Physical Models in Social Contexts: Wealth redistribution mechanisms

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Abstract:
In this paper we present simulation results for multi-agent models of economic systems. We interpret the stationary distributions of wealth and focus on measuring their inequality with the Gini coefficient. We introduce saving propensity of the agents both global and individual. In the first case, we find that inequality decreases with an increasing saving propensity and that the distribution of wealth evolves to a gamma distribution. An analogy is then made with the kinetic theory model and its equilibrium Maxwell-Boltzmann distribution. In the second case, the equilibrium distribution is found to have an incipient power-law tail. We also add a redistribution of wealth mechanism based on taxes and evaluate the reduction of wealth inequality that it entails. Last, we study a model with dynamic saving propensity rates and a mechanism that rewards those trades in which significant wealth is brought into play.

I. INTRODUCTION

The nature of economic inequality and the distribution of wealth in society has been an issue of research for economists and sociologists for a long time but also for physicists in the last decades. More than a century ago, Vilfredo Pareto quantified the high-end of the income distribution in a society and found it to follow a power law $P(m) \sim m^{-\nu-1}$. After him, multiple other studies corroborated this result obtaining values of ν ranging from 1 to 3 [1–4] . In the last two decades, research in the field of Econophysics has been intense and a number of models have been proposed to try to explain the functioning principles of trading systems. Many of these use a statistical mechanics approach: they make an analogy between a closed economic system in which agents perform pairwise money-conservative transactions and an ideal gas in which gas molecules exchange energy in pairwise collisions.

A pioneer model developed by Dragulescu and Yakovenko in 2000 [5] showed that if we let such a system of economic agents perform multiple arbitrary and random sharing but locally conserving transactions, the wealth distribution goes to the well-known equilibrium Gibbs' equilibrium distribution of statistical mechanics.

A few years later, Chatterjee et. al. upgraded this model by letting agents save a part of their wealth determined by a saving propensity rate parameter [1]. This and multiple subsequent studies found that when this parameter is the same for all agents, the wealth distributions in equilibrium go to gamma-shaped distributions [4, 6, 7]. Otherwise, when the saving propensity is not global but assumed to behave randomly, the higher end of the wealth distribution is found to follow the Pareto power law decay [6]. Furthermore, Manna et. al. showed that this observed Pareto law is essentially a convolution of the single agent distributions [8].

In this study, we present the results of the simulations of the mentioned models in sections II and III.

Further, with the goal of finding ways to reduce economic inequality, we developed a model which implements a wealth redistribution system based on taxes (section IV). Last, in section V we exhibit what we called a *learned saving propensity model* and that we devised aiming to come closer to what a real open economic system is. The main difference here is that this is now a non-conservative system. For other proposals of open systems see [9].

Over the paper, attention is specially drawn to the evaluation of inequality by means of the Gini coefficient. This coefficient ranges from 0 to 1, with 0 expressing perfect equality and 1 expressing maximal inequality among agents.

II. GROUNDS OF THE MODEL

We consider an ideal-gas model of a closed economic system where total wealth W and total number of agents N is fixed. Starting from a uniform distribution of wealth, we let the system dynamically evolve with time following pairwise wealth conservative transactions.

At each time step t, two agents that possess wealths $w_i(t)$ and $w_j(t)$ are randomly picked and they perform a transaction after which they possess $w_i(t+1)$ and $w_j(t+1)$, respectively. The wealth is locally conserved, i.e., $w_i(t) + w_j(t) = w_i(t+1) + w_j(t+1)$ and no debt is allowed, i.e., $w_i(t) > 0$.

Then, after a typical relaxation time ($t \sim 10^4$) a time-independent probability distribution $P(w)$ is obtained irrespective of the initial distribution. We determined this steady state by following the behaviour of the distribution every 100 steps.

Now, the trading rule applied to transactions is own of each model and is explained in its corresponding section.

III. SAVING PROPENSITY MODELS

Each of the two agents involved in this money conservative transaction saves a fraction λ of its wealth, which we call *saving propensity rate*, and trades the rest. Next, the sum of the wealth that the two agents bring into play is bipartitioned between them with a uniformly at random bipartion fraction ϵ.

A. Global saving propensities

In this version of the model, the saving propensity rate is the same for all agents and is fixed over time.

If i, j are the involved agents in the t-th transaction with wealths $w_i(t)$, $w_j(t)$, their wealths after this time step will turn to:

$$w_i(t+1) = \lambda w_i(t) + \epsilon(t)(1-\lambda)(w_i(t)+w_j(t)) \quad (1)$$
$$w_j(t+1) = \lambda w_j(t) + (1-\epsilon(t))(1-\lambda)(w_i(t)+w_j(t)).$$

This dynamics is then followed until the system reaches the steady state ($t \sim 10^4$).

In our study, the simulation was run for different global saving propensities ($\lambda \in [0.1, 0.9]$) obtaining equilibrium distributions such as the ones shown in Fig. 1 (a). It is important to clarify that this curves are an adjustment of a histogram. For this reason, some of them slightly exceed $w = 0$ which is in clear contradiction with the fact that debt is not allowed in this model.

It is notable that the lower the global saving propensity, the lower the mode of the distribution which gradually shifts from 0 to 1 with increasing λ. It attracts attention that the mode of the red curve ($\lambda = 0.8$) has a higher probability than the other distributions with lower λ. However, this doesn't affect the Gini coefficient (inset plot of Fig. 1 (a)).

Besides, when the global saving propensity is low enough ($\lambda \sim 0.5$), the range of wealth that the agents reach is wide enough to encounter the barrier $w_i = 0$. The distributions then have a narrow initial growth upto a most-probable value after which they fall-off with a power-law tail. This result suggests that when the global saving propensity is low, a few agents accumulate the same wealth as the sum of the rest.

In agreement with this, the Gini coefficient of the distributions drops when the global saving propensity rises, i.e, the inequality diminishes.

Additionally, the functional form of this distributions has been successfully fitted with a gamma distribution on the base of an analogy with the kinetic theory of gases. This becomes clear when we express the Maxwell-Boltzmann kinetic energy distribution of an ideal gas in D dimensions like:

$$f(K) = \frac{1}{\Gamma(D/2)k_BT}\left(\frac{K}{k_BT}\right)^{D/2-1} exp\left(-\frac{K}{k_BT}\right), \quad (2)$$

	Kinetic model	**Economic model**
variable	K (kinetic energy)	w (wealth)
units	N particles	N agents
interaction	collisions	trades
dimension	integer D	real number D_λ
temperature	$k_BT = 2 < K > /D$	$T_\lambda = 2 < w > /D_\lambda$
reduced variable	$\xi = \frac{K}{k_BT}$	$\xi = \frac{w}{T_\lambda}$
equilibrium distribution	$f(\xi) = \gamma_{D/2}$	$f(\xi) = \gamma_{D/2}$

TABLE I: Analogy between the kinetic model and the agent-based model, extracted from [6].

which is a gamma distribution of the reduced variable $\xi = \frac{K}{k_BT}$, i.e, $f(\xi) = \gamma_{D/2}(\xi)$.

In our model, the reduced variable would be $\xi = \frac{w}{T_\lambda}$, with $T_\lambda = 2 < w > /D_\lambda$. We briefly exhibit the analogy for the rest of the involved parameters in Table I. For a deeper analysis of this and its derivation, see [6].

B. Individual saving propensities

In this other version of the model, each agent has a unique saving propensity comprehended between 0 and 1. Let λ_i and λ_j be the saving propensity rates of the two agents that participate in the t-th transaction. Then, their wealth after this transaction will be:

$$\begin{aligned} w_i(t+1) &= \lambda_i w_i(t) + \epsilon(t)((1-\lambda_i)w_i(t) \\ &\quad +(1-\lambda_j)w_j(t)) \\ w_j(t+1) &= \lambda_j w_j(t) + (1-\epsilon(t))((1-\lambda_i)w_i(t) \\ &\quad +(1-\lambda_j)w_j(t)). \end{aligned} \quad (3)$$

We initialized the simulation assigning each agent a uniform at random saving propensity ($\lambda \in (0,1)$). We then let the system evolve to its steady state and looked at the final wealth of four agents with equispaced saving propensities ($\lambda_t = 0.2, 0.4, 0.6, 0.8$). We repeated the procedure 1000 times with the same distribution of saving propensities and plotted the distribution of the wealths at equilibrium of these tagged agents (Fig. 1 (b)). This gives us an idea of what is more likely going to be the wealth of an agent with a given λ_i in a society with distributed saving propensities. It is notable in Fig. 1 (b) that agents with a high saving propensity become wealthier than the mean while agents with a very low saving propensity are likely to loose all their wealth.

In addition, we averaged the equilibrium distribution of the system over different initial sets of individual saving propensities. With a given configuration λ_i, we let the system evolve until it reaches equilibrium, then a new set of random saving propensities is assigned to all agents, and the whole procedure is repeated 100 times. As a result of the average over the equilibrium distributions corresponding to the various λ_i configurations, one obtains a distribution with an incipient power-law tail (Fig. 2).

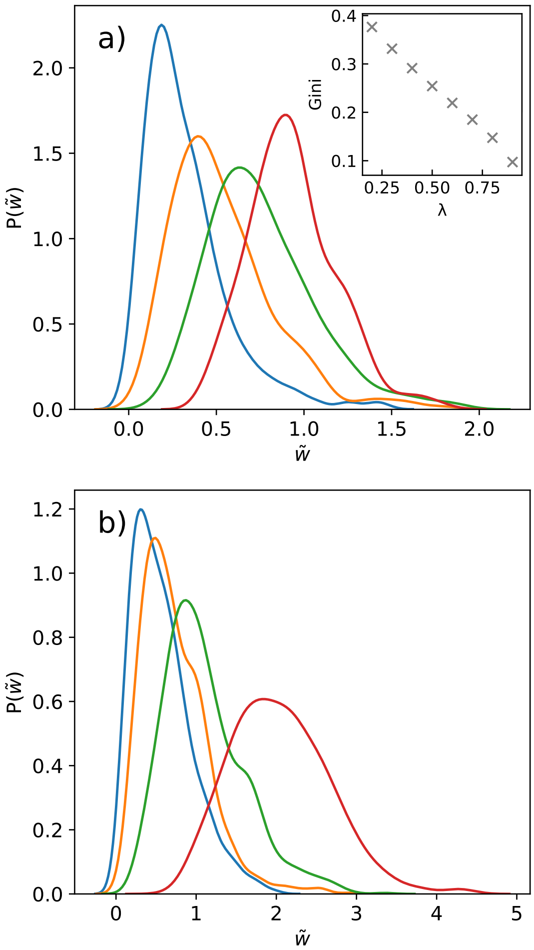

FIG. 1: (a) Equilibrium relative wealth distributions $P(\tilde{w})$ ($t = 10^4$) of a system of N=300 agents with a global saving propensity $\lambda = 0.2, 0.4, 0.6, 0.8$ for the blue, orange, green and red curves, respectively. The inset plot shows the Gini coefficient of the equilibrium distributions of the system with different global saving propensities. (b) Distribution of the equilibrium relative wealth of an agent with an individual saving propensity λ_i in a system where each agent has a different saving propensity $\lambda_i \in (0, 1)$. The blue, orange, green and red curves correspond to agents with $\lambda_i = 0.2, 0.4, 0.6, 0.8$, respectively.

IV. TAXATION MODELS

With the motivation of finding a way to overcome the wealth inequality that the propensity models exhibit, we here introduce taxes as a mechanism of wealth redistribution.

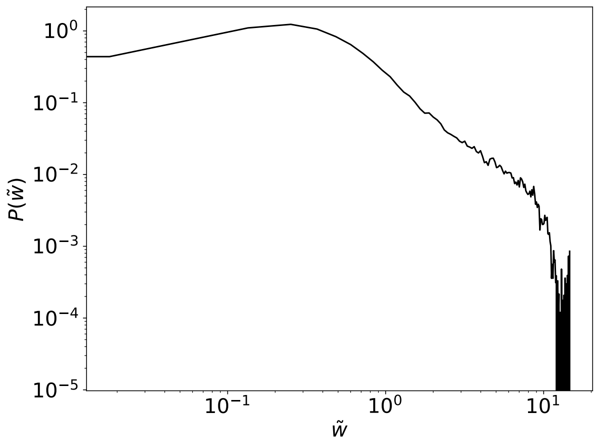

FIG. 2: Distribution of wealth in the case where the saving propensity is uniformely distributed in the range (0,1).

At each time step a part of the transacted wealth is not bipartioned between the two participating agents but rather partitioned equally between all of the agents. We define this transacted wealth as:

$$W_{tr}(t) = (1 - \lambda_i) w_i(t) + (1 - \lambda_j) w_j(t). \tag{4}$$

And how much of this is retained depends on the tax rate. The simplest version of the model is that in which the tax rate r is fixed over time and same for any transaction. Now, the agents participating in the t-th transaction see their wealth first varied by:

$$\begin{aligned} w_i(t+1) &= \lambda_i w_i(t) + \epsilon(t)(1-r)W_{tr}(t) \\ w_j(t+1) &= \lambda_j w_j(t) + (1-\epsilon(t))(1-r)W_{tr}(t). \end{aligned} \tag{5}$$

Next, the wealth collected by the taxes is redistributed equally among all agents:

$$\begin{gathered} w_k(t+1) = w_k(t) + rW_{tr}(t)/N \\ k = 1, ..., N. \end{gathered} \tag{6}$$

This upgrade of the saving propensity model leads to a significant change in the final distribution, specially when the saving propensity is sufficiently low. When so, it is stated that the mode of the distribution is displaced to the mean value $\tilde{w} = 1$ and that the distribution is narrowed (Fig. 3).

To evaluate the effect of varying both the saving propensity and tax rates, we ran the simulation fixing λ and varying r and repeated this for different λ values. From the obtained stationary destributions we measured the Gini coefficient and found it to decrease with increasing tax rate with any saving propensity (Fig. 4). This reinforces the statement that the inequality is reduced when taxes are added.

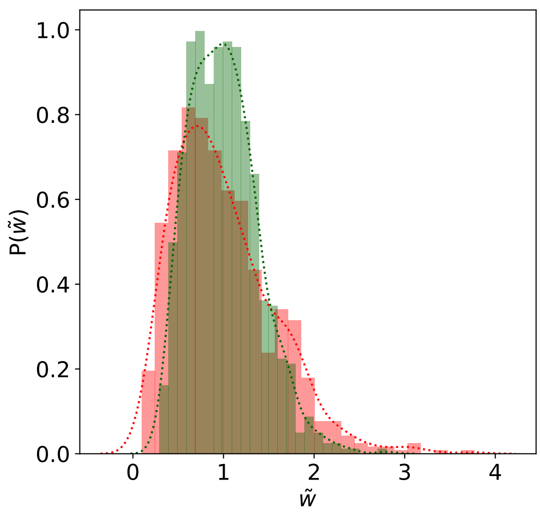

FIG. 3: Relative stationary wealth distribution when taxes are collected (red, $r = 0$) and when all transactions are free of taxes (green, $r = 0.35$). For this simulation, N=500 and $\lambda = 0.4$.

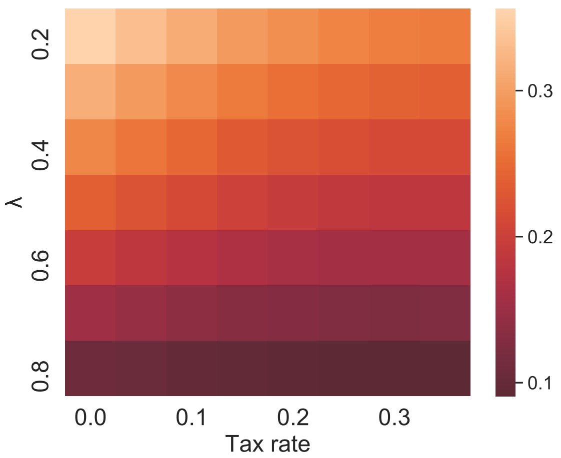

FIG. 4: Gini coefficient of the stationary distributions obtained for each pair of values of saving propensity λ and tax rate r.

V. LEARNED SAVING PROPENSITY MODELS

This last model is based on the saving propensity models but it includes some upgrades that aim to make it closer to a real economic system. From all the models that we described, it could be argued that real economic agents change their saving propensity over time and that this propensity is not independent of the agents' experience in previous transactions.

To implement this idea, we introduced a dynamic saving propensity such that in each transaction agents can choose between two values, $\lambda = 0.4$ and $\lambda = 0.7$. We consider that a successful choice is that which leaves the agent in a wealthier state after the transaction.

Each agent's first choice is made randomly. If it turns to be a successful choice, a "point" for this rate is added to the record of successes of this agent. In case of an unsuccessful choice, a "point" is added for the other rate. At each transaction, the participating agents choose the saving propensity for which they have accumulated more successes up to that given time step.

One can then argue that, if the wealth brought into play in a transaction is going to be randomly bipartitioned, it will always be a wiser idea to choose the high saving propensity rate.

In order to compensate this imbalance and propitiate agents to risk a greater rate of their wealth, a reward system was added to the model. If the transacted wealth $W_{tr}(t)$ is higher than a certain threshold W_r, the involved agents are rewarded with an increase of this quantity by a factor $\alpha > 1$. Namely, $W'_{tr}(t) = \alpha W_{tr}(t)$ and after this transaction:

$$
\begin{aligned}
w_i(t+1) &= \lambda_i(t) w_i(t) + \epsilon(t) W'_{tr}(t) \\
w_j(t+1) &= \lambda_j(t) w_j(t) + (1-\epsilon(t)) W'_{tr}(t).
\end{aligned} \tag{7}
$$

It is important to note that with this reward system the total wealth is not conserved anymore which is in fact a closer approach to what happens in a real open economic system.

So, hands on the simulations, there are two parameters to be regulated in this model: the factor α and the threshold W_r.

We first asserted that the introduction of the reward factor does have the desired effect of rising the number of 0.4 saving propensity rate choices. Next we used the Gini coefficient once again as a measure of inequality after the introduction of the reward system. In Fig. 5, each point corresponds to the Gini coefficient of the final rescaled distribution obtained performing the simulations for different reward factors and averaging each over six reward thresholds ($W_r \in [0.75, 1.75]$).

Two remarkable observations can be made. On the one hand, it is plain to see that adding a reward factor aggravates inequality. This rise in the Gini coefficient seems to be a consequence of the fact that a greater reward factor enhances a higher percentage of 0.4 saving propensity rate choices. And from the results in section III we know that a lower saving propensity leads to wealth distribution with a higher Gini coefficient.

On the other hand, as the standard deviation of the measures with different thresholds is small, we can say that where we put the threshold to reward is not relevant to the final distribution.

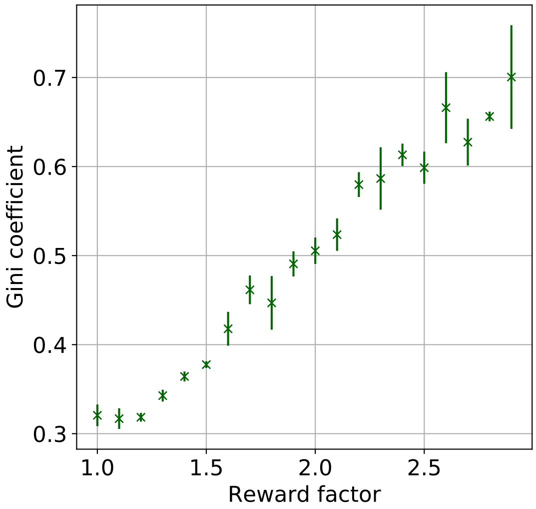

FIG. 5: Gini coefficient of the equilibrium distribution when different reward factors are applied. Each point is measured as the averaged coefficient over six measures with equispaced reward thresholds ranging from 0.75 to 1.75.

VI. CONCLUSIONS

We have simulated some multi-agent based models for the distribution of wealth, in which wealth is exchanged following different trading rules. We first introduced a saving propensity parameter which allows agents to save part of their wealth and trade with the rest. On the one hand, we find that when this saving propensity is global for all agents, the mode of the distribution shifts from 0 to 1 and the Gini coefficient drops with increasing saving propensity. On top of that, we fitted the resulting distributions with gamma distributions and used this result to make an interesting analogy between our economic system of agents exchanging wealth in trades and the kinetic theory model in which gas molecules exchange kinetic energy in collisions.

On the other hand, we do a random assignment of saving propensities to each agent. We find that in such a system, agents with a higher saving propensity are likely to accumulate more than the mean wealth in detraction of the agents with low saving propensities. Besides, when we average over several initial saving propensity distributions, the final probability density function is found to follow a power-law decay.

Further, we review a taxation model in which a part of what is traded at each transaction, is redistributed equally between all agents. We find that adding taxes to the previous model diminishes inequality as the stationary distribution narrows, the mode is displaced to the mean value and the Gini coefficient is reduced.

We finally study a model where the saving propensity of agents is dynamic: agents learn from their previous transactions and can choose wether they have a high or low saving propensity. Also in this model, if the wealth exchanged in a transaction is higher than a given threshold, the involved agents are rewarded increasing this quantity by a so-called reward factor. What the results of this model show is that the Gini coefficient rises when this reward factor is introduced, i.e., it is a source of inequality.

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