STABLE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS ARE SMOOTH UP TO DIMENSION 9

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Abstract. In this paper we prove the following long-standing conjecture: *stable solutions to semilinear elliptic equations are bounded (and thus smooth) in dimension* $n \leq 9$.

This result, that was only known to be true for $n \leq 4$, is optimal: $\log(1/|x|^2)$ is a $W^{1,2}$ singular stable solution for $n \geq 10$.

The proof of this conjecture is a consequence of a new universal estimate: we prove that, in dimension $n \leq 9$, stable solutions are bounded in terms only of their L^1 norm, independently of the nonlinearity. In addition, in every dimension we establish a higher integrability result for the gradient and optimal integrability results for the solution in Morrey spaces.

As one can see by a series of classical examples, all our results are sharp. Furthermore, as a corollary we obtain that extremal solutions of Gelfand problems are $W^{1,2}$ in every dimension and they are smooth in dimension $n \leq 9$. This answers to two famous open problems posed by Brezis and Brezis-Vázquez.

Contents

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1. Introduction

Given $\Omega \subset \mathbb{R}^n$ a bounded domain and $f : \mathbb{R} \to \mathbb{R}$, we consider $u : \Omega \to \mathbb{R}$ a solution to the semilinear equation

$$-\Delta u = f(u) \qquad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.1}$$

If we define $F(t) := \int_0^t f(s)\,ds$, then (1.1) corresponds to the Euler-Lagrange equation for the energy functional

$$\mathcal{E}[u] := \int_\Omega \Big(\frac{|\nabla u|^2}{2} - F(u) \Big)\, dx.$$

In other words, u is a critical point of $\mathcal{E}$, namely

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{E}[u+\epsilon\xi] = 0 \qquad \text{for all } \xi \in C_c^\infty(\Omega)$$

(the space of C^∞ functions with compact support in Ω). Consider the second variation of $\mathcal{E}$, that when $f \in C^1$ is given by

$$\frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \mathcal{E}[u+\epsilon\xi] = \int_\Omega \Big(|\nabla \xi|^2 - f'(u)\xi^2 \Big)\, dx.$$

Then, one says that u is a stable solution of equation (1.1) in Ω if the second variation is nonnegative, namely

$$\int_\Omega f'(u)\xi^2\, dx \le \int_\Omega |\nabla \xi|^2\, dx \qquad \text{for all } \xi \in C_c^\infty(\Omega).$$

Note that stability of u is considered within the class of functions agreeing with u near the boundary of Ω.

Our interest is in nonnegative nonlinearities f that grow at $+\infty$ faster than linearly. In this case it is well-known that, independently of the Dirichlet boundary conditions that one imposes on (1.1), the energy $\mathcal{E}$ admits no absolute minimizer.[1] However, we will see that in many instances there exist nonconstant stable solutions, such as local minimizers. The regularity of stable solutions to semilinear elliptic equations is a very classical topic in elliptic equations, initiated in the seminal paper of Crandall and Rabinowitz [15], which has given rise to a huge literature on the topic; see the monograph [18] for an extensive list of results and references.

Note that this question is a PDE analogue of another fundamental problem in mathematics, namely the regularity of stable minimal surfaces. As it is well known, stable minimal surfaces in $\mathbb{R}^n$ may not be smooth in dimension n larger than 7 [31, 2], and it is a fundamental open problem whether they are smooth in dimension $n \le 7$. Up to now this question has been solved only in dimension $n = 3$ by Fischer-Colbrie and Schoen [19] and Do Carmo and Peng [17].

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[1] To see this, take $v \in C_c^1(\Omega)$ with $v \ge 0$ and $v \not\equiv 0$, and given $M > 0$ consider

$$\mathcal{E}[u+Mv] = \frac{1}{2}\int_\Omega \big|\nabla(u+Mv)\big|^2\, dx - \int_\Omega F(u+Mv)\, dx.$$

Since f grows superlinearly at $+\infty$, it follows that $F(t) \gg t^2$ for t large. This leads to $\mathcal{E}[u+Mv] \to -\infty$ as $M \to +\infty$, which shows that the infimum of the energy among all functions with the same boundary data as u is $-\infty$.

Note that, also in our PDE problem, the dimension plays a key role. Indeed, when

$$n \geq 10, \quad u = \log \frac{1}{|x|^2}, \quad \text{and} \quad f(u) = 2(n-2)e^u, \tag{1.2}$$

we are in the presence of a singular $W_0^{1,2}(B_1)$ stable solution of (1.1) in $\Omega = B_1$ —as easily shown using Hardy's inequality. On the other hand,

- if $f(t) = e^t$ or $f(t) = (1+t)^p$ with $p > 1$,
- or more in general if $f \in C^2$ is positive, increasing, convex, and $\lim_{t\to+\infty} \frac{f(t)f''(t)}{f'(t)^2}$ exists[2],

then it is well-known since the 1970's that $W_0^{1,2}(\Omega)$ stable solutions are bounded (and therefore smooth, by classical elliptic regularity theory [21]) when $n \leq 9$, see [15]. Notice that among general solutions (not necessarily stable), an L^∞ bound only holds for subcritical and critical nonlinearities.[3]

All these results motivated the following long-standing[4]

Conjecture: *Let $u \in W_0^{1,2}(\Omega)$ be a stable solution to* (1.1). *Assume that f is positive, nondecreasing, convex, and superlinear at $+\infty$, and let $n \leq 9$. Then u is bounded.*

In the last 25 years, several attempts have been made in order to prove this result. In particular, partial positive answers to the conjecture above have been given (chronologically):

- by Nedev, when $n \leq 3$ [26];
- by Cabré and Capella when $\Omega = B_1$ and $n \leq 9$ [8];
- by Cabré when $n = 4$ and Ω is convex [6];
- by Villegas when $n = 4$ [34];
- by Cabré and Ros-Oton when $n \leq 7$ and Ω is a convex domain "of double revolution" [11];
- by Cabré, Sanchón, and Spruck when $n = 5$ and $\limsup_{t\to+\infty} \frac{f'(t)}{f(t)^{1+\varepsilon}} < +\infty$ for every $\varepsilon > 0$ [12].

The aim of this paper is to give a full proof of the conjecture stated above. Actually, as we shall see below, the interior boundedness of solutions requires no convexity or monotonicity of f. This fact was only known in dimension $n \leq 4$, by a result of the first author [6]. In addition, even more surprisingly, both in the interior and in the global settings we can prove that $W^{1,2}$ *stable solutions are universally bounded for $n \leq 9$*, namely they are bounded in terms only of their L^1 norm, with a constant that is independent of the nonlinearity f.

[2]The existence of the limit $c := \lim_{t\to+\infty} \frac{f(t)f''(t)}{f'(t)^2} \geq 0$ is a rather strong assumption. Indeed, as noticed in [15], if it exists then necessarily $c \leq 1$ (otherwise f blows-up in finite time). Now, when $c = 1$ the result follows by [15, Theorem 1.26], while $c < 1$ implies that $f(t) \leq C(1+t)^p$ for some p and then the result follows by [15, Lemma 1.17].

[3]We recall that a nonlinearity f is called subcritical (resp. critical/supercritical) if $|f(t)| \leq C(1+|t|)^p$ for some $p < \frac{n+2}{n-2}$ (resp. $p = \frac{n+2}{n-2}$/resp. $p > \frac{n+2}{n-2}$). While solutions to subcritical and critical equations are known to be bounded, in the supercritical case one can easily construct radially decreasing unbounded $W^{1,2}$ solutions.

[4]As we shall explain in Section 1.2, this conjecture is strongly related to an open problem stated by Brezis in the context of "extremal solutions" in [3].

1.1. **Main results.** In order to prove our result on the regularity of stable solutions up to the boundary we will be forced to work with nonlinearities f that are only locally Lipschitz (and not necessarily C^1). Hence, it is important for us to extend the definition of stability to this class of nonlinearities. For this, we need to choose a precise representative for f'.

Definition 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function, and let $u \in W^{1,2}(\Omega)$ be a weak solution to (1.1), in the sense that $f(u) \in L^1_{\rm loc}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f(u)\varphi \, dx \quad \text{for all } \varphi \in C^\infty_c(\Omega). \tag{1.3}$$

Then, we say that u is a stable solution in Ω if $f'_-(u) \in L^1_{\rm loc}(\Omega)$ and

$$\int_\Omega f'_-(u)\xi^2 \, dx \le \int_\Omega |\nabla \xi|^2 \, dx \qquad \text{for all } \xi \in C^\infty_c(\Omega), \tag{1.4}$$

where f'_- is defined as

$$f'_-(t) := \liminf_{h\to 0} \frac{f(t+h) - f(t)}{h} \qquad \text{for } t \in \mathbb{R}. \tag{1.5}$$

As we shall see later, in our proofs we only use (1.4) with test functions ξ that vanish in the set $\{|\nabla u| = 0\}$. Hence, as a consequence of Lemma A.3(i), in this situation the notion of stability is independent of the particular representative chosen for f'.

Our first main result provides a universal interior a priori bound on the C^α norm of solutions when $n \le 9$. Actually, in every dimension we can prove also a higher integrability result for the gradient (with respect to the natural energy space $W^{1,2}$). Since the result is local, we state it in the unit ball. Also, because stable solutions u can be approximated by smooth ones (at least when $u \in W^{1,2}_0(\Omega)$ and f is convex; see [18, Section 3.2.2]), we shall state the result as an a priori bound assuming that u is smooth.

Theorem 1.2. *Let B_1 denote the unit ball of $\mathbb{R}^n$. Assume that $u \in C^2(B_1)$ is a stable solution of*

$$-\Delta u = f(u) \quad \text{in } B_1,$$

with $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz and nonnegative.

Then

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \le C\|u\|_{L^1(B_1)}, \tag{1.6}$$

where $\gamma > 0$ and C are dimensional constants. In addition, if $n \le 9$ then

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \le C\|u\|_{L^1(B_1)}, \tag{1.7}$$

where $\alpha > 0$ and C are dimensional constants.

Remark 1.3. As mentioned before, it is remarkable that the interior estimates hold with bounds that are *independent* of the nonlinearity f. Note that, also in the global regularity result Theorem 1.5, we can prove a bound independent of f.

Combining the previous interior bound with the moving planes method, we obtain a universal bound on u when Ω is convex.

Corollary 1.4. *Let $n \le 9$ and let $\Omega \subset \mathbb{R}^n$ be any bounded convex C^1 domain. Assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and nonnegative. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a stable solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists a constant C, depending only on Ω, such that

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{L^1(\Omega)}. \tag{1.8}$$

We now state our second main result, which concerns the global regularity of stable solutions in general C^3 domains when the nonlinearity is convex and nondecreasing. As we shall explain in the next section, this result completely solves two open problems posed by Brezis and Brezis-Vázquez in [3, 5]. Again, we work with classical solutions and prove an a priori estimate. In this case it is crucial for us to assume f to be convex and nondecreasing. Indeed, the proof of regularity up to the boundary will rely on a very general closedness result for stable solutions with convex nondecreasing nonlinearities, that we prove in Section 4.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is nonnegative, nondecreasing, and convex. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a stable solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \le C\|u\|_{L^1(\Omega)}, \tag{1.9}$$

where $\gamma > 0$ is a dimensional constant and C depends only on Ω. In addition, if $n \le 9$ then

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C\|u\|_{L^1(\Omega)}, \tag{1.10}$$

where $\alpha > 0$ is a dimensional constant and C depends only on Ω.

As an immediate consequence of such a priori estimates, we will prove the long-standing conjecture stated above.

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain of class C^3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is nonnegative, nondecreasing, convex, and satisfies*

$$\frac{f(t)}{t} \ge \sigma(t) \longrightarrow +\infty \quad \text{as} \quad t \to +\infty$$

for some function $\sigma : \mathbb{R} \to \mathbb{R}$. Let $u \in W^{1,2}_0(\Omega)$ be any stable weak solution of (1.1) and assume that $n \le 9$. Then

$$\|u\|_{L^\infty(\Omega)} \le C,$$

where C is a constant depending only on σ and Ω.

The key point here is to prove the bounds for classical solutions (Theorem 1.5). Once this is done, a well known approximation argument (see [18, Theorem 3.2.1 and Corollary 3.2.1]) shows that the same bounds (1.9)-(1.10) hold for every $W^{1,2}_0(\Omega)$ stable weak solution u. Finally, to control $\|u\|_{L^1(\Omega)}$ in (1.9), we use Proposition B.1.

1.2. **Application: $W_0^{1,2}$ and L^∞ regularity of extremal solutions.** Let $f : [0, +\infty) \to \mathbb{R}$ satisfy $f(0) > 0$ and be nondecreasing, convex, and superlinear at $+\infty$ in the sense that

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty.$$

Given a constant $\lambda > 0$ consider the nonlinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. We say that u is a classical solution if $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$.

In the literature, this problem is usually referred to as the "Gelfand problem", or a "Gelfand-type problem". It was first presented by Barenblatt in a volume edited by Gelfand [20], and was motivated by problems occurring in combustion[5]. Later, it was studied by a series of authors; see for instance [3, 5, 18, 7] for a complete account on this topic.

The basic results concerning (1.11) can be summarized as follows (see for instance [3, Theorem 1 and Remark 1] or the book [18] by Dupaigne):

Theorem 1.7 (see [3, 5, 18])**.** *There exists a constant $\lambda^\star \in (0, +\infty)$ such that:*

(i) *For every $\lambda \in (0, \lambda^\star)$ there is a unique $W_0^{1,2}(\Omega)$ stable solution u_λ of* (1.11)*. Also, u_λ is a classical solution and $u_\lambda < u_{\lambda'}$ for $\lambda < \lambda'$.*
(ii) *For every $\lambda > \lambda^\star$ there is no classical solution.*
(iii) *For $\lambda = \lambda^\star$ there exists a unique L^1-weak solution $u^\star$, in the following sense: $u^\star \in L^1(\Omega)$, $f(u^\star)\mathrm{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$, and*

$$-\int_\Omega u^\star \Delta\zeta\, dx = \lambda^\star \int_\Omega f(u^\star)\zeta\, dx \qquad \text{for all } \zeta \in C^2(\overline{\Omega}) \text{ with } \zeta_{|\partial\Omega} = 0.$$

This solution is called the extremal solution *of* (1.11) *and satisfies $u_\lambda \uparrow u^\star$ as $\lambda \uparrow \lambda^\star$.*

The uniqueness of weak solution for $\lambda = \lambda^\star$ is a delicate result that was proved by Martel [25].

In [3, Open problem 1], Brezis asked the following:

Open problem 1: *Is there something "sacred" about dimension 10? More precisely, is it possible in "low" dimensions to construct some f (and some Ω) for which the extremal solution $u^\star$ is unbounded? Alternatively, can one prove in "low" dimension that $u^\star$ is smooth for every f and every Ω?*

To connect this to the conjecture stated before, note that Brezis' problem can be thought as an a priori bound for the stable solutions $\{u_\lambda\}_{\lambda<\lambda^\star}$. Hence, understanding the regularity of extremal solutions is equivalent to understanding a priori estimates for stable classical solutions.

[5]Originally, Barenblatt introduced problem (1.11) for the exponential nonlinearity $f(u) = e^u$ (arising as an approximation of a certain empirical law). Nowadays, the terminology of Gelfand or Gelfand-type problem applies to all f satisfying the assumptions above.

Note that, a priori, extremal solutions are merely in $L^1(\Omega)$. In particular, there is no classical notion of trace for $u^\star$ on $\partial\Omega$. It would be therefore desirable to know whether extremal solutions do belong to the natural energy space $W^{1,2}_0(\Omega)$. This important question was posed by Brezis and Vázquez in [5, Open problem 1]:

Open problem 2: *Does there exist some f and Ω for which the extremal solution is a weak*[6] *solution not in $W^{1,2}_0(\Omega)$?*

Concerning this problem, it has been proved that $u^\star$ belong to the energy space $W^{1,2}_0(\Omega)$ when $n \le 5$ by Nedev [26], for every n when Ω is convex also by Nedev [27], and finally when $n = 6$ by Villegas [34]. Here we prove that $u^\star \in W^{1,2}_0(\Omega)$ for every n and for every smooth domain Ω, thus giving a conclusive answer also to this second open problem.

Note that, thanks to the superlinearity of f, it follows by Proposition B.1 that the $L^1(\Omega)$ norms of the functions $\{u_\lambda\}_{\lambda<\lambda^\star}$ are uniformly bounded by a constant depending only on f and Ω. Hence, by applying Theorem 1.5 to the functions $\{u_\lambda\}_{\lambda<\lambda^\star}$ and letting $\lambda \uparrow \lambda^\star$, we immediately deduce that extremal solutions are always $W^{1,2}$ (actually even $W^{1,2+\gamma}$) in every dimension, and that they are universally bounded (and hence smooth) in dimension $n \le 9$. We summarize this in the following:

Corollary 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^3. Assume that $f : [0,+\infty) \to (0,+\infty)$ is nondecreasing, convex, and superlinear at $+\infty$, and let $u^\star$ denote the extremal solution of* (1.11)*.*

Then $u^\star \in W^{1,2+\gamma}_0(\Omega)$ for some dimensional exponent $\gamma > 0$. In addition, if $n \le 9$ then $u^\star$ is bounded and it is therefore a classical solution.

1.3. **The case $\mathbf{n \ge 10}$.** In view of the results described in the previous sections, it is natural to ask what can one say about stable solutions in dimension $n \ge 10$. Our strategy of proof can be used to provide optimal (or perhaps almost optimal) integrability estimates in Morrey spaces in every dimension, as stated next (see Section 7 for more details and for Morrey estimates for the gradient of stable solutions).

Recall that Morrey norms are defined as

$$\|w\|^p_{M^{p,\beta}(\Omega)} := \sup_{y\in\overline{\Omega},\ r>0} r^{\beta-n}\int_{\Omega\cap B_r(y)} |w|^p\,dx,$$

for $p \ge 1$ and $\beta \in (0,n)$.

Theorem 1.9. *Let $u \in C^2(B_1)$ be a stable solution of*

$$-\Delta u = f(u) \quad \text{in } B_1 \subset \mathbb{R}^n,$$

with $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz and nonnegative. Assume that $n \ge 10$ and define

$$p_n := \begin{cases} \infty & \text{if } n = 10,\\ \frac{2(n-2\sqrt{n-1}-2)}{n-2\sqrt{n-1}-4} & \text{if } n \ge 11. \end{cases} \tag{1.12}$$

Then

$$\|u\|_{M^{p,2+\frac{4}{p-2}}(B_{1/2})} \le C\|u\|_{L^1(B_1)} \qquad \text{for every} \quad p < p_n, \tag{1.13}$$

[6]In the sense of Theorem 1.7(iii).

where C depends only on n and p.

In addition, if $\Omega\subset\mathbb{R}^n$ is a bounded domain of class C^3 and $u\in C^0(\overline{\Omega})\cap C^2(\Omega)$ is a stable solution of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

then

$$\|u\|_{M^{p,2+\frac{4}{p-2}}(\Omega)} \le C\|u\|_{L^1(\Omega)} \qquad \text{for every} \quad p<p_n, \tag{1.14}$$

for some constant C depending only on p and Ω.

It is interesting to observe that the above result is essentially optimal. To see this we recall that, in dimension $n=10$, the function $u=\log(1/|x|^2)$ is an unbounded $W_0^{1,2}(B_1)$ stable solution in B_1 (see (1.2), and recall that it can be approximated by stable classical solutions by [18, Section 3.2.2]). Also, as shown in [5], for $n\ge 11$ the function $u(x)=|x|^{-2/(q_n-1)}-1$ is the extremal solution of

$$\begin{cases} -\Delta u = \lambda^\star(1+u)^{q_n} & \text{in } B_1\\ u>0 & \text{in } B_1\\ u=0 & \text{on } \partial B_1,\end{cases} \tag{1.15}$$

with $\lambda^\star=\frac{2}{q_n-1}\big(n-2-\frac{2}{q_n-1}\big)$ and $q_n:=\frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4}$. In particular, it is easy to see that $u\in M^{p,2+\frac{4}{p-2}}(B_{1/2})$ if and only if $p\le p_n$. It is an open question whether (1.13) holds with $p=p_n$ for a general stable solution u.

1.4. **Idea of the proofs.** The starting point is the stability inequality for u, i.e.,

$$\int_{B_1} f'(u)\xi^2\,dx \le \int_{B_1}|\nabla\xi|^2\,dx \qquad \text{for all } \xi\in C_c^\infty(B_1). \tag{1.16}$$

In order to get a strong information on u, one has to choose an appropriate test function ξ in (1.16). Most of the papers on this topic (including those of Crandall-Rabinowitz [15] and Nedev [26]) have considered $\xi=h(u)$ for some appropriate function h depending on the nonlinearity f. The main idea in the L^∞ estimate of [6] for $n\le 4$ was to take, instead, $\xi=|\nabla u|\varphi(u)$, and choose then a certain φ depending on the solution u itself.

Here, a first key idea in our proofs is to take a test function of the form

$$\xi=(x\cdot\nabla u)|x|^{(2-n)/2}\zeta,$$

with $0\le\zeta\le 1$ a smooth cut-off function equal to 1 in B_ρ and vanishing outside $B_{3\rho/2}$. Thanks to this, we can prove the following inequality (see Lemma 2.1): there exists a dimensional constant C such that

$$(n-2)(10-n)\int_{B_\rho}|x|^{-n}|x\cdot\nabla u|^2\,dx \le C\rho^{2-n}\int_{B_{3\rho/2}\setminus B_\rho}|\nabla u|^2\,dx \qquad \text{for all } 0<\rho<\tfrac{2}{3}. \tag{1.17}$$

From this inequality we see immediately that for $3\le n\le 9$ we get a highly nontrivial information. While of course one can always assume that $n\ge 3$ (if $n\le 2$ it suffices to add some superfluous variables to reduce to the case $n=3$), here we see that the assumption $n\le 9$ is crucial.

Thus, when $n\le 9$, the above inequality tells us that the radial derivative of u in a ball is controlled by the total gradient in an annulus. Still, it is important to notice that (1.17) does *not* lead to an L^∞ bound for general solutions u to $-\Delta u = f(u)$ in dimensions $n \le 9$.[7] Thus, we still need to use stability again in a crucial way.

If we could prove that for stable solutions the radial derivative $x\cdot\nabla u$ and the total derivative ∇u have comparable size in L^2 at every scale, then we could control the right hand side of (1.17) with $\int_{B_{3\rho/2}\setminus B_\rho} |x|^{-n}|x\cdot\nabla u|^2\,dx$. This would imply that

$$\int_{B_\rho} |x|^{-n}|x\cdot\nabla u|^2\,dx \le C\int_{B_{3\rho/2}\setminus B_\rho} |x|^{-n}|x\cdot\nabla u|^2\,dx,$$

and by a suitable iteration and covering argument we could conclude that $u \in C^\alpha$. This is indeed the core of our interior argument: we show that the radial derivative and the total derivative have comparable size in L^2 (at least whenever the integral of $|\nabla u|^2$ on balls enjoys a doubling property; see Lemma 3.1). This is based on a delicate compactness argument, which relies on a series of a priori estimates:

(1) curvature-type estimates for the level sets of u, which follow by taking $\xi = |\nabla u|\eta$ as a test function in the stability inequality; see Lemma 2.3;
(2) the higher $L^{2+\gamma}$ integrability of the gradient, which follows from (1) and a suitable Dirichlet energy estimate, (2.13), on each level set of u; see Proposition 2.4;
(3) a general compactness argument for superharmonic functions; see Lemma A.1;
(4) the non-existence of nontrivial 0-homogeneous superharmonic functions; see the proof of Lemma 3.1.

Combining all these ingredients, we prove Theorem 1.2.

For the boundary estimate we would like to repeat the interior argument described above near a boundary point. We note that, whenever the boundary is completely flat and contains the origin, since $x\cdot\nabla u$ vanishes on the flat boundary then one can still use the test function $\xi = (x\cdot\nabla u)|x|^{(2-n)/2}\zeta$ to deduce the analogue of (1.17). Actually, a suitable variant of this test function allows us to obtain a similar estimate even when the boundary is C^3-close to a hyperplane (see Lemma 6.2). In addition, when the boundary is C^3-close to a hyperplane, we are able to prove the higher $L^{2+\gamma}$ integrability of the gradient near the boundary (see Proposition 5.2), and from there we can conclude that the $W^{1,2}$ norm near the boundary can be controlled only in terms of the L^1 norm (see Proposition 5.5).

Unfortunately, even if the boundary is completely flat, one cannot repeat the argument used in the interior case to deduce that the radial derivative controls the total gradient near a boundary point —which was a crucial point in the interior case. Indeed, while in the interior case the proof relied on the non-existence of nontrivial 0-homogeneous superharmonic functions in a neighborhood of the origin (see the proof of Lemma 3.1), in the boundary case such superharmonic functions may exist! Hence, in this case we need to exploit in a stronger way the fact that u solves a semilinear equation (and not simply that u is superharmonic since $f \ge 0$). However, since our arguments are based on a compactness technique, we need bounds that are independent of the nonlinearity f.

A new key ingredient here is presented in Section 4: we are able to prove that, whenever the nonlinearity is convex and nondecreasing —but possibly taking the value $+\infty$ in an interval $[M,\infty)$— the class of stable solutions is closed under L^1_{loc} convergence (see Theorem 4.1).

[7]This can be seen by taking functions u in $\mathbb{R}^3$ depending only on two variables; see Remark 2.2.

Note that this is particularly striking, since no compactness assumptions are made on the nonlinearities!

With this powerful compactness theorem at hand, we are able to reduce ourself to a flat-boundary configuration, control the gradient by its radial component, and prove Theorem 1.5.

Finally, the case $n \geq 11$ is obtained by choosing the test function $\xi = (x \cdot \nabla u)|x|^{-a/2}\zeta$, where $a = a_n \in (0, n-2)$ are suitable exponents, while in the case $n = 10$ we choose $\xi = (x \cdot \nabla u)|x|^{-4}\big|\log|x|\big|^{-\delta/2}\zeta$, with $\delta > 0$.

The techniques and ideas introduced in this paper are robust enough to be used for proving analogues of our results in other nonlinear problems. This is done in a series of forthcoming works by Miraglio, Sanchón, and the first author [10] for the p-Laplacian, and by Sanz-Perela and the first author [13] for the fractional Laplacian.

1.5. **Structure of the paper.** In Section 2 we exploit the stability of u and choose a series of different test functions to deduce inequality (1.17) as well as a universal $W^{1,2+\gamma}$ bound in terms only of the L^1 norm of the solution. This is used in Section 3 to prove our interior estimate of Theorem 1.2.

In Section 4 we prove that the class of stable solutions with convex nondecreasing nonlinearities is closed in $L^1_{\rm loc}$, while in Section 5 we obtain a $W^{1,2+\gamma}$ bound near the boundary in terms of the L^1 norm when $\partial\Omega$ is a small C^3-deformation of a hyperplane. These results are used in Section 6 to prove Theorem 1.5 via a blow-up and covering argument.

Finally, in Section 7 we deal with the case $n \geq 10$ and prove Theorem 1.9.

In the appendices we collect a series of technical lemmata and we show a classical a priori estimate on the L^1 norm of solutions to Gelfand problems.

2. Interior $W^{1,2+\gamma}$ estimate

In this section we begin by proving a series of interior estimates that follow by choosing suitable test functions in the stability inequality. Then we show a universal $W^{1,2+\gamma}$ bound in terms only of the L^1 norm of the solution. This is done by first controlling $\|\nabla u\|_{L^{2+\gamma}}$ by $\|\nabla u\|_{L^2}$, and then $\|\nabla u\|_{L^2}$ by $\|u\|_{L^1}$.

Here and in the sequel, we shall use subscripts to denote partial derivatives (i.e., $u_i = \partial_i u$, $u_{ij} = \partial_{ij} u$, etc.).

As mentioned in the introduction, our first key estimate for stable solutions comes from considering the test function $\xi = (x \cdot \nabla u)\eta$, and then take $\eta = |x|^{(2-n)/2}\zeta$ for some cut-off function ζ. We split the computations in two steps since this will be useful in the sequel.

We denote by $C^{0,1}_c(B_1)$ the space of Lipschitz functions with compact support in B_1.

Lemma 2.1. *Let $u \in C^2(B_1)$ be a stable solution of $-\Delta u = f(u)$ in $B_1 \subset \mathbb{R}^n$, with f locally Lipschitz. Then, for all $\eta \in C^{0,1}_c(B_1)$ we have*

$$\int_{B_1} \Big(\big\{(n-2)\eta + 2x \cdot \nabla \eta\big\}\eta\,|\nabla u|^2 - 2(x \cdot \nabla u)\nabla u \cdot \nabla(\eta^2) - |x \cdot \nabla u|^2 |\nabla \eta|^2 \Big)\, dx \leq 0. \quad (2.1)$$

As a consequence, for all $\zeta \in C^{0,1}_c(B_1)$ we have

$$
\begin{aligned}
\frac{(n-2)(10-n)}{4} &\int_{B_1} |x|^{-n}|x\cdot\nabla u|^2\zeta^2\,dx \\
&\le \int_{B_1}(-2)|x|^{2-n}|\nabla u|^2\zeta(x\cdot\nabla\zeta)\,dx + \int_{B_1} 4|x|^{2-n}(x\cdot\nabla u)\zeta\,\nabla u\cdot\nabla\zeta\,dx \\
&\quad + \int_{B_1}(2-n)|x|^{-n}|x\cdot\nabla u|^2\zeta(x\cdot\nabla\zeta)\,dx + \int_{B_1}|x|^{2-n}|x\cdot\nabla u|^2|\nabla\zeta|^2\,dx.
\end{aligned}
\tag{2.2}
$$

In particular, if $3 \le n \le 9$, then for all $\rho < 2/3$ it holds

$$
\int_{B_\rho} |x|^{-n}|x\cdot\nabla u|^2\,dx \le C\rho^{2-n}\int_{B_{3\rho/2}\setminus B_\rho} |\nabla u|^2\,dx, \tag{2.3}
$$

where C is a dimensional constant.

Proof. We split the proof in three steps.

Step 1: *Proof of* (2.1). We note that, by approximation, (1.4) holds for all $\xi \in C^{0,1}_c(B_1)$. Hence, we can consider as test function in (1.4) a function of the form $\xi = \mathbf{c}\eta$, where $\mathbf{c} \in W^{2,p}_{\rm loc}(B_1)$ for some $p > n$, and $\eta \in C^{0,1}_c(B_1)$. Then, a simple integration by parts gives that

$$
\int_{B_1}\big(\Delta\mathbf{c} + f'_-(u)\mathbf{c}\big)\mathbf{c}\,\eta^2\,dx \le \int_{B_1}\mathbf{c}^2\,|\nabla\eta|^2\,dx. \tag{2.4}
$$

We now choose $\mathbf{c}(x) := x\cdot\nabla u(x)$ (this function belongs to $W^{2,p}_{\rm loc}(B_1)$ for every $p < \infty$ by Lemma A.3(ii)). Then, by a direct computation and using Lemma A.3(ii) again, we deduce that

$$
\Delta\mathbf{c} = x\cdot\nabla\Delta u + 2\sum_{i=1}^n u_{ii} = -f'_-(u)\mathbf{c} + 2\Delta u
$$

a.e. in B_1. Hence, substituting this identity in (2.4) we get

$$
\begin{aligned}
\int_{B_1}|x\cdot\nabla u|^2\,|\nabla\eta|^2\,dx &\ge \int_{B_1}\big(\Delta\mathbf{c}+f'_-(u)\mathbf{c}\big)\mathbf{c}\,\eta^2\,dx = 2\int_{B_1}(x\cdot\nabla u)\Delta u\,\eta^2\,dx \\
&= \int_{B_1}\Big(\mathrm{div}\big(2(x\cdot\nabla u)\nabla u - |\nabla u|^2x\big) + (n-2)|\nabla u|^2\Big)\eta^2\,dx \\
&= \int_{B_1}\Big(-2(x\cdot\nabla u)\nabla u\cdot\nabla(\eta^2) + |\nabla u|^2x\cdot\nabla(\eta^2) + (n-2)|\nabla u|^2\eta^2\Big)\,dx,
\end{aligned}
$$

and (2.1) follows.

Step 2: *Proof of* (2.2). Given $a < n$, we would like to take the function $\eta := |x|^{-a/2}\zeta$ with $\zeta \in C^{0,1}_c(B_1)$ as a test function in (2.1). Since, η is not Lipschitz for $a > 0$, we approximate it by the $C^{0,1}_c(B_1)$ function

$$
\eta_\varepsilon := \min\{|x|^{-a/2}, \varepsilon^{-a/2}\}\zeta
$$

for $\varepsilon \in (0,1)$, which agrees with η in $B_1 \setminus B_\varepsilon$. We have that $\eta_\varepsilon \to \eta$ and $\nabla\eta_\varepsilon \to \nabla\eta$ a.e. in B_1 as $\varepsilon \downarrow 0$. At the same time, every term in (2.1) with η replaced by η_ε is bounded in absolute value by $C|x|^{-a}|\nabla u|^2 \le \widetilde{C}|x|^{-a} \in L^1_{\rm loc}(B_1)$ (since $u \in C^2(B_1)$). Hence, the dominated convergence theorem gives that (2.1) also holds with $\eta := |x|^{-a/2}\zeta$.

Now, noticing that

$$x\cdot\nabla\eta = -\frac{a}{2}|x|^{-a/2}\zeta + |x|^{-a/2}x\cdot\nabla\zeta, \quad \nabla(\eta^2) = -a|x|^{-a-2}\zeta^2 x + 2|x|^{-a}\zeta\nabla\zeta \tag{2.5}$$

and

$$|\nabla\eta|^2 = \left|-\frac{a}{2}|x|^{-a/2-2}\zeta x + |x|^{-a/2}\nabla\zeta\right|^2 = \frac{a^2}{4}|x|^{-a-2}\zeta^2 + |x|^{-a}|\nabla\zeta|^2 - a|x|^{-a-2}\zeta(x\cdot\nabla\zeta), \tag{2.6}$$

(2.2) follows from (2.1) by choosing $a = n-2$.

Step 3: *Proof of* (2.3). Given $\rho \in (0, 2/3)$, we consider a Lipschitz function ζ, with $0 \le \zeta \le 1$, such that $\zeta_{|B_\rho} = 1$, $\zeta_{|\mathbb{R}^n\setminus B_{3\rho/2}} = 0$, and $|\nabla\zeta| \le C/\rho$. Using this function in (2.2) and noticing that $|x|$ is comparable to ρ inside $\mathrm{supp}(\nabla\zeta) \subset \overline{B}_{3\rho/2}\setminus B_\rho$, the result follows easily. □

Remark 2.2. To deduce our L^∞ estimate from (2.3), we will need to use again the stability of u. In fact, there exist $W^{1,2}$ weak solutions of semilinear equations (with $f > 0$) which satisfy (2.3) (in balls $B_\rho = B_\rho(y)$ centered at any point $y \in B_1(0)$) and are unbounded.

For instance, with $n = 3$ take $u(x_1, x_2, x_3) = \widetilde{u}(x_1, x_2)$, where $\widetilde{u}$ is unbounded but belongs to $W^{1,2}_{\rm loc}(\mathbb{R}^2)$. One can then verify that (2.3) holds inside every ball $B_\rho = B_\rho(y)$. At the same time, by taking $\widetilde{u}$ to be radially decreasing in $\mathbb{R}^2$, we can guarantee that $\widetilde{u}$ solves a semilinear equation (and hence also u) for some nonlinearity f. An example is $\widetilde{u}(\rho) = \log|\log\rho|$ in a small neighborhood of the origin, which leads to a smooth nonlinearity $f > 0$.

The key point to deduce boundedness from (2.3) will be a higher $L^{2+\gamma}$ integrability result for the gradient of the solution, that we establish in the remaining of this section.

Towards this, we exploit again the stability of u by choosing now, as another test function, $\xi = |\nabla u|\eta$ with η a cut-off. In the case when $u \in C^3$ this choice of test function and the following lemma are due to Sternberg and Zumbrun [32]. We verify next that the result holds also when f is locally Lipschitz.

Lemma 2.3. *Let $u \in C^2(B_1)$ be a stable solution of $-\Delta u = f(u)$ in $B_1 \subset \mathbb{R}^n$, with f locally Lipschitz. Then, for all $\eta \in C^{0,1}_c(B_1)$ we have*

$$\int_{B_1}\mathcal{A}^2\eta^2 dx \le \int_{B_1}|\nabla u|^2|\nabla\eta|^2 dx,$$

where[8]

$$\mathcal{A} := \begin{cases} \left(\sum_{ij} u_{ij}^2 - \sum_i\left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2\right)^{1/2} & \text{if } \nabla u \ne 0 \\ 0 & \text{if } \nabla u = 0. \end{cases} \tag{2.7}$$

[8]Even though we will not use it here, it is worth noticing that the quantity $\mathcal{A}$ controls the second fundamental form of the level sets of u. This was crucially used in [6], in combination with the Sobolev-type inequality of Michael-Simons and Allard, to prove regularity of stable solutions up to dimension $n \le 4$.

When $u \in C^3$ (and $f \in C^1$), this follows from the stability inequality (1.16) plus the fact that

$$|\nabla u|\big(\Delta|\nabla u| + f'(u)|\nabla u|\big) = \mathcal{A}^2 \quad \text{in} \quad \{\nabla u \neq 0\};$$

see [6] for a proof. We give here an aternative proof that does not require to compute $\Delta|\nabla u|$.

Proof of Lemma 2.3. We begin from the identity

$$-\Delta u_i = f'_-(u)u_i \qquad \text{for} \quad i = 1, \dots, n;$$

see Lemma A.3(ii). Multiplying this identity by $u_i\eta^2$ and integrating by parts, we obtain

$$\int_{B_1} \Big(|\nabla(u_i\eta)|^2 - (u_i)^2|\nabla\eta|^2\Big)\,dx = \int_{B_1} \nabla u_i \cdot \nabla(u_i\eta^2)\,dx = \int_{B_1} f'_-(u)u_i^2\eta^2\,dx,$$

so that summing over i we get

$$\int_{B_1} \Big(\sum_i \big|\nabla(u_i\eta)\big|^2 - |\nabla u|^2|\nabla\eta|^2\Big)\,dx = \int_{B_1} f'_-(u)|\nabla u|^2\eta^2\,dx. \tag{2.8}$$

On the other hand, testing the stability inequality (1.4) with the Lipschitz function $|\nabla u|\eta$, we obtain

$$\int_{B_1} |\nabla(|\nabla u|\eta)|^2\,dx \geq \int_{B_1} f'_-(u)|\nabla u|^2\eta^2\,dx. \tag{2.9}$$

Hence, combining (2.8) with (2.9) gives

$$\int_{B_1} |\nabla u|^2|\nabla\eta|^2\,dx \geq \int_{B_1} \Big(\sum_i \big|\nabla(u_i\eta)\big|^2 - |\nabla(|\nabla u|\eta)|^2\Big)\,dx.$$

Then, a direct computation shows that, inside the set $\{\nabla u \neq 0\}$,

$$\sum_i \big|\nabla(u_i\eta)\big|^2 - |\nabla(|\nabla u|\eta)|^2 = \bigg(\sum_{i,j} u_{ij}^2 - \sum_i \Big(\sum_j \frac{u_{ij}u_j}{|\nabla u|}\Big)^2\bigg)\eta^2 = \mathcal{A}^2\eta^2.$$

On the other hand, since ∇u is Lipschitz, then $D^2u = 0$ a.e. in $\{\nabla u = 0\}$ (see, e.g., [33, Theorem 1.56]). Therefore $\sum_i \big|\nabla(u_i\eta)\big|^2 - |\nabla(|\nabla u|\eta)|^2 = 0$ a.e. inside $\{\nabla u = 0\}$, concluding the proof. □

Next we prove a general result that gives, in every dimension, a higher integrability result for the gradient of stable solutions.

Proposition 2.4. *Let $u \in C^2(B_1)$ be a stable solution of $-\Delta u = f(u)$ in $B_1 \subset \mathbb{R}^n$, with f locally Lipschitz and nonnegative. Then*

$$\|\nabla u\|_{L^{2+\gamma}(B_{3/4})} \leq C\|\nabla u\|_{L^2(B_1)},$$

where $\gamma > 0$ and C are dimensional constants.

Proof. Without loss of generality, we can assume that $\|\nabla u\|_{L^2(B_1)} = 1$ (this normalization will be particularly convenient in Step 3). Let $\eta \in C_c^\infty(B_1)$ be a nonnegative cut-off function with $\eta \equiv 1$ in $B_{3/4}$.

Step 1: *We show that*

$$\int_{B_1} \big|\mathrm{div}(|\nabla u|\,\nabla u)\big|\eta^2\,dx \le C. \tag{2.10}$$

Set $\nu := -\frac{\nabla u}{|\nabla u|}$ in the set $\{|\nabla u| \neq 0\}$, and $\nu = 0$ in $\{|\nabla u| = 0\}$. We begin from the pointwise identity

$$\mathrm{div}(|\nabla u|\,\nabla u) = |\nabla u|\left(\sum_{ij}\frac{u_{ij}u_iu_j}{|\nabla u|^2} + \Delta u\right) = -|\nabla u|\,\mathrm{tr}\big(D^2u - (D^2u[\nu,\nu])\nu\otimes\nu\big) + 2|\nabla u|\Delta u \tag{2.11}$$

in the set $\{|\nabla u| \neq 0\}$. Also, we note that $\mathcal{A}^2$ (as defined in Lemma 2.3) is larger or equal than half the squared Hilbert-Schmidt norm of the matrix $D^2u-(D^2u[\nu,\nu])\,\nu\otimes\nu$,[9] and hence there exists a dimensional constant C such that

$$\big|\mathrm{tr}\big(D^2u - (D^2u[\nu,\nu])\nu\otimes\nu\big)\big| \le C\mathcal{A}. \tag{2.12}$$

Furthermore, thanks to Lemma 2.3 we obtain (note that, in the next integrals, we can indistinctly integrate in B_1 or in $B_1 \cap \{|\nabla u| \neq 0\}$)

$$\begin{aligned}
-\int_{B_1} 2|\nabla u|\,\Delta u\,\eta^2\,dx &= -\int_{B_1}|\nabla u|\,\mathrm{tr}\big(D^2u - (D^2u[\nu,\nu])\nu\otimes\nu\big)\,\eta^2\,dx - \int_{B_1}\mathrm{div}(|\nabla u|\,\nabla u)\,\eta^2\,dx\\
&\le C\left(\int_{B_1}|\nabla u|^2\eta^2\,dx\right)^{1/2}\left(\int_{B_1}\mathcal{A}^2\eta^2\,dx\right)^{1/2} + \int_{B_1}|\nabla u|\,\nabla u\cdot\nabla(\eta^2)\,dx \le C.
\end{aligned}$$

Hence, combining this bound with (2.11) and (2.12), and using again Lemma 2.3 together with the fact that $\Delta u \le 0$, we get

$$\begin{aligned}
\int_{B_1}\big|\mathrm{div}(|\nabla u|\,\nabla u)\big|\eta^2\,dx &\le \int_{B_1} -2|\nabla u|\,\Delta u\,\eta^2\,dx + C\int_{B_1}|\nabla u|\mathcal{A}\,\eta^2\,dx\\
&\le C + C\left(\int_{B_1}|\nabla u|^2\eta^2\,dx\right)^{1/2}\left(\int_{B_1}\mathcal{A}^2\eta^2\,dx\right)^{1/2} \le C,
\end{aligned}$$

as desired.

Step 2: *We show that, for a.e. $t \in \mathbb{R}$,*

$$\int_{\{u=t\}\cap B_{3/4}} |\nabla u|^2 d\mathcal{H}^{n-1} \le C. \tag{2.13}$$

We claim that, for a.e. $t \in \mathbb{R}$, we have

$$\int_{\{u=t\}\cap B_{3/4}} |\nabla u|^2 d\mathcal{H}^{n-1} \le \int_{\{u=t\}\cap B_1}|\nabla u|^2\eta^2 d\mathcal{H}^{n-1} = -\int_{\{u>t\}\cap B_1}\mathrm{div}\big(|\nabla u|\,\nabla u\,\eta^2\big)\,dx. \tag{2.14}$$

Note that this bound, combined with (2.10), implies (2.13). So, we only need to prove the validity of (2.14).

To show (2.14) some care is needed to deal with the divergence, since we cannot use Sard's theorem here (u is only C^2). Thus, to prove it, we consider $s \mapsto H_\epsilon(s)$ a smooth

[9] This is easily seen by writing $D^2u(x)$ in the orthonormal basis given by $\nu(x)$ and the principal directions of the level set of u at x.

approximation of the indicator function of $\mathbb{R}_+$, so that $H'_\epsilon(s) \rightharpoonup^* \delta_0$ as $\epsilon \to 0$. Then, for any given $t \in \mathbb{R}$ we can apply Lemma A.2 with $g = H'_\epsilon(u-t)|\nabla u|^2\eta^2$ to get

$$-\int_{B_1} H_\epsilon(u-t)\mathrm{div}\big(|\nabla u|\,\nabla u\,\eta^2\big)\,dx = \int_{B_1} H'_\epsilon(u-t)\,\nabla u\cdot\big(|\nabla u|\,\nabla u\,\eta^2\big)\,dx$$
$$= \int_{B_1} H'_\epsilon(u-t)|\nabla u|^3\eta^2\,dx = \int_{\mathbb{R}} H'_\epsilon(\tau-t)\bigg(\int_{\{u=\tau\}\cap B_1}|\nabla u|^2\eta^2 d\mathcal{H}^{n-1}\bigg)\,d\tau.$$

In particular, whenever t is a Lebesgue point for the L^1 function $\tau \mapsto \int_{\{u=\tau\}\cap B_1}|\nabla u|^2\eta^2 d\mathcal{H}^{n-1}$, letting $\epsilon \to 0$ we deduce (2.14), as claimed.

Step 3: *Conclusion.*
First note that, by the standard Sobolev-Poincaré inequality, for some dimensional $p > 2$ we have

$$\left(\int_{B_1}|u-\overline{u}|^p\,dx\right)^{\frac{1}{p}} \le C\left(\int_{B_1}|\nabla u|^2\,dx\right)^{\frac{1}{2}} = C, \tag{2.15}$$

where $\overline{u} := \fint_{B_1} u$. Thus, using (2.15) and Lemma A.2 with $g = \frac{|u-\overline{u}|^p}{|\nabla u|}1_{\{|\nabla u|\neq 0\}}$, we obtain

$$\int_{\mathbb{R}} dt \int_{\{u=t\}\cap B_1\cap\{|\nabla u|\neq 0\}} |t-\overline{u}|^p\,|\nabla u|^{-1}\,d\mathcal{H}^{n-1} = \int_{B_1}|u-\overline{u}|^p 1_{\{|\nabla u|\neq 0\}}\,dx \le C. \tag{2.16}$$

Also, since $p > 2$, we may choose dimensional constants $q > 1$ and $\theta \in (0, 1/3)$ such that $p/q = (1-\theta)/\theta$. Thus, defining

$$h(t) := \max\big\{1, |t-\overline{u}|\big\}$$

and using the coarea formula (Lemma A.2) and Hölder inequality (note that $p\theta - q(1-\theta) = 0$), we obtain

$$\int_{B_{3/4}}|\nabla u|^{3-3\theta}\,dx = \int_{\mathbb{R}} dt \int_{\{u=t\}\cap B_{3/4}\cap\{|\nabla u|\neq 0\}} h(t)^{p\theta-q(1-\theta)}|\nabla u|^{-\theta+2(1-\theta)}\,d\mathcal{H}^{n-1}$$
$$\le \left(\int_{\mathbb{R}} dt \int_{\{u=t\}\cap B_1\cap\{|\nabla u|\neq 0\}} h(t)^p|\nabla u|^{-1}\,d\mathcal{H}^{n-1}\right)^{\theta}\left(\int_{\mathbb{R}} dt \int_{\{u=t\}\cap B_{3/4}} h(t)^{-q}|\nabla u|^2\,d\mathcal{H}^{n-1}\right)^{1-\theta}.$$

Observe now that, thanks to (2.16) and the definition of $h(t)$, we have

$$\int_{\mathbb{R}} dt \int_{\{u=t\}\cap B_1\cap\{|\nabla u|\neq 0\}} h(t)^p|\nabla u|^{-1}\,d\mathcal{H}^{n-1} \le \int_{\overline{u}-1}^{\overline{u}+1} dt \int_{\{u=t\}\cap B_1\cap\{|\nabla u|\neq 0\}} |\nabla u|^{-1}\,d\mathcal{H}^{n-1} + C$$
$$\le |B_1| + C \le C.$$

Also, since $q > 1$ it follows that $\int_{\mathbb{R}} h(t)^{-q}dt$ is finite, and thus (2.13) implies that

$$\int_{\mathbb{R}} dt\, h(t)^{-q}\int_{\{u=t\}\cap B_{3/4}}|\nabla u|^2\,d\mathcal{H}^{n-1} \le C\int_{\mathbb{R}} h(t)^{-q}\,dt \le C.$$

Therefore, we have proved that

$$\int_{B_{3/4}}|\nabla u|^{3-3\theta}\,dx \le C$$

for some dimensional constants $\theta \in (0, 1/3)$ and C, as desired. ☐

We conclude this section with the following useful result.

Proposition 2.5. *Let $u \in C^2(B_1)$ be a stable solution of $-\Delta u = f(u)$ in $B_1 \subset \mathbb{R}^n$, with f locally Lipschitz and nonnegative. Then*

$$\|\nabla u\|_{L^2(B_{1/2})} \le C\|u\|_{L^1(B_1)},$$

where C is a dimensional constant.

Proof. Since $-\Delta u \ge 0$ we can apply Lemma A.1(i) to the constant sequence $v_k = u$ to get

$$\|\nabla u\|_{L^1(B_{1/2})} \le C\|u\|_{L^1(B_1)}.$$

Also, it follows from Proposition 2.4 that

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \le C\|\nabla u\|_{L^2(B_1)}.$$

Therefore, by Hölder and Young inequalities, for every $\epsilon > 0$ we have

$$\begin{aligned}\|\nabla u\|^2_{L^2(B_{1/2})} &\le \|\nabla u\|^{\frac{\gamma}{1+\gamma}}_{L^1(B_{1/2})}\|\nabla u\|^{\frac{2+\gamma}{1+\gamma}}_{L^{2+\gamma}(B_{1/2})} \le C\|u\|^{\frac{\gamma}{1+\gamma}}_{L^1(B_1)}\|\nabla u\|^{\frac{2+\gamma}{1+\gamma}}_{L^2(B_1)}\\ &\le \epsilon\|\nabla u\|^2_{L^2(B_1)} + \frac{C}{\epsilon}\|u\|^2_{L^1(B_1)}.\end{aligned}$$

Applying this estimate to the functions $u_{r,y}(x) := u(y + rx)$, where $B_r(y) \subset B_1$ (note that $u_{r,y}$ is a stable solution to the semilinear equation $-\Delta u_{r,y} = f_r(u_{r,y})$ in B_1 with $f_r(t) = r^2 f(t)$, so all the previous results apply to $u_{r,y}$ as well), we conclude that

$$\begin{aligned}r^{n+2}\int_{B_{r/2}(y)} |\nabla u|^2\,dx &\le \epsilon r^{n+2}\int_{B_r(y)} |\nabla u|^2\,dx + \frac{C}{\epsilon}\left(\int_{B_r(y)} |u|\,dx\right)^2\\ &\le \epsilon r^{n+2}\int_{B_r(y)} |\nabla u|^2\,dx + \frac{C}{\epsilon}\left(\int_{B_1} |u|\,dx\right)^2\end{aligned}$$

for every $\epsilon > 0$. By Lemma A.4 applied with $\sigma(B) := \|\nabla u\|^2_{L^2(B)}$, the result follows. ☐

3. Interior C^α estimate and global estimate in convex domains: proof of Theorem 1.2 and Corollary 1.4

We begin this section by proving that, under a doubling assumption on $|\nabla u|^2 dx$, the radial derivative of a stable solution controls its full derivative.

Lemma 3.1. *Let $u \in C^2(B_2)$ be a stable solution of $-\Delta u = f(u)$ in $B_2 \subset \mathbb{R}^n$, with f locally Lipschitz and nonnegative. Assume that*

$$\int_{B_1} |\nabla u|^2\,dx \ge \delta \int_{B_2} |\nabla u|^2\,dx$$

for some $\delta > 0$. Then there exists a constant C_δ, depending only on n and δ, such that

$$\int_{B_{3/2}} |\nabla u|^2\,dx \le C_\delta \int_{B_{3/2}\setminus B_1} |x \cdot \nabla u|^2\,dx.$$

Proof. Assume the result to be false. Then, there exists a sequence of stable solutions u_k (with $f_k \geq 0$ varying) such that

$$\int_{B_1} |\nabla u_k|^2\,dx \geq \delta \int_{B_2} |\nabla u_k|^2\,dx, \quad \int_{B_{3/2}} |\nabla u_k|^2\,dx = 1, \quad \text{and} \quad \int_{B_{3/2}\setminus B_1} |x\cdot\nabla u_k|^2\,dx \to 0. \tag{3.1}$$

Now, thanks to (3.1),

$$\int_{B_2} |\nabla u_k|^2\,dx \leq \frac{1}{\delta}\int_{B_1} |\nabla u_k|^2\,dx \leq \frac{1}{\delta}\int_{B_{3/2}} |\nabla u_k|^2\,dx = \frac{1}{\delta} \leq C. \tag{3.2}$$

Therefore, using Proposition 2.4 (rescaled from B_1 to B_2) we obtain

$$\int_{B_{3/2}} |\nabla u_k|^{2+\gamma}\,dx \leq C.$$

Hence, the sequence of superharmonic functions

$$v_k := u_k - \fint_{B_2} u_k$$

satisfies

$$\|v_k\|_{L^1(B_2)} \leq C\|v_k\|_{L^2(B_2)} \leq C$$

(thanks to Hölder and Poincaré inequalities, and by (3.2)), as well as

$$\|\nabla v_k\|_{L^2(B_{3/2})} = 1, \quad \|v_k\|_{W^{1,2+\gamma}(B_{3/2})} \leq C, \quad \text{and} \quad \int_{B_{3/2}\setminus B_1} |x\cdot\nabla v_k|^2\,dx \to 0.$$

Thus it follows from Lemma A.1 applied with $r = \frac{3}{2} < 2 = R$ that, up to a subsequence, $v_k \to v$ strongly in $W^{1,2}(B_{3/2})$ where v is a superharmonic function in $B_{3/2}$ satisfying

$$\|\nabla v\|_{L^2(B_{3/2})} = 1 \qquad \text{and} \qquad x\cdot\nabla v \equiv 0 \quad \text{a.e. in } B_{3/2}\setminus B_1.$$

From the fact that v is 0-homogeneous and superharmonic in the annulus $B_{3/2}\setminus B_1$, it follows that $v = c_0$ inside $B_{3/2}\setminus B_1$ for some constant $c_0 \in \mathbb{R}$. Indeed, by the mean value property (or by Theorem 8.17 of [21], since $u \in W^{1,1}_{\rm loc} \subset L^{\frac{n}{n-1}}_{\rm loc}$ by Lemma A.1), v is bounded from below in $B_{3/2}\setminus B_1$. As a consequence, by 0-homogeneity, $\inf_{B_{3/2}\setminus B_1} v = \inf_{B_{1/4}(x_0)} v$ for some point $x_0 \in \partial B_{5/4}$. Hence, by the strong maximum principle (Theorem 8.19 of [21]), v is constant in $B_{3/2}\setminus B_1$, as desired.

In particular, we have proved that $v_{|\partial B_1} = c_0$, so by the maximum principle for superharmonic functions we get $v \geq c_0$ inside B_1.

Combining all this together, we get that

$$v \geq c_0 \quad \text{in } B_{3/2} \qquad \text{and} \qquad v \equiv c_0 \quad \text{in } B_{3/2}\setminus B_1,$$

and by the strong maximum principle for superharmonic functions we get $v \equiv c_0$ in $B_{3/2}$, a contradiction with $\|\nabla v\|_{L^2(B_{3/2})} = 1$. □

The following lemma will be used a couple of times in the paper to prove geometric decay of certain integral quantities satisfying appropriate recurrence relations.

Lemma 3.2. *Let* $\{a_j\}_{j\geq 0}$ *and* $\{b_j\}_{j\geq 0}$ *be two sequences of nonnegative numbers satisfying* $a_0 \leq M$*,* $b_0 \leq M$*,*

$$b_j \leq b_{j-1} \quad \text{and} \qquad a_j + b_j \leq L a_{j-1} \qquad \text{for all } j \geq 1,$$

and

$$\text{if} \quad a_j \geq \frac{1}{2} a_{j-1} \quad \text{then} \quad b_j \leq L(b_{j-1} - b_j) \qquad \text{for all } j \geq 1, \tag{3.3}$$

for some positive constants M *and* L*. Then there exist* $\theta \in (0,1)$ *and* $C > 0$*, depending only on* L*, such that*

$$b_j \leq CM\theta^j \qquad \text{for all } j \geq 0.$$

Proof. Define, for $\varepsilon > 0$ to be chosen,

$$c_j := a_j^{\varepsilon} b_j.$$

We consider two cases, depending whether $a_j < \frac{1}{2} a_{j-1}$ or not.
- *Case 1:* If $a_j < \frac{1}{2} a_{j-1}$, then since $b_j \leq b_{j-1}$ we get

$$c_j = a_j^{\varepsilon} b_j \leq 2^{-\varepsilon} a_{j-1}^{\varepsilon} b_{j-1} = 2^{-\varepsilon} c_{j-1}.$$

- *Case 2:* If $a_j \geq \frac{1}{2} a_{j-1}$ we can apply (3.3) and we have $b_j \leq L(b_{j-1} - b_j)$ or, equivalently,

$$b_j \leq \frac{L}{1+L} b_{j-1}.$$

Therefore, using that $a_j \leq L a_{j-1}$, we have

$$c_j = a_j^{\varepsilon} b_j \leq L^{\varepsilon} a_{j-1}^{\varepsilon} \frac{L}{1+L} b_{j-1} = \theta^{1+\varepsilon} c_{j-1},$$

where we choose first $\varepsilon > 0$ such that $2^{-\varepsilon} = L^{1+\varepsilon}/(1+L)$ (this can be done since we may assume from the beginning that $L > 1/2$), and then we define $\theta := (2^{-\varepsilon})^{\frac{1}{1+\varepsilon}} = L/(1+L)^{\frac{1}{1+\varepsilon}}$.

Hence, we have proven that in both cases $c_j \leq \theta^{1+\varepsilon} c_{j-1}$ for some $\theta \in (0,1)$. By iterating this estimate we conclude that $c_j \leq \theta^{(1+\varepsilon)j} c_0$.

Finally, recalling that $b_j \leq L a_{j-1}$, $b_j \leq b_{j-1}$, $a_0 \leq M$, and $b_0 \leq M$, recalling the definition of c_{j-1} and c_0 we obtain

$$b_j^{1+\varepsilon} \leq L^{\varepsilon} a_{j-1}^{\varepsilon} b_{j-1} = L^{\varepsilon} c_{j-1} \leq \frac{L^{\varepsilon}}{\theta^{1+\varepsilon}} \theta^{(1+\varepsilon)j} c_0 \leq C \theta^{(1+\varepsilon)j} M^{1+\varepsilon}$$

and the lemma follows. □

We can now prove Theorem 1.2.

Proof of Theorem 1.2. We begin by noticing that, combining Propositions 2.4 and 2.5, we immediately get the bound

$$\|\nabla u\|_{L^{2+\gamma}(B_{3/8})} \leq C \|u\|_{L^1(B_1)}.$$

Hence (1.6) follows by a classical scaling and covering argument.

We are left with proving (1.7). For this we may assume that $3 \leq n \leq 9$. (Indeed, recall that in case $n \leq 2$ one can easily reduce to the case $n = 3$ by adding extra artificial variables.

Note that the stability condition is preserved under this procedure). Given $\rho \in (0,1)$, we define the quantities

$$\mathcal{D}(\rho) := \rho^{2-n} \int_{B_\rho} |\nabla u|^2 \, dx \qquad \text{and} \qquad \mathcal{R}(\rho) := \int_{B_\rho} |x|^{-n} |x \cdot \nabla u|^2 \, dx.$$

We split the proof of (1.7) in three steps.

Step 1: *We prove that there exists a dimensional exponent $\alpha > 0$ such that*

$$\mathcal{R}(\rho) \le C \rho^{2\alpha} \|\nabla u\|^2_{L^2(B_{1/2})}$$

for all $\rho \in (0, 1/4)$.

Recall that, by (2.3), for every $\rho \in (0, 1/4)$ it holds

$$\mathcal{R}(\rho) \le C \rho^{2-n} \int_{B_{3\rho/2} \setminus B_\rho} |\nabla u|^2 \, dx. \tag{3.4}$$

Hence, if $\mathcal{D}(\rho) \ge \frac{1}{2}\mathcal{D}(2\rho)$ then we can apply Lemma 3.1 with $\delta = 1/2$ to the function $u(\rho\,\cdot)$, and we deduce that

$$\rho^{2-n} \int_{B_{3\rho/2}} |\nabla u|^2 \, dx \le C \rho^{-n} \int_{B_{3\rho/2} \setminus B_\rho} |x \cdot \nabla u|^2 \, dx \le C\big(\mathcal{R}(3\rho/2) - \mathcal{R}(\rho)\big)$$

for some dimensional constant C. Combining this bound with (3.4) and using that $\mathcal{R}$ is nondecreasing, we deduce that

$$\mathcal{R}(\rho) \le C\big(\mathcal{R}(2\rho) - \mathcal{R}(\rho)\big) \qquad \text{provided } \mathcal{D}(\rho) \ge \tfrac{1}{2}\mathcal{D}(2\rho). \tag{3.5}$$

Thus, if we define $a_j := \mathcal{D}(2^{-j-2})$, $b_j := \mathcal{R}(2^{-j-2})$ we have, for some dimensional constant $L > 0$,

- $b_j \le b_{j-1}$ for all $j \ge 1$ (since $\mathcal{R}$ is nondecreasing);
- $a_j + b_j \le L a_{j-1}$ for all $j \ge 1$ (by (3.4));
- if $a_j \ge \frac{1}{2} a_{j-1}$ then $b_j \le L(b_{j-1} - b_j)$, for all $j \ge 1$ (by (3.5)).

Therefore, by Lemma 3.2 we deduce that

$$b_j \le C M \theta^j,$$

where $\theta \in (0,1)$ and $M := a_0 + b_0 \le C \|\nabla u\|^2_{L^2(B_{1/2})}$ (here we used again (3.4) in order to bound b_0).

Choosing $\alpha > 0$ such that $2^{-2\alpha} = \theta$, Step 1 follows easily.

Step 2: *We show that*

$$[u]_{C^\alpha(\overline{B}_{1/8})} \le C \|\nabla u\|_{L^2(B_{3/4})}, \tag{3.6}$$

where α and C are positive dimensional constants.

Applying Step 1 to the function $u_y(x) := u(x+y)$ with $y \in B_{1/4}$, since $B_{1/2}(y) \subset B_{3/4}$ we get

$$\int_{B_\rho(y)} |x-y|^{-n} |(x-y) \cdot \nabla u|^2 \, dx \le C \rho^{2\alpha} \int_{B_{3/4}} |\nabla u|^2 \, dx \qquad \text{for all } \rho \le 1/2.$$

In particular,

$$\rho^{2-n}\int_{B_\rho(y)}\left|\frac{x-y}{|x-y|}\cdot\nabla u\right|^2 dx \le C\rho^{2\alpha}\int_{B_{3/4}}|\nabla u|^2\,dx \qquad \text{for all } y\in B_{1/4},\ \rho\le 1/2.$$

Then, given $z\in B_{1/8}$, we can average the above inequality with respect to $y\in B_{\rho/4}(z)$ to get

$$\rho^{2-n}\int_{B_{\rho/8}(z)}|\nabla u|^2\,dx \le C\rho^{2\alpha}\int_{B_{3/4}}|\nabla u|^2\,dx \qquad \text{for all } \rho\le 1/2.$$

Since $z\in B_{1/8}$ is arbitrary, by classical estimates on Morrey spaces (see for instance [21, Theorem 7.19]) we deduce (3.6).

Step 3: *Proof of* (1.7).

Note that, using Proposition 2.5 and a standard scaling and covering argument, we have

$$\|\nabla u\|_{L^2(B_{3/4})}\le C\|u\|_{L^1(B_1)}.$$

Hence, it follows by Step 2 that $[u]_{C^\alpha(\overline{B}_{1/8})}\le C\|u\|_{L^1(B_1)}$. Also, by classical interpolation estimates, we have the bound

$$\|u\|_{L^\infty(B_{1/8})}\le C\left([u]_{C^\alpha(\overline{B}_{1/8})}+\|u\|_{L^1(B_{1/8})}\right).$$

Combining these estimates, we conclude that

$$\|u\|_{C^\alpha(\overline{B}_{1/8})}\le C\|u\|_{L^1(B_1)}.$$

Finally, (1.7) follows by a classical scaling and covering argument. □

We conclude the section by proving global regularity in convex domains.

Proof of Corollary 1.4. First of all, since $f\ge 0$ we have that u is superharmonic, so by the maximum principle $u\ge 0$ in Ω.

Since Ω is a bounded convex domain of class C^1, by the classical moving planes method there exists $\rho_0>0$, depending only on Ω, such that

$$u(x)\le \max_{\Gamma_0} u \qquad \text{for all } x\in N_0, \tag{3.7}$$

where $N_0:=\Omega\cap\{y\,:\,\mathrm{dist}(y,\partial\Omega)<\rho_0\}$ and $\Gamma_0:=\{y\in\Omega\,:\,\mathrm{dist}(y,\partial\Omega)=\rho_0\}$.[10]

Hence, it follows by Theorem 1.2 that $u\le C\|u\|_{L^1(\Omega)}$ inside $\Omega\setminus N_0$, where C depends only on Ω and ρ_0. Thus, recalling (3.7), we conclude that $0\le u\le C\|u\|_{L^1(\Omega)}$ inside Ω. □

[10]Here we are using that, in any convex C^1 domain, we can start the classical moving planes method at any boundary point.

We note that the classical moving planes method is usually stated for strictly convex C^1 domains. If Ω is merely convex (instead of strictly convex), then the boundary may contain a piece of a hyperplane. Still, by a simple contradiction argument one can show that, given any boundary point, there exist hyperplanes that separate a small cap around this point from their reflected points, and such that the reflected points are contained inside Ω. This suffices to use the moving planes method in a neighborhood of any boundary point.

4. A general closedness result for stable solutions with convex nondecreasing nonlinearities

The goal of this section is to establish a very strong closedness property for stable solutions to equations with convex, nondecreasing, and nonnegative nonlinearities. As mentioned in the introduction, in addition to its own interest, this result will play a crucial role in the proof of the global regularity result of Theorem 1.5.

Define

$$\mathcal{C} := \big\{ f : \mathbb{R} \to [0, +\infty] \; : \; f \text{ is lower semicontinuous, nondecreasing, and convex} \big\}.$$

Note that functions $f \in \mathcal{C}$ are nonnegative but are allowed to take the value $+\infty$. This fact is important, since limits of nondecreasing convex nonlinearities $f_k : \mathbb{R} \to \mathbb{R}$ could become $+\infty$ in an interval $[M, \infty)$; this is why, in $\mathcal{C}$, we must allow f to take the value $+\infty$.

For $f \in \mathcal{C}$ and $t \in \mathbb{R}$ such that $f(t) < +\infty$, the following is the definition and a property for $f'_-(t)$:

$$f'_-(t) := \lim_{h \downarrow 0} \frac{f(t) - f(t-h)}{h} \geq \frac{f(t_2) - f(t_1)}{t_2 - t_1} \quad \text{for all } t_1 < t_2 \leq t. \tag{4.1}$$

If $f(t) = +\infty$ for some t, then we simply set $f'_-(t) = +\infty$.

Given an open set $U \subset \mathbb{R}^n$, we define

$$\mathcal{S}(U) := \left\{ u \in C^0(U) \cap W^{1,2}_{\mathrm{loc}}(U) : \begin{array}{c} u \text{ is a stable weak solution of} \\ -\Delta u = f(u) \text{ in } U, \text{ for some } f \in \mathcal{C} \end{array} \right\}. \tag{4.2}$$

The meaning of weak solution is that of Definition 1.1. In particular, since $f(u) \in L^1_{\mathrm{loc}}(U)$ then $f(u)$ is finite a.e., and since f is nondecreasing we deduce that $f < +\infty$ in $[\inf_U u, \sup_U u)$. Note also that, similarly, since $f'_- \geq 0$ and f is convex, we have that $f'_- < +\infty$ in $[\inf_U u, \sup_U u)$.

The following theorem states that, given an open set $U \subset \mathbb{R}^n$, the set $\mathcal{S}(U)$ is closed in $L^1_{\mathrm{loc}}(U)$. This is particularly surprising since no bound is required on the nonlinearities.

Theorem 4.1. *Let $U \subset \mathbb{R}^n$ be an open set. Let $u_k \in \mathcal{S}(U)$, and assume that $u_k \to u$ in $L^1_{\mathrm{loc}}(U)$ for some $u \in L^1_{\mathrm{loc}}(U)$.*

Then, $u \in \mathcal{S}(U)$ and the convergence $u_k \to u$ holds in $W^{1,2}_{\mathrm{loc}}(U)$. If, in addition, $n \leq 9$ then the convergence also holds in $C^0(U)$.

For the proof of this result we shall use the interior estimates of Theorem 1.2. However we proved these interior estimates for C^2 solutions, while solutions in the class $\mathcal{S}(U)$ are in general only continuous —notice that it may happen that $f(u(x_0)) = f(\sup_U u) = +\infty$ for some $x_0 \in U$. Thus, we will need to prove first that the interior estimates of Theorem 1.2 extend to all weak solutions in the class $\mathcal{S}(B_1)$ (see Corollary 4.3 below). For this, we need the following useful approximation result.

Proposition 4.2. *Let $f \in \mathcal{C}$, and assume that $u \in C^0(\overline{B_1}) \cap W^{1,2}(B_1)$ is a stable weak solution of $-\Delta u = f(u)$ in B_1, with $f(u) \in L^1(B_1)$.*

Then, either $u \in C^2(B_1)$ or there exists a family of stable solutions $u_\varepsilon \in C^2(B_1) \cap W^{1,2}(B_1)$, with $\varepsilon \in (0, 1]$, of

$$\begin{cases} -\Delta u_\varepsilon = (1-\varepsilon) f(u_\varepsilon) & \text{in } B_1 \\ u_\varepsilon = u & \text{on } \partial B_1 \end{cases}$$

such that $u_\varepsilon \uparrow u$ *as* $\varepsilon \downarrow 0$.

Proof. If $f'_-(\sup_{B_1} u) < +\infty$ then f is Lipschitz on $[\inf_{B_1} u, \sup_{B_1} u]$ (by convexity of f) and the result is classical, see [18, Section 3.2.2]. Hence, we can assume that $f'_-(\sup_{B_1} u) = +\infty$.

Notice also that, considering $\widetilde{u} = u + C$ and $\widetilde{f}(t) = f(t - C)$ if necessary, we may assume that $u \ge 0$.

Step 1: *Construction of* u_ε.

For $\varepsilon = 1$, the function u_1 can be defined as the harmonic extension of u. Indeed, since $u \in W^{1,2}(B_1)$, the Dirichlet energy $\int_{B_1} |\nabla v|^2$ admits a minimizer u_1 in the convex set $\{v \in W^{1,2}(B_1) : v - u \in W_0^{1,2}(B_1)\}$. In particular, u_1 is stable.

To construct u_ε for $\varepsilon \in (0,1)$ we start a monotone iteration by taking $u_\varepsilon^{(0)} = u_1$ and define, for $j \ge 1$, the function $u_\varepsilon^{(j)}$ as the solution to the linear problem

$$\begin{cases} -\Delta u_\varepsilon^{(j)} = (1-\varepsilon) f(u_\varepsilon^{(j-1)}) & \text{in } B_1 \\ u_\varepsilon^{(j)} = u & \text{on } \partial B_1. \end{cases}$$

Note that, since $f \ge 0$, it follows by the maximum principle that $u_\varepsilon^{(1)} \ge u_\varepsilon^{(0)}$. Also, since f is nondecreasing, the inequality

$$-\Delta(u_\varepsilon^{(j)} - u_\varepsilon^{(j-1)}) = (1-\varepsilon)\big(f(u_\varepsilon^{(j-1)}) - f(u_\varepsilon^{(j-2)})\big) \qquad \text{for all } j \ge 2$$

proves, by induction on j, that $u_\varepsilon^{(j)} \ge u_\varepsilon^{(j-1)}$. On the other hand, since $u_\varepsilon^{(0)} = u_1 \le u$ and

$$\begin{aligned} -\Delta(u - u_\varepsilon^{(j)}) &= f(u) - (1-\varepsilon) f(u_\varepsilon^{(j-1)}) = \varepsilon f(u) + (1-\varepsilon)\big(f(u) - f(u_\varepsilon^{(j-1)})\big) \\ &\ge (1-\varepsilon)\big(f(u) - f(u_\varepsilon^{(j-1)})\big) \end{aligned} \tag{4.3}$$

for all $j \ge 1$, again by induction we prove that $u_\varepsilon^{(j)} \le u$ for all $j \ge 0$. Hence, we can define

$$u_\varepsilon := \lim_{j\to\infty} u_\varepsilon^{(j)} \le u.$$

Note that by (4.3) we have $-\Delta(u - u_\varepsilon) \ge \varepsilon f(u) \ge 0$. Also, $f(u)$ cannot be identically zero (since by assumption $f'_-(\sup_{B_1} u) = +\infty$), and thus it follows by the Harnack inequality that, for all $r \in (0,1)$ and $\varepsilon > 0$, there exists $\delta_{\varepsilon,r} > 0$ such that $u_\varepsilon \le u - \delta_{\varepsilon,r}$ in B_r.

We now recall that, as already observed before, the fact that $-\Delta u = f(u)$ with $u \in C^0(U)$ and $f \in \mathcal{C}$ leads to $f < +\infty$ in $[\inf_U u, \sup_U u)$. Hence, since $f \ge 0$ is convex and nondecreasing, we obtain that

$$\|f\|_{C^{0,1}((-\infty,t])} \le C(f,t) < \infty \qquad \text{for all } t < \sup_{B_1} u.$$

Thus, since $u_\varepsilon^{(j)} \le u_\varepsilon \le \sup_{B_1} u - \delta_{\varepsilon,r}$ in B_r, by standard elliptic regularity (see for instance [21, Chapter 6]) we obtain that $u_\varepsilon^{(j)} \in C^{2,\beta}_{\rm loc}(B_1)$ for all $\beta \in (0,1)$, uniformly in j. In particular, this proves that $u_\varepsilon \in C^2(B_1)$ and that $-\Delta u_\varepsilon = (1-\varepsilon) f(u_\varepsilon)$ in B_1.

Notice also that the functions $u_\varepsilon^{(j)}$ and u_ε are in $W^{1,2}(B_1)$, uniformly in j and ε. Indeed, since $u \in W^{1,2}(B_1)$ and

$$\int_{B_1} \nabla u \cdot \nabla(u - u_\varepsilon^{(j)})\, dx = \int_{B_1} f(u)(u - u_\varepsilon^{(j)})\, dx,$$

$$\int_{B_1} \nabla u_\varepsilon^{(j)} \cdot \nabla(u - u_\varepsilon^{(j)})\, dx = \int_{B_1} (1-\varepsilon) f(u_\varepsilon^{(j-1)})(u - u_\varepsilon^{(j)})\, dx \geq 0,$$

then

$$\int_{B_1} |\nabla(u - u_\varepsilon^{(j)})|^2\, dx \leq \int_{B_1} f(u)(u - u_\varepsilon^{(j)})\, dx \leq \|f(u)\|_{L^1(B_1)} \|u\|_{L^\infty(B_1)}.$$

Taking $j \to \infty$, we deduce that the same inequality holds for u_ε.

Step 2: *The solutions u_ε are stable.*

It simply follows from the stability of u and the pointwise inequality $f'_-(u) \geq (1-\varepsilon) f'_-(u_\varepsilon)$ in B_1 that

$$\int_{B_1} |\nabla \xi|^2\, dx \geq \int_{B_1} f'_-(u)\xi^2\, dx \geq \int_{B_1} (1-\varepsilon) f'_-(u_\varepsilon)\xi^2\, dx$$

for all $\xi \in C_c^\infty(B_1)$. Thus, u_ε is stable.

Step 3: $u_\varepsilon \uparrow u$ *as* $\varepsilon \downarrow 0$.

Note that, by construction, $u_\varepsilon < u$. Assume by contradiction that $u_\varepsilon \uparrow u^* \leq u$ as $\varepsilon \downarrow 0$ and $u^* \not\equiv u$. Then

$$-\Delta u^* = f(u^*) \quad \text{in } B_1, \qquad u - u^* \in W_0^{1,2}(B_1), \quad u - u^* \geq 0, \qquad u - u^* \not\equiv 0,$$

and thus $u - u^* > 0$ by the strong maximum principle. Testing the stability inequality for u with $u - u^*$ we obtain

$$\int_{B_1} \big(f(u) - f(u^*)\big)(u - u^*)\, dx = \int_{B_1} |\nabla(u - u^*)|^2\, dx \geq \int_{B_1} f'_-(u)(u - u^*)^2\, dx.$$

Recalling (4.1) and that $u > u^*$, this implies $f'_-(u)(u-u^*)^2 = \big(f(u) - f(u^*)\big)(u - u^*)$ a.e. in B_1 and (since f is convex) that f is linear in the interval $[u^*(x), u(x)]$ for all $x \in B_1$. Now, by continuity of u and u^*, since f is convex and lower semicontinuous, this forces f to be linear on the whole interval $[\inf_{B_1} u^*, \sup_{B_1} u]$. This contradicts $f'_-(\sup_{B_1} u) = +\infty$, concluding the proof. □

As a consequence, we find the following.

Corollary 4.3. *The interior estimates of Theorem 1.2 extend to all weak solutions in the class $\mathcal{S}(B_1)$.*

Proof. Just note that Theorem 1.2 can be applied to each approximating u_ε given by Proposition 4.2, and then the same estimates hold for u letting $\varepsilon \downarrow 0$. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1. By assumption we have a sequence $u_k \in \mathcal{S}(U)$ of weak solutions of $-\Delta u_k = f_k(u_k)$, with $f_k \in \mathcal{C}$ and U an open set of $\mathbb{R}^n$, such that $u_k \to u$ in $L^1_{\rm loc}(U)$. Then, by Corollary 4.3 and Lemma A.1, the previous convergence also holds in $W^{1,2}_{\rm loc}(U)$. Also, up to a subsequence, we can assume that $u_k \to u$ a.e. If $n \leq 9$, the same results give that $u_k \to u$ locally uniformly in U. However, since in order to prove $u \in \mathcal{S}(U)$ we are not assuming $n \leq 9$, we cannot use this information.

Step 1: *A compactness estimate on f_k.*

Let $M := \sup_U u \in (-\infty, +\infty]$, and let $m < M$. We claim that

$$\limsup_{k\to\infty} f_k(m) < \infty. \tag{4.4}$$

Indeed, let $x_0 \in U$ be a Lebesgue point for u such that

$$m < u(x_0) < M,$$

and set $\delta := u(x_0) - m > 0$. Since x_0 is a Lebesgue point, there exists $\varepsilon_0 > 0$ such that $\overline{B}_{2\varepsilon_0}(x_0) \subset U$ and

$$\fint_{B_\varepsilon(x_0)} |u(x) - u(x_0)|\, dx \le \frac{\delta}{2} \qquad \text{for all } \varepsilon \in (0, 2\varepsilon_0].$$

In particular, for k sufficiently large we have

$$m \le \fint_{B_\varepsilon(x_0)} u_k\, dx \le \fint_{B_\varepsilon(x_0)} |u_k|\, dx \le |u(x_0)| + \delta \qquad \text{for all } \varepsilon \in (0, 2\varepsilon_0].$$

Hence, since f_k is nondecreasing and convex, applying Jensen's inequality and Lemma A.1(a) we get

$$\begin{aligned} f_k(m) &\le f_k\Big(\fint_{B_{\varepsilon_0}(x_0)} u_k\, dx\Big) \le \fint_{B_{\varepsilon_0}(x_0)} f_k(u_k)\, dx = \fint_{B_{\varepsilon_0}(x_0)} (-\Delta u_k)\, dx \\ &\le C\varepsilon_0^{-2} \fint_{B_{2\varepsilon_0}(x_0)} |u_k|\, dx \le C\varepsilon_0^{-2}\big(|u(x_0)| + \delta\big) \end{aligned}$$

for a dimensional constant C and all k sufficiently large, proving (4.4).

Notice now that, since

$$(f_k)'_-(m) \le \frac{f_k(m+\delta) - f_k(m)}{\delta} \le \frac{f_k(m+\delta)}{\delta}$$

and $m + \delta = u(x_0) < M$, (4.4) applied with m replaced by $m + \delta$ gives that $f_k \to f$ locally uniformly in $(-\infty, m]$ for every $m < M$. As a consequence, since f_k are nonnegative, nondecreasing, and convex, we obtain that they converge to a function f in $(-\infty, M)$, with $f < +\infty$ on $(-\infty, M)$. Extending f to all $\mathbb{R}$ defining $f(M) := \lim_{t\uparrow M} f(t)$, and $f(t) := +\infty$ for $t > M$, it is easy to check that $f \in \mathcal{C}$.

Step 2: $-\Delta u = f(u)$ *in* U.

For every $\xi \in C_c^\infty(U)$ we have

$$\begin{aligned} \int_U \nabla u \cdot \nabla \xi\, dx &= -\int_U u\Delta\xi\, dx = -\lim_k \int_U u_k \Delta\xi\, dx = \lim_k \int_U \nabla u_k \cdot \nabla\xi\, dx \\ &= \lim_k \int_U f_k(u_k)\, \xi\, dx. \end{aligned} \tag{4.5}$$

Note that, since since $f_k \to f$ locally uniformly on $(-\infty, M)$ and $u_k \to u$ a.e., it follows that

$$f_k(u_k) \to f(u) \qquad \text{a.e. inside } \{u < M\}. \tag{4.6}$$

In the following, $\eta \in C_c^\infty(U)$ denotes a nonnegative cut-off function such that $\eta = 1$ on the support of ξ.

Case 1: $M = +\infty$. We have

$$\int_{\mathrm{supp}(\xi)} f_k(u_k)u_k\,dx \le \int_U f_k(u_k)|u_k|\eta\,dx = \int_U(-\Delta u_k)|u_k|\eta\,dx = \int_U \nabla u_k\cdot\nabla(|u_k|\eta)dx \le C$$

for some constant C independent of k, where the last bound follows from the $W^{1,2}_{\mathrm{loc}}$ boundedness of u_k. In particular, given a continuous function $\varphi:\mathbb{R}\to[0,1]$ such that $\varphi=0$ on $(-\infty,0]$ and $\varphi=1$ on $[1,+\infty)$, we deduce that

$$\begin{aligned}\int_{\mathrm{supp}(\xi)} f_k(u_k)\,\varphi(u_k-j)\,dx &\le \int_{\mathrm{supp}(\xi)\cap\{u_k>j\}} f_k(u_k)\,dx\\ &\le \frac{1}{j}\int_{\mathrm{supp}(\xi)\cap\{u_k>j\}} f_k(u_k)u_k\,dx \le \frac{C}{j} \qquad \text{for all } j>1.\end{aligned}\tag{4.7}$$

Therefore, by Fatou's Lemma (since $u_k\to u$ a.e. and $f_k(u_k)\to f(u)$ a.e. by (4.6) and $M=+\infty$), we also have

$$\int_{\mathrm{supp}(\xi)} f(u)\,\varphi(u-j)\,dx \le \frac{C}{j} \qquad \text{for all } j>1. \tag{4.8}$$

Furthermore, using again that $u_k\to u$ a.e. and $f_k(u_k)\to f(u)$ a.e., by dominated convergence we get

$$f_k(u_k)\,[1-\varphi(u_k-j)] \to f(u)\,[1-\varphi(u-j)] \qquad \text{in } L^1(\mathrm{supp}(\xi)).$$

This, combined with (4.7) and (4.8), gives that

$$\begin{aligned}\limsup_{k\to\infty}\int_{\mathrm{supp}(\xi)} |f_k(u_k)-f(u)|\,dx &\le \limsup_{k\to\infty}\int_{\mathrm{supp}(\xi)} f_k(u_k)\,\varphi(u_k-j)\,dx\\ &\qquad + \int_{\mathrm{supp}(\xi)} f(u)\,\varphi(u-j)\,dx \le \frac{2C}{j}.\end{aligned}$$

By the arbitrariness of j, this proves that

$$f_k(u_k)\to f(u) \qquad \text{in } L^1(\mathrm{supp}(\xi)).$$

Recalling (4.5), this concludes the proof of Step 2 in the case $M=+\infty$.

Case 2: $M<+\infty$. Let $\delta>0$. Since $(u_k-M-\delta)_+\ge\delta$ inside $\{u_k>M+2\delta\}$ and $-\Delta u_k = f_k(u_k)\ge 0$, we have

$$\begin{aligned}\delta\int_{\mathrm{supp}(\xi)\cap\{u_k>M+2\delta\}} f_k(u_k)\,dx &= \delta\int_{\mathrm{supp}(\xi)\cap\{u_k>M+2\delta\}} -\Delta u_k\,dx\\ &\le \int_{\mathrm{supp}(\xi)\cap\{u_k>M+2\delta\}} -\Delta u_k\,(u_k-M-\delta)_+\,dx\\ &\le \int_U -\Delta u_k\,(u_k-M-\delta)_+\eta\,dx\\ &= \int_{U\cap\{u_k>M+\delta\}} |\nabla u_k|^2\eta\,dx + \int_U \nabla u_k\cdot\nabla\eta\,(u_k-M-\delta)_+\,dx.\end{aligned}$$

Note that, thanks to the higher integrability estimate (1.6) applied to u_k (recall Corollary 4.3), the functions u_k are uniformly bounded in $W^{1,2+\gamma}(\mathrm{supp}(\eta))$. Thus, since $1_{\{u_k>M+\delta\}}\to 0$ and

$(u_k - M - \delta)_+ \to 0$ a.e., we deduce from Hölder's inequality that the last two integrals tend to 0 as $k \to \infty$, and therefore

$$\lim_{k\to\infty} \int_{\mathrm{supp}(\xi)\cap\{u_k>M+2\delta\}} f_k(u_k)\,dx = 0. \tag{4.9}$$

On the other end, we note that $f_k(u_k - 3\delta) \le f_k(M-\delta) \le C$ inside $\mathrm{supp}(\xi)\cap\{u_k \le M+2\delta\}$. Hence, thanks to (4.9) and the uniform convergence of f_k to f on $(-\infty, M-\delta]$, we get (recall that $u \le M$ a.e.)

$$\begin{aligned}
&\lim_k \int_U f_k(u_k)\,\xi\,dx = \lim_k \int_{U\cap\{u_k\le M+2\delta\}} f_k(u_k)\,\xi\,dx\\
&= \lim_k \left\{ \int_{U\cap\{u_k\le M+2\delta\}} f_k(u_k-3\delta)\,\xi\,dx + \int_{U\cap\{u_k\le M+2\delta\}} \big(f_k(u_k) - f(u_k-3\delta)\big)\,\xi\,dx \right\}\\
&= \int_U f(u-3\delta)\xi\,dx + \lim_k \int_{U\cap\{u_k\le M+2\delta\}} \big(f_k(u_k) - f(u_k-3\delta)\big)\xi\,dx.
\end{aligned}$$

Now, by (4.1), the definition of f'_-, and the stability of u_k, we have (recall that $\eta = 1$ on the support of ξ)

$$\left|\int_U \big(f_k(u_k) - f(u_k-3\delta)\big)\xi\,dx\right| \le 3\delta \int_U (f_k)'_-(u_k)|\xi|\,dx \le 3\delta\|\xi\|_\infty \int_U (f_k)'_-(u_k)\eta^2\,dx \le C\delta$$

and therefore, letting $\delta \to 0$, by monotone convergence we find

$$\lim_k \int_U f_k(u_k)\,\xi\,dx = \int_U f(u)\,\xi\,dx.$$

Recalling (4.5), this proves that $-\Delta u = f(u)$ inside U in the case $M < +\infty$.

Step 3: *u is stable.*

Thanks to the convexity of f_k, it follows from (4.1) and the stability inequality for u_k that, for any $\delta > 0$,

$$\int_U \frac{f_k(u_k-2\delta) - f_k(u_k-3\delta)}{\delta}\,\xi^2\,dx \le \int_U |\nabla\xi|^2\,dx \qquad \text{for all } \xi \in C_c^\infty(U).$$

Hence, since $u_k \to u$ a.e. in U and $f_k \to f$ locally uniformly in $(-\infty, m]$ for all $m < M$, and since f_k is nondecreasing, it follows by Fatou's lemma applied to the sequence $1_{\{u_k\le\min\{j,M\}+\delta\}}\,\delta^{-1}\big(f_k(u_k-2\delta) - f_k(u_k-3\delta)\big)$ that, for any $j > 1$,

$$\begin{aligned}
&\int_{U\cap\{u\le\min\{j,M\}\}} \frac{f(u-2\delta) - f(u-3\delta)}{\delta}\,\xi^2\,dx \le\\
&\qquad \le \liminf_k \int_{U\cap\{u_k\le\min\{j,M\}+\delta\}} \frac{f_k(u_k-2\delta) - f_k(u_k-3\delta)}{\delta}\,\xi^2\,dx\\
&\qquad \le \liminf_k \int_U \frac{f_k(u_k-2\delta) - f_k(u_k-3\delta)}{\delta}\,\xi^2\,dx\\
&\qquad \le \int_U |\nabla\xi|^2\,dx \qquad \text{for all } \xi \in C_c^\infty(U).
\end{aligned}$$

Since

$$\frac{f(t-2\delta)-f(t-3\delta)}{\delta} \uparrow f'_-(t) \qquad \text{as } \delta \to 0, \quad \text{for all } \ t \le \min\{j, M\},$$

the result follows by the monotone convergence theorem, letting first $\delta \to 0$ and then $j \to +\infty$. □

5. Boundary $W^{1,2+\gamma}$ estimate

In this section we prove a uniform $W^{1,2+\gamma}$ bound near the boundary in terms only of the L^1 norm of the solution. As in the interior case (see Section 2), this is done by first controlling $\|\nabla u\|_{L^{2+\gamma}}$ with $\|\nabla u\|_{L^2}$, and then $\|\nabla u\|_{L^2}$ with $\|u\|_{L^1}$.

We begin by introducing the notion of a small deformation of a half-ball. It will be useful in several proofs, particularly in that of Lemma 6.2. Given $\rho > 0$, we denote by B_ρ^+ the upper half-ball in the e_n direction, namely

$$B_\rho^+ := B_\rho \cap \{x_n > 0\}.$$

Definition 5.1. Given $\vartheta \ge 0$, we say that $\Omega \subset \mathbb{R}^n$ is a ϑ-deformation of B_2^+ if

$$\Omega = \Phi(B_2 \cap \{x_n > 0\})$$

for some $\Phi \in C^3(B_2; \mathbb{R}^n)$ satisfying $\Phi(0) = 0$, $D\Phi(0) = \mathrm{Id}$, and

$$\|D^2\Phi\|_{L^\infty(B_2)} + \|D^3\Phi\|_{L^\infty(B_2)} \le \vartheta.$$

Here, the norms of $D^2\Phi$ and $D^3\Phi$ are computed with respect to the operator norm.

Note that, given a bounded C^3 domain, one can cover its boundary with finitely many small balls so that, after rescaling these balls, the boundary of the domain is given by a finite union of ϑ-deformations of B_2^+ (up to isometries) with ϑ arbitrarily small.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a ϑ-deformation of B_2^+ for $\vartheta \in [0, \frac{1}{100}]$. Let $u \in C^2(\overline{\Omega} \cap B_1)$ be a nonnegative stable solution of $-\Delta u = f(u)$ in $\Omega \cap B_1$, with $u = 0$ on $\partial\Omega \cap B_1$. Assume that f is locally Lipschitz, nonnegative, and nondecreasing. Then*

$$\|\nabla u\|_{L^{2+\gamma}(\Omega \cap B_{3/4})} \le C\|\nabla u\|_{L^2(\Omega \cap B_1)},$$

where $\gamma > 0$ and C are dimensional constants.

The proof will make us of the following lemma, which is based on a Pohozaev-type identity.

Lemma 5.3. *Under the assumptions of Proposition 5.2 we have*

$$\|u_\nu\|_{L^2(\partial\Omega \cap B_{7/8})} \le C\|\nabla u\|_{L^2(\Omega \cap B_1)}, \tag{5.1}$$

where u_ν is the normal derivative of u at $\partial\Omega$ and C is a dimensional constant.

Proof. Take a cut-off function $\eta \in C_c^2(B_1)$ such that $\eta = 1$ in $B_{7/8}$, and consider the vector-field $\mathbf{X}(x) := x + \boldsymbol{e}_n$. Multiplying the identity

$$\mathrm{div}\big(|\nabla u|^2\mathbf{X} - 2(\mathbf{X} \cdot \nabla u)\nabla u\big) = (n-2)|\nabla u|^2 - 2(\mathbf{X} \cdot \nabla u)\Delta u$$

by η^2 and integrating in $\Omega\cap B_1$, since $u=0$ on $\partial\Omega$ and $u\ge 0$ in $\Omega\cap B_1$ (and hence the exterior unit normal ν is given by $-\frac{\nabla u}{|\nabla u|}$), we obtain

$$-\int_{\partial\Omega\cap B_1}(\mathbf{X}\cdot\nu)|\nabla u|^2\eta^2\,d\mathcal{H}^{n-1}-\int_{\Omega\cap B_1}\big(|\nabla u|^2\mathbf{X}-2(\mathbf{X}\cdot\nabla u)\nabla u\big)\cdot\nabla\eta^2\,dx=$$
$$=\int_{\Omega\cap B_1}\big((n-2)|\nabla u|^2-2(\mathbf{X}\cdot\nabla u)\Delta u\big)\eta^2\,dx.$$

Note that, since Ω is a small deformation of B_2^+, we have $-\mathbf{X}\cdot\nu\ge\frac12$ on $\partial\Omega\cap B_1$. Hence, since $F(t):=\int_0^t f(s)ds$ satisfies $\mathbf{X}\cdot\nabla(F(u))=f(u)\mathbf{X}\cdot\nabla u=-\Delta u\,\mathbf{X}\cdot\nabla u$, we obtain

$$\frac12\int_{\partial\Omega\cap B_1}|\nabla u|^2\eta^2 d\mathcal{H}^{n-1}\le C\int_{\Omega\cap B_1}|\nabla u|^2\,dx+2\int_{\Omega\cap B_1}\mathbf{X}\cdot\nabla(F(u))\eta^2\,dx$$
$$=C\int_{\Omega\cap B_1}|\nabla u|^2\eta^2\,dx-2\int_{\Omega\cap B_1}F(u)\,\mathrm{div}(\eta^2\mathbf{X})\,dx.$$

We now observe that, since f is nondecreasing, $0\le F(t)\le f(t)t$ for all $t\ge 0$. Hence, noticing that the function $g:=|\mathrm{div}(\eta^2\mathbf{X})|$ is Lipschitz, we can bound

$$-\int_{\Omega\cap B_1}F(u)\,\mathrm{div}(\eta^2\mathbf{X})\,dx\le\int_{\Omega\cap B_1}u\,f(u)\,g\,dx=-\int_{\Omega\cap B_1}u\,\Delta u\,g\,dx$$
$$=\int_{\Omega\cap B_1}\big(|\nabla u|^2g+u\,\nabla u\cdot\nabla g\big)\,dx\le C\int_{\Omega\cap B_1}\big(u^2+|\nabla u|^2\big)\,dx,$$

and we conclude using Poincaré inequality (since u vanishes on $\partial\Omega\cap B_1$). □

We next give the:

Proof of Proposition 5.2. The key idea is to use a variant of

$$\xi=\big(|\nabla u|-u_n\big)\eta$$

as test function in the stability inequality (note that this function vanishes on the boundary if $\partial\Omega\cap B_1\subset\{x_n=0\}$ is flat).

Step 1: *We prove that, whenever $B_\rho(z)\subset B_{7/8}$,*

$$\int_{\Omega\cap B_{\rho/2}(z)}\rho^4\mathcal{A}^2\,dx\le C\int_{\Omega\cap B_\rho(z)}\left(\rho^3|D^2u||\nabla u|+\rho^2|\nabla u|^2\right)\,dx, \tag{5.2}$$

where $\mathcal{A}$ is as in Lemma 2.3.

By scaling and a covering argument, it is enough to prove the result for $z=0$ and $\rho=1$.[11] Observe that, thanks to Lemma A.3(iii), $\nabla u\in(W^{2,p}\cap C^1)(\overline{\Omega}\cap B_{7/8})$ for all $p\in(1,\infty)$.

Since Ω is a ϑ-deformation of B_2^+ with $\vartheta\le 1/100$, Φ is a diffeomorphism. Let

$$\mathbf{Y}:=\nabla(\boldsymbol{e}_n\cdot\Phi^{-1})=\nabla((\Phi)^{-1})^n$$

[11]For this, note that when $B_\rho(z)\subset\Omega$ then (5.2) follows from Lemma 2.3. Note also that if $z\in\partial\Omega\cap B_{7/8}$ then, within a small ball centered at z, Ω is (after a translation, rotation, and dilation) a ϑ-deformation of B_2^+.

be the gradient of the pushforward of the n-coordinate $x_n : B_1^+ \to \mathbb{R}$ through Φ. Note that $\mathbf{Y}$ is orthogonal to $\partial\Omega$. We define $\boldsymbol{N} = \mathbf{Y}/|\mathbf{Y}|$, and note that $\boldsymbol{N}$ belongs to $C^2(\overline{\Omega})$ and that $\boldsymbol{N} = -\nu$ on $\partial\Omega$.

Consider the following convex $C^{1,1}$ regularization of the absolute value: for $r > 0$ small, we set

$$\phi_r(z) := |z| 1_{\{|z|>r\}} + \Big(\frac{r}{2} + \frac{|z|^2}{2r}\Big) 1_{\{|z|<r\}}. \tag{5.3}$$

Then $\phi_r(\nabla u) \in (W^{2,p} \cap C^1)(\overline{\Omega} \cap B_{7/8})$ for all $p < \infty$. Moreover, since u is nonnegative and superharmonic, unless $u \equiv 0$ (in which case there is nothing to prove) then it follows by the Hopf lemma that $|\nabla u| \geq c > 0$ on $\partial\Omega \cap B_{7/8}$, for some constant c. Hence, since ∇u is C^1 up to the boundary, for $r > 0$ small enough we have

$$\phi_r(\nabla u) = |\nabla u| \quad \text{in a neighborhood of } \partial\Omega \text{ inside } B_{7/8}. \tag{5.4}$$

After choosing $r > 0$ small enough such that (5.4) holds, we set

$$\mathbf{c} := \phi_r(\nabla u) - \boldsymbol{N} \cdot \nabla u,$$

and we take $\eta \in C_c^2(B_{7/8})$ with $\eta = 1$ in $B_{1/2}$. Note that $\mathbf{c} \equiv 0$ on $\partial\Omega \cap B_{7/8}$, and $\mathbf{c} \in (W^{2,p} \cap C^1)(\overline{\Omega} \cap B_{7/8})$. Then, since $\mathbf{c}$ vanishes on $\partial\Omega \cap B_{7/8}$, thanks to an approximation argument we are allowed to take $\xi = \mathbf{c}\eta$ as a test function in the stability inequality (1.4). Thus, with this choice, integration by parts yields

$$\int_{\Omega\cap B_1} \big(\Delta\mathbf{c} + f'_-(u)\mathbf{c}\big)\, \mathbf{c}\, \eta^2 \, dx \leq \int_{\Omega\cap B_1} \mathbf{c}^2 |\nabla\eta|^2 \, dx. \tag{5.5}$$

Note now that

$$\begin{aligned}\big(\Delta\mathbf{c} + f'_-(u)\mathbf{c}\big)\, \mathbf{c} = \big(&\Delta[\phi_r(\nabla u)] + f'_-(u)\phi_r(\nabla u)\big)\phi_r(\nabla u) \\ &- \big(\Delta(\boldsymbol{N}\cdot\nabla u) + f'_-(u)\boldsymbol{N}\cdot\nabla u\big)\big(\phi_r(\nabla u) - \boldsymbol{N}\cdot\nabla u\big) \\ &- \big(\Delta[\phi_r(\nabla u)] + f'_-(u)\phi_r(\nabla u)\big)\boldsymbol{N}\cdot\nabla u.\end{aligned} \tag{5.6}$$

Since $\Delta\nabla u = -f'_-(u)\nabla u$ (see Lemma A.3(ii)), we have

$$\big(\Delta[\phi_r(\nabla u)] + f'_-(u)\phi_r(\nabla u)\big)\phi_r(\nabla u) = f'_-(u)\phi_r(\nabla u)\Big(\phi_r(\nabla u) - \textstyle\sum_j u_j (\partial_j\phi_r)(\nabla u)\Big) \tag{5.7}$$

$$+\, \phi_r(\nabla u) \textstyle\sum_{i,j,k} (\partial^2_{jk}\phi_r)(\nabla u) u_{ij} u_{ik}. \tag{5.8}$$

Note that, inside the set $\{|\nabla u| \leq r\}$, the term (5.8) is nonnegative since ϕ_r is convex, while the term (5.7) is equal to

$$f'_-(u)\phi_r(\nabla u)\Big(\frac{r}{2} - \frac{|\nabla u|^2}{2r}\Big)$$

and therefore it is also nonnegative (all three factors are nonnegative). On the other hand, inside the set $\{|\nabla u| > r\}$, the term (5.7) vanishes, while the term (5.8) equals $\mathcal{A}^2$. Therefore, we conclude that

$$\big(\Delta[\phi_r(\nabla u)] + f'_-(u)\phi_r(\nabla u)\big)\phi_r(\nabla u) \geq \mathcal{A}^2\, 1_{\{|\nabla u|>r\}}, \tag{5.9}$$

where $\mathcal{A}^2$ is as in (2.7).

Coming back to (5.6), we note that

$$\Delta(\boldsymbol{N}\cdot\nabla u)+f'_-(u)\boldsymbol{N}\cdot\nabla u=\sum_i \Delta\boldsymbol{N}^i u_i+2\sum_{ij}\boldsymbol{N}^i_j u_{ij}, \tag{5.10}$$

so it follows from the bound $|\phi_r(\nabla u)|\le|\nabla u|+r$ that

$$\begin{aligned}\left|\int_{\Omega\cap B_1}\big(\Delta(\boldsymbol{N}\cdot\nabla u)+f'_-(u)\boldsymbol{N}\cdot\nabla u\big)\big(\phi_r(\nabla u)-\boldsymbol{N}\cdot\nabla u\big)\eta^2\,dx\right|\le\\ \le C\int_{\Omega\cap B_1}(|\nabla u|+r)\left(|D^2u|+|\nabla u|\right)\,dx.\end{aligned} \tag{5.11}$$

Also, since $\eta\in C^2_c(B_{7/8})$, integrating by parts and recalling (5.4) we have

$$\begin{aligned}\int_{\Omega\cap B_1}\Delta[\phi_r(\nabla u)]\,\boldsymbol{N}\cdot\nabla u\,\eta^2\,dx&=\int_{\Omega\cap B_1}\phi_r(\nabla u)\,\Delta(\boldsymbol{N}\cdot\nabla u)\,\eta^2\,dx\,+\\&+\int_{\Omega\cap B_1}\left(2\phi_r(\nabla u)\,\nabla(\boldsymbol{N}\cdot\nabla u)\cdot\nabla(\eta^2)+|\nabla u|\,\boldsymbol{N}\cdot\nabla u\,\Delta(\eta^2)\right)\,dx\\&+\int_{\partial\Omega\cap B_1}\left(|\nabla u|_\nu\,\boldsymbol{N}\cdot\nabla u\,\eta^2-|\nabla u|(\boldsymbol{N}\cdot\nabla u\,\eta^2)_\nu\right)d\mathcal{H}^{n-1}.\end{aligned} \tag{5.12}$$

Since $\boldsymbol{N}=\frac{\nabla u}{|\nabla u|}=-\nu$ on the boundary, it follows that on $\partial\Omega\cap B_1$ it holds

$$|\nabla u|_\nu\boldsymbol{N}\cdot\nabla u=-\frac{\sum_{ij}u_{ij}u_ju_j}{|\nabla u|}\qquad\text{and}\qquad|\nabla u|(\boldsymbol{N}\cdot\nabla u)_\nu=-\sum_{i,j}\boldsymbol{N}^i_ju_iu_j-\frac{\sum_{ij}u_{ij}u_ju_j}{|\nabla u|},$$

and therefore, thanks to Lemma 5.3,

$$\begin{aligned}\left|\int_{\partial\Omega\cap B_1}\left(|\nabla u|_\nu\,\boldsymbol{N}\cdot\nabla u\,\eta^2-|\nabla u|(\boldsymbol{N}\cdot\nabla u\,\eta^2)_\nu\right)\,d\mathcal{H}^{n-1}\right|&\le C\int_{\partial\Omega\cap B_{7/8}}|u_\nu|^2\,d\mathcal{H}^{n-1}\\&\le C\int_{\Omega\cap B_1}|\nabla u|^2dx.\end{aligned} \tag{5.13}$$

Thus, combining (5.12) and (5.10), and then using (5.13), we conclude that

$$\left|\int_{\Omega\cap B_1}\big(\Delta[\phi_r(\nabla u)]+f'_-(u)\phi_r(\nabla u)\big)\boldsymbol{N}\cdot\nabla u\,\eta^2\,dx\right|\le C\int_{\Omega\cap B_1}(|\nabla u|+r)\left(|D^2u|+|\nabla u|\right)\,dx.$$

Combining this bound with (5.5), (5.6), (5.9), and (5.11), we finally obtain

$$\int_{\Omega\cap B_1}\mathcal{A}^2\eta^2 1_{\{|\nabla u|>r\}}\,dx\le C\int_{\Omega\cap B_1}(|\nabla u|+r)^2+(|\nabla u|+r)\left(|D^2u|+|\nabla u|\right)\,dx.$$

Recalling that $\eta=1$ in $B_{1/2}$, letting $r\downarrow 0$ this proves (5.2) for $z=0$ and $\rho=1$, as desired.

Step 2: *We prove that*

$$\|\mathcal{A}\|^2_{L^2(\Omega\cap B_{7/8})}\le C\|\nabla u\|^2_{L^2(\Omega\cap B_1)}. \tag{5.14}$$

It suffices to prove that, for every $B_\rho(z) \subset B_{7/8}$ and $\varepsilon > 0$, we have

$$\rho^2 \|\mathcal{A}\|^2_{L^2(\Omega\cap B_{2\rho/5}(z))} \le \varepsilon \rho^2 \|\mathcal{A}\|^2_{L^2(\Omega\cap B_\rho(z))} + \frac{C}{\varepsilon}\|\nabla u\|^2_{L^2(\Omega\cap B_\rho(z))} \tag{5.15}$$

where $\mathcal{A}$ is as in (2.7). Indeed, it follows from Lemma A.4 applied with $\sigma(B) := \|\mathcal{A}\|^2_{L^2(\Omega\cap B)}$ that (5.15) leads to (5.14) with $\Omega \cap B_{7/8}$ replaced by $\Omega \cap B_{7/16}$. A covering and scaling argument then gives (5.14) with $\Omega \cap B_{7/8}$ in the left hand side.

To prove (5.15), we argue as at the beginning of Step 1 to note that we may assume $z = 0$ and $\rho = 1$.

We observe that, for any given $\eta \in C^2_c(B_{7/8})$ with $\eta \equiv 1$ in $B_{4/5}$, it follows from (2.11) and (2.12) that

$$-\int_{\Omega\cap B_1} \operatorname{div}\big(|\nabla u|\nabla u\big)\eta^2\,dx \ge \int_{\Omega\cap B_1} \big(-2\Delta u - C\mathcal{A}\big)|\nabla u|\,\eta^2\,dx. \tag{5.16}$$

Hence, since $|D^2u| \le |\Delta u| + C\mathcal{A}$ and $\Delta u \le 0$, using (5.16) we get

$$\int_{\Omega\cap B_1} |D^2u|\,|\nabla u|\,\eta^2\,dx \le \left|\frac{1}{2}\int_{\Omega\cap B_1} \operatorname{div}\big(|\nabla u|\nabla u\big)\eta^2\,dx\right| + C\int_{\Omega\cap B_1} \mathcal{A}\,|\nabla u|\,\eta^2\,dx. \tag{5.17}$$

On the other hand, using Lemma 5.3 we obtain

$$\begin{aligned}\left|\int_{\Omega\cap B_1} \operatorname{div}\big(|\nabla u|\nabla u\big)\eta^2\,dx\right| &= \left|-\int_{\partial\Omega\cap B_1} |u_\nu|^2\eta^2\,d\mathcal{H}^{n-1} - \int_\Omega |\nabla u|\nabla u\cdot\nabla(\eta^2)\,dx\right| \\ &\le C\int_{\Omega\cap B_1} |\nabla u|^2\,dx.\end{aligned} \tag{5.18}$$

Thus, combining (5.17) and (5.18), we get

$$\int_{\Omega\cap B_1} |D^2u|\,|\nabla u|\,\eta^2\,dx \le C\int_{\Omega\cap B_1} \mathcal{A}\,|\nabla u|\,\eta^2\,dx + C\int_{\Omega\cap B_1} |\nabla u|^2\,dx. \tag{5.19}$$

Recalling that $\eta \equiv 1$ in $B_{4/5}$, (5.19) and (5.2) yield, for every $\varepsilon \in (0,1)$,

$$\begin{aligned}\int_{\Omega\cap B_{2/5}} \mathcal{A}^2\,dx &\le C\|\nabla u\|^2_{L^2(\Omega\cap B_{4/5})} + C\int_{\Omega\cap B_{4/5}} |D^2u|\,|\nabla u|\,dx \\ &\le C\|\nabla u\|^2_{L^2(\Omega\cap B_1)} + C\int_{\Omega\cap B_1} \mathcal{A}\,|\nabla u|\,dx \le \frac{C}{\varepsilon}\|\nabla u\|^2_{L^2(\Omega\cap B_1)} + \varepsilon\int_{\Omega\cap B_1} \mathcal{A}^2\,dx,\end{aligned}$$

which proves (5.15).

Step 3: *We show that*

$$\int_{\Omega\cap B_{4/5}} \big|\operatorname{div}(|\nabla u|\,\nabla u)\big|\,dx \le C\int_{\Omega\cap B_1} |\nabla u|^2\,dx.$$

As in the previous step, we take $\eta \in C^2_c(B_{7/8})$ with $\eta \equiv 1$ in $B_{4/5}$. Then it suffices to combine (2.11), (2.12), (5.19), and (5.14), to get

$$\begin{aligned}\int_{\Omega\cap B_{4/5}} \left|\mathrm{div}(|\nabla u|\,\nabla u)\right| dx &\le \int_{\Omega\cap B_{4/5}} -2|\nabla u|\,\Delta u\,dx + C\int_{\Omega\cap B_{4/5}} |\nabla u|\,\mathcal{A}\,dx\\ &\le C\int_{\Omega\cap B_{7/8}} \mathcal{A}^2\,dx + C\int_{\Omega\cap B_1}|\nabla u|^2\,dx + C\left(\int_{\Omega\cap B_{4/5}}|\nabla u|^2\,dx\right)^{1/2}\left(\int_{\Omega\cap B_{4/5}}\mathcal{A}^2\,dx\right)^{1/2}\\ &\le C\int_{\Omega\cap B_1}|\nabla u|^2\,dx,\end{aligned}$$

as desired.

Step 4: *Conclusion.*

Here it is convenient to assume, after multiplying u by a constant, that $\|\nabla u\|_{L^2(\Omega\cap B_1)} = 1$.

Thanks to Step 3, we can repeat the same argument as the one used in Step 2 in the proof of Proposition 2.4 to deduce that, for a.e. $t > 0$,

$$\int_{\Omega\cap\{u=t\}\cap B_{3/4}} |\nabla u|^2\,d\mathcal{H}^{n-1} \le C\int_{\Omega\cap B_1}|\nabla u|^2\,dx = C. \tag{5.20}$$

Also, since u vanishes on $\partial\Omega\cap B_1$, setting $h(t) = \max\{1,t\}$, by the Sobolev embedding we deduce that

$$\int_{\mathbb{R}^+} dt\int_{\Omega\cap\{u=t\}\cap B_1\cap\{|\nabla u|\neq 0\}} h(t)^p\,|\nabla u|^{-1}\,d\mathcal{H}^{n-1} \le |\Omega\cap B_1\cap\{u<1\}| + \int_{\Omega\cap B_1} u^p\,dx \le C, \tag{5.21}$$

for some $p > 2$. Hence, choosing dimensional constants $q > 1$ and $\theta \in (0,1/3)$ such that $p/q = (1-\theta)/\theta$, we can write

$$\begin{aligned}\int_{\Omega\cap B_{3/4}} |\nabla u|^{3-3\theta}\,dx &= \int_{\mathbb{R}^+} dt\int_{\Omega\cap\{u=t\}\cap B_{3/4}\cap\{|\nabla u|\neq0\}} h(t)^{p\theta-q(1-\theta)}|\nabla u|^{-\theta+2(1-\theta)}d\mathcal{H}^{n-1}\\ &\le \left(\int_{\mathbb{R}^+} dt\int_{\Omega\cap\{u=t\}\cap B_1\cap\{|\nabla u|\neq0\}} h(t)^p|\nabla u|^{-1}d\mathcal{H}^{n-1}\right)^{\theta}\times\\ &\qquad\times\left(\int_{\mathbb{R}^+} dt\int_{\Omega\cap\{u=t\}\cap B_{3/4}\cap\{|\nabla u|\neq0\}} h(t)^{-q}|\nabla u|^2 d\mathcal{H}^{n-1}\right)^{1-\theta},\end{aligned} \tag{5.22}$$

and by (5.21) and the very same argument as the one used at the end of Step 3 in the Proof of Proposition 2.4 (now using (5.20)) we obtain

$$\int_{\Omega\cap B_{3/4}} |\nabla u|^{3-3\theta}\,dx \le C,$$

which concludes the proof. □

Remark 5.4. Note that, in Step 4 of the previous proof, one may also take any exponent $p > 2$, and then $\theta = 1/3$ and $q = p/2 > 1$. With these choices, if we normalize u so that $\|u\|_{L^p(\Omega\cap B_1)} = 1$ (instead of the normalization $\|\nabla u\|_{L^2(\Omega\cap B_1)} = 1$ made in Step 4 of the previous proof), setting $h(t) := \max\{1,t\}$ it follows from (5.22), (5.21), and the inequality in (5.20), that

$$\int_{\Omega\cap B_{3/4}} |\nabla u|^2\,dx \le C\Big(\int_{\Omega\cap B_1}|\nabla u|^2\,dx\Big)^{2/3} \qquad \text{whenever } \|u\|_{L^p(\Omega\cap B_1)}=1, \tag{5.23}$$

where we used that $\int_{\mathbb{R}^+} h(t)^{-q}\,dt \le C$.

In the general case, applying this estimate to $u/\|u\|_{L^p(\Omega\cap B_1)}$, we deduce that

$$\int_{\Omega\cap B_{3/4}} |\nabla u|^2\,dx \le C\left(\int_{\Omega\cap B_1}|u|^p\,dx\right)^{\frac{2}{3p}}\left(\int_{\Omega\cap B_1}|\nabla u|^2\,dx\right)^{\frac{2}{3}} \tag{5.24}$$

for every $p>2$.

As a consequence of this remark, we deduce the following important a priori estimate.

Proposition 5.5. *Under the assumptions of Proposition 5.2, there exists a dimensional constant C such that*

$$\|\nabla u\|_{L^2(\Omega\cap B_{1/2})} \le C\|u\|_{L^1(\Omega\cap B_1)}. \tag{5.25}$$

Proof. By Remark 5.4, we can choose $p\in(2,2^*)$ (here 2^* is the Sobolev exponent, or any number less than infinity if $n=2$) and then $\zeta\in(0,1)$ such that $p=\zeta 2^*+(1-\zeta)$, to obtain

$$\begin{aligned}\|\nabla u\|_{L^2(\Omega\cap B_{3/4})} &\le C\|u\|^{1/3}_{L^p(\Omega\cap B_1)}\|\nabla u\|^{2/3}_{L^2(\Omega\cap B_1)} \le C\|u\|^{\zeta/3}_{L^{2^*}(\Omega\cap B_1)}\|u\|^{(1-\zeta)/3}_{L^1(\Omega\cap B_1)}\|\nabla u\|^{2/3}_{L^2(\Omega\cap B_1)}\\ &\le C\|\nabla u\|^{(2+\zeta)/3}_{L^2(\Omega\cap B_1)}\|u\|^{(1-\zeta)/3}_{L^1(\Omega\cap B_1)} \le \varepsilon\|\nabla u\|_{L^2(\Omega\cap B_1)} + \frac{C}{\varepsilon}\|u\|_{L^1(\Omega\cap B_1)}.\end{aligned}$$

Hence, applying this estimate to the functions $u_{r,y}(x) := u(y+rx)$ for all balls $B_r(y)\subset B_1$ (as in the proof of Proposition 2.5), we can use Lemma A.4 with $\sigma(B)=\|\nabla u\|_{L^2(\Omega\cap B)}$ to conclude. □

6. Boundary C^α estimate for $n\le 9$, and proof of Theorem 1.5

In order to prove Theorem 1.5, as observed at the beginning of Section 5, every bounded domain of class C^3 can be covered by finitely many balls so that, after rescaling the balls to have size 1, inside each ball the boundary is a ϑ-deformation of B_2^+ for some $\vartheta\le\frac{1}{100}$. Hence, by applying Propositions 5.2 and 5.5, we deduce that there exists a neighborhood of $\partial\Omega$ in which the $W^{1,2+\gamma}$-norm of u is controlled by $\|u\|_{L^1(\Omega)}$. Combining this information with (1.6) and a covering argument, we conclude the validity of (1.9). Hence, we are left with proving (1.10).

By the same reasoning as the one we just did, but now using (1.7) instead of (1.6), to show (1.10) when $n\le 9$ it suffices to obtain a uniform C^α control near the boundary when $\partial\Omega$ is a small ϑ-deformation of B_2^+ (recall Definition 5.1). Hence, to conclude the proof of Theorem 1.5, it suffices to show the following:

Theorem 6.1. *Let $n\le 9$, $\vartheta\in[0,\frac{1}{100}]$, and $\Omega\subset\mathbb{R}^n$ be a ϑ-deformation of B_2^+. Assume that $u\in C^0(\overline{\Omega}\cap B_1)\cap C^2(\Omega\cap B_1)$ is a nonnegative stable solution of*

$$-\Delta u = f(u) \quad \text{in } \Omega\cap B_1 \qquad \text{and} \qquad u=0 \quad \text{on } \partial\Omega\cap B_1$$

for some nonnegative, nondecreasing, convex function $f : \mathbb{R} \to \mathbb{R}$. *Then*

$$\|u\|_{C^\alpha(\overline{\Omega}\cap B_{1/2})} \le C\|u\|_{L^1(\Omega\cap B_1)},$$

where $\alpha > 0$ *and* C *are dimensional constants.*

To prove this theorem, we first need the boundary analogue of the key interior estimate (2.1).

Lemma 6.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a* ϑ*-deformation of* B_2^+ *for* $\vartheta \in [0, \frac{1}{100}]$, *and let* $u \in C^2(\overline{\Omega} \cap B_1)$ *be a nonnegative stable solution of* $-\Delta u = f(u)$ *in* $\Omega \cap B_1$, *with* $u = 0$ *on* $\partial\Omega \cap B_1$. *Assume that* f *is locally Lipschitz.*

Then there exists a dimensional constant C *such that, for all* $\eta \in C_c^{0,1}(B_1)$,

$$\int_{\Omega\cap B_1} \Big(\big\{(n-2)\eta + 2x\cdot\nabla\eta\big\}\, \eta\, |\nabla u|^2 - 2(x\cdot\nabla u)\nabla u\cdot\nabla(\eta^2) - |x\cdot\nabla u|^2|\nabla\eta|^2 \Big)\, dx$$
$$\le C\vartheta \int_{\Omega\cap B_1} |\nabla u|^2 \big(\eta^2 + |x|\,|\nabla(\eta^2)| + |x|^2|\nabla\eta|^2\big)\, dx.$$

Proof. The key idea is to use a variant of $\xi = (x\cdot\nabla u)\eta$ as test function in the stability inequality (note that this function vanishes on the boundary if $\partial\Omega \cap B_1 = \{x_n = 0\} \cap B_1$ is flat).

We consider the vector-field

$$\mathbf{X}(x) = (D\Phi)(\Phi^{-1}(x))\cdot\Phi^{-1}(x) \qquad \text{for all } x \in \Omega\cap B_1,$$

with Φ as in Definition 5.1. Note that $\mathbf{X}$ is tangential to $\partial\Omega$ since, for $x \in \partial\Omega\cap B_1$, $\Phi^{-1}(x)$ is tangent to the flat boundary of B_1^+. Hence, since $u = 0$ on $\partial\Omega\cap B_1$, we deduce that $\mathbf{X}\cdot\nabla u = 0$ on $\partial\Omega\cap B_1$. Also, since Ω is a ϑ-deformation of B_2^+, it is easy to check that

$$|\mathbf{X} - x| \le C\vartheta|x|^2, \qquad |\nabla\mathbf{X} - \mathrm{Id}| \le C\vartheta|x|, \qquad |D^2\mathbf{X}| \le C\vartheta, \tag{6.1}$$

where C is a dimensional constant. The bound on $D^2\mathbf{X}$ follows by a direct computation, while the two first ones follow by integrating the latter and using that $\nabla\mathbf{X}(0) = \mathrm{Id}$ and $\mathbf{X}(0) = 0$.

Set $\mathbf{c} := \mathbf{X}\cdot\nabla u$, and take $\eta \in C_c^2\big(B_1\big)$. Note that $\mathbf{c} \equiv 0$ on $\partial\Omega\cap B_1$ and $\mathbf{c} \in (W^{2,p}_{\mathrm{loc}} \cap C^1)(\overline{\Omega}\cap B_1)$ for all $p < \infty$ (thanks to Lemma A.3). Hence, arguing as usual by approximation, one is allowed to take $\xi = \mathbf{c}\eta$ as a test function in the stability inequality (1.4). Thus, using that $\mathbf{c}$ vanishes on $\partial\Omega\cap B_1$, integration by parts yields

$$\int_{\Omega\cap B_1} \big\{\Delta\mathbf{c} + f'_-(u)\mathbf{c}\big\}\,\mathbf{c}\,\eta^2\, dx \le \int_{\Omega\cap B_1} \mathbf{c}^2|\nabla\eta|^2\, dx. \tag{6.2}$$

By a direct computation it follows that

$$\begin{aligned}
\Delta\mathbf{c} &= \mathbf{X}\cdot\nabla\Delta u + 2\nabla\mathbf{X} : D^2u + \Delta\mathbf{X}\cdot\nabla u\\
&= -f'_-(u)\,\mathbf{X}\cdot\nabla u + 2(\nabla\mathbf{X})^s : D^2u + \Delta\mathbf{X}\cdot\nabla u\\
&= -f'_-(u)\,\mathbf{c} + 2\mathrm{div}\big((\nabla\mathbf{X})^s\nabla u\big) + \big[\Delta\mathbf{X} - 2\mathrm{div}\big((\nabla\mathbf{X})^s\big)\big]\cdot\nabla u,
\end{aligned}$$

where $(\nabla\mathbf{X})^s := \frac{1}{2}(\nabla\mathbf{X} + (\nabla\mathbf{X})^*)$ is the symmetrized version of $\nabla\mathbf{X}$ and we used that $\nabla\mathbf{X} : D^2u = (\nabla\mathbf{X})^s : D^2u$ (since D^2u is a symmetric matrix).

Hence, substituting this identity in (6.2) and using (6.1) we get

$$\int_{\Omega\cap B_1} |\mathbf{X}\cdot\nabla u|^2\,|\nabla\eta|^2\;dx \geq 2\int_{\Omega\cap B_1} (\mathbf{X}\cdot\nabla u)\,\mathrm{div}\big((\nabla\mathbf{X})^s\nabla u\big)\,\eta^2\,dx - C\vartheta\int_{\Omega\cap B_1}|\nabla u|^2\,\eta^2\,dx. \tag{6.3}$$

Noticing that $|\nabla\mathbf{X}-\mathrm{Id}|+|(\nabla\mathbf{X})^s-\mathrm{Id}|+|\mathrm{div}\mathbf{X}-n|+|\nabla(\nabla\mathbf{X})^s|\leq C\vartheta$ (as a consequence of (6.1)), we see that

$$\begin{aligned}
&\mathrm{div}\big(2(\mathbf{X}\cdot\nabla u)\,[(\nabla\mathbf{X})^s\nabla u]-\big\{[(\nabla\mathbf{X})^s\nabla u]\cdot\nabla u\big\}\mathbf{X}\big)\\
&\quad=2(\mathbf{X}\cdot\nabla u)\,\mathrm{div}\big((\nabla\mathbf{X})^s\nabla u\big)+2[\nabla\mathbf{X}\nabla u]\cdot[(\nabla\mathbf{X})^s\nabla u]\\
&\qquad-\mathrm{div}\mathbf{X}\,\big\{[(\nabla\mathbf{X})^s\nabla u]\cdot\nabla u\big\}-\big\{[\mathbf{X}\cdot\nabla(\nabla\mathbf{X})^s]\cdot\nabla u\big\}\cdot\nabla u\big\}\\
&\quad=2(\mathbf{X}\cdot\nabla u)\,\mathrm{div}\big((\nabla\mathbf{X})^s\nabla u\big)+(2-n)|\nabla u|^2+O(\vartheta|\nabla u|^2).
\end{aligned}$$

Hence, using this identity in (6.3), and taking into account (6.1) and that $\mathbf{X}\cdot\nabla u=0$ and $\mathbf{X}\cdot\nu=0$ on $\partial\Omega\cap B_1$, we get

$$\begin{aligned}
&\int_{\Omega\cap B_1}|x\cdot\nabla u|^2\,|\nabla\eta|^2\;dx+C\vartheta\int_{\Omega\cap B_1}|\nabla u|^2\,\eta^2\,dx+C\vartheta\int_{\Omega\cap B_1}|\nabla u|^2|x|^2|\nabla\eta|^2\,dx\\
&\quad\geq\int_{\Omega\cap B_1}\Big(\mathrm{div}\big(2(\mathbf{X}\cdot\nabla u)\,[(\nabla\mathbf{X})^s\nabla u]-\big\{[(\nabla\mathbf{X})^s\nabla u]\cdot\nabla u\big\}\mathbf{X}\big)+(n-2)|\nabla u|^2\Big)\eta^2\,dx\\
&\quad=\int_{\Omega\cap B_1}\Big(-2(\mathbf{X}\cdot\nabla u)[(\nabla\mathbf{X})^s\nabla u]\cdot\nabla(\eta^2)+\big\{[(\nabla\mathbf{X})^s\nabla u]\cdot\nabla u\big\}\mathbf{X}\cdot\nabla(\eta^2)\Big)\,dx\\
&\qquad+\int_{\Omega\cap B_1}(n-2)|\nabla u|^2\eta^2\,dx\\
&\quad\geq\int_{B_1}\Big(-2(x\cdot\nabla u)\nabla u\cdot\nabla(\eta^2)+|\nabla u|^2x\cdot\nabla(\eta^2)+(n-2)|\nabla u|^2\eta^2\Big)\,dx\\
&\qquad-C\vartheta\int_{\Omega\cap B_1}|\nabla u|^2\,|x|\,|\nabla(\eta^2)|\,dx.
\end{aligned}$$

This proves the result for $\eta\in C^2_c(B_1)$, and the general case follows by approximation. ☐

To prove Theorem 6.1 we will use a blow-up argument that will rely on the following Liouville-type result in a half-space. In the blown-up domains, the constant ϑ in Lemma 6.2 will tend to zero. Recall that the class $\mathcal{S}(U)$, for $U\subset\mathbb{R}^n$, was defined in (4.2). We use the notation $\mathbb{R}^n_+:=\mathbb{R}^n\cap\{x_n>0\}$.

Proposition 6.3. *When $3\leq n\leq 9$, there exists a dimensional constant $\alpha_n>0$ such that the following holds. Assume that $u:\mathbb{R}^n_+\to\mathbb{R}$ belongs to $W^{1,2}_{\mathrm{loc}}\big(\overline{\mathbb{R}^n_+}\big)\cap C^0_{\mathrm{loc}}(\mathbb{R}^n_+)$, $u\in\mathcal{S}(\mathbb{R}^n_+)$, and $u=0$ on $\{x_n=0\}$ in the trace sense. Suppose in addition that, for some $\alpha\in(0,\alpha_n)$ and $\gamma>0$, denoting $u_R(x):=u(Rx)$ we have*

$$\|\nabla u_R\|_{L^{2+\gamma}(B^+_{3/2})}\leq C_1\|\nabla u_R\|_{L^2(B^+_2)}\leq C_2R^\alpha\qquad \textit{for all }\ R\geq 1 \tag{6.4}$$

with constants C_1 and C_2 independent of R, and that u satisfies

$$\int_{\mathbb{R}^n_+}\Big(\big\{(n-2)\eta+2x\cdot\nabla\eta\big\}\,\eta\,|\nabla u|^2-2(x\cdot\nabla u)\nabla u\cdot\nabla(\eta^2)-|x\cdot\nabla u|^2|\nabla\eta|^2\Big)\,dx\leq 0 \tag{6.5}$$

for all $\eta \in C^{0,1}_c\big(\overline{\mathbb{R}^n_+}\big)$*. Then* $u \equiv 0$*.*

Proof. Let us define, for $\rho > 0$,

$$\mathcal{D}(\rho) := \rho^{2-n} \int_{B_\rho^+} |\nabla u|^2\, dx \qquad \text{and} \qquad \mathcal{R}(\rho) := \int_{B_\rho^+} |x|^{-n} |x \cdot \nabla u|^2\, dx.$$

We divide the proof in three steps. As we shall see, for the validity of Step 1 the assumption $3 \le n \le 9$ is crucial.

Step 1: *We prove that, for all* $\rho > 0$,

$$\mathcal{R}(\rho) \le C\rho^{2-n} \int_{B_{2\rho}^+ \setminus B_\rho^+} |\nabla u|^2\, dx \tag{6.6}$$

for some dimensional constant $C > 0$*.*

Let $\psi \in C^\infty_c(B_2)$ be some radial decreasing nonnegative cut-off function with $\psi \equiv 1$ in B_1, and set $\psi_\rho(x) := \psi(x/\rho)$. Then, as in the interior case, for $a < n$ and $\varepsilon \in (0, \rho)$ we use the Lipschitz function $\eta_\varepsilon(x) := \min\{|x|^{-a/2}, \varepsilon^{-a/2}\}\psi_\rho(x)$ as a test function in (6.5). Hence, noting that $\nabla\psi_\rho$ has size C/ρ and vanishes outside of the annulus $B_{2\rho} \setminus B_\rho$, and throwing away the term $\int_{\mathbb{R}^n_+ \cap B_\varepsilon} (n-2)\eta_\varepsilon^2 |\nabla u|^2 dx$, we obtain

$$\int_{\mathbb{R}^n_+ \setminus B_\varepsilon} \Big\{(n-2-a)|\nabla u|^2 + \Big(2a - \frac{a^2}{4}\Big)|x\cdot\nabla u|^2 |x|^{-2}\Big\}|x|^{-a}\psi_\rho^2\, dx \le C(n,a)\rho^{-a} \int_{B_{2\rho}^+ \setminus B_\rho} |\nabla u|^2\, dx.$$

Choosing $a := n-2$, since $2a - \frac{a^2}{4} = (n-2)\big(2 - \frac{n-2}{4}\big) = \frac{1}{4}(n-2)(10-n) > 0$ for $3 \le n \le 9$ we obtain

$$\int_{\mathbb{R}^n_+ \setminus B_\varepsilon} |x|^{-n}|x\cdot\nabla u|^2\, \psi_\rho^2\, dx \le C\rho^{2-n} \int_{B_{2\rho}^+ \setminus B_\rho} |\nabla u|^2\, dx.$$

Recalling that $\psi_\rho^2 \equiv 1$ in B_ρ, the claim follows by letting $\varepsilon \downarrow 0$.

Step 2: *We prove that there exists a dimensional constant* C *such that, if for some* $R \ge 1$ *we have*

$$\int_{B_1^+} |\nabla u_R|^2\, dx \ge \frac{1}{2} \int_{B_2^+} |\nabla u_R|^2\, dx,$$

then

$$\int_{B_{3/2}^+} |\nabla u_R|^2\, dx \le C \int_{B_{3/2}^+ \setminus B_1^+} |x|^{-n}|x\cdot\nabla u_R|^2\, dx.$$

The proof is by compactness. We assume by contradiction that we have a sequence $u_k := u_{R_k}/\|\nabla u_{R_k}\|_{L^2(B_{3/2}^+)} \in \mathcal{S}(B_2^+) \cap W^{1,2}_{\rm loc}(\overline{\mathbb{R}^n_+})$, with $u_k = 0$ on $\{x_n = 0\}$, satisfying

$$\int_{B_1^+} |\nabla u_k|^2\, dx \ge \frac{1}{2} \int_{B_2^+} |\nabla u_k|^2\, dx, \tag{6.7}$$

$$\int_{B_{3/2}^+} |\nabla u_k|^2\, dx = 1, \qquad \text{and} \qquad \int_{B_{3/2}^+ \setminus B_1^+} |x|^{-n}|x\cdot\nabla u_k|^2\, dx \to 0. \tag{6.8}$$

Note that, since $\int_{B_2^+} |\nabla u_k|^2\,dx \le 2$, thanks to Lemma A.1 and our interior $W^{1,2+\gamma}$ estimate there exists a function u such that, up to a subsequence, $u_k \to u$ strongly in $W^{1,2}_{\rm loc}(B_2^+)$. On the other hand, using the first bound in (6.4), for every $\delta \in (0,1)$ we have

$$\begin{aligned}\int_{B_{3/2}^+\cap\{x_n\le\delta\}} |\nabla u_k|^2\,dx &\le \Big(\int_{B_{3/2}^+\cap\{x_n\le\delta\}} |\nabla u_k|^{2+\gamma}\,dx\Big)^{2/(2+\gamma)} \big|B_{3/2}^+\cap\{x_n\le\delta\}\big|^{\gamma/(2+\gamma)}\\ &\le C\delta^{\gamma/(2+\gamma)}.\end{aligned}$$

This means that the mass of $|\nabla u_k|^2$ near the boundary can be made arbitrarily small by choosing δ small enough. Combining this information with the convergence of $u_k \to u$ in $W^{1,2}_{\rm loc}(B_2^+)$ we deduce that $u_k \to u$ strongly in $W^{1,2}(B_{3/2}^+)$. Moreover, by Theorem 4.1 we obtain that $u \in \mathcal{S}(B_{3/2}^+)$, and taking the limit in (6.8) we obtain

$$\int_{B_{3/2}^+} |\nabla u|^2\,dx = 1 \qquad \text{and} \qquad x\cdot\nabla u \equiv 0 \quad \text{in } B_{3/2}^+\setminus B_1^+.$$

Moreover, since the trace operator is continuous in $W^{1,2}(B_{3/2}^+)$, we deduce that $u = 0$ on $\{x_n = 0\}\cap B_{3/2}$.

Hence, we have found a function $u \in \mathcal{S}(B_{3/2}^+)$ which is 0-homogeneous in the half annulus $B_{3/2}^+\setminus\overline{B_1^+}$. In particular, since u is a weak solution of $-\Delta u = f(u)$ in $B_{3/2}^+$ with $-\Delta u = f(u) \in L^1_{\rm loc}\cap C^0(B_{3/2}^+\setminus\overline{B_1^+})$, this is only possible if $f\equiv 0$ (this follows from the fact that Δu is (-2)-homogeneous while $f(u)$ is 0-homogeneous). It follows that u is a 0-homogeneous harmonic function in the half annulus $B_{3/2}^+\setminus\overline{B_1^+}$ vanishing on $\partial(B_{3/2}^+\setminus\overline{B_1^+})\cap\{x_n=0\}$. Hence, as in the proof of Lemma 3.1, the supremum and infimum of u are attained at interior points, and thus u must be zero by the strong maximum principle. Furthermore, exactly as in the proof of Lemma 3.1, the superharmonicity of u combined with the fact that u vanishes in $B_{3/2}^+\setminus B_1^+$ gives that u vanishes in $B_{3/2}^+$. This contradicts the fact that $\int_{B_{3/2}^+}|\nabla u|^2\,dx = 1$ and concludes the proof.

Step 3: *Conclusion.*

Exactly as in Step 1 of the proof of Theorem 1.2, using Lemma 3.2 (combined with Steps 1 and 2 above) we deduce that

$$\|x\cdot\nabla u_{rR}\|_{L^2(B_1^+)} \le Cr^{\alpha_n}\|\nabla u_R\|_{L^2(B_1^+)} \qquad \text{for all } r\in(0,1/2) \text{ and } R\ge 1,$$

where C and $\alpha_n > 0$ are dimensional constants. Hence, since by assumption $\|\nabla u_R\|_{L^2(B_1^+)} \le CR^\alpha$ with $\alpha < \alpha_n$, given a constant $M > 0$, we choose $r = M/R$ and let $R\to\infty$ to find

$$\|x\cdot\nabla u_M\|_{L^2(B_1^+)} = 0.$$

Since $u_M \in \mathcal{S}(B_1^+)$ and $u_M = 0$ on $\{x_n = 0\}\cap\overline{B_1^+}$, as in the previous Step 2 we conclude that $u_M \equiv 0$. Since $M > 0$ is arbitrary, the proof is finished. □

We can now prove Theorem 6.1.

Proof of Theorem 6.1. Note that, as in the interior case, we may assume that $3 \le n \le 9$ by adding superfluous variables and considering a "cylinder" with base Ω. Also, by Lemma A.3, $u \in C^2(\overline{\Omega} \cap B_1)$.

Recalling that Ω is a ϑ-deformation of B_2^+ with $\vartheta \in [0, \frac{1}{100}]$, it suffices to prove that there exists a dimensional constant C such that

$$r^{2-n} \int_{\Omega \cap B_r} |\nabla u|^2 \, dx \le C \, r^{\alpha_n} \|\nabla u\|^2_{L^2(\Omega \cap B_1)} \qquad \text{for all } \ r \in (0,1), \tag{6.9}$$

where α_n is given by Proposition 6.3. Indeed, given $r \in (0, \frac{1}{4})$ there exists a dimensional constant $c \in (0,1)$ such that $B_{cr}(re_n) \subset \Omega$, and the L^∞ estimate from (1.7) applied in this ball, together with the inclusion $B_{cr}(re_n) \subset B_{2r}$, give

$$u(re_n) \le Cr^{-n} \int_{B_{cr}(re_n)} u \, dx \le Cr^{-n} \int_{\Omega \cap B_{2r}} u \, dx.$$

Thus, once (6.9) is proven, it follows from this, the Sobolev inequality, and Proposition 5.5, that

$$\begin{aligned} u(re_n) &\le Cr^{-n} \int_{\Omega \cap B_{2r}} u \, dx \le C \Big(r^{2-n} \int_{\Omega \cap B_{2r}} |\nabla u|^2 \, dx \Big)^{1/2} \\ &\le C \, r^{\alpha_n/2} \|\nabla u\|_{L^2(\Omega \cap B_{1/2})} \le C \, r^{\alpha_n/2} \|u\|_{L^1(\Omega \cap B_1)} \end{aligned}$$

for all $r \in (0, 1/4)$. Applying this estimate to the functions $u_y(z) := u(y+z)$ with $y \in \partial\Omega \cap B_{1/2}$, we deduce that

$$u(x) \le C \, \mathrm{dist}(x, \partial\Omega)^{\alpha_n/2} \|u\|_{L^1(\Omega \cap B_1)} \qquad \text{for all } \ x \in \Omega \cap B_{1/2}.$$

Combining this growth control with (1.7) it follows by a standard argument that

$$\|u\|_{C^\beta(\overline{\Omega} \cap B_{1/2})} \le C \|u\|_{L^1(\Omega \cap B_1)},$$

where $\beta := \min\{\alpha_n/2, \alpha\}$, with α as in (1.7). Hence, we only need to prove (6.9).

We argue by contradiction, similarly to [29, 28]. Assume that there exist a sequence of radii $r_k \in (0,1)$ and of stable solutions u_k with nonlinearities f_k in domains Ω_k, with u_k, f_k, Ω_k satisfying the hypotheses of the theorem and such that

$$r_k^{2-n} \int_{\Omega_k \cap B_{r_k}} |\nabla u_k|^2 \, dx \ge k \, r_k^{\alpha_n} \|\nabla u_k\|^2_{L^2(\Omega_k \cap B_1)} \tag{6.10}$$

for all $k \in \mathbb{N}$. Then, for $r \in (0,1)$ we define the nonincreasing function

$$\Theta(r) := \sup_k \sup_{s \in (r,1)} \frac{s^{2-n} \int_{\Omega_k \cap B_s} |\nabla u_k|^2 \, dx}{s^{\alpha_n} \|\nabla u_k\|^2_{L^2(\Omega_k \cap B_1)}}$$

and note that Θ is finite since obviously $\Theta(r) \le r^{2-n-\alpha_n} < \infty$ for all $r > 0$. By (6.10) and since Θ is nonincreasing we have $\Theta(r) \uparrow +\infty$ as $r \downarrow 0$. Also, by the definition of Θ, for any given $m \in \mathbb{N}$ there exists $\overline{r}_m \in (1/m, 1)$ and k_m such that

$$\Theta(\overline{r}_m) \ge \frac{\overline{r}_m^{\,2-n} \int_{\Omega_{k_m} \cap B_{\overline{r}_m}} |\nabla u_{k_m}|^2 \, dx}{\overline{r}_m^{\,\alpha_n} \|\nabla u_{k_m}\|^2_{L^2(\Omega_{k_m} \cap B_1)}} \ge \frac{9}{10} \Theta(1/m) \ge \frac{9}{10} \Theta(\overline{r}_m). \tag{6.11}$$

Since $\Theta(1/m) \uparrow \infty$ as $m \uparrow \infty$, it follows that $\overline{r}_m \downarrow 0$.

Consider the sequence of functions

$$\overline{u}_m := \frac{u_{k_m}(\overline{r}_m\,\cdot\,)}{\overline{r}_m^{\alpha_n}\,\Theta(\overline{r}_m)\,\|\nabla u_{k_m}\|^2_{L^2(\Omega_{k_m}\cap B_1)}},$$

and denote $\widetilde{\Omega}_m := \frac{1}{\overline{r}_m}\Omega_{k_m}$. Then $\widetilde{\Omega}_m \to \mathbb{R}^n_+$ locally uniformly as $m \to \infty$, and for all $R \in [1, 1/\overline{r}_m)$ we have

$$\begin{aligned} R^{2-n}\int_{\widetilde{\Omega}_m\cap B_R} |\nabla \overline{u}_m|^2\,dx &= \frac{(R\overline{r}_m)^{2-n}\int_{\Omega_{k_m}\cap B_{R\overline{r}_m}} |\nabla u_{k_m}|^2\,dx}{\overline{r}_m^{\alpha_n}\,\Theta(\overline{r}_m)\,\|\nabla u_{k_m}\|^2_{L^2(\Omega_{k_m}\cap B_1)}} \\ &\le \frac{(R\overline{r}_m)^{2-n}\int_{\Omega_{k_m}\cap B_{R\overline{r}_m}} |\nabla u_{k_m}|^2\,dx}{(R\overline{r}_m)^{\alpha_n}\,\Theta(R\overline{r}_m)\,\|\nabla u_{k_m}\|^2_{L^2(\Omega_{k_m}\cap B_1)}} R^{\alpha_n} \le R^{\alpha_n}, \end{aligned} \tag{6.12}$$

where we used that $\Theta(R\overline{r}_m) \le \Theta(\overline{r}_m)$ since $R \ge 1$.

On the other hand, using (6.11) we have

$$\int_{\widetilde{\Omega}_m\cap B_1} |\nabla \overline{u}_m|^2\,dx \ge \frac{9}{10}. \tag{6.13}$$

Now, similarly to Step 2 in the proof of Proposition 6.3, thanks to Proposition 5.2 and Lemma A.1 there exists a function $\overline{u}$ such that, up to a subsequence, $\overline{u}_m \to \overline{u}$ strongly in $W^{1,2}_{\rm loc}(\mathbb{R}^n_+)$. In addition, since $\widetilde{\Omega}_m \to \mathbb{R}^n_+$, using again Proposition 5.2 we see that, for every $R \ge 1$ and $\delta \in (0,1)$,

$$\begin{aligned} \int_{\widetilde{\Omega}_m\cap B_R\cap\{x_n\le\delta\}} |\nabla \overline{u}_m|^2\,dx &\le \Big(\int_{\widetilde{\Omega}_m\cap B_R\cap\{x_n\le\delta\}} |\nabla \overline{u}_m|^{2+\gamma}\,dx\Big)^{\frac{2}{2+\gamma}} \big|\widetilde{\Omega}_m\cap B_R\cap\{x_n\le\delta\}\big|^{\frac{2}{2+\gamma}} \\ &\le C(R)\,\big(\delta^{\gamma/(2+\gamma)} + o_m(1)\big), \end{aligned}$$

where $o_m(1) \to 0$ as $m \to \infty$. Hence, as $m \to \infty$ the mass of $|\nabla \overline{u}_m|^2$ near the boundary can be made arbitrarily small by choosing δ small enough, and combining this information with the convergence of $\overline{u}_m \to \overline{u}$ in $W^{1,2}_{\rm loc}(\mathbb{R}^n_+)$ we deduce that $\overline{u}_m \to \overline{u}$ strongly in $W^{1,2}(B_R^+)$ for all $R \ge 1$.

Moreover, by Theorem 4.1 we obtain that $\overline{u} \in \mathcal{S}(\mathbb{R}^n_+)$, and taking the limit in (6.12) and (6.13) we obtain

$$\int_{B_1^+} |\nabla \overline{u}|^2\,dx \ge \frac{9}{10} \qquad \text{and} \qquad \|\nabla \overline{u}_R\|^2_{L^2(B_1^+)} = R^{2-n}\int_{B_R^+} |\nabla \overline{u}|^2\,dx \le R^{\alpha_n} \quad \text{for all } R \ge 1,$$

where $\overline{u}_R := \overline{u}(R\,\cdot\,)$. Moreover, since the trace operator is continuous in $W^{1,2}(B_R^+)$, we have $\overline{u} = 0$ on $\{x_n = 0\}$. The last bound (applied with R replaced by $2R$) and Proposition 5.2 give that $\overline{u}$ satisfies the hypothesis (6.4) in Proposition 6.3 with $\alpha = \alpha_n/2$.

Therefore, to show that $\overline{u}$ satisfies the assumptions Proposition 6.3 with $\alpha = \alpha_n/2$, it only remains to prove that (6.5) holds (with u replaced by $\overline{u}$). This is a consequence of Lemma 6.2:

since $\frac{1}{R}\widetilde{\Omega}_m$ is a ϑ-deformation of B_2^+ with $\vartheta = CR\overline{r}_m$, for all $\eta \in C_c^{0,1}\big(\overline{B_1^+}\big)$ we have

$$\int_{(\frac{1}{R}\widetilde{\Omega}_m)\cap B_1} \Big(\big\{(n-2)\eta + 2x\cdot\nabla\eta\big\}\,\eta\,|\nabla\overline{u}_{m,R}|^2 - 2(x\cdot\nabla\overline{u}_{m,R})\nabla\overline{u}_{m,R}\cdot\nabla(\eta^2)\Big)\,dx$$
$$- \int_{(\frac{1}{R}\widetilde{\Omega}_m)\cap B_1} |x\cdot\nabla\overline{u}_{m,R}|^2|\nabla\eta|^2\,dx \le CR\overline{r}_m \int_{(\frac{1}{R}\widetilde{\Omega}_m)\cap B_1} |\overline{u}_{m,R}|^2\,dx,$$

and hence, by letting $m\to\infty$, we deduce that

$$\int_{B_1^+} \Big(\big\{(n-2)\eta + 2x\cdot\nabla\eta\big\}\,\eta\,|\nabla\overline{u}_R|^2 - 2(x\cdot\nabla\overline{u}_R)\nabla\overline{u}_R\cdot\nabla(\eta^2) - |x\cdot\nabla\overline{u}_R|^2|\nabla\eta|^2\Big)\,dx \le 0$$

for all $\eta\in C_c^{0,1}\big(\overline{B_1^+}\big)$. Since this holds for all $R>1$, this proves that (6.5) holds for every $\eta\in C_c^{0,1}\big(\overline{\mathbb{R}^n_+}\big)$ with u replaced by $\overline{u}$. Thus, it follows by Proposition 6.3 that $\overline{u}\equiv 0$, a contradiction since $\int_{B_1^+}|\nabla\overline{u}|^2\,dx \ge \frac{9}{10}$. □

As explained at the beginning of this section, Theorem 1.5 follows immediately from Theorem 6.1. Thus, it only remains to give the:

Proof of Corollary 1.6. Since $u\in W_0^{1,2}(\Omega)$, it follows from (1.3) and a standard approximation argument that

$$\int_\Omega f(u)\mathrm{dist}(\cdot,\partial\Omega)\,dx \le C\|\nabla u\|_{L^2(\Omega)}.$$

Thus, thanks to the approximation argument in [18, Theorem 3.2.1 and Corollary 3.2.1], u can be written as the limit of classical solutions $u_\varepsilon\in C_0^2(\overline{\Omega})$ of $-\Delta u_\varepsilon = (1-\varepsilon)f(u_\varepsilon)$ in Ω, as $\varepsilon\downarrow 0$. Thus, applying Theorem 1.5 to the functions u_ε, using Proposition B.1, and letting $\varepsilon\downarrow 0$, the result follows. □

7. Estimates for $n\ge 10$: Proof of Theorem 1.9

In this section we show how our method also gives sharp information in higher dimensions. We first deal with the interior case, and we prove a strengthened version of Theorem 1.9. Recall the definition of the Morrey space $M^{p,\beta}(\Omega)$ given in Section 1.3. Here $p\ge 1$ and $\beta\in(0,n)$.

Theorem 7.1. *Let $u\in C^2(B_1)$ be a stable solution of*

$$-\Delta u = f(u) \quad \textit{in } B_1,$$

with $f:\mathbb{R}\to\mathbb{R}$ locally Lipschitz and nonnegative, and assume that $n\ge 10$. Then

$$\|u\|_{M^{\frac{2\beta}{\beta-2},\beta}(B_{1/4})} + \|\nabla u\|_{M^{2,\beta}(B_{1/4})} \le C\|u\|_{L^1(B_1)} \quad \textit{for every} \quad \beta\in(n-2\sqrt{n-1}-2,n),$$

for some constant C depending only on n and β. In particular (1.13) *holds.*

Recall that, in the radially symmetric case,

$$\text{if } u \text{ is radial and } \nabla u\in M^{2,\beta}(B_{1/4}),\ \text{then } u\in L^{\frac{2n}{\beta-2}}(B_{1/8}); \tag{7.1}$$

indeed this follows from [9] after cutting-off u outside $B_{1/8}$ to have compact support in $B_{1/4}$.

Thus, Theorem 7.1 together with (7.1) yield the following L^p bound for radial solutions:

$$\|u\|_{L^p(B_{1/8})} \le C\|u\|_{L^1(B_1)} \qquad \text{for every} \quad p < \overline{p}_n := \frac{2n}{n - 2\sqrt{n-1} - 4}. \tag{7.2}$$

Hence, in the radial case we recover the L^p estimates established by Capella and the first author in [8], which are known to be sharp: (7.2) cannot hold for $p = \overline{p}_n$.

Unfortunately, as shown recently by Charro and the first author in [9], the embedding (7.1) is false for non-radial functions,[12] and thus it is not clear whether (7.2) holds in the nonradial case, too. From $\nabla u \in M^{2,\beta}$, the best one can say is $u \in M^{\frac{2\beta}{\beta-2},\beta} \subset L^{\frac{2\beta}{\beta-2}}$ as stated in Theorem 7.1.

Proof of Theorem 7.1. We split the proof into two cases.

Case 1: Assume first $n \ge 11$. Then, repeating the proof of Lemma 2.1, in Step 2 we can take an exponent a satisfying

$$8 < a < 2(1 + \sqrt{n-1}) < n - 2. \tag{7.3}$$

Then, choosing $0 \le \zeta \le 1$ such that $\zeta_{|B_{1/4}} = 1$, $\zeta_{|\mathbb{R}^n \setminus B_{1/2}} = 0$, and $|\nabla \zeta| \le C$, we obtain

$$\int_{B_{1/4}} \Big\{(n-2-a)|\nabla u|^2 + \Big(2a - \frac{a^2}{4}\Big)|x\cdot\nabla u|^2|x|^{-2}\Big\}|x|^{-a}\,dx \le C(n,a)\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2\,dx.$$

Since $2a - a^2/4 < 0$, the left hand side above can be bounded from below by

$$(n - 2 + a - a^2/4)\int_{B_{1/4}} |\nabla u|^2|x|^{-a}\,dx,$$

and because $n - 2 + a - a^2/4 > 0$ (thanks to the choice of a in (7.3)), we deduce that

$$\int_{B_{1/4}} |\nabla u|^2|x|^{-a}\,dx \le C\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2\,dx \le C\|u\|^2_{L^1(B_1)},$$

where the last inequality follows from Proposition 2.5.

Applying this estimate to the functions $u_y(x) := u(y+x)$ with $y \in \overline{B}_{1/4}$, it follows that

$$\rho^{-a}\int_{B_\rho(y)} |\nabla u|^2\,dx \le \int_{B_{1/4}(y)} |\nabla u(x)|^2|x-y|^{-a}\,dx \le C\|u\|^2_{L^1(B_1)} \quad \text{for all } \ y \in \overline{B}_{1/4},\ \rho \in (0, \tfrac{1}{4}).$$

This proves that $\nabla u \in M^{2,\beta}(B_{1/4})$ for every $\beta := n - a > n - 2\sqrt{n-1} - 2$.

Now, after cutting-off u outside of $B_{1/8}$ to have compact support in $B_{1/4}$, we can apply [1, Proposition 3.1 and Theorems 3.1 and 3.2] (see also the proof in [9, Section 4]) and, since $\beta \in (2, n)$, we deduce that $u \in M^{\frac{2\beta}{\beta-2},\beta}(B_{1/8})$. This estimate in $B_{1/8}$ can also be stated in $B_{1/4}$, as in Theorem 7.1, after an scaling and covering argument. Taking $p = \frac{2\beta}{\beta-2}$, this leads to (1.13).

Case 2: Assume now $n = 10$. Then, repeating the proof of Lemma 2.1, in Step 2 we take

$$\eta = |x|^{-4}\big|\log|x|\big|^{-\delta/2}\zeta, \tag{7.4}$$

[12]When $\beta \in (2, n)$ is an integer, this can be easily shown considering functions in $\mathbb{R}^n$ depending only on β Euclidean variables; see [9]. We thus encounter here the same obstruction as in Remark 2.2.

with $\delta > 0$ small. Then, choosing $0 \le \zeta \le 1$ such that $\zeta_{|B_{1/4}} = 1$, $\zeta_{|\mathbb{R}^n \setminus B_{1/2}} = 0$, and $|\nabla \zeta| \le C$, we obtain

$$\int_{B_{1/4}} \delta \big|\log|x|\big|^{-1-\delta} |\nabla u|^2 |x|^{-8}\,dx$$
$$+ \int_{B_{1/4}} \Big\{ 2\delta \big|\log|x|\big|^{-1-\delta} |x\cdot\nabla u|^2 |x|^{-2} - (\delta^2/4)\big|\log|x|\big|^{-2-\delta} |x\cdot\nabla u|^2|x|^{-2} \Big\} |x|^{-8}\,dx$$
$$\le C(n,\delta) \int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2\,dx.$$

Now, using that

$$(\delta^2/4)\big|\log|x|\big|^{-2-\delta} \le 2\delta\big|\log|x|\big|^{-1-\delta} \quad \text{in } B_{1/4},$$

we deduce

$$\int_{B_{1/4}} \big|\log|x|\big|^{-1-\delta} |\nabla u|^2|x|^{-8}\,dx \le C(n,\delta)\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2\,dx.$$

Finally, since for every $\varepsilon > 0$ we have

$$|x|^{-8+\varepsilon} \le C(n,\delta,\varepsilon)\big|\log|x|\big|^{-1-\delta}|x|^{-8} \quad \text{in } B_{1/4},$$

we find that

$$\int_{B_{1/4}} |\nabla u|^2|x|^{-a}\,dx \le C(n,\delta,a)\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2\,dx$$

for all $a := 8-\varepsilon < 8$. The rest of the proof is then analogous to the case $n \ge 11$. □

We now deal with a boundary version of the same theorem. We first consider a domain Ω that is a ϑ-deformation of B_2^+ (see Definition 5.1).

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^n$ be a ϑ-deformation of B_2^+ for $\vartheta \in [0, \frac{1}{100}]$, and let $u \in C^0(\overline{\Omega} \cap B_1) \cap C^2(\Omega \cap B_1)$ be a stable solution of*

$$-\Delta u = f(u) \quad \text{in } \Omega\cap B_1 \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega \cap B_1,$$

with $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz and nonnegative. Assume that $n \ge 10$. Then

$$\|u\|_{M^{\frac{2\beta}{\beta-2},\beta}(\Omega\cap B_{1/2})} + \|\nabla u\|_{M^{2,\beta}(\Omega\cap B_{1/2})} \le C\|u\|_{L^1(\Omega\cap B_1)} \quad \text{for all} \quad \beta \in (n-2\sqrt{n-1}-2, n),$$

for some constant C depending only on n and β.

Proof. Assume $n \ge 11$; the case $n = 10$ can be handled similarly (as done in the proof of Theorem 7.1). In this case we start from Lemma 6.2 and, as in Step 1 in the proof of Proposition 6.3, we let $\psi \in C_c^\infty(B_1)$ be some radial decreasing nonnegative cut-off function with $\psi \equiv 1$ in $B_{1/2}$. In Lemma 6.2 we use the test function $\eta(x) := |x|^{-a/2}\psi(x)$ with $a < n$. Then, since the domain $\rho^{-1}(\Omega \cap B_\rho)$ is a $(c\rho\vartheta)$-deformation of B_2^+ (for some dimensional constant c), we deduce that

$$\int_{\Omega\cap B_\rho} \Big\{ (n-2-a-C\rho\vartheta)|\nabla u|^2 + \Big(2a - \frac{a^2}{4}\Big)(x\cdot\nabla u)^2|x|^{-2}\Big\}|x|^{-a}\,dx$$
$$\le C(n,a)\rho^{-a}\int_{\Omega\cap B_{2\rho}\setminus B_\rho} |\nabla u|^2\,dx.$$

Hence, given a satisfying (7.3), we can take ρ_0 sufficiently small (depending on n and a) so that $n-2+a-a^2/4-C\rho\vartheta>0$ for all $\rho\le\rho_0$. This allows us to argue as in the proof of Theorem 7.1 to get

$$\int_{\Omega\cap B_{\rho_0}}|\nabla u|^2|x|^{-a}\,dx\le C\int_{\Omega\cap B_{1/2}}|\nabla u|^2\,dx\le C\|u\|^2_{L^1(\Omega\cap B_1)}$$

by Proposition 5.5. We now conclude as in Theorem 7.1. □

We finally give the:

Proof of Theorem 1.9. The estimate (1.13) follows from Theorem 7.1 by taking $p=\frac{2\beta}{\beta-2}$ and using a covering argument. On the other hand, (1.14) follows from Theorems 7.1 and 7.2, using again a covering argument. □

Appendix A. Technical lemmata

The next lemma is a regularity and compactness result for superharmonic functions. For an integrable function v to be superharmonic, we mean it in the distributional sense. For all our applications of the lemma one could further assume that $v\in W^{1,2}(B_R)$ and that $-\Delta v\ge 0$ is meant in the usual $W^{1,2}$ weak sense (which, in this case, is equivalent to the distributional sense), but we do not need this additional hypothesis.

Lemma A.1. *Let $v\in L^1(B_R)$ be superharmonic in a ball $B_R\subset\mathbb{R}^n$, and let $r\in(0,R)$. Then:*

(a) *The distribution $-\Delta v=|\Delta v|$ is a nonnegative measure in B_R, $v\in W^{1,1}_{\rm loc}(B_R)$,*

$$\int_{B_r}|\Delta v|\le\frac{C}{(R-r)^2}\int_{B_R}|v|\,dx,\qquad \text{and}\qquad \int_{B_r}|\Delta v|\le\frac{C}{R-r}\int_{B_R}|\nabla v|\,dx,$$

where $C>0$ is a dimensional constant. In addition,

$$\int_{B_r}|\nabla v|\,dx\le C(n,r,R)\int_{B_R}|v|\,dx$$

for some constant $C(n,r,R)$ depending only on n, r, and R.

Assume now that $v_k\in L^1(B_R)$, $k=1,2,...$, is a sequence of superharmonic functions with $\sup_k\|v_k\|_{L^1(B_R)}<\infty$. Then:

(b1) *Up to a subsequence, $v_k\to v$ strongly in $W^{1,1}(B_r)$ to some superharmonic function v.*

(b2) *In addition, if for some $\gamma>0$ we have $\sup_k\|v_k\|_{W^{1,2+\gamma}(B_r)}<\infty$, then $v_k\to v$ strongly in $W^{1,2}(B_r)$.*

Proof. (a) By assumption we know that

$$\langle-\Delta v,\xi\rangle=\int_{B_R}v(-\Delta\xi)\,dx\ge 0\quad\text{for all nonnegative }\xi\in C^\infty_c(B_R).\tag{A.1}$$

Let $0<r<\rho<R$ and choose a nonnegative function $\chi\in C^\infty_c(B_R)$ with $\chi\equiv 1$ in B_ρ. Now, for all $\eta\in C^\infty_c(B_\rho)$, using (A.1) with the test functions $\|\eta\|_{C^0}\chi\pm\eta\ge 0$ in B_R, we deduce that $\pm\langle-\Delta v,\eta\rangle\le\|\eta\|_{C^0}\|v\|_{L^1(B_R)}\|\Delta\chi\|_{C^0}\le C\|\eta\|_{C^0}$. Thus, $-\Delta v$ is a nonnegative measure in B_ρ, for all $\rho<R$.

Let us now take $\rho = \frac{1}{2}(r+R)$, and consider χ as before satisfying $|\nabla\chi| \le \frac{C}{R-r}$ and $|D^2\chi| \le \frac{C}{(R-r)^2}$. Then, since $-\Delta v \ge 0$, we have

$$\int_{B_r} |\Delta v| \le -\int_{B_R} \Delta v\,\chi = -\int_{B_R} v\,\Delta\chi\,dx \le \frac{C}{(R-r)^2}\|v\|_{L^1(B_R)}. \tag{A.2}$$

To prove that $v \in W^{1,1}_{\rm loc}(B_R)$, we define on $\mathbb{R}^n$ the measure $\mu := \chi\,(-\Delta v)$, and we consider the fundamental solution $\Phi = \Phi(x)$ of the Laplacian in $\mathbb{R}^n$ —that is, $\Phi(x) = c\log|x|$ if $n=2$ and $\Phi(x) = c_n|x|^{2-n}$ if $n \ge 3$.

Define the $L^1_{\rm loc}(\mathbb{R}^n)$ function $\widetilde{v} := \Phi * \mu$. Since $\Phi \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$ it is easy to check (using the definition of weak derivatives) that $\widetilde{v} \in W^{1,1}(B_R)$ and $\nabla\widetilde{v} = \nabla\Phi * \mu$. Furthermore, from (A.2) (with r replaced by ρ), one easily deduces that

$$\|\widetilde{v}\|_{W^{1,1}(B_\rho)} \le C\|v\|_{L^1(B_R)}, \tag{A.3}$$

where the constant C depends only on n, r, and R (recall that $\rho = \frac{1}{2}(r+R)$).

On the other hand, using (A.3), we see that $w := v - \widetilde{v}$ satisfies

$$\|w\|_{L^1(B_\rho)} \le \|v\|_{L^1(B_\rho)} + \|\widetilde{v}\|_{L^1(B_\rho)} \le C\|v\|_{L^1(B_R)}$$

and

$$\Delta w = 0 \quad \text{in } B_\rho.$$

By standard interior estimates for harmonic functions, this leads to

$$\|w\|_{C^2(\overline{B}_r)} \le C\|w\|_{L^1(B_\rho)} \le C\|v\|_{L^1(B_R)}.$$

In particular, recalling (A.3), we have shown that $v \in W^{1,1}(B_r)$ and $\|v\|_{W^{1,1}(B_r)} \le C\|v\|_{L^1(B_R)}$.

Finally, exactly as in (A.2), we have

$$\int_{B_r} |\Delta v| \le -\int_{B_R} \Delta v\,\chi = \int_{B_R} \nabla v \cdot \nabla\chi\,dx \le \frac{C}{R-r}\|\nabla v\|_{L^1(B_R)},$$

finishing the proof of (a).

(b1) Let now v_k be a bounded sequence in $L^1(B_R)$. Define μ_k, $\widetilde{v}_k$, w_k as we did in the proof of (a), but with v replaced by v_k. Note that the operators $\mu \mapsto \Phi * \mu$ and $\mu \mapsto \nabla\Phi * \mu$ are compact from the space of measures (with finite mass and support in B_R) to $L^1(B_r)$. This is proved in a very elementary way in [4, Corollary 4.28] when these operators are considered from $L^1(\mathbb{R}^n)$ to $L^1(B_r)$, but the same exact proof works for measures.

Thus, up to a subsequence, $\widetilde{v}_k$ converges in $W^{1,1}(B_r)$. Since $w_k = v_k - \widetilde{v}_k$ are harmonic and uniformly bounded in $L^1(B_\rho)$, up to a subsequence also w_k converges in $W^{1,1}(B_r)$. Therefore, we deduce that a subsequence of v_k converges in $W^{1,1}(B_r)$, which proves (b1).

(b2) If in addition we have $\sup_k \|v_k\|_{W^{1,2+\gamma}(B_r)} < \infty$, using Hölder inequality we obtain

$$\|\nabla(v_k - v)\|_{L^2(B_r)} \le \|\nabla(v_k - v)\|_{L^1(B_r)}^{\frac{\gamma}{2(1+\gamma)}}\|\nabla(v_k - v)\|_{L^{2+\gamma}(B_r)}^{\frac{2+\gamma}{2(1+\gamma)}} \le C\|v_k - v\|_{W^{1,1}(B_r)}^{\frac{\gamma}{2(1+\gamma)}} \to 0,$$

which shows that $v_k \to v$ strongly in $W^{1,2}(B_r)$. □

We now discuss a result about the composition of Lipschitz functions with C^2 functions. This result is far from being sharp in terms of the assumptions, but it suffices for our purposes. For its proof (as well as for other results proved in this paper) we shall need the coarea formula, which we recall here for the convenience of the reader (we refer to [24, Theorem 18.8] for a proof):

Lemma A.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u : \Omega \to \mathbb{R}$ be a Lipschitz function. Then, for every function $g : \Omega \to \mathbb{R}$ such that either g is nonnegative or $g \in L^1(\Omega)$, the integral of g over $\{u = t\}$ is well defined in $(-\infty, +\infty]$ for a.e. $t \in \mathbb{R}$ and*

$$\int_\Omega g\,|\nabla u|\,dx = \int_{\mathbb{R}} \left(\int_{\{u=t\}} g\,d\mathcal{H}^{n-1} \right) dt.$$

We recall that, given a locally Lipschitz function f, we defined

$$f'_-(t) := \liminf_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

Lemma A.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solve $-\Delta u = f(u)$ in Ω, where $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Then:*

(i) *Inside the region $\{\nabla u \neq 0\}$ the function $f'(u)$ is well-defined and it coincides a.e. with $f'_-(u)$.*
(ii) *$u \in W^{3,p}_{\mathrm{loc}}(\Omega)$ for every $p < \infty$ and $-\Delta \nabla u = f'(u)\nabla u = f'_-(u)\nabla u$ in the weak sense and also a.e. in Ω.*
(iii) *If $\partial\Omega \cap B_1$ is of class C^3 and $u|_{\partial\Omega\cap B_1} = 0$, then $u \in (W^{3,p}_{\mathrm{loc}} \cap C^2)(\overline{\Omega} \cap B_1)$ for every $p < \infty$.*

Proof. The first point is a simple application of the coarea formula. Indeed, if we set $M := \|u\|_{L^\infty(\Omega)}$, given any Borel set $E \subset \{\nabla u \neq 0\}$ we can apply Lemma A.2 with $g = \frac{1_E}{|\nabla u|} f'_-(u)$ (in fact we apply the lemma to both g^+ and g^-, the positive and negative part of g, obtaining finite quantities for both $\int_\Omega g^\pm |\nabla u|\,dx$ since $f'_-(u) \in L^\infty(\Omega)$) to get

$$\int_E f'_-(u)\,dx = \int_{-M}^{M} f'_-(t) g_E(t)\,dt, \qquad \text{with} \quad g_E(t) := \int_{\{u=t\}\cap E} \frac{1}{|\nabla u|}\,d\mathcal{H}^{n-1}. \tag{A.4}$$

Then, since

$$\int_{\mathbb{R}} g_E(t)\,dt \le \int_{\mathbb{R}} \left(\int_{\{u=t\}\cap\{|\nabla u|\neq 0\}} \frac{1}{|\nabla u|}\,d\mathcal{H}^{n-1} \right) dt = \big|\Omega \cap \{\nabla u \neq 0\}\big| < \infty,$$

it follows that the function g_E belongs to $L^1(\mathbb{R})$. Thus, since $f'_-(t)$ belongs to $L^\infty([-M, M])$, this proves that the right hand side in (A.4) is independent of the specific representative chosen for f', and therefore so is the left hand side. Since E is arbitrary and $f'_-(t) = f'(t)$ a.e., (i) follows.

To prove (ii) we first notice that, since $f(u)$ is Lipschitz inside Ω (because both u and f are so), it follows that $f(u) \in W^{1,p}$ and by interior elliptic regularity (see for instance [21, Chapter 9]) that $u \in W^{3,p}_{\mathrm{loc}}(\Omega)$ for every $p < \infty$. This means that $\nabla u \in W^{2,p}$, and therefore it suffices to show that the identities in (ii) hold a.e. (because then they automatically hold in the weak sense).

Now, in the region $\{\nabla u = 0\}$, we have

$$f'(u)\nabla u = f'_-(u)\nabla u = 0 \qquad \text{and} \qquad \Delta\nabla u = 0 \quad \text{a.e.}$$

(see, e.g., [33, Theorem 1.56]), so the result is true there.

On the other hand, in the region $\{\nabla u \neq 0\}$, for $h > 0$ and $1 \leq i \leq n$, let

$$\delta_i^h w := \frac{w(\cdot + h\boldsymbol{e}_i) - w}{h}.$$

Since $-\Delta u = f(u)$ in Ω, given $\Omega' \subset\subset \Omega$, for $h > 0$ sufficiently small we have

$$-\Delta \delta_i^h u = \delta_i^h\big(f(u)\big) \qquad \text{inside } \Omega'. \tag{A.5}$$

Thus, if we define by $D_f \subset \mathbb{R}$ the set of differentiability points of f, we see that

$$\delta_i^h[f(u)] \to f'(u)\partial_i u = f'_-(u)\partial_i u \qquad \text{for all } x \in \Omega' \text{ such that } u(x) \in D_f$$

as $h \to 0$. On the other hand, if we set $N := \mathbb{R} \setminus D_f$, since N has measure zero (because f is differentiable a.e., being Lipschitz) it follows from Lemma A.2 applied with $g = \frac{1}{|\nabla u|}\, 1_{\Omega' \cap \{\nabla u \neq 0\}}\, 1_N \circ u$ that

$$\int_{\Omega' \cap \{\nabla u \neq 0\}} 1_N(u(x))\,dx = \int_{\mathbb{R}} 1_N(t) \bigg(\int_{\{u=t\} \cap \Omega' \cap \{\nabla u \neq 0\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} \bigg)\, dt = 0,$$

which proves that $u(x) \notin N$ for a.e. $x \in \Omega' \cap \{\nabla u \neq 0\}$.

Hence, we have shown that $\delta_i^h[f(u)] \to f'(u)\partial_i u$ for a.e. $x \in \Omega' \cap \{\nabla u \neq 0\}$ (and so also in L^p for any $p < \infty$, by dominated convergence). Letting $h \to 0$ in (A.5) we deduce that $-\Delta \nabla u = f'(u)\nabla u$ a.e. in Ω', since we already checked the equality a.e. in $\{\nabla u = 0\}$. Recalling that $\Omega' \subset\subset \Omega$ is arbitrary, this proves (ii).

Finally, (iii) follows by elliptic regularity up to the boundary (see for instance [21, Chapter 9] or [23, Section 9.2]). □

We conclude this section with a general abstract lemma due to Simon [30] (see also [14, Lemma 3.1]):

Lemma A.4. *Let $\beta \in \mathbb{R}$ and $C_0 > 0$. Let $\sigma : \mathcal{B} \to [0, +\infty]$ be a nonnegative function defined on the class $\mathcal{B}$ of open balls $B \subset \mathbb{R}^n$ and satisfying the following subadditivity property:*

$$\text{if} \quad B \subset \bigcup_{j=1}^N B_j \quad \text{then} \quad \sigma(B) \leq \sum_{j=1}^N \sigma(B_j).$$

Assume also that $\sigma(B_1) < \infty$.

Then, there exists $\delta > 0$, depending only on n and β, such that if

$$r^\beta \sigma\big(B_{r/4}(y)\big) \leq \delta r^\beta \sigma\big(B_r(y)\big) + C_0 \quad \text{whenever } B_r(y) \subset B_1,$$

then

$$\sigma(B_{1/2}) \leq C C_0,$$

where C depends only on n and β.

Appendix B. A universal bound on the L^1 norm

In this section we recall a classical and simple a priori estimate on the L^1 norm of solutions when f grows at infinity faster than a linear function with slope given by the first eigenvalue of the Laplacian.

Proposition B.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^3, and let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ solve*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for some $f : \mathbb{R} \to [0, +\infty)$ satisfying

$$f(t) \geq At - B \qquad \text{for all } t \geq 0, \quad \text{with } A > \lambda_1 \text{ and } B \geq 0, \tag{B.1}$$

where $\lambda_1 = \lambda_1(\Omega) > 0$ is the first Dirichlet eigenvalue of the Laplacian in Ω. Then there exists a constant C, depending only on A, B, and Ω, such that

$$\|u\|_{L^1(\Omega)} \leq C.$$

Proof. First of all we note that, since $f \geq 0$, then $u \geq 0$ inside Ω by the maximum principle.

Let $\Phi_1 > 0$ be the first Dirichlet eigenfunction of the Laplacian in Ω, so that $-\Delta\Phi_1 = \lambda_1\Phi_1$ in Ω and $\Phi_1 = 0$ on $\partial\Omega$. Then

$$\lambda_1 \int_\Omega u\,\Phi_1\,dx = -\int_\Omega u\,\Delta\Phi_1\,dx = -\int_\Omega \Delta u\,\Phi_1\,dx = \int_\Omega f(u)\,\Phi_1\,dx. \tag{B.2}$$

Thanks to assumption (B.1), we have

$$\int_\Omega f(u)\,\Phi_1\,dx \geq A\int_\Omega u\,\Phi_1\,dx - B\int_\Omega \Phi_1\,dx,$$

that combined with (B.2) gives

$$(A - \lambda_1)\int_\Omega u\,\Phi_1\,dx \leq B\int_\Omega \Phi_1\,dx.$$

Note that, using (B.2) again, this implies that

$$\int_\Omega f(u)\,\Phi_1\,dx = \lambda_1\int_\Omega u\,\Phi_1\,dx \leq \lambda_1\frac{B}{A-\lambda_1}\int_\Omega \Phi_1\,dx.$$

Since Φ_1 (which is defined up to a multiplicative constant) can be taken to be comparable to $\operatorname{dist}(\cdot, \partial\Omega)$, this proves that

$$\int_\Omega \operatorname{dist}(x, \partial\Omega)\,f(u(x))\,dx \leq C \tag{B.3}$$

for some constant C depending only on A, B, and Ω.

Consider now ϕ the solution of

$$\begin{cases} -\Delta\phi = 1 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, since $0 \leq \phi \leq C\operatorname{dist}(\cdot, \partial\Omega)$, using (B.3) we get

$$\int_\Omega u\,dx = -\int_\Omega u\,\Delta\phi\,dx = \int_\Omega f(u)\,\phi\,dx \leq C,$$

as desired. □

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