(\tau)}, \quad (3.43)$$

with suitable adaptation of conditions, we obtain (3.39) with  $\bar{g}_q$  given in (3.43). This is [20, Theorem 7.6, upper bound of (167), p. 188].

### 3.2. Lower bounds for hitting probabilities

The aim of this section is to establish lower bounds on hitting probabilities of Gaussian processes in terms of  $\mathfrak{g}$ -capacities.

#### The single $q$ -anisotropic case

Let  $M$  be a  $D$ -dimensional Gaussian stochastic process, as given in (3.1), with i.i.d. components; let  $K \subset \mathbb{R}^d$  be a compact set of positive Lebesgue measure. We will use the notation  $\sigma_x^2 := \text{Var}(M_1(x))$ ,  $\sigma_{x,\bar{x}}^2 := \text{Cov}(M_1(x), M_1(\bar{x}))$ ,  $\rho_{x,\bar{x}} = \text{Corr}(M_1(x), M_1(\bar{x}))$ ,  $f(x) = E(M(x))$ ,  $\tilde{M}(x) = M(x) - f(x)$ .

We introduce the following set of conditions.

*Hypotheses ( $H_M$ )*

1. There exist positive constants  $c_1, c_2$  such that for all  $x \in K$ ,

$$c_1 \leq \sigma_x^2 \leq c_2. \quad (3.44)$$

2.  $\rho_{x,\bar{x}} < 1$  for all  $x, \bar{x} \in K$ .
3. There exist  $\eta > 0$  and  $c_3 > 0$  such that for all  $x, \bar{x} \in K$ ,

$$|\sigma_x^2 - \sigma_{\bar{x}}^2| \leq c_3 \|M_1(x) - M_1(\bar{x})\|_{L^2(\Omega)}^{1+\eta}. \quad (3.45)$$

4. There exists a gauge function  $q$  such that for all  $x, \bar{x} \in K$ ,

$$\|\tilde{M}_1(x) - \tilde{M}_1(\bar{x})\|_{L^2(\Omega)} \asymp q(|x - \bar{x}|), \quad |f(x) - f(\bar{x})| \leq C q(|x - \bar{x}|). \quad (3.46)$$

**Remark 3.5.** Let  $\text{Var}(M_1(\bar{x})|M_1(x))$  denote the conditional covariance of  $M_1(\bar{x})$  given  $M_1(x)$ . The conditions 1 to 3 in  $(H_M)$  imply

$$\text{Var}(M_1(\bar{x})|M_1(x)) \asymp \|M_1(x) - M_1(\bar{x})\|_{L^2(\Omega)}^2. \quad (3.47)$$

Indeed, if  $M$  is centred, this is [6, Lemma 3.2, (1)] (with  $\tau_{\bar{x},x}^2 := \text{Var}(M_1(\bar{x})|M_1(x))$  there). Going through the proof we see that the property of being centred is not used. Along with (3.46), we deduce

$$\text{Var}(M_1(\bar{x})|M_1(x)) = \text{Var}(\tilde{M}_1(\bar{x})|\tilde{M}_1(x)) \asymp q^2(|x - \bar{x}|). \quad (3.48)$$

Associated with the gauge function  $q$  we define

$$v_q(\tau) = \int_{q^{-1}(\tau)}^{\text{diam}(K)} [q(\rho)]^{-D} \rho^{d-1} d\rho, \quad \tau \in \mathbb{R}_+. \quad (3.49)$$

This section is devoted to prove the following statement.

**Theorem 3.4.** Let  $K \subset \mathbb{R}^d$  be a compact set of positive Lebesgue measure. Fix  $N > 0$  and let  $A \subset B_N(0) \subset \mathbb{R}^D$  be a Borel set. Assume that conditions  $(H_M)$  hold. Furthermore, suppose that

$$\sup_{\tau \in [0, \text{diam}(K)]} v_q(\tau) g_q(\tau) \in (0, \infty), \quad (3.50)$$

where  $g_q$  is defined in (3.29). Then there exists a constant  $C := C(f, K, N, d, D) > 0$  such that

$$P\{M(K) \cap A \neq \emptyset\} \geq C \text{Cap}_{(g_q)^{-1}}(A). \quad (3.51)$$

*Proof.* We adapt the method used for example in [3, Theorem 2.1] inspired in [11, pp. 204-206].

For any  $x \in K$  and a probability measure  $\mu$  on  $A$ , define

$$\begin{aligned} \bar{\nu}_n(x, \omega) &= \int_A (2\pi n)^{D/2} \exp\left(-\frac{n|M(x) - y|^2}{2}\right) \mu(dy) \\ &= \int_A \mu(dy) \int_{\mathbb{R}^D} d\xi \exp\left(-\frac{|\xi|^2}{2n} + i\langle \xi, M(x) - y \rangle\right). \end{aligned} \quad (3.52)$$

Consider the sequence of random measures on  $K$ ,  $(\nu_n, n \geq 1)$ , with corresponding densities  $(\bar{\nu}_n(x, \omega), n \geq 1)$ . Set  $\nu_n(K)(\omega) = \int_K \bar{\nu}_n(x, \omega) dx$ . We aim to prove:

- (i) There exists  $C_1 > 0$  such that for any  $n \geq 1$ ,  $E(\nu_n(K)) \geq C_1$ .
- (ii) There exists  $C_2 > 0$  such that for any  $n \geq 1$ ,  $E[(\nu_n(K))^2] \leq C_2 \mathcal{E}_{(g_q)^{-1}}$ .

By Paley-Zygmund inequality, this will imply

$$P\{\nu_n(K) > 0\} \geq \frac{[E(\nu_n(K))]^2}{E[(\nu_n(K))^2]} \geq \frac{C_1}{C_2 \mathcal{E}_{(g_q)^{-1}}}.$$

Using an argument based on weak convergence of finite measures, we deduce (3.51).

*Proof of (i).* By Fubini's theorem,

$$\begin{aligned} E(\nu_n(K)) &= \int_K dx \int_A \mu(dy) \int_{\mathbb{R}^D} d\xi \exp\left(-\frac{|\xi|^2}{2n} - i\langle \xi, y \rangle\right) E(\exp(i\langle \xi, M(x) \rangle)) \\ &= \int_K dx \int_A \mu(dy) \left(\frac{2\pi}{1/n + \sigma_x^2}\right)^{D/2} \exp\left(-\frac{|y - f(x)|^2}{2[1/n + \sigma_x^2]}\right). \end{aligned}$$

The last equality is obtained computing first the characteristic function  $E(\exp(i\langle \xi, M(x) \rangle))$  and then, the Fourier inversion formula.

Let  $N_0 = N + \sup_{x \in K} |f(x)|$ . Applying (3.44), and since on the set  $A$ ,  $|y - f(x)| \leq N_0$ , the above computations yield

$$\begin{aligned} E(\nu_n(K)) &\geq \int_K dx \int_A \mu(dy) \left(\frac{2\pi}{1 + \sigma_x^2}\right)^{D/2} \exp\left(-\frac{N_0^2}{2\sigma_x^2}\right) \\ &\geq |K| \left(\frac{2\pi}{1 + c_2}\right)^{D/2} \exp\left(-\frac{N_0^2}{2c_1}\right) := C_1. \end{aligned}$$

This ends the proof of (i). Notice that  $C_1 := C_1(f, K, N, D)$ .

*Proof of (ii).* For any  $x, \bar{x} \in K$ ,  $y, \bar{y} \in A$ , set

$$\begin{aligned} I(x, \bar{x}, y, \bar{y}) &= \int_{\mathbb{R}^D \times \mathbb{R}^D} e^{-i\langle (\xi, \bar{\xi}), (y, \bar{y}) \rangle} \exp\left(-\frac{|(\xi, \bar{\xi})|^2}{2n}\right) \exp(i\langle (\xi, \bar{\xi}), (M(x), M(\bar{x})) \rangle) d\xi d\bar{\xi}. \end{aligned}$$

Using (3.52), the definition of  $\nu_n(K)$  and Fubini's theorem, we see that

$$E\left[(\nu_n(K))^2\right] = \int_{K \times K} dx d\bar{x} \int_{A \times A} \mu(dy) \mu(d\bar{y}) E(I(x, \bar{x}, y, \bar{y})). \quad (3.53)$$

With elementary computations based on the properties of the exponential function, we deduce  $I(x, \bar{x}, y, \bar{y}) = \prod_{j=1}^D I_j(x, \bar{x}, y, \bar{y})$ , with

$$\begin{aligned} I_j(x, \bar{x}, y, \bar{y}) &= \int_{\mathbb{R}^2} d\xi_j d\bar{\xi}_j e^{-i\langle (\xi_j, \bar{\xi}_j), (y_j, \bar{y}_j) \rangle} \exp\left(-\frac{|(\xi_j, \bar{\xi}_j)|^2}{2n}\right) \exp(i\langle (\xi_j, \bar{\xi}_j), (M_j(x), M_j(\bar{x})) \rangle) \end{aligned}$$

Since the factors in the product above are i.i.d random variables, from (3.53) we obtain

$$E\left[(\nu_n(K))^2\right] = \int_{K \times K} dx d\bar{x} \int_{A \times A} \mu(dy) \mu(d\bar{y}) \prod_{j=1}^D [E(I_j(x, \bar{x}, y, \bar{y}))]. \quad (3.54)$$

Let  $\Gamma_{x, \bar{x}}$  denote the covariance matrix of the 2-dimensional Gaussian random vector  $(M_j(x), M_j(\bar{x}))$  (which is the same as for  $(\tilde{M}_j(x), \tilde{M}_j(\bar{x}))$ ), and set  $\Gamma_{x, \bar{x}}^n = \frac{1}{n} \text{Id}_2 + \Gamma_{x, \bar{x}}$ . Computing  $E(\exp(i\langle (\xi_j, \bar{\xi}_j), (M_j(x), M_j(\bar{x})) \rangle))$



and then applying the Fourier inversion formula, we obtain

$$E(I_j(x, \bar{x}, y, \bar{y})) \quad (3.55)$$

$$\begin{aligned} &= \int_{\mathbb{R}^2} d\xi_j d\bar{\xi}_j e^{-i\langle (\xi_j, \bar{\xi}_j), (y_j - f_j(x), \bar{y}_j - f_j(\bar{x})) \rangle} \exp\left(-\frac{1}{2}(\xi_j, \bar{\xi}_j)\Gamma_{x, \bar{x}}^n(\xi_j, \bar{\xi}_j)^\top\right) \\ &= \frac{2\pi}{(\det \Gamma_{x, \bar{x}}^n)^{1/2}} \exp\left(-\frac{1}{2}(y_j - f_j(x), \bar{y}_j - f_j(\bar{x}))(\Gamma_{x, \bar{x}}^n)^{-1}(y_j - f_j(x), \bar{y}_j - f_j(\bar{x}))^\top\right). \end{aligned} \quad (3.56)$$

Explicit computations show

$$\begin{aligned} &(y_j - f_j(x), \bar{y}_j - f_j(\bar{x}))(\Gamma_{x, \bar{x}}^n)^{-1}(y_j - f_j(x), \bar{y}_j - f_j(\bar{x}))^\top \\ &\geq \frac{E\left[\left((y_j - f_j(x))\tilde{M}_j(x) - (\bar{y}_j - f_j(\bar{x}))\tilde{M}_j(\bar{x})\right)^2\right]}{\det \Gamma_{x, \bar{x}}^n}. \end{aligned}$$

Hence, applying Lemma 3.4 we deduce

$$E(I_j(x, \bar{x}, y, \bar{y})) \leq C \frac{1}{(\det \Gamma_{x, \bar{x}}^n)^{1/2}} \exp\left(-\frac{c|(y_j - \bar{y}_j) - (f_j(x) - f_j(\bar{x}))|^2}{2 \det \Gamma_{x, \bar{x}}^n}\right). \quad (3.57)$$

Using this estimate in (3.54), we obtain

$$\begin{aligned} E\left[(\nu_n(K))^2\right] &\leq C \int_{K \times K} dx d\bar{x} \int_{A \times A} \mu(dy) \mu(d\bar{y}) \frac{1}{(\det \Gamma_{x, \bar{x}}^n)^{D/2}} \\ &\quad \times \exp\left(-\frac{c|(y - \bar{y}) - (f(x) - f(\bar{x}))|^2}{2 \det \Gamma_{x, \bar{x}}^n}\right). \end{aligned} \quad (3.58)$$

Since  $\Gamma_{x, \bar{x}}$  is nonnegative definite,

$$\det \Gamma_{x, \bar{x}}^n \geq \det \Gamma_{x, \bar{x}} = \sigma_x^2 \text{Var}(M(\bar{x})|M(x)) \geq Cq^2(|x - \bar{x}|), \quad (3.59)$$

where the last inequality follows from (3.44) and (3.48). This estimate along with (3.46) implies

$$\sup_{x, \bar{x} \in K} \frac{|f(x) - f(\bar{x})|^2}{\det \Gamma_{x, \bar{x}}^n} \leq C < \infty. \quad (3.60)$$

Apply the inequality  $|(y - \bar{y}) - (f(x) - f(\bar{x}))|^2 \geq \frac{1}{2}|y - \bar{y}|^2 - |f(x) - f(\bar{x})|^2$  and (3.60) on the right-hand side of (3.58) to deduce,

$$E\left[(\nu_n(K))^2\right] \leq C \int_{K \times K} dx d\bar{x} \int_{A \times A} \mu(dy) \mu(d\bar{y}) \frac{1}{(\det \Gamma_{x, \bar{x}}^n)^{D/2}} \exp\left(-\frac{c|y - \bar{y}|^2}{2 \det \Gamma_{x, \bar{x}}^n}\right). \quad (3.61)$$

If  $\det \Gamma_{x, \bar{x}}^n \geq |y - \bar{y}|^2$ , the integrand is bounded from above by the factor  $(\det \Gamma_{x, \bar{x}}^n)^{-D/2}$ . If on the contrary,  $\det \Gamma_{x, \bar{x}}^n < |y - \bar{y}|^2$ , the integrand is bounded (up to a multiplicative constant) by  $|y - \bar{y}|^{-D}$ ,

because the function  $z \mapsto z^{D/2}e^{-cz}$  is bounded over  $\mathbb{R}_+$ . In this way,

$$\begin{aligned} E \left[ (\nu_n(K))^2 \right] &\leq C \int_{K \times K} dx \, d\bar{x} \int_{A \times A} \mu(dy) \, \mu(d\bar{y}) \frac{1}{\max \left( (\det \Gamma_{x, \bar{x}}^n)^{D/2}, |y - \bar{y}|^D \right)} \\ &\leq C \int_{K \times K} dx \, d\bar{x} \int_{A \times A} \mu(dy) \, \mu(d\bar{y}) \frac{1}{\max (q^D(|x - \bar{x}|), |y - \bar{y}|^D)}, \end{aligned} \quad (3.62)$$

where in the second inequality we have applied (3.59).

Our next goal is to prove

$$\int_{K \times K} \frac{dx \, d\bar{x}}{\max (q^D(|x - \bar{x}|), |y - \bar{y}|^D)} \leq C(K, d) [g_q(|y - \bar{y}|)]^{-1}. \quad (3.63)$$

Indeed,

$$\begin{aligned} &\int_{(K \times K) \cap \{q(|x - \bar{x}|) \leq |y - \bar{y}|\}} \frac{dx \, d\bar{x}}{\max (q^D(|x - \bar{x}|), |y - \bar{y}|^D)} \\ &\leq C(K, d) |y - \bar{y}|^{-D} \int_0^{q^{-1}(|y - \bar{y}|)} \rho^{d-1} d\rho = C(K, d) [g_q(|y - \bar{y}|)]^{-1}, \end{aligned}$$

and

$$\begin{aligned} &\int_{(K \times K) \cap \{q(|x - \bar{x}|) > |y - \bar{y}|\}} \frac{dx \, d\bar{x}}{\max (q^D(|x - \bar{x}|), |y - \bar{y}|^D)} \\ &\leq C(K, d) \int_{q^{-1}(|y - \bar{y}|)}^{\text{diam}(K)} [q(\rho)]^{-D} \rho^{d-1} d\rho = C(K, d) v_q(|y - \bar{y}|) \leq \tilde{C}(K, d) [g_q(|y - \bar{y}|)]^{-1}, \end{aligned}$$

where the last equality holds because of hypothesis (3.50).

Hence,

$$E \left[ (\nu_n(K))^2 \right] \leq C(K, d) \mathcal{E}_{(g_q)^{-1}}(\mu), \quad (3.64)$$

and the right-hand side does not depend of  $n$ .

The proof of the theorem is complete.  $\square$

Consider the gauge functions  $q$  introduced in Examples 3.1. In Lemma 5.2 we compute the corresponding functions  $v_q$  (defined in (3.49)) and find  $g_q$  satisfying condition (3.50).

We end this section with a technical result used in the proof of Theorem 3.4.

**Lemma 3.4.** *Assume  $(H_M)$ . Then for any  $a, b \in \mathbb{R}$  and  $x, \bar{x} \in K$ , there exists a constant  $c > 0$  such that*

$$E[(a\tilde{M}_1(x) - b\tilde{M}_1(\bar{x}))^2] \geq c(a - b)^2. \quad (3.65)$$

*Proof.* Property (3.65) is equivalent to say that the matrix

$$N_{x, \bar{x}} \begin{pmatrix} \sigma_x^2 - c & -(\sigma_{x, \bar{x}}^2 - c) \\ -(\sigma_{x, \bar{x}}^2 - c) & \sigma_{\bar{x}}^2 - c \end{pmatrix}$$

is nonnegative definite. Computing  $\det N_{x,\bar{x}}$ , we see that this holds if and only if  $(\det \Gamma_{x,\bar{x}}) \left( \|\tilde{M}_1(x) - \tilde{M}_1(\bar{x})\|_{L^2(\Omega)}^2 \right)^{-1} \geq c$ . Applying Remark 3.5, and using (3.44), we obtain

$$\frac{\det \Gamma_{x,\bar{x}}}{\|\tilde{M}_1(x) - \tilde{M}_1(\bar{x})\|_{L^2(\Omega)}^2} = \frac{\sigma_x^2 \text{Var}(\tilde{M}_1(\bar{x}) | \tilde{M}_1(x))}{\|\tilde{M}_1(x) - \tilde{M}_1(\bar{x})\|_{L^2(\Omega)}^2} \asymp 1.$$

□

### The multiple $q$ -anisotropic case

Let  $U = \{U(t, x) = (U_1(t, x), \dots, U_D(t, x)), (t, x) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\}$  be a  $D$ -dimensional Gaussian stochastic process with i.i.d. components, and  $I \subset \mathbb{R}^{d_1}$ ,  $J \subset \mathbb{R}^{d_2}$  be compact sets of positive Lebesgue measure. We will use the notation  $\sigma_{t,x}^2 := \text{Var}(U_1(t, x))$ ,  $\sigma_{(t,x),(s,y)}^2 := \text{Cov}(U_1(t, x), U_1(s, y))$ ,  $\rho_{(t,x),(s,y)} = \text{Corr}(U_1(t, x), U_1(s, y))$ ,  $f(t, x) = E(U(t, x))$  and  $\tilde{U}(t, x) = U(t, x) - f(t, x)$ .

By analogy with assumptions  $(H_M)$  in the discussion on single  $q$ -anisotropic processes, we introduce the following set of conditions.

*Hypotheses  $(H_U)$*

1. There exist positive constants  $c_1, c_2$  such that for all  $(t, x) \in I \times J$ ,

$$c_1 \leq \sigma_{t,x}^2 \leq c_2. \quad (3.66)$$

2.  $\rho_{(t,x),(s,y)} < 1$  for all  $(t, x), (s, y) \in I \times J$ .
3. There exist  $\eta > 0$  and  $c_3 > 0$  such that for all  $(t, x), (s, y) \in I \times J$ ,

$$|\sigma_{t,x}^2 - \sigma_{s,y}^2| \leq c_3 \|U_1(t, x) - U_1(s, y)\|_{L^2(\Omega)}^{1+\eta}. \quad (3.67)$$

4. There exist gauge functions  $q_1, q_2$  such that for all  $(t, x), (s, y) \in I \times J$ ,

$$\|\tilde{U}_1(t, x) - \tilde{U}_1(s, y)\|_{L^2(\Omega)} \asymp q_1(|t - s|) + q_2(|x - y|). \quad (3.68)$$

$$|f(t, x) - f(s, y)| \leq C (q_1(|t - s|) + q_2(|x - y|)). \quad (3.69)$$

**Remark 3.6.** Using similar arguments as in Remark 3.5, now applied to the process  $U$ , we deduce

$$\text{Var}(U_1(t, x) | U_1(s, y)) \asymp \|U_1(t, x) - U_1(s, y)\|_{L^2(\Omega)}^2 \asymp (q_1(|t - s|) + q_2(|x - y|))^2, \quad (3.70)$$

for any  $(t, x), (s, y) \in I \times J$ .

Set  $d_I = \text{diam}(I)$ ,  $d_J = \text{diam}(J)$ ,  $c_{I,J} = \max(q_1(d_I), q_2(d_J))$ . Assuming that  $q_1$  and  $q_2$  are differentiable, for  $\tau \in [0, c_{I,J}]$ , define

$$\bar{v}_q(\tau) = \int_{\tau}^{c_{I,J}} \rho^{-D+1} [q_1^{-1}(\rho)]^{d_1-1} [q_2^{-1}(\rho)]^{d_2-1} [\dot{q}_1(q_1^{-1}(\rho))]^{-1} [\dot{q}_2(q_2^{-1}(\rho))]^{-1} d\rho. \quad (3.71)$$

To highlight the analogy between  $\bar{v}_q$  and the function  $v_q$  defined in (3.49), we observe that if  $q$  in (3.49) is differentiable, with the change of variable  $\rho \mapsto q(\rho)$  we have

$$v_q(\tau) = \int_{\tau}^{q(\text{diam}(K))} \rho^{-D} [q^{-1}(\rho)]^{d-1} [\dot{q}(q^{-1}(\rho))]^{-1} d\rho.$$

Our purpose is to prove the following result.

**Theorem 3.5.** *Let  $I \subset \mathbb{R}^{d_1}$  and  $J \subset \mathbb{R}^{d_2}$  be compact sets of positive Lebesgue measure. Fix  $N > 0$  and let  $A \subset B_N(0) \subset \mathbb{R}^D$  be a Borel set. Assume that conditions  $(H_U)$  hold. Furthermore, suppose that on  $(0, c_{I,J})$ , the gauge functions  $q_i$ ,  $i = 1, 2$ , are differentiable with decreasing derivatives  $\dot{q}_i$ , and*

$$\sup_{\tau \in [0, c_{I,J}]} \bar{v}_q(\tau/2) \bar{g}_q(\tau) \in (0, \infty), \quad (3.72)$$

where  $\bar{g}_q$  is the function defined in (3.38).

Then there exists a constant  $C := C(f, I, J, N, d_1, d_2, D) > 0$  such that

$$P\{U(I \times J) \cap A \neq \emptyset\} \geq C \text{Cap}_{(\bar{g}_q)^{-1}}(A). \quad (3.73)$$

*Proof.* The approach to the proof is the same as that of Theorem 3.4. To avoid repetitions, we only provide details on the relevant differences.

For any  $(t, x) \in I \times J$  and a probability measure  $\mu$  on  $A$ , define

$$\begin{aligned} \bar{v}_n((t, x), \omega) &= \int_A (2\pi n)^{D/2} \exp\left(-\frac{n|U(t, x) - y|^2}{2}\right) \mu(dy) \\ &= \int_A \mu(dy) \int_{\mathbb{R}^D} d\xi \exp\left(-\frac{|\xi|^2}{2n} + i\langle \xi, U(t, x) - y \rangle\right), \quad n \geq 1, \end{aligned} \quad (3.74)$$

and let  $\nu_n(I \times J)(\omega) = \int_{I \times J} \bar{v}_n((t, x), \omega) dt dx$ .

Applying (3.66), similarly as for the proof of (i) in Theorem 3.4, we obtain

$$E(\nu_n(I \times J)) \geq \bar{C}_1, \quad (3.75)$$

with  $\bar{C}_1 = C_1(f, I, J, N, D)$ .

With similar computations as those used to derive (3.62), we have

$$\begin{aligned} &E\left[(\nu_n(I \times J))^2\right] \\ &\leq C \int_{(I \times J)^2} dt dx d\bar{t} d\bar{x} \int_{A \times A} \mu(dy) \mu(d\bar{y}) \frac{1}{\max([q_1(|t - \bar{t}|) + q_2(|x - \bar{x}|)]^D, |y - \bar{y}|^D)}. \end{aligned} \quad (3.76)$$

For  $h \geq 0$ , set

$$I := \int_{(I \times J)^2} dt dx d\bar{t} d\bar{x} [\max([q_1(|t - \bar{t}|) + q_2(|x - \bar{x}|)]^D, h^D)]^{-1}. \quad (3.77)$$

Apply the change of variables  $(t, \bar{t}) \mapsto (t, t - \bar{t})$ ,  $(x, \bar{x}) \mapsto (x, x - \bar{x})$ , to deduce

$$I \leq |I \times J| \int_{B_{d_1}(0)} dr \int_{B_{d_2}(0)} dz [\max([q_1(|r|) + q_2(|z|)]^D, h^D)]^{-1}, \quad (3.78)$$

where  $|I \times J|$  denotes the Lebesgue measure of  $I \times J$ .

Let  $I_1$  denote the integral in (3.78) over the set of points  $(r, z)$  satisfying  $q_1(|r|) + q_2(|z|) \leq h$ . Changing to polar coordinates, we see that

$$\begin{aligned} I_1 &= h^{-D} \int_{B_{d_I}(0)} dr \int_{B_{d_J}(0)} dz \mathbf{1}_{\{q_1(|r|) + q_2(|z|) \leq h\}} \\ &\leq h^{-D} \left( \int_{B_{d_I}(0)} dr \mathbf{1}_{\{q_1(|r|) \leq h\}} \right) \left( \int_{B_{d_J}(0)} dz \mathbf{1}_{\{q_2(|z|) \leq h\}} \right) \\ &\leq C(d_1, d_2) h^{-D} \left( \int_0^{q_1^{-1}(h)} \rho^{d_1-1} d\rho \right) \left( \int_0^{q_2^{-1}(h)} \rho^{d_2-1} d\rho \right) = C(d_1, d_2) [\bar{g}_q(h)]^{-1}. \end{aligned} \quad (3.79)$$

Next, we denote by  $I_2$  the integral in (3.78) over the set of points  $(r, z)$  such that  $q_1(|r|) + q_2(|z|) > h$ . Applying two changes of variables: first polar coordinates,  $r \mapsto (\rho_1, \theta_1)$ ,  $z \mapsto (\rho_2, \theta_2)$ , and then  $\rho_i \mapsto q_i(\rho_i)$ ,  $i = 1, 2$ , we obtain

$$\begin{aligned} I_2 &= C(d_1, d_2) \int_0^{d_I} d\rho_1 \int_0^{d_J} d\rho_2 \mathbf{1}_{\{q_1(\rho_1) + q_2(\rho_2) > h\}} [q_1(\rho_1) + q_2(\rho_2)]^{-D} \rho_1^{d_1-1} \rho_2^{d_2-1} \\ &= C(d_1, d_2) \int_0^{q_1(d_I)} d\tau_1 \int_0^{q_2(d_J)} d\tau_2 \mathbf{1}_{\{\tau_1 + \tau_2 > h\}} (\tau_1 + \tau_2)^{-D} \\ &\quad \times (q_1^{-1}(\tau_1))^{d_1-1} (q_2^{-1}(\tau_2))^{d_2-1} [\dot{q}_1(q_1^{-1}(\tau_1))]^{-1} [\dot{q}_2(q_2^{-1}(\tau_2))]^{-1} \\ &\leq C(d_1, d_2, D) \int_0^{q_1(d_I)} d\tau_1 \int_0^{q_2(d_J)} d\tau_2 \mathbf{1}_{\{|\tau_1, \tau_2| > h/2\}} [|\tau_1, \tau_2|]^{-D} \\ &\quad \times (q_1^{-1}(\tau_1))^{d_1-1} (q_2^{-1}(\tau_2))^{d_2-1} [\dot{q}_1(q_1^{-1}(\tau_1))]^{-1} [\dot{q}_2(q_2^{-1}(\tau_2))]^{-1}, \end{aligned}$$

where in the last inequality we have used  $|(\tau_1, \tau_2)| \leq \tau_1 + \tau_2 \leq 2|(\tau_1, \tau_2)|$  ( $|\cdot|$  is the Euclidean norm). Changing  $(\tau_1, \tau_2)$  into polar coordinates, because for  $i = 1, 2$ ,  $q_i$  are increasing and  $\dot{q}_i$  decreasing, we deduce

$$\begin{aligned} I_2 &\leq C(d_1, d_2, D) \int_{h/2}^{c_{I,J}} \rho^{-D+1} [q_1^{-1}(\rho)]^{d_1-1} [q_2^{-1}(\rho)]^{d_2-1} \\ &\quad \times [\dot{q}_1(q_1^{-1}(\rho))]^{-1} [\dot{q}_2(q_2^{-1}(\rho))]^{-1} d\rho \\ &= C(d_1, d_2, D) \bar{v}_q(h/2) \leq C(I, J) [\bar{g}_q(h)]^{-1}, \end{aligned} \quad (3.80)$$

where the last inequality follows from the assumption (3.72).

Thus, from (3.76) by applying (3.79) and (3.80) with  $h := |y - \bar{y}|$ , we obtain

$$E \left[ (\nu_n(I \times K))^2 \right] \leq C(I, J, d_1, d_2, D) \mathcal{E}_{(\bar{g}_q)^{-1}}(\mu). \quad (3.81)$$

We conclude in a similar way as in the proof of Theorem 3.4.  $\square$

In Lemma 5.2, we give two examples where Theorem 3.5 can be applied.

By the definition of capacity, we have (see e.g. [12, p.529])

$$\text{Cap}_{\mathfrak{g}}(\{z\}) > 0 \quad \text{if and only if} \quad \mathfrak{g}(0) < \infty. \quad (3.82)$$

Take  $A = \{z\}$ ,  $z \in \mathbb{R}^D$ , in Theorems 3.4 and 3.5. If  $\{z\}$  is polar for the process  $M$  restricted to the compact  $K$  (respectively, for the process  $U$  restricted to the compact  $I \times J$ ), then necessarily,  $\text{Cap}_{(g_q)^{-1}}(\{z\}) = 0$  (respectively,  $\text{Cap}_{(\bar{g}_q)^{-1}}(\{z\}) = 0$ ). According to (3.82) this is equivalent to  $g_q(0) = 0$  (respectively,  $\bar{g}_q(0) = 0$ ). Together with Corollaries 3.1 and 3.2 we obtain the following result on polarity of points.

**Proposition 3.1.** *A singleton  $\{z\}$  is polar for the process  $M$  restricted to the compact  $K$  (respectively, for the process  $U$  restricted to the compact  $I \times J$ ) if and only if  $\lim_{\tau \downarrow 0} g_q(\tau) = 0$  (respectively,  $\lim_{\tau \downarrow 0} \bar{g}_q(\tau) = 0$ ).*

### 3.3. Examples

Under the unifying umbrella provided by Theorems 3.2, 3.3, 3.4 and 3.5, we present in this section a selection of known results on hitting probabilities. We defer to Section 4 the new application to the multiple  $q$ -anisotropic process that has motivated this work.

**Example 3.1.** *Fix compact sets  $I \subset \mathbb{R}^{d_1}$ ,  $J \subset \mathbb{R}^{d_2}$  of positive Lebesgue measure, and  $\varepsilon \in (0, 1)$ . Assume that the process  $U$  defined in (3.33) is Gaussian with i.i.d. components. Suppose that there exist  $\nu_1, \nu_2 \in (0, 1)$  and for any  $(s, y), (t, x) \in (I \times J)^{(2\varepsilon)}$ ,*

$$\|U(t, x) - U(s, y)\|_{L^2(\Omega)} \asymp (|t - s|^{\nu_1} + |x - y|^{\nu_2}). \quad (3.83)$$

By Kolmogorov's continuity theorem,  $\{U(t, x)\}_{(t, x) \in I \times J}$  has continuous sample paths, a.s.

The condition (3.34) holds with  $q_i(\tau) = \tau^{\nu_i}$ ,  $i = 1, 2$  (see Remark 3.3). The function  $\bar{g}_q$  defined in (3.38) is

$$\bar{g}_q(\tau) = r^{D - \left(\frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}\right)},$$

and it is increasing if  $D > \left(\frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}\right)$  (see Lemma 5.1).

Assume that the process  $U$  satisfies  $\sigma_{I, J}^2 := \inf_{(t, x) \in (I \times J)^{(2\varepsilon)}} \text{Var}(U(t, x)) > 0$ ; then from Theorem 3.3 we deduce the following:

There exists a constant  $C := C(I \times J, \sigma_{I, J}, D, d_1, d_2)$  such that for any Borel set  $A \subset \mathbb{R}^D$ ,

$$P(U(I \times J) \cap A \neq \emptyset) \leq C \mathcal{H}_{D - \left(\frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}\right)}(A). \quad (3.84)$$

If  $D \leq \left(\frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}\right)$ , by definition of the Hausdorff measure,  $\mathcal{H}_{D - \left(\frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}\right)}(A) = \infty$ . Thus (3.84) is still valid, but non informative.

By Lemma 5.2 (1.) we deduce the validity of (3.72). Therefore, assuming that  $U$  satisfies  $(H_U)$ , we see that the hypotheses of Theorem 3.5 are satisfied. Thus, for any bounded Borel set  $A \subset B_N(0) \subset \mathbb{R}^D$  there exists  $c := c(I, J, d_1, d_2, D)$  such that

$$P\{U(I \times J) \cap A \neq \emptyset\} \geq c \text{Cap}_{D - \left(\frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}\right)}(A). \quad (3.85)$$

With (3.84) and (3.85), we recover a version of [20, Theorem 7.6] on hitting probabilities in the classical centred anisotropic case.

Remark 3.4 motivates an extension of (3.84). Indeed, let us replace the upper bound in (3.83) by

$$\|U(t, x) - U(s, y)\|_{L^2(\Omega)} \leq C \left( \sum_{j=1}^{d_1} |t_j - s_j|^{\delta_j} + \sum_{k=1}^{d_2} |x_k - y_k|^{\nu_k} \right). \quad (3.86)$$

Then, taking  $q_j(\tau) = \tau^{\delta_j}$ ,  $j = 1, \dots, d_1$ , and  $q_j(\tau) = \tau^{\nu_j}$ ,  $j = d_1 + 1, \dots, d_1 + d_2$ , we have

$$\bar{g}_q(\tau) = \frac{\tau^D}{\prod_{j=1}^{d_1} \tau^{1/\delta_j} \prod_{k=1}^{d_2} \tau^{1/\nu_k}}.$$

Consequently, letting  $Q := \left(\sum_{j=1}^{d_1} \frac{1}{\delta_j}\right) + \left(\sum_{k=1}^{d_2} \frac{1}{\nu_k}\right)$ , we obtain

$$P(U(I \times J) \cap A \neq \emptyset) \leq C(I \times J, \sigma_{I,J}, D, d_1, d_2) \mathcal{H}_{D-Q}(A). \quad (3.87)$$

The following examples provide illustrations of the preceding results.

1. Consider the random field solution,  $\{u(t, x), (t, x) \in [0, T] \times [0, L]\}$ , to the system of linear stochastic heat equations

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u_i(t, x) = \dot{W}_i(t, x), \quad (t, x) \in (0, T] \times [0, L], \quad i = 1, \dots, D,$$

with null initial value and vanishing Dirichlet boundary conditions. The processes  $(\dot{W}_i(t, x))_i$  are independent space-time white noises. Here,  $d_1 = d_2 = 1$ ,  $\nu_1 = 1/4$  and  $\nu_2 = 1/2$  (see e.g. [5]). Hence, (3.84) and (3.85) hold with  $\mathcal{H}_{D-6}(A)$  and  $\text{Cap}_{D-6}(A)$ , respectively. We recover [5, Theorems 2.1 and 3.1].

2. For any  $k \geq 1$ , let  $\{u(t, x), (t, x) \in (0, T] \times \mathbb{R}^k\}$  be the random field solution to the system of linear stochastic wave equations

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u_i(t, x) = \dot{W}_i(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^k, \quad i = 1, \dots, D,$$

with null initial conditions. Assume that  $(\dot{W}_i(t, x))_i$  are independent noises white in time, with a stationary spatial covariance given by a Riesz kernel of order  $\beta \in (0, k \wedge 2)$ . In this example,  $d = 1 + k$ . According to the results in [6], on any set  $I \times J = [t_0, T] \times [-M, M]^k$  ( $t_0, M > 0$ ), the hypotheses introduced above are satisfied with  $\nu_1 = \nu_2 = \frac{2-\beta}{2}$  in (3.83). Therefore,  $\mathcal{H}_{g_q}(A) = \mathcal{H}_{D-2(1+k)/(2-\beta)}(A)$  and  $\text{Cap}_{g_q}(A) = \text{Cap}_{D-2(1+k)/(2-\beta)}(A)$ . We therefore recover [6, Theorems 4.4 and 4.5].

**Example 3.2.** Fix a compact set  $K \subset \mathbb{R}^d$  of positive Lebesgue measure, and  $\varepsilon \in (0, 1)$ . Let the process  $M$  in (3.1) be Gaussian, centred, with i.i.d. components. Suppose there exists  $\nu \in (0, 1)$  such that, for any  $x, y \in K^{(2\varepsilon)}$ ,

$$\|M(x) - M(y)\|_{L^2(\Omega)} \asymp |x - y|^\nu. \quad (3.88)$$

By Kolmogorov's continuity theorem, the sample paths of  $\{M(x)\}_{x \in K}$  are continuous a.s. Appealing to Remark 3.1, we see that (3.2) holds with  $q(\tau) = \tau^\nu$  and hence,  $g_q(\tau) = \tau^{D-\frac{d}{\nu}}$ . Assume that  $\nu >$

$d/D$  (which, according to Lemma 5.1 (1.), ensures that  $g_q$  is increasing) and furthermore,  $\sigma_K^2 := \inf_{x \in K^{(\eta)}} \text{Var}(M(x)) > 0$  (for  $\eta > 0$  small enough). From Theorem 3.2, we deduce, for any Borel set  $A \subset \mathbb{R}^D$ ,

$$P(M(K) \cap A \neq \emptyset) \leq C \mathcal{H}_{D-\frac{d}{\nu}}(A), \quad (3.89)$$

with  $C := C(K, \sigma_K, d, D)$ . If  $\eta \leq d/D$ , the right-hand side in (3.89) is infinite (by definition). Thus (3.89) still holds.

In addition to the above assumptions, suppose that  $M$  satisfies  $(H_M)$  and observe that by Lemma 5.2 (1.), (3.50) holds. Fix a bounded Borel set  $A \subset B_N(0) \subset \mathbb{R}^D$ . Then applying Theorem 3.4 we obtain,

$$P(M(K) \cap A \neq \emptyset) \geq c \text{Cap}_{D-\frac{d}{\nu}}(A), \quad (3.90)$$

with  $c := c(K, N, d, D)$ .

Consider the example of a system of linear stochastic Poisson equations on an open set  $O \subset \mathbb{R}^d$  ( $d = 1, 2, 3$ ):

$$-\Delta u_i(x) = \sum_{j=1}^d \sigma_{i,j} \dot{W}^j(x), \quad x \in O, \quad i = 1, \dots, D, \quad u|_{\partial O} = 0, \quad (3.91)$$

where  $\dot{W} = (\dot{W}^j)_j$  is a  $d$ -dimensional white noise and  $(\sigma_{i,j})_{1 \leq i, j \leq d}$  is a non-singular deterministic matrix. Assume: (i)  $O = (0, b)$  if  $d = 1$ ; (ii)  $O = B_1(0)$  if  $d = 2, 3$ .

1. Case  $d = 1, 3$ . From [16, Lemmas 5.4 and 5.7] we see that the random field solution to (3.91),  $M(x) = (u_i(x))_i$ , satisfies (3.88) with  $\nu = 1$  and  $\nu = 1/2$  for  $d = 1$  and  $d = 3$ , respectively. This is condition 4 of hypotheses  $(H_M)$ . As for conditions 1–3 of  $(H_M)$ , they are established in [16] (see (25) on p. 1871, Lemma 5.1 and the proof of Theorem 5.1, respectively). Let  $D > 1$  if  $d = 1$  and  $D > 6$  if  $d = 3$ . Therefore from (3.89) and (3.90), we recover [16, Theorems 5.10 and 5.11] (with  $d := k$  there), respectively.
2. Case  $d = 2$ . Let  $r_0 > 0$  be such that  $\bar{B}_{r_0}(0)$  is strictly contained in  $O$ . Claim 1 in [16, Lemma 5.5] states that, there exists a constant  $C$  (depending on  $r_0$ ) and for all  $x, y \in B_{r_0}(0)$  with  $|x - y| \leq e^{-1}$ ,

$$\|u(x) - u(y)\|_{L^2(\Omega)} \leq C|x - y| |\log|x - y||. \quad (3.92)$$

Thus, applying Remark 3.1 (with  $M = u$ ), we deduce that (3.2) holds with  $q(\tau) = \tau \log(\frac{c}{\tau})$ . The function  $q$  is of the form considered in Lemma 5.1 (3.) with  $\nu = \delta = 1$ . Furthermore, if  $D > 2$ ,  $\tau \mapsto \tau \log(\frac{c}{\tau})$  is an increasing function on the interval  $(0, c \exp(-D/(D-2)))$ . Therefore, applying Theorem 3.2 we obtain

$$P(M(K) \cap A \neq \emptyset) \leq C(K, \sigma_K, d, D) \mathcal{H}_{g_q}(A), \quad (3.93)$$

for any compact set  $K \subset \bar{B}_{r_0}(0)$ , with  $g_q(\tau) = \tau \log(\frac{c}{\tau})$ . Comparing with [16, Theorem 5.10], we see that (3.93) provides a sharper estimate.

#### 4. A linear heat equation with fractional noise

We consider a non-negative definite distribution in  $\mathcal{S}'(\mathbb{R}^d)$  given by an absolutely continuous measure  $\Lambda(dx) = f(x)dx$ . Let  $\mu(d\xi) = (\mathcal{F}^{-1}f)(\xi)d\xi$ ; by the Bochner-Schwarz theorem, the measure  $\mu$  is non-negative, tempered and symmetric; it is called *spectral measure*. We assume that for any non-negative



measurable function  $h$ ,

$$\int_{\mathbb{R}^d} h(\xi)\mu(d\xi) \asymp \int_{\mathbb{R}^d} h(\xi)|\xi|^{-\alpha}d\xi, \quad \text{for some } \alpha \in [0, d). \quad (4.1)$$

In an abridged form, we will write this property as  $\mu(d\xi) \asymp |\xi|^{-\alpha}d\xi$ .

Fix  $\alpha \in [0, d)$ ,  $H \in (0, 1)$  and let  $\{W^{H,\alpha}(t, A), t \in [0, T], A \in \mathcal{B}(\mathbb{R}^d)\}$  be a centred Gaussian field with covariance

$$E(W^{H,\alpha}(t, A)W^{H,\alpha}(s, B)) = R_H(t, s) \int_A \int_B f(z-w)dzdw, \quad (4.2)$$

where  $R_H(t, s) := \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$  is the covariance of a fractional Brownian motion with Hurst index  $H$ . In this section, we restrict to the case  $H \in (1/2, 1)$ .

If  $\alpha > 0$ ,  $W^{H,\alpha}$  is called a *fractional-colored* noise because it is a fractional Brownian motion in time and has a non trivial spatial covariance. Consider the particular case  $f(x) = \delta_{\{0\}}(x)$ . Then,  $\mu(d\xi) = d\xi$  and (4.1) trivially holds with  $\alpha = 0$ . This corresponds to the *fractional-white* noise, whose covariance according to (4.2) is

$$E(W^{H,0}(t, A)W^{H,0}(s, B)) = R_H(t, s)|A \cap B|,$$

where  $|\cdot|$  denotes the Lebesgue measure.

The Riesz and the Bessel kernels are examples of functions  $f$  that satisfy the above assumptions (see e.g. [17, Ch. V]).

Consider the linear stochastic heat equation

$$\frac{\partial v}{\partial t} = \Delta v + \dot{W}^{H,\alpha}, \quad (t, x) \in (0, T] \times \mathbb{R}^d; \quad v(0, x) = v_0(x), \quad x \in \mathbb{R}^d. \quad (4.3)$$

The random field solution to this equation is the Gaussian stochastic process

$$v(t, x) = I_0(t, x) + u(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^d, \quad (4.4)$$

where

$$I_0(t, x) = \int_{\mathbb{R}^d} G(t, x-y)v_0(y)dy, \quad u(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)W^{H,\alpha}(ds, dy), \quad (4.5)$$

with  $G(t, x) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right) \mathbb{1}_{\{t \geq 0\}}$ .

Suppose that the function  $G(t, x-\cdot)v_0(\cdot)$  belongs to  $L^1(\mathbb{R}^d)$ , to ensure that  $x \mapsto I_0(t, x)$  is well defined for all  $t \in [0, T]$ . Furthermore, assume

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{2H}} < \infty. \quad (4.6)$$

Owing to [18, Theorem 2.5] (see also [1, Sec. 2]), this is a necessary and sufficient condition for  $(u(t, x))$  given in (4.5) to define a  $L^2(\Omega)$  random field, and in this case,  $\sup_{(t,x) \in [0, T] \times \mathbb{R}^d} E(u(t, x)^2) < \infty$ . Assuming (4.1), we can check that (4.6) holds if and only if  $0 < d - \alpha < 4H$ . In the remaining of the section, we will assume this constraint.

Throughout this section, we will make use the following expression for the variance of  $u(t, x)$ ,  $(t, x) \in (0, \infty) \times \mathbb{R}^d$ :

$$\begin{aligned} \sigma_{t,x}^2 &:= E(|u(t, x)|^2) = \alpha_H \int_0^t d\tau \int_0^t d\sigma |\tau - \sigma|^{2H-2} \\ &\quad \times \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw f(z-w) G(t-\tau, x-z) G(t-\sigma, x-w) \\ &= \frac{\alpha_H}{(2\pi)^d} \int_0^t d\tau \int_0^t d\sigma |\tau - \sigma|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(\tau+\sigma)|\xi|^2}. \end{aligned} \quad (4.7)$$

where  $\alpha_H = H(2H-1)$ . The first equality can be found in [18, Sec 2.5.1], while the second one follows from Parseval's identity, since  $\mathcal{F}(G(t, \cdot))(\xi) = e^{-t|\xi|^2}$ . From the second equality in (4.7), we see that  $u$  is stationary in  $x$  ( $\sigma_{t,x}^2$  does not depend on  $x$ ).

For its further use, we prove some properties relative to  $\sigma_{t,x}^2$ .

**Lemma 4.1.** *1. For any  $0 < t_0 < T$ , there exist  $0 < c < C < \infty$  such that, for any  $(t, x) \in [t_0, T] \times \mathbb{R}^d$ ,  $c \leq \sigma_{t,x}^2 \leq C$ .*

*2. For any  $(t, x) \in (0, \infty) \times \mathbb{R}^d$ , the mapping  $t \mapsto \sigma_{t,x}^2$  is differentiable.*

*Proof.* Use (4.1) in the last expression of the array (4.7) and then, the change of variable  $\xi \mapsto (\tau + \sigma)^{\frac{1}{2}} \xi$  along with (4.6). Applying the change of variables,  $\tau \mapsto \frac{\tau}{t}$ ,  $\sigma \mapsto \frac{\sigma}{t}$ , we see that  $\sigma_{t,x}^2$  is bounded from below (respectively, from above) by

$$c_{\alpha,d,H} \int_0^t d\tau \int_0^t d\sigma \frac{|\tau - \sigma|^{2H-2}}{(\tau + \sigma)^{(d-\alpha)/2}} = t^{2H-(d-\alpha)/2} c_{\alpha,d,H} \int_0^1 d\tau \int_0^1 d\sigma \frac{|\tau - \sigma|^{2H-2}}{(\tau + \sigma)^{(d-\alpha)/2}}. \quad (4.8)$$

Let  $C_{\alpha,d,H} = \int_0^1 d\tau \int_0^1 d\sigma \frac{|\tau - \sigma|^{2H-2}}{(\tau + \sigma)^{\frac{d-\alpha}{2}}}$  and observe that, since  $4H - (d - \alpha) > 0$ ,  $C_{\alpha,d,H} < \infty$ . From the above computations, we deduce that the lower inequality (respectively, the upper inequality) in the first claim holds with  $c \leq t_0^{2H - \frac{(d-\alpha)}{2}} C_{\alpha,d,H}$  (respectively, with  $C \geq T^{2H - \frac{(d-\alpha)}{2}} C_{\alpha,d,H}$ ).

Claim 2. follows from the expression (4.7).  $\square$

#### 4.1. Equivalence for the canonical metric

The canonical pseudo-distance associated with the process  $u$  is defined by

$$\mathfrak{d}((t, x), (s, y)) = \|u(t, x) - u(s, y)\|_{L^2(\Omega)}. \quad (4.9)$$

The goal is to prove Theorem 4.1, which gives an equivalent pseudo-distance for  $\mathfrak{d}$ .

We start by recalling some related results. According to [18, Theorems 2.2 and 2.6], there exist positive constants  $c_1, c_2$ , which depend on  $\alpha, d, H$ , and  $T$ , such that for all  $t, s \in [0, T]$  and  $x \in \mathbb{R}^d$ ,

$$\|u(t, x) - u(s, x)\|_{L^2(\Omega)}^2 \asymp |t - s|^{2H - \frac{d-\alpha}{2}}. \quad (4.10)$$

If  $\alpha \in (0, d)$ , according to [19, Theorem 4], for any fixed  $t_0 \in (0, T]$ , there exist positive constants  $c_3, c_4$  such that for any  $t \in [t_0, T]$ ,  $x, y \in [-M, M]^d$ ,

$$\|u(t, x) - u(t, y)\|_{L^2(\Omega)}^2 \asymp \left( \log \frac{1}{|x - y|} \right)^\beta |x - y|^{2 \wedge (4H - (d - \alpha))}. \quad (4.11)$$

where  $\beta = 1$ , if  $4H - (d - \alpha) \geq 2$ , and  $\beta = 0$ , otherwise.

Let  $W^\alpha$  be a centered Gaussian process with covariance

$$E(W^\alpha(t, A)W^\alpha(s, B)) = (t \wedge s) \int_A \int_B f(z - w) dz dw.$$

The stochastic integral in (4.5) can be written as an integral with respect to  $W^\alpha$  (see e.g. [18, (2.31)]):

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} G(t - r, x - z) W^{\alpha, H}(dr, dz) \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} d\tau G(t - \tau, x - z) (\tau - r)_+^{H - \frac{3}{2}} \right) W^\alpha(dr, dz). \end{aligned} \quad (4.12)$$

Using this property, we generalize the lower bound in (4.10), as follows.

**Proposition 4.1.** *There exists a positive constant  $c_1$  which depends on  $\alpha, d, H$ , and  $T$ , such that, for all  $t, s \in [0, T]$  and  $x, y \in \mathbb{R}^d$ ,*

$$\|u(t, x) - u(s, y)\|_{L^2(\Omega)}^2 \geq c_1 |t - s|^{2H - \frac{(d - \alpha)}{2}}. \quad (4.13)$$

*Proof.* Assume, without loss of generality, that  $0 \leq s < t \leq T$ . Then, from (4.12), the Itô isometry (see

[18, Sec. 2.3.1]), and Parseval's identity, we obtain,

$$\begin{aligned}
& \|u(t, x) - u(s, y)\|_{L^2(\Omega)}^2 \\
&= E\left(\left|\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} d\tau [G(t - \tau, x - z)1_{(\tau \leq t)} - G(s - \tau, y - z)1_{(\tau \leq s)}] (\tau - r)_+^{H - \frac{3}{2}}\right) \right.\right. \\
&\quad \left.\left. \times W^\alpha(dr, dz)\right|^2\right) \\
&= \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw \\
&\quad \left(\int_{\mathbb{R}} d\tau [G(t - \tau, x - z)1_{(\tau \leq t)} - G(s - \tau, y - z)1_{(\tau \leq s)}] (\tau - r)_+^{H - \frac{3}{2}}\right) \\
&\quad \times \left(\int_{\mathbb{R}} d\tau [G(t - \tau, x - w)1_{(\tau \leq t)} - G(s - \tau, y - w)1_{(\tau \leq s)}] (\tau - r)_+^{H - \frac{3}{2}}\right) f(z - w) \\
&= (2\pi)^{-d} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} \mu(d\xi) \\
&\quad \left|\mathcal{F}\left(\int_{\mathbb{R}} d\tau [G(t - \tau, x - \cdot)1_{(\tau \leq t)} - G(s - \tau, y - \cdot)1_{(\tau \leq s)}] (\tau - r)_+^{H - \frac{3}{2}}\right)(\xi)\right|^2 \\
&= (2\pi)^{-d} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} \mu(d\xi) \left|\int_{\mathbb{R}} d\tau [e^{-i\xi x} e^{-(t-\tau)|\xi|^2} 1_{(\tau \leq t)} - e^{-i\xi y} e^{-(s-\tau)|\xi|^2} 1_{(\tau \leq s)}]\right. \\
&\quad \left. \times (\tau - r)_+^{H - \frac{3}{2}}\right|^2. \tag{4.14}
\end{aligned}$$

Split the domain of integration of the variable  $r$  into the subdomains  $[s, t]$  and  $[s, t]^c$ , and observe that on  $[s, t]$ , the term  $1_{(\tau \leq s)}(\tau - r)_+$  equals zero. Since the integrand is non negative, we have

$$\begin{aligned}
& \|u(t, x) - u(s, y)\|_{L^2(\Omega)}^2 \\
&\geq (2\pi)^{-d} \int_s^t dr \int_{\mathbb{R}^d} \mu(d\xi) \left(\int_{\mathbb{R}} d\tau e^{-(t-\tau)|\xi|^2} 1_{(\tau \leq t)} (\tau - r)_+^{H - \frac{3}{2}}\right)^2. \tag{4.15}
\end{aligned}$$

Computing the integrals, we see that this is bounded below by a constant multiple of  $(t - s)^{2H - \frac{d - \alpha}{2}}$ , where the constant depends on  $\alpha, d$  and  $H$ .  $\square$

The next proposition extends (4.11) to cover the range  $\alpha \in [0, d)$ . The proof is the same as that of [19, Theorem 4], where  $\alpha \in (0, d)$ . For the sake of completeness, we provide the details and see that the arguments can be adapted to cover the case  $\alpha = 0$ .

**Proposition 4.2.** *Let  $M > 0$ . There exists positive constants  $c_3, c_4$ , that depend on  $\alpha, d, H, M$ , such that for any  $t > 0, x, y \in [-M, M]^d$ ,*

$$\begin{aligned}
& c_3(t^{2H} \wedge 1) \left(\log \frac{2e\sqrt{d}M}{|x - y|}\right)^\beta |x - y|^{2 \wedge (4H - (d - \alpha))} \\
& \leq \|u(t, x) - u(t, y)\|_{L^2(\Omega)}^2 \leq c_4(t^{2H} + 1) \left(\log \frac{2e\sqrt{d}M}{|x - y|}\right)^\beta |x - y|^{2 \wedge (4H - (d - \alpha))}, \tag{4.16}
\end{aligned}$$

where  $\beta = 1$ , if  $4H - (d - \alpha) = 2$ , and  $\beta = 0$ , otherwise.

*Proof.* Similarly as in (4.7), using Parseval's identity, we have

$$\begin{aligned} & \|u(t, x) - u(t, y)\|_{L^2(\Omega)}^2 \\ &= \frac{\alpha_H}{(2\pi)^d} \int_0^t d\tau \int_0^t d\sigma |\tau - \sigma|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(\tau+\sigma)|\xi|^2} (1 - \cos[(x-y) \cdot \xi]). \end{aligned} \quad (4.17)$$

According to [2, Prop 4.3], there exist positive constants  $c_{1,H}, c_{2,H}$  such that

$$\begin{aligned} c_{1,H}(t^{2H} \wedge 1) \left( \frac{1}{1 + |\xi|^2} \right)^{2H} &\leq \int_0^t d\tau \int_0^t d\sigma |\tau - \sigma|^{2H-2} e^{-2(\tau+\sigma)|\xi|^2} \\ &\leq c_{2,H}(t^{2H} + 1) \left( \frac{1}{1 + |\xi|^2} \right)^{2H}. \end{aligned} \quad (4.18)$$

Recall (4.1). After having applied the change of variables  $\xi \mapsto \frac{\eta}{|x-y|}$ , from (4.1), (4.17) and (4.18) we deduce

$$\begin{aligned} & c_{1,d,H}(t^{2H} \wedge 1) |x-y|^{4H-d-\alpha} \int_{\mathbb{R}^d} d\eta \frac{\left(1 - \cos \left[ \left( \frac{x-y}{|x-y|} \right) \cdot \eta \right]\right)}{|\eta|^\alpha (|x-y|^2 + |\eta|^2)^{2H}} \leq \|u(t, x) - u(t, y)\|_{L^2(\Omega)}^2 \\ & \leq c_{2,d,H}(t^{2H} + 1) |x-y|^{4H-d-\alpha} \int_{\mathbb{R}^d} d\eta \frac{\left(1 - \cos \left[ \left( \frac{x-y}{|x-y|} \right) \cdot \eta \right]\right)}{|\eta|^\alpha (|x-y|^2 + |\eta|^2)^{2H}}, \end{aligned} \quad (4.19)$$

for all  $t > 0$  and  $x, y \in \mathbb{R}^d$ , with some positive and finite constants  $c_{1,d,H}, c_{2,d,H}$ .

Next we give lower and upper bounds for the terms on the left hand side and the right hand side of (4.19), respectively.

*Lower bounds.* By Schwarz's inequality,  $\overline{B_1(0)} \subset \left\{ \eta \in \mathbb{R}^d : \left| \left( \frac{x-y}{|x-y|} \right) \cdot \eta \right| \leq 1 \right\}$ . Moreover, for  $|\theta| \leq 1$ ,  $1 - \cos \theta \geq \frac{\theta^2}{4}$ . Consequently,

$$\mathcal{I} := \int_{\mathbb{R}^d} d\eta \frac{\left(1 - \cos \left[ \left( \frac{x-y}{|x-y|} \right) \cdot \eta \right]\right)}{|\eta|^\alpha (|x-y|^2 + |\eta|^2)^{2H}} \geq \frac{1}{4} \int_{\overline{B_1(0)}} d\eta \frac{\left( \left( \frac{x-y}{|x-y|} \right) \cdot \eta \right)^2}{|\eta|^\alpha (|x-y|^2 + |\eta|^2)^{2H}}. \quad (4.20)$$

Shrink the ball  $\overline{B_1(0)}$  to the spherical sector defined by the constraint  $\varphi \in [0, \pi/4]$  on the angle. Then, pass to spherical coordinates and, without loss of generality, suppose that  $(x-y)/|x-y|$  is the unit vector  $(1, 0, \dots, 0)$  in  $\mathbb{R}^d$ . Since  $\frac{x-y}{|x-y|} \cdot \eta = |\eta| \cos \varphi$ , where  $\varphi \in [0, \pi/4]$  is the angle between  $(x-y)/|x-y|$  and  $\eta$ , we obtain,

$$\mathcal{I} \geq C \int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(|x-y|^2 + \rho^2)^{2H}}.$$

We estimate this integral by distinguishing three cases.

*Case 1.*  $0 < 4H - (d - \alpha) < 2$ . Since  $|x-y|^2 + \rho^2 \leq 4dM^2 + 1$ ,

$$\int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(|x-y|^2 + \rho^2)^{2H}} \geq \int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(4dM^2 + 1)^{2H}} = \frac{1}{(d - \alpha + 2)(4dM^2 + 1)^{2H}}.$$

*Case 2.*  $4H - (d - \alpha) = 2$ . Because  $|x - y| \leq 2\sqrt{d}M$ , we clearly have

$$\begin{aligned} \int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(|x-y|^2 + \rho^2)^{2H}} &\geq c_{\alpha,d,H,M} \int_{\frac{|x-y|}{2e\sqrt{d}M}}^1 d\rho \rho^{d-\alpha-4H+1} \\ &= c_{\alpha,d,H,M} \log\left(\frac{2e\sqrt{d}M}{|x-y|}\right). \end{aligned}$$

*Case 3.*  $4H - (d - \alpha) > 2$ . Using a similar argument as for case 2,

$$\begin{aligned} \int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(|x-y|^2 + \rho^2)^{2H}} &\geq c_{\alpha,d,H,M} \int_{\frac{|x-y|}{2e\sqrt{d}M}}^1 d\rho \rho^{d-\alpha+1-4H} \\ &= c_{\alpha,d,H,M} |x-y|^{d-\alpha-4H+2}. \end{aligned}$$

*Upper bounds.* Apply the inequality  $1 - \cos(\theta) \leq 2 \wedge \theta^2$  and then, use spherical coordinates to see that the integral  $\mathcal{I}$  defined in (4.20) satisfies

$$\mathcal{I} \leq \int_{\mathbb{R}^d} d\eta \frac{(2 \wedge |\eta|^2)}{|\eta|^\alpha (|x-y|^2 + |\eta|^2)^{2H}} = c_d \int_0^\infty d\rho \frac{(1 \wedge \rho^2) \rho^{d-\alpha-1}}{(|x-y|^2 + \rho^2)^{2H}} := c_d \mathcal{J}. \quad (4.21)$$

We estimate  $\mathcal{J}$  by considering three cases, as we did for the lower bounds.

*Case 1.*  $0 < 4H - (d - \alpha) < 2$ . Since  $|x - y|^2 + \rho^2 \geq \rho^2$ , we have

$$\mathcal{J} \leq \int_0^1 d\rho \rho^{d-\alpha-4H+1} + \int_1^\infty d\rho \rho^{d-\alpha-4H-1} = c_{\alpha,d,H}.$$

*Case 2.*  $4H - (d - \alpha) = 2$ . Splitting the domain of integration of  $\mathcal{J}$ , we obtain

$$\begin{aligned} \mathcal{J} &\leq \int_0^{|x-y|} d\rho \frac{\rho^{d-\alpha+1}}{|x-y|^{4H}} + \int_{|x-y|}^{2e\sqrt{d}M} d\rho \rho^{d-\alpha-4H+1} + \int_{2e\sqrt{d}M}^\infty d\rho \rho^{d-\alpha-4H-1} \\ &= \frac{1}{(d-\alpha+2)} + \log\left(\frac{2e\sqrt{d}M}{|x-y|}\right) + \frac{(2e\sqrt{d}M)^2}{2} \leq c_{\alpha,d,H,M} \log\left(\frac{2e\sqrt{d}M}{|x-y|}\right). \end{aligned}$$

*Case 3.*  $4H - (d - \alpha) > 2$ . Using the inequalities  $1/(|x-y|^2 + \rho^2) \leq 1/|x-y|^2$  and  $1/(|x-y|^2 + \rho^2) \leq 1/\rho^2$ , on  $\{0 \leq \rho \leq |x-y|\}$  and  $\{|x-y| < \rho < \infty\}$ , respectively, we have

$$\mathcal{J} \leq |x-y|^{-4H} \int_0^{|x-y|} d\rho \rho^{d-\alpha+1} + \int_{|x-y|}^\infty d\rho \rho^{d-\alpha-4H+1} = c_{\alpha,d,H} |x-y|^{d-\alpha-4H+2}.$$

From (4.19), and using the lower and upper bounds obtained before, we deduce (4.16).  $\square$

We end this section by proving the equivalence for the canonical pseudo-distance (4.9). It is a consequence of (4.10) and Proposition 4.2.

**Theorem 4.1.** Fix  $M > 0$  and  $t_0 \in (0, T]$ . There exists positive constants  $c_5, c_6$  depending on  $\alpha, d, t_0, H, M, T$  such that for any  $t, s \in [t_0, T]$  and  $x, y \in [-M, M]^d$ ,

$$\|u(t, x) - u(s, y)\|_{L^2(\Omega)}^2 \asymp |t - s|^{2H - \frac{d-\alpha}{2}} + \left( \log \frac{2e\sqrt{d}M}{|x-y|} \right)^\beta |x - y|^{2\wedge(4H-(d-\alpha))}, \quad (4.22)$$

where  $\beta = 1$ , if  $4H - (d - \alpha) = 2$ , and  $\beta = 0$ , otherwise.

The upper bound holds for any  $t, s \in [0, T]$ .

*Proof.* The estimate from above is a consequence of the upper bounds in (4.10) and (4.16), which hold for any  $t, s \in [0, T]$ .

We prove the estimates from below by distinguishing two cases.

*Case 1.*  $|t - s|^{2H - \frac{d-\alpha}{2}} < \frac{c_3(t_0^{2H} \wedge 1)}{4c_2} \left( \log \frac{2e\sqrt{d}M}{|x-y|} \right)^\beta |x - y|^{2\wedge(4H-(d-\alpha))}$ . Applying the triangle inequality and then, using the lower bound in (4.16) and the upper bound in (4.10), we obtain

$$\begin{aligned} \|u(t, x) - u(s, y)\|_{L^2(\Omega)}^2 &\geq \frac{1}{2} \|u(t, x) - u(t, y)\|_{L^2(\Omega)}^2 - \|u(t, y) - u(s, y)\|_{L^2(\Omega)}^2 \\ &\geq \frac{c_3(t_0^{2H} \wedge 1)}{2} \left( \log \frac{2e\sqrt{d}M}{|x-y|} \right)^\beta |x - y|^{2\wedge(4H-(d-\alpha))} - c_2 |t - s|^{2H - \frac{d-\alpha}{2}} \\ &\geq \frac{c_3(t_0^{2H} \wedge 1)}{8} \left( \log \frac{2e\sqrt{d}M}{|x-y|} \right)^\beta |x - y|^{2\wedge(4H-(d-\alpha))} + \frac{c_2}{2} |t - s|^{2H - \frac{d-\alpha}{2}}. \end{aligned}$$

*Case 2.*  $|t - s|^{2H - \frac{d-\alpha}{2}} \geq \frac{c_3(t_0^{2H} \wedge 1)}{4c_2} \left( \log \frac{2e\sqrt{d}M}{|x-y|} \right)^\beta |x - y|^{2\wedge(4H-(d-\alpha))}$ . By Proposition 4.1,

$$\begin{aligned} \|u(t, x) - u(s, y)\|_{L^2(\Omega)}^2 &\geq c_1 |t - s|^{2H - \frac{d-\alpha}{2}} \\ &\geq \frac{c_1}{2} |t - s|^{2H - \frac{d-\alpha}{2}} + \frac{c_3(t_0^{2H} \wedge 1)}{8c_2} \left( \log \frac{2e\sqrt{d}M}{|x-y|} \right)^\beta |x - y|^{2\wedge(4H-(d-\alpha))}. \end{aligned}$$

The proof is complete. □

**Remark 4.1.** Assume that  $v_0 \in C^\zeta(\mathbb{R}^d)$ , for some  $\zeta \in (0, 1]$ . Then the function

$$[0, T] \times \mathbb{R}^d \ni (t, x) \longrightarrow I_0(t, x) = \int_{\mathbb{R}^d} G(t, x - y) v_0(y) dy,$$

is globally Hölder continuous, jointly in  $(t, x)$ , with exponents  $(\zeta/2, \zeta)$  (see e.g. [10]).

Furthermore, the upper bound estimate in (4.22) and the classical Kolmogorov's continuity criterion ensures the existence of a version of the process  $(v(t, x))$  with continuous (and even Hölder continuous) sample paths, jointly in  $(t, x)$ .

We end this section giving some properties of the covariance function of the process  $(u(t, x))$  that will be used in Section 4.2.

**Lemma 4.2.** Fix  $M > 0$  and  $t_0 \in (0, T]$ .

1. There exists  $\eta > 0$  and  $C > 0$ , depending on  $\alpha, d, t_0, H, M, T$ , such that, for all  $s, t \in [t_0, T]$  and  $x, y \in [-M, M]^d$ ,

$$|\sigma_{t,x}^2 - \sigma_{s,y}^2| \leq C \|u(t, x) - u(s, y)\|_{L^2(\Omega)}^{1+\eta}. \quad (4.23)$$

2. For any  $(t, x), (s, y) \in [t_0, T] \times \mathbb{R}^d$  such that  $(t, x) \neq (s, y)$ ,

$$\rho_{(t,x),(s,y)} < 1.$$

*Proof.* 1. Assume, without loss of generality, that  $0 < s \leq t$ . For all  $x, y \in \mathbb{R}^d$ , from (4.7) and similarly as in (4.8), we deduce

$$\begin{aligned} & \left( \frac{\alpha_H}{(2\pi)^d} \right)^{-1} |\sigma_{t,x}^2 - \sigma_{s,y}^2| = \left( \frac{\alpha_H}{(2\pi)^d} \right)^{-1} (\sigma_{t,x}^2 - \sigma_{s,y}^2) \\ & = \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(\tau+\sigma)|\xi|^2} \left( \int_0^t d\tau \int_0^t d\sigma |\tau - \sigma|^{2H-2} - \int_0^s d\tau \int_0^s d\sigma |\tau - \sigma|^{2H-2} \right) \\ & \leq c_{\alpha,d,H} \left( \int_s^t d\tau \int_s^t d\sigma \frac{|\tau - \sigma|^{2H-2}}{(\tau + \sigma)^{\frac{d-\alpha}{2}}} + 2 \int_0^s d\tau \int_s^t d\sigma \frac{|\tau - \sigma|^{2H-2}}{(\tau + \sigma)^{\frac{d-\alpha}{2}}} \right). \end{aligned} \quad (4.24)$$

Apply polar coordinates  $(\tau, \sigma) \mapsto (\rho \cos \theta, \rho \sin \theta)$  and then, the mean value theorem, to see that

$$\begin{aligned} & \int_s^t d\tau \int_s^t d\sigma \frac{|\tau - \sigma|^{2H-2}}{(\tau + \sigma)^{\frac{d-\alpha}{2}}} \\ & \leq \int_{\sqrt{2}s}^{\sqrt{2}t} d\rho \rho^{2H - \frac{d-\alpha}{2} - 1} \left( \int_0^{\frac{\pi}{2}} d\theta \frac{|\cos \theta - \sin \theta|^{2H-2}}{(\cos \theta + \sin \theta)^{\frac{d-\alpha}{2}}} \right) \\ & \leq \frac{2^{H - \frac{(d-\alpha)}{4}} T^{2H - \frac{(d-\alpha)}{2} - 1} (t-s)}{\left(2H - \frac{(d-\alpha)}{2}\right)^2} \int_0^{\frac{\pi}{2}} d\theta \frac{|\cos \theta - \sin \theta|^{2H-2}}{(\cos \theta + \sin \theta)^{\frac{d-\alpha}{2}}} \leq C(\alpha, d, H, T)(t-s). \end{aligned}$$

Since  $0 < 2H - \frac{(d-\alpha)}{2} < 2$ , we have  $\eta_1 := \left(H - \frac{(d-\alpha)}{4}\right)^{-1} - 1 > 0$ , and we deduce,

$$\int_s^t d\tau \int_s^t d\sigma \frac{|\tau - \sigma|^{2H-2}}{(\tau + \sigma)^{\frac{d-\alpha}{2}}} \leq C(H, d, T)(t-s)^{\frac{4H - (d-\alpha)}{4}(1+\eta_1)}. \quad (4.25)$$

As for the second integral on the last line of (4.24), we have

$$\int_0^s d\tau \int_s^t d\sigma \frac{|\tau - \sigma|^{2H-2}}{(\tau + \sigma)^{\frac{d-\alpha}{2}}} \leq \int_0^s d\tau \int_s^t d\sigma (\sigma - \tau)^{2H - \frac{(d-\alpha)}{2} - 2}, \quad (4.26)$$

because  $\tau \leq \sigma$  implies  $\tau + \sigma \geq \sigma - \tau$ .

Our next goal is to obtain estimates from above on the right-hand side of (4.26) in terms of powers of  $(t-s)$ . For this, we consider three cases.



Case 1.  $0 < 4H - (d - \alpha) < 2$ .

$$\begin{aligned} \int_0^s d\tau \int_s^t d\sigma (\sigma - \tau)^{2H - \frac{(d-\alpha)}{2} - 2} &= \frac{s^{2H - \frac{(d-\alpha)}{2}} + (t-s)^{2H - \frac{(d-\alpha)}{2}} - t^{2H - \frac{(d-\alpha)}{2}}}{(2H - \frac{(d-\alpha)}{2})(1 + \frac{(d-\alpha)}{2} - 2H)} \\ &\leq \frac{(t-s)^{2H - \frac{(d-\alpha)}{2}}}{(2H - \frac{(d-\alpha)}{2})(1 + \frac{(d-\alpha)}{2} - 2H)} = C(\alpha, d, H)(t-s)^{2H - \frac{d-\alpha}{2}} \\ &= C(\alpha, d, H)(t-s)^{\frac{4H - (d-\alpha)}{4}(1+\eta_2)}, \end{aligned} \quad (4.27)$$

with  $\eta_2 = 1$

Case 2.  $0 < 4H - (d - \alpha) = 2$ .

$$\begin{aligned} \int_0^s d\tau \int_s^t d\sigma (\sigma - \tau)^{-1} &= t \log(t) - s \log(s) + (t-s) \log((t-s)^{-1}) \\ &\leq 2[(t \log t - s \log s) \vee ((t-s) \log((t-s)^{-1}))] \\ &\leq 2(t-s)[(\log T + 1) \vee \log((t-s)^{-1})], \end{aligned}$$

where in the last inequality we have applied the mean value theorem. This yields, for any  $\gamma \in (0, 1)$ ,

$$\begin{aligned} \int_0^s d\tau \int_s^t d\sigma (\sigma - \tau)^{-1} &\leq 2(|\log T| + 2)[(t-s)^\gamma \vee (t-s)] \leq C(T) (t-s)^\gamma \\ &= C(T)(t-s)^{\frac{4H - (d-\alpha)}{4}(1+\eta_3)}, \end{aligned} \quad (4.28)$$

with  $\eta_3 = 2\gamma - 1$

Case 3.  $2 < 4H - (d - \alpha) < 4$ .

$$\begin{aligned} \int_0^s d\tau \int_s^t d\sigma (\sigma - \tau)^{2H - 2 - \frac{(d-\alpha)}{2}} &= \frac{t^{2H - \frac{(d-\alpha)}{2}} - s^{2H - \frac{(d-\alpha)}{2}} - (t-s)^{2H - \frac{(d-\alpha)}{2}}}{\left(2H - \frac{(d-\alpha)}{2}\right) \left(2H - 1 - \frac{(d-\alpha)}{2}\right)} \\ &\leq \frac{t^{2H - \frac{(d-\alpha)}{2}} - s^{2H - \frac{(d-\alpha)}{2}}}{\left(2H - \frac{(d-\alpha)}{2}\right) \left(2H - 1 - \frac{(d-\alpha)}{2}\right)} \leq \frac{T^{2H - 1 - \frac{(d-\alpha)}{2}}}{2H - 1 - \frac{(d-\alpha)}{2}} (t-s) \\ &\leq C(\alpha, d, H, T)(t-s)^{\frac{4H - (d-\alpha)}{4}(1+\eta_4)}, \end{aligned} \quad (4.29)$$

with  $\eta_4 = \eta_1 = \left(H - \frac{(d-\alpha)}{4}\right)^{-1} - 1$ .

Set  $\eta = \min(\eta_i, i = 1, 2, 3)$ . Appealing to Theorem 4.1, and using (4.24), (4.25), (4.27), (4.28) and (4.29), we obtain

$$|\sigma_{t,x}^2 - \sigma_{s,y}^2| \leq C(\alpha, d, H, T)(t-s)^{(H - \frac{d-\alpha}{4})(1+\eta)} \leq c_5^{-1} C(\alpha, d, H, T) \|u(t, x) - u(s, y)\|_{L^2(\Omega)}^{1+\eta},$$

with  $c_5$  as in (4.22). The proof of Claim 1. is complete.

Next, we prove Claim 2 of the Lemma. Assume that  $\rho_{(t,x),(s,y)} = 1$  and hence, that there exists  $\lambda \in \mathbb{R}$  such that

$$\|u(t, x) - \lambda u(s, y)\|_{L^2(\Omega)} = 0. \quad (4.30)$$

We will see that this assumption leads to a contradiction.

*Case 1.*  $s < t$ . Apply (4.14) with  $u(s, y)$  replaced by  $\lambda u(s, y)$  to obtain

$$\begin{aligned} \|u(t, x) - \lambda u(s, y)\|_{L^2(\Omega)}^2 &= (2\pi)^{-d} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} \mu(d\xi) \\ &\times \left| \int_{\mathbb{R}} d\tau \left[ e^{-2(t-\tau)|\xi|^2} \mathbf{1}_{(\tau \leq t)} - \lambda e^{-2(s-\tau)|\xi|^2} \mathbf{1}_{(\tau \leq s)} \right] (\tau - r)_+^{H-\frac{3}{2}} \right|^2. \end{aligned}$$

As in (4.15), this is bounded from below by a constant multiple of

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_s^t dr \left( \int_r^t d\tau e^{-2(t-\tau)|\xi|^2} (\tau - r)^{H-\frac{3}{2}} \right)^2.$$

A direct computation shows that  $\int_s^t dr \left( \int_r^t d\tau e^{-2(t-\tau)|\xi|^2} (\tau - r)^{H-\frac{3}{2}} \right)^2 \neq 0$ . Since we are assuming (4.30), we reach a contradiction.

We notice that, in the case under consideration, the arguments hold for any  $(t, x), (s, y) \in [0, \infty) \times \mathbb{R}^d$ .

*Case 2.*  $s = t \in [t_0, T]$ ,  $x \neq y$ . Apply (4.17) with  $u(t, y)$  replaced by  $\lambda u(t, y)$  to see that

$$\begin{aligned} \|u(t, x) - \lambda u(s, y)\|_{L^2(\Omega)}^2 &= \frac{\alpha_H}{(2\pi)^d} \int_0^t d\tau \int_0^t d\sigma |\tau - \sigma|^{2H-2} \\ &\times \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(\tau+\sigma)|\xi|^2} (1 + \lambda^2 - 2\lambda \cos[(x - y) \cdot \xi]). \end{aligned}$$

Using the lower bound estimates in (4.1) and (4.18), we deduce

$$\begin{aligned} \|u(t, x) - \lambda u(s, y)\|_{L^2(\Omega)}^2 &\geq C(\alpha, d, t_0, H) \int_{\mathbb{R}^d} (1 + \lambda^2 - 2\lambda \cos[(x - y) \cdot \xi]) \\ &\times \frac{|\xi|^{-\alpha}}{(1 + |\xi|^2)^{2H}} d\xi. \end{aligned}$$

By assumption, the integral on the right-hand side must be zero. However, this integral is bounded from below by the integral on the spherical sector of the ball  $B_1(0)$  where  $2 \cos[(x - y) \cdot \xi] \leq \frac{1+\lambda^2}{2}$ . Consequently,

$$0 = \|u(t, x) - \lambda u(t, y)\|_{L^2(\Omega)}^2 \geq C(\alpha, d, t_0, H) \frac{1 + \lambda^2}{2} \int_0^1 \frac{r^{d-\alpha-1}}{(1+r^2)^{2H}} dr.$$

Since  $\int_0^1 \frac{r^{d-\alpha-1}}{(1+r^2)^{2H}} dr > 0$ , this is a contradiction. This ends the proof of Claim 2.  $\square$

## 4.2. Hitting probabilities

Consider the random field  $U = \{U(t, x) = (U_1(t, x), \dots, U_D(t, x)), (t, x) \in [0, T] \times \mathbb{R}^d\}$ , where the components are independent copies of the random variable  $v(t, x)$  defined in (4.4). The process  $U$  is the

random field solution to the system of SPDEs

$$\begin{cases} \frac{\partial U_j}{\partial t}(t, x) = \Delta U_j(t, x) + \dot{W}_j^{H, \alpha}, & (t, x) \in (0, T] \times \mathbb{R}^d, \\ U_j(0, x) = v_0(x), & x \in \mathbb{R}^d, \end{cases}$$

$j = 1, \dots, D$ , where  $(W_j^{H, \alpha}, j = 1, \dots, D)$  are independent copies of the fractional-colored noise  $W^{H, \alpha}$  introduced at the beginning of Section 4. We will write  $U_j(t, x) = I_0(t, x) + u_j(t, x)$ . In the sequel, we assume that  $v_0$  is such that the function  $(t, x) \mapsto I_0(t, x)$  is continuous (see Remark 4.1 for sufficient conditions).

Throughout this section, we will consider the compact sets  $I = [t_0, T]$  and  $J = [-M, M]^d$ , with  $t_0 \in (0, T]$ ,  $M > 0$ , and the gauge functions defined in  $\mathbb{R}_+$ ,

$$q_1(\tau) = \tau^{H - \frac{d-\alpha}{4}}, \quad q_2(\tau) = \begin{cases} \tau^{1 \wedge (2H - \frac{d-\alpha}{2})}, & \text{if } 4H - (d - \alpha) \neq 2, \\ \tau \left( \log \frac{2e\sqrt{d}M}{\tau} \right)^{\frac{1}{2}}, & \text{if } 4H - (d - \alpha) = 2. \end{cases} \quad (4.31)$$

If  $4H - (d - \alpha) \neq 2$ , the functions  $q_1$  and  $q_2$  belong to the class of examples considered in Lemma 5.1 (1.), with  $\nu_1 := H - \frac{d-\alpha}{4}$ ,  $\nu_2 := 1 \wedge (2H - \frac{d-\alpha}{2})$ . If  $D > \left( \frac{1}{H - \frac{d-\alpha}{4}} + \frac{d}{1 \wedge (2H - \frac{d-\alpha}{2})} \right)$ , the function

$$\bar{g}_q(\tau) = \tau^{D - \left( \frac{1}{H - \frac{d-\alpha}{4}} + \frac{d}{1 \wedge (2H - \frac{d-\alpha}{2})} \right)} \quad (4.32)$$

(see (3.38)) is strictly increasing.

Furthermore, we prove in Lemma 5.2 (1.) that the function  $\bar{v}_q(\tau)$  defined in (3.71) satisfies the condition (3.72) with  $\bar{g}_q$  given in (4.32).

If  $4H - (d - \alpha) = 2$ ,  $q_1$  and  $q_2$  belong to the class of examples considered in Lemma 5.1 (2.) with  $\nu_1 := H - \frac{d-\alpha}{4}$ ,  $\nu_2 = 1$ ,  $\delta = \frac{1}{2}$ . If  $D > \frac{1}{H - \frac{d-\alpha}{4}} + d$ , the function

$$\bar{g}_q(\tau) = \tau^{D - \frac{1}{H - \frac{d-\alpha}{4}}} (q_2^{-1}(\tau))^{-d}, \quad (4.33)$$

is strictly increasing on a small interval  $(0, \rho_0)$ . Moreover, according to Lemma 5.2 (3.), this function satisfies the condition (3.72), where  $\bar{v}_q(\tau)$  is defined in (3.71).

We now give the main theorem on hitting probabilities for the process  $U$ .

**Theorem 4.2.** *Let  $t_0 > 0$ ,  $I = [t_0, T]$ ,  $J = [-M, M]^d$ . Suppose that the function  $I \times J \ni (t, x) \mapsto I_0(t, x)$  satisfies the condition (3.69).*

1. *Case  $4H - (d - \alpha) \neq 2$ . Assume  $D > \left( \frac{1}{H - \frac{d-\alpha}{4}} + \frac{d}{1 \wedge (2H - \frac{d-\alpha}{2})} \right)$  and let  $\bar{g}_q$  be as in (4.32).*

(a) *There exists a constant  $C := C(I, J, D, d)$  such that for any Borel set  $A \subset \mathbb{R}^D$ ,*

$$P(U(I \times J) \cap A \neq \emptyset) \leq C \mathcal{H}_{\bar{g}_q}(A). \quad (4.34)$$

(b) Fix  $N > 0$  and let  $A \subset B_N(0) \subset \mathbb{R}^D$  be a Borel set. There exists a constant  $c := c(I, J, N, D, d)$  such that

$$P(U(I \times J) \cap A \neq \emptyset) \geq c \text{Cap}_{(\bar{g}_q)^{-1}}(A). \quad (4.35)$$

2. Case  $4H - (d - \alpha) = 2$ . Assume  $D > \frac{1}{H - \frac{d-\alpha}{4}} + d$  and let  $\bar{g}_q$  be as in (4.33).

(a) There exist a constant  $C := C(I, J, D, d)$  such that for any Borel set  $A \subset \mathbb{R}^D$ ,

$$P(U(I \times J) \cap A \neq \emptyset) \leq C \mathcal{H}_{\bar{g}_q}(A). \quad (4.36)$$

(b) Fix  $N > 0$  and let  $A \subset B_N(0) \subset \mathbb{R}^D$  be a Borel set. There exists a constant  $c := c(I, J, N, D, d)$  such that

$$P(U(I \times J) \cap A \neq \emptyset) \geq c \text{Cap}_{(\bar{g}_q)^{-1}}(A). \quad (4.37)$$

*Proof.* (i) *Upper bounds.* The inequalities (4.34) and (4.36) are obtained applying Theorem 3.3. Indeed, the random field  $U$  is Gaussian and has i.i.d. components and has continuous sample paths, a.s. Lemma 4.1 (1.) gives the non degeneracy condition  $\sigma_{I,J}^2 > 0$  on the variances. Furthermore from the discussion at the begining of this section, we see that the hypotheses on the gauge functions and the corresponding  $\bar{g}_q$  are satisfied. Finally, the upper bound in (4.22) implies the validity of (3.36) and therefore, by Remark 3.3, that of (3.34). Hence, in the two cases,  $U$  satisfies the hypotheses of Theorem 3.3.

Observe that, when  $4H - (d - \alpha) \neq 2$ , if  $D - 1/(H - \frac{d-\alpha}{4}) + d/(1 \wedge (2H - \frac{d-\alpha}{2})) < 0$ , we have  $\mathcal{H}_{\bar{g}}(A) = \infty$ ; thus (4.34) still holds but is not informative.

(ii) *Lower bounds.* The inequalities (4.35) and (4.37) are obtained applying Theorem 3.5. For this, we first check that the process  $U$  satisfies the hypotheses ( $H_U$ ) of Section 3.2. Indeed, (3.66) is Lemma 4.1 (1.), and conditions 2 and 3 are proved in Lemma 4.2. Theorem 4.1 tells us that (3.68) is satisfied with  $q_1$  and  $q_2$  given in (4.31). Hence, ( $H_U$ ) is satisfied. The conditions required on the gauge functions and the corresponding functions  $\bar{g}_q$  and  $\bar{v}_q$  are proved in Lemmas 5.1 and 5.2 (3.). Thus, in the two cases,  $U$  satisfies the hypotheses of Theorem 3.5.

The proof of the theorem is complete.  $\square$

## 5. Auxiliary lemmas

In the next lemmas,  $q, q_1, q_2$  are gauge functions and  $g_q, \bar{g}_q$  the functions defined in (3.29), (3.38), respectively. For convenience we recall their respective expressions:

$$g_q(\tau) = \frac{\tau^D}{(q^{-1}(\tau))^d}, \quad \bar{g}_q(\tau) = \frac{\tau^D}{(q_1^{-1}(\tau))^{d_1} (q_2^{-1}(\tau))^{d_2}}, \quad \tau \in \mathbb{R}_+,$$

where  $\bar{g}_q$ , stands for  $\bar{g}_{(q_1, q_2)}$ . Observe that if  $q_1 = q_2 := q$  then  $\bar{g}_q = g_q$  with  $d := d_1 + d_2$ .

**Lemma 5.1.** Fix  $\rho_0 > 0$ . Assume that  $q_1, q_2$  are differentiable in  $(0, \rho_0)$ . Then  $\bar{g}_q$  is strictly increasing on  $(0, \rho_0)$  if and only if

$$D > \tau \left( \frac{d_1}{q_1^{-1}(\tau) \dot{q}_1(q_1^{-1}(\tau))} + \frac{d_2}{q_2^{-1}(\tau) \dot{q}_2(q_2^{-1}(\tau))} \right), \quad \tau \in (0, \rho_0), \quad (5.1)$$

or equivalently, if and only if for any  $\tau \in (0, q_2^{-1}(\rho_0))$ ,

$$D > q_2(\tau) \left( \frac{d_1}{q_1^{-1}(q_2(\tau))\dot{q}_1(q_1^{-1}(q_2(\tau)))} + \frac{d_2}{\tau\dot{q}_2(\tau)} \right). \quad (5.2)$$

When  $q_1 = q_2 := q$ , the condition (5.1) is

$$D > d \frac{\tau}{q^{-1}(\tau)\dot{q}(q^{-1}(\tau))}, \quad \tau \in (0, \rho_0) \iff D > d \frac{q(\tau)}{\tau\dot{q}(\tau)}, \quad \tau \in (0, q^{-1}(\rho_0)), \quad (5.3)$$

whith  $d = d_1 + d_2$ .

For the gauge functions listed below, we have the following.

1. Let  $q_i(\tau) = \tau^{\nu_i}$ ,  $\tau \geq 0$ ,  $\nu_i > 0$ ,  $i = 1, 2$ . Assume that  $D > \frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}$ . Then condition (5.1) holds on  $\mathbb{R}_+$  and therefore,  $\bar{g}_q$  is strictly increasing. Moreover, (3.42) is satisfied if and only if  $D > \frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}$ . In particular, if  $q_1(\tau) = q_2(\tau) = \tau^\nu$  and  $d_1 + d_2 = d$ , the function  $g_q$  is strictly increasing on  $\mathbb{R}_+$  whenever  $D > d/\nu$ . The condition (3.30) is satisfied if and only if  $D > d/\nu$  holds.
2. Let  $q_1(\tau) = \tau^{\nu_1}$ ,  $q_2(\tau) = \tau^{\nu_2} \left(\log \frac{c}{\tau}\right)^\delta$ ,  $\tau \geq 0$ ,  $\nu_1, \nu_2, \delta > 0$ . Assume that  $D > \frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}$  and  $\nu_2 \geq \delta$ . Set  $\eta := (\nu_1\nu_2D - \nu_2d_1 - \nu_1d_2)/(\nu_1D - d_1)$ . Then, on the interval  $(0, c \min(e^{-1}, \exp(-d/\eta)))$ , the condition (5.2) holds and therefore,  $\bar{g}_q$  is strictly increasing on this interval. The condition (3.42) holds if and only if  $D > \frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}$ .
3. Let  $q(\tau) = \tau^\nu \left(\log(c/\tau)\right)^\delta$ ,  $\tau \geq 0$ ,  $\nu, \delta > 0$ . Suppose  $d/D < \nu$ . If  $\nu - \delta < d/D < \nu$  then  $g_q$  is strictly increasing on  $\tau \in (0, c \exp(-\delta/(\nu - d/D)))$ . If  $d/D \leq \nu - \delta$  then  $g_q$  is strictly increasing on  $\tau \in (0, c/e)$ . Furthermore, the condition (3.30) holds if and only if  $D > d/\nu$ .

All these examples consist of infinitely differentiable functions and, if  $\nu, \nu_1, \nu_2 \in (0, 1)$ , the first order derivatives are decreasing on  $(0, r_0)$ . For  $\tau \mapsto \tau^\nu$ ,  $r_0 = \infty$ , while for  $\tau \mapsto \tau^\nu \left(\log(c/\tau)\right)^\delta$ ,  $r_0 = c \exp(-(1-\delta)/(1-\nu))$ .

*Proof.* Imposing the constraint  $\dot{g}(\tau) > 0$  for any  $\tau \in (0, \rho_0)$ , yields (5.1). The equivalent form (5.2) is obtained by the change of variable  $\tau \mapsto q_2^{-1}(\tau)$ . Taking  $q_1 = q_2 = q$ , yields (5.3).

The results on monotonicity concerning the three examples can be argued by elementary computations on the expressions (5.1), (5.2) and (5.3), respectively.

In the examples discussed in 1. and under the given conditions, the validity of (3.30) is trivial. Let  $q$  be as in 3. The inverse  $q^{-1}$  is given by the relation

$$q^{-1}(\tau) = c \exp \left[ \frac{\delta}{\nu} W_{-1} \left( -\frac{\nu}{\delta} c^{-\frac{\nu}{\delta}} \tau^{\frac{1}{\delta}} \right) \right], \quad (5.4)$$

where  $W_{-1}$  is the real branch of the multi-valued Lambert function  $W(z)$  defined for  $z \in (-e^{-1}, -1)$ , satisfying  $W(z) \leq -1$ . According to [4, Theorem 1],

$$-1 - \sqrt{2z} - z < W_{-1}(-e^{-z-1}) < -1 - \sqrt{2z} - \frac{2}{3}z, \quad z > 0. \quad (5.5)$$

Applying this result, we see that

$$q^{-1}(\tau) \asymp c_1 \tau^{\frac{1}{\nu}} \exp \left( -\sqrt{2} \frac{\delta}{\nu} \left( -\log \left( c_2 \tau^{\frac{1}{\delta}} \right) \right)^{\frac{1}{2}} \right),$$

where  $c_1, c_2$  are constants depending on  $\nu, \eta$ ; consequently,

$$g_q(\tau) \asymp \tau^{D-\frac{d}{\nu}} \exp \left[ C_1 \left( \log \frac{1}{c_2 \tau^{\frac{1}{\delta}}} \right)^{\frac{1}{2}} \right].$$

Assuming  $D > \frac{d}{\nu}$ , the limit of the right-hand side of the above equivalence tends to zero as  $\tau \downarrow 0$ . Therefore, (3.30) holds.

With similar arguments, one checks that in Example 2., (3.42) holds.

Finally, after computation of the second derivatives and the analysis of their sign, we obtain the last statement.  $\square$

In the next lemma we study properties of the functions  $v_q$  and  $\bar{v}_q$  defined in (3.49) and (3.71), respectively, for the particular cases of gauge functions relevant to this article.

**Lemma 5.2.** 1. Let  $q_i(\tau) = \tau^{\nu_i}$ ,  $\tau \geq 0$ , with  $\nu_i > 0$ ,  $i = 1, 2$ . Let  $\chi = \frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}$ . Then

$$\bar{v}_q(\tau) = \begin{cases} (\nu_1 \nu_2 (D - \chi))^{-1} \left[ \tau^{-(D-\chi)} - c_{I,J}^{-(D-\chi)} \right] & \text{if } \chi \neq D, \\ (\nu_1 \nu_2)^{-1} \log \left( \frac{c_{I,J}}{\tau} \right), & \text{if } \chi = D. \end{cases} \quad (5.6)$$

Therefore, up to multiplicative constants,  $\bar{v}_q$  is bounded above by the Bessel-Riesz potential kernel of order  $\beta := D - \chi$ .

If  $\chi < D$ , the function  $\bar{g}_q$ , which in this particular example is  $\bar{g}_q(\tau) = \tau^{D-\chi}$ , satisfies the condition (3.72).

In the particular case  $q(\tau) := q_1(\tau) = q_2(\tau) = \tau^\nu$ ,  $\tau \geq 0$ ,  $\nu > 0$ , we have  $\chi = \frac{d}{\nu}$  with  $d = d_1 + d_2$ , and  $\bar{v}_q = \nu^{-1} v_q$ . Therefore,

$$v_q(\tau) = \begin{cases} (\nu D - d)^{-1} \left[ \tau^{-(D-d/\nu)} - c_{I,J}^{-(D-d/\nu)} \right], & \text{if } d/\nu \neq D, \\ \nu^{-1} \log \left( \frac{c_{I,J}}{\tau} \right), & \text{if } d/\nu = D. \end{cases} \quad (5.7)$$

Hence, if  $d/\nu < D$ , the function  $g_q$  satisfies (3.50).

2. Let  $q(\tau) = \tau^\nu \left( \log \frac{c}{\tau} \right)^\delta$ ,  $\tau \geq 0$ , with  $\nu > 0$ ,  $\delta > 0$ . Then,

$$v_q(\tau) \asymp 1, \text{ if either } d/\nu > D \text{ or } (d/\nu = D, 1 - \delta D < 0), \quad (5.8)$$

while if either  $d/\nu = D, 1 - \delta D \geq 0$  or  $d/\nu < D$ ,

$$\lim_{\tau \downarrow 0} v_q(\tau) = \infty. \quad (5.9)$$

Furthermore, the function  $g_q$  satisfies (3.50) only if  $d/\nu < D$ .

3. Let  $q_1(\tau) = \tau^{\nu_1}$ ,  $q_2(\tau) = \tau^{\nu_2} \left( \log \frac{c}{\tau} \right)^\delta$ ,  $\tau \geq 0$ , with  $\nu_i, \delta > 0$ ,  $i = 1, 2$ . Then, if  $D > \frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}$  and  $d_2 \geq \nu_2$ , the function  $\bar{g}_q$ , which in this case is

$$\bar{g}_q(\tau) = \tau^{D-\frac{d_1}{\nu_1}} \left( q_2^{-1}(\tau) \right)^{-d_2}, \quad (5.10)$$

satisfies (3.72).

*Proof.* 1. Computing the integral (3.71) for the particular choice of gauge functions  $q_1, q_2$ , we obtain (5.6). Up to multiplicative constants, this is indeed bounded by the Bessel-Riesz potential kernel of order  $\beta := D - \chi$ .

If  $\chi < D$ ,  $\bar{v}_q(\tau) \leq ((\nu_1\nu_2(D - \chi) \bar{g}_q(\tau))^{-1})$ , and therefore (3.72) holds. Particularizing to  $q(\tau) := q_1(\tau) = q_2(\tau) = \tau^\nu$  yields (5.7) and its consequences.

2. Properties (5.8) and (5.9) are proved using (5.4) and (5.5).

Since  $v_q$  and  $g_q$  are continuous functions on  $(0, \infty)$ , the condition (3.50) is equivalent to  $\lim_{\tau \downarrow 0} v_q(\tau)g_q(\tau) \in (0, \infty)$ . Furthermore, because  $\tau \mapsto q(\tau)$  is strictly increasing and  $q(0) = 0$ , this is equivalent to  $\lim_{\tau \downarrow 0} v_q(q(\tau))g_q(q(\tau)) = l_0 \in (0, \infty)$ .

Consider first the cases: (i)  $D < d/\nu$ ; (ii)  $D = d/\nu$  and  $1 - \delta D < 0$ . Since  $v_q \asymp 1$  and  $\lim_{\tau \downarrow 0} g_q(\tau) = 0$ . We deduce  $\lim_{\tau \downarrow 0} v_q(\tau)g_q(\tau) = 0$ , and therefore (3.50) is not satisfied.

Next, we consider: (iii)  $D = d/\nu$  and  $1 - \delta D \geq 0$ ; (iv)  $D > d/\nu$ .

Using the definitions of  $v_q$  and  $g_q$ , we have

$$v_q(g(\tau))g_q(g(\tau)) = \left[ \int_{\tau}^{q(\text{diam}(A))} \left( \log \frac{c}{\rho} \right)^{-\delta D} \rho^{-\nu D + d - 1} d\rho \right] \left[ \frac{(\tau)^d}{(q(\tau))^D} \right]^{-1}.$$

Then, computing the limit (for example, applying the L'Hospital's rule), we obtain

$$\lim_{\tau \downarrow 0} v_q(g(\tau))g_q(g(\tau)) = (D\nu - d)^{-1}.$$

Hence, in the case (iii) (3.50) is not satisfied, while in the case (iv) it is.

3. From (5.5) we deduce that the function  $\bar{g}_q$  given in (5.10) satisfies  $\bar{g}_q(\tau) \asymp \bar{g}_q(\tau/2)$ . Moreover, since  $\bar{v}_q$  and  $\bar{g}_q$  are continuous away from zero, we see that the condition (3.72) is equivalent to

$$\lim_{\tau \downarrow 0} \bar{v}_q(\tau)\bar{g}_q(\tau) \in (0, \infty). \quad (5.11)$$

Substituting in (3.71) the gauge functions  $q_1(\tau)$  and  $q_2(\tau)$  by  $\tau^{\nu_1}$  and  $\tau^{\nu_2} \left( \log \frac{c}{\tau} \right)^\delta$ , respectively, we obtain

$$\bar{v}_q(\tau) = \nu_1^{-1} \int_{\tau}^{cI, J} \rho^{-D + \frac{d_1}{\nu_1}} (q_2^{-1}(\rho))^{d_2 - \nu_2} \left( \log \frac{c}{q_2^{-1}(\rho)} \right)^{1 - \delta} \left( \nu_2 \log \frac{c}{q_2^{-1}(\rho)} - \delta \right)^{-1}.$$

Computing the derivative of the reciprocal of  $\bar{g}_q$ , we see that

$$\begin{aligned} \frac{d}{d\tau} \left( (\bar{g}_q(\tau))^{-1} \right) &= \tau^{\frac{d_1}{\nu_1} - D - 1} (q_2^{-1}(\tau))^{d_2 - 1} \left[ \left( \frac{d_1}{\nu_1} - D \right) q_2^{-1}(\tau) \right. \\ &\quad \left. + d_2 \tau (q_2^{-1}(\tau))^{1 - \nu_2} \left( \log \frac{c}{q_2^{-1}(\tau)} \right)^{1 - \delta} \left( \nu_2 \log \frac{c}{q_2^{-1}(\tau)} - \delta \right)^{-1} \right]. \end{aligned} \quad (5.12)$$

Apply the L'Hospital's rule to obtain

$$\lim_{\tau \downarrow 0} [\bar{v}_q(\tau)\bar{g}_q(\tau)]^{-1} = \lim_{\tau \downarrow 0} \frac{\frac{d}{d\tau} \left( (\bar{g}_q(\tau))^{-1} \right)}{\frac{d\bar{v}_q}{d\tau}(\tau)} = \lim_{\tau \downarrow 0} (L_1(\tau) + L_2(\tau)),$$

where using (5.12), we have

$$L_1(\tau) = \frac{\tau^{\frac{d_1}{\nu_1}-D-1} (q_2^{-1}(\tau))^{d_2} \left(\frac{d_1}{\nu_1} - D\right)}{\frac{d\bar{v}_q}{d\tau}(\tau)},$$

$$L_2(\tau) = \frac{d_2 \tau^{\frac{d_1}{\nu_1}-D} (q_2^{-1}(\tau))^{d_2-\nu_2} \left(\log \frac{c}{q_2^{-1}(\tau)}\right)^{1-\delta} \left(\nu_2 \log \frac{c}{q_2^{-1}(\tau)} - \delta\right)^{-1}}{\frac{d\bar{v}_q}{d\tau}(\tau)}.$$

Since  $\frac{1}{2}\nu_2 \log \frac{c}{q_2^{-1}(\tau)} \leq \nu_2 \log \frac{c}{q_2^{-1}(\tau)} - \delta \leq \nu_2 \log \frac{c}{q_2^{-1}(\tau)}$ , as  $\tau \downarrow 0$ , we find:

$$\lim_{\tau \downarrow 0} L_1(\tau) = D\nu_1\nu_2 - d_1\nu_2, \quad \lim_{\tau \downarrow 0} L_2(\tau) = -d_2\nu_1.$$

Consequently,

$$\lim_{\tau \downarrow 0} \bar{v}_q(\tau)\bar{g}_q(\tau) = (D\nu_1\nu_2 - (d_1\nu_2 + d_2\nu_1))^{-1}.$$

This implies (5.11).

The proof of the lemma is complete.  $\square$

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