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Polarized K3 Surfaces of Low Degree

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1 Introduction

K3 surfaces have been an active topic of research for a few years now. They represent one of the most interesting lowest dimensional non-trivial complex surfaces in many aspects, while still having enough structure to allow a thorough study. They are of interest to specialists from many fields, including number theory, algebraic geometry, physics and more.

From a geometrical point of view, K3 surfaces are one of the four kinds of projective surfaces that have Kodaira dimension 0, the other three being abelian surfaces (projective algebraic varieties with a group structure), hyperelliptic surfaces and Enriques surfaces. They have historically been used as examples on which to test conjectures, as happened with the Weil conjecture proven by Deligne. This is due to their particular mixture of having a non-trivial structure (for example their intersection forms and arithmetic structure) and their well-behaved deformation theory.

On the other hand, from the point of view of number theory, K3 surfaces have a lot of interest as well. Even though in this report we will focus on complex K3 surfaces, these surfaces may also be considered over other fields and much of what we claim is still true in that setting. In particular we can consider the case of number fields, particularly $\mathbb{Q}$. Indeed, some of the constructions we make can be used to compute the number of rational points (since K3 surfaces are not rational, but their Kodaira dimension is 0, they still have an interesting arithmetic, like elliptic curves).

Physicists in recent years have been interested in what is known as Calabi-Yau manifolds. Their interest lies mainly in that they provide solutions to Einstein's field equation when the space is "empty". They are mostly interested in dimension greater than 3, but in the (complex) 2-dimensional case, the Calabi-Yau property reduces to that of K3 surfaces. In fact, depending on the definition you take, other complex manifolds with null Kodaira dimension may also be considered as Calabi-Yau manifolds. However, the only examples that are both simply connected and compact, which are desirable properties in applications to string theory, are K3 surfaces.

The first examples of K3 surfaces that were studied were smooth quartic surfaces of projective 3-space. In a similar vein, complete intersections of appropriate degree are also K3 surfaces, as we will see in Section 3.2. Hence, a natural question is how "generic" are these representations within the space of polarized K3 surfaces of the same degree. This will be answered somewhat satisfactorily through application of Reider's method, which is the main theorem we will be proving in this report.

Plan of the TFG. This report is intended to serve as an introduction to the theory of K3 surfaces and, in particular, those equipped with a polarization. It starts by giving a recap of all prerequisites in algebraic geometry that are needed to understand the latter proofs from an undergraduate level. Of course, a complete account is impossible and many proofs are left out, though various references are provided when needed.

We start by giving some basic definitions and results of sheaves, and proceed to introduce the algebra of sheaves and their cohomological theory.

After that, we are ready to give a proper introduction to algebraic geometry. The discussion of general schemes is avoided as polarized K3 surface are necessarily projective and we can rely on the theory of projective (sub)varieties.

We will also make reference to one of the key theorems that allow us to osculate between analytic and algebraic geometry: the GAGA (Géometrie Algébrique et Géométrie Analytique). It will allow us for the most part to speak indistinguishably about (smooth) algebraic varieties over $\mathbb{C}$ and analytical complex manifolds. It will also allow us to transport some of the consequences of the existence of certain purely analytical constructions (like the exponential sequence) to the algebraic geometry setting.

After finally establishing our foundations, we will proceed to introduce some important tools, including the theory of divisors, which behaves particularly well in our case, the intersection product and Chern Classes. We will also apply some of the tools we have learned in our computation of the cohomology of line bundles on projective space.

After this we will finally introduce the concept of a K3 surface, giving some of its main properties as well as the aforementioned relevant examples. From there we will jump into the classification problem. First of all, we will present the case of degree 2, which uses techniques a bit different from the rest given that its natural morphism does not factor through an embedding to $\mathbf{P}^n$. Then we will focus on the main theorem: Reider's Method. We will then be ready to explore our cases of interest: the polarizations of degree $4, 6$ and 8. After leaving a brief mention of the cases of higher degree, we finalize by giving some closing remarks in our conclusion.

2 Preliminary results

2.1 Sheaves

One of our most basic tools are called sheaves, and they allow us to talk about global behaviour that can be reconstructed locally. In order to be able to discuss about sheaves, we first have to consider presheaves.

Definition 2.1. A **presheaf** $\mathcal{F}$ over a topological space X consists of:

- An "object" $\mathcal{F}(U)$ (typically a set, group or ring) on each open set U.
- A "morphism" (function of sets, group morphism or ring morphism respectively) $\mathrm{res}_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every $V \subseteq U$ such that $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$.

We call $\mathcal{F}(X)$ the *global sections*, $\mathcal{F}(U)$ the *sections over* U and $\mathrm{res}_{V,U}$ the *restriction from* U *to* V. In simpler terms, if we consider open subsets of X together with their partial order given by inclusion as the category $O(X)$, then we can simply consider a presheaf as a contravariant functor from $O(X)$ to a fixed category C, generally *Set*, *Ab*, *Ring* or R-Mod, though the latter will be less important, since the notion of $\mathcal{O}_X$-modules will provide a useful generalization for us.

A morphism between presheaves is a natural transformation between functors, that is, a family of morphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that $\phi_U \circ \mathrm{res}_{V,U} = \mathrm{res}_{V,U} \circ \phi_U$ whenever it makes sense.

Example 2.2. For a motivating example, consider any continuous function $f : Y \to X$. We may consider $\mathcal{F}_f(U)$ to be the set of sections of f

$$\mathcal{F}_f(U) := \{s : U \to f^{-1}(U) \mid f \circ s = \mathrm{Id}_X\},$$

with the restriction morphism being the function sending each section to its literal restriction. Indeed this trivially forms a presheaf. Even more, if we endow Y with some additional structure we can transform this presheaf into one of groups or vector spaces. This actually forms a functor $Top/X \to Pre_{Set}(X)$, which will be useful for our motivation of the definition of vector bundles.

This last example is more than just a presheaf, it is also a sheaf, which we will now define.

Definition 2.3. A **sheaf** is a presheaf such that given a cover $\{U_i\}_{i\in I}$ of an open set U and a set of compatible sections $s_i \in \mathcal{F}(U_i)$ in the sense that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for any $i, j \in I$, we have that they can be "glued together" uniquely, or in other words, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$

This definition helps to restrict the "freeness" of presheaves, giving them geometrical (or perhaps more aptly local) meaning. A morphism of sheaves is just the same as a morphism of presheaves. An important point is that of mono/epimorphisms. The first one can be defined as a morphism such that all particular morphisms on open sets are injective, as one would expect. However, the same is not true for epimorphisms. Though we could take Proposition 2.6 as a definition, we will instead rely on the categorical definition, which is the following:

Definition 2.4. Given a morphism $f : \mathcal{F} \to \mathcal{G}$ we say that it is an **epimorphism** if for all morphisms $g, h : \mathcal{G} \to \mathcal{I}$ we have $g \circ f = h \circ f \implies g = h$.

And we may also define monomorphism by reversing the direction of all morphisms and the order of composition, which gives us an equivalent definition to the previous one. We mention this definition as it is the one that best shows the structural reasoning behind these concepts.

For our other, more practical definition, we have to make precise the idea that sheaves carry only local information. This will be done through stalks:

Definition 2.5. The **stalk** of a sheaf $\mathcal{F}$ at a point $p \in X$ is the space of sections together with open sets (s, U) such that $s \in \mathcal{F}(U)$ and $(s, U) \cong (t, V)$ if and only if $\mathrm{res}_{U,W}(s) = \mathrm{res}_{V,W}(t)$ for some $W \subseteq U, V$ such that $p \in U, V, W$.

This can also be expressed as the colimit of the full diagram containing all open neighbourhoods of p, and is usually denoted $\mathcal{F}_p$. It can be difficult to prove things about sheaves directly from the definition, as for the most part we won't always know how a sheaf looks on all open subsets. Luckily, a lot of the information we care about is contained in the stalks. Most importantly:

Proposition 2.6. *A morphism between sheaves $f : \mathcal{F} \to \mathcal{G}$ is a monomorphism/epimorphism/isomorphism if and only if the induced morphism $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is a monomorphism/epimorphism/isomorphism for all $p \in X$.*

Proof. For this proof, some special sheaves that help us classify morphisms are used. In the case of monomorphisms, we will consider the sheaves given by

$$\mathcal{F}_U(V) = \begin{cases} \{*\} & V \subseteq U \\ \emptyset & \text{otherwise} \end{cases}$$

A morphism from this sheaf to any presheaf (and in particular sheaf) corresponds to a choice of section $s \in \mathcal{F}(U)$, so it is trivial that monomorphisms can be checked on open sets, and by taking limits on stalks.

On the other hand epimorphisms are done in a similar fashion, using a similar construction. Indeed, consider the skyscraper sheaf at $\{0,1\}$, simply defined as $i_{p*}\{0,1\}$ where i_p is the inclusion $i_p : \{p\} \to X$ and construct two auxiliary morphisms: one constant in 1 and one that is 1 only in the image of our morphism.

The isomorphism case is obtained by merely considering that for every section $s \in \mathcal{G}(U)$ and point $p \in X$ there is an open set $U_p \subseteq U$ such that $p \in U_p$ and there is a unique $s'_p \in \mathcal{F}$ such that $f(U_p)(s'_p) = s|_{U_p}$. But then we can glue them all together and show that there is a unique $s' \in \mathcal{F}(U)$ such that $f(U)(s') = s$. We may thus define $f^{-1}(U)(s) := s'$. It is direct to check that this is a morphism of sheaves. □

Note that in many cases to impose that a property is satisfied on all open sets is a stronger condition than on stalks. By the previous proposition it is trivial to check that a morphism is an isomorphism if and only if is is both a monomorphism and an epimorphism.

Now, some natural presheaves that may be of importance to us fail to be a sheaf. For example the tensor product that we will introduce later on and many other algebraic constructions. For this, we have a concept called sheafification.

Definition 2.7. Given any presheaf $\mathcal{F}$, its **sheafification** is a sheaf $\mathcal{F}^{sh}$ together with a morphism $i : \mathcal{F} \to \mathcal{F}^{sh}$ such that for any morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves it decomposes uniquely as $\mathcal{F} \xrightarrow{i} \mathcal{F}^{sh} \xrightarrow{\hat{f}} \mathcal{G}$.

The proof of its existence can easily be found in any introductory text on sheaves or in the short account of [Voi02, p. 85-87].

In particular the morphism that we have defined conserves stalks. As such, we can interpret sheafification as the process of striping a presheaf of its global properties, conserving only local ones. The property that we have given to sheafification essentially tells us that it interacts with other sheaves in the same way that the original presheaf did.

Note that, even though we do have an explicit expression for the sheafification of any presheaf, in general, it is not particularly useful, so we tend to treat its sections as a black box, relying mainly on the universal property.

An important example of the need of sheafification comes when considering abelian sheaves. In that case, we may define the kernel of a morphism in the obvious way, and it is indeed a sheaf with the expected properties (for example it is 0 if and only if the morphism is mono) but the image presheaf defined in the obvious way is not a sheaf and must be sheafified. This is very important, as it means that taking global section is left exact but not necessarily right exact, which will lead us to have an interesting cohomology theory. Also note that:

Proposition 2.8 ([Uen01, Proposition 4.7])**.** *A short sequence of abelian sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact if and only if the induced sequence on stalks $0 \to \mathcal{F}_p \to \mathcal{G}_p \to \mathcal{H}_p \to 0$ is exact for all $p \in X$.*

Lastly, an important way of constructing sheaves is by defining them on an open basis in a compatible way, in particular we have:

Proposition 2.9. *If $\mathcal{U}$ is a basis of the topology of X, and we define the values of a sheaf $\mathcal{F}_{\mathcal{U}}$ on $\mathcal{U}$ such that all conditions are satisfied by restricting the open sets that appear within the definition to $\mathcal{U}$ (instead of the intersection we consider any contained open set), then there is a unique sheaf $\mathcal{F}$ on X that coincides with $\mathcal{F}_{\mathcal{U}}$ on this basis.*

Proof. Because any open set can be considered as a union of elements of the basis, we can impose that the value at any open set U is the collection of compatible elements from a covering of U by $\mathcal{U}$. Checking that this is a sheaf is direct.

Uniqueness is easy to check, as any element $s \in \mathcal{F}(U)$ must give a collection of compatible elements in any open covering composed of elements of $\mathcal{U}$. $\square$

This proposition will be used in later sections.

2.2 $\mathcal{O}$-Modules

We have briefly mentioned in the previous section something called "$\mathcal{O}$-modules" which helps us to generalize the concept of sheaf of modules over a ring, but instead of a ring we consider a sheaf of rings $\mathcal{O}$. More precisely, we make the following definition:

Definition 2.10. Given a sheaf of rings $\mathcal{O}$, an **$\mathcal{O}$-module** $\mathcal{F}$ is a sheaf such that for every open set U, $\mathcal{F}(U)$ is a $\mathcal{O}(U)$-module. The restriction morphism must be compatible in the expected way, that is to say $\mathrm{res}_{V,U}(s \cdot f) = \mathrm{res}_{V,U}(s) \cdot \mathrm{res}_{V,U}(f)$ (in other words it is linear by considering $\mathcal{F}(V)$ as a $\mathcal{O}(U)$-module by restriction along $\mathrm{res}_{V,U}$).

As we will see we can do a lot of commutative algebra in this setting. And just as we commonly fix a base ring and study its algebras and modules and how they interact, here we must fix a ringed space.

Definition 2.11. A **ringed space** is a topological space X together with a sheaf of rings $\mathcal{O}_X$ over X. In the case that the ring is local on stalks, we say that we are working with a **locally ringed space**.

A morphism between ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous function $f : X \to Y$ together with a ring sheaf morphism $\phi : \mathcal{O}_Y \to f_*\mathcal{O}_X$. In the case of locally ringed spaces we also require that for all points $p \in X$ ϕ_p is a local morphism, meaning that the antiimage of the maximal ideal of the codomain is the maximal ideal of the domain.

Indeed many of the usual properties that we care about for rings can be redefined for ringed spaces, though this won't be terribly important as all the spaces that we are going to work with are very well behaved in this regard. We only mention some aspects of locally ringed spaces because they will be particularly important in our arguments.

Fixing a locally ringed space $(X, \mathcal{O}_X)$ we will now give a summary of some important classes of $\mathcal{O}_X$-modules, which generalize the corresponding concepts in classical algebra.

Definition 2.12. An $\mathcal{O}_X$-module $\mathcal{F}$ is of **finite type** if there is an open cover $\{U_i\}_{i\in I}$ of X, such that for every $i \in I$ there is an epimorphism from $\mathcal{O}_X^n|_{U_i}$ to $\mathcal{F}|_{U_i}$.

Definition 2.13. An $\mathcal{O}_X$-module $\mathcal{F}$ is **quasi-coherent** if there is an open cover $\{U_i\}_{i\in I}$ such that there exists A_i, B_i not necessarily finite sets such that there is an exact sequence $\mathcal{O}_X^{A_i}|_{U_i} \to \mathcal{O}_X^{B_i}|_{U_i} \to \mathcal{F}|_{U_i} \to 0$

Definition 2.14. An $\mathcal{O}_X$-module $\mathcal{F}$ is **coherent** if it is of finite type and for every morphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$ its kernel is of finite type.

Definition 2.15. An $\mathcal{O}_X$-module $\mathcal{F}$ is **locally free** if there exists an open cover $\{U_i\}_{i\in I}$ of X such that for every $i \in I$, we have $\mathcal{F}|_{U_i} \cong \mathcal{O}_X^n|_{U_i}$. The natural number n is called the rank of the locally free sheaf.

We will speak more about locally free sheaves and their importance in another section, though we mention that, since we are dealing with local properties, they act as free modules usually do. Being of finite type is akin to finite generation in usual modules (in fact it is sometimes called by that name). There is a notion of coherence on commutative algebra, but it is more useful to think of it as finite presentation, and in the cases we are most interested in (due to noetherianity) it is enough to consider it as being of finite type.

There is also the weaker condition called quasicoherence that does have some importance in the more general context of the theory, but we will only be using it for technical results.

It is important to note that most constructions regarding modules over rings can also be transported into the setting of sheaves. Some useful ones include:

- The **tensor product** of modules, which requires sheafification to be truly realized. This is important as it means that, in general, it is not trivial to understand the sections of a tensor product, even if we know quite well the two sheaves forming it.

- The **exterior algebra** which is crucial for the definition of many important sheaves, and it will be used extensively in proofs. Note that, though it can be defined more generally, it is by far better behaved in locally free spaces, so we will use it basically on those.
- The **internal hom**, that is to say the module of linear functions between modules. More precisely it can be defined as $\mathcal{H}om(\mathcal{F},\mathcal{G})(U) := \{\phi : \mathcal{F}|_U \to \mathcal{G}|_U\}$ with obvious restriction maps. When reduced to maps of $\mathcal{O}_X$-modules it gives us another $\mathcal{O}_X$-module. In particular, this allows us to define the **dual module** by considering $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X)$

2.3 Sheaf Cohomology

Inspired by the theory of topological cohomology, which proved to be a powerful topological (and homotopical) invariant, we can also define a cohomology theory on sheaves. It follows most of the rules we regularly use in singular cohomology. In fact given some "niceness" conditions (such as being locally contractible or even weaker conditions) on the space, we get that for A an abelian group $H^n_{\text{sing}}(X, A) \cong H^n(X, \underline{A})$ where $\underline{A}$ is the sheafification of the constant functor to A, often just called the constant sheaf over A. The constant sheaf of an algebraic structure (modules, groups, rings...) becomes the sheaf version of that structure. In our case, $\underline{A}$ is an abelian sheaf.

However, although in the previous paragraph we have mentioned cohomology over abelian sheaves, we will mostly be working with cohomology over $\mathcal{O}_X$-modules. Having said that, they both coincide as groups, but the latter additionally is an $\mathcal{O}_X(X)$-module.

Now given any $\mathcal{O}_X$-module $\mathcal{F}$ we define the cohomology as a family of functors $H^n(X,F)$: $Sh_{\mathcal{O}_X}(X) \to \mathcal{O}_X(X)$-Mod (from now on, we will omit the X if it is clear by context). An actual definition of these functors would require us to talk in one way or another about injective sheaf resolutions and derived functors, which while useful in their own right don't help us in clarifying the concept of cohomology of sheaves.

Working with sheaf cohomology requires a lot of technical results that we are not interested in tackling. We will be skipping most proofs and some interesting results. To get a more complete picture on this subject I recommend [Har77, Chapter 3].

Having given this reference we will instead focus on giving some important properties and more down to earth ways to actually compute them (in fact, one of these ways served as the original definition of sheaf cohomology). Some of these important properties are:

1. (Long exact cohomology sequence) Given a short exact sequence of $\mathcal{O}_X$-modules
$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$
we have a long exact sequence:
$$0 \to H^0(\mathcal{F}) \xrightarrow{H^0(f)} H^0(\mathcal{G}) \xrightarrow{H^0(g)} H^0(\mathcal{H}) \xrightarrow{\sigma_0} H^1(\mathcal{F}) \xrightarrow{H^1(f)} \dots$$
Where σ_* is called the boundary morphism.

2. (Global sections) The 0th cohomology group is defined to be the group of global sections: $H^0(X, \mathcal{F}) := \mathcal{F}(X)$.

3. (Cup product) As with singular cohomology, we have a cup product $\cup : H^i(\mathcal{F}) \otimes H^j(\mathcal{G}) \to H^{i+j}(\mathcal{F} \otimes \mathcal{G})$. In particular, we have $\cup : H^i(\mathcal{O}_X) \otimes H^j(\mathcal{O}_X) \to H^{i+j}(\mathcal{O}_X)$ making $H^*(\mathcal{O}_X)$ a graded ring, known as the cohomology ring.

When it comes to actually computing these cohomology groups from scratch, that is, without using the long exact cohomology sequence, flabby resolutions and Čech cohomology will be our main tools (and they are in fact strongly related).

Definition 2.16. A **resolution** of a sheaf $\mathcal{F}$ is a long exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \dots$. If all $\mathcal{G}^i$ share some property, the resolution is said to hold it as well.

We mentioned briefly that cohomology can be defined through an injective resolution, by considering the cochain complex $0 \to I^0(X) \to I^1(X) \to \dots$ and calculating chain cohomology. Now, while I have mentioned that delving into injective sheaves is not very enlightening, there are other kinds of resolutions that can be used to compute sheaf cohomology in the same way.

Definition 2.17. A sheaf $\mathcal{F}$ is said to be **acyclic** if all of its higher cohomology groups are trivial, that is, $H^i(\mathcal{F}) = 0 \; \forall i > 0$.

Proposition 2.18 ([Voi02, Proposition 4.32])**.** *Given an acyclic resolution its cohomology calculated in the aformentioned way coincides with injective cohomology.*

While this may seem useful at first, identifying a sheaf as acyclic from the definition requires us to compute its cohomology, so we are really just shifting the problem of finding a simple way to identify acyclic sheaves. A perhaps more useful notion (through the following result) is:

Definition 2.19. A sheaf is flasque/flabby if all of its restriction morphisms are surjections. This is equivalent to all restrictions from the global sections being surjective.

Proposition 2.20 ([God58, Théorème 3.1.3])**.** *All flabby sheaves are acyclic.*

As such, we can think of injective sheaves as being better suited for theoretical arguments, while acyclic sheaves are the most general case and flabby sheaves are easier to check.

Flabbyness is indeed a much better property in this sense, as it is conserved through many operations, which makes proving that a particular sheaf is flabby both directly and indirectly relatively simpler. This makes them ideal to compute sheaf cohomology.

We now move onto Čech cohomology, which is a method for computing sheaf cohomology directly through chain complexes. It is not quite as direct as the computations of singular cohomlogy, as it is initially taken on an open cover and then is refined by taking finer covers (more precisely the filtered limit through all open covers). Moreover, it is not immediately clear that it coincides with sheaf cohomology, though it does in our cases of interest. To begin, we define the Čech complex:

Definition 2.21. The **Čech complex** of $\mathcal{F}$ over the open cover $\mathcal{U} = \{U_i\}_{i \in I}$ is defined as:

1. The groups are defined as:

$$C^n(\mathcal{U}, \mathcal{F}) := \prod_{i_0, \ldots, i_n \in I} \mathcal{F}\Big(\bigcap_{j=0}^{n} U_{i_j}\Big)$$

2. The differentials of the complex are given by $d : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$ with

$$d(\alpha)_{i_0,\ldots,i_n} = \sum_{k=0}^{n} (-1)^k \alpha_{i_0,\ldots,i_{k-1},i_{k+1},\ldots,i_n}|_{\bigcap_{j=0}^{n} U_j}$$

The Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$, denoted by $\hat{\mathrm{H}}^*(\mathcal{U}, \mathcal{F})$, is the chain cohomology of the Čech complex.

Definition 2.22. The Čech cohomology of $\mathcal{F}$, $\hat{\mathrm{H}}^*(X, \mathcal{F})$, is defined as the direct limit over all open covers of X ordered by refinement.

Proving that the Čech complex is indeed a complex is a simple computation. We may notice that the last step of calculating $\hat{\mathrm{H}}^*(X, \mathcal{F})$ is quite computationally unpleasant. However, given a sufficiently nice cover $\mathcal{U}$ we may have $\hat{\mathrm{H}}^*(X, \mathcal{F}) = \hat{\mathrm{H}}^*(\mathcal{U}, \mathcal{F})$.

Of course, its usefulness relies on its comparison with sheaf cohomology. In general, they needn't be the same, but in some nice cases we have a natural isomorphism of functors, giving us a fairly simple computational tool. An example of such a condition is the following:

Theorem 2.23. *Given an open cover $\{U_i\}_{i \in I}$ such that all U_i are affine and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, Čech cohomology on $\mathcal{U}$ coincides with sheaf cohomology (and, by sandwich, with Čech cohomology).*

This is proven in [Har77, Theorem 4.5, p. 222]. Note that there it is proven for a "Noetherian separated scheme", but as we are working with projective varieties we needn't worry about such subtleties. We will use this result in our section on projective cohomology.

It is also interesting to note that, even though Čech cohomology in general does not always coincide with sheaf cohomology, it always does for the first degree H^1. The common yoga is that the first cohomology group has better properties than the rest, and it is no coincidence that it is the one that usually can be interpreted more geometrically (or as an obstruction), but we won't get into that here.

2.4 Complex geometry and algebraic geometry

The definitions in Sections 2.1, 2.2 and 2.3 have a topological background. In this section we will delve into the main geometrical definitions needed for this report.

Polarized K3 surfaces, which we will be our object of study, lie in the intersection of algebraic and analytical geometry. Hence, a good understanding of both will be required. An important property that we will be using is the ability to change between interpreting our surfaces as algebraic or analytic spaces, while still retaining most of the structure (basically its cohomology). But before all of this, we require some definitions:

Definition 2.24. A **complex manifold** or complex analytical variety is a topological space X together with a holomorphic atlas. That is to say, an open cover $\{U_i\}_{i\in I}$ such that all opens of the cover are equipped with an homeomorphism $\phi_i : U_i \to V_i \subseteq \mathbb{C}^{n_i}$ with V_i open and for all $i, j \in I$ the map $\phi_i \circ \phi_j^{-1} : \phi_j(U_i) \to \phi_i(U_j)$ is a holomorphic morphism in the usual sense. (Note: it can be shown with local homology that n_i is constant on connected components).

A holomorphic morphism $M \to N$ between complex manifolds is simply a continuous map such that restricted to an element of the atlas of N and then further by an element of the atlas of M we get a holomorphic map. We may define a structural sheaf on these spaces, $\mathcal{O}_X$, by considering $\mathcal{O}_X(U)$ to be the ring of holomorphic maps from U to $\mathbb{C}$. As such, it has the structure of a locally ringed space. It is easy to check that holomorphic maps are simply the ringed space morphisms between them.

We can also consider complex geometry from an algebraic point of view, by taking closed sets to be determined not by a local property, but rather by a global one: they are the zero sets of ideals of $\mathbb{C}[x_1, \ldots, x_n]$. Notice that the maximal ideals correspond to the usual notion of points on $\mathbb{C}^n$ thanks to the Nullstellensatz (see, for example, [AM69, p. 85, Chapter 7]).

As such, we may be tempted to consider a space with maximal ideals as points. But, as we commonly see in commutative algebra maximal ideals only carry part of the information. For the rest, we will also have to consider prime ideals, which correspond to the "generic" points of an irreducible subvariety, that is, an "imaginary" point that topologically is infinitely close to any other point of the subvariety.

More precisely, its topology, called the Zariski topology, is defined by its closed sets, by associating to every ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ the closed set $V(I) := \{\rho \in \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]) | I \subseteq \rho\}$. This space is called the affine n-dimensional complex space, denoted $\mathbb{A}^n_{\mathbb{C}}$. Note that this space has as an open basis the sets $D(f) := \mathbb{A}^n_{\mathbb{C}} - V((f))$, called the standard opens.

This topological space can also be equipped with the structure of a locally ringed space. We define its structural sheaf on standard opens as

$$\mathcal{O}_{\mathbb{A}^n_{\mathbb{C}}}(D(f)) := \mathbb{C}[x_1, \ldots, x_n]_f = \{g/f^n | n \in \mathbb{N}, g \in C[x_1, \ldots, x_n]\}$$

This is enough thanks to Proposition 2.9. We thus call any section of this sheaf a regular function. It is important to note that there may be regular functions that aren't rational functions when looking at non standard open sets.

Note that our toplogical space is not even $T1$. Indeed, when looking at complex varieties we are used to the concept of opens giving us a sense of neighbourhood, so that they may be "arbitrarily" small in some sense, while in Zariski topology open sets are usually quite big.

An algebraic variety is similar to an analytical one in that it can be glued together from open subspaces of some base space, in this case n-dimensional affine space. Unfortunately, going through with this path would require to make technical discussions about certain important properties of these spaces that fall out of the scope of this report. Therefore, we prefer to focus on one particular algebraic variety which is very well behaved, the projective space, and only consider closed subvarieties of it.

The algebraic projective space with the homogeneous polynomials taking the place of regular ones follows this condition, with the charts $D_+(x_i) = \{x_i \neq 0\}$. Indeed in each of these we can "set" $x_i = 1$ and recover the affine n-dimensional complex space as a ringed space. We now can define:

Definition 2.25. A projective algebraic variety is a closed subset of algebraic projective space.

Algebraic projective space comes naturally equipped with a structural sheaf, called the *sheaf of regular functions*. On an open set $D_+(f)$ it is the set of all rational homogeneous polynomials of total degree 0 such that the denominator is non zero in U. Thus, we may consider the structural sheaf on any projective variety as the pullback of the projective structure sheaf through its immersion. As such they are ringed spaces and we define morphisms between them as ringed space morphisms.

There is a more general definition of algebraic variety, but in this report we won't be delving any deeper into the formalism.

An evidence of the rigidity of holomorphic functions in comparison to smooth ones is the connection between analytical complex geometry and algebraic geometry, best exemplified through the GAGA theorems. To begin with, we must first explain the analytification process.

This is quite simple, as it follows directly the intuition of what makes analytic and algebraic varieties similar: having a covering made by $\mathbb{C}^n$ or affine spaces. Starting with a projective algebraic variety, the idea is simply to consider all its closed points, which as in affine space are just the "usual" points in $\mathbb{C}^n$ (as can be seen thanks to t Nullstellensatz) and endow it with the analytical topology. This transforms an algebraic variety into an analytical one and it comes equipped with a continuous function from the latter to the former (the inclusion), as algebraic varieties are closed in the analytical topology.

This induces a functor between sheaves, and the first GAGA theorem states that on projective algebraic varieties as we have defined previously it restricts to an equivalence of categories over the coherent $\mathcal{O}_X$-modules that preserves the global sections functor and thus, in particular, the cohomology theory. For a more detailed account on the GAGA Theorem and its proof we would like to mention the following course: [Ked09, GAGA, Lectures 30–33].

An application of this powerful result comes in the form of Chow's theorem, which states that every complex analytical subspace of projective space is also a projective algebraic variety. This result really shows the difference in rigidity between differential and complex geometry.

2.5 Vector bundles

Among all possible sheaves there is a particular family that will be of our interest: locally free sheaves. Intuitively, they fulfil the role of free modules in algebra. In a sense coherent modules can be seen as a good generalization of them, giving a better category while conserving enough "finiteness" for our purposes.

Note that in this section we are considering vector bundles over complex manifolds, leaving the algebraic point of view aside. Since all locally free sheaves in the space we are working with are coherent, the GAGA theorem allows us to translate them algebraically (Note: the coherence of locally free sheaves is non trivial and is equivalent to the coherence of $\mathcal{O}_X$).

However, though on a theoretical level it is a useful characterization, locally free sheaves have been used long ago via an equivalent notion, that of vector bundles.

Definition 2.26. A **vector bundle** is a complex manifold E together with an holomorphic map $\pi : E \to X$ such that:

- The fiber $\pi^{-1}(p)$ has the structure of a complex vector space for all $p \in X$.
- There exists an open cover $\{U_i\}_{i \in I}$ such that there is an homeomorphism $\phi_i : U_i \times \mathbb{C}^{n_i} \to \pi^{-1}(U_i)$.
- The prior morphisms make the following diagram commute:

$$\begin{tikzcd} U_i \times \mathbb{C}^{n_i} \arrow[rr, "\phi_i"] \arrow[dr] & & \pi^{-1}(U_i) \arrow[dl, "\pi"] \\ & U_i & \end{tikzcd}$$

- The homeomorphism ϕ_i is $\mathbb{C}$-linear when restricted to any fiber $p \in X$.

There exists a more lax definition, that of a topological vector bundle. Even though they don't respect the holomorphic structure of the complex manifold, they are also very useful, for example, when we consider properties that depend only on the smooth or the topological structure. When speaking of a vector bundle we usually refer to it just by the object E, even if the morphism $\pi : E \to X$ is as important as the object. However, we will usually refer to it with the same terminology as the sheaf it represents, because of the following proposition:

Proposition 2.27. *There is a one to one correspondence between isomorphism classes of vector bundles and isomorphism classes of locally free sheaves given in one direction by taking the sheaf of sections of the projection.*

Proof. It is easy to see that the local trivialization of a vector bundle induces a open cover of X where the sheaf of sections associated to the vector bundle is free of the same rank.

For the other direction we consider an open cover $\{U_i\}_{i\in I}$ such that the sheaf $\mathcal{F}$ is free on each U_i with rank r. Note that the rank is constant on connected components, and that $\mathcal{F}|_{U_i\cap U_j}$ is also free for non disjoint U_i and U_j. In fact we are given two isomorphisms $f_i, f_j : \mathcal{F}|_{U_i\cap U_j} \to \mathcal{O}_X|_{U_i\cap U_j}^{\oplus r}$, by restricting the trivialization. As such, we may consider the automorphism $f_i \circ f_j^{-1}$ over $\mathcal{O}_X|_{U_i\cap U_j}^{\oplus r}$. This is equivalent to choosing a basis $\{s_k\}_{1\leq k\leq r}$ of $\mathcal{O}_X(X)^r$. Then we define the automorphism f over $U_i \cap U_j \times \mathbb{C}^r$ by $f(x, \vec{z}) = (x, \sum_{k=1}^r z_k s_k(x))$.

We can now construct the space $\coprod_{i\in I} U_i \times \mathbb{C}^{n_i}/\sim$. The equivalence relation $\sim$ identifies for any non disjoint U_i and U_j through the previously defined automorphism. Then, considering the obvious projection to X, it is easy to show that these operations are inverses one from the other (up to isomorphism). $\square$

Because of the previous proposition, we will use the terms "vector bundle" and "locally free sheaf" interchangeably. Of particular interest are vector bundles of rank 1, usually called "line bundles" or "invertible sheaves". The second name comes from the following fact:

Proposition 2.28. *Let L be a line bundle. Then, there is a natural isomorphism $L \otimes L^\vee \to \mathcal{O}_X$.*

Proof. Consider the morphism defined on open sets by $\phi_U : L(U) \times L^\vee(U) \to \mathcal{O}_X(U)$ defined as $\phi_U(x, f) := f(x)$. This map is bilinear, and as such induces a morphism $\phi'_U : L(U) \otimes L^\vee(U) \to \mathcal{O}_X(U)$.

It is easily checked that this is indeed a morphism among presheaves (that is to say that these maps commute with restriction), and as such through sheafification we obtain a morphism $\phi'' : L \otimes L^\vee \to \mathcal{O}_X$. The first map was natural over L, and since all subsequent transformations come from universal properties and functors, so is ϕ''.

But now because of local freeness, there is an open cover $\{U_i\}_{i\in I}$ such that $L|_{U_i} \cong \mathcal{O}_X|_{U_i}$, and because of naturality of $\phi|''_{U_i}$ we may restrict ourselves to the case $L|_{U_i} = \mathcal{O}_X|_{U_i}$. The map considered at the level of stalks has an inverse $\mu : \mathcal{O}_{X,p} \to \mathcal{O}_{X,p} \otimes \mathcal{O}_{X,p}^\vee$ acting as $\mu(x) = x \otimes \mathrm{Id}$. Hence, the map is an isomorphism. $\square$

This proof is a simple example of the potential technical complexity of working with sheaves. To define a function between sheaves you usually have to define it first between presheaves and then use sheafification (or use the universal properties of the corresponding constructions). In that very first step it becomes necessary to check compatibility of the morphism defined openwise. To resolve all these issues we could add many technical lemmas. For the sake of brevity we will cut many details to leave room for a more high level analysis of the next proofs.

Now note that Proposition 2.28 gives us the inverse of a group law. Indeed, if we consider isomorphism classes of line bundles, these are closed under tensoring, they have a right and left identity $\mathcal{O}_X$ and they have a right and left inverse $L^\vee$. Then, by the associativity and commutativity of the tensor product, we have the structure of an abelian group, known as the **Picard group** (we will denote it $\mathrm{Pic}(X)$).

To be fair, strictly by the definition we have just given we may have some trouble in truly considering the Picard group a proper group, as it may not even form a set (though this is not really necessary for some of the theory). Luckily we needn't worry about that thanks to some equivalences that will be introduced at a later stage.

We are interested in a few important definitions for vector bundles/line bundles that will come into play latter on:

Definition 2.29. A point $p \in X$ is a **basepoint** of a vector bundle E (or possibly any sheaf) if for every global section $s \in E(X)$, $s(p) = 0$.

What is important is that, given a line bundle L, if U is the set of all non-basepoints of X, then we may define a morphism $\phi_L : U \to \mathbb{P}(H^0(X, L)^\vee)$, given by $p \to \mathrm{Ker}(\cdot_p)$. This will be the key point for our classification of K3 surfaces later on. For this to be useful, we would like:

Definition 2.30. A line bundle L is **basepoint-free** L if it has no basepoints.

Definition 2.31. A line bundle is **very ample** if it is basepoint-free and the morphism ϕ_L is a closed immersion.

Of course this last definition is very useful, but also quite badly behaved. It is difficult to check when a line bundle is very ample and it is not a property that is commonly preserved. As such, we will use a weakening of this condition:

Definition 2.32. A line bundle is **ample** if there exists an integer m such that $L^{\otimes m}$ is very ample.

Tangent bundle

An important vector bundle that can always be constructed in analytical spaces is that of the tangent bundle. Indeed, its fiber at each point is the tangent space at the corresponding point. Note that because of the rigidity of the holomorphic structure, this construction can be split into two variants.

The first one is the topological tangent bundle, defined as:

Definition 2.33. Given a n-dimensional complex manifold X with atlas $\{\phi_i : U_i \to \mathbb{C}^n\}_{i\in I}$, the **topological tangent bundle** has as underlying set $TX_t := \amalg_{p\in X} T_pX$, with the obvious projection morphism π sending all elements of T_pX to p.

To define its topology we need to consider the extended functions $\tilde{\phi_i} : \pi^{-1}(U_i) \to \mathbb{C}^{2n}$, and we simply define its open sets as those sets U such that $\tilde{\phi_i}(U \cap \pi^{-1}(U_i))$ is open for all $i \in I$.

The topological tangent bundle doesn't take into account the holomorphic structure and it is simply a topological vector bundle. By contrast, it does naturally inherit a differential structure, which we may use as a starting point to endow it with a complex structure. It is still important, but we will mostly be interested in the holomorphic vector bundle, which is a proper holomorphic vector bundle.

As the topological tangent bundle has an induced smooth structure from its morphism, we can consider the tensor product $TM_t \otimes_{\mathbb{R}} \mathbb{C}$. This doesn't quite give it a holomorphic structure yet. Notice how we have $dim_{\mathbb{C}}(X) = n \implies dim_{\mathbb{R}}(X) = 2n \implies dim_{\mathbb{R}}(TX_t) = 4n \implies dim_{\mathbb{R}}(TX_t \otimes_{\mathbb{R}} \mathbb{C}) = 8n \implies dim_{\mathbb{C}}(TX_t \otimes_{\mathbb{R}} \mathbb{C}) = 4n$, which is double what we would expect. It is here where we can impose the holomorphic structure through the Cauchy-Riemman equations, which essentially impose that a function $f : \mathbb{C} \supseteq U \to \mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}) = 0$. As such, taking local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ we may consider taking the subbundle spanned by $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$. This is known as the **holomorphic tangent bundle**.

While this vector bundle plays an important role in geometry, for our purposes we will be more interested in its dual bundle, the **cotangent bundle** usually denoted by Ω_X, and its determinant known as the **canonical line bundle** denoted by lower case ω_X. The first one appears in various exact sequences that will become relevant, and the second one is important because it give us a concept of "curvature". Indeed, the fact that the canonical line bundle is trivial is one of the two basic conditions for being a K3 surface, and it becomes quite important when making computations.

2.6 Weil and Cartier divisors

In this section we will introduce the concept of a Weil divisor. This will be useful to give the theory of line bundles a geometric flair, simplifying conceptually further arguments.

Definition 2.34. A **prime divisor** is an irreducible Zariski closed subvariety of X of codimension 1.

Definition 2.35. A **Weil divisor** of X is a finite linear combination over $\mathbb{Z}$ prime divisors. That is, they are elements of the free commutative group generated by irreducible closed subspaces of codimension 1, which is denoted by $\mathrm{Div}_W(X)$.

While we are obviously interested in the group structure by definition, we are also interested in certain well behaved Weil divisors that we may consider to be realized as a closed subvariety with multiplicity.

Definition 2.36. A Weil divisor D is said to be **effective** if all of its coefficients are non negative.

We can consider these to be the "positive" divisors, and as such we may define $D \geq D' \iff D - D'$ effective. They will play a key role when giving "numerical" conditions for various phenomena.

We have mentioned that Weil divisors are related to line bundles, but it is clear that $\mathrm{Div}_W(X)$ is far too big to be isomorphic to the Picard group. For that, we need to mod out the principal divisors.

Definition 2.37. Given any $f \in \mathcal{O}_X(X)^*$ and any irreducible subvariety Z, we define its order as $\mathrm{ord}_Z(f) := \mathrm{length}(\mathcal{O}_{X,Z}/(f))$.

Note that here we are using that, since Z is an irreducible subvariety, it has an associated prime in $\mathbb{P}^N$, and by our definition it is also a point in the corresponding ringed space. This allows us to consider easily $\mathcal{O}_{X,Z}$.

Definition 2.38. Given any $f \in \mathcal{O}_X(X)^*$, we define its **principal** divisor by $div(f) := \sum \mathrm{ord}_Z(f) \cdot Z$ where the sum is taken over all prime divisors Z. Note that $f|_{\mathcal{O}_{X,Z}} \neq 0$ implies $\mathrm{ord}_Z(f) = 0$, and there are only a finite number of Z such that $f|_{\mathcal{O}_{X,Z}} = 0$.

Given this definition, and the fact that $\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g)$ we can see that principal divisors form a subgroup of Weil divisors, and thus we can consider taking the quotient. In our case we obtain a group isomorphic to the Picard group.

This isomorphism usually factors through another type of divisors that fall much closer to line bundles. Those are Cartier divisors.

Definition 2.39. We define the **sheaf of rational functions** $\mathcal{M}_X$ as the sheafification of $\mathcal{M}_X^{\mathrm{pre}}(U) := T_U^{-1}\mathcal{O}_X(U)$ with T the set of all sections such that they are not zero divisors at any stalk. This is equivalent to taking the field of fractions in the case that all $\mathcal{O}_X(U)$ are integral domains.

When X is an irreducible variety this is just the constant sheaf of the field of fractions of $\mathcal{O}_X(X)$.

Definition 2.40. A **Cartier divisor** is composed of an open cover $\{U_i\}_{i\in I}$ together with a collection $s_i \in \mathcal{M}_X^*(U_i)$ of sections, with the restriction that $s_i|_{U_i \cap U_j} = \mu_{i,j} \cdot s_j|_{U_i\cap U_j}$ with $\mu_{i,j} \in \mathcal{O}_X^*(U_i \cap U_j)$. Two Cartier divisors are equivalent if they are equal on a refinement of their open covers.

This definition is equivalent to simply saying that a Cartier divisor is a global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$. Its connection to invertible sheaves can be more naturally understood via Weil divisors. Indeed we may identify every Cartier divisor with the sheaf obtained by gluing together the sheaves $\frac{1}{s_i} \cdot \mathcal{O}_X|_{U_i}$. Note how the given open cover provides a trivialization making it a line bundle, which we refer to as $\mathcal{O}_X(D)$ for D a Cartier Divisor.

Though it is already implicit in the second definition we have given, we define the product of two Cartier divisors $D = (U_i, s_i)_{i\in I}$ and $D' = (V_j, m_j)_{j\in J}$ as

$$D + D' := (U_i \cap V_j, s_i|_{U_i\cap V_j} \cdot m_j|_{U_i\cap V_j})$$

We denote the group of all Cartier divisors by $\mathrm{Div}_C(X)$.

Just as before, this group is way bigger than what we want, and as such we define the principal Cartier divisors:

Definition 2.41. A **principal** Cartier divisor is obtained from a global section $s \in \mathcal{M}_X^*(X)$ as (X, s). We call this subgroup $\operatorname{Div}_C^p(X)$.

In other words, they are the image of the morphism $H^0(X, \mathcal{M}_X^*) \to H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$, induced by the exact sequence:

$$0 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathcal{M}_X^*/\mathcal{O}_X^* \to 0$$

By taking the quotient with the principal Cartier divisors we obtain once more the Picard group. What is interesting is that, though not generally true, in our particular case Cartier divisors correspond exactly with Weil divisors, as we will soon see.

Before moving on, we may take a second look at the previous exact sequence. In particular we may notice that there is an inclusion $\operatorname{Div}_C(X)/\operatorname{Div}_C^p(X) \to H^1(X, \mathcal{M}_X^*)$. This inclusion is actually an isomorphism, though this can be difficult to see. We will instead show directly the isomorphism $\operatorname{Pic}(X) \cong H^1(X, \mathcal{M}_X^*)$.

Now, let us take a look at all these equivalences we have been claiming in the last paragraphs. First of all we will see that:

Proposition 2.42. *The quotient between Cartier divisors and principal Cartier divisors is isomorphic to the Picard group.*

Proof. We have already constructed a map from Cartier divisors to the Picard group, by considering the associated line bundle. We will show that this map is a group morphism which is exhaustive and has $\operatorname{Div}_C^p(X)$ as kernel.

For the first condition, simply note that for any $s, s' \in \mathcal{M}_X^*(U)$ for U any open set, then $\frac{1}{ss'}\mathcal{O}_X|_U \cong \frac{1}{s}\mathcal{O}_X|_U \otimes \frac{1}{s'}\mathcal{O}_X|_U$, showing what we wanted.

Showing the second condition is done by picking a trivialization $(U_i, \phi_i)_{i \in I}$ of some line bundle L. We first see that we can consider L to be a subsheaf of $\mathcal{M}_X$. Indeed we may consider ϕ_i. Then we obtain that the isomorphisms $\phi_i : \mathcal{O}_X|_{U_i} \to L|_{U_i}$ which pick sections $s_i \in L(U_i)^* \subseteq \mathcal{M}_X(X)^*$. As such we see that the Cartier divisor (U_i, s_i) is an preimage of L with respect to the aforementioned morphism.

The third condition is the easiest to see, as giving an isomorphism $\phi : \mathcal{O}_X \to \mathcal{O}_X(D)$, is equivalent to choosing a global section $s \in \mathcal{O}_X(D)(X) \subseteq \mathcal{M}_X(X)^*$. As such, we may consider the principal divisor $div(s)$. The other direction is obvious as $\frac{1}{s}\mathcal{O}_X \xrightarrow{\cdot s} \mathcal{O}_X$ is an isomorphism. $\square$

The following proposition uses that the ambient variety is smooth.

Proposition 2.43. *The group of Cartier divisors is isomorphic to the group of Weil divisors, and the isomorphism preserves principal divisors.*

Proof. From a Cartier divisor $D = (U_i, s_i)_{i\in I}$ you can always get a Weil divisor, by considering $div(D) := \sum_Z \mathrm{ord}_Z(f_{i_Z})Z$ where Z runs over all prime divisors and i_Z is any index such that $Z \cap U_{i_Z} \neq \emptyset$. Note that the value is independent of our choice of i_Z.

For an inverse to this morphism we require the local rings $\mathcal{O}_{X,p}$ to be UFDs, which is not a problem in our case as our varieties are smooth (see [Mum99, Section III.7]). Then, consider the sheaf of regular functions that vanish at D, $\mathcal{O}_X(-D)$. This sheaf is locally a prime ideal of height 1 (that is, the only prime ideal strictly contained in it is (0)). This comes naturally as height correspond with codimension (given that our varieties are smooth).

Now, because for every point $p \in X$, $\mathcal{O}_{X,p}$ is a UFD, if we have a height 1 prime ideal $\mathcal{P}$, and we consider a non-zero element $x \in \mathcal{P}$, then some irreducible element $a|x$ is in $\mathcal{P}$. As such $(a) \subseteq \mathcal{P} \implies (a) = \mathcal{P}$, so $\mathcal{P}$ is principal.

If we consider now $\mathcal{O}_{X,p}(-D) = (s_p)$, then in particular we have $s_p \in \mathcal{M}^*_X(X)$, and it must be that there is some open set U_x that intersects only divisors that contain x. As such, they coincide in the stalk, so we may consider the Cartier divisor (U_x, f_x). Proving that these are inverses is just a straightforward translation of properties. $\square$

This shows in particular that the quotient of Weil divisors with principal Weil divisors also gives us the Picard group. We can thus ask how the properties of the relevant line bundles interact with their associated divisors. In particular, given a very ample line bundle L, the immersion ϕ_L gives also an inclusion of $\mathrm{Div}_W(X)$ into the cycles of dimension 1 in projective space. In this context, the divisors associated to L are the intersections of the image of X into $\mathbf{P}^n$ and a hyperplane $H \subset \mathbf{P}^n$. On the other hand, there is no good analog for a description of ampleness. This shows that, while very ampleness can be seen as a geometric property, ampleness is purely algebraic.

Lastly, we prove the last equivalence of the Picard group that we will be using :

Proposition 2.44. *There is an isomorphism* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*_X)$.

Proof. We will make use of Čech cohomology for this. Consider any cover $\mathcal{U} = \{U_i\}_{i\in I}$ on X. We will show that $\hat{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}^*_X)$ is isomorphic to the group of $\mathcal{U}$ line bundles, that is, those that have $\mathcal{U}$ as a trivializing open cover. By taking the limit over all covers it is clear that we obtain the group of line bundles $\mathrm{Pic}(X)$.

Consider the first Čech group, which is by definition the product of all

$$C^1(\mathcal{U}, \mathcal{O}^*_X) = \prod_{i,j\in I} \mathcal{O}^*_X(U_i \cap U_j).$$

Now we want to impose the condition $d(\alpha) = 0$, which in this case translates to $\alpha_{i,j} \cdot \alpha_{j,k} = \alpha_{i,k}$. Note that $\alpha_{i,j} \in \mathcal{O}^*_X(U_i \cap U_j)$ defines an automorphism over $\mathcal{O}_X|_{U_i\cap U_j}$ and our previous property corresponds simply to the cocycle condition, which allows us to define a line bundle L_α.

This allows us to define a morphism from $C^1(\mathcal{U}, \mathcal{O}^*_X)$ to $\mathrm{Pic}(X)$. That it is exhaustive is clear. Indeed, we can simply consider the isomorphisms $\phi_i : L|_{U_i} \to \mathcal{O}_X|_{U_i}$ and compose them as

$\phi_i|_{U_i \cap U_j} \circ \mathrm{res}_{U_i, U_i \cap U_j} \circ \phi_j^{-1}$ to obtain an isomorphism over $\mathcal{O}_X|_{U_i \cap U_j}$ and thus an element of $\mathcal{O}_X^*(U_i \cap U_j)$.

It remains to be shown that its kernel is equal to the image of

$$d_0 : C^0(\mathcal{U}, \mathcal{O}_X^*) = \prod_{i \in I} \mathcal{O}_X^*(U_i) \to C^1(\mathcal{U}, \mathcal{O}_X^*).$$

This comes naturally from considering an isomorphism $\phi : L_\alpha \to \mathcal{O}_X$. Indeed we note that $\phi_i|_{U_i \cap U_j} \circ \phi_j^{-1}|_{U_i \cap U_j} = \alpha_{i,j}$ by naturality. In the other direction, given ϕ_i, we can define an isomorphism with $\mathcal{O}_X$ directly. □

2.7 Cohomology of $\mathbb{P}^n$

Since in this report we will be dealing with polarized K3 surfaces, we need to explicitly introduce $\mathbb{P}^n$. Indeed, in a future theorem we will see that all polarizations of K3 surfaces (except in degree 2) provide an embedding into one of these spaces, so understanding $\mathbb{P}^n$ well will be crucial for further arguments. Most importantly, we will compute the cohomology of its line bundles through the use of Čech cohomology.

We will be borrowing this proof straight from [Har77]. It is added here for its importance and as an example of how to use Čech cohomology in computations. Before we can start our calculations, it will be convenient for us to introduce the concept of grading.

Definition 2.45. A ($\mathbb{Z}$-)**graded ring** is a ring R together with a decomposition of its additive group as $R = \sum_{i \in \mathbb{Z}} R_i$ such that the product restricted to these subsets defines a function $(\cdot) : R_i \times R_j \to R_{i+j}$.

We will actually be working with graded algebras over $\mathbb{C}$, which just means that you fix a ring morphism $\mathbb{C} \to R_0$. We will consider $\mathbb{C}[x_0, \ldots, x_n]$ to be graded by total degree, with 0 belonging to all degrees and negative degrees being trivial. In the case of a localization, we will consider $1/f$ to have degree $\deg(1/f) = -\deg(f)$. It will be important for the next arguments to keep track of the grading. To simplify terminology we will be refering to $\mathbb{C}[x_0, \ldots, x_n]$ as S.

We have mentioned that we are interested in computing the cohomology of line bundles over $\mathbb{P}^n$, but we haven't defined any yet. We thus begin by defining Serre's twisting sheaf. We will use the affine cover $\mathcal{U} = \{U_i\}_{i \in I}$, where $U_i := D(x_i)$.

Definition 2.46. We define **Serre's twisting sheaf** on n-dimensional projective space as the sheaf obtained by setting $\mathcal{O}_{\mathbb{P}^n}(1)(D(f)) := (S_f)_1$ with the obvious restriction maps.

Going back to the definition of the structural sheaf on projective space, we can consider a more general construction, which we will call $\mathcal{O}_{\mathbb{P}^n}(*)$, defined as $\mathcal{O}_{\mathbb{P}^n}(*)(D(f)) := S_f$. We note that this inherits from S a grading on the level of sheaves, and it allows us to define $\mathcal{O}_{\mathbb{P}^n}(r) := \mathcal{O}_{\mathbb{P}^n}(*)_r$, though this is usually just defined as $\mathcal{O}_{\mathbb{P}^n}(r) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes r}$, which shows more clearly that it is indeed a line bundle.

In fact, Serre's twisting sheaves compose all line bundles over $\mathbb{P}^n$ since $\mathcal{O}_{\mathbb{P}^n}(1)$ generates its Picard group (see, for example, [Kem93, Lemma 5.5.4]). We commonly refer to $\mathcal{O}_{\mathbb{P}^n}(-1)$ as the *tautological line bundle*. For any $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$, we use the terminology $\mathcal{F}(r) := \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(r)$.

Now, note that by the way the Çech complex is constructed, cohomology commute with direct sums, and in this case it will preserve our grading. Indeed, we can now construct the Čech complex directly as:

$$C^r(\mathcal{U}, \mathcal{F}) = \prod_{0 \leq i_1, \ldots, i_r \leq n} S_{x_{i_1}, \ldots, x_{i_r}}.$$

As such we can now compute the cohomology groups directly. First of all, let us consider the first morphism in the complex

$$d^0 : \prod_{k=0}^{n} S_{x_k} \to \prod_{k,l=0}^{n} S_{x_k, x_l}.$$

Its kernel is formed by those elements such that $d^0(\alpha)_{k,l} = \alpha_k|_{S_{x_k, x_l}} - \alpha_l = 0$ which is equivalent to $\alpha_k = \mu|_{S_{x_k}}$ for $\mu \in S$, i.e., such that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*)) \cong S$. Note that all maps presented naturally preserve the grading, and as such we have $H^0(X, \mathcal{O}_{\mathbb{P}^n}(r)) = S_r$. Note that as we briefly mentioned in Section 2.4, this fact together with GAGA implies that all analytical subvarieties of $\mathbb{P}^n$ are also algebraic (closed in the Zariski topology).

We will now consider the other easy case: the last non zero morphism in the complex

$$d^{n-1} : \prod_{0 \leq i_1, \ldots, i_{n-1} \leq n} S_{x_{i_1}, \ldots, x_{i_{n-1}}} \to S_{x_{i_1}, \ldots, x_{i_r}}.$$

The nth cohomology group of $\mathcal{O}_{\mathbb{P}^n}(*)$ is thus the cokernel of this morphism. The image can be seen simply as the free $\mathbb{C}$-module generated by all $x_0^{m_0} \cdots x_n^{m_n}$ such that at least one of the m_i is non-negative. As such the cokernel is generated by the elements of the form $x_0^{m_0} \cdots x_n^{m_n}$ with $m_i < 0$. Now if we take a look at the degrees, we see that all elements have degree less or equal to $-n-1$, with exactly one with that degree. As such we have in particular that: $H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0$ for all $r \geq -n$.

Now, for the harder case, the values $0 < i < n$, we will proceed by induction over n to show that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*)) = 0$. Note that the initial case $n = 1$ is trivial. Then, the proof relies mainly on the fact that, by definition, $\mathcal{O}_{\mathbb{P}^n}(*)$ is invariant under tensoring by any Serre's twisting sheaf. In particular, we may consider the global section $x_n \in \mathcal{O}_{\mathbb{P}^n}(1)$ and consider the induced exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\cdot x_n} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0,$$

which through tensoring with $\mathcal{O}_{\mathbb{P}^n}(*)$ gives us:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(*)(-1) \xrightarrow{\cdot x_n} \mathcal{O}_{\mathbb{P}^n}(*) \to \mathcal{O}_{\mathbb{P}^{n-1}}(*) \to 0.$$

Now we take cohomology and we use the induction hypothesis. This immediately gives us that for $1 < i < n-1$ we have that multiplication by x_n is an isomorphism on $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*))$ as a

$H^0(X, \mathcal{O}_{\mathbb{P}^n}(*))$-module. For the extremal cases we use our previous computations to see that we have an exact sequence:

$$0 \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*)(-1)) \xrightarrow{\cdot x_n} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^{n-1}}(*)) \to 0$$

As $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*)(-1)) \cong S$ and $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^{n-1}}(*)) \cong S/(x_n)$. Thus since at the level of H^0 the sequence is exact: $0 \to S \to S \to S/(x_n) \to 0$, multiplication by x_n is an automorphism of $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*))$.

For the last case we have the sequence:

$$H^{n-1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^{n-1}}(*)) \xrightarrow{\phi} H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*)) \xrightarrow{\cdot x_n} H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*)) \to 0,$$

and we need to see that ϕ is injective. This is quite simple, as the second morphism has as kernel the elements of the form $x_0^{m_0} \cdots x_n^{m_n}$ with $m_i < 0$ and $m_n = -1$. But as $H^{n-1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^{n-1}}(*))$ is generated by the elements of the form $x_0^{m_0} \cdots x_{n-1}^{m_{n-1}}$ with $m_i < 0$, and ϕ act as division by x_n, ϕ is indeed injective. This finally gives us that $H^{n-1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*))$ also has $\cdot x_n$ as an automorphism.

Hence, since multiplication by x_n is an automorphism, $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*)) \cong H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*))_{x_n}$ the localization at x_n. Then, via Čech cohomology, we get that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(*))_{x_n} \cong H^i(U_n, \mathcal{O}_{\mathbb{P}^n}(*)|_{U_n})$ which is zero, as U_n is affine. This last part can be proven in various ways, as $\mathbb{A}^n_{\mathbb{C}}$ behaves as you would expect from the "trivial" space, in the sense that it has trivial cohomology for all quasicoherent sheaves (see for example [Har77, Theorem 3.5, p. 215]).

2.8 Intersection product

Another important tool that we need to introduce before we can properly begin to work with K3 surfaces is the intersection product. Normally we would just give the definition immediately and later explain all its properties, but in this case this will be complicated.

There are quite a few ways to define the intersection product, but the main ones are:

- (Topological) The **intersection product** of two divisors can be obtained by considering the composed morphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ and then taking the toplogical cup product. We will better explore this definition in the next section, as we have not defined the morphism $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ yet.
- (Cohomological) This is perhaps the most general and direct, though it is less clear what are we counting than in the other definitions. We define

  $$(L.L') := \chi(X, \mathcal{O}_X) - \chi(X, L^\vee) - \chi(X, L'^\vee) + \chi(X, L^\vee \otimes L'^\vee)$$

  (remember that $\chi(\mathcal{F}) := \sum_i (-1)^i h^i(\mathcal{F})$).
- (Geometrical) This one relies on the fact that we want this form to be linear, and as such we can define it purely for irreducible subvarieties C and D. As such, given any

point $p \in C \cap D$, we may consider $(C.D)_p := \text{length}(\mathcal{O}_{X,p}/(f,g))$ where f, g are the local representatives of C and D respectively.

More interestingly, there is also an axiomatic characterization. It states:

Theorem 2.47. *There is a unique pairing* $\text{Div}(X) \times \text{Div}(X) \to \mathbb{Z}$ *such that:*

- *If C and D are two non-singular curves meeting transversally, then $C.D = |C \cap D|$.*
- *The product is symmetric: $C.D = D.C$.*
- *It is linear: $(C_1 + C_2).D = C_1.D + C_2.D$.*
- *It annihilates if either of the divisors is principal, that is to say it extend to a pairing* $\text{Pic}(X) \times \text{Pic}(X) \to \mathbb{Z}$

The proof of this theorem can be found in [Har77, p. 357–360].

It is this last one that we will be using most often. From now on we will adopt a slight abuse of notation, so given a line bundle L and a divisor D we may talk about $L.D := L.\mathcal{O}(D)$. First of all, we will be able to make a few important definitions.

Definition 2.48. A line bundle is **numerically effective** (or nef) if for every smooth curve $C \subseteq X$, $L.C \geq 0$.

The notion of numerical effectiveness is actually directly weaker than that of ampleness, even if it may not be apparent at first.

Proposition 2.49. *Given an ample line bundle L and an effective divisor D, $L.D > 0$. In particular L is nef.*

Proof. By definition of ampleness we have that there exists an m such that mL is very ample, and if we borrow [Har77, Theorem 7.2, p. 48-49] we can see that for any curve C on $\mathbb{P}^n$ and hyperplane H they have non empty intersection. As such, since C has a finite number of irreducible components, there must be some H which doesn't contain any of them. Therefore, we have $m(L.C) = (mL).C = H.C > 0$ which trivially implies $L.C > 0$ as desired. □

2.9 Chern Classes

Another necessary tool for our discussion is that of Chern classes. They act as powerful and "well behaved" invariants attached to vector bundles. Moreover, they are important in some computations, mainly through the generalized Riemann-Roch theorem that will appear latter on. Though our use in this report will be somewhat tangential, there is quite a bit of theory about Chern classes and other invariants that can help to clear some of the concepts we will be using. Also note that, throughout this section, we will refer to topological/smooth vector bundles instead of holomorphic ones as the Chern classes are a topological invariant.

To begin with, we will look at the simplest case, the first Chern class of a line bundle, which merits special attention. For this short section we will be considering X as an analytical complex manifold. For this, we will make use of the exponential exact sequence:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

As we haven't really given any examples on how one proves that a sequence of sheaves is exact, we will take this simple example as a chance to illustrate it:

Proposition 2.50. *The sequence* $0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$ *is exact as* $\mathcal{O}_X$*-modules in the analytic topology.*

Proof. As previously mentioned, exactness can be checked on stalks. Injectivity of the first morphism and exactness in the middle is easy to see by basic complex analysis, and it holds on every open set. Much more interesting is showing exhaustivity of the exponential, as it is exclusive to stalks. For every germ (U, f) over a point p there is a representative open set V such that $f(V)$ doesn't intersect all rays from the origin, as $f(p) \neq 0$, and as such there is a branch of the logarithm such that $\forall x \in V,\ f(x) = \exp \ln(f(x))$. $\square$

We may now take a look at its long cohomology sequence, which is what we are interested in. In particular:

$$\dots H^0(X, \mathcal{O}_X^*) \to H^1(X, \underline{\mathbb{Z}}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \underline{\mathbb{Z}}) \to \dots$$

Now in general we cannot say much about this, other than that we define c_1 to be the first chern class of a line bundle in $H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$ (recall Proposition 2.44). The surjectivity of $\dots H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{Z})$ depends on the particular structure of X, though in the cases that we will be interested in, it will always be the case. Finally, the injectivity of the first Chern class hinges on the triviality of $H^1(X, \mathcal{O}_X)$.

We are also interested in the second Chern class, which since we will be dealing with surfaces it is the highest non trivial one. Its codomain is $H^4(X, \mathbb{Z}) \cong Z$ (Note: depending on the definition, Chern classes may be defined in other groups through which it factors). We will define Chern classes in a more general setting though, as it makes certain explanations a lot more natural.

To do this we will require the assistance of another construction. Just as we define the projectivization of any vector space, we can do the same for any vector bundle.

Definition 2.51. Given a vector bundle E, we can define its projectivization as the fiber bundle which on stalks it is equal to $\mathbf{P}(E)_p := \mathbf{P}(E_p)\ \forall p \in X$ with topology given by the projection from E.

Now, just as with the definition of projective space earlier, the projectivization of a vector bundle has some vector bundles of its own.

Definition 2.52. The **universal subbundle** over $\mathbf{P}(E)$ for E a vector bundle is defined as $S = \{(l, v) \in E \times_X \mathbf{P}(E) | v \in l\}$ with the obvious projection onto $\mathbf{P}(E)$.

This gives us an exact sequence of vector bundles over $\mathbf{P}(E)$:

$$0 \to S \to E \times_X \mathbf{P}(E) \to Q \to 0,$$

where Q is by definition the **quotient bundle**. What is interesting is that, as we have explained before, we have the graded cohomology ring $H^*(\cdot, \underline{\mathbb{Z}})$. In particular, we can see that by contravariance on cohomology $H^*(\mathbf{P}(E), \underline{\mathbb{Z}})$ is an $H^*(X, \underline{\mathbb{Z}})$-algebra, which is generated by $z = c_1(S^\vee)$. This can be shown by various ways, one of which can be found in [BT82, p. 270].

Thus, we obtain an exhaustive morphism of graded $H^*(X, \underline{\mathbb{Z}})$-algebras,

$$H^*(X, \underline{\mathbb{Z}})[Z] \to H^*(\mathbf{P}(E), \underline{\mathbb{Z}})$$

which sends $Z \to z$. Note that as $H^*(\mathbf{P}(E), \underline{\mathbb{Z}})$ vanishes from a certain point onwards, the map is not injective. As such we have that

$$H^*(\mathbf{P}(E), \underline{\mathbb{Z}}) \cong H^*(X, \underline{\mathbb{Z}})[Z]/Z^n + c_1(E)Z^{n-1} + \cdots + c_n(E).$$

This is how we define our Chern classes. Note that since all the monomials must be of the same degree and $z \in H^2(\mathbf{P}(E), \underline{\mathbb{Z}})$, we have $c_i(E) \in H^{2i}(X, \underline{\mathbb{Z}})$.

The first thing we wish to check is that indeed both of our definitions of c_1 coincide on line bundles. In this case $\mathbf{P}(L) = X$ and as such $S = L$, implying that the kernel of the previously defined morphism must be $z + c_1(L)$ where z is the first Chern class of the dual in the previous sense, and thus we get the desired equality.

We define the total Chern class as $c(E) := 1 + c_1(E) + \cdots + c_n(E)$. Note that if we keep track of the grading we can obtain any of the Chern classes from $c(E)$. An important result is the following:

Lemma 2.53 (Splitting Lemma)**.** *Given a vector bundle $E \to X$, there is a continuous map $f : Y \to X$ from some space Y such that:*

- *The maps $f^* : H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$ are injective.*
- *The vector bundle f^*E is a direct sum of line bundles.*

Proof. The space Y is simply $\mathbb{P}(E)$ with its usual projection, and all the properties can be checked directly, see for reference [Voi02, Lemma 11.24, p 277]. □

It is very important to keep in mind that this bundles are being treated as differentiable objects. There is no reason to believe that $\mathbb{P}(E) \to X$ is holomorphic as we have defined it and the statement is certainly not true in the holomorphic case (though there is an analogous statement).

We can now use this result to proof the following property, by assuming that E is a direct sum of line bundles.

Proposition 2.54 (Whitney sum formula)**.** *The total Chern class is additive in the sense that if* $0 \to E' \to E \to E'' \to 0$ *is an exact sequence of vector bundles, then* $c(E) = c(E')c(E'')$*.*

Proof. By the splitting Lemma we may assume that $E = \bigoplus_{i=1}^n L_i$. We will need to show then that $c(E) = \prod_{i=1}^n (1 + c_1(L_i))$. This we do directly from the definition, by observing that in the spaces we are interested in $H^*(\mathbf{P}(E), \underline{\mathbb{Z}}) \cong \bigoplus_{i=1}^n H^*(\mathbf{P}(L_i), \underline{\mathbb{Z}})$. It then becomes clear that the kernel is simply the intersection of the kernels of the projection which is given by $\prod_{i=1}^n (Z + c_1(L_i))$ as we wanted. □

Another useful result uses our original description of the first Chern class:

Proposition 2.55. *Given a vector bundle* E*, we have* $c_1(E) = c_1(\det(E))$*.*

Proof. Note that by our initial description of the first chern class, if L_1 and L_2 are both line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. Now we can once more assume that $E = \bigoplus_{i=1}^n L_i$, and by the Whitney sum formula we have $c_1(E) = \sum_{i=1}^n c_1(L_i) = c_1(\bigotimes_{i=1}^n L_i) = c_1(\det(E))$ as intended. □

This will become important in future computations. It also shows partly why Chern classes are so useful. In fact, Chern classes can be characterized axiomatically by fixing trivial cases, imposing some naturality conditions and the Whitney sum formula. We have only defined Chern classes for vector bundles, but this may be extended to any coherent sheaf. Indeed, consider a locally free resolution:

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to \mathcal{F} \to 0$$

We can simply impose that the Whitney sum formula applies, meaning that $c(L) := \prod_{i=1}^n c(E_i)^{-1^i}$. Note that all total chern classes are invertible in the cohomological ring, as $c_0 = 1$ is invertible. It is a known fact that on a smooth variety

3 K3 surfaces

And so we finally arrive at the main definition of this report: K3 surfaces. Just as with everything we've done so far, it has both an analytic and algebraic definition, which in the projective case are equivalent through GAGA.

Definition 3.1. A **complex K3 surface** is a compact connected complex manifold of dimension 2 such that its canonical bundle is trivial ($\omega_X \cong \mathcal{O}_X$) and $H^1(X, \mathcal{O}_X) = 0$. Henceforth we will be restricting further by considering it to be connected and compact.

These are the spaces we will be interested in, though we need to add some extra structure. Ideally, we might want to classify projective K3 surfaces directly as subvarieties of projecive space (that is to say, together withh a very ample line bundle). Instead we provide a better behaved definition of a polarization:

Definition 3.2. A **polarization** is a primitive and ample line bundle L over our K3 surface. We call $L^2 = L.L = 2g - 2$ its **degree** with g being called its **genus**.

The main focus of this report is the classification of K3 surfaces of low genus (degree).

An important thing to notice is that we will consider the classification of polarized K3 surfaces, but a K3 surface can admit different polarizations of different degree. Indeed, classifying K3 surfaces (without fixing the polarization) can be obtained via the Torelli theorem, which characterizes K3 surfaces through their Hodge structure, although the classification space is less well behaved.

3.1 Basic properties

First of all, one of the motivations for studying K3 surfaces that we have previously mentioned is that they have Kodaira dimensión 0, so we give the definition:

Definition 3.3. The **Kodaira dimension** of a projective algebraic variety can be formally defined as:

$$\kappa(X) = \max_m \{\dim \varphi_{\omega_X^{\otimes m}}(X)\}$$

Birational geometry is a branch of algebraic geometry that studies varieties up to birational equivalence, that is, such that their function fields are isomorphic. The Kodaira dimension is the most basic birational invariant of a variety.

Of course directly from the definition we can see that K3 surfaces indeed do have Kodaira dimension 0 as the canonical bundle is trivial.

Because of the triviality of the canonical bundle, Serre Duality gives us an isomorphism

$$H^i(X, \mathcal{O}_X) \cong H^{n-i}(X, \omega_X \otimes \mathcal{O}_X^\vee)^\vee \cong H^{n-i}(X, \mathcal{O}_X)^\vee.$$

Thus besides the fact that for any K3 surface $H^1(X, \mathcal{O}_X) = 0$, since holomorphic functions from a compact connected complex manifold to $\mathbb{C}$ are constant, we get that

$$H^2(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X) \cong \mathbb{C}.$$

We will now announce the Riemann-Roch for line bundles on surfaces. A more general result is referenced later on, but this one will suffice for now.

Theorem 3.4 (E.g. [Bea96, Theorem I.12]). *Given a line bundle L on a non-singular projective surface L we have the following identity:*

$$\chi(X, L) = \chi(X, \mathcal{O}_X) + \frac{L.(L \otimes \omega_X^\vee)}{2}$$

Because ω_X is trivial for K3 surfaces, this just means that $\chi(X, L)$ depends only on L^2, which is useful for computations. A very important fact about K3 surfaces that can be seen as a application of this theorem is the following:

Proposition 3.5. *The Picard group of a K3 surface X is torsion free.*

Proof. We will use proof by contradiction. Assume that we have some line bundle L such that $L^n \cong \mathcal{O}_X$. Then by the Riemann-Roch, we have that $\chi(X, L) = \chi(X, \mathcal{O}_X) + \frac{L^2}{2}$. Now because L^n is trivial $L^2 = 0$, and by definition of K3 surface and Serre Duality we have $h^0(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$, so $\chi(X\mathcal{O}_X) = 2$ and as such $\chi(X, L) = 2$.

But by definition and Serre Duality we have that $\chi(X, L) = h^0(X, L) + h^0(X, L^\vee) - h^1(X, L)$ and so one of L and $L^\vee$ has a non trivial global section, so we can assume that of L. But then it is clear that if $0 \neq s \in H^0(X, L)$, then $s^n \in H^0(X, L^n) = H^0(X, \mathcal{O}_X)$ is non-zero, and thus nowhere vanishing as it is on $\mathcal{O}_X$. But then s must also be nowhere vanishing. Now we may use the following lemma, showing the L is indeed trivial. $\square$

Lemma 3.6. *A line bundle L is trivial if, and only if, it has a nowhere vanishing global section $s \in L(X)$. That is to say: for every $p \in X$, $s(p) \neq 0$.*

Proof. For one direction consider the global section $1 \in \mathcal{O}_X(X)$. For the other, the global section we just defined gives us a morphism $f : X \times \mathbb{C} \to L$ defined by $(p, z) \to s(p) \cdot z$, and this is both trivially an isomorphism ($s(p) \neq 0 \implies \pi^{-1}(p) = s(p)\mathbb{C}$) and if $pr_1 : X \times \mathbb{C} \to X$ is the first projection then $\pi \circ f = pr_1$ by virtue of s being a global section. $\square$

This allows us to use elimination, that is to say $L^n = M^n \implies L = M$, which we will use implicitly in later proofs. We may add that not only does either L or $L^\vee$ have global sections but also we have that of A is an ample line bundle then $A.L > 0$ implies $h^0(X, L) > 0$, as necessarily one of L or $L^\vee$ is effective, and if $A.L > 0$, then $A.L^\vee < 0$ implies $L^\vee$ is not effective. Also note that $A.L = 0$ implies that $A.L^\vee = 0$ and as one of them is effective, it must be 0.

Another important property of the Picard group for K3 surfaces is that the intersection product is even, that is, for any element $L \in \mathrm{Pic}(X)$, L^2 is even. This is implicit in applying the Riemann-Roch to K3 surfaces, as it can be rewritten as $L^2 = 2(\chi(X, L) - 2)$.

Lastly, we mention this theorem that will let us compute $h^0(X, L)$ directly as $\chi(X, L)$ for L an ample line bundle.

Theorem 3.7 (Kodaira vanishing). *Given a complex manifold X and an ample line bundle L, $\omega_X \otimes L$ is acyclic.*

Although we will use it in very particular cases, the proof of this result is beyond the scope of this report. In any case, the proof can be found in [Voi02, Theorem 7.12].

3.2 Examples

One of the simplest examples of polarized K3 surfaces is that of a smooth quartic hypersurface of $\mathbb{P}^3$ with Serre's twisted sheaf $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^3}(1)_{|X}$. In fact, we will soon easily generalize these arguments to complete intersections of degree $(d_1, \dots, d_n)$ such that $\sum_i d_i = n+3$ in the appropriate projective space.

The relevance of the projective space and, more generally, of the complex grassmannians cannot be understated. Indeed, most polarized K3 surfaces of low degree can be embedded into one of these spaces. In fact we will soon show that for low enough degree the function ϕ_L induced by our polarization is better behaved than one might expect at first, being a Zariski closed embedding. We will proof the "closed embedding" part, as being Zariski comes naturally as a corollary of the GAGA theorem.

Indeed, let us begin by showing that complete intersections of degree $(d_1, ..., d_n)$ such that $\sum_i d_i = n+3$ are K3 surfaces. First of all, note that if any of the d_i are of degree 1, then one can consider the others as forming a $n-1$ complete intersection that accomplishes the condition, so we may assume without lack of generality that $d_i \geq 2$. Hence, $n+3 = \sum_i d_i \geq 2n$ implies $n \leq 3$, giving us only three cases: a quartic in $\mathbb{P}^3$, the complete intersection of a cubic and a quadric and finally the complete intersection of three quadrics.

To show this we may take into account our previous calculation of the cohomology of line bundles on $\mathbb{P}^n$ and we will use the following short exact sequence:

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^{n+2}} \to i_* \mathcal{O}_X \to 0$$

To do so, we will first need to compute the ideal sheaf $\mathcal{I}_X$ of any complete intersection of the form $(d_1, \dots, d_n)$. Indeed, the ideal sheaf of a degree d_1 hypersurface is simply $\mathcal{O}_{\mathbb{P}^{n+2}}(-d_1) = \mathcal{O}_{\mathbb{P}^{n+2}}(d_1)^\vee$, given that it can be described as the zero locus of a section of $\mathcal{O}_{\mathbb{P}^{n+2}}(d_1)$. More generally, given any vector bundle E of rank r on a non-singular space X with dimension m (in this particular case $\mathbb{P}^{n+2}$) and a global section $s \in E(X)$, we may consider

$$Z = \ker(s) = \{x \in X | s(x) = 0\}.$$

This definition gives us a sequence, known as the **Koszul complex**:

$$0 \to \Lambda^r E^\vee \to \cdots \to \Lambda E^\vee \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{1}$$

This sequence will be exact if $\operatorname{codim}_X Z = r$, which we may see as a generalization of being a complete intersection. One way to see this is to consider the vector bundle E as a $m+r$-dimensional complex variety. In this case a global section is just an immersion giving an m-dimensional subvariety, so we may consider intersecting two of these subvarieties given by sections. Because of their vector space structure we can assume that one of them is the zero section. Then Z may be identified with the intersection of these two subvarieties, giving us in the maximal case a subvariety with codimension $2r$ in E and r in X. As such, we may think of Z as a self-intersection of X in a suitable ambient space.

In our particular case, we may consider $E = \sum_{i=1}^{n} \mathcal{O}_{\mathbb{P}^{n+2}}(d_i)$. By the fact that $\Lambda^m(M \oplus N) = \bigoplus_{i+j=m} \Lambda^i M \otimes \Lambda^j N$ we have that $\Lambda^m E$ is a sum of line bundles, which by our calculations on cohomology over $\mathbb{P}^{n+2}$ we know that they have trivial cohomology groups for indexes $0 < i < n+2$.

Therefore, by partitioning the long exact sequence into short ones,

$$0 \to I_j \to \Lambda^{n-j} E \to I_{j+1} \to 0,$$

with $I_1 = \Lambda^n E$ and $I_{n+1} = \mathcal{O}_Z$ (we adopt the convention that $\Lambda^0 E^\vee := \mathcal{O}_{\mathbb{P}^{n+2}}$), we may prove that the I_j within them have trivial cohomology groups for $0 < i < n+3-j$ and in particular for $I_{n+1} = \mathcal{O}_Z$ we have that $H^1(\mathcal{O}_Z) = 0$. Indeed, we have

$$H^i(\Lambda^{n-j} E^\vee) \to H^i(I_{j+1}) \to H^{i+1}(I_j)$$

and by induction on j, $H^i(\Lambda^{n-j} E^\vee) = H^{i+1}(I_j) = 0$ for all $0 < i < n+3-j$ as required.

To check that the canonical bundle is trivial, we first take the sequence (1), split it at the ideal sheaf, and tensor it with $\mathcal{O}_Z$, obtaining $\Lambda^2 E^\vee \otimes \mathcal{O}_Z \to E^\vee \otimes \mathcal{O}_Z \to I_{Z/\mathbb{P}^{n+2}} \otimes \mathcal{O}_Z \to 0$. Since by definition Z is annihilated at s, and the morphism $\Lambda^2 E^\vee \to E^\vee$ is obtained by contraction by s, it becomes 0. Hence, $E^\vee \otimes \mathcal{O}_Z \cong I_{Z/\mathbb{P}^{n+2}} \otimes \mathcal{O}_Z$ obtaining thus by the adjunction formula:

$$\omega_X \cong \omega_{\mathbb{P}^{n+2}} \otimes \det(\mathcal{I} \otimes \mathcal{O}_Z)^\vee \cong \mathcal{O}_{\mathbb{P}^{n+2}}(n+3) \otimes \mathcal{O}_{\mathbb{P}^{n+2}}(-n-3) \cong \mathcal{O}_{\mathbb{P}^{n+2}}$$

This proof may also be performed directly by considering all three cases separately, the first two being direct and the third one requiring some extra steps. I have chosen to give this proof to illustrate the relation between K3 surfaces and global sections of vector bundles on projective space.

Note we have only proved that they are K3 surfaces. As will be the norm, we give them a polarization L by restricting the Serre's twisting sheaf, i.e. $L \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)$, since the restriction of an ample line bunlde is ample. Recall that our classification deals with polarized K3 surfaces, not only classifies K3 surfaces up to biholomorphism. Indeed some quartics of $\mathbb{P}^3$ could very well admit other polarizations not being the restriction of the Serre's twisting sheaf.

4 Constructions of K3 surfaces

We will divide our classification according to the degree of the polarization. To begin with our classification, we are interested in the case $L^2 = 2$, the one with minimal self intersection. We will work with it separately as it works differently from all the other cases.

Then we will continue increasing the degree. However, perhaps it is important to mention that the Kodaira dimension of, roughly speaking, the space that parametrizes all polarized

K3 surfaces of degree d (properly speaking we should introduce moduli spaces) is positive (in fact, maximal) for d sufficiently big (at least for d greater or equal than 62). Hence, birational geometry tells us that we cannot expect to construct the general K3 surface of degree ≥ 62 as we will do with those of low degree.

4.1 The case $L^2 = 2$

Note that in this case we have $\operatorname{coDom}\phi_L = \mathbb{P}^2$ which is itself a surface. Thus unlike the other cases where we expect polarized K3 surfaces to be subobjects of $\mathbb{P}^n$, here we will have that they form covers (in fact double covers) over $\mathbb{P}^2$, ramified over a sextic curve. First of all, let us make some definitions.

Definition 4.1. A map between complex spaces $f : X \to Y$ is a **finite map** if at any point $p \in Y$ the fiber $f^{-1}(p)$ is finite.

Definition 4.2. We call **covering** to a surjective map between connected projective spaces which is finite and proper (the preimage of a compact subset is compact).

Note that our definition of finite maps is only true because we are working on projective spaces, and all maps we construct are "projective morphisms". We won't be delving into this part of the theory, as we won't need it.

Now, given a covering $(X, \pi : X \to \mathbf{P}^2)$, we may consider the rank 2 vector bundle $\pi_*\mathcal{O}_X$. It has the usual map $\mathcal{O}_{\mathbf{P}^2} \to \pi_*\mathcal{O}_X$, but also the trace map $\operatorname{Tr} : \pi_*\mathcal{O}_X \to \mathcal{O}_{\mathbf{P}^2}$, which when composed with the previous morphism gives us an automorphism of $\mathcal{O}_{\mathbf{P}^2}$, which is easily checked to be multiplication by 2, the degree. Hence the morphism $\frac{1}{2}\operatorname{Tr}$ gives us a splitting of $\pi_*\mathcal{O}_X$ into $\mathcal{O}_{\mathbf{P}^2} \oplus A$, where A is the quotient sheaf.

On the other hand, we have the induced morphism $\pi^*\omega_{\mathbf{P}^2} \to \omega_X$, which gives us a global section of $\omega_X \otimes (\pi^*\omega_{\mathbf{P}^2})^\vee$, whose zero divisor will consist of the points known as the **ramification** points. The vanishing locus of this section is a divisor R called the **ramification divisor**. The projection of R into $\mathbf{P}^2$ through the projection π is called the **branching divisor** B.

These two formulas together imply that $\omega_X \cong \pi^*\omega_{\mathbf{P}^2} \otimes \mathcal{O}_X(R) \cong \pi^*\mathcal{O}_{\mathbf{P}^2}(-3) \otimes \mathcal{O}_X(R)$.

Then, since $\mathcal{O}_X(2R) = \pi^*\mathcal{O}_B$, for X to be a K3 surface we must have $\mathcal{O}_{\mathbf{P}^2}(B) \cong \mathcal{O}_{\mathbf{P}^2}(6)$, meaning that B is a smooth sextic (smoothness is required for X to be smooth).

Now, we want to proof that given a basepoint-free polarization L of X of degree 2, ϕ_L gives a $2:1$ covering of $\mathbf{P}^2$. This comes almost directly by considering that for any hyperplane $H \subset \mathbf{P}^2$, $\pi^*(H) \equiv L$ implying that $2 = \pi^*(H)^2 = 2H^2$. In particular this means that over a generic point two hypersurfaces intersect twice over X, proving that it is a double cover (we obtain properness trivially as we only consider compact surfaces).

4.2 Reider's Method

For the sake of our classification, we first need a way to know when L is very ample. Unfortunately the definition given above is not particularly helpful. We would like to have something similar to a numeric condition on being very ample that we can actually compute. This will be obtained through Reider's method.

To begin, we want to consider a 0-cycle $Z \subseteq X$ such that it is a section of a rank 2 vector bundle with determinant L. Note that given such a vector bundle, we can construct an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\cdot s} E \to I_Z \otimes L \to 0.$$

The aim of the first part of Reider's method will be to go in the opposite direction, to construct from Z and L an appropriate $E(Z, L)$.

More precisely, we want an exact sequence $0 \to \mathcal{O}_X \to E(Z, L) \to I_Z \otimes L \to 0$ which by definition gives us an element of $\mathrm{Ext}^1(I_Z \otimes L, \mathcal{O}_X)$. Now, we know that

$$\dim(\mathrm{Ext}^1(I_Z \otimes L, \mathcal{O}_X)) = \dim(\mathrm{Ext}^1(I_Z \otimes L, \mathcal{O}_X)^\vee)$$

and that

$$\mathrm{Ext}^1(I_Z \otimes L, \mathcal{O}_X)^\vee \cong \mathrm{Ext}^1(\mathcal{O}_X, I_Z \otimes L \otimes \omega_X) \cong H^1(I_Z \otimes L \otimes \omega_X)$$

(recall that in the case of K3 surfaces $\omega_X \cong \mathcal{O}_X$, we just write it to show how the argument works in a more general setting). The first isomorphism is given by Serre Duality and the second by the natural equivalence $H^0(X, \cdot) \cong \mathrm{Hom}_{Sh(X)}(\mathcal{O}_X, \cdot)$ for $\mathcal{O}_X$-modules, since a global section s uniquely determines a morphism from $\mathcal{O}_X$ sending the unit to s and determining the morphism on opens by restriction. Note that we take Ext in the category of coherent sheaves, so that we may assume that $E(Z, L)$ is coherent.

Now, this implies that $H^0(I_Z \otimes L) > 0$ is a necessary condition, and we are thus left wondering what a sufficient condition might look like. This condition turns out to be essentially that $E(Z, L)$ is "unique", in the following sense.

Theorem 4.3. *Given a line bundle L and a 0-cycle Z such that $h^1(I_Z \otimes L \otimes \omega_X) = 1$ and for any $Z' \subset Z$ we have $h^1(I_{Z'} \otimes L \otimes \omega_X) = 0$, there exists a unique exact sequence $0 \to \mathcal{O}_X \to E(Z, L) \to I_Z \otimes L \to 0$ such that $E(Z, L)$ is a rank 2 vector bundle.*

To prove this theorem it will be important to find some sufficient condition for being locally free we can work with. We will accomplish this in two steps.

For the first one, we will show that it is enough to show that E is projective on stalks. Note that because for all $p \in X$ we have that $\mathcal{O}_{X,p}$ is a local ring, projective is equivalent to free of finite rank. So what we really need is:

Proposition 4.4. *An $\mathcal{O}_X$-module $\mathcal{F}$ is locally free if and only if it is free on stalks and of finite presentation.*

Proof. The left to right implication is trivial, so we proceed with the other direction. We wish to obtain for each $p \in X$ an open set $p \in U \subseteq X$ together with an isomorphism $\phi : \mathcal{O}_X^n|_U \to \mathcal{F}|_U$. Note that $(\phi, U) \in \mathcal{H}om(\mathcal{O}_X^n, \mathcal{F})_p$. So what we require is that $\mathcal{H}om$ commutes with stalks.

This is not true in general, but it is enough that the first module is of finite presentation.

Lemma 4.5. *If $\mathcal{G}$ are both $\mathcal{O}_X$-modules, with $\mathcal{G}$ of finite presentation, we have $\mathcal{H}om(\mathcal{G}, \mathcal{H})_p \cong \mathrm{Hom}(\mathcal{G}_p, \mathcal{H}_p)$.*

Proof. Indeed, if we consider the exact sequence $\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{G} \to 0$, we may apply the contravariant functor $\mathcal{H}om(\cdot, \mathcal{H})$, which is left exact, and obtain the exact sequence:

$$0 \to \mathcal{H}om(\mathcal{G}, \mathcal{H}) \to \mathcal{H}^{\oplus n} \to \mathcal{H}^{\oplus m}$$

If we now take the stalk at p we obtain:

$$0 \to \mathcal{H}om(\mathcal{G}, \mathcal{H})_p \to \mathcal{H}_p^{\oplus n} \to \mathcal{H}_p^{\oplus m}$$

However, we may also take the stalk at p of the first sequence and apply the contravariant left exact functor $\mathrm{Hom}(\cdot, \mathcal{H})$ instead, obtaining:

$$0 \to \mathrm{Hom}(\mathcal{G}_p, \mathcal{H}_p) \to \mathcal{H}_p^{\oplus n} \to \mathcal{H}_p^{\oplus m}$$

Now as there is a natural morphism $\mathcal{H}om(\mathcal{G}, \mathcal{H})_p \to \mathrm{Hom}(\mathcal{G}_p, \mathcal{H}_p)$, by the five lemma and some easy comprobations we see it forms an isomorphism. □

Now because both $\mathcal{O}_X^n$ and $\mathcal{F}$ are of finite presentation, and the isomorphism previously mentioned is natural on both $\mathcal{G}$ and $\mathcal{H}$, we have that the isomorphism $\mathcal{H}om(\mathcal{O}_X^n, \mathcal{F})_p \times \mathcal{H}om(\mathcal{F}, \mathcal{O}_X^n)_p \cong \mathrm{Hom}(\mathcal{O}_{X_p}^n, \mathcal{F}_p) \times \mathrm{Hom}(\mathcal{F}_p, \mathcal{O}_{X,p}^n)$ preserves composition, and in particular our isomorphism preserves isomorphisms.

From this, because of freedom on stalks we have an isomorphism in $\mathrm{Hom}(\mathcal{O}_{X,p}^n, \mathcal{F}_p)$ and this yields an element $(\phi, U) \in \mathcal{H}om(\mathcal{O}_X^n, \mathcal{F})_p$ together with an element $(\phi^{-1}, V) \in \mathcal{H}om(\mathcal{F}, \mathcal{O}_X^n)_p$, such that $\phi|_{U \cap V}$ is an isomorphism as we wanted. □

For the second step, we are going to show that E being projective on stalks is implied by being reflexive, that is to say that $E \cong E^{\vee\vee}$ through the usual morphism. This is only possible because we are working on complex surfaces, in higher dimensional spaces there are analogous results for vector bundles of appropriate rank with additional conditions.

To show projectivity, we can use the Auslander–Buchsbaum formula which states the following:

Theorem 4.6 (Auslander–Buchsbaum formula). *Given a commutative Noetherian local ring R, if M is an R-module of finite projective dimension, then:*

$$\mathrm{pd}_R(M) + \mathrm{depth}(M) = \mathrm{depth}(R)$$

First of all it is important to note that on a regular ring (which $\mathcal{O}_{X,p}$ is as we are on a smooth variety) all finitely generated modules have a finite free (and thus projective) resolution. This is what allows us to apply the Auslander-Buchsbaum formula.

Now, since $\mathcal{O}_{X,p}$ is regular and, in particular, it is Cohen-Macaulay (which is to say, its depth is equal to its dimension), and as we are in a surface this implies that $\operatorname{depth}(R) = 2$. It is at this point that we may use reflexivity of E_p to see that $\operatorname{depth}(M)$ has the S_2 property, and thus in particular $\operatorname{depth}(M) \geq 2$. This implies that $\operatorname{pd}_{\mathcal{O}_{X,p}}(E_p) = 0$, which means that E_p is projective (it has a 0-length projective resolution).

We skimmed over this part of the proof as it requires more algebra than we've been using. For a more complete account see [Sch10].

We are now ready to give the proof of Theorem 4.3.

Proof. First of all note that if we show that E is locally free. Then by taking a point $p \notin Z$ and taking stalks on the exact sequence, we obtain by definition that $(I_Z \otimes L)_p \cong I_{Z,p} \otimes L_p \cong \mathcal{O}_{X,p} \otimes \mathcal{O}_{X,p} \cong \mathcal{O}_{X,p}$, and as rank is additive we obtain that E has rank 2.

In the case that the sheaf is reflexive, the previous discussion gives us the desired result. Hence we are interested in ruling out the case where $E(Z, L)$ is not reflexive.

Since $E^{\vee\vee}$ is easily shown to be reflexive, the previous discussion shows that it is a vector bundle. Now, given that a morphism $\mathcal{O}_X \to E$ is, as we have explained, simply determined by the choice of a global section $s \in E(X)$, we may use the canonical morphism $E \to E^{\vee\vee}$ to construct an associated section on $E^{\vee\vee}$. We call the zero locus of the given section Z' and thanks to the construction that we mentioned beforehand and the obvious morphisms we obtain a diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & I_Z \otimes L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & E^{\vee\vee} \arrow[r] \arrow[d] & I_{Z'} \otimes L \arrow[r] \arrow[d] & 0 \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
 & & 0 & 0, &
\end{tikzcd}$$

which is exact in all rows and the left and central columns, giving us the third column by the nine lemma. Then, the injectivity of $L \otimes I_Z \to L \otimes I_{Z'}$ implies that $Z' \subseteq Z$, but then the existence of $E^{\vee\vee}$ contradicts our special condition, as $h^1(I_{Z'} \otimes L) \geq 1$. ☐

The construction of this intermediate vector bundle will be crucial to our argument. Essentially we will be using it as a pivot point. Just as how we have defined it as an extension of $\mathcal{O}_X$ and L, we can now consider exact sequences of vector bundles in which it fits. Note that this is equivalent to choosing a line subbundle and a torsion-free quotient.

Now let us assume that we have a non trivial (not a multiple of the identity) endomorphism $\phi : E \to E$. Then it is true that:

Proposition 4.7. *Given a rank 2 vector bundle E with a non trivial endomorphism ϕ, we can find line bundles M,N and a 0-cycle A such that we have $0 \to M \to E \to I_A \otimes N \to 0$ and such that $c_1(E).(M \otimes N^\vee) \geq 0$ (this is known as E having the* ***weak Bogomolov property****).*

Proof. Choosing any $p \in X$ and any eigenvalue λ of ϕ_p, we can consider $\phi' := \phi - \lambda \operatorname{Id}$ which has non trivial kernel. We may as such assume that ϕ has non trivial kernel.

Hence, we have $0 \to \ker(\phi) \to E \to \operatorname{im}(\phi) \to 0$. Now because ϕ is an endomorphism, it follows that both $\ker(\phi)$ and $\operatorname{im}(\phi)$ are subsheaves of E, and in particular they are torsion-free. This implies that $\ker(\phi)$ is reflexive, and thus a vector bundle (by virtue of being coherent). As $\operatorname{im}(\phi)$, which is the quotient bundle of the inclusion $\ker(\phi) \hookrightarrow E$, is torsion-free we see that $\ker(\phi)$ is a line bundle.

We may now split the argument into two possible cases:

- $\ker(\phi) = \ker(\phi^2)$. This means that the morphism $\phi|_{\operatorname{im}(\phi)}$ is injective, so the sequence splits, implying that $\operatorname{im}(\phi)$ is also a line bundle. Since the sequence splits we can consider the exact sequence in both directions, and we choose whichever gives us the Weak Bogomolov property.
- $\ker(\phi) \subset \ker(\phi^2)$ is a proper subset. In such a case $ker(\phi^2) = E$, which implies $\operatorname{im}(\phi) \subseteq \ker(\phi)$. This gives us a morphism $\mathcal{O}_X(c_1(\operatorname{im}(\phi))) \to \ker(\phi)$ or equivalently a global section of $\ker(\phi) \otimes \mathcal{O}_X(c_1(\operatorname{im}(\phi)))^\vee$. Note that because of ampleness of L, $L.(\ker(\phi) \otimes \mathcal{O}_X(c_1(\operatorname{im}(\phi)))^\vee) > 0$. Note that we are using that there is an isomorphism $\operatorname{im}(\phi) \cong \mathcal{O}_X(c_1(\operatorname{im}(\phi))) \otimes I_A$ for some 0-cycle A. $\square$

Now, the Euler characteristic of a coherent sheaf $\mathcal{F}$ becomes particularly important in the case of K3 surfaces. Indeed, by Serre Duality we have that $\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) + h^0(\mathcal{F}^\vee)$, so $\chi(\mathcal{F}) > 0$ (resp. < 0) implies $h^0(\mathcal{F}) > 0$ or $h^0(\mathcal{F}^\vee) > 0$ (resp. $h^1(\mathcal{F}) > 0$), among other things.

Therefore, a method to compute $\chi(\mathcal{F})$ without relying on knowing all the cohomology would be extremely useful. Luckily, a generalization of the classical Riemann-Roch for coherent sheaves on K3 surfaces allows us to do just that.

Theorem 4.8 (Hirzebruch-Riemann-Roch)**.** *If X is a K3 surface and $\mathcal{F}$ is a coherent sheaf, then $\chi(\mathcal{F}) = \frac{c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})}{2} + 2\operatorname{rk}(\mathcal{F})$.*

A proof can be found in [Gat02, Section 10.5], together with the computation of the Todd class of a K3 surface $\operatorname{td}(X) = (1, 0, 2)$.

Our first application will be finding a non trivial endomorphism of our rank 2 vector bundle. This needn't be true even with the conditions of Theorem 4.3, but the conditions of Reider's method are settled to be enough. We start laying the computational groundwork.

We can use that $\mathrm{Hom}(E, E) \cong H^0(E \otimes E^\vee)$, and as such we may use the Hirzebruch-Riemann-Roch Theorem to show that $\chi(E \otimes E^\vee) > 4$. As $E \otimes E^\vee \cong (E \otimes E^\vee)^\vee$, this implies $\dim_{\mathcal{O}_X(X)} \mathrm{Hom}(E, E) > 1$. Indeed, the rank of $E \otimes E^\vee$ is simply $2\,\mathrm{rk}(E)$, that by convention is denoted by $2(r+1)$, and we only need to find the values of $c_1(E \otimes E^\vee)$, $c_2(E \otimes E^\vee)$. For the first one we use that $c_1(V \otimes W) = c_1(V)^{\mathrm{rk}(W)} + c_1(W)^{\mathrm{rk}(V)}$. Hence $c_1(E \otimes E^\vee)$ is trivial, and so we only need compute the second Chern class.

For the second Chern class, since the first class vanishes, it coincides with the opposite of the second Chern character. Hence

$$\begin{aligned} c_2(E \otimes E^\vee) &= -(2\,\mathrm{ch}_2(E) \cdot (r+1) - c_1^2(E)) = -(c_1^2(E)(r+1) - 2c_2(E)(r+1) - c_1(E)^2) \\ &= -r(2g-2) + 2(r+1)d, \end{aligned}$$

where we have used that $\mathrm{rk}(E) = r+1$, $c_1(E)^2 = 2g-2$ and $c_2(E) = \deg Z = d$. Indeed, it is trivial to see that $c_1(E) = c_1(L)$ by Proposition 2.55 as $\det(E) = L$. More interesting is the case of $c_2(E)$. It is obtained by looking at our defining exact sequence $0 \to \mathcal{O}_X \to E \to I_Z \otimes L \to 0$ and using the additivity of Chern classes.

Therefore, putting it all together we obtain that

$$\chi(E \otimes E^\vee) = 2 - 2(g - (r+1)(r-d+g)).$$

We express it in these particular way because $\rho(g, r, d) := g - (r+1)(r-d+g)$ is usually known as the Brill-Noether number, and was already known and used previously, although we won't delve into this as well. What matters is that we finally have a computational formula for $\chi(E \otimes E^\vee)$.

We finally have all the pieces needed to put together Reider's method, as well as apply it to our case.

Theorem 4.9 (Reider's Method). *Fix an ample line bundle L and a 0-cycle Z of degree d such that $h^1(I_Z \otimes L \otimes \omega_X) = 1$ and for $Z' \subset Z$ we have $h^1(I_{Z'} \otimes L \otimes \omega_X) = 0$. If $E(Z, L)$ has the Weak Bogomolov Property, then there exists a divisor D such that:*

- $Z \subseteq D$
- $L.D \geq 0$
- $L.(L - 2D) \geq 0$
- $D.(L - D) \leq d$

Proof. We begin by considering the sequence from the Weak Bolgomolov Property together with the defining sequence of $E(Z, L)$, through the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & M \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "s'"] \arrow[dr, "s"'] & E(Z,L) \arrow[r] \arrow[d] & I_Z \otimes L \arrow[r] & 0 \\
 & & I_A \otimes N \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

We are mainly interested in the map s. First of all, if we assume that $s = 0$, then s' must factor through the kernel, that is $\mathcal{O}_X$ is included in M. Since they are both line bundles and the cokernel needs to be torsion free, we have $\mathcal{O}_X \cong M$. Hence, $I_Z \otimes L \cong I_A \otimes N$, and so by the Weak Bogomolov Property $L.(M \otimes N^{-1}) \geq 0 \implies L.L^{-1} \geq 0 \implies L.L \leq 0$ contradicting ampleness.

Therefore, s gives us a non-trivial section of $I_A \otimes N$, and by the inclusion $I_A \otimes N \hookrightarrow N$ we can consider a non-trivial section of N. We will call D to its zero locus. Now it is just a matter of computation:

- First of all, we may notice that s factors by definition through s', and as the zero locus of s' is Z, we obtain that $Z \subseteq D$.
- Because D is effective, we have that $L.D \geq 0$ by ampleness of L.
- By definition $L \cong M \otimes N$. So we obtain that the Weak Bogomolov Property can be translated into $0 \leq L.(M \otimes N^{-1}) = L.((L - D) - D) = L.(L - 2D)$.
- By definition we have $d = c_2(E) = c_2(M) + c_2(I_A \otimes N) + c_1(I_A \otimes N).c_1(M) = 0 + \deg(A) + N.M$, implying $d \geq N.M = D.(L - D)$. □

The strength of the previous result might be somewhat misleading. Indeed, by using the Hodge Index Theorem we can show just how restrictive it really is. We will use the following form of the Hodge Index Theorem.

Theorem 4.10 (Hodge Index Theorem). *Given an ample line bundle L over a surface X and a divisor D such that $L.D = 0$, then either $D \cong 0$ or $D^2 < 0$.*

For a proof of the Hodge Index Theorem, see for example [Har77, Theorem V.19] or [Voi02, Theorem 6.33] for a more general version (not necessarily on surfaces).

Proposition 4.11. *Given an ample line bundle L and a divisor D such that the conclusion of Reider's Method follows, we have one of the following:*

$$\text{Either } \begin{cases} 0 < L.D \le \min(2d, \frac{L^2}{2}) \text{ and} \\ \max(0, L.D - d) \le D^2 \le \frac{(L.D)^2}{L^2} \end{cases} \quad \text{with equality iff } (L.D)L \cong (L^2)D$$
$$\text{Or } \begin{cases} 0 < L.D \le \min(d-1, \frac{L^2}{2}) \text{ and} \\ L.D - d \le D^2 < 0. \end{cases}$$

Proof. First of all, note that in any case $L.(L-2D) \ge 0 \iff L.D \le \frac{L^2}{2}$ and $L.D \ge 0$ already gives us an important part of the required inequalities. Now we can divide this in two cases:

- $D^2 < 0$. This is the simplest case, as from $D.(L-D) \le d$ we obtain both inequalities we need, namely $L.D \le d + D^2 \le d - 1$ and $D^2 \ge L.D - d$.
- $D^2 \ge 0$. To attack this case we will extensively use the Hodge Index Theorem. First of all see that $L.((L.D)L - (L.L)D) = 0$, and as such $((L.D)L - (L.L)D)^2 \le 0 \iff (L.L)(D.D) \le (L.D)^2$, with equality if and only if $(L.D)L = (L.L)D$. For the other non trivial inequality we note that from the previous equation gives us $(L.D - d)(L.D) \le (D^2)(L.D) \le (D^2)(\frac{L^2}{2}) \le \frac{(L.D)^2}{2}$ and isolating $(L.D)$ (using its positivity) gives us $L.D \le 2d$ as wanted. □

So fixing d there is a finite amount of possible values for $L.D$ and, in consequence, for D^2. This is of course a more general construction than we will need. We are mainly interested in the following application:

Theorem 4.12. *If L is a polarization on a K3 surface X such that $L^2 \ge 4$, then it is either very ample or there exists an effective divisor D such that $D^2 = 0$ and either $L.D = 1, 2$.*

Proof. Being an ample line bundle mainly relies on showing that given any 2 points $p, q \in X$, they are separated by ϕ_L. For this, we will use Reider's Method on the 0-cycle $Z = p + q$.

First, we need a translation between properties of the points composing Z and the conditions for Reider's Method. This is obtained thanks to the exact sequence $0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$, which we can tensor by $\omega_X \otimes L$ to obtain $0 \to I_Z \otimes \omega_X \otimes L \to \omega_X \otimes L \to (\omega_X \otimes L)|_Z \to 0$ and now consider the corresponding long cohomology sequence:

$$0 \to H^0(X, I_Z \otimes L) \to H^0(X, L) \to H^0(Z, L) \to H^1(X, I_Z \otimes L) \to H^1(X, L).$$

From here we can see that if $Z = p$, then $H^0(Z, L) \cong \mathbb{C}$ and (by ampleness of L and Kodaira vanishing 3.7) $H^1(X, L) \cong 0$. Since p is a base point of L if and only if $H^0(X, L) \to H^0(Z, L)$ is the zero map, then having a base point is equivalent to $h^1(I_Z \otimes L) = 1$.

Now first assume we have a basepoint $p \in X$ of L, then we consider $Z = p$. By the previous result we have that $h^1(X, I_Z \otimes L) = 1$ and thus we have a rank 2 vector bundle $E = E(Z, L)$.

We then compute $\chi(E \otimes E^\vee) = 2 - 2(g - r(r - d + g - 1)) = 2g + 2 = L^2 + 4 \geq 4$ and thus E has a nontrivial endomorphism and by Proposition 4.7 we can apply Reider's Method, with given divisor D. We see first that $D^2 < 0 \implies L.D \leq d - 1 = 0$ contradiction. Now if $D^2 = 2$ (it can't get any greater as $L^2 \geq 2$ and $L.D \leq 2$), then we have $D \cong L$ but $L.D = L^2 \leq \frac{L^2}{2}$ is a contradiction. Thus $D^2 = 0$, $L.D \leq D^2 + d$ giving us $L.D = 1$ as we wanted.

Similarly, if $Z = p + q$, then $H^0(Z, L) \cong \mathbb{C}^2$ and, if both p and q are not base points, by the prior paragraph the dimension of $H^0(X, L) \to H^0(Z, L)$ is at least 1. For it to be exactly 1 is then equivalent to p and q being annihilated by the same sections, and thus not being separated by ϕ_L.

We just need to use our previous computation to show that $E = E(Z, L)$ has the Weak Bogomolov Property so that we may apply Reider's Method. Note that $\chi(E \otimes E^\vee) = 2 - 2(g - (r + 1)(r - d + g)) = 2 - 2(g - 2(g - 1)) = 2g - 2 = L.L \geq 4$.

As such, we can now apply Reider's Method to obtain a divisor D following all the previous conditions. We can now use Proposition 4.11 and check all possible cases.

In the case that $D^2 < 0$ we have $0 < L.D \leq 1$ so $L.D = 1$. But then $D^2 \geq L.D - d = 1 - 2 = -1$ and the fact that D^2 is even gives us a contradiction, so this case is impossible.

In the case that $D^2 \geq 0$ we have $0 < L.D \leq 4$. Now as $D^2 \leq \frac{(L.D)^2}{L^2} \leq (L.D)\frac{L.D}{L^2} \leq \frac{L.D}{2}$ we have that either $D^2 = 0 or D^2 = 2$.

Note that in the latter case we must have equality, meaning that $L.D = 4$, $L^2 = 8$ and $4L = 8D \implies L = 2D$ contradicting primality of L. Thus the only case that matters is $D^2 = 0$. In this case $0 < L.D \leq D^2 + d \implies L.D = 1, 2$. □

With this result we can finally start classifying some surfaces.

4.3 Classification of degrees $L^2 = 4, 6, 8$

From this point onwards we will assume that our polarized K3 surface has a very ample line bundle, which as we have seen thanks to Reider's Method is a mild assumption for K3 surfaces. For example, if we assume that there is an isomorphism in the Picard groups, $\mathrm{Pic}(X) \cong \mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, then it is easy to see that L must generate $\mathrm{Pic}(X)$ because of primitivity. As such, by Theorem 4.12 we may see that L must be very ample. We could also consider a more general case and study the case where we have an effective divisor D such that $L.D = 2$ and $D^2 = 0$, but we won't do it here. To explore this case one can look at [Deb19].

For this section we are going to make a simple observation on the meaning of the existence of certain global sections. First of all, since L is very ample, we have mentioned that the zero divisor of one of its global sections (in the non-degenerate case) can be seen as the divisor associated with the intersection of a hyperplane of $\mathbb{P}^n$ with $\phi_L(X)$. Notice however that hyperplanes are simply the global sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, and that $L = \phi_L^*(\mathcal{O}_{\mathbb{P}^n}(1))$.

More generally, for any line bundle A we may consider $|\phi_L^*(A)|$ to be the complete intersections of elements of $|A|$ and X. We may now observe that $\phi_L^*(\mathcal{O}_{\mathbb{P}^n}(n)) = \phi_L^*(\mathcal{O}_{\mathbb{P}^n}(1)^{\otimes n}) = \phi_L^*(\mathcal{O}_{\mathbb{P}^n}(1))^{\otimes n} = L^{\otimes n}$. Note that this relies on the fact that the inverse image functor commutes with the tensor product, which is easy to check by looking at stalks.

The first and easiest case by far is that of $L^2 = 4$. Indeed, we just need to show that X is generically a quartic surface. Because L is very ample, X can be seen as a smooth hypersurface of $\mathbb{P}^3$, and thus an effective divisor, meaning that there exists a global section $s \in \mathcal{O}_{\mathbb{P}^3}(X) = \mathcal{O}_{\mathbb{P}^3}(m)$ such that $Z(s) = X$. Therefore, we may now make the same calculations as in the proof that all smooth quartics are K3 at the beginning of Section 4 to show that indeed $m = 4$ as we wanted. While this direct approach works here, some subtlety is required for the next 2 cases.

In the case $L^2 = 6$ we start by doing some computations. Indeed, notice that by Theorem 3.7, we have that ample line bundles are acyclic, and thus that $h^0(X, L) = \chi(X, L) = \frac{L^2}{2} + 2$, from which we may see that:

- $h^0(X, L) = 5$
- $h^0(X, L^{\otimes 2}) = 14$, $\dim(S^2 H^0(X, L)) = 15$, so $\dim(\ker(S^2 H^0(X, L) \to H^0(X, L^{\otimes 2}))) = 1$.
- $h^0(X, L^{\otimes 3}) = 29$, $\dim(S^3 H^0(X, L)) = 35$, so $\dim(\ker(S^3 H^0(X, L) \to H^0(X, L^{\otimes 3}))) = 6$.

From this, we extract that X is contained in a quadric Q and in 6 linearly independent cubics. Note that X is irreducible and not contained in any hyperplane, so the quadric must be irreducible. But now the space of cubics containing Q is of dimension 5, so there must be some cubic C which does not contain Q. Hence $X \subseteq Q \cap C$. But now, the degree of the subvariety is necessarily $\deg(Q)\deg(C) = 6$, which is the same degree as X, so X is the complete intersection of Q and C.

Essentially the same can be done for the case $L^2 = 8$, in this case being even simpler as we have:

- $h^0(X, L) = 6$
- $h^0(X, L^{\otimes 2}) = 18$, $\dim(S^2 H^0(X, L)) = 21$, so $\dim(\ker(S^2 H^0(X, L) \to H^0(X, L^{\otimes 2}))) = 3$,

which implies that as $8 = 2^3$ is the degree of X, it is the complete intersection of three quadrics.

More precisely we have proven that

Theorem 4.13. *Let X be a K3 surface and let L be an ample line bundle over X.*

1. *Let $L^2 = 4$ and suppose there is no divisor D on X such that $D^2 = 0$ and $L.D \in \{1, 2\}$. Then $\phi_L : X \to \mathbb{P}^3$ induces an isomorphism on a quartic hypersurface.*

2. *Let $L^2 = 6$ and suppose there is no divisor D on X such that $D^2 = 0$ and $L.D \in \{1, 2\}$. Then $\phi_L : X \to \mathbb{P}^4$ induces an isomorphism on the intersection of a quadric and a cubic.*

3. *Let $L^2 = 8$ and suppose that L is very ample. Then $\phi_L : X \to \mathbb{P}^3$ induces an isomorphism on the complete intersection of three quadrics.*

Proof. To complete the proof we only need to observe that in the first two cases Reider's method ensures that L is very ample. □

Remark 4.14. *In the case $L^2 = 8$ we cannot characterize numerically the fact that L is very ample and, in general, we can only impose numerical condition such that L induces an embedding onto the intersection of three quadrics or a degree to finite morphism onto the Veronese surfaces inside $\mathbb{P}^5$. This is not a concern if we consider primitive line bundles.*

4.4 Classification of degrees $L^2 \geq 10$

Unfortunately this trick stops working from here on. The key idea will still be useful for the next few cases though. Indeed, we just need to show that our immersion into $\mathbb{P}^n$ factorizes through the **Grassmaniann** which is better behaved for our intentions. Roughly speaking, the Grassmaniann $\mathrm{Gr}(k, m)$ is the space parametrizing the k-dimensional linear subspaces of $\mathbb{C}^m$.

We leave [Muk88] as a reference for those who are interested in proceeding with the classification.

5 Conclusions

We have presented a narrow path leading us to the classification of the lowest degree polarized K3 surfaces, but there are a lot of alternative paths and other interesting problems around. For example, we skipped over the theory of schemes, which can help clarify some of our proofs. Even though we did mention the definitions of both projective and affine schemes, we have not delved into the parallelisms between the theory of coherent sheaves on affine schemes and the theory of modules over rings, and similarly for morphisms between them.

On the other hand, if one wishes to continue learning about the higher degree classification of polarized K3 surfaces, it becomes indispensable to study some representation theory and it would be very interesting to study the theory of moduli spaces over varieties, since the results of this report and the higher degree case can be better expressed in this language. More importantly, one can prove that moduli spaces of polarized K3 surfaces of degree at least 62 have positive Kodaira dimension, so there cannot exist a "construction" as we presented in this report.

Another important topic that we have not discussed is that of Lie groups and their actions on complex manifolds. Understanding these can also give us another path through various proofs, and is essential to fully understand the Grassmaniann and its natural vector bundles.

On a personal note, the biggest shock when studying for this project was learning about the power of the GAGA theorem. Being able to switch between studying algebraic and analytic spaces while maintaining a substantial amount of structure was both useful and a testament to the restrictiveness of holomorphic structure on compact spaces. I found this very interesting, as I considered the analytical topology to not be related in any fundamental way with the more intrinsic (from an algebraic point of view) Zariski topology.

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