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Option Price Decomposition for Local and Stochastic Volatility Jump Diffusion Models

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Declaration

The views expressed in this doctoral thesis are the ones of the author and do not necessarily represent those of VidaCaixa, S.A. I do hereby declare that the entire thesis is my original work and that I have used only the cited sources.

Raúl Merino

Abstract

In this thesis, an option price decomposition for local and stochastic volatility jump diffusion models is studied. On the one hand, we generalise and extend the Alòs decomposition to be used in a wide variety of models such as a general stochastic volatility model, a stochastic volatility jump diffusion model with finite activity or a rough volatility model. Furthermore, we note that in the case of local volatility models, specifically, spot-dependent models, a new decomposition formula must be used to obtain good numerical results. In particular, we study the CEV model. On the other hand, we observe that the approximation formula can be improved by using the decomposition formula recursively. Using this decomposition method, the call price can be transformed into a Taylor type formula containing an infinite series with stochastic terms. New approximation formulae are obtained in the Heston model case, finding better approximations.

Abstract (Catalan)

En aquesta tesi, s'estudia una descomposició del preu d'una opció per a models de volatilitat local i volatilitat estocàstica amb salts. D'una banda, generalitzem i estenem la descomposició d'Alòs per a ser utilitzada en una àmplia varietat de models com, per exemple, un model de volatilitat estocàstica general, un model volatilitat estocàstica amb salts d'activitat finita o un model de volatilitat 'rough'. A més a més, veiem que en el cas dels models de volatilitat local, en particular, els models dependents del 'spot', s'ha d'utilitzar una nova fórmula de descomposició per a obtenir bons resultats numèrics. En particular, estudiem el model CEV. D'altra banda, observem que la fórmula d'aproximació es pot millorar utilitzant la fórmula de descomposició de forma recursiva. Mitjançant aquesta tècnica de descomposició, el preu d'una opció de compra es pot transformar en una fórmula tipus Taylor que conté una sèrie infinita de termes estocàstics. S'obtenen noves fórmules d'aproximació en el cas del model de Heston, trobant una millor aproximació.

Abstract (Spanish)

En esta tesis, se estudia una descomposición del precio de una opción para los modelos de volatilidad local y volatilidad estocástica con saltos. Por un lado, generalizamos y ampliamos la descomposición de Alòs para ser utilizada en una amplia variedad de modelos como, por ejemplo, un modelo de volatilidad estocástica general, un modelo de volatilidad estocástica con saltos de actividad finita o un modelo de volatilidad 'rough'. Además, vemos que en el caso de los modelos de volatilidad local, en particular, los modelos dependientes del 'spot', se debe utilizar una nueva fórmula de descomposición para obtener buenos resultados numéricos. En particular, estudiamos el modelo CEV. Por otro lado, observamos que la fórmula de aproximación se puede mejorar utilizando la fórmula de descomposición de forma recursiva. Mediante esta técnica de descomposición, el precio de una opción de compra se puede transformar en una fórmula tipo Taylor que contiene una serie infinita de términos estocásticos. Se obtienen nuevas fórmulas de aproximación en el caso del modelo de Heston, encontrando una mejor aproximación.

Publications

This thesis is based on the following articles:

- Merino R, Vives J (2015) A generic decomposition formula for pricing vanilla options under stochastic volatility models. International Journal of Stochastic Analysis, Vol. 2015, Article ID 103647, 11 pages. DOI 10.1155/2015/103647

  In this paper, a general decomposition formula is studied. The results are partially exposed in Chapter 4. In addition, some additional results are presented in Appendices A and B.

- Merino R, Vives J (2017) Option price decomposition in spot-dependent volatility models and some applications. International Journal of Stochastic Analysis, Vol. 2017, Article ID 8019498, 16 pages. DOI 10.1155/2017/8019498

  In this work, we apply the Itô decomposition presented in Alòs (2012) to spot-dependent volatility models, a type of local volatility models. The results obtained are presented in Chapter 8 and Chapter 9.

- Merino R, Pospíšil J, Sobotka T, Vives J (2018) Decomposition formula for jump diffusion models. International Journal of Theoretical and Applied Finance, Vol. 21, Number 8, 1850052, DOI 10.1142/S0219024918500528

  Here, an extension of the decomposition formula for a general functional is presented. The results are applied to find a decomposition formula for stochastic volatility models with finite activity jumps. The results obtained are part of Chapter 4 and all the Chapter 6

- Gulisashvili A, Lagunas M, Merino R, Vives J (2020) Higher order approximation of call option prices under stochastic volatility models. Journal of Computational Finance, Vol. 24, No. 1, pp:1-20, DOI 10.21314/JCF.2020.387

  Here, a higher order approximation of the decomposition formula is obtained and applied to the Heston model, improving previous approximations. The results are part of Chapter 5.

- Merino R, Pospíšil J, Sobotka T, Sottinen T, Vives J (2020) Decomposition formula for rough fractional stochastic volatility models. Manuscript submitted to International Journal of Theoretical and Applied Finance.

  In this work, we study the decomposition formula when the volatility process follows an exponential model driven by a Volterra process. The particular cases of a fractional Brownian motion and Brownian motion are studied. The results are exposed in Chapter 7.

The irreducible price of learning is realizing that you do not know. One may go further and point out — as any scientist, or artist, will tell you — that the more you learn, the less you know; but that means that you have begun to accept, and are even able to rejoice in, the relentless conundrum of your life.

James Baldwin (Esquire Magazine, October 1st 1980)

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Contents

List of Figures

List of Tables

CHAPTER 1

Introduction

1.1 A brief history

Financial market instruments can be divided into two different categories. On the one hand, we have the 'prime source' assets, which we will refer to as 'underlyings', and which can be stocks, bonds, commodities, foreign currencies, etc. On the other hand, their 'derivative' contracts, financial claims that promise some payment or delivery in the future, depending on the behaviour of the underlying.

The most typical financial derivatives are futures (or forwards) and options. A future (or forward) is a legal agreement to buy or sell a particular asset at a predetermined price and at a specified time in the future. Meanwhile, an option contract gives the right but not the obligation to buy (or sell) a particular asset at a predetermined price and at a specified time in the future.

Many people think that derivative contracts, such as futures and options, are inventions of the modern economy. However, derivative contracts emerged as soon as humans could make credible promises. They were the first instruments to guarantee the supply of basic products, facilitate trade and insure farmers against the loss of crops. The first written evidence of a derivative contract was in law 48 of the Hammurabi code, roughly between 1782 to 1750 BCE.

One of the first stories related to the speculation of derivatives is due to Thales of Mileto. Thales made a deposit at the local olive presses. As nobody knew for sure whether the harvest would be good or bad, Thales purchased the rights to the presses at a relatively low rate. When the harvest proved to be abundant, the demand for the presses was high, Thales charged a high price for their use and reaped a considerable profit.

Although the use of financial contracts evolved, for an extended reading see Kummer and Pauleto (2012), it was not until 1900 that the history of mathematical modelling of financial markets began. Louis Bachelier introduced the first model in his thesis 'Théorie de la spéculation', Bachelier (1900), being one the cornerstones of modern pricing theory.

On the Bachelier thesis, he realised that there was an equilibrium between buyers and sellers.

It seems that the market, the aggregate of speculators, can believe in neither a market rise nor a market fall, since, for each quoted price, there are as many buyers as sellers.

In particular, he realised the need to use martingales to describe price movements.

The mathematical expectation of speculators are null.

The method of obtaining option prices is similar to the modern approach. However, the argument is very different. Bachelier derived the model using an equilibrium argument, while non-arbitrage arguments are used today.

Despite the modern techniques used by Bachelier in his thesis, it remained unknown for several decades. Apparently, unaware of Bachelier's work, in 1953, Kendall, Kendall (1953), analysed 22 series of prices at weekly intervals with the purpose of finding a model that would fit the stock data. Before the work went far, he realised that between the intervals, there were random changes discarding a systematic effect. He was also the first to notice the time dependence of the empirical variance. A few years later, in 1959, Osborne, Osborne (1959), found that the logarithm returns follow a Brownian motion.

In the middle of 1950, the statistician Jimmy Savage recovered Bachelier's work and sent it to different friends. Fortunately, one of those postcards came to Paul Samuelson, who was concerned with problems of valuation of options and warrants. Paul Samuelson was inspired by Bachelier's work and related the option pricing with the use of martingales in Samuelson (1965).

In the year 1973, the world's first listed options exchange opened in Chicago, the Chicago Board Options Exchange (CBOE). The same year, the famous Black-Scholes-Merton model was published using no-arbitrage assumptions, see Black and Scholes (1973) and Merton (1973).

The Black-Scholes-Merton model is an analytical formula that describes parsimoniously market option prices. The main drivers are the ratio between the stock price and the strike, also known as moneyness, the level of interest rates and the constant volatility. This last feature is the main problem of the Black-Scholes-Merton model. Despite that, in practice, the Black-Scholes-Merton model is used as a marking model to quote volatilities of traded options prices. These volatilities are called implied volatilities, i.e. it is the constant volatility input needed in the Black-Scholes-Merton formula to match a given market price.

Over the following years, new models appeared trying to adapt the Black-Scholes-Merton model to the observable patterns of stock returns. The Constant Elasticity Variance model, also known as the CEV model, was presented in an unpublished note in Cox (1975), see also Beckers (1980). The main purpose of the CEV model was to explain the inverse relationship between the level of the stock price and the variance of its returns. The following year, the first Jump Diffusion model was published in Merton (1976). Merton extended the Black-Scholes-Merton model to stochastic processes with non-continuous sample paths, explaining possible 'abnormal' vibrations in price due to the arrival of important new information about the stock with a non-marginal effect of the stock price. Those models were able to introduce skew in the implied volatility.

On October 19, 1987, also known as Black Friday, one of the biggest financial crisis occurred. All international markets experienced large losses, the Down Jones lost 22.6% and a large increase of volatility was observed. Shortly after the collapse, options traders noted that deep out of the money put options were unusually high compared to put options closer to the current price. This phenomenon was called the 'volatility smile'.

Stochastic volatility (SV) models appeared as a useful tool to explain in a self-consistent way the volatility surface observed by traders. In Johnson (1979), we find one of the firsts approaches where variance follows a stochastic process. To obtain a partial differential equation, a perfect correlation between the asset price and the variance was assumed, although no solution was found for the option price. Johnson and Shanno (1985), Wiggins (1987) and Scott (1987) tried several numerical solutions to obtain option prices. Using a different approach, Hull and White (1987) obtained a price approximation as a Taylor series expansion when the asset price and the variance are uncorrelated. In Stein and Stein (1991), a model where the volatility is driven by an arithmetic Ornstein-Ublenbeck process uncorrelated with the asset price was proposed and a solution based on numerical integration was obtained to calculate options prices. In 1993, Heston (1993), proposed a model with an arbitrary correlation between the asset and the volatility process driven by a CIR process, Cox et al. (1985). He obtained semi-analytical formulae for European plain vanilla options. The Heston model has become one of the most popular stochastic volatility models, due to its analytical tractability and its good statistical properties.

The volatility surface not only presents a 'volatility smile', but short-term options are traded with higher implied volatilities. Traders, aware of the possibility of a large market movement, request higher premiums. Stochastic volatility models are not rich enough to reproduce these movements in the short term. To improve them, stochastic volatility jump diffusion (SVJ) models appeared. The first SVJ model is credited to Bates (1996) who incorporated a stochastic variance process postulated by Heston (1993) alongside Merton (1976) - style jumps. The variance of stock prices follows a CIR process and the stock prices themselves are assumed to be of a jump diffusion type with log-normal jump sizes. In particular, this model should improve the market fit for short-term maturity options, while the original Heston (1993) approach would often need unrealistically high volatility of variance parameter to fit the short-term smile reasonably well, see Bayer et al. (2016) and Mrázek et al. (2016).

An SVJ model with a non-constant interest rate was introduced by Scott (1997). Several other authors studied SVJ models which have a different distribution for jump sizes, for example, Yan and Hanson (2006) utilised log-uniform jump amplitudes, or Kou (2002) a double exponential jump diffusion. These models can be generalised using an Exponential Lévy model. Figueroa-López has done extensive work proposing different short-time expansions: one regarding the volatility smile, Figueroa-López et al. (2012a), and the others regarding the option price, for example, Figueroa-López et al. (2012b) and Figueroa-López et al. (2016) among others. Naturally, one can extend SVJ models by adding jumps into the variance process, see, for example, a model introduced by Duffie et al. (2000). However, according to several empirical studies, these models tend to overfit market prices and, despite having more parameters than the original Bates (1996) model, they may not provide a better fit to market option prices.

Despite the hectic research in stochastic volatility models, adding a stochastic volatility structure to the Black-Scholes-Merton model complicates the calculation of option prices, these models need to be calibrated. In other words, find the correct parameters to minimise the error between the model option prices and the market option prices, which, in general, is a difficult and complex task. Derman and Kani (1994), Dupire (1994) and Rubinstein (1994) proposed a different model, the local volatility model. They defined a unique instantaneous volatility that is a deterministic function of time and the asset price consistent with market option prices.

Local volatility models are self-consistent, arbitrage-free and can be calibrated precisely to the whole volatility surface. However, as was pointed out by Hagan et al. (2002), the dynamic behavior of smiles and skews predicted by these models are exactly contrary to the behaviour observed on the market, obtaining worse hedges than using the Black-Scholes-Merton model. In Hagan et al. (2002), the SABR model was introduced and it can be classified as stochastic local volatility model. This model consists of modelling the asset price with the Constant Elasticity Variance model with an exponential stochastic volatility process. The main success of Hagan et al. (2002) was to obtain a model able to fit the 'volatility smile' in a parsimonious way, obtaining an easy to implement approximation formula of the implied volatility. For a general summary, see Gatheral (2006).

Although many SV and SVJ models have been proposed, it seems that none of them can be considered as the universal best market practice approach. Several models might perform well for calibration to complex volatility surfaces, but can suffer from over-fitting or they might not be robust in the sense described by Pospíšil et al. (2018). In addition, a model with a good fit to implied volatility surface might not be in-line with the observed time series properties.

The last trend is focused on modelling the volatility with a Volterra process, in particular, the fractional Brownian motion. Pioneers of the fractional SV models, see for example Comte and Renault (1998) and also Comte et al. (2012), assumed a long-memory volatility process. They replaced the Brownian motion by a fractional Brownian motion with Hurst parameter ranged within $H \in (1/2, 1)$ which implies that the spot variance evolution is represented by a persistent process, i.e. it would have a long-memory property. In Alòs et al. (2007), we observe the first approach considering a full range for the Hurst parameter with a mean reverting fractional stochastic volatility model. The models with the Hurst parameter $H \in (0, 1/2)$ are called rough volatility models. Bayer et al. (2016) and Gatheral et al. (2018) found a consistency between the realised volatility time series and the rough fractional volatility. That means that it should be a more consistent model to price market options. A large number of papers have been published lately about this approach, for example, in Funahashi and Kijima (2017), a two factor fractional volatility model, combining a rough term ($H < \frac{1}{2}$) and a persistent one ($H > \frac{1}{2}$), was presented or in Alòs et al. (2019), an approximation for target-volatility options under log-normal fractional SABR model was studied using the Malliavin calculus techniques.

1.2 Objectives

From the beginning of derivative contracts up until today, we have wondered how to price a derivative product. Although the works of Bachelier and Black-Scholes-Merton are the cornerstone of the pricing models, unfortunately, the dynamics of these models are not rich enough to adapt to the market. For this reason, new models have emerged, each with richer, but also more complex dynamics. The valuation of derivatives under these more complex models is a more elaborate task compared to the Black-Scholes-Merton model. To face this challenge, various techniques and numerical methods have been proposed which is an important mathematical challenge from an academic and practical point of view.

Many authors have introduced semi-closed form formulae using various transformation techniques of the pricing partial (integro) differential equations: Heston (1993), Bates (1996), Scott (1997), Lewis (2000), Albrecher et al. (2007), Baustian et al. (2017) to name a few. These pricing methods are typically efficient tools to evaluate non-path dependent derivatives.

Some other authors considered approximation techniques that were pioneered by Hull and White (1987). In recent years, the Hull and White (1987) pricing formula was reinvented using techniques of the Malliavin calculus because a future average volatility that is used in the formula is a non-adapted stochastic process. The main goal was to generalise the Hull and White formula. More precisely, the price of a European option can be decomposed as the Black-Scholes-Merton option price plus other correction terms. In Alòs (2006), Alòs et al. (2007) and Alòs et al. (2008), a general jump diffusion model with no prescribed volatility process is analysed. There have been several extensions thereof, for example by assuming Lévy processes in Jafari and Vives (2013), see also the survey in Vives (2016).

In Alòs (2012), a new approach of dealing with the Hull and White formula for the Heston model has been proposed. Instead of expanding option prices around the Hull and White term by means of anticipating stochastic calculus, a classical Itô formula is used to expand the prices around the Black-Scholes-Merton formula. The main idea of this approach is to use an adapted projection of the future volatility. Using this technique an exact decomposition formula for call option prices is obtained as well as an approximate formula. Using the approximate formula, in Alòs et al. (2015) a valuable intuition on the behaviour of smiles and term structures under the Heston model is derived.

The main objective of this thesis is to study the decomposition formula presented in Alòs (2012) and extend it to different models. The idea of being able to decompose the price of a call option by the Black-Scholes-Merton formula plus other corrections dependent on the Black-Scholes-Merton derivatives is interesting. In practice, the Black-Scholes-Merton model is a reference model and its derivatives are usually calculated. Therefore, all the necessary tools to use a decomposition are already available. Furthermore, for years practitioners have been calculating derivatives, having an idea of how they work, so it might be easier to understand how the price varies from a Black-Scholes-Merton model to a stochastic volatility model or how the stochastic volatility model works. In addition, the decomposition is numerically efficient in time and accuracy.

As initial research, we had the idea of extending the decomposition formula to a general stochastic volatility model, without assuming a lognormal process for the asset price dynamics. In this case, we saw that the decomposition formula had an extra term. Playing with the decomposition and using the CEV model as a toy model, we realised that the previous decomposition was numerically not good enough. It was necessary to develop a new decomposition formula for the case of models with spot-dependent volatility. Despite not having constant volatility, the new decomposition has more terms to correct the Black-Scholes-Merton formula.

Later, we focused on the stochastic volatility jump diffusion models. In this case, we observed that conditioning on the jump process, we were able to use the previous ideas of the decomposition formulae to obtain a new decomposition for jumps with finite activity. In particular, we realised that the approximation formula error for the Bates model was of the same magnitude as the Heston model. As in Alòs et al. (2015), we were able to approximate the implied volatility dynamics.

Over recent years, academic research about fractional volatility models has increased. A natural step was to try to understand if it was possible to use the decomposition formula with these models. One of the main complexities is the fact that these types of models are not Markovian. The rBergomi model is studied. In the case of fractional volatility models, the numerical effort to calculate the decomposition is much higher than in the Brownian motion models.

Apart from extending the decomposition formula to other models, we notice that using it recursively we were able to obtain better approximation formulae. This was interesting because in Alòs et al. (2015) the error term of approximation formula for the Heston model was quantified by $O(\nu^2\left(|\rho|+\nu\right)^2)$. In the previous expression, ν is the vol-vol parameter and ρ is the correlation coefficient in the Heston model. However, in the above-mentioned approximation formula, some terms of order ν^2 were ignored, whereas other terms of the same order were kept. This may be considered as a drawback in the approximation formula obtained in Alòs et al. (2015). We were able to find new approximation formulae of order $O(\nu^3(|\rho|+\nu))$ and $O(\nu^4(1+|\rho|))$. In the case of zero correlation, we derived an approximation formula with an error of order $O(\nu^6)$.

All the approximation formulae have been tested on practical examples. Therefore the quality of the decomposition has been contrasted numerically with a reference model.

1.3 Outline

The thesis is organised as follows. In Chapter 2, we introduce the basic framework and rudiments necessary for the thesis. In Section 2.1, we describe the economic side and in Section 2.2 we give a brief introduction of the main mathematical concepts needed for the thesis.

Chapter 3 is devoted to models. First of all, we explain the framework in which the Black-Scholes-Merton formula can be derived. Then, we give a brief introduction to different models used in the thesis. In Section 3.5, we introduce some general notation that we are going to use throughout the dissertation.

From a conceptual point of view, Chapter 4 is one of the most important. Here, the decomposition formula is explained as well as a new decomposition formula given for a general stochastic volatility model. Some examples are given. Furthermore, how to find an approximation formula is demonstrated. No upper-bounds of the error term are calculated as the stochastic volatility structure is needed.

The Heston model is addressed in Chapter 5. It is proved what the error term is in the approximation formula from Alòs (2012). This was calculated in Alòs et al. (2015). Then, we observe how using the decomposition formula recursively can lead us to new approximations. The error term for this new approximation formula is calculated. In Section 5.4, we perform numerical experiments contrasting the quality of our methodology.

In Chapter 6, we focus on stochastic volatility models with finite activity jumps. We observe that applying a conditional expectation over the jump process transforms the model into a stochastic volatility model. Some examples are given depending on the jump process in the case that the volatility is of Heston type. Numerical examples about the quality of the approximation are given. In Section 6.3, following the steps of Alòs et al. (2015), an approximation of the implied volatility is given. We have also contrasted the quality of the implied derived calculation with a reference model.

Chapter 7 is the most complex mathematically. On the one hand, the use of a fractional volatility model complicates things, whereas on the other hand, markovianity is lost, making all the calculations more complex. In the first part of the chapter, we develop a decomposition formula as general as possible for a Volterra process. Then, a fractional Brownian motion is assumed and an alternative of rBergomi model used. In Section 7.2, we study the numerical efficiency of the method. Unfortunately, the method is not robust for high vol-vol parameters or very large maturities. Despite that, it can be used in a hybrid calibration jointly with the Bennedsen et al. (2017) MC scheme.

In Chapter 8 a new decomposition formula in the case of spot-dependent volatilities models is given. We show how the approximation formula should be, but, without specifying the volatility structure it is not possible to know the error form. In the following chapter, Chapter 9 , we study the case of the CEV model. An approximation formula is obtained and an error form of $(\beta - 1)^2$ found where β is the elasticity parameter. In addition, an implied volatility approximation is calculated. In Section 9.2, a numerical study about the performance of the model is conducted. We compare the approximation formula and the approximation of the implied volatility with different reference models. In Section 9.4, we observe that if the call option prices are generated by a CEV model, we can recover the parameters by a quadratic linear regression. Although from a practical point of view it is unrealistic, it is interesting.

Finally, in Chapter 10, we explain why we have focused on several models and we summarise the results obtained in the thesis.

There are two appendices to this thesis. These appendices are not directly related to the thesis, and therefore can be considered as some exercises. In Appendix A, we extend the decomposition formula using Malliavin calculus for a general stochastic volatility model.

In Appendix B, an expression of the derivative of the implied derivative ATM is found, in the Itô sense as well as in the Malliavin sense. These results have not been used in the thesis.

CHAPTER 2

Preliminaries

In this chapter, we introduce the basic framework and rudiments necessary for the thesis. The chapter is divided into two distinct sections. In Section 2.1, the basic concepts of derivatives are explained while in Section 2.2, a summary of the mathematical foundations are expounded.

2.1 Derivative preliminaries

Often, people need to get into financial arrangements to exchange a set of distinct cash-flows at different times. These arrangements are financial contracts and each person who enters into the agreement is called a counterparty. For example, when someone needs to buy a house, they get a mortgage loan where they receive a large sum of money in exchange for future monthly payments.

When a financial contract depends on the evolution of another asset, it is called a derivative contract. This is because the price of the financial contract is 'derived' from the evolution of the asset, also named underlying. We will refer to the underlying price as S_t. For example, a derivatives contract would allow us to receive an amount of cash today in exchange for 50 pounds of rice at a future time. This product can be interesting for farmers, being able to advance part of the income before harvesting the rice. The simplest financial derivatives are futures or forwards.

Definition 2.1.1. *A futures contract is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price. The time when the asset is delivered is called maturity time and is denoted by T. The specified price is known as the delivery price K. The payoff of the contract is $S_T - K$. The value of the contract at agreement time is zero.*

A futures contract is a zero-sum game, one counterpart earns money and the other loses it. To price this contract, there is no need of probabilities or models. By the non-arbitrage hypothesis, it can be proved that a futures contract is fair when the delivery price is the forward price.

Definition 2.1.2. *The forward price of an asset which current price is* S_t *and maturity date* T *is*

$$F(t,T) = S_t e^{r(T-t)}$$

where r *is the risk-free interest rate.*

The profit and loss of a future is given by the following graphic.

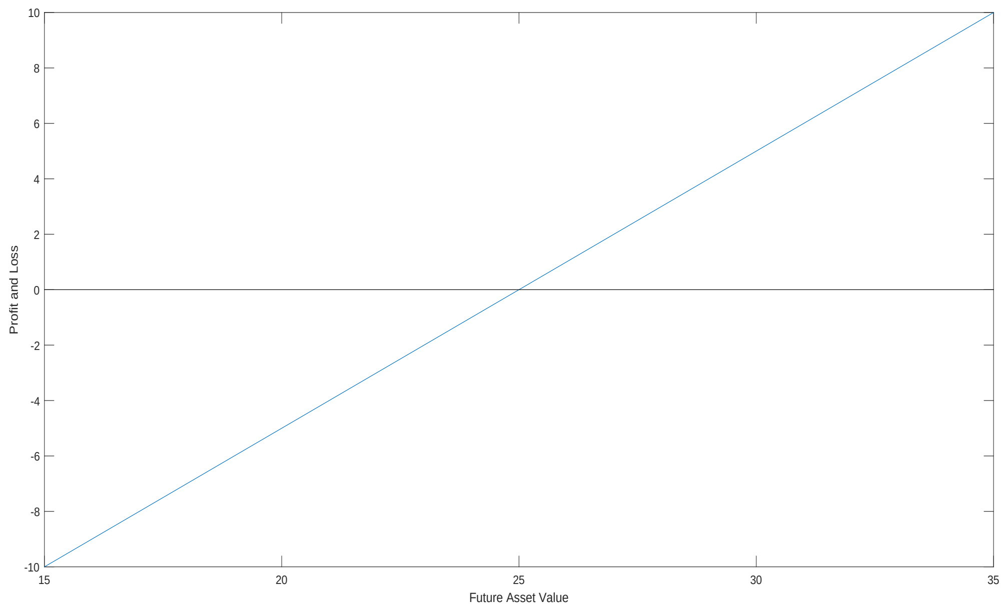


Figure 2.1: Long future with delivery price 25 euros.

Not all the derivatives contracts can be priced without the use of a model. In fact, almost none can. For example, when adding the possibility to exercise or not the contract at maturity time depending on the evolution of the stock, a model must be specified. These types of products are known as options. The most basic options are European Plain Vanilla options.

Definition 2.1.3. *A European call option is a legal agreement that gives the holder the right, but not the obligation, to buy one unit of an underlying asset for a predetermined strike price* K *on the maturity date* T*. If* S_T *is the price of the underlying asset at maturity* T*, then the value, or payoff, of a call option is*

$$(S_T - K)_+ = \begin{cases} S_T - K & \text{if } S_T > K, \\ 0 & \text{if } S_T \leq K. \end{cases}$$

In an event where the asset price at maturity is greater than the predetermined price, the holder will exercise the option and obtain a profit. Otherwise, if the asset price at maturity is below the predetermined price, it would be a loss and the holder would decide not to exercise the option.

The profit and loss of a call option is given by the following graphic.

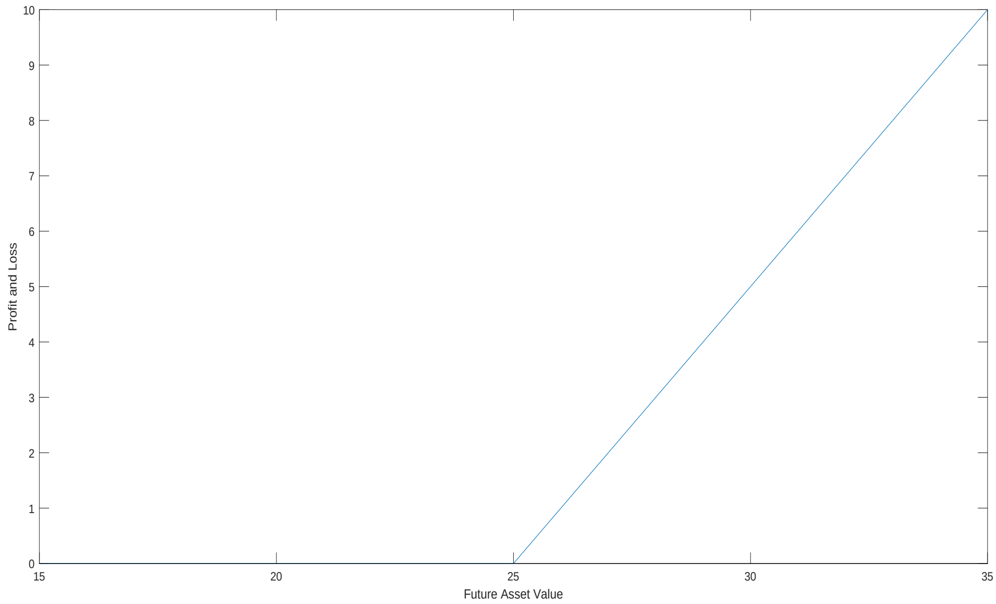


Figure 2.2: Call option with strike K=25 euros.

Definition 2.1.4. *A European put option is a legal agreement that gives the holder the right, but not the obligation, to sell one unit of an underlying asset for a predetermined strike price K on the maturity date T. If S_T is the price of the underlying asset at maturity T, then the value, or payoff, of a put option is*

$$(K - S_T)_+ = \begin{cases} K - S_T & \text{if } S_T < K, \\ 0 & \text{if } S_T \geq K. \end{cases}$$

Investors would buy a call option if they believe the asset price will increase. The use of put options is more natural, the investors buy insurance, so therefore, an investor would buy a put option to obtain protection against a possible market downturn.

The profit and loss of a put option is given by the following graphic.

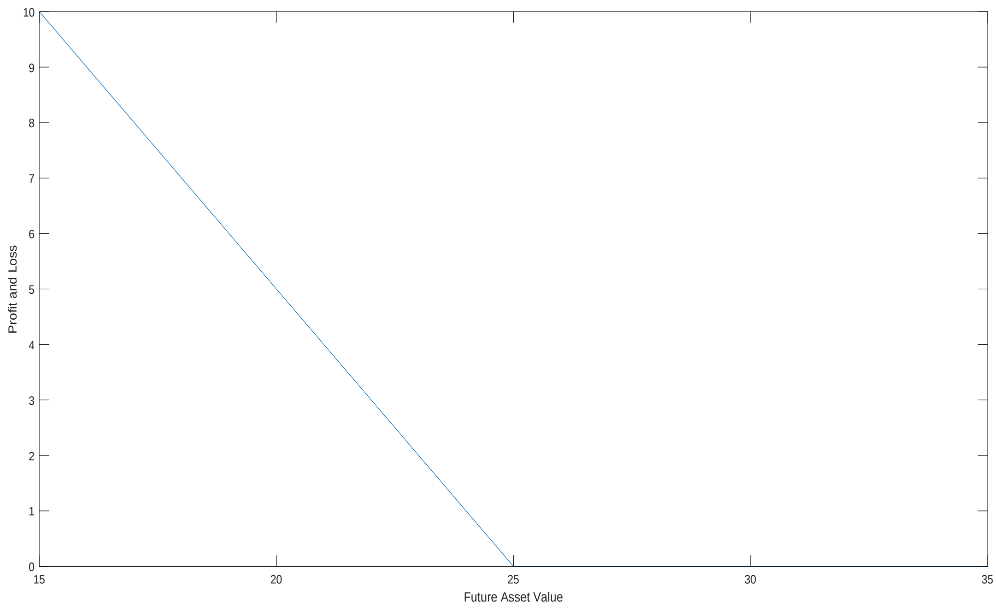


Figure 2.3: Put option with strike K=25 euros.

The European Plain Vanilla options are the most basic options, but there are other types more 'exotic' types.

- **American options:** allows the holder of the option to execute it at any time before the expiration.

- **Path-independent:** The payoff depends on the value of the asset at maturity. For example, a Gap option:

$$(S_T - K_1)_+ = \begin{cases} S_T - K_1 & \text{if } S_T > K_2, \\ 0 & \text{if } S_T \leq K_2. \end{cases}$$

- **Path-dependent:** The payoff depends on the evolution of the asset price value. For instance, a Lookback option:

$$\left(\max_{0\leq t\leq T} S_t - K\right)_+ = \begin{cases} \max_{0\leq t\leq T} S_t - K & \text{if } \max_{0\leq t\leq T} S_t > K, \\ 0 & \text{if } \max_{0\leq t\leq T} S_t \leq K. \end{cases}$$

When options have non-typical features, they are called exotic options. American options are usually quoted for stocks, but the others are tailormade. When this happens, they are traded Over The Counter (OTC).

Options can be classified depending on how the asset price is with respect to the strike today.

Definition 2.1.5. *An option is*

- *In the money (that is, ITM) if we would execute the option today, the option would have a positive value, i.e.* $S_t > K$.
- *At the money (i.e. ATM) when the current asset price* S_t *is at the same level as the strike* K, *i.e.* $S_t = K$.
- *Out the money (i.e. OTM) if we would execute the option today, the option would have zero value, i.e.* $S_t < K$.

Remark 2.1.6. *We refer to ATMF when it is measured against the Forward value instead of the Spot price. For example, an option is ATMF when* $F(t,T) = S_t e^{r(T-t)} = K$.

Definition 2.1.7. *The moneyness of an option is the ratio between the current asset price and the strike, i.e.* S_t/K.

The most liquid options are European plain vanilla options for indexes (Standard & Poor's 500, Eurostoxx, Ibex, ...) and American options for stocks. These options are used to calibrate the models. Then, once the model is set, we price other exotic derivatives.

This thesis is based on European plain vanilla options.

2.2 Mathematical Preliminaries

In this section, we give a brief introduction to some concepts of stochastic calculus.

2.2.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where

- Ω is the sample space,
- $\mathcal{F}$ is a collection of subsets of Ω, also known as events.
- $\mathbb{P}$ specifies the probability of each event $A \in \mathcal{F}$.

The collection $\mathcal{F}$ is a σ-field, that it is, $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under the operations of countable union and taking complements. The probability $\mathbb{P}$ must satisfy the usual axioms of probability.

In order to describe the evolution of market prices, stochastic processes are used to model the random behavior of the assets.

Definition 2.2.1. *A stochastic process $(X_t)_{t\geq 0}$ is a family of real random variables on the probability space $(\Omega,\ \mathcal{F})$. Usually, index t represents time.*

Note that a stochastic process can be also seen as a random map: for each $w \in \Omega$, we can associate the map:

$$\begin{array}{rcl} \mathbb{R}_+ & \mapsto & \mathbb{R} \\ t & \mapsto & X_t(w) \end{array}$$

named trajectory of the process. If the trajectories are continuous, then the process is said to be continuous.

Definition 2.2.2 (Filtration)**.** *A filtration $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of sub-σ-fields of $\mathcal{F}$. We say that (X_t) is an adapted process if for all t, X_t is $\mathcal{F}_t$-measurable.*

Given a process $(X_t)_{t\geq 0}$, we can define the natural filtration $\mathcal{F}_t := \sigma(X_u, 0 \leq u \leq t)$, for which the process is adapted to. We consider the completion of the filtration adding all the $\mathbb{P}$-null sets of Ω. Recall that A is a $\mathbb{P}$-null set if $A \subseteq \Omega$ and if exist $B \in \mathcal{F}$ having $\mathbb{P}(B) = 0$ and $A \subseteq B$. We will refer to this filtration as the one generated by the process X, without making explicit that it is the completion.

We introduce the concept of martingale which it is the mathematical representation of a 'fair game'.

Definition 2.2.3 (Martingale)**.** *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$ on this space. A stochastic process $(X_t)_{t\geq 0}$ adapted to the filtration is a martingale if $\mathbb{E}\left[|X_t|\right] < +\infty$ $\forall t \geq 0$ and $\mathbb{E}[X_u|\mathcal{F}_t] = X_t$, for any $u \geq t$.*

It follows from this definition that, if $(X_t)_{t\geq 0}$ is a martingale, then $\mathbb{E}\left[X_t\right] = \mathbb{E}\left[X_0\right]$ for any t.

2.2.2 Brownian motion

A particular example of stochastic process is the Brownian motion, also known as Wiener process. It is the core element in most financial models.

Definition 2.2.4 (Brownian motion/Wiener process)**.** *A Brownian motion, or Wiener process, is a stochastic process $(X_t)_{t\geq 0}$ with:*

1. *The sample trajectories $t \mapsto X_t$ are continuous with probability 1.*
2. *For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments*

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$$

are independent.

3. It has stationary increments. For $t_1 < t_2$, $X_{t_2} - X_{t_1} \sim X_{t_2 - t_1} - X_0$.

The definition of the Brownian motion induces its own distribution.

Theorem 2.2.5. *If $(X_t)_{t\geq 0}$ is a Brownian motion, then $X_t - X_0$ is a normal random variable with mean rt and variance $\sigma^2 t$, where r and σ are constant real numbers.*

Proof. See proof in Corcuera (2018), Theorem 2.2.1. □

Definition 2.2.6. *A standard Brownian motion is a Brownian motion such that $X_0 = 0$ a.s., $r = 0$ and $\sigma^2 = 1$. We will refer to standard Brownian motion as $(W_t)_{t\geq 0}$.*

From now on, we will refer to the Brownian motion as the standard Brownian motion.

Proposition 2.2.7. *If (W_t) is a Brownian motion, then (W_t) is a martingale.*

Proof. See proof in Lamberton and Lapeyre (1996), Proposition 3.3.3. □

2.2.3 Stochastic Integral

When we work with the different stochastic models to express the evolution of an asset, we would like to use this type of integrals:

$$\int_0^T f(t)\, \mathrm{d}W_t.$$

Unfortunately, the definition of this type of integrals fails because although the paths of Brownian motion are continuous, they are not of bounded variation.

To define this type of integrals, we will construct a stochastic integral by defining it for a set of the most basic processes, the simple process. Later, we will extend it to a larger class.

From now on, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbb{P})$ and $(W_t)_{t\geq 0}$ a $\mathbb{F}$-Brownian motion. We assume that $T > 0$.

Definition 2.2.8 (Simple process)**.** *A process $(H_t)_{0\leq t\leq T}$ is called a simple process if it can be written as*

$$H_t(\omega) = \sum_{i=1}^{p} a_i(\omega) \mathbf{1}_{(t_{i-1}, t_i]}(t)$$

where $0 = t_0 < t_1 < \cdots < t_p = T$ and a_i is $F_{t_{i-1}}$-measurable and bounded.

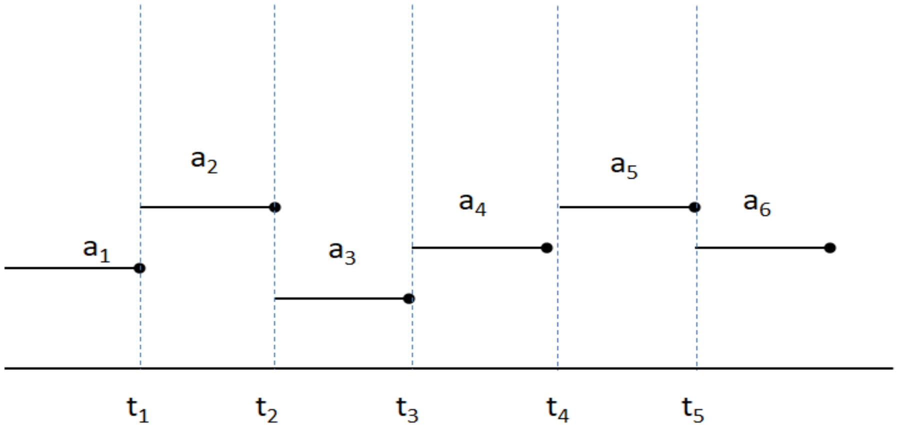


Figure 2.4: Simple process

Definition 2.2.9 (Stochastic integral)**.** *The stochastic integral of a simple process H is a continuous process $(I(H)_t)_{0\leq t\leq T}$ defined for any $t \in (t_{i-1}, t_i]$ as*

$$I(H)_t = \sum_{1\leq i\leq k} a_i \left(W_{t_i} - W_{t_{i-1}}\right) + a_{k+1}\left(W_t - W_{t_k}\right).$$

The stochastic integral $I(H)_t$ can be written as

$$I(H)_t = \sum_{1\leq i\leq p} a_i \left(W_{t_i\wedge t} - W_{t_{i-1}\wedge t}\right)$$

that proves the continuity of $t \mapsto I(H)_t$. We write $\int_0^t H_s dW_s$ for $I(H)_t$.

Note that by definition

$$\int_t^T H_s \, \mathrm{d}W_s = \int_0^T H_s \, \mathrm{d}W_s - \int_0^t H_s \, \mathrm{d}W_s.$$

We have defined the stochastic integral for simple processes. We are going to extend the concept to a larger class of adapted process. The next lemma helps identify a space of functions for which we can reasonably extend the definition.

Proposition 2.2.10. *If $(H_t)_{0\leq t\leq T}$ is a simple process:*

1. $\left(\int_0^t H_s \, \mathrm{d}W_s\right)_{0\leq t\leq T}$ *is a continuous $\mathbb{F}_t$-martingale.*
2. *(Itô isometry)* $\mathbb{E}\left[\left(\int_0^t H_s \, \mathrm{d}W_s\right)^2\right] = \mathbb{E}\left(\int_0^t H_s^2 \, \mathrm{d}s\right)$.
3. $\mathbb{E}\left[\sup_{t\leq T}\left|\int_0^t H_s \, \mathrm{d}W_s\right|^2\right] \leq 4\mathbb{E}\left[\int_0^T H_s^2 \, \mathrm{d}s\right]$.

Proof. See proof in Lamberton and Lapeyre (1996), Proposition 3.4.2. □

We define the class of adapted processes $\mathcal{H}$

$$\mathcal{H} = \left\{ H = (H_t)_{0 \leq t \leq T},\ \mathbb{F}\text{-adapted process such that } \mathbb{E}\left[\int_0^T H_s^2 \, \mathrm{d}s\right] < \infty \right\}.$$

If H is a process in the class $\mathcal{H}$. The integral is defined as the L^2 limit

$$\int_0^t H_s \, \mathrm{d}s = \lim_{n \longrightarrow \infty} \int_0^t H_s^{(n)} \, \mathrm{d}W_s.$$

where $(H^{(n)})_{n \geq 1}$ is a sequence of simple processes such that

$$\lim_{n \longrightarrow \infty} \|H^n - H\|^2 = \lim_{n \longrightarrow \infty} \int_0^t \mathbb{E}\left[H_s^n - H_s\right]^2 \mathrm{d}s = 0.$$

Theorem 2.2.11. *Suppose that $(W_t)_{t \geq 0}$ is Brownian motion and let $\mathbb{F}$ denote its natural filtration. There exist a linear mapping, J, from $\mathcal{H}$ to the space of continuous $\mathbb{F}$-martingales defined on $[0, T]$ such that*

1. *If $(H_t)_{t \leq T}$ is a simple process and $t \leq T$,*

$$J(H)_t = \int_0^t H_s \, \mathrm{d}W_s.$$

2. *If $t \leq T$,*

$$\mathbb{E}\left[J(H)_t^2\right] = \int_0^t \mathbb{E}\left[H_s^2\right] \mathrm{d}s.$$

3.

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} J(H)_t^2\right] \leq 4 \int_0^T \mathbb{E}\left[H_s^2\right] \mathrm{d}s.$$

Proof. See proof in Etheridge (2011), Theorem 4.2.7. □

We have defined the stochastic integral respect to Brownian motion. It can be extended to any process $(X_t)_{t \geq 0}$ that can be written as $X_t = W_t + A_t$ where $(W_t)_{t \geq 0}$ is a Brownian motion and $(A_t)_{t \geq 0}$ is a continuous process of bounded variation. In that case, we can define the integral as the sum of two integrals: one integral with respect to the Brownian motion and another respect to the bounded variation process. The latter is defined in the classical sense.

Definition 2.2.12. *Suppose that $(M_t)_{t \geq 0}$ is a continuous martingale and $(A_t)_{t \geq 0}$ is a process of bounded variation. Then the process $(X_t)_{t \geq 0}$ defined by $X_t = M_t + A_t$ is a semimartingale.*

A semimartingale is any process that can be decomposed in this way. If $A_0 = 0$, then the decomposition is unique as can be seen in Protter (2004), Chapter II, Theorem 9.

2.2.4 Itô calculus

In the previous section, we have defined a stochastic integral. Itô calculus is the differential calculus based on this types of stochastic integrals. The Itô formula is the stochastic calculus counterpart of the 'chain rule'. We are going to define to which type of processes the Itô formula is applicable.

Definition 2.2.13 (Itô Process)**.** *An Itô process $(X_t)_{0\leq t\leq T}$ is a process of type:*

$$X_t = X_0 + \int_0^t K_s \,\mathrm{d}s + \int_0^t H_s \,\mathrm{d}W_s,$$

where for all $t \leq T$, we have that

1. *X_0 is $\mathcal{F}_0$-measurable.*
2. *(K_t) and (H_t) are $\mathbb{F}$-adapted process.*
3. *$\int_0^T |K_s| \,\mathrm{d}s < \infty$ $\mathbb{P}$ a.s.*
4. *$\int_0^T |H_s|^2 \,\mathrm{d}s < \infty$ $\mathbb{P}$ a.s.*

In Corcuera (2018), Proposition 2.4.6, we can see that the decomposition of an Itô process is unique.

The following theorem is known as the Itô formula. It is one of the fundamental theorems of stochastic calculus.

Theorem 2.2.14 (Itô's formula)**.** *Let $(X_t)_{0\leq t\leq T}$ be an Itô process*

$$X_t = X_0 + \int_0^t K_s \,\mathrm{d}s + \int_0^t H_s \,\mathrm{d}W_s,$$

and $f(t,x) \in \mathcal{C}^{1,2}$, then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \partial_s f(s, X_s) \,\mathrm{d}s + \int_0^t \partial_x f(s, X_s) \,\mathrm{d}X_s \\ &+ \frac{1}{2} \int_0^t \partial_x^2 f(s, X_s) \,\mathrm{d}\,[X, X]_s \end{aligned}$$

where

$$[X, X]_t = \int_0^t H_s^2 \,\mathrm{d}s.$$

Proof. See Protter (2004), Chapter II, Theorem 32. □

Suppose that we want find the solution of $(S_t)_{0\leq 0\leq T}$ for the equation

$$S_t = x_0 + \int_0^t \mu S_s \, \mathrm{d}s + \int_0^t \sigma S_s \, \mathrm{d}W_s. \tag{2.1}$$

Written in a differential form

$$\mathrm{d}S_t = S_t \left(\mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t\right), \quad S_0 = x_0. \tag{2.2}$$

Assuming S_t is non-negative, we apply Itô formula to $f(t, S_t) = \log(S_t)$, and we obtain

$$\log(S_t) = \log(S_0) + \int_0^t (\mu \, \mathrm{d}s + \sigma \, \mathrm{d}W_s) + \frac{1}{2}\int_0^t \left(\frac{-1}{S_s^2}\right) \sigma^2 S_s^2 \, \mathrm{d}s.$$

Then, we have that

$$S_t = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

is a solution of Equation (2.1).

Proposition 2.2.15 (Integration by parts)**.** *Let X and Y be two Itô processes such that*

$$X_t = X_0 + \int_0^t K_s \, \mathrm{d}s + \int_0^t H_s \, \mathrm{d}W_s$$

and

$$Y_t = Y_0 + \int_0^t \widetilde{K}_s \, \mathrm{d}s + \int_0^t \widetilde{H}_s \, \mathrm{d}W_s.$$

Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \; Y_s + \int_0^t Y_s \, \mathrm{d}X_s + [X, Y]_t$$

where

$$[X, Y]_t := \int_0^t H_s \widetilde{H}_s \, \mathrm{d}s.$$

Proof. See proof in Lamberton and Lapeyre (1996), Proposition 3.4.12. □

In this thesis, we are going to model the evolution of the asset price and its volatility with different random processes. Therefore, the multi-dimensional case is useful.

Definition 2.2.16 (Multi-dimensional Brownian motion)**.** *We call a standard n-dimensional Brownian motion an $\mathbb{R}^n$-valued process $(W_t = (W_t^1, \cdots, W_t^n))_{t\geq 0}$ adapted to $\mathbb{F}$, where all the $(W_t^i)_{t\geq 0}$ are independent standard Brownian motions.*

Then, we can define an Itô process in a multi-dimensional framework.

Definition 2.2.17 (Multi-dimensional Itô process). *We call* $(X_t)_{0\leq t\leq T}$ *with* $X = (X^1, \cdots, X^n)$ *a multi-dimensional Itô process if*

$$X_t^i = X_0^i + \int_0^t K_s^i \,\mathrm{d}s + \sum_{j=1}^{p} \int_0^t H_s^{i,j} \,\mathrm{d}W_s^j$$

where for all i, j*:*

1. K_t^i *and all the processes* $(H_t^{i,j})$ *are adapted to the natural filtration.*
2. $\int_0^T |K_s^i| \,\mathrm{d}s < \infty$ $\mathbb{P}$ *a.s.*
3. $\int_0^T (H_s^{i,j})^2 \,\mathrm{d}s < \infty$ $\mathbb{P}$ *a.s.*

There is a very useful analogue of Itô formula in many dimensions.

Theorem 2.2.18 (Multi-dimensional Itô formula). *Let* $(X_t^1, \cdots, X_t^n)$ *be an Itô process*

$$X_t^i = X_0^i + \int_0^t K_s^i \,\mathrm{d}s + \sum_{j=1}^{p} \int_0^t H_s^{i,j} \,\mathrm{d}W_s^j$$

and $f(t, x_1, \cdots, x_n) \in \mathcal{C}^{1,2,\cdots,2}$*, then*

$$\begin{aligned} f(t, X_t^1, \cdots, X_t^n) &= f(0, X_0^1, \cdots, X_0^n) + \int_0^t \partial_s f(s, X_s^1, \cdots, X_s^n) \,\mathrm{d}s \\ &+ \sum_{i=1}^{n} \int_0^t \partial_{x_i} f(s, X_s^1, \cdots, X_s^n) \,\mathrm{d}X_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^{n} \partial^2_{x_i,x_j}(s, X_s^1, \cdots, X_s^n) \,\mathrm{d}\left[X^i, X^j\right]_s \end{aligned}$$

with

1. $\mathrm{d}X_s^i = K_s^i \,\mathrm{d}s + \sum_{j=1}^{p} H_s^{i,j} \,\mathrm{d}W_s^j$*,*
2. $\mathrm{d}\left[X^i, X^j\right]_s = \sum_{m=1}^{p} H_s^{i,m} H_s^{j,m} \,\mathrm{d}s.$

Proof. See Sanz-Solé (2012), Theorem 3.2. □

2.2.5 Stochastic differential equations

The evolution of different phenomena is explained by differential equations, the most typical example is the equations of motion. Finance is no exception, but, in particular, to price financial derivatives, its stochastic counterpart is used: the Stochastic Differential Equations (SDE). We will consider the $\mathbb{R}^n$ case. We consider the following type of equations:

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}W_t \tag{2.3}$$

with the initial condition $X_0 = x_0$, where $x_0 \in \mathbb{R}^n$, $W = (W_t^1, \cdots, W_t^p)$ is $\mathbb{R}^p$-valued Brownian motion, the function $a : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a n-dimensional function, and the function $b : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a matrix of dimension $n \times p$.

The existence and uniqueness of a solution of a stochastic differential equation is given by the following theorem.

Theorem 2.2.19. *[Existence and uniqueness of SDE] Consider a SDE such* (2.3). *Suppose that:*

1. *The coefficient functions a(t, x), b(t, x) are Lipschitz with respect to x and uniformly continuous with respect to t.*
2. *A constant C>0 exists such that functions a(t, x), b(t, x) satisfy*

$$|a(t, x)| + |b(t, x)| \leq C\,(1 + |x|)\,.$$

Then, there exist a unique solution to stochastic differential equation.

Proof. The proof for one dimension is in Lamberton and Lapeyre (1996), Theorem 3.5.3. The multi-dimensional case is in Øksendal (2003), Theorem 5.2.1. □

It is important that the solutions of stochastic differential equations satisfy the Markov property. The intuitive meaning of the Markov property is that the future behavior of a process $(X_t)_{t\geq 0}$ after t does only depend on X_t. In other words, the behavior does not depend on the history of the process before t.

Definition 2.2.20 (Markov Property)**.** *A stochastic process (X_t) satisfies the Markov property if, for any bounded measurable function $f : \mathbb{R} \mapsto \mathbb{R}$ and any $s < t$,*

$$\mathbb{E}\left[f(X_t)|\mathcal{F}_s\right] = \mathbb{E}\left[f(X_t)|X_s\right]. \tag{2.4}$$

Theorem 2.2.21. *Consider a SDE such* (2.3) *where the coefficients fulfil Theorem 2.2.19. Then, the solution of X_t has the Markov property.*

In Theorem 2.2.11, it is seen that under certain conditions of a simple function, we can obtain a $\mathbb{F}$-martingale. The Brownian martingale representation theorem tell us that all $\mathbb{F}$-martingales can be represented as Itô integrals.

Definition 2.2.22. *A $\mathbb{F}$-martingale $(M_t)_{t\geq 0}$ is said to be square-integrable if* $\mathbb{E}\left[|M_t|^2\right] < \infty$ *for each* $t > 0$.

Theorem 2.2.23 (Brownian Martingale Representation)**.** *Let $\mathbb{F}$ the natural filtration of a Brownian motion $(W_t)_{t\geq 0}$. Let $(M_t)_{t\geq 0}$ be a square-integrable martingale with respect the filtration $\mathbb{F}$. Then, there exist an $\mathbb{F}$-predictable process $(\theta)_{t\geq 0}$ such that with $\mathbb{P}$-probability one*

$$M_t = M_0 + \int_0^t \theta_s \, \mathrm{d}W_s. \tag{2.5}$$

Proof. See proof in Etheridge (2011), Theorem 4.6.2. □

In order to price and hedge derivatives, we introduce the Girsanov Theorem. This theorem allow us to change the probability measure. The process of changing the martingale measure can be viewed as a reweighting of the probabilities under our original measure.

Recall that two probability measures are equivalent if they have the same sets of probability zero.

Theorem 2.2.24 (Girsanov's Theorem)**.** *Let W be a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Suppose that θ is an adapted $\mathbb{R}^d$-valued process such that*

$$\mathbb{E}\left[\exp\left(\int_0^T \theta_t \, \mathrm{d}W_t - \frac{1}{2}\int_0^T \theta_t^2 \, \mathrm{d}t\right)\right] = 1. \tag{2.6}$$

Define a probability measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ by means of the Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^T \theta_t \, \mathrm{d}W_t - \frac{1}{2}\int_0^T \theta_t^2 \, \mathrm{d}t\right) \tag{2.7}$$

$\mathbb{P}$-a.s. Then the process $\widetilde{W}$ given by

$$\widetilde{W}_t = W_t - \int_0^t \theta_u \, \mathrm{d}u \tag{2.8}$$

for all $t \in [0, T]$, follows a standard d-dimensional Brownian motion on the space $(\Omega, \mathbb{F}, \widetilde{\mathbb{P}})$

Proof. The proof of the one dimensional case can be found in Protter (2004), Chapter II, Theorem 42. The multi-dimensional case is in Øksendal (2003), Theorem 8.6.4. □

An important statement that we are going to use throughout the thesis is the Feynman-Kac formula. This formula allows us to express the price of an option as a solution of partial differential equation. This is a consequence of the deep connection between stochastic differential equations and certain parabolic differential equations.

Theorem 2.2.25 (The Feynman-Kac Formula)**.** *Consider the partial differential equation*

$$\partial_t u(x,t) + \mu(x,t)\partial_x u(x,t) + \frac{1}{2}\sigma^2(x,t)\partial_x^2 u(x,t) - V(x,t)u(x,t) + f(x,t) = 0,$$

defined for all $x \in \mathbb{R}$ and $t \in [0,T]$, subject to the terminal condition

$$u(x,T) = \varphi(x),$$

where $\mu, \sigma, \varphi, V, f$ are known functions, T is a parameter and $u : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ is the unknown solution. Then, the Feynman-Kac formula tells us that the solution can be written as the conditional expectation

$$u(x,t) = \mathbb{E}^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^r V(X_\tau,\tau)d\tau} f(X_r,r)\,\mathrm{d}r + e^{-\int_t^T V(X_\tau,\tau)\,\mathrm{d}\tau}\varphi(X_T)|X_t = x\right]$$

under the probability measure $\mathcal{Q}$ such that X_t is an Itô process driven by the equation

$$\mathrm{d}X_t = \mu(X,t)dt + \sigma(X,t)dW_t^{\mathcal{Q}},$$

where $W_t^{\mathcal{Q}}$ is a Wiener process under $\mathcal{Q}$ and the initial condition for X_t is $X_t = x$.

Proof. See proof in Etheridge (2011), Theorem 4.8.1. □

2.2.6 Volterra Process and Fractional Brownian Motion

Volterra Gaussian processes are a generalisation of the Brownian motion. More precisely, a Volterra Gaussian process is a Wiener integral process with respect to the Brownian motion.

A Gaussian Volterra process $Y = (Y_t, t \geq 0)$ is defined by

$$Y_t = \int_0^t K(t,s)\ \mathrm{d}W_s, \tag{2.9}$$

where $K(t,s)$ is a kernel such that for all $0 < s < t \leq T$

$$\int_s^T K^2(t,s)\,\mathrm{d}t < \infty, \qquad \int_0^t K^2(t,s)\,\mathrm{d}s < \infty, \tag{2.10}$$

and

$$\mathcal{F}_t^Y = \mathcal{F}_t^W. \tag{2.11}$$

Denote the autocovariance function of process Y_t as

$$r(t,s) := \mathbb{E}[Y_t Y_s], \quad t,s \geq 0, \tag{2.12}$$

and the variance function, i.e. the second moment, as

$$r(t) := r(t,t) = \mathbb{E}[Y_t^2], \quad t \geq 0. \tag{2.13}$$

Let us now focus on a very important example of Gaussian Volterra processes: the *fractional Brownian motion* (fBm), a process with covariance function

$$r(t,s) := \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t,s \geq 0, \tag{2.14}$$

and, in particular, with variance

$$r(t) := r(t,t) = t^{2H}, \quad t \geq 0. \tag{2.15}$$

where $H \in (0,1)$ is the Hurst parameter.

One of the first applications of fractional Brownian motion is credited to Hurst (1951) who modelled the long term storage capacity of reservoirs along the Nile river. However, the origen of this concept goes back to Kolmogorov (1940), who studied Wiener spirals and some other curves in Hilbert spaces. Later, Lévy (1953) used the Riemann–Liouville fractional integral to define the process as

$$\widetilde{B}_t^H := \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \, \mathrm{d}W_s,$$

where H may be any positive number. This type of integral turned out to be ill-suited to the applications of fractional Brownian motion because of its over-emphasis on the origin for many applications. Mandelbrot and Van Ness (1968) introduced the Weyl's representation of the fractional Brownian motion

$$B_t^H := \frac{1}{\Gamma(H+1/2)} \left[Z_t + \int_0^t (t-s)^{H-1/2} \, \mathrm{d}W_s \right], \tag{2.16}$$

where

$$Z_t := \int_{-\infty}^0 \left[(t-s)^{H-1/2} - (-s)^{H-1/2}\right] \mathrm{d}W_s$$

and W_t is the standard Wiener process. Nowadays, the most widely used representation of fBm is the one by Molchan and Golosov (1969)

$$B_t^H := \int_0^t K_H(t,s) \, \mathrm{d}W_s, \tag{2.17}$$

where for $H > \frac{1}{2}$

$$K_H(t,s) := C_H \left[s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} \, \mathrm{d}u \right] \tag{2.18}$$

and for $H \leq \frac{1}{2}$

$$K_H(t,s) := C_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{H-\frac{1}{2}} \int_s^t z^{H-\frac{3}{2}} (z-s)^{H-\frac{1}{2}} \, \mathrm{d}z \right] \quad (2.19)$$

with

$$C_H := \sqrt{\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma\left(2 - 2H\right)}}.$$

To understand the connection between Molchan-Golosov and Mandelbrot-Van Ness representations of fBm we refer readers to the paper by Jost (2008).

Despite of the aforementioned arguments, Alòs et al. (2000) proposed to consider a process $\hat{B}_t = \int_0^t (t-s)^{H-1/2} \, \mathrm{d}W_s$ instead of B_t^H in fractional stochastic calculus, since Z_t has absolutely continuous trajectories. Since B_t^H is not a semimartingale, the process $\hat{B}_t = \Gamma(H + 1/2) B_t^H - Z_t$ is also not a semimartingale. Later on, Thao (2006) introduced the so called approximate fractional Brownian motion process as

$$\hat{B}_t^\varepsilon = \int_0^t (t - s + \varepsilon)^{H-1/2} \, \mathrm{d}W_s, \quad H \in (0,1), H \neq \frac{1}{2}, \varepsilon > 0,$$

and showed that for every $\varepsilon > 0$ the process $\hat{B}_t^\varepsilon$ is a semimartingale and it converges to $\hat{B}_t$ in $L^2(\Omega)$ when ε tends to zero. This convergence is uniform with respect to $t \in [0,T]$, see Theorem 2.1 in Thao (2006).

2.2.7 Poisson process

Frequently, Brownian motion is used to model asset behavior. In these cases, the price is a continuous function of time. In the market, we sometimes observe abrupt variations in the price called 'jumps'. To model this kind of phenomena, we must introduce the Poisson process, a discontinuous stochastic process. The Poisson process is a popular process to model the number of times an event occurs in an interval of time or space. For instance:

- The number of patients arriving in an emergency room between an interval of hours.
- The number of photons hitting a detector in a particular time interval.

Definition 2.2.26. *Let $(T_i)_{i \geq 1}$ be a sequence of independent, identically exponentially distributed random variables with parameter λ. We set $\tau_n = \sum_{i=1}^n T_i$. We call Poisson process with intensity λ the process N_t defined by*

$$N_t = \sum_{n \geq 1} \mathbf{1}_{\{\tau_n \leq t\}} = \sum_{n \geq 1} n \mathbf{1}_{\{\tau_n \leq t < \tau_{n+1}\}}.$$

The distribution of the increments of Poisson process is defined in the following proposition.

Proposition 2.2.27. *If $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ then, for any $t>0$, the random variable N_t follows a Poisson law with parameter λ, that is,*

$$P\left(N_t = n\right) = \exp\left(-\lambda t\right) \frac{\left(\lambda t\right)^n}{n!}.$$

In particular, we have

$$\begin{aligned} \mathbb{E}\left[N_t\right] &= \lambda t, \\ Var\left[N_t\right] &= \lambda t. \end{aligned}$$

Moreover, for $s>0$,

$$\mathbb{E}\left[s^{N_i}\right] = \exp\left(\lambda t(s-1)\right).$$

Proof. See proof in Lamberton and Lapeyre (1996), Proposition 7.1.3. □

The Poisson process satisfies the following conditions.

Proposition 2.2.28. *Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity λ. The process $(N_t)_{t\geq 0}$ is a process with independent and stationary increments.*

Proof. See proof in Shreve (2004), Theorem 11.2.3. □

Definition 2.2.29. *If $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ then the compensated Poisson process is*

$$(N_t - \lambda t)_{t\in\mathbb{R}_+}.$$

The compensated Poisson process is a martingale. In particular, it has centered increments

$$\mathbb{E}\left[N_t - \lambda t\right] = 0.$$

CHAPTER 3

Models

Since Bachelier's thesis, various models have been introduced in order to price derivatives and replicate the statistical phenomena observed in the markets. In this thesis, we will use different types of models. In this chapter, we are going to give a brief summary of the basic properties of each model.

3.1 Black-Scholes-Merton formula

In 1973, Black-Scholes, Black and Scholes (1973), and Merton, Merton (1973), proposed a formula to price options. Today it is the most popular formula, becoming the bechmark model. This is not only for its simplicity, but also for the techniques used to derive the formula. Using Partial Differential Equations and non-arbitrage assumptions, a pricing formula can be obtained. However, it can also be derived it from the probabilistic side using Stochastic Differential Equations and Martingale properties.

3.1.1 The Black-Scholes-Merton framework

Consider a fixed time T and $t \in [0, T]$. In a simple financial model, we assume there are two assets: a bank account B_t which is a riskless asset and a risky asset S_t. The price of B_t is given by

$$B_t = e^{rt} \tag{3.1}$$

where $r \geq 0$ is the risk-free interest rate. It is the solution of the Ordinary Differential Equation:

$$\mathrm{d}B_t = rB_t \, \mathrm{d}t. \tag{3.2}$$

The price of the risky asset S_t is given by

$$\mathrm{d}S_t = \mu S_t \, \mathrm{d}t + \sigma S_t \, \mathrm{d}W_t \tag{3.3}$$

where μ is the growth rate of the asset, $\sigma > 0$ is the volatility and W_t is a Brownian motion.

As we have seen, using Itô's formula, the stochastic differential equation (3.3) has a closed-form solution given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right) \tag{3.4}$$

where $S_0 \geq 0$ is the current price.

We are interested in find the price today of a contingency at time T, C_T. In order to find this price, we seek a self-financing portfolio whose value at time T is exactly C_T. In the absence of arbitrage, the value of the claim must be the same as the cost of constructing the replication portfolio.

An investment strategy can be performed based on the amounts purchased in the bank account and the risky asset. It can be represented as a random vector $\vartheta = (\psi, \phi)$ with values in $\mathbb{R}^2$, adapted to the natural filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. The value of the portfolio at time t is given by

$$V_t(\vartheta) = \psi_t B_t + \phi_t S_t, \qquad V_0 = v_0.$$

A self-financing strategy is defined by an adapted process ϑ where

1. $\int_0^T |\psi_t|\,\mathrm{d}t < \infty$, a.s.
2. $\int_0^T \phi_t^2\,\mathrm{d}t < \infty$, a.s.
3. $V_t = V_0 + \int_0^t dV_s$ a.s., for all $t \in [0, T]$.

The discounted price of the risky asset it is denoted by $\widetilde{S}_t = \exp(-rt)\,S_t$.

Proposition 3.1.1. *Let ϑ an adapted process and $\widetilde{V}_t(\vartheta) = \exp(-rt)\,V_t(\vartheta)$. The process ϑ defines a self-financing strategy if*

$$\mathrm{d}\widetilde{V}_t(\vartheta) = \phi_t\,\mathrm{d}\widetilde{S}_t \tag{3.5}$$

a.s. for all $t \in [0, T]$.

Proof. See proof in Lamberton and Lapeyre (1996), Proposition 4.1.2. □

Note that the price of a portfolio is given by

$$\widetilde{V}_t(\vartheta) = V_0 + \int_0^t \phi_t\,\mathrm{d}\widetilde{S}_t. \tag{3.6}$$

The expected value of the portfolio is

$$\mathbb{E}\left[\widetilde{V}_t(\vartheta)\right] = V_0 + \mathbb{E}\left[\int_0^t \phi_t\,\mathrm{d}\widetilde{S}_t\right]. \tag{3.7}$$

If we were able to find a probability measure $\mathcal{Q}$ under which the discount asset price $\widetilde{S}_t$ is martingale, it will be obtained that

$$\begin{aligned}\mathbb{E}^{\mathcal{Q}}\left[V_t(\vartheta)\right] &= V_0 + \mathbb{E}^{\mathcal{Q}}\left[\int_0^t \phi_t \,\mathrm{d}\widetilde{S}_t\right] \\ &= V_0.\end{aligned}$$

Then, for finding the price of a discounted contingency, it is only necessary to calculate $\mathbb{E}^{\mathcal{Q}}\left[\widetilde{C}_T\right] = V_0$.

Lemma 3.1.2. *There is a probability measure $\mathcal{Q}$, equivalent to $\mathcal{P}$, under which the discounted asset price $(\widetilde{S}_t)_{t\geq 0}$ is a martingale. Moreover, the Radon-Nikodyn derivative of $\mathcal{Q}$ with respect to $\mathcal{P}$ is given by*

$$L_t = \frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}\mathcal{P}} = \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right),$$

where $\theta = (\mu - r)/\sigma$. Then, the discounted asset price is given by

$$\widetilde{S}_t = S_0 \exp\left(\sigma X_t - \frac{1}{2}\sigma^2 t\right)$$

where $X_t = W_t + \frac{1}{\sigma}\left(\mu - r\right)t$.

Proof. Recall (3.3), so

$$\mathrm{d}\widetilde{S}_t = \widetilde{S}_t\left(-r\,\mathrm{d}t + \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right).$$

Therefore

$$\mathrm{d}\widetilde{S}_t = \widetilde{S}_t \sigma\,\mathrm{d}X_t.$$

Using the Girsanov Theorem, Theorem 2.2.24, $(X_t)_{t\geq 0}$ is a $\mathcal{Q}$-Brownian motion and $(\widetilde{S}_t)_{t\geq 0}$ is a $\mathcal{Q}$-martingale. Moreover,

$$\widetilde{S}_t = S_0 \exp\left(\sigma X_t - \frac{1}{2}\sigma^2 t\right).$$

□

Remark 3.1.3. *We call the quantity $\frac{\mu - r}{\sigma}$ the market price of the asset risk.*

A Fundamental Theorem of Asset Pricing can be developed in the Black-Scholes-Merton framework.

Theorem 3.1.4 (Fundamental Theorem of Asset Pricing)**.** *Let $\mathcal{Q}$ be the measure under which the discount asset price is a martingale. Suppose that a claim at time T is given by a non-negative random variable C_T adapted to $\mathcal{F}_T$. If $\mathbb{E}^{\mathcal{Q}}[C_T^2] < \infty$, then the claim is replicable and the value at time t of any replicating portfolio is given by*

$$V_t = \mathbb{E}^{\mathcal{Q}}\left[\exp\left(-r(T-t)\right) C_T | \mathcal{F}_t\right].$$

In particular, the fair price at time zero for the option is

$$V_0 = \mathbb{E}^{\mathcal{Q}}\left[\exp\left(-rT\right) C_T\right] = \mathbb{E}^{\mathcal{Q}}\left[\widetilde{C}_T\right].$$

Proof. See Etheridge (2011), Theorem 5.1.5. □

3.1.2 The Black-Scholes-Merton price

We have introduced a framework to price options under a risk-neutral measure. In the case of European options, the price of the option can be obtained explicitly.

Proposition 3.1.5. *Let us denote the value of an option at time t as V_t. A European option whose payment at maturity is $C_T = f(S_T)$ has a value that is given by*

$$V_t = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma y\sqrt{T-t}\right)\right) \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y.$$

Proof. From Theorem 3.1.4, we know that the value at time t is

$$\mathbb{E}_t^{\mathcal{Q}}\left[\exp\left(-r(T-t)\right) f(S_T)\right], \tag{3.8}$$

where $\mathcal{Q}$ is the martingale measure. Under this measure

$$X_t = W_t + \frac{(\mu - r)\,t}{\sigma}$$

is a Brownian motion and

$$\mathrm{d}\widetilde{S}_t = \sigma \widetilde{S}_t \,\mathrm{d}X_t.$$

Solving this equation,

$$\widetilde{S}_T = \widetilde{S}_t \exp\left(\sigma\left(X_t - X_t\right) - \frac{1}{2}\sigma^2\left(T-t\right)\right).$$

We can substitute into 3.8 to obtain

$$V_t = \mathbb{E}_t^{\mathcal{Q}}\left[e^{-r(T-t)} f\left(S_t e^{r(T-t)} \exp\left(\sigma(X_T - X_t) - \frac{1}{2}\sigma^2(T-t)\right)\right)\right].$$

Since under $\mathcal{Q}$, conditional on $\mathcal{F}_t$, $X_T - X_t$ is a normally distributed random variable with mean zero and variance $(T-t)$, we can evaluate this as

$$\begin{aligned} V_t &= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} e^{-r(T-t)} f\left(S_t e^{r(T-t)} \exp\left(\sigma z - \frac{1}{2}\sigma^2 (T-t)\right)\right) \exp\left(-\frac{z^2}{2(T-t)}\right) \mathrm{d}z \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma y \sqrt{T-t}\right)\right) \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y. \end{aligned}$$

□

The price of a call option can be calculated explicitly.

Proposition 3.1.6 (European call option)**.** *The price of a European call option with payoff* $f(S_T) = (S_T - z)_+$ *is*

$$C_{BS}(t, T, x, z, r, y) = x\Phi(d_+) - ze^{-r\tau}\Phi(d_-)$$

where x *is the current price,* y *is the constant volatility,* z *is the strike price,* $\tau = T - t$ *is the time to maturity,* r *is the interest rate, and* Φ *denotes the cumulative distribution function of the standard normal law. The symbols* d_+ *and* d_- *stand for the following functions*

$$d_{\pm} = \frac{\ln(x/z) + (r \pm \frac{y^2}{2})(T-t)}{y\sqrt{T-t}}$$

Proof. Using $f(S_T) = (S_T - z)_+$ in Proposition 3.1.5, it is obtained that

$$C_{BS}(t, T, x, z, r, y) = \mathbb{E}\left[\left(x \exp\left(y\sqrt{\tau}Z - y^2\tau/2\right) - z\exp(-r\tau)\right)_+\right], \tag{3.9}$$

where $Z \sim \mathcal{N}(0,1)$. Finding the values of Z where the integrand is non-zero, it is observed that

$$x \exp\left(y\sqrt{\tau}Z - y^2\tau/2\right) > z\exp(-r\tau)$$

and it is equivalent to have that

$$Z > \frac{\ln(z/x) + (\frac{y^2}{2} - r)\tau}{y\sqrt{\tau}}.$$

The integrand in (3.9) is non-zero if $Z + d_2 \geq 0$. Then

$$\begin{aligned} C_{BS}(t, T, x, z, r, y) &= \mathbb{E}\left[\left(x \exp\left(y\sqrt{\tau}Z - \frac{1}{2}y^2\tau\right) - K\exp(-r\tau)\right)_+\right] \\ &= \mathbb{E}\left[\left(x \exp\left(y\sqrt{\tau}Z - \frac{1}{2}y^2\tau\right) - K\exp(-r\tau)\right) 1_{\{Z+d_2\geq 0\}}\right] \\ &= \int_{-d_2}^{\infty} \left(x \exp\left(y\sqrt{\tau}\omega - \frac{1}{2}y^2\tau\right) - K\exp(-r\tau)\right) \exp\left(-\frac{\omega^2}{2}\right) \mathrm{d}\omega \\ &= \int_{-\infty}^{d_2} \left(x \exp\left(-y\sqrt{\tau}\omega - \frac{1}{2}y^2\tau\right) - K\exp(-r\tau)\right) \exp\left(-\frac{\omega^2}{2}\right) \mathrm{d}\omega \\ &= x\int_{-\infty}^{d_2} \exp\left(-y\sqrt{\tau}\omega - \frac{1}{2}y^2\tau\right) \exp\left(-\frac{\omega^2}{2}\right) \mathrm{d}\omega - K\exp(-r\tau)\,\Phi(d_2). \end{aligned}$$

Substituting $w = \omega + y\sqrt{\tau}$ in the first integral in the last line, the price of a European call option is obtained . □

Remark 3.1.7. *The European put option price can be found analogously.*

Remark 3.1.8. *By convenience, we will focus on a European call option, but all the formulae of the thesis can be applied to European put options.*

Notice that the price of a European call options depends just on one unknown parameter, the volatility σ. In the Black-Scholes-Merton framework, the volatility is assumed to be constant. The question is how to estimate the volatility. If we focus on the data, the volatility estimate is different depending on the length of the sample, or, for example, if we use a weighted average estimation. To adjust the model to market prices, the most common approach is to estimate the volatility from quoted market prices. To do this, one can estimate what is the volatility that adjusts the Black-Scholes-Merton price formula with the market quote. This is called the implied volatility.

Definition 3.1.9. *Given a market quoted European option C with strike K and maturity time T, the implied volatility, $IV = \sigma(K,T)$, is the volatility we use in the Black-Scholes-Merton formula to obtain the market price of the option. That is*

$$C_{Market} = C_{BS}\left(0, T, S_0, K, r, \sigma(K,T)\right) \tag{3.10}$$

where S_0 is the asset price at the trading time.

There is a unique solution for the implied volatility. The Black-Scholes-Merton formula is monotonically increasing in σ, with lower and upper limits depending on if it is a call option or a put option. Therefore, by the inverse function theorem, for every price there is an implied volatility.

The empirical implied volatility is not constant over time or strike. This is due to the shape of the volatility surface. In particular, the volatility skewness and the volatility smile of the surface.

Despite the popularity of the model, the dynamics of the model does not fit the market data or the statistical properties of financial time series. Some of the drawbacks are

- The assumption of constant volatility is not able to reproduce the volatility surface. It is possible to observe a volatility smile or skewness.
- The log-returns are normally distributed whereas the empirical time series shows negative skewness and excess of kurtosis.
- The normal distribution is not able to produce extreme events, i.e. fat tails.
- Empirical observation show an inverse relationship between the level of the stock and the volatility that the model does not take into account.

- The simulated path are continuous and have no jumps.
- The model does not produce volatility clustering.

The objective of this thesis is to express the observed price of a European call option as the Black-Scholes-Merton price plus corrective terms. For that reason, it is convenient for us to describe the following adaptations of the Black-Scholes-Merton formula.

- Black-Scholes-Merton formula respect to the variance.

$$C_{\widehat{BS}}(t,T,x,z,r,y) = x\Phi(\widehat{d}_+) - ze^{-r\tau}\Phi(\widehat{d}_-)$$

  where, in this case, y is the constant variance. The symbols $\widehat{d}_+$ and $\widehat{d}_-$ stand for the following functions

$$\widehat{d}_\pm = \frac{\ln(x/z) + (r \pm \frac{y}{2})(T-t)}{\sqrt{y(T-t)}}.$$

- Black-Scholes-Merton formula respect to the log-price.

$$C_{\overline{BS}}(t,T,x,z,r,y) = e^x\Phi(\overline{d}_+) - ze^{-r\tau}\Phi(\overline{d}_-)$$

  where x is the log-price. The symbols $\overline{d}_+$ and $\overline{d}_-$ stand for the following functions

$$\overline{d}_\pm = \frac{x - \ln(z) + (r \pm \frac{y^2}{2})(T-t)}{y\sqrt{T-t}}.$$

- Black-Scholes-Merton formula respect to the log-price and the variance.

$$C_{\widetilde{BS}}(t,T,x,z,r,y) = e^x\Phi(\widetilde{d}_+) - ze^{-r\tau}\Phi(\widetilde{d}_-)$$

  where x is the log-price and y is the constant variance. The symbols $\widetilde{d}_+$ and $\widetilde{d}_-$ stand for the following functions

$$\widetilde{d}_\pm = \frac{x - \ln(z) + (r \pm \frac{y}{2})(T-t)}{\sqrt{y(T-t)}}.$$

The price is the same in all the formulae, i.e. $C_{BS} = C_{\widehat{BS}} = C_{\overline{BS}} = C_{\widetilde{BS}}$. The only difference is the price dependency on the variables.

3.1.3 Upper bound for the Black-Scholes-Merton derivatives

For the methodology we want to develop, it will be convenient to find an upper-bound estimation of the Black-Scholes-Merton derivatives with respect to the price or the log-price.

In particular, this lemma is a combination of the lemmas used in Alòs (2012) and Merino and Vives (2017).

Lemma 3.1.10. *The following upper-bounds are valid depending on the version of Black-Scholes-Merton formula is used:*

(i) For any $n \geq 2$, and for any positive quantities x, y, p and q, we have

$$|x^p (\ln x)^q x^n \partial_x^n C_{BS}(t, T, x, z, r, y)| \leq \frac{C}{(y\sqrt{\tau})^{n-1}}$$

where C is a constant that depends on p, q and n.

(ii) For any $n \geq 0$ and for any positive quantities x, we have

$$\left|\partial_x^n \left(\partial_x^2 - \partial_x\right) C_{\overline{BS}}(t, T, x, z, r, y)\right| \leq \frac{C}{(y\sqrt{\tau})^{n+1}}.$$

where C is a constant that depends on x.

Proof. To prove this lemma we are going to use the functional relationship of the variables.

(i) We have that

$$\partial_x^2 C_{BS}(t, T, x, z, r, y) = \frac{\phi(d_+)}{xy\sqrt{\tau}},$$

therefore

$$x^2 \partial_x^2 C_{BS}(t, T, x, z, r, y) = \frac{x\phi(d_+)}{y\sqrt{\tau}}.$$

Note that for $n \geq 2$, it can be obtained the following expression

$$\partial_x^n C_{BS}(t, T, x, z, r, y) = \frac{\phi(d_+)}{(xy\sqrt{\tau})^{n-1}} P_{n-2}(d_+, y\sqrt{\tau})$$

where P_{n-2} is a polynomial of order $n-2$. Then,

$$x^n \partial_x^n C_{BS}(t, T, x, z, r, y) = \frac{x\phi(d_+)}{(y\sqrt{\tau})^{n-1}} P_{n-2}(d_+, y\sqrt{\tau}).$$

The exponential decreasing on x of the Gaussian kernel compensates the possible increasing of x and $\ln x$ and find the upper bound

$$|x^p (\ln x)^q x^n \partial_x^n C_{BS}(t, T, x, z, r, y)| \leq \frac{C}{(y\sqrt{\tau})^{n-1}}$$

where C is a constant that depends on p, q and n.

(ii) Similarly, we have that

$$\left(\partial_x^2 - \partial_x\right) C_{\overline{BS}}\left(t, T, x, z, r, y\right) = \frac{e^x \phi(\bar{d}_+)}{y\sqrt{\tau}}.$$

In particular,

$$\partial_x^n \left(\partial_x^2 - \partial_x\right) C_{\overline{BS}}\left(t, T, x, z, r, y\right) = \frac{e^x \phi(\bar{d}_+)}{(y\sqrt{\tau})^{n+1}} P_n(\bar{d}_+, y\sqrt{\tau}).$$

As it happens in the former case, the gaussian kernel controls the possible increase of x, having

$$\left|\partial_x^n \left(\partial_x^2 - \partial_x\right) C_{\overline{BS}}\left(t, T, x, z, r, y\right)\right| \leq \frac{C}{(y\sqrt{\tau})^{n+1}}.$$

□

3.2 Spot-dependent volatility models

Despite the success of the Black-Scholes-Merton formula, one of the first alternatives was a model where the volatility was dependent on the level of the asset price.

Let $S = \{S_t, t \in [0, T]\}$ be a positive price process under a market chosen risk neutral measure that follows the model

$$\mathrm{d}S_t = rS_t \,\mathrm{d}t + \upsilon(S_t) S_t \,\mathrm{d}W_t \tag{3.11}$$

where W is a standard Brownian motion, r is the constant interest rate and $\upsilon : [0, \infty) \to [0, \infty)$ is a function of $C^2([0, \infty))$ such that $\upsilon(S_t)$ is a square-integrable process adapted to the filtration generated by W. The process $\upsilon(S_t)$ satisfies enough conditions to ensure the existence and uniqueness of a solution of (3.11).

The Feynman-Kac equation for this model satisfies $\mathcal{L}_y C_{SD}\left(t, x, r, y\right) = 0$ for any t, x, y, and r where

$$\mathcal{L}_y C_{SD}\left(t, x, r, y\right) := \partial_t + \frac{1}{2} y^2 \partial_x^2 + rx\partial_x - r. \tag{3.12}$$

Changing variables, it can be seen that $\widehat{\mathcal{L}}_y C_{\widehat{SD}}(t, x, r, y) = 0$ for any t, x, y, where

$$\mathcal{L}_y C_{\widehat{SD}}\left(t, x, r, y\right) := \partial_t + \frac{1}{2} y \partial_x^2 + rx\partial_x - r. \tag{3.13}$$

Remark 3.2.1. *Note that in the case of $\mathcal{L}_y C_{SD}$, we have that $y = \upsilon(S_t) S_t$. But, in the case of $\mathcal{L}_y C_{\widehat{SD}}$, we have $y = \upsilon^2(S_t) S_t^2$.*

3.2.1 CEV

In 1975, John Cox, Cox (1975), developed the first alternative model to the Black-Scholes-Merton price. Instead of using constant volatility, a model where the volatility is a direct inverse function of the asset price was proposed. The model is able to capture the leverage effect: when the asset price declines, the asset volatility increases. The Constant Elasticity of Variance, CEV, model, is a diffusion process that under a market chosen risk neutral measure follows

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t^{\beta}\,\mathrm{d}W_t. \tag{3.14}$$

The parameter $\beta \geq 0$ is called the elasticity of the volatility and $\sigma \geq 0$ is a scale parameter. This model is a type of spot dependent volatility model, when the volatility structure is $\upsilon(S_t) := \sigma S_t^{\beta-1}$.

In this model, the returns are not necessarily normally distributed. In fact, the distribution of the model depends on the possible values of β. For example

- when $\beta = 1$, the model reduces to the Osborne-Samuelson model.
- when $\beta = 0$, the model reduces to Bachelier model.
- when $\beta = \frac{1}{2}$, the model reduces to Cox-Ingersoll-Ross model.

There is a closed-form formula for European Plain Vanilla options, see Cox (1975) and Emanuel and MacBeth (1982). An Approximation of the implied volatility is given in Hagan and Woodward (1999).

One difference between the Black-Scholes-Merton model and the CEV model is that the latter is capable of exhibiting a skew in implied volatility. The skew is controlled by the parameter β.

3.3 Stochastic volatility models

Stochastic volatility models are a natural extension of the Black-Scholes-Merton model. In this class of models, not only the price is a random variable, also the volatility. Several models have been proposed, but the most used in the industry are the Heston model, Heston (1993), and the SABR model, Hagan et al. (2002). Although the latest trend among academics is to use rough volatility models.

In these models, we consider $S = \{S(t), t \in [0, T]\}$ be a strictly positive price process that follows

$$\mathrm{d}S_t \;=\; \mu S_t\,\mathrm{d}t + \sigma_t S_t\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right) \tag{3.15}$$

where W and $\widetilde{W}$ are independent Brownian motions, $\rho \in (-1, 1)$ is the correlation between the asset price and the volatility, μ is the growth rate, and σ_t is a square-integrable process adapted to the filtration generated by W. It is assumed that the paths of the process σ are positive P-a.s. We assume on σ sufficient conditions to ensure the existence and uniqueness of the solution of (3.15). The Brownian motions W and $\widetilde{W}$ are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbb{P})$. Denote by $\mathbb{F}^W$ and $\mathbb{F}^{\widetilde{W}}$ the filtrations generated by W and $\widetilde{W}$, respectively. Set $\mathbb{F}_t := \mathbb{F}^W \vee \mathbb{F}^{\widetilde{W}}$. Notice that we we do not assume any concrete volatility structure.

3.3.1 Stochastic volatility framework

Before introducing the different models with stochastic volatility, we need to set up a correct environment. We need to find a measure where the discounted prices are martingale. Therefore, if we price any derivative, under that measure, with time to maturity T as

$$V_t = \mathbb{E}\left[\exp\left(-r(T-t)\right) C_T | \mathcal{F}_t\right],$$

there is no-arbitrage opportunity. Then, V_t is the price of the claim.

To construct the equivalent martingale in a general stochastic volatility model, we rewrite model 3.15 to the following one

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma_t S_t \,\mathrm{d}W_t, \\ \mathrm{d}\sigma_t &= \bullet \,\mathrm{d}t + \circ \,\mathrm{d}\widetilde{Z}_t, \\ \mathrm{d}\widetilde{Z}_t &= \rho \,\mathrm{d}W_t + \sqrt{1-\rho^2} \,\mathrm{d}Z_t \end{aligned}$$

where W and Z are independent Brownian motion and σ is a square-integrable process strictly positive. Note that the parameters of the volatility equation are not specified. Instead there are the symbols $\bullet$ and $\circ$. Here, a parameter or a function is allowed, being only required to fulfil the conditions of existence and uniqueness of a solution.

The presence of two Brownian motions induces the model incompleteness and, as a consequence, there is a non-unique martingale measure. Suppose that the market imposes a martingale measure under which the derivative contracts are correctly priced. As we have seen in Fouque et al. (2000), we can construct an equivalent martingale measure. The first step is to shift the drift of the asset price, as we did in the Black-Scholes-Merton framework, see Proposition 3.1.5, by defining the following Brownian motion

$$\widetilde{W}_t^* = W_t + \int_0^t \frac{\mu - r}{\sigma_s} \,\mathrm{d}s.$$

We have to proceed in a similar way with the volatility process. We define an arbitrary adapted, square-integrable process (γ_t), then, the Brownian motion Z_t can be transformed to the form

$$Z_t^* = Z_t + \int_0^t \gamma_s \,\mathrm{d}s.$$

Let $\mathcal{Q}_\gamma$ be defined by

$$\frac{d\mathcal{Q}_\gamma}{d\mathcal{P}} = \exp\left(-\int_0^t \frac{\mu - r}{\sigma_s}\,\mathrm{d}W_s - \int_0^t \gamma_s\,\mathrm{d}Z_s - \frac{1}{2}\int_0^t \left(\left(\frac{\mu - r}{\sigma_s}\right)^2 + \gamma_s^2\right)\mathrm{d}s\right).$$

By Girsanov's theorem, each measure $\mathcal{Q}_\gamma$ is a risk neutral measure. The process (γ_t) is called the market price of volatility risk. Notice that the process (γ_t) parametrizes the space of risk neutral measures.

Under a risk-neutral measure, the asset price dynamics are as follows:

$$\begin{aligned}
\mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}W_t,\\
\mathrm{d}\sigma_t &= \left(\bullet - \circ\left(\rho\frac{\mu - r}{\sigma_s} + \gamma_t\sqrt{1-\rho^2}\right)\right)\mathrm{d}t + \circ\,\mathrm{d}\widetilde{Z}_t,\\
\mathrm{d}\widetilde{Z}_t &= \rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}Z_t.
\end{aligned}$$

In general, when models are used in practice, the parameters are re-parameterized to absorb the market risk price. As a result, we can use the generic expression for asset prices in a stochastic volatility model as

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma_t S_t\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right) \tag{3.16}$$

being W and $\widetilde{W}$ two independent Brownian motions.
From now on, we will consider that we are always in a risk-neutral measure.

The Feynman-Kac equation for this model satisfies $\mathcal{L}_y C_{SV}(t,x,r,y) = 0$ for any t, x, y, and r where

$$\mathcal{L}_y C_{SV}(t,x,r,y) := \partial_t + \frac{1}{2}y^2\partial_x^2 + rx\partial_x - r. \tag{3.17}$$

Changing variables, it can be seen that $\mathcal{L}_y\widehat{C_{SV}}(t,x,y) = 0$ for any t, x, y, where

$$\mathcal{L}_y C_{\widehat{SV}}(t,x,r,y) := \partial_t + \frac{1}{2}y\partial_x^2 + rx\partial_x - r. \tag{3.18}$$

The Feynman-Kac formulas coincidence with (3.12) and (3.13).

Without loss of generality, we can re-write the model (3.16) using the log-price $X_t = \log(S_t)$ as

$$\mathrm{d}X_t = \left(r - \frac{1}{2}\sigma_t^2\right)\mathrm{d}t + \sigma_t\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right). \tag{3.19}$$

After changing the variable from the price process to the log-price process, we obtain a slightly different Feynman-Kac formula

$$\mathcal{L}_y C_{\overline{SV}}(t,x,r,y) := \partial_t + \frac{1}{2}y^2\partial_x^2 + \left(r - \frac{1}{2}y^2\right)\partial_x - r. \tag{3.20}$$

where $\mathcal{L}_y C_{\overline{SV}}(t,x,r,y) = 0$ for any t, x, r and y. Analogously,

$$\mathcal{L}_y C_{\widetilde{SV}}(t,x,r,y) := \partial_t + \frac{1}{2} y \partial_x^2 + \left(r - \frac{1}{2} y\right) \partial_x - r \tag{3.21}$$

satisfies $\mathcal{L}_y C_{\widetilde{SV}}(t,x,r,y) = 0$ for any t, x, r and y.

Remember that all these different nomenclatures represent the same stochastic volatility model. It is just a way of writing and how the dependence of the variables with respect to the option price is represented. For example, using a log-price version makes it easier to calculate the derivatives with respect to the log-price. This can be interesting and useful.

3.3.2 The Heston Model

The Heston model, Heston (1993), is a stochastic volatility model in which the instantaneous variance follows a mean reverting square root process. The model dynamics for the underlying asset can take into account the asymmetry and the excess of kurtosis observed in financial asset returns as well as fit market prices. It became popular for being able to capture the 'volatility smile' and having semi-analytical closed form.

The price process S satisfies the following system of stochastic differential equations

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sigma_t S_t \left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right), \\ \mathrm{d}\sigma_t^2 &= \kappa\left(\theta - \sigma_t^2\right)\mathrm{d}t + \nu\sqrt{\sigma_t^2}\,\mathrm{d}W_t. \end{aligned} \tag{3.22}$$

Here, the process σ^2 models the stochastic variance of the asset price, $\theta > 0$ is the long-run mean level of the variance, $\kappa > 0$ is the rate at which σ_t^2 reverts to the mean θ, $\nu > 0$ is the volatility of volatility parameter, and r is the interest rate. The initial conditions for the volatility process σ and the price process S will be denoted by $\sigma_0 > 0$ and $s_0 > 0$, respectively. We will assume that the Feller condition $2\kappa\theta \geq \nu^2$, the non-hitting condition, is satisfied. Although the Feller condition is not necessary in the development of the thesis, it is an essential condition to ensure positivity of σ and necessary to the model to have an economic explanation.

3.3.3 The SABR Model

The SABR model, Hagan et al. (2002), is a stochastic volatility model in which the asset price follows the Constant Elasticity Variance model with an exponential stochastic volatility. The model is able to capture the 'volatility smile' in a parsimonious way, obtaining an easy implementation of the dynamics of the implied volatility. The model produces very stable hedges. Note that in this model, the volatility increases exponentially for long term options, so adjusting the entire volatility surface requires time-dependent parameters.

The price process S satisfies the following system of stochastic differential equations

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sigma_t S_t^\beta \left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right), \\ \mathrm{d}\sigma_t &= \xi\sigma_t\,\mathrm{d}W_t. \end{aligned} \tag{3.23}$$

Here, the process σ models the stochastic volatility of the asset price, β is the elasticity parameter, $\xi > 0$ is the volatility of volatility parameter, and r is the interest rate. The initial conditions for the volatility process σ and the price process S will be denoted by $\sigma_0 > 0$ and $s_0 > 0$, respectively.

Note that SABR model is usually written in terms of the Forward price which is equivalent to have $r = 0$.

3.3.4 General Volterra volatility model

The latest trend in volatility modelling is the use of Volterra process. This approach started in Comte and Renault (1998). Since then, different models have been proposed, most of them considering the Fractional Brownian motion. At that moment, the research was focused on the long memory exhibit when $H > 1/2$. In Alòs et al. (2007), we find the first approach considering rough volatility, that is, the case $H < \frac{1}{2}$. In recent years, since the publication of Bayer et al. (2016) and Gatheral et al. (2018), it seems that rough models have better properties to fit the volatility surface and reproduce the statistical properties of volatility time series.

Exponential Volterra volatility model

One of the easiest examples of Volterra volatility model it is the exponential volatility model, that is the Volterra version of the SABR model.

We assume

$$\sigma_t = g(t, Y_t) = \sigma_0 \exp\left\{\xi Y_t - \frac{1}{2}\alpha\xi^2 r(t)\right\}, \quad t \geq 0, \tag{3.24}$$

where $(Y_t, t \geq 0)$ is a Gaussian Volterra process satisfying assumptions (2.10) and (2.11), $r(t)$ is its autocovariance function (2.13), and $\sigma_0 > 0$, $\xi > 0$ and $\alpha \in [0, 1]$ are model parameters.

Exponential fractional volatility model

Let us now consider the *exponential fractional volatility process*

$$\sigma_t := \sigma_0 \exp\left\{\xi B_t^H - \frac{1}{2}\alpha\xi^2 r(t)\right\}, \quad t \geq 0, \tag{3.25}$$

where $(B_t^H, t \geq 0)$ is one of the above mentioned representations of fBm. We are especially interested in the 'rough' models, i.e. when $H < 1/2$. In this case, we call the model (3.16) with volatility process (3.25) the *rough fractional stochastic volatility model* (αRFSV). Note that if $\alpha = 1$, we get the rBergomi model, see Bayer et al. (2016) and Gatheral et al. (2018), if $\alpha = 0$, we get the original exponential fractional volatility model. Values of α between 0 and 1 give us a new degree of freedom that can be viewed as a weight between these two models.

3.3.5 Generic stochastic volatility model

We define a general stochastic volatility model where all the other models can be included. In order to do so, we will define the variance through a function dependent on time, the asset price and volatility process θ: $[0, T] \times \mathbb{R}_+^2 \to \mathbb{R}_+$. We define the price process under a market chosen risk neutral measure as

$$\mathrm{d}S_t = \mu(t, S_t)\,\mathrm{d}t + \theta(t, S_t, \sigma_t)\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right) \tag{3.26}$$

where μ: $[0, T] \times \mathbb{R}_+ \to \mathbb{R}$ is the drift and θ: $[0, T] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is the volatility structure. We assume on μ, θ and σ sufficient conditions to ensure the existence and uniqueness of the solution of (3.26). In this model it is not possible to change the price variable for the log-price variable.

The Feynman-Kac formula for this model satisfies $\mathcal{L}_y C_{GSV}(t, x, r, y) = 0$ where

$$\mathcal{L}_\theta C_{GSV}(t, x, r, \theta(t, x, y)) := \partial_t + \frac{1}{2}\theta^2(t, x, y)\partial_x^2 + \mu(t, x)\partial_x - r \tag{3.27}$$

for any t, x, r and y. Analogously, $\mathcal{L}_y C_{\widetilde{GSV}}(t, x, r, y) = 0$ where

$$\mathcal{L}_y C_{\widetilde{GSV}}(t, x, r, \theta(t, x, y)) := \partial_t + \frac{1}{2}\theta(t, x, y)\partial_x^2 + \mu(t, x)\partial_x - r \tag{3.28}$$

for any t, x, r and y.

3.4 Jump diffusion models

The volatility surface is difficult to calibrate. In the long term, the volatility surface is flat, but in the short term there is not only a smile, but a peak. Traders ask for a premium in the short term in case large movements on the asset price would happen. In order to obtain these prices, the implied volatility has to increase. Even considering stochastic volatility models, these movements can only be obtained by adding a jump process to the model.

Considering a price process, $S = \{S_t, t \in [0, T]\}$, under a market chosen risk neutral measure, we have

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma_t S_t\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right) + S_{t-}\,\mathrm{d}Z_t, \tag{3.29}$$

where

$$Z_t = \int_0^t \int_{\mathbb{R}} (e^y - 1)\widetilde{N}(\,\mathrm{d}s,\,\mathrm{d}y) \tag{3.30}$$

where N and $\tilde{N}$ denote the Poisson measure and the compensated Poisson measure, respectively. We can associate to measure N a compound Poisson process J, independent of W and $\widetilde{W}$, with intensity $\lambda \geq 0$ and jump amplitudes given by random variables Y_i, independent copies of a random variable Y with law given by Q. Recall that this compound Poisson process can be written as

$$J_t := \int_0^t \int_{\mathbb{R}} yN(\,\mathrm{d}s,\,\mathrm{d}y) = \sum_{i=1}^{n_t} Y_i, \tag{3.31}$$

where n_t is a $\lambda-$ Poisson process. Denote by $k := \mathbb{E}_Q(e^Y - 1)$.

As we have been seen previously and, without any loss of generality, it is sometimes convenient to use as underlying process the log-price process $X_t^J = \log S_t, t \in [0,T]$, that satisfies

$$\mathrm{d}X_t^J = \left(r - \lambda k - \frac{1}{2}\sigma_t^2\right)\,\mathrm{d}t + \sigma_t\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right) + \,\mathrm{d}J_t. \tag{3.32}$$

We introduce also the corresponding continuous process,

$$\mathrm{d}X_t^C = \left(r - \lambda k - \frac{1}{2}\sigma_t^2\right)\,\mathrm{d}t + \sigma_t\left(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right). \tag{3.33}$$

The volatility process σ is a square-integrable process assumed to be adapted to the filtration generated by W and J and its trajectories are assumed to be a.s. square integrable, right continuous with left limits and strictly positive a.e.

Under these type of models, we denote by $\mathbb{F}^W$, $\mathbb{F}^{\widetilde{W}}$ and $\mathbb{F}^N$ the filtrations generated by the independent processes W, $\widetilde{W}$ and J respectively. Moreover, we define $\mathbb{F} := \mathbb{F}^W \vee \mathbb{F}^{\widetilde{W}} \vee \mathbb{F}^N$.

The Feynman-Kac formula for this model has taken into account the jump compensation, we have then

$$\mathcal{L}_y C_{\overline{SVJ}}(t,x,r,y) := \partial_t + \frac{1}{2}y^2\partial_x^2 + \left(r - \lambda k - \frac{1}{2}y^2\right)\partial_x - r \tag{3.34}$$

that satisfies $\mathcal{L}_y C_{\overline{SVJ}}(t,x,r,y) = 0$. Analogously,

$$\mathcal{L}_y C_{\widetilde{SVJ}}(t,x,r,y) := \partial_t + \frac{1}{2}y\partial_x^2 + \left(r - \lambda k - \frac{1}{2}y\right)\partial_x - r \tag{3.35}$$

satisfies $\mathcal{L}_y C_{\widetilde{SVJ}}(t,x,r,y) = 0$ for any t, x,r and y.

3.4.1 Merton model

Robert C. Merton, Merton (1976), was the first one to propose a jump diffusion process. He proposed a mixture of continuous and jump diffusion processes. His work was motivated because there is an empirical evidence that sometimes we observe changes on the asset prices of extraordinary magnitude that cannot be considered as outliers.

Merton considered the following jump diffusion constant volatility price process

$$\mathrm{d}S_t = (r - \lambda k)\, S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t + S_{t-}(e^Y - 1)\,\mathrm{d}N_t, \tag{3.36}$$

where r and σ are positive constants, N is a standard Poisson process with intensity λ and Y describes jump sizes according to the log-normal distribution with parameters μ_J and σ_J. Note that $k = \exp\{\mu_J + \frac{\sigma_J^2}{2}\} - 1$.

A price of a European call option with time to maturity $T - t$ can be expressed by

$$V(t, S_t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}(\lambda\,(T-t))^n}{n!} BS(t, S_{(n)}, \sigma_{(n)}), \tag{3.37}$$

where

$$\begin{aligned} S_{(n)} &= S_t \exp\left\{n\mu_J + \frac{1}{2} n\sigma_J^2 - \lambda k\tau\right\}, \\ k &= \exp\left\{\mu_J + \frac{1}{2}\sigma_J^2\right\} - 1, \\ \sigma_{(n)} &= \sqrt{\sigma^2 + \frac{n\sigma_J^2}{T-t}}. \end{aligned}$$

This formula was proposed by Matsuda (2004), see equation (27) on page 21, and it is an alternative version of the original Merton (1976) formula.

3.4.2 Bates model

The Bates model, Bates (1996), is the first stochastic volatility jump diffusion model. He combined the ideas of Merton, Merton (1976), and Heston, Heston (1993). Under a risk neutral measure, the model has the following dynamics

$$\begin{aligned} \mathrm{d}S_t &= rS_t \,\mathrm{d}t + \sigma_t S_t \left(\rho \,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_t\right) + S_{t-}\,\mathrm{d}Z_t, \\ \mathrm{d}\sigma_t^2 &= \kappa\left(\theta - \sigma_t^2\right)\,\mathrm{d}t + \nu\sqrt{\sigma_t^2}\,\mathrm{d}W_t. \end{aligned}$$

3.5 General notation

We will use the following notation:

- We use the notation $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ for any $\{\mathcal{F}_t, t \geq 0\}$.
- Sometimes, to simplify notation, we refer to the time to maturity as $\tau_t = T - t$.
- We define the operators
  - $\Lambda := x\partial_x$,
  - $\Gamma := x^2\partial_x^2$,
  - $\Gamma^2 = \Gamma \circ \Gamma$

  when considering models with respect to the price process.
- We define the operators
  - $\widetilde{\Lambda} := \partial_x$,
  - $\widetilde{\Gamma} := (\partial_x^2 - \partial_x)$,
  - $\widetilde{\Gamma}^2 = \widetilde{\Gamma} \circ \widetilde{\Gamma}$

  when considering models with respect to the log-price process.
- Given two continuous semi-martingales X and Y, we define

$$L[X,Y]_t := \mathbb{E}_t\left[\int_t^T \sigma_u \, \mathrm{d}[X,Y]_u\right] \tag{3.38}$$

  and

$$D[X,Y]_t := \mathbb{E}_t\left[\int_t^T \mathrm{d}[X,Y]_u\right]. \tag{3.39}$$

CHAPTER 4

Decomposition formula for stochastic volatility models.

In Hull and White (1987), a decomposition formula for options prices under uncorrelated stochastic volatility was presented, but it was not until Willard (1997) and Fouque et al. (2000) that some techniques for the correlated case were shown. In Alòs (2006), the Hull and White formula was generalised, including the correlated case, by means of the Malliavin calculus. It is natural to use the Malliavin calculation due to the fact that the future average volatility used in the formula is a non-adapted stochastic process. More precisely, the price of a European option can be decomposed as the Black-Scholes-Merton option price plus other correction terms. In Alòs et al. (2007) and Alòs et al. (2008), a general jump diffusion model with no prescribed volatility process is analysed. Some other extensions for Lévy models were presented in Jafari and Vives (2013), see also the survey in Vives (2016).

In the preprint Alòs (2003), a new approach was developed to decompose the Hull and White formula using Itô calculus. The paper went unnoticed, even until today, although it already presented the ideas that were applied later to the particular case of the Heston model in Alòs (2012). The main ideas of the method is to use the adapted projection of the average future variance on the Black-Scholes-Merton formula and then apply the Itô formula to the payoff. As a result, we can decompose the expectation of the option on several terms. The main term of the decomposition is the alternative Black-Scholes-Merton price with respect to the adapted projection of the average future variance and some corrections based on the Black-Scholes-Merton derivatives. These corrections are difficult to calculate numerically. Using the Itô formula recursively, the option price can be approximated by 'freezing' the Black-Scholes-Merton derivatives, as a consequence new terms will emerge that can be considered the error of the approximation.

We follow the ideas developed in Alòs (2012) to extend the decomposition formula to the general case. An interesting point of this methodology is that we can construct a decomposition without having to specify the volatility process explicitly, which enables us to obtain a very flexible decomposition formula. As we mentioned in Chapter 3, we can consider alternative expressions of the Black-Scholes-Merton formula, for example, using the price or the log-price process, the variance or the volatility process. We will see how these small changes in the way we write the formula affect to the decomposition formula.

Some of the results of this chapter are already proved. For example, in Merino and Vives

(2015), a decomposition formula for the Black-Scholes-Merton formula under the model (3.26). Later on, a generalised decomposition formula for a generic functional respect to the log-price process, model (3.19), was proved in Merino et al. (2018).

4.1 Decomposition formula

It is known that for uncorrelated stochastic volatility models, i.e. the price and volatility process are independent, the following formula is valid for a European style call option

$$V_t = E_t[BS(t, S_t, \bar{\sigma}_t)]. \tag{4.1}$$

Here, the symbol $\bar{\sigma}^2(t)$ stands for the average future variance defined by

$$\bar{\sigma}_t^2 := \frac{1}{T-t} \int_t^T \sigma_s^2 \, \mathrm{d}s. \tag{4.2}$$

The equality in (4.1) is called the Hull and White formula (see, e.g., Fouque et al. (2000), page 51). For correlated models, that is, models where $\rho \neq 0$, there is a generalisation of the Hull and White formula (see, for example, formula (2.31) in Fouque et al. (2000)). However, the latter formula is significantly more complicated than the formula in (4.1).

Another way of generalising the Hull and White formula was suggested in Alòs (2006). The idea used in Alòs (2006) is to obtain an expansion of the random variable V_t with the leading term equal to $E_t[BS(t, S_t, \bar{\sigma}_t)]$ and extra terms obtained using Malliavin calculus techniques. In Alòs (2003), and later, in Alòs (2012), a similar formula was found, in which the leading term contains the adapted projection of the average future variance, that is, the quantity

$$v_t^2 := \mathbb{E}_t(\bar{\sigma}_t^2) = \frac{1}{T-t} \int_t^T \mathbb{E}_t[\sigma_s^2] \, \mathrm{d}s, \tag{4.3}$$

instead of the future variance $\bar{\sigma}^2$. The previous remark illustrates an important idea of switching from an anticipative process $t \mapsto \bar{\sigma}_t$ to a non-anticipative (adapted) process $t \mapsto v_t$. This idea was further elaborated in Alòs (2012) in the case of the Heston model, which lead to a Hull and White type formula with the leading term equal to $BS(t, S_t, v_t)$ and two more terms. In Merino and Vives (2015), the latter call price expansion was generalised to any stochastic volatility model.

We consider the adapted projection of the future variance

$$a_t^2 := \int_t^T \mathbb{E}_t[\sigma_u^2] \, \mathrm{d}u, \tag{4.4}$$

then the average future variance can be re-written as

$$v_t^2 = \frac{a_t^2}{T-t}.$$

Let us define

$$M_t = \int_0^T \mathbb{E}_t\left[\sigma_s^2\right] \mathrm{d}s. \tag{4.5}$$

We can re-write the adapted projection of the average future variance as

$$v_t^2 := \mathbb{E}_t(\bar{\sigma}_t^2) = \frac{1}{T-t}\left[M_t - \int_0^t \sigma_s^2 \,\mathrm{d}s\right].$$

Then, it is obtained that

$$\begin{aligned} \mathrm{d}v_t^2 &= \frac{\mathrm{d}t}{(T-t)^2}\left[M_t - \int_0^t \sigma_s^2\,\mathrm{d}s\right] + \frac{1}{T-t}\left[\mathrm{d}M_t - \sigma_t^2\,\mathrm{d}t\right] \\ &= \frac{1}{T-t}\left[\mathrm{d}M_t + \left(v_t^2 - \sigma_t^2\right)\,\mathrm{d}t\right]. \end{aligned}$$

Remark 4.1.1. *In order to generalise the decomposition formula, it is assumed that the maturity, T, the strike, z, and the interest rates, r, are fixed. Therefore, any payoff can be specified by a function $A(t,T,x,z,r,y) = A(t,x,y)$. The function $A(t,x,y)$ belongs to the space $\mathcal{C}^{1,2,2}((0,T)\times(0,\infty)\times(0,\infty))$ if A is one time differentiable with respect to t on $(0,T)$ and two times differentiable with respect to x and y on $(0,\infty)$. It is also assumed that the derivatives are continuous.*

Then we have the following decomposition formula.

Theorem 4.1.2 (Functional decomposition under a SV model)**.** *Let S_t be a price process defined in (3.26), $\{B_t, t\in[0,T]\}$ be a continuous semi-martingale with respect to the filtration $\mathcal{F}_t^W$, let $A(t,x,y)$ be a continuous function on the space $[0,T]\times[0,\infty)\times[0,\infty)$ such that $A\in\mathcal{C}^{1,2,2}((0,T)\times(0,\infty)\times(0,\infty))$. Let us also assume that $\mathcal{L}_y A_{\widehat{GSV}}(t,x,r,y) = 0$ and v_t^2 and M_t are as above. Then, for every $t\in[0,T]$, the following formula holds:*

$$\begin{aligned} \mathbb{E}_t\left[e^{-r(T-t)}A(T,S_T,v_T^2)B_T\right] &= A(t,S_t,v_t^2)B_t \\ &+ \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y A(u,S_u,v_u^2)B_u\frac{v_u^2-\sigma_u^2}{T-u}\,\mathrm{d}u\right] \\ &+ \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}A(u,S_u,v_u^2)\,\mathrm{d}B_u\right] \\ &+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x^2 A(u,S_u,v_u^2)B_u\left(\theta(u,S_u,\sigma_u)^2 - v_u^2 S_u^2\right)\,\mathrm{d}u\right] \\ &+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y^2 A(u,S_u,v_u^2)\frac{B_u}{(T-u)^2}\,\mathrm{d}[M,M]_u\right] \\ &+ \rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{x,y}^2 A(u,S_u,v_u^2)B_u\frac{\theta(u,S_u,\sigma_u)}{T-u}\,\mathrm{d}[W,M]_u\right] \end{aligned}$$

$$+ \quad \rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x A(u,S_u,v_u^2)\theta(u,S_u,\sigma_u)\,\mathrm{d}[W,B]_u\right]$$
$$+ \quad \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y A(u,S_u,v_u^2)\frac{1}{T-u}\,\mathrm{d}[M,B]_u\right].$$

Proof. Applying the Itô formula to the process $e^{-rt}A(t,S_t,v_t^2)B_t$, we obtain:

$$\begin{aligned}
e^{-rT}A(T,S_T,v_T^2)B_T &= e^{-rt}A_{BS}(t,S_t,v_t^2)B_t \\
&- r\int_t^T e^{-ru}A(u,S_u,v_u^2)B_u\,\mathrm{d}u \\
&+ \int_t^T e^{-ru}\partial_u A(u,S_u,v_u^2)B_u\,\mathrm{d}u \\
&+ \int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)B_u\,\mathrm{d}S_u \\
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,v_u^2)B_u\,\mathrm{d}v_u^2 \\
&+ \int_t^T e^{-ru}A(u,S_u,v_u^2)dB_u \\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_x^2 A(u,S_u,v_u^2)B_u\,\mathrm{d}[S,S]_u \\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_y^2 A(u,S_u,v_u^2)B_u\,\mathrm{d}[v^2,v^2]_u \\
&+ \int_t^T e^{-ru}\partial_{x,y}^2 A(u,S_u,v_u^2)B_u\,\mathrm{d}[S,v^2]_u \\
&+ \int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)\,\mathrm{d}[S,B]_u \\
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,v_u^2)\,\mathrm{d}[v^2,B]_u.
\end{aligned}$$

Developing the co-variations, we see

$$\begin{aligned}
e^{-rT}A(T,S_T,v_T^2)B_T &= e^{-rt}A(t,S_t,v_t^2)B_t \\
&- r\int_t^T e^{-ru}A(u,S_u,v_u^2)B_u\,\mathrm{d}u \\
&+ \int_t^T e^{-ru}\partial_u A(u,S_u,v_u^2)B_u\,\mathrm{d}u \\
&+ \int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)B_u\mu(u,S_u)\,\mathrm{d}u \\
&+ \rho\int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)B_u\theta(u,S_u,\sigma_u)\,\mathrm{d}W_u
\end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{1-\rho^2}\int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)B_u\theta(u,S_u,\sigma_u)\,\mathrm{d}\widetilde{W}_u\\
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,v_u^2)\frac{B_u}{T-u}\,\mathrm{d}M_u\\
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,v_u^2)B_u\frac{v_u^2-\sigma_u^2}{T-u}\,\mathrm{d}u\\
&+ \int_t^T e^{-ru}A(u,S_u,v_u^2)\,\mathrm{d}B_u\\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_x^2 A(u,S_u,v_u^2)B_u\theta(u,S_u,\sigma_u)^2\,\mathrm{d}u\\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_y^2 A(u,S_u,v_u^2)\frac{B_u}{(T-u)^2}\,\mathrm{d}[M,M]_u\\
&+ \rho\int_t^T e^{-ru}\partial_{x,y}^2 A(u,S_u,v_u^2)\frac{B_u\theta(u,S_u,\sigma_u)}{T-u}\,\mathrm{d}[W,M]_u\\
&+ \sqrt{1-\rho^2}\int_t^T e^{-ru}\partial_{x,y}^2 A(u,S_u,v_u^2)\frac{B_u\theta(u,S_u,\sigma_u)}{T-u}\,\mathrm{d}[\widetilde{W},M]_u\\
&+ \rho\int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)\theta(u,S_u,\sigma_u)\,\mathrm{d}[W,B]_u\\
&+ \sqrt{1-\rho^2}\int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)\theta(u,S_u,\sigma_u)\,\mathrm{d}[\widetilde{W},B]_u\\
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,v_u^2)\frac{1}{T-u}\,\mathrm{d}[M,B]_u.
\end{aligned}
$$

We add and subtract the term

$$\frac{1}{2}\int_t^T e^{-ru}S_u^2\partial_x^2 A(u,S_u,v_u^2)B_u v_u^2\,\mathrm{d}u$$

having

$$
\begin{aligned}
e^{-rT}A(T,S_T,v_T^2)B_T &= e^{-rt}A(t,S_t,v_t^2)B_t\\
&- r\int_t^T e^{-ru}A(u,S_u,v_u^2)B_u\,\mathrm{d}u\\
&+ \int_t^T e^{-ru}\partial_u A(u,S_u,v_u^2)B_u\,\mathrm{d}u\\
&+ \int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)B_u\mu(u,S_u)\,\mathrm{d}u\\
&+ \rho\int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)B_u\theta(u,S_u,\sigma_u)\,\mathrm{d}W_u\\
&+ \sqrt{1-\rho^2}\int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)B_u\theta(u,S_u,\sigma_u)\,\mathrm{d}\widetilde{W}_u
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,v_u^2)\frac{B_u}{T-u}\,\mathrm{d}M_u \\
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,v_u^2)B_u\frac{v_u^2-\sigma_u^2}{T-u}\,\mathrm{d}u \\
&+ \int_t^T e^{-ru}A(u,S_u,v_u^2)\,\mathrm{d}B_u \\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_x^2 A(u,S_u,v_u^2)B_u\theta(u,S_u,\sigma_u)^2\,\mathrm{d}u \\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_y^2 A(u,S_u,v_u^2)\frac{B_u}{(T-u)^2}\,\mathrm{d}[M,M]_u \\
&+ \rho\int_t^T e^{-ru}\partial_{x,y}^2 A(u,S_u,v_u^2)B_u\frac{\theta(u,S_u,\sigma_u)}{T-u}\,\mathrm{d}[W,M]_u \\
&+ \rho\int_t^T e^{-ru}\partial_x A(u,S_u,v_u^2)\theta(u,S_u,\sigma_u)\,\mathrm{d}[W,B]_u \\
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,v_u^2)\frac{1}{T-u}\,\mathrm{d}[M,B]_u \\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_x^2 A(u,S_u,v_u^2)B_u v_u^2 S_u^2\,\mathrm{d}u \\
&- \frac{1}{2}\int_t^T e^{-ru}\partial_x^2 A(u,S_u,v_u^2)B_u v_u^2 S_u^2\,\mathrm{d}u.
\end{aligned}
$$

Grouping the blue terms, the corresponding Feynman-Kac formula $\mathcal{L}_y A_{\widehat{GSV}}(t,x,r,y)$ is obtained so those terms vanish. Multiplying by e^{-rt} and using conditional expectations, we have that

$$
\begin{aligned}
& e^{-r(T-t)}\mathbb{E}_t\left[A(T,S_T,v_T^2)B_T\right] \\
= \;& A(t,S_t,v_t^2)B_t \\
+ \;& \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y A(u,S_u,v_u^2)B_u\frac{v_u^2-\sigma_u^2}{T-u}\,\mathrm{d}u\right] \\
+ \;& \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}A(u,S_u,v_u^2)\,\mathrm{d}B_u\right] \\
+ \;& \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x^2 A(u,S_u,v_u^2)B_u\left(\theta(u,S_u,\sigma_u)^2-v_u^2S_u^2\right)\,\mathrm{d}u\right] \\
+ \;& \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y^2 A(u,S_u,v_u^2)\frac{B_u}{(T-u)^2}\,\mathrm{d}[M,M]_u\right] \\
+ \;& \rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{x,y}^2 A(u,S_u,v_u^2)B_u\frac{\theta(u,S_u,\sigma_u)}{T-u}\,\mathrm{d}[W,M]_u\right] \\
+ \;& \rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x A(u,S_u,v_u^2)\theta(u,S_u,\sigma_u)\,\mathrm{d}[W,B]_u\right]
\end{aligned}
$$

$$+ \quad \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \partial_y A(u, S_u, v_u^2) \frac{1}{T-u} \, \mathrm{d}[M, B]_u \right].$$

□

The following statement can be derived from Theorem 4.1.2.

Corollary 4.1.3. *Let function A and process B as in Theorem 4.1.2. Suppose that the function A satisfy*

$$\partial_y A(t, x, y) = \frac{1}{2} x^2 \partial_x^2 A(t, x, y)(T - t). \tag{4.6}$$

Let $A_t := A(t, S_t, v_t^2)$ $\forall t \in [0, T]$. Then, for every $t \in [0, T]$, the following formula holds:

$$\begin{aligned}
e^{-r(T-t)} \mathbb{E}_t [A_T B_T] &= A_t B_t \\
&+ \frac{1}{2} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \Gamma A_u B_u \left(v_u^2 - \sigma_u^2 \right) \mathrm{d}u \right] \\
&+ \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} A_u \, \mathrm{d}B_u \right] \\
&+ \frac{1}{2} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \Gamma A_u B_u \left(\frac{\theta(u, S_u, \sigma_u)^2}{S_u^2} - v_u^2 \right) \mathrm{d}u \right] \\
&+ \frac{1}{8} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \Gamma^2 A_u B_u \, \mathrm{d}[M, M]_u \right] \\
&+ \frac{\rho}{2} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \partial_x \Gamma A_u B_u \theta(u, S_u, \sigma_u) \, \mathrm{d}[W, M]_u \right] \\
&+ \rho \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \Lambda A_u \theta(u, S_u, \sigma_u) \, \mathrm{d}[W, B]_u \right] \\
&+ \frac{1}{2} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \Gamma A_u \, \mathrm{d}[M, B]_u \right].
\end{aligned}$$

Proof. Substituting (4.6) in Theorem 4.1.2 and using the definitions of Λ and Γ the proof is straightforward. □

Remark 4.1.4. *Note that $C_{\widehat{SV}}$ satisfies all the conditions of Corollary 4.1.3.*

Remark 4.1.5. *Being $C_{SV}(t, S_t, \sigma_t)$ the price of a call option under the model (3.26), notice that*

$$V_T = C_{SV}(T, S_T, \sigma_T) = C_{\widehat{BS}}(T, S_T, v_T).$$

Then,

$$V_t = C_{SV}(t, S_t, \sigma) = e^{-r(T-t)} \mathbb{E}_t \left[C_{\widehat{BS}}(T, S_T, v_T) \right].$$

Assuming that $A = C_{\widehat{BS}}$ and $B \equiv 1$, the price under the model (3.26) has the decomposition

$$
\begin{aligned}
V_t &= C_{BS}(t, S_t, v_t) \\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{BS}(u, S_u, v_u)\left(v_u^2 - \sigma_u^2\right) \mathrm{d}u\right] \\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{BS}(u, S_u, v_u)\left(\frac{\theta(u, S_u, \sigma_u)^2}{S_u^2} - v_u^2\right) \mathrm{d}u\right] \\
&+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{BS}(u, S_u, v_u)\, \mathrm{d}[M, M]_u\right] \\
&+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{BS}(u, S_u, v_u)\theta(u, S_u, \sigma_u)\, \mathrm{d}[W, M]_u\right]
\end{aligned}
$$

and it is equal to $C_{SV}(t, S_t, \sigma)$. Note that $C_{BS}(t, S_t, v_t) = C_{\widehat{BS}}(t, S_t, v_t)$ and this equivalence is maintained for any derivative with respect to the price.

Example 4.1.6 (Lognormal asset price)**.** *If we assume that the asset prices follow a lognormal distribution as in model (3.16), that is $\theta(u, S_u, \sigma_u) = \sigma_u S_u$, then, the following formula holds:*

$$
\begin{aligned}
V_t &= C_{BS}(t, S_t, v_t) \\
&+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{BS}(u, S_u, v_u)\, \mathrm{d}[M, M]_u\right] \\
&+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{BS}(u, S_u, v_u)\sigma_u\, \mathrm{d}[W, M]_u\right].
\end{aligned}
$$

Example 4.1.7 (CEV asset price)**.** *If we assume a different distribution, for example, that the asset price follows a CEV model for the asset price, that is $\theta(u, S_u, \sigma_u) = \sigma_u S_u^{\beta}$ with $\beta \geq 0$, then, the following formula is obtained*

$$
\begin{aligned}
V_t &= C_{BS}(t, S_t, v_t) \\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{BS}(u, S_u, v_u)\left(S_u^{2(\beta-1)} - 1\right)\sigma_u^2\, \mathrm{d}u\right] \\
&+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{BS}(u, S_u, v_u)\, \mathrm{d}[M, M]_u\right] \\
&+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{BS}(u, S_u, v_u)\sigma_u S_u^{\beta-1}\, \mathrm{d}[W, M]_u\right].
\end{aligned}
$$

In the case that the volatility follows an exponential volatility process, this is known as the SABR model.

We have found a decomposition method for a general case without specifying the structure of the volatility process. The decomposition depends on the Black-Scholes-Merton price and its derivatives with respect to the price. It is well known that it is easier to calculate the derivatives with respect to the log-price. We can try to make a change of variable of the previous formula, but instead, we do an analogous proof of Theorem 4.1.2.

Corollary 4.1.8 (Functional decomposition under SV model with respect the log-price)**.** *Let X_t be a log-price process defined in (3.19), $\{B_t, t \in [0,T]\}$ be a continuous semi-martingale with respect to the filtration $\mathcal{F}_t^W$, let $A(t,x,y)$ be a continuous function on the space $[0,T] \times [0,\infty) \times [0,\infty)$ such that $A \in \mathcal{C}^{1,2,2}((0,T) \times (0,\infty) \times (0,\infty))$. Let us also assume that $\mathcal{L}_y A_{\widetilde{SV}}(t,x,r,y) = 0$ and v_t^2 and M_t are as above. Then, for every $t \in [0,T]$, the following formula holds:*

$$
\begin{aligned}
\mathbb{E}_t\left[e^{-r(T-t)}A(T,X_T,v_T^2)B_T\right] &= A(t,X_t,v_t^2)B_t \\
&+ \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y A(u,X_u,v_u^2)B_u \frac{1}{T-u}\left(v_u^2-\sigma_u^2\right)\mathrm{d}u\right] \\
&+ \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}A(u,X_u,v_u^2)\,\mathrm{d}B_u\right] \\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^2-\partial_x\right)A(u,X_u,v_u^2)B_u\left(\sigma_u^2-v_u^2\right)\mathrm{d}u\right] \\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y^2 A(u,X_u,v_u^2)B_u\frac{1}{(T-u)^2}\,\mathrm{d}[M,M]_u\right] \\
&+ \rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{x,y}^2 A(u,X_u,v_u^2)B_u\frac{\sigma_u}{T-u}\,\mathrm{d}[W,M]_u\right] \\
&+ \rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x A(u,X_u,v_u^2)\sigma_u\,\mathrm{d}[W,B]_u\right] \\
&+ \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y A(u,X_u,v_u^2)\frac{1}{T-u}\,\mathrm{d}[M,B]_u\right].
\end{aligned}
$$

Proof. The proof is analogous to Theorem 4.1.2. It is done by using the corresponding Feynman-Kac formula $\mathcal{L}_y A_{\widetilde{SV}}(t,x,r,y)$. □

Remark 4.1.9. *The drift of the process is always absorbed by the Feynman-Kac formula. In fact, if the process instead of being X_t as defined in (3.19), it is X_t^C defined in (3.33), then, the proof is analogous and the decomposition is the same, but instead of assuming $\mathcal{L}_y A_{\widetilde{SV}}(t,x,r,y) = 0$, it has to be assumed $\mathcal{L}_y A_{\widetilde{SVJ}}(t,x,r,y) = 0$. The difference between the two operators is the drift. Note that the Feynman-Kac is always associated to the price process.*

From the previous theorem, the following corollary can be easily derived.

Corollary 4.1.10. *Let function A and process B as in Theorem 4.1.3. Suppose that the function A satisfy*

$$\partial_y A(t,x,y) = \frac{(T-t)}{2}\left(\partial_x^2 - \partial_x\right)A(t,x,y). \tag{4.7}$$

Let $A_t := A(t, X_t, v_t^2)$ $\forall t \in [0,T]$. *Then, for every* $t \in [0,T]$, *the following formula holds:*

$$\begin{aligned} e^{-r(T-t)} E_t\left[A_T B_T\right] &= A_t B_t \\ &+ \frac{\rho}{2} E_t\left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}\widetilde{\Gamma} A_u B_u \sigma_u d[W,M]_u\right] \\ &+ \frac{1}{8} E_t\left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^2 A_u B_u d[M,M]_u\right] \\ &+ \rho E_t\left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda} A_u \sigma_u d[W,B]_u\right] \\ &+ \frac{1}{2} E_t\left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma} A_u d[M,B]_u\right] \\ &+ E_t\left[\int_t^T e^{-r(u-t)} A_u dB_u\right]. \end{aligned}$$

Proof. Substituting (4.7) in Theorem 4.1.8 and using the definitions of $\widetilde{\Lambda}$ and $\widetilde{\Gamma}$ the proof is straightforward. □

Remark 4.1.11. *Note that* $C_{\widetilde{SV}}$ *satisfies all the conditions of Corollary 4.1.10.*

Remark 4.1.12. *Being* $C_{SV}(t, S_t, \sigma_t)$ *the price of a call option under the model* (3.19), *notice that*

$$V_T = C_{SV}(T, S_T, \sigma_T) = C_{\widetilde{BS}}(T, X_T, v_T).$$

Then,

$$V_t = C_{SV}(t, S_t, \sigma) = e^{-r(T-t)} \mathbb{E}_t\left[C_{\widetilde{BS}}(T, X_T, v_T)\right].$$

Assuming that $A = C_{\widetilde{BS}}$ *and* $B \equiv 1$. *Therefore, the price under the model* (3.26) *has the following decomposition*

$$\begin{aligned} V_t &= C_{BS}(t, S_t, v_t) \\ &+ \frac{1}{8} \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\, \mathrm{d}[M,M]_u\right] \\ &+ \frac{\rho}{2} \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}\widetilde{\Gamma} C_{\overline{BS}}(t, X_t, v_t) \sigma_u\, \mathrm{d}[W,M]_u\right]. \end{aligned}$$

and it is equal to $C_{SV}(t, S_t, \sigma)$. *Note that* $C_{\overline{BS}}(t, X_t, v_t) = C_{\widetilde{BS}}(t, X_t, v_t)$ *and this equivalence it maintained for any derivative with respects to the price.*

Remark 4.1.13. *The price decomposition formula under the model* (3.19) *was proved in Alòs (2012).*

Remark 4.1.14. *There are several differences between Alòs (2012) and Alòs (2003). In Alòs (2012), the decomposition formula is derived with to respect the log-price, the volatility structure is specified using the Heston model and an approximation method is developed.*

Remark 4.1.15. *Notice that the decomposition of Example 4.1.6 and Corollary 4.1.12 are the same. The decomposition is expressed with respect to the log-price or the price. Then, it can be appreciated the following equivalence:*

- $\Lambda C_{BS}(t, S_t, v_t) = \widetilde{\Lambda} C_{\overline{BS}}(t, X_t, v_t)$,
- $\Gamma C_{BS}(t, S_t, v_t) = \widetilde{\Gamma} C_{\overline{BS}}(t, X_t, v_t)$.

One may argue why we need to find a decomposition price with respect to the log-price over the price process. The reasons for this are numerical and practical. On the one hand it is easier to calculate all the derivatives of the decomposition and on the other hand, the equivalence (4.7) is commutative with respect to the log-price, while the equivalence (4.6) is not with respect to the price.

Another argument is why we need to calculate the decomposition with respect to the variance. It seems strange at first sight, especially because the Black-Scholes-Merton formula is expressed with respect to the volatility. If we want to calculate it with respect to v_t,

$$v_t = \sqrt{\frac{1}{T-t}\left[M_t - \int_0^t \sigma_s^2 \, \mathrm{d}s\right]},$$

we need to calculate $\mathrm{d}v_t$. To do that calculation, we need to use the Itô formula,

$$\begin{aligned}\mathrm{d}v_t &= \frac{\left[M_t - \int_0^t \sigma_s^2 \, \mathrm{d}s\right] \, \mathrm{d}t}{2v_t(T-t)^2} + \frac{\mathrm{d}M_t - \sigma_t^2 \, \mathrm{d}t}{2v_t(T-t)} - \frac{\mathrm{d}[M,M]}{4v_t^3(T-t)^2} \\ &= \frac{\mathrm{d}M_t + (v_t^2 - \sigma_t^2) \, \mathrm{d}t}{2v_t(T-t)} - \frac{\mathrm{d}[M,M]}{4v_t^3(T-t)^2}.\end{aligned}$$

In this formula, a new terms appears

$$-\frac{\mathrm{d}[M,M]}{4v_t^3(T-t)^2},$$

which is not possible to compensate in Theorems 4.1.2 or Corollary 4.1.8. Therefore, an extra terms appears.

We have found exact decompositions formulae to express the call option price under a stochastic volatility framework. Despite this, the new conditional expectations are not possible to evaluate analytically. Consequently, we will have to apply a numerical method to approximate those terms. For this reason, the decomposition formula with respect to the volatility process, which has an additional term, may not be a good idea, considering that we will have to approximate an additional term and that might increase the error of the approximation formula.

4.2 Approximation formula

In the previous section, two different decomposition formulas have been obtained; one concerning the price process, model (3.15), and the other concerning the log-price process, model (3.19). Unfortunately, although the decomposition formula is very compact, it is not analytically tractable. Therefore, numerical methods must be applied to calculate it. Following the ideas of Alòs (2012) and Alòs et al. (2015), an approximation formula can be constructed by recursively using Corollary 4.1.3 or 4.1.10 for each term. Using it, by Itô calculus, we can 'freeze' the Black-Scholes-Merton derivative and take the derivative out of the conditional expectation. By doing so, an approximation formula can be obtained, but to measure the error of the approximation, a volatility structure must be defined.

Recall Corollary 4.1.12 (or Example 4.1.6). The following decomposition formula is obtained for any stochastic volatility model where the price process follows a lognormal dynamics:

$$\begin{aligned} V_t &= C_{BS}(t, X_t, v_t) \\ &+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^2 C_{\overline{BS}}(u, X_u, v_u)\,\mathrm{d}[M,M]_u\right] \\ &+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma} C_{\overline{BS}}(u, X_u, v_u)\sigma_u\,\mathrm{d}[W,M]_u\right] \\ &= C_{BS}(t, S_t, v_t) + (I) + (II). \end{aligned}$$

As it has been stated previously, the terms (I) and (II) are not easy to evaluate. If we can find simpler estimates of those terms and control the error, then we will find a valid approximation formula. We show in the following Corollary, how it can be done.

Corollary 4.2.1 (1st Approximation Formula under SV model)**.** *Let X_t be the log-price defined in (3.19). Then, the call option fair value V_t can be expressed using the processes $L[X,Y]_t$ and $D[X,Y]_t$ defined by (3.38) and (3.39), respectively. In particular,*

$$\begin{aligned} V_t &= C_{\overline{BS}}(t, S_t, v_t) \\ &+ \frac{1}{8}\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}[M,M]_u\right] \\ &+ \frac{\rho}{2}\widetilde{\Lambda}\widetilde{\Gamma} C_{\overline{BS}}(t, X_t, v_t)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\sigma_u\,\mathrm{d}[W,M]_u\right] \\ &+ \epsilon_t \end{aligned}$$

where ϵ_t is the error term which is given by

$$\epsilon_t = \frac{\rho^2}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(u, X_u, v_u) L[W,M]_u \sigma_u\,\mathrm{d}[W,M]_u\right]$$

$$
\begin{aligned}
&+ \frac{\rho}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)L[W,M]_u \, \mathrm{d}[M,M]_u\right] \\
&+ \frac{\rho^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma} C_{\overline{BS}}(u,X_u,v_u)\sigma_u \, \mathrm{d}[W,L[W,M]]_u\right] \\
&+ \frac{\rho}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u) \, \mathrm{d}[M,L[W,M]]_u\right] \\
&+ \frac{\rho}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)D[M,M]_u\sigma_u \, \mathrm{d}[W,M]_u\right] \\
&+ \frac{1}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^4 C_{\overline{BS}}(u,X_u,v_u)D[M,M]_u \, \mathrm{d}[M,M]_u\right] \\
&+ \frac{\rho}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)\sigma_u \, \mathrm{d}[W,D[M,M]]_u\right] \\
&+ \frac{1}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u) \, \mathrm{d}[M,D[M,M]]_u\right].
\end{aligned}
$$

that depends on the volatility structure.

Proof. The main idea in the proof is to apply the Corollary 4.1.10 to the terms (I) and (II) in a smart way.

The term (I) is decomposed using Corollary 4.1.10 with $A_t = \widetilde{\Lambda}\widetilde{\Gamma} C_{\overline{BS}}(t,X_t,v_t)$ and $B_t = \frac{\rho}{2}L[W,M]_t$. This gives

$$
\begin{aligned}
(I) \;=\; &\frac{\rho}{2}\widetilde{\Lambda}\widetilde{\Gamma} C_{\overline{BS}}(t,X_t,v_t)L[W,M]_t \qquad (4.8)\\
&+ \frac{\rho^2}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)L[W,M]_u\sigma_u \, \mathrm{d}[W,M]_u\right] \\
&+ \frac{\rho}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)L[W,M]_u \, \mathrm{d}[M,M]_u\right] \\
&+ \frac{\rho^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma} C_{\overline{BS}}(u,X_u,v_u)\sigma_u \, \mathrm{d}[W,L[W,M]]_u\right] \\
&+ \frac{\rho}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u) \, \mathrm{d}[M,L[W,M]]_u\right].
\end{aligned}
$$

Applying the same idea to the term (II) and choosing $A_t = \Gamma^2 C_{\overline{BS}}(t,X_t,v_t)$ and $B_t = \frac{1}{8}D[M,M]_t$, we obtain

$$
\begin{aligned}
(II) \;=\; &\frac{1}{8}\widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)D[M,M]_t \qquad (4.9)\\
&+ \frac{\rho}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)D[M,M]_u\sigma_u \, \mathrm{d}[W,M]_u\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^4 C_{\overline{BS}}(u,X_u,v_u)D[M,M]_u\,\mathrm{d}[M,M]_u\right]\\
&+ \quad \frac{\rho}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)\sigma_u\,\mathrm{d}[W,D[M,M]]_u\right]\\
&+ \quad \frac{1}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)\,\mathrm{d}[M,D[M,M]]_u\right].
\end{aligned}
$$

□

An analogous representation can be found with respect to the price process. In the following chapters, as the derivatives with respect to the log-price process are easier to calculate, we are going to focus on developing formulae with respect to the log-price.

CHAPTER 5

The Heston model

The Heston model, Heston (1993), is an industry-standard model which can explain the volatility surface observed in the market. This is a stochastic volatility model and therefore there are two sources of randomness, in contrast to the Black-Scholes-Merton model, where volatility is constant and there is only one source of randomness. The Heston model stands out from the class of stochastic volatility models mainly for two reasons. Firstly, the process for the volatility is mean-reverting, which is what we observe in the markets. Secondly, there exists a semi-analytical solution for European options. Several methods have been developed to improve the accuracy and the computational time to compute option prices, most of them using Fourier transformations. For example: Heston (1993); Lewis (2000); Kahl and Jäckel (2005); Fang and Oosterlee (2009); Ortiz-Gracia and Oosterlee (2016) among others. For a summary of methods for pricing and calibrating the Heston model, see Mrázek and Pospíšil (2017). We are interested in the approach taken by Alòs (2012) and Alòs et al. (2015).

In Alòs (2012) and Alòs et al. (2015), an approximation formula for option prices was developed based on the decomposition formulae explained in the previous chapter. In Gulisashvili et al. (2020), the approximation formulae are expanded in different orders, improving its numerical efficiency. The new expression is a Taylor-type formula containing an infinite series with stochastic terms. The results obtained are applied to the Heston model and its numerical efficiency is studied. We will focus on a decomposition using the log-price process as was mentioned at the end of the previous chapter.

5.1 Auxiliary lemmas

We will need several results from Alòs et al. (2015) and Gulisashvili et al. (2020) in the proofs. Recall that the following notation will be used in the sequel:

$$\varphi(t) := \int_t^T e^{-\kappa(z-t)}\,\mathrm{d}z = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right).$$

The next statements, Lemmas 5.1.1-5.1.5, are true under the Heston model. Lemmas 5.1.1-5.1.2 are proven in Alòs et al. (2015) and Lemmas 5.1.3-5.1.5 are straightforward.

Lemma 5.1.1. *In the Heston model, the following results are valid*

1. *If $s \geq t$, then*

$$\mathbb{E}_t(\sigma_s^2) = \theta + (\sigma_t^2 - \theta)e^{-\kappa(s-t)} = \sigma_t^2 e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}).$$

In particular, the previous quantity is bounded from below by $\sigma_t^2 \wedge \theta$ and from above by $\sigma_t^2 \vee \theta$.

2. $\mathbb{E}_t\left(\int_t^T \sigma_s^2\, \mathrm{d}s\right) = \theta\,(T-t) + \frac{\sigma_t^2 - \theta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right).$

3. $\mathrm{d}M_t = \nu\sigma_t\varphi(t)\,\mathrm{d}W_t.$

4. $\frac{\rho}{2}L[W,M]_t := \frac{\rho}{2}E_t\left(\int_t^T \sigma_s d\left\langle M, W\right\rangle_s\right) = \frac{\rho}{2}\nu\int_t^T E_t\left(\sigma_s^2\right)\left(\int_s^T e^{-\kappa(u-s)}\,\mathrm{d}u\right)\mathrm{d}s$

$$= \frac{\rho\nu}{2\kappa^2}\left\{\theta\kappa\,(T-t) - 2\theta + \sigma_t^2 + e^{-\kappa(T-t)}\left(2\theta - \sigma_t^2\right) - \kappa\,(T-t)\,e^{-\kappa(T-t)}\left(\sigma_t^2 - \theta\right)\right\}.$$

5. $\frac{1}{8}D[M,M]_t := \frac{1}{8}E_t\left(\int_t^T d\left\langle M, M\right\rangle_s\right) = \frac{1}{8}\nu^2\int_t^T E_t\left(\sigma_s^2\right)\left(\int_s^T e^{-\kappa(u-s)}\,\mathrm{d}u\right)^2\mathrm{d}s$

$$\begin{aligned} &= \frac{\nu^2}{8\kappa^2}\Bigg\{\theta\,(T-t) + \frac{(\sigma_t^2 - \theta)}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \\ &\quad -\frac{2\theta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) - 2\left(\sigma_t^2 - \theta\right)(T-t)\,e^{-\kappa(T-t)} \\ &\quad + \frac{\theta}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right) + \frac{(\sigma_t^2 - \theta)}{\kappa}\left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}\right)\Bigg\}. \end{aligned}$$

6. $\frac{\rho}{2}\,\mathrm{d}L[W,M]_t = \frac{\rho\nu^2}{2}\left(\int_t^T e^{-\kappa(z-t)}\varphi(z)\,\mathrm{d}z\right)\sigma_t\,\mathrm{d}W_t - \frac{\rho\nu}{2}\varphi(t)\sigma_t^2\,\mathrm{d}t.$

7. $\frac{1}{8}\,\mathrm{d}D[M,M]_t = \frac{\nu^3}{8}\left(\int_t^T e^{-\kappa(z-t)}\varphi(z)^2\,\mathrm{d}z\right)\sigma_t\,\mathrm{d}W_t - \frac{\nu^2}{8}\varphi(t)^2\sigma_t^2\,\mathrm{d}t.$

In the following statement, lower bounds are provided for the adapted projection of the future integrated variance.

Lemma 5.1.2. *For every $s \in [0,T]$,*

(i) $\int_s^T E_s(\sigma_u^2)\,\mathrm{d}u \;\geq\; \frac{\theta\kappa}{2}\varphi(s)^2.$

(ii) $\int_s^T E_s\left(\sigma_u^2\right)\,\mathrm{d}u \;\geq\; \sigma_s^2\varphi(s).$

To improve the approximation formula, we will need the following lemmas.

Lemma 5.1.3. *In the Heston model, the following formulas hold*

$$
\begin{aligned}
L[W, L[W, M]]_t &= \mathbb{E}_t \left[\int_t^T \sigma_u \, \mathrm{d}[L[W, M], W]_u \right] \\
&= \nu^2 \int_t^T \mathbb{E}_t \left[\sigma_u^2 \right] \left(\int_u^T e^{-\kappa(z-u)} \varphi(z) dz \right) \mathrm{d}u \\
&= \frac{\nu^2}{2\kappa^3} \Bigg\{ 2 \Big[\sigma_t^2 + \theta \left(\kappa\tau - 3 \right) \Big] + e^{-\kappa\tau} \Big[\theta \left(\kappa^2\tau^2 + 4\kappa\tau + 6 \right) \\
&\qquad - \sigma_t^2 \left(\kappa^2\tau^2 + 2\kappa\tau + 2 \right) \Big] \Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}L[W, L[W, M]]_t &= \nu^3 \left[\int_t^T \left(\int_u^T e^{-\kappa(z-u)} \varphi(z) \, \mathrm{d}z \right) e^{-\kappa(u-t)} \, \mathrm{d}u \right] \sigma_t \, \mathrm{d}W_t \\
&\quad - \nu^2 \left(\int_t^T e^{-\kappa(z-t)} \varphi(z) \, \mathrm{d}z \right) \sigma_t^2 \, \mathrm{d}t.
\end{aligned}
$$

Similarly, the next lemma is used to increase the order of the approximation in the proof of Theorem 5.3.4.

Lemma 5.1.4. *The following equalities hold true in the Heston model*

$$
\begin{aligned}
D[M, \frac{1}{8} D[M, M]]_t &= \frac{\nu^3}{8} \int_t^T \mathbb{E}_t \left[\sigma_u^2 \right] \left(\int_t^T e^{-\kappa(z-u)} \varphi(z)^2 \, \mathrm{d}z \right) \mathrm{d}u \\
&= \frac{\nu^3}{16\kappa^4} \Bigg\{ \theta \Big[(2\kappa\tau - 7) + 2e^{-\kappa\tau} \left(\kappa^2\tau^2 + 2\kappa\tau + 4 \right) - e^{-2\kappa\tau} \Big] \\
&\qquad - 2\sigma_t^2 e^{-\kappa\tau} \Big[\kappa^2\tau^2 - 2cosh \left(\kappa\tau \right) + 2 \Big] \Bigg\}, \\
D[M, L[W, M]]_t &= \nu^3 \int_t^T \mathbb{E}_t \left[\sigma_u^2 \right] \varphi(u) \left(\int_t^T e^{-\kappa(z-t)} \varphi(z) \, \mathrm{d}z \right) \mathrm{d}u \\
&= \frac{\nu^3}{4\kappa^4} \Bigg\{ \theta \Big[2\kappa\tau e^{-2\kappa\tau} + (4\kappa\tau - 13) \\
&\qquad + 2e^{-\kappa\tau} \left(\kappa^2\tau^2 + 6\kappa\tau + 4 \right) + 5e^{-2\kappa\tau} \Big] \\
&\qquad + 2\sigma_t^2 \Big[e^{-\kappa\tau} \left(-\kappa^2\tau^2 - 4\kappa\tau + 2 \right) - 2e^{-2\kappa\tau} \left(\kappa\tau + 2 \right) + 2 \Big] \Bigg\},
\end{aligned}
$$

and

$$L[W,L[W,L[W,M]]]_t = \nu^3 \int_t^T \mathbb{E}_t\left[\sigma_u^2\right]\left(\int_u^T\left(\int_s^T e^{-\kappa(z-s)}\varphi(z)\,\mathrm{d}z\right)e^{-\kappa(s-u)}\,\mathrm{d}s\right)\mathrm{d}u$$
$$= \frac{\nu^3 e^{-\kappa\tau}}{6\kappa^4}\Big\{\theta\Big[6e^{\kappa\tau}(\kappa\tau-4)+\kappa^3\tau^3+6\kappa^2\tau^2+18\kappa\tau+24\Big]$$
$$+\sigma_t^2\Big[6e^{\kappa\tau}-6-\kappa^3\tau^3-3\kappa^2\tau^2-6\kappa\tau\Big]\Big\}.$$

The statement formulated below is used to prove the decomposition formula when it is assumed that the correlation coefficient is equal to zero.

Lemma 5.1.5. *The following equality holds in the Heston model*

$$D[M,\frac{1}{8}D[M,M]]_t = \frac{\nu^4}{8}\int_t^T \mathbb{E}_t\left[\sigma_u^2\right]\varphi(u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)^2\,\mathrm{d}z\right)\mathrm{d}u$$
$$= \frac{\nu^4}{48\kappa^5}\Big\{\theta\Big[3e^{-\kappa\tau}\left(2\kappa^2\tau^2+6\kappa\tau+5\right)$$
$$+6e^{-2\kappa\tau}(\kappa\tau+1)+(6\kappa\tau-22)+e^{-3\kappa\tau}\Big]$$
$$+3\sigma_t^2\Big[e^{-\kappa\tau}\left(-2\kappa^2\tau^2-2\kappa\tau+1\right)-2e^{-2\kappa\tau}(2\kappa\tau+1)+2-e^{-3\kappa\tau}\Big]\Big\}$$

and

$$\mathrm{d}D[M,\frac{1}{8}D[M,M]]_t = \frac{\nu^5\sigma_t}{8}\,\mathrm{d}W_t\int_t^T e^{-\kappa(u-t)}\varphi(u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)^2\,\mathrm{d}z\right)\mathrm{d}u$$
$$-\frac{\nu^4}{8}\sigma_t^2\varphi(t)\left(\int_t^T e^{-\kappa(z-t)}\varphi(z)^2\,\mathrm{d}z\right)\mathrm{d}t.$$

5.2 Original approximation formula for the Heston Model

In Alòs (2012), an approximation formula with a general error term was obtained for call option prices in the Heston model. This error term was quantified in Alòs et al. (2015), where it was shown that the error term has the form $O(\nu^2(|\rho|+\nu)^2)$. In the following theorem, we prove this upper-bound for the Heston model approximation formula.

Theorem 5.2.1 (1st General Approximation Formula for the Heston Model)**.** *Let S_t be the price process defined in* (3.22) *and taking into account the change $X_t = \log(S_t)$. Then, we can express the call option fair value V_t using processes $L[X,Y]_t$ and $D[X,Y]_t$ defined by* (3.38) *and* (3.39)*, respectively. In particular,*

$$V_t = C_{\overline{BS}}(t,X_t,v_t)$$

$$
\begin{aligned}
&+ \quad \frac{\rho}{2}\widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}(u,X_u,v_u)L[W,M]_t \\
&+ \quad \frac{1}{8}\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)D[M,M]_t \\
&+ \quad \epsilon_t
\end{aligned}
$$

where ϵ_t is the error term satisfying

$$|\epsilon_t| \leq \left(\nu^2\left(|\rho|+\nu\right)^2\right)\left(\frac{1}{r}\wedge(T-t)\right)\Pi(\kappa,\theta),$$

and $\Pi(\kappa,\theta)$ is a positive constant depending on κ and θ.

Proof. Using formula (4.8), it is obtained

$$
\begin{aligned}
(I) &= \frac{\rho\nu}{2}\widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right) \qquad (5.1)\\
&+ \frac{\rho^2\nu^2}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^2C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)\,\mathrm{d}u\right]\\
&+ \frac{\rho\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^3C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)^2\,\mathrm{d}u\right]\\
&+ \frac{\rho^2\nu^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)\,\mathrm{d}z\right)\sigma_u^2\,\mathrm{d}u\right]\\
&+ \frac{\rho\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^2C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)\,\mathrm{d}z\right)\sigma_u^2\varphi(u)\,\mathrm{d}u\right]\\
&= \widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)L[W,M]_t+(I.I)+(I.II)+(I.III)+(I.IV).
\end{aligned}
$$

Applying Lemma 3.1.10 (ii) and the equivalence $a_u = v_u\sqrt{T-u}$, then

$$
\begin{aligned}
&\left|(I)-\frac{\rho\nu}{2}\widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\right|\\
\leq\ & \frac{\rho^2\nu^2}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)\,\mathrm{d}u\right]\\
+\ & \frac{|\rho|\,\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6}+\frac{2}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)^2\,\mathrm{d}u\right]\\
+\ & \frac{\rho^2\nu^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{1}{a_u^3}\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)\,\mathrm{d}z\right)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{|\rho|\,\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)\,\mathrm{d}z\right)\sigma_u^2\varphi(u)\,\mathrm{d}u\right].
\end{aligned}
$$

Using that $\varphi(t)$ is a decreasing function,

$$\left|(I)-\frac{\rho\nu}{2}\widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\right|$$

$$\begin{aligned}
&\leq \frac{\rho^2\nu^2}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)^2\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6}+\frac{2}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)ds\right)\sigma_u^2\varphi(u)^3\,\mathrm{d}u\right]\\
&+ \frac{\rho^2\nu^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{1}{a_u^3}\varphi(u)^2\sigma_u^2\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\sigma_u^2\varphi(u)^3\,\mathrm{d}u\right].
\end{aligned}$$

Lemma 5.1.2 (ii), gives $\sigma_t^2 \leq \frac{a_t^2}{\varphi(t)}$, therefore

$$\begin{aligned}
&\left|(I)-\frac{\rho\nu}{2}\widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\right|\\
&\leq \frac{\rho^2\nu^2}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u}+1\right)\varphi(u)\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^2}+\frac{2}{a_u}+1\right)\varphi(u)^2\,\mathrm{d}u\right]\\
&+ \frac{\rho^2\nu^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{1}{a_u}\varphi(u)\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^2}+\frac{1}{a_u}\right)\varphi(u)^3\,\mathrm{d}u\right].
\end{aligned}$$

Moreover, Lemma 5.1.2 (i) implies that that $a_t \geq \frac{\sqrt{\theta\kappa}}{\sqrt{2}}\varphi(t)$ and

$$\begin{aligned}
&\left|(I)-\frac{\rho\nu}{2}\widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\right|\\
&\leq \frac{\rho^2\nu^2}{4}\left[\int_t^T e^{-r(u-t)}\left(\frac{3\sqrt{2}}{\sqrt{\theta\kappa}\varphi(u)}+1\right)\varphi(u)\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{16}\left[\int_t^T e^{-r(u-t)}\left(\frac{2\left(1+4\varphi(u)\right)}{\theta\kappa\varphi(u)^2}+\frac{2\sqrt{2}\left(1+2\varphi(u)\right)}{\sqrt{\theta\kappa}\varphi(u)}+1\right)\varphi(u)^2\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{4}\left[\int_t^T e^{-r(u-t)}\left(\frac{2}{\theta\kappa\varphi(u)^2}+\frac{\sqrt{2}}{\sqrt{\theta\kappa}\varphi(u)}\right)\varphi(u)^3\,\mathrm{d}u\right].
\end{aligned}$$

Next, using the estimate $\varphi(t) \leq \frac{1}{\kappa}$, we obtain

$$\left|(I)-\frac{\rho\nu}{2}\widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\right|$$

$$\leq \frac{\rho^2\nu^2}{4\kappa}\left(\frac{3\sqrt{2\kappa}}{\sqrt{\theta}}+1\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right]$$
$$+ \frac{|\rho|\,\nu^3}{16\kappa^3}\left(\frac{2\kappa(\kappa+4)}{\theta}+\frac{2\sqrt{2\kappa}(\kappa+2)}{\sqrt{\theta}}+\kappa\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right].$$

Following the process with the term (II), using formula (4.9), we see

$$\begin{aligned}
(II) &= \widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right) \quad (5.2)\\
&+ \frac{\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^4 C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\sigma_u^2\varphi^2(u)\,\mathrm{d}u\right]\\
&+ \frac{\rho\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\sigma_u^2\varphi(u)\,\mathrm{d}u\right]\\
&+ \frac{\rho\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)^2\,\mathrm{d}z\right)\sigma_u^2\,\mathrm{d}u\right]\\
&+ \frac{\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)^2\,\mathrm{d}z\right)\varphi(u)\sigma_u^2\,\mathrm{d}u\right]\\
&= \widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)D[M,M]_t + (II.I) + (II.II) + (II.III) + (II.IV).
\end{aligned}$$

Applying Lemma 3.1.10 (ii) and the equivalence $a_u = v_u\sqrt{T-u}$, then

$$\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\right|$$
$$\begin{aligned}
&\leq \frac{\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^7}+\frac{3}{a_u^6}+\frac{3}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\sigma_u^2\varphi^2(u)\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6}+\frac{2}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\sigma_u^2\varphi(u)\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)^2\,\mathrm{d}z\right)\sigma_u^2\,\mathrm{d}u\right]\\
&+ \frac{\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{2}{a_u^4}+\frac{1}{a_u^3}\right)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)^2\,\mathrm{d}z\right)\varphi(u)\sigma_u^2\,\mathrm{d}u\right].
\end{aligned}$$

Using that $\varphi(t)$ is a decreasing function,

$$\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\right|$$
$$\begin{aligned}
&\leq \frac{\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^7}+\frac{3}{a_u^6}+\frac{3}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)^4\,\mathrm{d}u\right]\\
&+ \frac{|\rho|\,\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6}+\frac{2}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)^3\,\mathrm{d}u\right]
\end{aligned}$$

$$+ \frac{|\rho|\,\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\sigma_u^2\varphi(u)^3\,\mathrm{d}u\right]$$
$$+ \frac{\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{2}{a_u^4}+\frac{1}{a_u^3}\right)\sigma_u^2\varphi(u)^4\,\mathrm{d}u\right].$$

Lemma 5.1.2 (ii), gives $\sigma_t^2 \leq \frac{a_t^2}{\varphi(t)}$, so

$$\begin{aligned}
&\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\right| \\
\leq\ & \frac{\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{3}{a_u^2}+\frac{3}{a_u}+1\right)\varphi(u)^3\,\mathrm{d}u\right] \\
+\ & \frac{|\rho|\,\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^2}+\frac{2}{a_u}+1\right)\varphi(u)^2\,\mathrm{d}u\right] \\
+\ & \frac{|\rho|\,\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^2}+\frac{1}{a_u}\right)\varphi(u)^2\,\mathrm{d}u\right] \\
+\ & \frac{\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^3}+\frac{2}{a_u^2}+\frac{1}{a_u}\right)\varphi(u)^3\,\mathrm{d}u\right].
\end{aligned}$$

Moreover, Lemma 5.1.2 (i) implies that $a_t \geq \frac{\sqrt{\theta\kappa}}{\sqrt{2}}\varphi(t)$ and

$$\begin{aligned}
&\left|(II) - \Gamma^2 C_{\overline{BS}}(t, X_t, v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\right| \\
\leq\ & \frac{\nu^4}{64}\left[\int_t^T e^{-r(u-t)}\left(\frac{4\sqrt{2}}{\theta^2\kappa^2\sqrt{\theta\kappa}\varphi(u)^5}+\frac{8\sqrt{2}}{\theta\kappa\sqrt{\theta\kappa}\varphi(u)^3}+\frac{22}{\theta\kappa\varphi(u)^2}+\frac{7\sqrt{2}}{\sqrt{\theta\kappa}\varphi(u)}+1\right)\varphi(u)^3\,\mathrm{d}u\right] \\
+\ & \frac{|\rho|\,\nu^3}{16}\left[\int_t^T e^{-r(u-t)}\left(\frac{6}{\theta\kappa\varphi(u)^2}+\frac{4\sqrt{2}}{\sqrt{\theta\kappa}\varphi(u)}+1\right)\varphi(u)^2\,\mathrm{d}u\right].
\end{aligned}$$

Next, using the estimate $\varphi(t) \leq \frac{1}{\kappa}$, we observe

$$\begin{aligned}
&\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\int_t^T \mathbb{E}_t\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)\right| \\
\leq\ & \frac{\nu^4}{64\kappa^3}\left(\frac{4\kappa^2\sqrt{2\kappa}}{\theta^2\sqrt{\theta}}+\frac{8\kappa\sqrt{2\kappa}}{\theta\sqrt{\theta}}+\frac{22\kappa}{\theta}+\frac{7\sqrt{2\kappa}}{\sqrt{\theta}}+1\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right] \\
+\ & \frac{|\rho|\,\nu^3}{16\kappa^2}\left(\frac{6\kappa}{\theta}+\frac{4\sqrt{2\kappa}}{\sqrt{\theta}}+1\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right].
\end{aligned}$$

□

Remark 5.2.2. *Note the in Alòs (2012), Alòs et al. (2015), and Merino et al. (2018), the above theorem is written in terms of the processes U_t and R_t. We have the following equivalence*

$$U_t = \frac{\rho}{2}L[W,M]_t \quad \text{and} \quad R_t = \frac{1}{8}D[M,M]_t. \tag{5.3}$$

We have to change the notation to be able to generalise the process and improve it.

5.3 Higher order approximation formulas for the Heston Model

In the above section, we have seen that the error term for the approximation of the Heston model has the form $O(\nu^2(|\rho|+\nu)^2)$. However, in the above-mentioned approximation formula, some terms of order ν^2 were ignored, whereas other terms of the same order were maintained. This may be considered as a drawback of the previous approximation formula.

We can improve this formula using Corollary 4.1.10 recursively to approximate the exact call price decomposition obtained in Remark 4.1.12 by an infinite series of stochastic terms. The first two terms in the new expansion are the same as in Theorem 5.2.1. Moreover, our result is consistent with the one obtained in Alòs et al. (2020), but presented and obtained in a different way. Using the new general approximation formula in the case of the Heston model, we add two more significant terms to the above-mentioned expansion, to reach an error of the form $O(\nu^3(|\rho|+\nu))$ (see Theorem 5.3.3), and seven more significant terms to obtain an error estimate of the form $O(\nu^4(1+|\rho|\nu))$ (see Theorem 5.3.4). In the particular case of zero correlation, we derive an approximation formula with four terms with an error of order $O(\nu^6)$.

We will next explain how to get an infinite expansion of the call price V_t. The starting point in the construction of an infinite expansion of V_t is the formula in Corollary 4.1.10. In the previous section, Corollary 4.1.10 was applied to the main terms of Remark 4.1.12. Only the main two terms in the expansion were maintained, while the remaining terms were ignored. The main idea used is to apply Corollary 4.1.10 to each new term, obtaining an infinite series with stochastic terms. By selecting which terms to keep in the approximation formula and which ones to discard, the approximation error can be controlled.

The process described above leads to the following expansion of V_t:

$$\begin{aligned} V_t &= C_{\overline{BS}}(t,X_t,v_t) \\ &+ \widetilde{\Lambda}\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)\left(\frac{\rho}{2}L[W,M]_t\right) + \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2C_{\overline{BS}}(t,X_t,v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2 \\ &+ \widetilde{\Gamma}^2C_{\overline{BS}}(t,X_t,v_t)\left(\frac{1}{8}D[M,M]_t\right) + \frac{1}{2}\widetilde{\Gamma}^4C_{\overline{BS}}(t,X_t,v_t)\left(\frac{1}{8}D[M,M]_t\right)^2 \\ &+ \widetilde{\Lambda}\widetilde{\Gamma}^3C_{\overline{BS}}(t,X_t,v_t)\left(\frac{\rho}{2}L[W,M]_t\right)\left(\frac{1}{8}D[M,M]_t\right) + \dots \end{aligned} \tag{5.4}$$

Note that it has some similarity to a Taylor-type formula, but is more complicated due to the intrinsic stochastic component.

Remark 5.3.1. *In Alòs et al. (2020), an exact representation of V_t is given in terms of a forest of iterated integrals, also called diamonds. The expansion of the call price found in the present section is equivalent to the one obtained in Alòs et al. (2020).*

Remark 5.3.2. *As mentioned above, the same representation can be found with respect to the price process.*

Theorem 5.3.3 (2nd order approximation formula)**.** *For every $t \in [0, T]$,*

$$\begin{aligned} V_t &= C_{\overline{BS}}(t, X_t, v_t) \\ &+ \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{1}{8} D[M, M]_t\right) \\ &+ \widetilde{\Lambda}\widetilde{\Gamma} C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2} L[W, M]_t\right) \\ &+ \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2} L[W, M]_t\right)^2 \\ &+ \rho\widetilde{\Lambda}^2\widetilde{\Gamma} C_{\overline{BS}}(t, X_t, v_t) L[W, \frac{\rho}{2} L[W, M]]_t \\ &+ \epsilon_t, \end{aligned}$$

where ϵ_t is the error term satisfying

$$|\epsilon_t| \leq \nu^3 \left(|\rho| + |\rho|^3 + \nu\right)\left(\frac{1}{r} \wedge (T - t)\right)\Pi(\kappa, \theta),$$

and $\Pi(\kappa, \theta)$ is a positive constant depending on κ and θ.

Proof. The main idea is to apply Corollary 4.1.10 to the call price formula iteratively. We also apply this corollary to the new terms that appear in the iterative procedure, and incorporate all the terms of order $O(\nu^3)$ into the error term.

The starting point are the expressions 5.1 and 5.2.

An upper bound for the term (I).

The idea now is to discard terms of order $O(\nu^3)$, and apply Corollary 4.1.10 to the terms of smaller order.

Upper bounds for the terms (I.II) and (I.IV).

First, we note that the terms (I.II) and (I.IV) are of order $O(\nu^3)$. Therefore, those terms can be incorporated into the error term. Then,

$$(I.II) + (I.IV) = \frac{\rho\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^3 C_{\overline{BS}}(u, X_u, v_u)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)^2\,\mathrm{d}u\right]$$

$$+ \quad \frac{\rho\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)\,\mathrm{d}z\right)\sigma_u^2\varphi(u)\,\mathrm{d}u\right].$$

Next, using Lemma 3.1.10 (ii), $a_u = v_u\sqrt{T-u}$ and the fact that $\varphi(t)$ is a decreasing function, we obtain

$$\begin{aligned}
&|(I.II)+(I.IV)|\\
\leq\ & C\frac{\rho\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6}+\frac{2}{a_u^5}+\frac{1}{a_u^4}\right)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\,\mathrm{d}s\right)\sigma_u^2\varphi(u)^3\,\mathrm{d}u\right]\\
+\ & C\frac{\rho\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\sigma_u^2\varphi^3(u)\,\mathrm{d}u\right].
\end{aligned}$$

It follows from Lemma 5.1.2 (ii) that $\sigma_t^2 \leq \frac{a_t^2}{\varphi(t)}$. Hence

$$|(I.II)+(I.IV)| \;\leq\; C\frac{\rho\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{5}{a_u^2}+\frac{6}{a_u}+1\right)\varphi(u)^2\,\mathrm{d}u\right].$$

Now, Lemma 5.1.2 (i) implies that $a_t \geq \frac{\sqrt{\theta\kappa}}{\sqrt{2}}\varphi(t)$ and

$$|(I.II)+(I.IV)| \;\leq\; C\frac{\rho\nu^3}{16}\left[\int_t^T e^{-r(u-t)}\left(\frac{10}{\theta\kappa}+\frac{6\sqrt{2}\varphi(u)}{\sqrt{\theta\kappa}}+\varphi(u)^2\right)\mathrm{d}u\right].$$

Finally, using the estimate $\varphi(t) \leq \frac{1}{\kappa}$, we see that

$$|(I.II)+(I.IV)| \;\leq\; C\frac{\rho\nu^3}{16}\left(\frac{10}{\theta\kappa}+\frac{6\sqrt{2}}{\kappa\sqrt{\theta\kappa}}+\frac{1}{\kappa^2}\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right].$$

An upper bound for the term (I.I).

Here, we note that the term (I.I) is of the order $O(\nu^2)$. Therefore, in order to improve the approximation, we should apply Corollary 4.1.10 to this term. Choosing

$$A_t = \Lambda^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t) \text{ and } B = \frac{1}{2}\left(\frac{\rho}{2}L[W,M]_t\right)^2,$$

we observe that

$$\begin{aligned}
&\left|(I.I)-\frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2\right|\\
=\ & \frac{\nu\rho^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^3\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)L^2[W,M]_u\sigma_u^2\varphi(u)\,\mathrm{d}u\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\rho^2\nu^2}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^4 C_{\overline{BS}}(u,X_u,v_u)L^2[W,M]_u\sigma_u^2\varphi^2(u)\,\mathrm{d}u\right]\\
&+ \frac{\rho^3\nu^2}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^3\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)L[W,M]_u\sigma_u^2\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)\,\mathrm{d}z\right)\,\mathrm{d}u\right]\\
&+ \frac{\nu^3\rho^2}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)L[W,M]_u\sigma_u^2\varphi(u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)\,\mathrm{d}z\right)\,\mathrm{d}u\right]\\
&+ \frac{\rho^2\nu^4}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T e^{-\kappa(z-u)}\varphi(z)\,\mathrm{d}z\right)^2\sigma_u^2\,\mathrm{d}u\right].
\end{aligned}
$$

Note that $L[W,M]_t \leq \nu a_t^2\varphi(t)$. It follows that

$$
\begin{aligned}
&\left|(I.I) - \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2\right|\\
\leq\ & \frac{\nu^3\rho^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^7 - 2\partial_x^6 + \partial_x^5\right)\widetilde{\Gamma} C_{\overline{BS}}(u,X_u,v_u)a_u^4\varphi^3(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{\rho^2\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^8 - 3\partial_x^7 + 3\partial_x^6 - \partial_x^5\right)\widetilde{\Gamma} C_{\overline{BS}}(u,X_u,v_u)a_u^4\varphi^4(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{\rho^3\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^5 - \partial_x^4\right)\widetilde{\Gamma} C_{\overline{BS}}(u,X_u,v_u)a_u^2\varphi^3(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{\nu^4\rho^2}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^6 - 2\partial_x^5 + \partial_x^4\right)\widetilde{\Gamma} C_{\overline{BS}}(u,X_u,v_u)a_u^2\varphi^4(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{\rho^2\nu^4}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^4 - \partial_x^3\right)\widetilde{\Gamma} C_{\overline{BS}}(u,X_u,v_u)\varphi^4(u)\sigma_u^2\,\mathrm{d}u\right].
\end{aligned}
$$

Next, using Lemma 3.1.10 (ii), we see that

$$
\begin{aligned}
&\left|(I.I) - \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2\right|\\
\leq\ & \frac{C\nu^3\rho^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4} + \frac{2}{a_u^3} + \frac{1}{a_u^2}\right)\varphi^3(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{C\rho^2\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5} + \frac{3}{a_u^4} + \frac{3}{a_u^3} + \frac{1}{a_u^2}\right)\varphi^4(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{C\rho^3\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4} + \frac{1}{a_u^3}\right)\varphi^3(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{C\nu^4\rho^2}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5} + \frac{2}{a_u^4} + \frac{1}{a_u^3}\right)\varphi^4(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{C\rho^2\nu^4}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5} + \frac{1}{a_u^4}\right)\varphi^4(u)\sigma_u^2\,\mathrm{d}u\right].
\end{aligned}
$$

Now, Lemma 5.1.2 (ii) implies that $\sigma_t^2 \leq \frac{a_t^2}{\varphi(t)}$ and

$$\begin{aligned}
&\left|(I.I) - \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2\right| \\
\leq\ & \frac{C\nu^3\rho^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{3}{a_u^2} + \frac{4}{a_u} + 1\right)\varphi^2(u)\,\mathrm{d}u\right] \\
+\ & \frac{C\rho^2\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{13}{a_u^3} + \frac{19}{a_u^2} + \frac{7}{a_u} + 1\right)\varphi^3(u)\,\mathrm{d}u\right].
\end{aligned}$$

Moreover, Lemma 5.1.2 gives $a_t \geq \frac{\sqrt{\theta\kappa}}{\sqrt{2}}\varphi(t)$ and

$$\begin{aligned}
&\left|(I.I) - \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2\right| \\
\leq\ & \frac{C\nu^3\rho^3}{16}\left[\int_t^T e^{-r(u-t)}\left(\frac{6}{\theta\kappa} + \frac{4\sqrt{2}\varphi(u)}{\sqrt{\theta\kappa}} + \varphi^2(u)\right)\mathrm{d}u\right] \\
+\ & \frac{C\rho^2\nu^4}{64}\left[\int_t^T e^{-r(u-t)}\left(\frac{26\sqrt{2}}{\theta\kappa\sqrt{\theta\kappa}} + \frac{38\varphi^2(u)}{\theta\kappa} + \frac{7\sqrt{2}\varphi^2(u)}{\sqrt{\theta\kappa}} + \varphi^3(u)\right)\mathrm{d}u\right].
\end{aligned}$$

Finally, using the estimate $\varphi(t) \leq \frac{1}{\kappa}$, we obtain

$$\begin{aligned}
&\left|(I.I) - \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2\right| \\
\leq\ & \frac{C\nu^3\rho^3}{16}\left(\frac{6}{\theta\kappa} + \frac{4\sqrt{2}}{\kappa\sqrt{\theta\kappa}} + \frac{1}{\kappa^2}\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right] \\
+\ & \frac{C\rho^2\nu^4}{64}\left(\frac{26\sqrt{2}}{\theta\kappa\sqrt{\theta\kappa}} + \frac{38}{\theta\kappa^3} + \frac{7\sqrt{2}}{\kappa^3\sqrt{\theta\kappa}} + \frac{1}{\kappa^3}\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right].
\end{aligned}$$

An upper bound for the term (I.III).

The term (I.III) is of order $O(\nu^2)$, and it has to be taken into account in the approximation. Here we apply Corollary 4.1.10 with

$$A_t = \widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t, X_t, v_t) \text{ and } B = \rho L[W, \frac{\rho}{2}L[W,M]]_t.$$

Then, we see that

$$\begin{aligned}
&\left|(I.III) - \rho\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t, X_t, v_t)L[W, \frac{\rho}{2}L[W,M]]_t\right| \\
=\ & \frac{\rho^3\nu}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^3\widetilde{\Gamma}^2 C_{\overline{BS}}(u, X_u, v_u)L[W, L[W,M]]_u\sigma_u^2\varphi(u)\,\mathrm{d}u\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\rho^2\nu^2}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u)L[W,L[W,M]]_u\sigma_u^2\varphi^2(u)\,\mathrm{d}u\right]\\
&+ \frac{\rho^3\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^3\widetilde{\Gamma} C_{\overline{BS}}(u,X_u,v_u)\sigma_u^2\left(\int_u^T\left(\int_s^T e^{-\kappa(z-s)}\varphi(z)dz\right)e^{-\kappa(s-u)}\,\mathrm{d}s\right)\mathrm{d}u\right]\\
&+ \frac{\rho^2\nu^4}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u)\sigma_u^2\varphi(u)\left(\int_u^T\left(\int_s^T e^{-\kappa(z-s)}\varphi(z)dz\right)e^{-\kappa(s-u)}\,\mathrm{d}s\right)\mathrm{d}u\right].
\end{aligned}
$$

It is easy to see that $L[W,L[W,M]]_t = \nu^2 a_t^2\varphi^2(t)$. It follows that

$$
\begin{aligned}
&\left|(I.III) - \rho\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)L[W,\frac{\rho}{2}L[W,M]]_t\right|\\
\le\ & \frac{\rho^3\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^5-\partial_x^3\right)\widetilde{\Gamma}C_{\overline{BS}}(u,X_u,v_u)a_u^2\varphi^3(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{\rho^2\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^6-2\partial_x^5+\partial_x^4\right)\widetilde{\Gamma}C_{\overline{BS}}(u,X_u,v_u)a_u^2\varphi^4(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{\rho^3\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x^3\widetilde{\Gamma}C_{\overline{BS}}(u,X_u,v_u)\sigma_u^2\varphi^3(u)\,\mathrm{d}u\right]\\
+\ & \frac{\rho^2\nu^4}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^4-\partial_x^3\right)\widetilde{\Gamma}C_{\overline{BS}}(u,X_u,v_u)\sigma_u^2\varphi^4(u)\,\mathrm{d}u\right].
\end{aligned}
$$

Next, using Lemma 3.1.10 (ii),

$$
\begin{aligned}
&\left|(I.III) - \rho\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)L[W,\frac{\rho}{2}L[W,M]]_t\right|\\
\le\ & \frac{C\rho^3\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^2}\right)\varphi^3(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{C\rho^2\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+2\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\varphi^4(u)\sigma_u^2\,\mathrm{d}u\right]\\
+\ & \frac{C\rho^3\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{1}{a_u^4}\sigma_u^2\varphi^3(u)\,\mathrm{d}u\right]\\
+\ & \frac{C\rho^2\nu^4}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{1}{a_u^4}\right)\sigma_u^2\varphi^4(u)\,\mathrm{d}u\right].
\end{aligned}
$$

Now, Lemma 5.1.2 (ii), gives $\sigma_t^2 \le \frac{a_t^2}{\varphi(t)}$ and

$$
\begin{aligned}
&\left|(I.III) - \rho\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)L[W,\frac{\rho}{2}L[W,M]]_t\right|\\
\le\ & \frac{C\rho^3\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^2}+1\right)\varphi^2(u)\,\mathrm{d}u\right]\\
+\ & \frac{C\rho^2\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^3}+2\frac{1}{a_u^2}+\frac{1}{a_u^1}\right)\varphi^3(u)\,\mathrm{d}u\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{C\rho^3\nu^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{1}{a_u^2}\varphi^2(u)\,\mathrm{d}u\right] \\
&+ \quad \frac{C\rho^2\nu^4}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^3}+\frac{1}{a_u^2}\right)\varphi^3(u)\,\mathrm{d}u\right].
\end{aligned}
$$

Moreover, Lemma 5.1.2 (i) implies that $a_t \geq \frac{\sqrt{\theta\kappa}}{\sqrt{2}}\varphi(t)$ and

$$
\begin{aligned}
&\left|(I.III) - \rho\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)L[W,\frac{\rho}{2}L[W,M]]_t\right| \\
\leq \quad & \frac{C\rho^3\nu^3}{4}\left[\int_t^T e^{-r(u-t)}\left(\frac{2}{\theta\kappa\varphi(u)}+1\right)\varphi^2(u)\,\mathrm{d}u\right] \\
+ \quad & \frac{C\rho^2\nu^4}{16}\left[\int_t^T e^{-r(u-t)}\left(\frac{2\sqrt{2}}{\theta\kappa\sqrt{\theta\kappa}\varphi^2(u)}+\frac{4}{\theta\kappa\varphi(u)}+\frac{\sqrt{2}}{\sqrt{\theta\kappa}}\right)\varphi^2(u)\,\mathrm{d}u\right] \\
+ \quad & \frac{C\rho^3\nu^3}{4}\left[\int_t^T e^{-r(u-t)}\frac{2}{\theta\kappa}\,\mathrm{d}u\right] \\
+ \quad & \frac{C\rho^2\nu^4}{8}\left[\int_t^T e^{-r(u-t)}\left(\frac{2\sqrt{2}}{\theta\kappa\sqrt{\theta\kappa}\varphi(u)}+\frac{2}{\theta\kappa}\right)\varphi(u)\,\mathrm{d}u\right].
\end{aligned}
$$

Next, using the estimate $\varphi(t) \leq \frac{1}{\kappa}$, we observe

$$
\begin{aligned}
&\left|(I.III) - \rho\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)L[W,\frac{\rho}{2}L[W,M]]_t\right| \\
\leq \quad & \frac{C\rho^3\nu^3}{4}\left[\int_t^T e^{-r(u-t)}\left(\frac{2}{\theta}+1\right)\frac{1}{\kappa^2}\,\mathrm{d}u\right] \\
+ \quad & \frac{C\rho^2\nu^4}{16}\left[\int_t^T e^{-r(u-t)}\left(\frac{2\sqrt{2\kappa}}{\theta\sqrt{\theta}}+\frac{4}{\theta}+\frac{\sqrt{2}}{\sqrt{\theta\kappa}}\right)\frac{1}{\kappa^2}\,\mathrm{d}u\right] \\
+ \quad & \frac{C\rho^3\nu^3}{4}\left[\int_t^T e^{-r(u-t)}\frac{2}{\theta\kappa}\,\mathrm{d}u\right] \\
+ \quad & \frac{C\rho^2\nu^4}{8}\left[\int_t^T e^{-r(u-t)}\left(\frac{2\sqrt{2}}{\theta\sqrt{\theta\kappa}}+\frac{2}{\theta\kappa}\right)\frac{1}{\kappa}\,\mathrm{d}u\right].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left|(I.III) - \rho\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)L[W,\frac{\rho}{2}L[W,M]]_t\right| \\
\leq \quad & \frac{C\rho^3\nu^3}{4\kappa^2}\left(\frac{2\kappa+2}{\theta}+1\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right] \\
+ \quad & \frac{C\rho^2\nu^4}{16\kappa^2}\left(\frac{6\sqrt{2\kappa}}{\theta\sqrt{\theta}}+\frac{8}{\theta}+\frac{\sqrt{2}}{\sqrt{\theta\kappa}}\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right].
\end{aligned}
$$

An upper bound for the term (II).

In this case, all the terms can be incorporated into the error term. Using the fact that $\varphi(t)$ is a decreasing function, then

$$
\begin{aligned}
&\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) D[M,M]_t\right| \\
\leq\ & \frac{\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^6 - 3\partial_x^5 + 3\partial_x^4 - \partial_x^3\right)\widetilde{\Gamma} C_{\overline{BS}}(u, X_u, v_u) a_u^2 \sigma_u^2 \varphi^4(u)\,\mathrm{d}u\right] \\
+\ & \frac{\rho\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^5 - 2\partial_x^4 + \partial_x^3\right)\widetilde{\Gamma} C_{\overline{BS}}(u, X_u, v_u) a_u^2 \sigma_u^2 \varphi^3(u)\,\mathrm{d}u\right] \\
+\ & \frac{\rho\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^3 - \partial_x^2\right)\widetilde{\Gamma} C_{\overline{BS}}(u, X_u, v_u) \sigma_u^2 \varphi(u)^3\,\mathrm{d}u\right] \\
+\ & \frac{\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^4 - 2\partial_x^3 + \partial_x^2\right)\widetilde{\Gamma} C_{\overline{BS}}(u, X_u, v_u) \varphi^4(u) \sigma_u^2\,\mathrm{d}u\right].
\end{aligned}
$$

Next, using Lemma 3.1.10 (ii) and using $a_u = v_u\sqrt{T-u}$,

$$
\begin{aligned}
&\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) D[M,M]_t\right| \\
\leq\ & \frac{C\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5} + \frac{3}{a_u^4} + \frac{3}{a_u^3} + \frac{1}{a_u^2}\right)\sigma_u^2 \varphi^4(u)\,\mathrm{d}u\right] \\
+\ & \frac{C\rho\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4} + \frac{2}{a_u^3} + \frac{1}{a_u^2}\right)\sigma_u^2 \varphi^3(u)\,\mathrm{d}u\right] \\
+\ & \frac{C\rho\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4} + \frac{1}{a_u^3}\right)\sigma_u^2 \varphi(u)^3\,\mathrm{d}u\right] \\
+\ & \frac{C\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5} + \frac{2}{a_u^4} + \frac{1}{a_u^3}\right)\varphi^4(u) \sigma_u^2\,\mathrm{d}u\right].
\end{aligned}
$$

It follows from Lemma 5.1.2 (ii) that $\sigma_t^2 \leq \frac{a_t^2}{\varphi(t)}$ and

$$
\begin{aligned}
&\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) D[M,M]_t\right| \\
\leq\ & \frac{C\nu^4}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^3} + \frac{3}{a_u^2} + \frac{3}{a_u} + 1\right)\varphi^3(u)\,\mathrm{d}u\right] \\
+\ & \frac{C\rho\nu^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^2} + \frac{2}{a_u} + 1\right)\varphi^2(u)\,\mathrm{d}u\right] \\
+\ & \frac{C\rho\nu^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^2} + \frac{1}{a_u}\right)\varphi^2(u)\,\mathrm{d}u\right] \\
+\ & \frac{C\nu^4}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^3} + \frac{2}{a_u^2} + \frac{1}{a_u}\right)\varphi^3(u)\,\mathrm{d}u\right].
\end{aligned}
$$

Therefore, Lemma 5.1.2 (i) gives $a_t \geq \frac{\sqrt{\theta\kappa}}{\sqrt{2}}\varphi(t)$ and

$$
\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) D[M,M]_t\right|
$$

$$
\begin{aligned}
&\le \frac{C\nu^4}{64}\left[\int_t^T e^{-r(u-t)}\left(\frac{2\sqrt{2}}{\theta\kappa\sqrt{\theta\kappa}\varphi^3(u)}+\frac{6}{\theta\kappa\varphi^2(u)}+\frac{3\sqrt{2}}{\sqrt{\theta\kappa}\varphi(u)}+1\right)\varphi^3(u)\,\mathrm{d}u\right]\\
&+ \frac{C\rho\nu^3}{16}\left[\int_t^T e^{-r(u-t)}\left(\frac{2}{\theta\kappa\varphi^2(u)}+\frac{2\sqrt{2}}{\sqrt{\theta\kappa}\varphi(u)}+1\right)\varphi^2(u)\,\mathrm{d}u\right]\\
&+ \frac{C\rho\nu^3}{8}\left[\int_t^T e^{-r(u-t)}\left(\frac{2}{\theta\kappa\varphi(u)}+\frac{\sqrt{2}}{\sqrt{\theta\kappa}}\right)\varphi(u)\,\mathrm{d}u\right]\\
&+ \frac{C\nu^4}{16}\left[\int_t^T e^{-r(u-t)}\left(\frac{2\sqrt{2}}{\theta\kappa\sqrt{\theta\kappa}\varphi^2(u)}+\frac{4}{\theta\kappa\varphi(u)}+\frac{\sqrt{2}}{\sqrt{\theta\kappa}}\right)\varphi^2(u)\,\mathrm{d}u\right].
\end{aligned}
$$

Finally, we observe that the estimate $\varphi(t) \le \frac{1}{\kappa}$ implies that

$$
\begin{aligned}
&\left|(II) - \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) D[M,M]_t\right|\\
&\le \frac{C\nu^4}{64\kappa^3}\left(\frac{2\kappa\sqrt{2\kappa}(1+4\kappa)}{\theta\sqrt{\theta}}+\frac{22\kappa}{\theta}+\frac{7\sqrt{2\kappa}}{\sqrt{\theta}}+1\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right]\\
&+ \frac{C\rho\nu^3}{16\kappa^2}\left(\frac{6\kappa+2\sqrt{2\kappa}}{\theta}+\frac{2\sqrt{2\kappa}}{\sqrt{\theta}}+1\right)\left[\int_t^T e^{-r(u-t)}\,\mathrm{d}u\right].
\end{aligned}
$$

This completes the proof of Theorem 5.3.3. □

The next assertion contains an approximation formula with the error term of the form $O(\nu^4(1+|\rho|))$.

Theorem 5.3.4 (3rd order approximation formula)**.** *For every $t \in [0,T]$,*

$$
\begin{aligned}
V_t &= C_{\overline{BS}}(t, X_t, v_t)\\
&+ \widetilde{\Lambda}\widetilde{\Gamma} C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2}L[W,M]_t\right) + \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2\\
&+ \frac{1}{6}\widetilde{\Lambda}^3\widetilde{\Gamma}^3 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^3 + \widetilde{\Lambda}\widetilde{\Gamma}^3 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{\rho}{2}L[W,M]_t\right)\left(\frac{1}{8}D[M,M]_t\right)\\
&+ \rho\widetilde{\Lambda}^2\widetilde{\Gamma} C_{\overline{BS}}(t, X_t, v_t) L[W, \frac{\rho}{2}L[W,M]]_t + \rho\widetilde{\Lambda}\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) L[W, \frac{1}{8}D[M,M]]_t\\
&+ \frac{1}{2}\widetilde{\Lambda}\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) D[M, \frac{\rho}{2}L[W,M]]_t\\
&+ \rho\widetilde{\Lambda}^3\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\frac{\rho}{2}L[W,M]_t L[W, \frac{\rho}{2}L[W,M]]_t\\
&+ \rho\widetilde{\Lambda}^3\widetilde{\Gamma} C_{\overline{BS}}(t, X_t, v_t) L[W, \rho L[W, \frac{\rho}{2}L[W,M]]]_t\\
&+ \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t)\left(\frac{1}{8}D[M,M]_t\right)\\
&+ \epsilon_t.
\end{aligned}
$$

where ϵ_t is the error term satisfying

$$|\epsilon_t| \leq \nu^4 \left(1 + \rho^2 \left(1+\rho^2\right) + |\rho|\, \nu \left(1+\rho^2\right)\right) \left(\frac{1}{r} \wedge (T-t)\right) \Pi(\kappa, \theta),$$

and $\Pi(\kappa,\theta)$ is a positive constant depending on κ and θ.

Proof. The proof follows the same arguments as the previous proof. We will next provide a sketch of the proof and skip the lengthy computations. The main idea employed in the proof is to keep applying Corollary 4.1.10 to all the terms with order lower than $O(\nu^4)$, and to estimate the new terms which appear. Our next goal is to describe the terms that have to be decomposed.

The term (I.I) is decomposed into a series of new terms. We approximate (I.I) by the following expression:

$$A_t = \widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) \text{ and } B_t = \frac{1}{2}\left(\frac{\rho}{2} L[W,M]_t\right)^2.$$

This gives

$$\begin{aligned}
&- \frac{\rho^2}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u) L[W,M]_u \sigma_u \,\mathrm{d}[W,M]_u\right] \\
&= \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2 \\
&+ \frac{\rho^3}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^3\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u) L^2[W,M]_u \sigma_u \,\mathrm{d}[W,M]_u\right] \\
&+ \frac{\rho^2}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^4 C_{\overline{BS}}(u,X_u,v_u) L^2[W,M]_u \,\mathrm{d}[M,M]_u\right] \\
&+ \frac{\rho^3}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^3\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u) L[W,M]_u \sigma_u \,\mathrm{d}[W,L[W,M]]_u\right] \\
&+ \frac{\rho^2}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^3 C_{\overline{BS}}(u,X_u,v_u) L[W,M]_u \,\mathrm{d}[M,L[W,M]]_u\right] \\
&+ \frac{\rho^2}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(u,X_u,v_u) \,\mathrm{d}[L[W,M],L[W,M]]_u\right] \\
&= \frac{1}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}^2 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{\rho}{2}L[W,M]_t\right)^2 + (I.I.I) + \ldots + (I.I.V).
\end{aligned}$$

The terms (I.II) and (II.I) are approximated by

$$A_t = \widetilde{\Lambda}\widetilde{\Gamma}^3 C_{\overline{BS}}(t,X_t,v_t) \text{ and } B_t = \left(\frac{\rho}{2}L[W,M]_t\right)\left(\frac{1}{8}D[M,M]_t\right),$$

while the term (I.III) is approximated by

$$A_t = \widetilde{\Lambda}^2\widetilde{\Gamma} C_{\overline{BS}}(t,X_t,v_t) \text{ and } B_t = \frac{\rho^2}{2}L[W,L[W,M]]_t.$$

It follows that

$$\begin{aligned}
&- \frac{\rho^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(u,X_u,v_u)\sigma_u \,\mathrm{d}[W,L[W,M]]_u\right] \\
&= \frac{\rho^2}{2}\widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)L[W,L[W,M]]_t \\
&+ \frac{\rho^3}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^3\widetilde{\Gamma}^2C_{\overline{BS}}(u,X_u,v_u)L[W,L[W,M]]_u\sigma_u \,\mathrm{d}[W,M]_u\right] \\
&+ \frac{\rho^2}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^3C_{\overline{BS}}(u,X_u,v_u)L[W,L[W,M]]_u \,\mathrm{d}[M,M]_u\right] \\
&+ \frac{\rho^3}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^3\widetilde{\Gamma}C_{\overline{BS}}(u,X_u,v_u)\sigma_u \,\mathrm{d}[W,L[W,L[W,M]]]_u\right] \\
&+ \frac{\rho^2}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}^2\widetilde{\Gamma}^2C_{\overline{BS}}(u,X_u,v_u) \,\mathrm{d}[M,L[W,L[W,M]]]_u\right] \\
&= \widetilde{\Lambda}^2\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t)\frac{\rho^2}{2}L[W,L[W,M]]_t + (I.III.I) + \ldots (I.III.IV).
\end{aligned}$$

The term (I.IV) is approximated by

$$A_t = \widetilde{\Lambda}\widetilde{\Gamma}^2C_{\overline{BS}}(t,X_t,v_t) \text{ and } B_t = \frac{\rho}{4}D[M,L[W,M]]_t,$$

while the term (II.III) is approximated by

$$A_t = \widetilde{\Lambda}\widetilde{\Gamma}^2C_{\overline{BS}}(t,X_t,v_t) \text{ and } B_t = \rho L[W,\frac{1}{8}D[M,M]]_t.$$

Similarly, the term (I.I.I) is approximated by

$$A_t = \widetilde{\Lambda}^3\widetilde{\Gamma}^3C_{\overline{BS}}(t,X_t,v_t) \text{ and } B_t = \frac{1}{6}\left(\frac{\rho}{2}L[W,M]_t\right)^3.$$

The terms (I.I.III) and (I.III.I) are approximated by the following expressions:

$$A_t = \widetilde{\Lambda}^3\widetilde{\Gamma}^2C_{\overline{BS}}(t,X_t,v_t) \text{ and } B_t = \frac{\rho^3}{4}L[W,M]_tL[W,L[W,M]]_t.$$

Moreover, the term (I.III.III) is approximated by

$$A_t = \widetilde{\Lambda}^3\widetilde{\Gamma}C_{\overline{BS}}(t,X_t,v_t) \text{ and } B_t = \frac{\rho^3}{2}L[W,L[W,L[W,M]]]_t.$$

We estimate each of the new terms appearing in the proof exactly as in the previous one. □

For the uncorrelated Heston model, we obtain a similar expansion with fewer terms and a better error estimate.

Theorem 5.3.5. *Suppose $\rho = 0$, then for every $t \in [0, T]$, the following formula holds:*

$$\begin{aligned} V_t &= C_{\overline{BS}}(t, X_t, v_t) \\ &+ \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) \left(\frac{1}{8} D[M, M]_t \right) + \frac{1}{2} \widetilde{\Gamma}^4 C_{\overline{BS}}(t, X_t, v_t) \left(\frac{1}{8} D[M, M]_t \right)^2 \\ &+ \frac{1}{2} \widetilde{\Gamma}^3 C_{\overline{BS}}(t, X_t, v_t) D[M, \frac{1}{8} D[M, M]]_t \\ &+ \epsilon_t \end{aligned}$$

where ϵ_t is the error term satisfying

$$|\epsilon_t| \leq \nu^6 \left(\frac{1}{r} \wedge (T - t) \right) \Pi(\kappa, \theta),$$

and $\Pi(\kappa, \theta)$ is positive constant depending on κ and θ.

Proof. It is easy to see that for the uncorrelated Heston model, the model reduces to

$$\begin{aligned} V_t &= C_{\overline{BS}}(t, X_t, v_t) \\ &+ \frac{1}{8} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^2 C_{\overline{BS}}(u, X_u, v_u) \, \mathrm{d}[M, M]_u \right]. \end{aligned}$$

Using Corollary 4.1.10, we obtain a special case of formula (4.9). In particular, it is obtained

$$\begin{aligned} (II) &= \frac{1}{8} \widetilde{\Gamma}^2 C_{\overline{BS}}(t, X_t, v_t) D[M, M]_t \\ &+ \frac{1}{64} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^4 C_{\overline{BS}}(u, X_u, v_u) D[M, M]_u \, \mathrm{d}[M, M]_u \right] \\ &+ \frac{1}{16} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^3 C_{\overline{BS}}(u, X_u, v_u) \, \mathrm{d}[M, D[M, M]]_u \right] \\ &= (A) + (B). \end{aligned}$$

An upper bound for (A)

We apply Corollary 4.1.10 to the expression denoted by (A). Choosing

$$A_t = \widetilde{\Gamma}^4 C_{\overline{BS}}(t, X_t, v_t) \text{ and } B = \frac{1}{2} \left(\frac{1}{8} D[M, M]_t \right)^2,$$

it follows that

$$\begin{aligned} (A) &= \frac{1}{2} \widetilde{\Gamma}^4 C_{\overline{BS}}(t, X_t, v_t) \left(\frac{1}{8} D[M, M]_t \right)^2 \\ &+ \frac{1}{1024} \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^6 C_{\overline{BS}}(u, X_u, v_u) \left(D[M, M]_u \right)^2 \, \mathrm{d}[M, M]_u \right]. \end{aligned}$$

In particular, for the Heston model, we obtain

$$\begin{aligned}(A) &= \frac{1}{2}\widetilde{\Gamma}^4 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{1}{8}D[M,M]_t\right)^2\\ &+ \frac{\nu^6}{1024}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^6 C_{\overline{BS}}(u,X_u,v_u)\left(\int_u^T \mathbb{E}_u\left(\sigma_s^2\right)\varphi(s)^2\,\mathrm{d}s\right)^2\sigma_u^2\varphi^2(u)\,\mathrm{d}u\right].\end{aligned}$$

Next, using Lemma 3.1.10 (ii), applying the equivalence $a_u = v_u\sqrt{T-u}$, and using the fact that $\varphi(t)$ is a decreasing function, then

$$\begin{aligned}&\left|(A) - \frac{1}{2}\widetilde{\Gamma}^4 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{1}{8}D[M,M]_t\right)^2\right|\\ \leq\ & C\frac{\nu^6}{1024}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^7}+\frac{5}{a_u^6}+\frac{10}{a_u^5}+\frac{10}{a_u^4}+\frac{5}{a_u^3}+\frac{1}{a_u^2}\right)\sigma_u^2\varphi(u)^4\,\mathrm{d}u\right].\end{aligned}$$

It follows from Lemma 5.1.2 (ii) that $\sigma_t^2 \leq \frac{a_t^2}{\varphi(t)}$ and

$$\begin{aligned}&\left|(A) - \frac{1}{2}\widetilde{\Gamma}^4 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{1}{8}D[M,M]_t\right)^2\right|\\ \leq\ & C\frac{\nu^6}{1024}\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{5}{a_u^4}+\frac{10}{a_u^3}+\frac{10}{a_u^2}+\frac{5}{a_u}+1\right)\varphi(u)^3\,\mathrm{d}u.\end{aligned}$$

Now, Lemma 5.1.2 (i) implies that $a_t \geq \frac{\sqrt{\theta\kappa}}{\sqrt{2}}\varphi(t)$. Applying the estimate $\varphi(t) \leq \frac{1}{\kappa}$, it is obtained that

$$\begin{aligned}&\left|(A) - \frac{1}{2}\widetilde{\Gamma}^4 C_{\overline{BS}}(t,X_t,v_t)\left(\frac{1}{8}D[M,M]_t\right)^2\right|\\ \leq\ & C\frac{\nu^6}{1024\kappa^3}\left(\frac{4\sqrt{2}\kappa^2\sqrt{\kappa}}{\theta^2\sqrt{\theta}}+\frac{10\kappa^2}{\theta^2}+\frac{20\sqrt{2}\kappa\sqrt{\kappa}}{\theta\sqrt{\theta}}+\frac{20\kappa}{\theta}+\frac{5\sqrt{2}\kappa}{\sqrt{\theta\kappa}}+1\right)\int_t^T e^{-r(u-t)}\,\mathrm{d}u.\end{aligned}$$

An upper bound for (B)

We apply the Corollary 4.1.10 to the expression denoted by (B). Choosing

$$A_t = \widetilde{\Gamma}^3 C_{\overline{BS}}(t,X_t,v_t) \text{ and } B = \frac{1}{16}D[M,D[M,M]]_t,$$

it follows that

$$\begin{aligned}(B) &= \frac{1}{16}\widetilde{\Gamma}^3 C_{\overline{BS}}(t,X_t,v_t)D[M,D[M,M]]_t\\ &+ \frac{1}{128}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^5 C_{\overline{BS}}(u,X_u,v_u)D[M,D[M,M]]_u\,\mathrm{d}[M,M]_u\right]\end{aligned}$$

$$+ \quad \frac{1}{24}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^4 C_{\overline{BS}}(u,X_u,v_u)\,\mathrm{d}[M,D[M,D[M,M]]]_u\right].$$

In the case of the Heston model, setting $a_u := v_u\sqrt{T-u}$ and using Lemma 3.1.10 (ii) and the fact that $\varphi(t)$ is a decreasing function, then

$$\begin{aligned}
&\left|(B) - \frac{1}{16}\widetilde{\Gamma}^3 C_{\overline{BS}}(t,X_t,v_t)D[M,D[M,M]]_t\right| \\
\leq\ & C\frac{\nu^6}{128}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^7}+\frac{4}{a_u^6}+\frac{6}{a_u^5}+\frac{4}{a_u^4}+\frac{1}{a_u^3}\right)\sigma_u^2\varphi(u)^6\,\mathrm{d}u\right] \\
+\ & C\frac{\nu^6}{24}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^7}+\frac{3}{a_u^6}+\frac{3}{a_u^5}+\frac{1}{a_u^4}\right)\sigma_u^2\varphi^6(u)\,\mathrm{d}u\right].
\end{aligned}$$

Now, Lemma 5.1.2 (ii) implies that $\sigma_t^2 \leq \frac{a_t^2}{\varphi(t)}$ and

$$\begin{aligned}
&\left|(B) - \frac{1}{16}\widetilde{\Gamma}^3 C_{\overline{BS}}(t,X_t,v_t)D[M,D[M,M]]_t\right| \\
\leq\ & C\frac{\nu^6}{128}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{4}{a_u^4}+\frac{6}{a_u^3}+\frac{2}{a_u^4}+\frac{1}{a_u}\right)\varphi(u)^5\,\mathrm{d}u\right] \\
+\ & C\frac{\nu^6}{24}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{3}{a_u^4}+\frac{3}{a_u^3}+\frac{1}{a_u^2}\right)\varphi^5(u)\,\mathrm{d}u\right].
\end{aligned}$$

It follows from Lemma 5.1.2 (i) that $a_t \geq \frac{\sqrt{\theta\kappa}}{\sqrt{2}}\varphi(t)$. In addition, the inequality $\varphi(t) \leq \frac{1}{\kappa}$ implies that

$$\begin{aligned}
&\left|(B) - \frac{1}{16}\widetilde{\Gamma}^3 C_{\overline{BS}}(t,X_t,v_t)D[M,D[M,M]]_t\right| \\
\leq\ & C\frac{\nu^6}{128}\left(\frac{4\sqrt{2}}{\theta^2\kappa^2\sqrt{\theta\kappa}}+\frac{16}{\theta^2\kappa^3}+\frac{12\sqrt{2}}{\theta\kappa^3\sqrt{\theta\kappa}}+\frac{4}{\theta\kappa^4}+\frac{\sqrt{2}}{\kappa^4\sqrt{\theta\kappa}}\right)\int_t^T e^{-r(u-t)}\,\mathrm{d}u \\
+\ & C\frac{\nu^6}{24}\left(\frac{4\sqrt{2}}{\theta^2\kappa^2\sqrt{\theta\kappa}}+\frac{12}{\theta^2\kappa^3}+\frac{6\sqrt{2}}{\theta\kappa^3\sqrt{\theta\kappa}}+\frac{2}{\theta\kappa^4}\right)\int_t^T e^{-r(u-t)}\,\mathrm{d}u.
\end{aligned}$$

□

Remark 5.3.6. *For each approximation formula for call option prices under the Heston model, we have found a positive constant $\Pi(\kappa,\theta)$ depending on κ and θ. Although we have used the same function to specify this constant, it is different for each approximation.*

5.4 Numerical results

In this section, we compare the performance of the call option price approximation formula proposed in Alòs (2012) and Alòs et al. (2015) with the new approximation formulae obtained before. To simplify the notation, we call the formula obtained in Alòs (2012) and Alòs et al. (2015), the formula with error estimate $O(\nu^2)$, while the two formulae obtained in the present paper are referred as the formulae with error estimates $O(\nu^3)$ and $O(\nu^4)$, respectively (see Theorems 5.3.3 and 5.3.4). We also make a similar comparison in the uncorrelated case. Here, we compare the formula with error estimate $O(\nu^4)$ established in Alòs (2012) and Alòs et al. (2015) with the new formula with an error estimate $O(\nu^6)$ found in the present paper (see Theorem 5.3.5). As a benchmark price, we choose a call option price obtained using a Fourier transform based pricing formula. This is one of the standard approaches to pricing European options under stochastic volatility models. In particular, we use a semi-closed form solution with one numerical integration as a reference price (see Mrázek and Pospíšil (2017))[1]. The comparison between approximations is made with two important aspects in mind: the practical precision of the pricing formula and the efficiency of the formula expressed in terms of the computational time needed for particular pricing tasks.

Analytical approximations of the implied volatility exist in the literature, for example, in Forde et al. (2012) and Lorig et al. (2017). We will compare these approximations with the implied volatilities obtained from the approximation formula with error estimate $O(\nu^4)$ for the correlated case and the formula with error estimate $O(\nu^6)$ for the uncorrelated case.

Our next goal is to illustrate the quality of our new approximation formulae for the call option price in the Heston model for various values of ρ and ν while keeping the other parameters fixed. Concretely, we choose the following parameters: $S_0 = 100$, $r = 0.001$, $v_0 = 0.25$, $\kappa = 1.5$, and $\theta = 0.2$. We understand the error in the price as the relative error in a $\log_{10}$ scale. The blue line illustrates the approximation with an error estimate $O(\nu^2)$, the red line is the approximation with an error estimate $O(\nu^3)$, while the yellow line corresponds to the approximation with an error estimate $O(\nu^4)$.

Figure 5.1 shows approximations of the call option price when the vol-vol, ν, and the absolute value of the correlation, ρ, are both small. In this case, $\nu = 5\%$ and $\rho = -0.2$. We observe that the approximation formula with an error estimate $O(\nu^3)$, in general, performs better than the formula with an error estimate $O(\nu^2)$. However, in some cases, there are exceptions, such as the ITM options for $\tau = 3$. The call option price approximation with an error estimate $O(\nu^4)$ is much better, with an error around $10^{-7} - 10^{-10}$.

[1]With a slight modification mentioned in Gatheral (2006) in order not to suffer from the "Heston trap" issues.

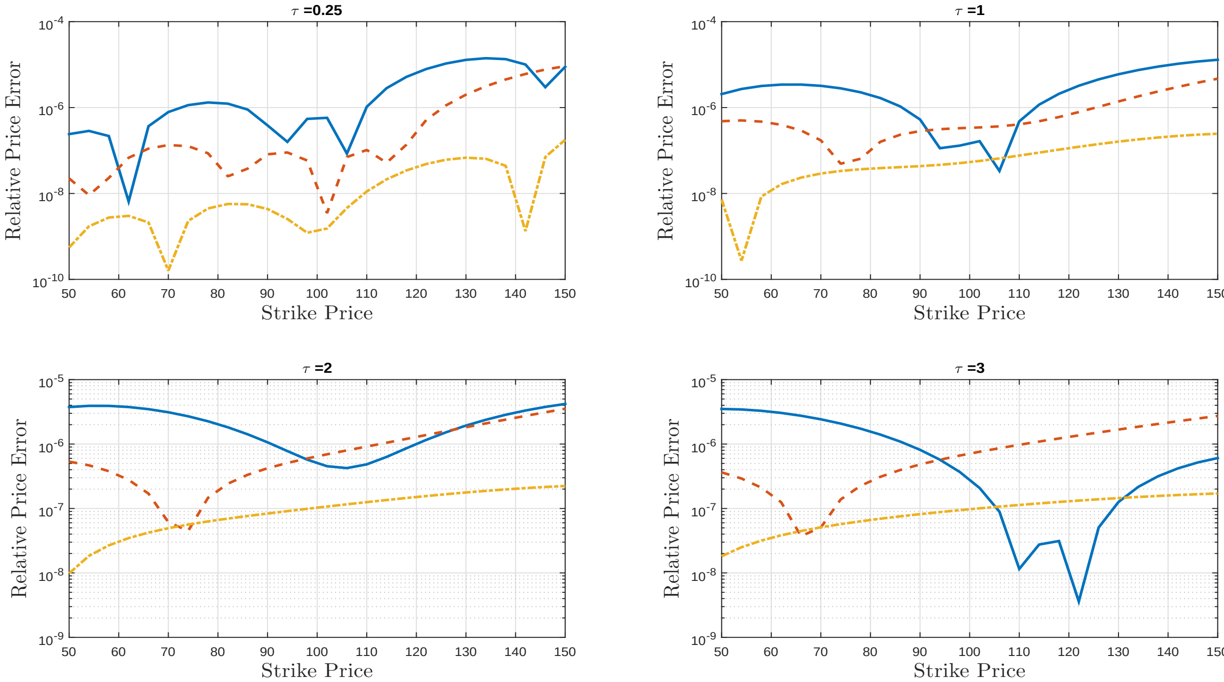


Figure 5.1: Heston model: Comparison of the three different approximation formulae and reference prices for $\nu = 5\%$ and $\rho = -0.2$.

In Figure 5.2, we discuss the case where ν is small while $|\rho|$ is close to one. In this case, $\nu = 5\%$ and $\rho = -0.8$. We observe that the new approximation formulae perform better than the formula previously known. The approximation error is in the range $10^{-4} - 10^{-8}$ for the formula with an error estimate $O(\nu^3)$ and $10^{-7} - 10^{-10}$ for the one with an error estimate $O(\nu^4)$.

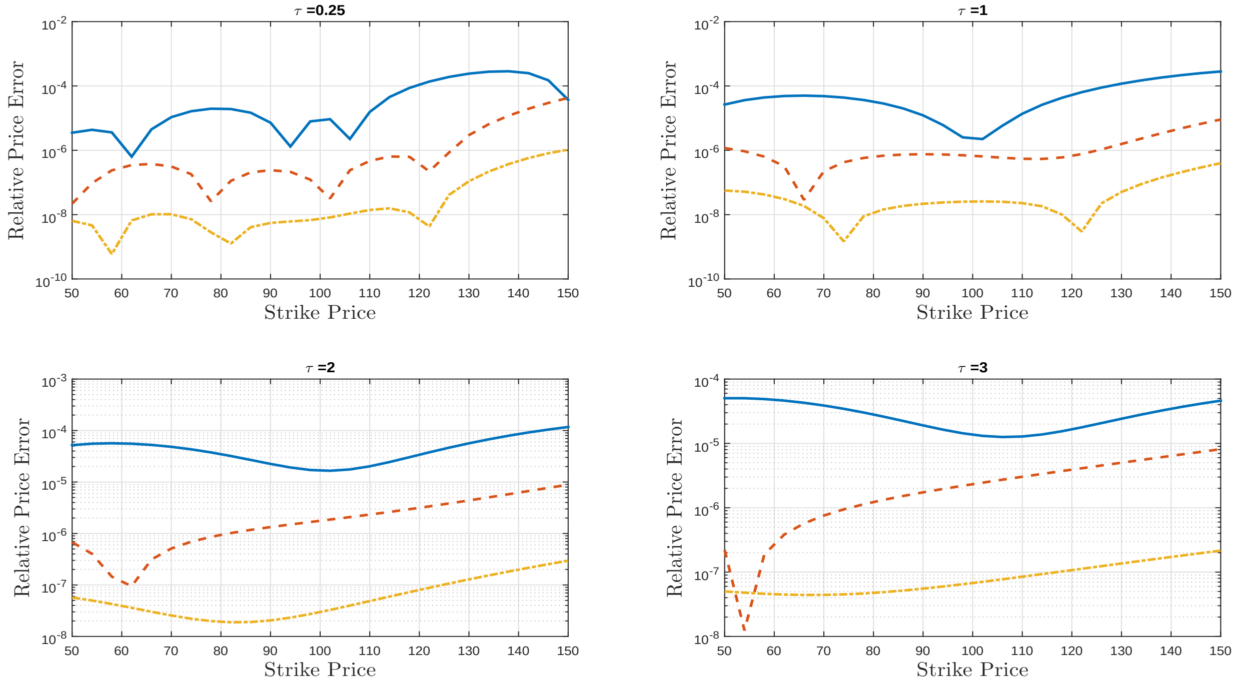


Figure 5.2: Heston model: Comparison of the three different approximation formulae and reference prices for $\rho = -0.8$ and $\nu = 5\%$.

Figure 5.3 refers to the case of high vol-vol and low absolute correlation. In this case, $\nu = 50\%$ and $\rho = -0.2$. Here, we note that the three approximation formulae show similar performance. The approximation formula where the error estimate is $O(\nu^4)$ seems to perform a little better, but not significantly.

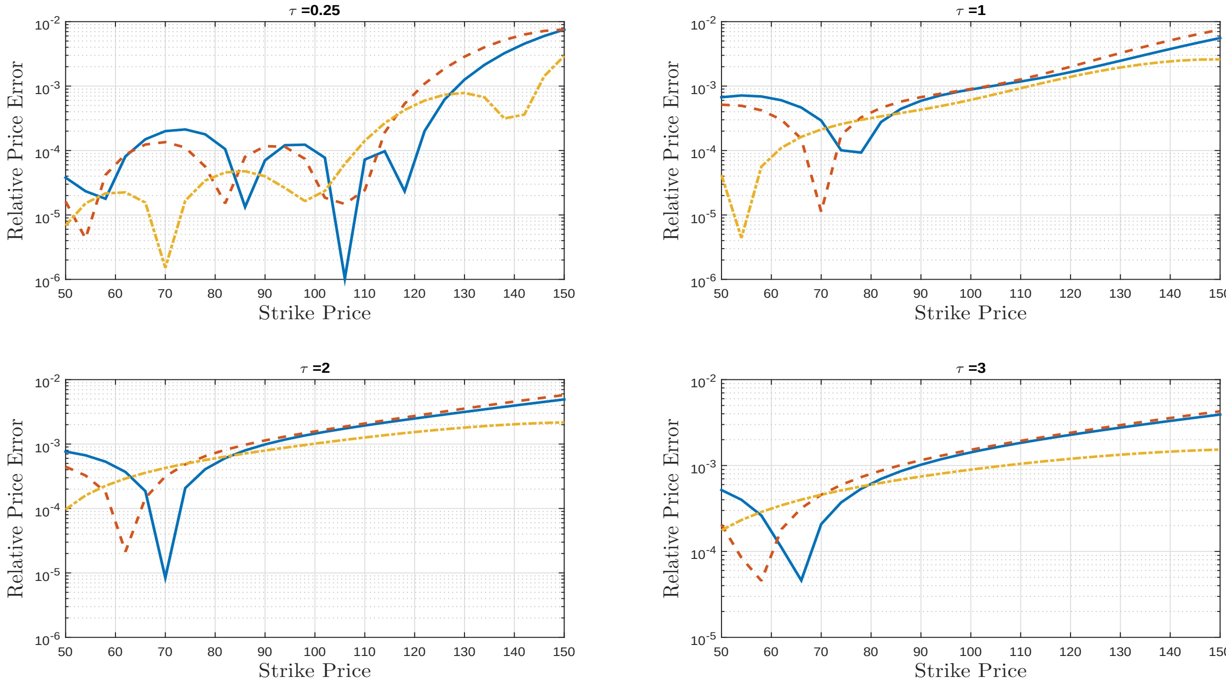


Figure 5.3: Heston model: Comparison of the three different approximation formulae and reference prices for $\rho = -0.2$ and $\nu = 50\%$.

Figure 5.4 illustrates the performance of the formulae when both parameters are not suitable for the approximation, e.g., when $\nu = 50\%$ and $\rho = -0.8$. Here, we observe that the approximations have a similar quality. The approximation formula with an error estimate $O(\nu^4)$ seems to perform better than the other formulae, while the formula with an error estimate $O(\nu^3)$ performs better only in the short term.

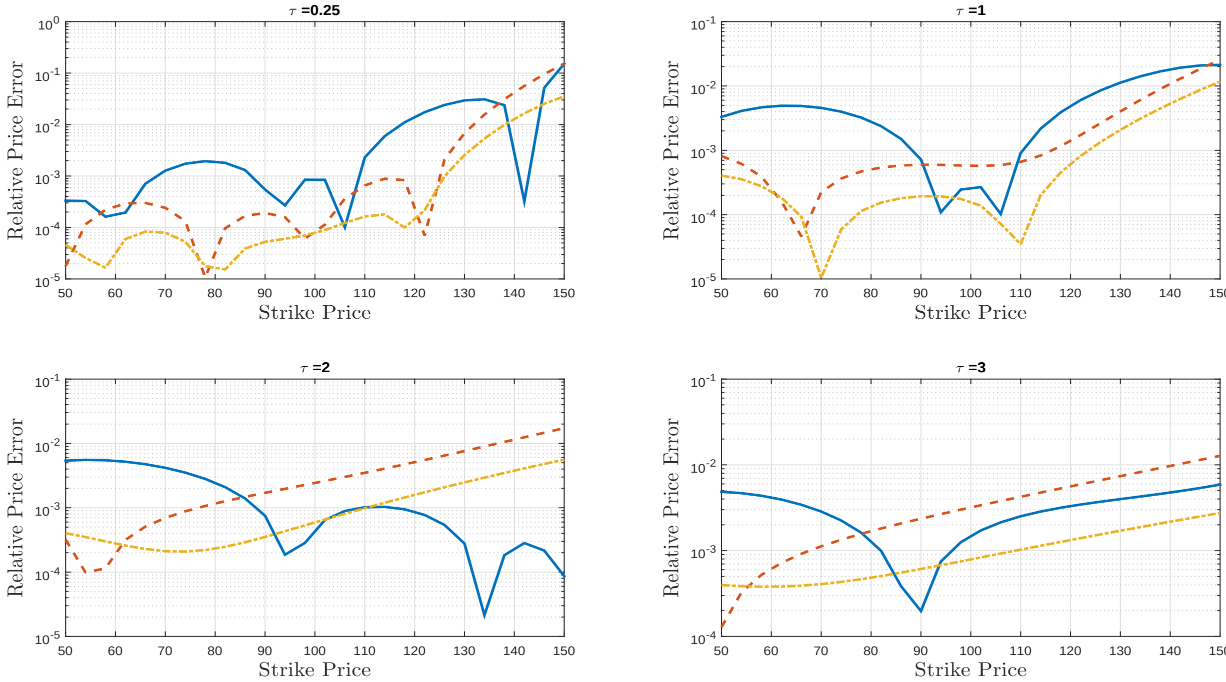


Figure 5.4: Heston model: Comparison of the three different approximation formulae and reference prices for $\rho = -0.8$ and $\nu = 50\%$.

Comparing Figure 5.4 with Figure 5.3, we observe that the new approximation formulae are more efficient in the former figure than in the latter ones. This can be explained by the fact that most of the terms in the expansion include the parameter ρ. When $|\rho|$ is small, the new approximations are closer to the known ones than when $|\rho|$ is close to one.

We have already observed that the approximation formulae obtained in the present paper perform better than the previously known formula when $|\rho|$ is close to one and ν is small. On the other hand, the improvement in the performance is not significant for large ν. This can be fixed by adding more terms. As an example, we compare the benchmark prices with their approximations in the uncorrelated case. In Figures 5.5 and 5.6, the blue line is the approximation with an error estimate $O(\nu^4)$, while the red line is the approximation with an error estimate $O(\nu^6)$.

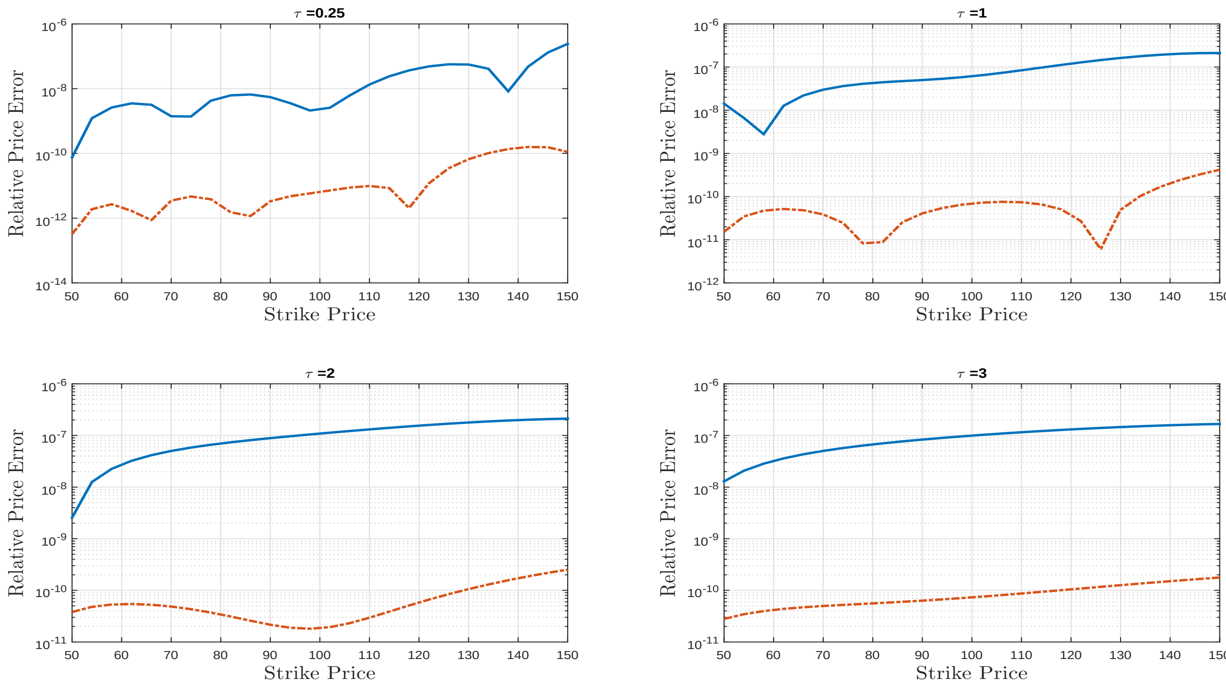


Figure 5.5: Heston model: Comparison of the two different approximation formulae and reference prices for $\rho = 0$ and $\nu = 5\%$.

In Figure 5.5, we illustrate the case of low vol-vol. In this case, $\nu = 5\%$ and $\rho = 0$. The formulae with error estimates $O(\nu^4)$ and $O(\nu^6)$ have a very small error, while the new approximation behaves much better. Figure 5.6 shows the approximations when ν is large. In this case, $\nu = 50\%$ and $\rho = 0$. We can see that the new approximation behaves better, especially in the long term.

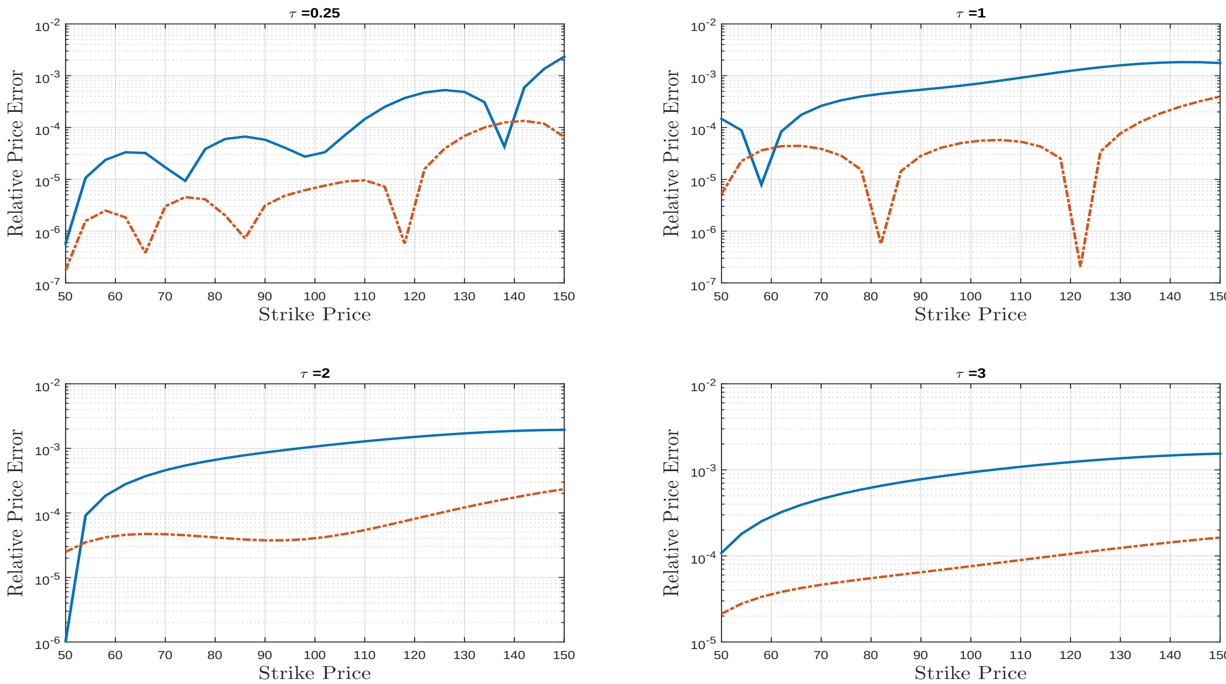


Figure 5.6: Heston model: Comparison of the two different approximation formulae and reference prices for $\rho = 0$ and $\nu = 50\%$.

One of the main advantages of the proposed option price approximations is its computational efficiency. To compare the amount of time each method spent on computations, we replicate the computational effort to perform in different calibration situations with three pricing tasks. We used a batch of 100 various call options with different strikes and times to maturity, including OTM, ATM, and ITM options with short-, mid- and long-term times to maturity. The first task was to evaluate the option prices in the batch with respect to 100 (uniformly) randomly sampled parameter sets. This task has a similar number of price evaluations to a market calibration task with a very good initial guess. Further on, we repeated the same trials for 1.000 and 10.000 parameter sets to mimic the number of evaluations for a typical local-search calibration and a global-search calibration, respectively (for more information about calibration tasks see e.g. Mikhailov and Nögel (2003) and Mrázek et al. (2016)).

Our results are listed in Table 5.1. The call prices were analytically calculated in all the cases. We observe that for the trials of 100 and 1.000 sets, the amount of time spent on computations are rather similar. For the trial of 10.000 sets, the experiment based on the approximation with an error order of $O(\nu^4)$ was a little bit slower than the other experiments.

Table 5.1: Heston model: Efficiency of the call price approximations

Pricing approach	Task	Time† [sec]	Speed-up factor
Heston-Lewis	#1	3.63	-
	#2	33.52	-
	#3	336.59	-
Approximation of order $O(\nu^2)$	#1	0.08	45×
	#2	0.76	44×
	#3	7.41	45×
Approximation of order $O(\nu^3)$	#1	0.08	45×
	#2	0.78	43×
	#3	7.77	43×
Approximation of order $O(\nu^4)$	#1	0.10	36×
	#2	0.91	37×
	#3	8.87	38×

† The results were obtained on a PC with Intel Core i7-7700HQ CPU @2.80 GHz 2.80GHz and 16 GB RAM.

The table shows that the approximations, where the error order is $O(\nu^2)$ or $O(\nu^3)$ are around 43-45 times faster than the approximation based on the fast Fourier transform methodology, while the approximation with the error of order $O(\nu^4)$ is around 36 times faster than the latter one. Therefore, the approximations with error order $O(\nu^2)$ or $O(\nu^3)$ are around 1.14-1.25 times less time-consuming than the $O(\nu^4)$-approximation.

Our next goal is to compare the approximation formulae presented in this paper with other analytical approximation methods. In Forde et al. (2012), based on saddlepoint methods, they derive a small-maturity expansion formula for prices that are transformed into a closed-form implied volatility for the Heston model. In Lorig et al. (2017), they derive an explicit implied volatility for local-stochastic volatility models, including the Heston case, using a perturbation technique for parabolic equations. We choose the following values for the Heston parameters: $S_0 = 100$, $r = 0$, $v_0 = 0.20$, $\kappa = 1.15$, $\theta = 0.04$, $\nu = 0.2$ and $\rho = -0.4$. We understand the error in the implied volatility as the absolute error in a $\log_{10}$ scale. The blue line illustrates the approximation with an error estimate $O(\nu^4)$, the red line is the 3rd order approximation of the implied volatility by Lorig et al. (2017), the yellow line is the 2nd order approximation of the implied volatility by Lorig et al. (2017), while the purple line corresponds to the approximation by Forde et al. (2012). We compare our methodology with Forde et al. (2012) only for maturities less than 1 year.

In Figure 5.7, we observe that our approximation is more accurate in almost all the cases. As it was expected, Forde et al. (2012) approximation is competitive for short-term maturities, but the error increases with the time to maturity and the 3rd order approximation of the implied volatility behaves in general better than the 2nd order approximation. We observe that the results of our approximation are very close to the 3rd order expansion of the implied volatility by Lorig et al. (2017).

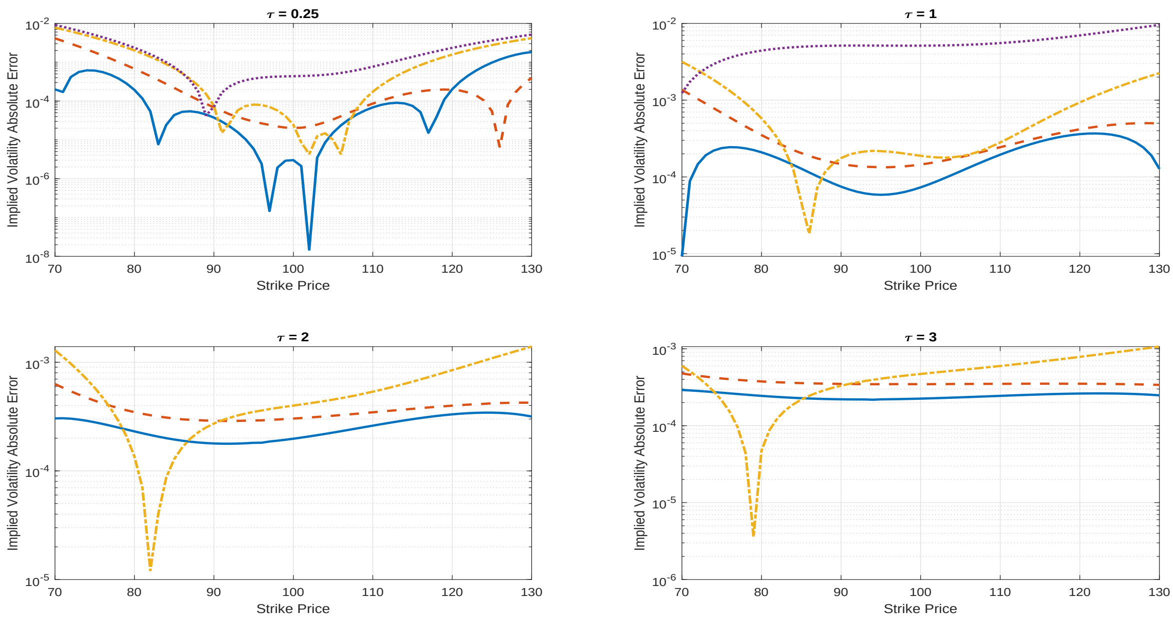


Figure 5.7: Heston model: Comparison with other analytical approximation methods.

In Figure 5.8, we compare all the approximations when $\rho = 0$. We observe that the 2nd order approximation and 3rd order approximation coincide. In general, our approximation is better than the other methods, especially when the time to maturity increases.

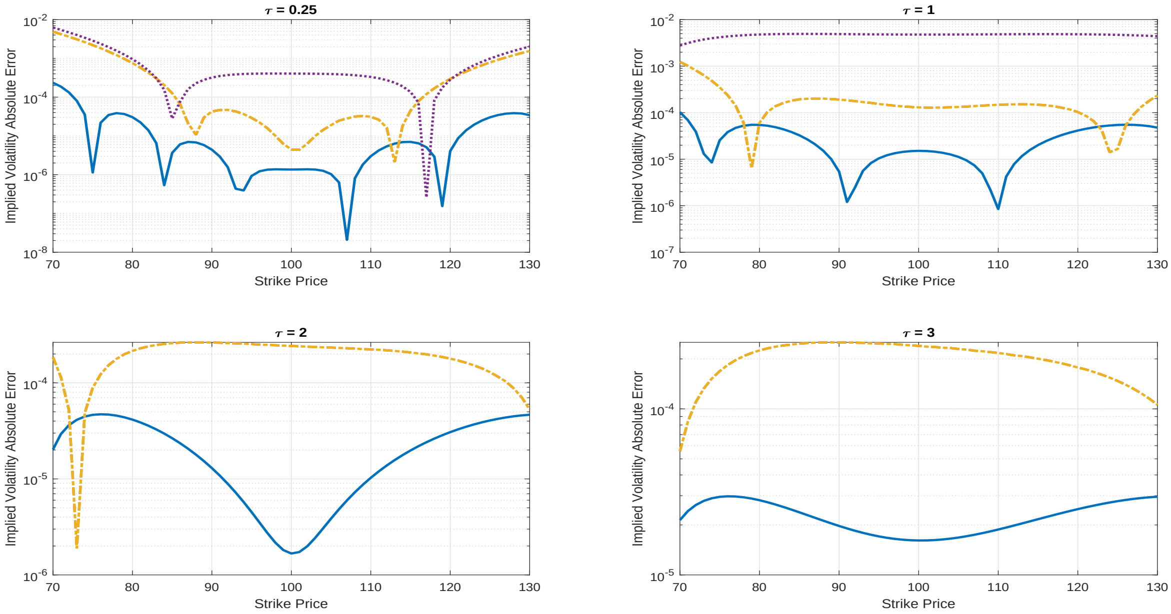


Figure 5.8: Heston model: Comparison with other analytical approximation methods when $\rho = 0$.

CHAPTER 6

Jump Diffusion Models

In Chapter 4, we saw how we can obtain a decomposition formula when the underlying model is a stochastic volatility model. In the previous chapter, we applied these ideas to Heston model. Unfortunately, frequently, when we look at the implied volatility market data, we can see a short-term spike in the volatility that these models cannot produce. There is a class of financial models that can reproduce this phenomenon, and these models combine the ideas of adding a jump diffusion process with a stochastic volatility structure. These types of models are stochastic volatility jump diffusion (SVJ) models.

Developing the ideas presented in the previous chapter, the decomposition formula can be extended to SVJ models with finite activity jumps. Assuming a Heston type volatility structure, an approximation price of a European call option can be found. The approximation is explicit and is numerically efficient. It is also possible to find an approximation of the implied volatility. To evaluate the precision and efficiency of this methodology, a numerical comparison has been made for the Bates model, comparing the approximation with the Fourier transformation introduced by Baustian et al. (2017). In contrast to Alòs et al. (2007) and Alòs et al. (2008) where a Hull and White formula is obtained to study the short-term behaviour of implied volatility, this chapter focuses on how to efficiently approach the valuation of European call option prices and the calculation of a parametric approximation of the surface of implied volatility. The ideas in this chapter are based on Merino et al. (2018).

6.1 A decomposition formula for SVJ models.

In previous chapters, a generic decomposition formula has been obtained for stochastic volatility models with continuous sample paths. Now, we need to extend this methodology for a general jump diffusion model with finite activity jumps. The main idea is to adapt the pricing process in such a way as to be able to apply the decomposition technique effectively. In our case, this would translate into conditioning the finite number of jumps n_T. If we denote $J_n = \sum_{i=0}^{n} Y_i$, using the integrability of Black-Scholes-Merton function, we can obtain the following conditioning formula for European options with payoff at maturity

$T : C_{\widetilde{BS}}(T, X_T^J, v_T)$.

Recall the X_t^C, defined in (3.33), is the continuous counterpart of the stochastic volatility model defined in (3.32).

We define $p_n(\lambda T)$ as the Poisson probability mass function with intensity λT. I.e. p_n takes the following form:

$$p_n(\lambda(T-t)) := \frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}. \tag{6.1}$$

We can re-write the pricing formula in the following way:

$$\begin{aligned}
V_t &= e^{-r(T-t)}\mathbb{E}_t\left[C_{\widetilde{BS}}(T, X_T^J, v_T)\right] \\
&= e^{-r(T-t)}\sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right)\mathbb{E}_t\left[C_{\widetilde{BS}}\left(T, X_T^C + \sum_{i=0}^{n_T} Y_i, v_T\right)\Big| n_T = n\right] \\
&= e^{-r(T-t)}\sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right)\mathbb{E}_t\left[C_{\widetilde{BS}}\left(T, X_T^C + J_n, v_T\right)\right] \\
&= e^{-r(T-t)}\sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right)\mathbb{E}_t\left[\mathbb{E}_{J_n}\left[C_{\widetilde{BS}}(T, X_T^C + J_n, v_T)\right]\right] \\
&= e^{-r(T-t)}\sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right)\mathbb{E}_t\left[G_n(T, X_T^C, v_T)\right]
\end{aligned}$$

where

$$G_n(T, \tilde{X}_T^J, v_T) := \mathbb{E}_{J_n}\left[C_{\widetilde{BS}}(T, X_T^C + J_n, v_T)\right]. \tag{6.2}$$

Notice that we have gone from a problem with a jump diffusion model with stochastic volatility to one without jumps. Combining the generic SV decomposition formula, Remark 4.1.12 alongside Remark 4.1.9, and conditioning with respect to the number of jumps, we obtain the basis for our decomposition formula.

Corollary 6.1.1 (SVJ decomposition formula)**.** *Let S_t be the price process defined in* (3.29)*, considering the change of variables $X_t^J = \log(S_t)$ to obtain the log-price process defined in* (3.32)*, and G_n be the function defined in* (6.2)*. Then we can express the call option fair value V_t using the Poisson mass function p_n and a martingale process M_t defined by* (4.5)*. In particular,*

$$\begin{aligned}
V_t &= \sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right) G_n(t, X_t^C, v_t) \qquad (6.3)\\
&+ \frac{1}{8}\sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Gamma}^2 G_n(u, X_u^C, v_u)d[M,M]_u\right] \\
&+ \frac{\rho}{2}\sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\widetilde{\Lambda}\widetilde{\Gamma} G_n(u, X_u^C, v_u)\sigma_u d[W,M]_u\right].
\end{aligned}$$

Proof. Using Remark 4.1.12, taking into account Remark 4.1.9, to $A(t, X_t^J, v_t^2) := G_n(t, X_t^C, v_t)$ and $B_t \equiv 1$. It can be seen that

$$\partial_{\sigma^2} BS(t,x,\sigma) = \frac{(T-t)}{2}\left(\partial_x^2 - \partial_x\right) BS(t,x,\sigma)$$

and

$$\partial_{\sigma^2}^2 BS(t,x,\sigma) = \frac{(T-t)^2}{4}\left(\partial_x^2 - \partial_x\right)^2 BS(t,x,\sigma).$$

Then, the corollary follows immediately. □

Remark 6.1.2. *For clarity, in the following we will refer to the terms of the previous decomposition as*

$$V_t = \sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right) G_n(t, X_t^C, v_t) + \sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right)\left[(I_n) + (II_n)\right]. \tag{6.4}$$

Note that if $\rho = 0$, we have

$$V_t = \sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right) G_n(t, X_t^C, v_t) + \sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right)(I_n). \tag{6.5}$$

The term (II_n) is the correction due to the dependence between the stock and volatility processes meanwhile (I_n) is the correction of the vol-vol of the volatility model.

As we mentioned in previous chapters, the expression above is elegant and compact, but it cannot be calculated directly. As with the Heston model, we will apply Remark 4.1.12, in this case jointly with Remark 4.1.9, recursively to obtain an approximation formula. In this case, we are going to develop the approximation formula with the same error magnitude as in Alòs (2012), even though it would be possible to extend the approach to higher orders as has been done in the Heston model.

Corollary 6.1.3 (Computationally suitable SVJ decomposition)**.** *Let S_t be the price process defined in (3.29), considering the change of variables $X_t^J = \log(S_t)$ to obtain the log-price process defined in (3.32), and G_n be the function defined in (6.2). Then we can express the call option fair value V_t using the Poisson mass function p_n and a martingale process M_t defined by (4.5). In particular,*

$$\begin{aligned} V_t &= \sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right) G_n(t, X_t^C, v_t) \\ &+ \sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right) \widetilde{\Lambda}\widetilde{\Gamma} G_n(t, X_t^C, v_t) L[W,M]_t \\ &+ \sum_{n=0}^{\infty} p_n\left(\lambda(T-t)\right) \widetilde{\Gamma}^2 G_n(t, X_t^C, v_t) D[M,M]_t \end{aligned} \tag{6.6}$$

$$+ \sum_{n=0}^{\infty} p_n \left(\lambda(T-t)\right) \epsilon_t^n$$

where ϵ_t^n *are error terms.*

Proof. The main idea in the proof is to apply the Remark 4.1.12, considering Remark 4.1.9, to the terms (I_n) and (II_n).

The term (I_n) is decomposed with

$$A(t, X_t, v_t^2) := \Lambda\Gamma G_n(t, X_t^C, v_t) \text{ and } B_t = \frac{\rho}{2} L[W, M]_t.$$

This gives

$$\begin{aligned}
& \frac{\rho}{2}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}\widetilde{\Gamma} G_n(u, X_u^C, v_u)\sigma_u d[W, M]_u \right] - \widetilde{\Lambda}\widetilde{\Gamma} G_n(t, X_t^C, v_t) L\left[W, M\right]_t \\
= \; & \frac{1}{8}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}\widetilde{\Gamma}^3 G_n(u, X_u^C, v_u) L\left[W, M\right]_u d[M, M]_u \right] \\
+ \; & \frac{\rho}{2}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}^2\widetilde{\Gamma}^2 G_n(u, X_u^C, v_u) L\left[W, M\right]_u \sigma_u d[W, M]_u \right] \\
+ \; & \rho\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}^2\widetilde{\Gamma} G_n(u, X_u^C, v_u)\sigma_u d[W, L\left[W, M\right]]_u \right] \\
+ \; & \frac{1}{2}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}\widetilde{\Gamma}^2 G_n(u, X_u^C, v_u) d[M, L\left[W, M\right]]_u \right].
\end{aligned}$$

The term (II_n) is decomposed with

$$A(t, X_t, v_t^2) := \widetilde{\Gamma}^2 G_n(t, X_t^C, v_t) \text{ and } B_t = \frac{1}{8} D[M, M]_t.$$

This gives

$$\begin{aligned}
& \frac{1}{8}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^2 G_n(u, X_u^C, v_u) d[M, M]_u \right] - \widetilde{\Gamma}^2 G_n(t, X_t^C, v_t) D\left[M, M\right]_t \\
= \; & \frac{1}{8}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^4 G_n(u, X_u^C, v_u) D\left[M, M\right]_u d[M, M]_u \right] \\
+ \; & \frac{\rho}{2}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}\widetilde{\Gamma}^3 G_n(u, X_u^C, v_u) D\left[M, M\right]_u \sigma_u d[W, M]_u \right] \\
+ \; & \rho\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Lambda}\widetilde{\Gamma}^2 G_n(u, X_u^C, v_u)\sigma_u d[W, D\left[M, M\right]]_u \right] \\
+ \; & \frac{1}{2}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} \widetilde{\Gamma}^3 G_n(u, X_u^C, v_u) d[M, D\left[M, M\right]]_u \right].
\end{aligned}$$

and then the statement follows immediately. □

Remark 6.1.4. *Note that the Corollary above is very similar to Corollary 4.2.1. The jump process has been decomposed into a sum of infinite functions. Numerically, this method is efficient because the weight of each function decreases exponentially.*

As we have seen in the previous chapter, this formula can be efficiently evaluated, while the neglected error terms do not significantly limit a practical use of the formula. The main ingredients to get SVJ approximate pricing formula are expressions for $L[W,M]_t$, $D[M,M]_t$ and $G_n(t, X_t^C, v_t)$. Now we provide some insight into how the latter term can be expressed under various jump-diffusion settings.

Remark 6.1.5. *In particular, we have a closed formula for a log-normal jump diffusion model (e.g. Bates SVJ model):*

$$G_n(t, X_t^C, v_t) = C_{\widetilde{BS}}\left(t, X_t^C, \sqrt{v_t^2 + n\frac{\sigma_J^2}{T-t}}\right)$$

where we modified the risk-free rate used in the Black-Scholes-Merton formula to

$$r^* = r - \lambda\left(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1\right) + n\frac{\mu_J + \frac{1}{2}\sigma_J^2}{T-t}.$$

A very similar formula for the Merton case is deduced by Hanson (2007). More details will follow in the next sections. Under general (finite-activity) jump diffusion settings, we will need to compute

$$\int_{\mathbb{R}} C_{\widetilde{BS}}\left(t, X_t^C + y, v_t\right) f_{J_n}(y)dy$$

where $f_{J_n} = (f_Y^{*n})(y)$ is the convolution of the law of n jumps.

Here we provide a list of known results for various popular models.

1. *Kou (2002) double exponential model:*

$$\begin{aligned} f^{*(n)}(u) &= e^{-\eta_1 u}\sum_{k=1}^{n} P_{n,k}\eta_1^k \frac{1}{(k-1)!}u^{k-1}\mathbf{1}_{\{u\geq 0\}} \\ &+ e^{-\eta_2 u}\sum_{k=1}^{n} Q_{n,k}\eta_2^k \frac{1}{(k-1)!}(-u)^{k-1}\mathbf{1}_{\{u< 0\}} \end{aligned}$$

where

$$P_{n,k} = \sum_{i=k}^{n-1}\binom{n-k-1}{i-k}\binom{n}{i}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-k}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i} p^i q^{n-i}$$

for all $1 \leq k \leq n-1$, and

$$Q_{n,k} = \sum_{i=k}^{n-1}\binom{n-k-1}{i-k}\binom{n}{i}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-k} p^{n-i} q^i$$

for all $1 \leq k \leq n-1$. In addition, $P_{n,n} = p^n$ and $Q_{n,n} = q^n$.

2. *Yan and Hanson (2006) model uses log-uniform jump sizes and hence the density is of the form (Killmann and von Collani, 2001):*

$$f^{*(n)}(u) = \begin{cases} \frac{\sum_{i=0}^{\tilde{n}(n,u)} (-1)^i \binom{n}{i} (u-na-i(b-a))^{n-1}}{(n-1)!(b-a)^n} & \text{if } na \le u \le nb \\ 0 & \text{otherwise.} \end{cases}$$

where $\tilde{n}(n,u) := \left[\frac{u-na}{b-a}\right]$ *is the largest integer less than* $\frac{u-na}{b-a}$.

6.2 SVJ models of the Heston type

In this section, we apply the previous generic results to derive a pricing formula for SVJ models with the Heston variance process. The aim is not to provide pricing solution for all known/studied models, but rather to detail the derivation for a selected model.

6.2.1 Approximation of the SVJ models of the Heston type

Now we have all the tools needed to introduce the main practical result - the pricing formula

Corollary 6.2.1 (Heston-type SVJ pricing formula). *Let* $G_n(t, X_t^C, v_t)$ *takes the expression as in Remark 6.1.5 for a particular jump-type setting, let* $L[W,M]_t$ *and* $D[M,M]_t$ *be defined as in Lemma 5.1.1. Then the European option fair value is expressed as*

$$\begin{aligned} V_t &= \sum_{n=0}^{\infty} p_n(\lambda(T-t)) G_n(t, X_t^C, v_t) \\ &+ \frac{\rho}{2} \sum_{n=0}^{\infty} p_n(\lambda(T-t)) \widetilde{\Lambda}\widetilde{\Gamma} G_n(t, X_t^C, v_t) L[W,M]_t \\ &+ \frac{1}{8} \sum_{n=0}^{\infty} p_n(\lambda(T-t)) \widetilde{\Gamma}^2 G_n(t, X_t^C, v_t) D[M,M]_t \\ &+ \sum_{n=0}^{\infty} p_n(\lambda(T-t)) \epsilon_t^n \end{aligned}$$

where ϵ_t^n *are error terms. The upper bound for any* ϵ_t^n *is given by*

$$\epsilon_t^n \le \nu^2 (|\rho| + \nu)^2 \left(\frac{1}{r} \wedge (T-t)\right) \Pi(\kappa, \theta)$$

where $\Pi(\kappa,\theta)$ *is a positive function. Therefore, the total error*

$$\epsilon_t = \sum_{n=0}^{\infty} p_n(\lambda(T-t)) \epsilon_t^n$$

is bounded by the same constant.

Proof. We apply the Heston volatility model dynamics into Corollary 6.1.3. Using the integrability of the Black-Scholes-Merton function, Fubini Theorem and the fact that the upper bound of Lemma 3.1.10 (ii) does not depend on the log-price, the upper bound can be used for every G_n function. Using Lemma 5.1.1 and Lemma 5.1.2 we prove the corollary. The proof is analogous to the Corollary 5.2.1. □

Remark 6.2.2 (Approximate fractional SVJ model)**.** *For the model introduced by Pospíšil and Sobotka (2016), one can derive a very similar decomposition as in Corollary 6.2.1. In fact, only the terms $L[W, M]_t$ and $D[M, M]_t$ have to be changed while the other terms remain the same.*

6.2.2 Numerical analysis of the SVJ models of the Heston type

We compare the newly obtained approximation formula for option prices under Bates model with the market standard approach for pricing European options under SVJ models - the Fourier-transform based pricing formula. The comparison is performed with two important aspects in mind: the practical precision of the pricing formula when neglecting the total error term ϵ and the efficiency of the formula expressed in terms of the computational time needed for particular pricing tasks.

In particular, we utilise a semi-closed form solution with one numerical integration as a reference price (Baustian et al., 2017) alongside a classical solution derived by Bates (1996)[1]. The numerical integration errors according to Baustian et al. (2017) should typically be well beyond 10^{-10}, hence we can take the numerically computed prices as the reference prices for the comparison.

Due to the theoretical properties of the total error term ϵ, we illustrate the approximation quality for several values of ρ and ν while keeping other parameters fixed. Concretely, we choose the following parameters: $S_0 = 100$, $r = 0.001$, $\tau = 0.3$, $v_0 = 0.25$, $\kappa = 1.5$, $\theta = 0.2$, $\lambda = 0.05$, $\mu_J = -0.05$ and $\sigma_J = 0.5$.

[1] With a slight modification mentioned in Gatheral (2006) to not suffer the "Heston trap" issues.

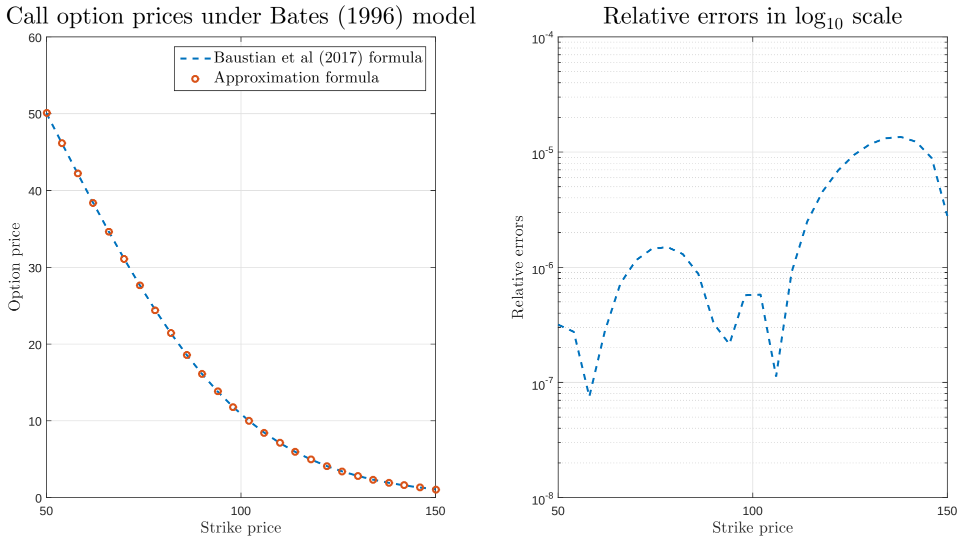


Figure 6.1: Bates model: Short-time price comparison for low ν and ρ.

In Figure 6.1, we inspect a mode of low volatility of the spot variance ν and low absolute value of the instantaneous correlation ρ between the two Brownian motions. In this case, $\nu = 5\%$, $\rho = -0.2$. The errors for an option price smile that corresponds to $\tau = 0.3$ are within $10^{-4} - 10^{-6}$ range, while slightly better absolute errors were obtained At-The-Money. Increasing either the absolute value of ρ or volatility ν should, in theory, worsen the computed error measures. However, if only one of the values is increased, we are still able to keep the errors below 10^{-3} in most of the cases, see Figure 6.2 where it is changed to $\rho = -0.8$.

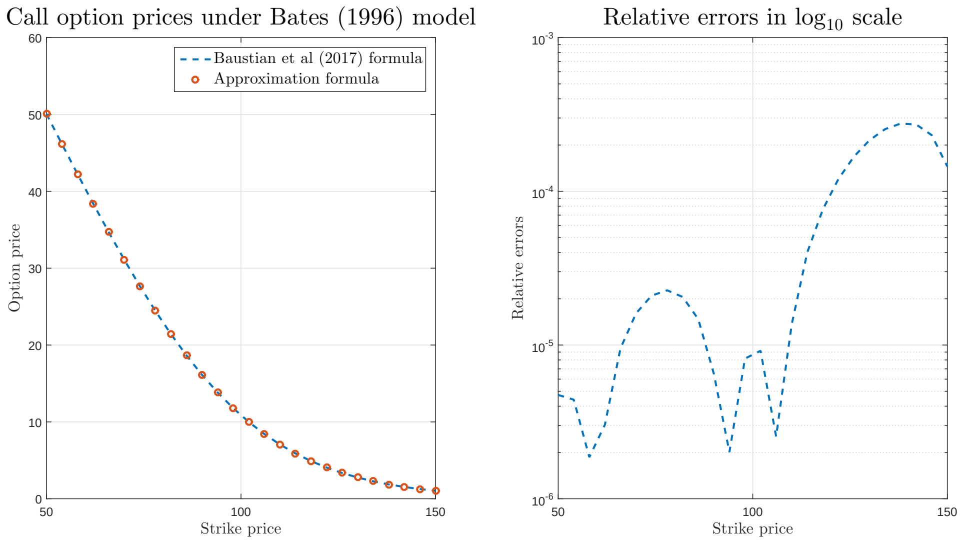


Figure 6.2: Bates model: Short-time price comparison for low ν and high ρ.

Last but not least, we illustrate the approximation quality for parameters that are not well-suited to the approximation. This is done by setting $\nu = 50\%$, correlation $\rho = -0.8$ and a smile with respect to $\tau = 3$. The errors obtained are depicted by Figure 6.3. Despite the values of parameters, the shape of the option price curve remains fairly similar to the one obtained by a more precise semi-closed formula.

The main advantage of the proposed pricing approximation lies in its computational efficiency – which might be advantageous for many tasks in quantitative finance that need fast evaluation of derivative prices. To inspect the time consumption, we set up three pricing tasks. We use a batch of 100 call options with different strikes and times to maturities that involves all types of options[2]. In the first task, we evaluate prices for the batch with respect to 100 (uniformly) randomly sampled parameter sets. This should encompass a similar number of price evaluations as a market calibration task with a very good initial guess. Further on, we repeat the same trials only for 1.000 and 10.000 parameter sets, to mimic the number of evaluations for a typical local-search calibration and a global-search calibration respectively, for more information about calibration tasks see e.g. Mikhailov and Nögel (2003) and Mrázek et al. (2016).

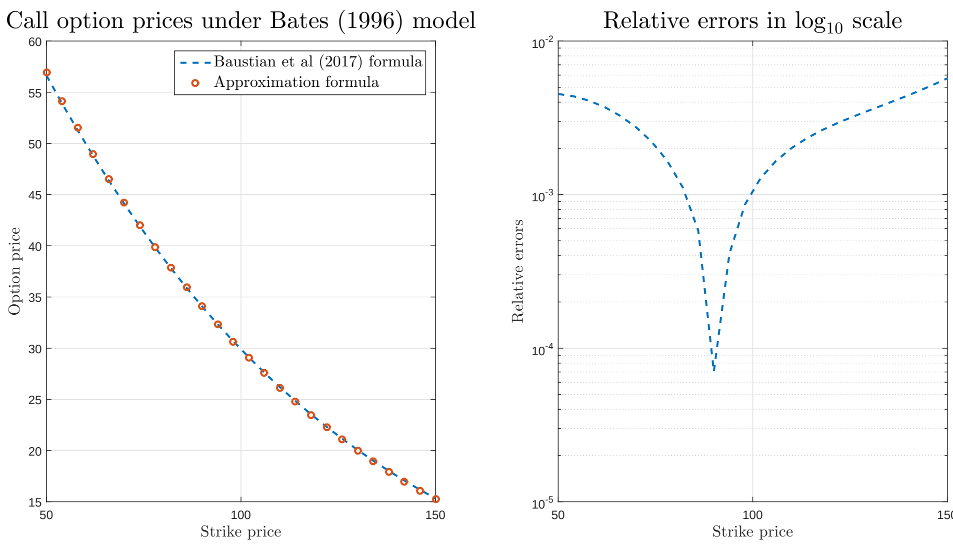


Figure 6.3: Bates model: Price comparison for high ν and ρ.

The obtained computational times are listed in Table 6.1. Unlike the formulae with numerical integration, the proposed approximation has an almost linear dependency of computational time on the number of evaluated prices. The results also vary based on the randomly generated parameter values for numerical schemes much more than for the approximation – this is caused by adaptivity of numerical quadratures that were used[3].

[2]It includes OTM, ATM, ITM options with short-, mid- and long-term times to maturities

[3]For both Baustian et al. (2017) and Gatheral (2006) formulae we use an adaptive Gauss-Kronrod(7,15) quadrature.

The newly proposed approximation is typically 3× faster compared to the classical two integral pricing formula and the computational time consumption does not depend on the model- nor on market-parameters.

Table 6.1: Bates model: Efficiency of the pricing formulae

Pricing approach	Task	Time† [sec]	Speed-up factor
Approximation formula	#1	0.97	3.23×
	#2	10.03	2.94×
	#3	99.67	2.83×
Baustian et al. (2017)	#1	2.09	1.52×
	#2	17.28	1.71×
	#3	135.95	2.01×
Gatheral (2006)	#1	3.18	-
	#2	29.48	-
	#3	281.72	-

† The results were obtained on a PC with Intel Core i7-6500U CPU and 8 GB RAM.

6.3 The approximated implied volatility surface for SVJ models of the Heston type

We have computed a bound for the error between the exact price and the approximated pricing formula for the SVJ models of the Heston type. Now, we are going to derive an approximation of the implied volatility surface alongside the corresponding ATM implied volatility profiles. These approximations can help us to understand the volatility dynamics of studied models in a better way. Without any loss of generality, we assume that $t = 0$.

6.3.1 Deriving an approximated implied volatility surface for SVJ models of the Heston type

The price of a European call option with strike K and maturity T is an observable quantity which will be referred to as $P_0^{obs} = P^{obs}(K, T)$. Recall that the *implied volatility* is defined as the value $I(T, K)$ that satisfies

$$C_{BS}(0, S_0, I(T, K)) = P_0^{obs}.$$

Define $\widehat{v}_0$ such that

$$C_{BS}(0, S_0, \widehat{v}_0) := \sum_{n=0}^{\infty} p_n(\lambda T)\mathbb{E}_{J_n}\left[C_{\overline{BS}}(0, x + J_n, v_0)\right].$$

Using the results from the previous section, we are going to derive an approximation to the implied volatility as in Fouque et al. (2003), using the idea to expand the implied volatility function $I(T, K)$ with respect to two scales $\{\delta^k\}_{k=0}^{\infty}$ and $\{\epsilon^k\}_{k=0}^{\infty}$ converging to 0. See also Alòs et al. (2015).

Let $\epsilon = \rho\nu$ and $\delta = \nu^2$. Then, we expand $I(T, K)$ with respect to these two scales and $\widehat{v}_0$ as

$$I(T, K) = \widehat{v}_0 + \rho\nu I_1(T, K) + \nu^2 I_2(T, K) + O((\rho\nu + \nu^2)).$$

We will denote by $\widehat{I}(T, K) = \widehat{v}_0 + \rho\nu I_1(T, K) + \nu^2 I_2(T, K)$ the approximation to the implied volatility and by $\widehat{V}(0, x, v_0)$ the approximation to the option price obtained in Corollary 6.2.1. According to that:

$$\begin{aligned}
\widehat{V}(0, x, v_0) &= \sum_{n=0}^{\infty} p_n(\lambda T)\mathbb{E}_{J_n}\left[C_{\widetilde{BS}}(0, x + J_n, v_0)\right] \\
&+ \frac{\rho}{2}\sum_{n=0}^{\infty} p_n(\lambda T)\mathbb{E}_{J_n}\left[\widetilde{\Lambda}\widetilde{\Gamma} C_{\widetilde{BS}}(0, x + J_n, v_0)\right] L[W, M]_0 \\
&+ \frac{1}{8}\sum_{n=0}^{\infty} p_n(\lambda T)\mathbb{E}_{J_n}\left[\widetilde{\Gamma}^2 C_{\widetilde{BS}}(0, x + J_n, v_0)\right] D[M, M]_0.
\end{aligned}$$

To simplify the notation, we define

$$\gamma_n := \frac{d_+^2(x, r, \sigma) - d_+^2(x + J_n, r, \sigma)}{2}$$

and

$$\begin{aligned}
D_1(x, n, \sigma, T) &:= \mathbb{E}_{J_n}\left[\frac{e^{J_n+\gamma_n}}{\sigma T}\left(1 - \frac{d_+(x + J_n, r, \sigma)}{\sigma\sqrt{T}}\right)\right], \\
D_2(x, n, \sigma, T) &:= \mathbb{E}_{J_n}\left[\frac{e^{J_n+\gamma_n}}{\sigma^3 T^2}\left(d_+^2(x + J_n, r, \sigma)\right.\right. \\
&\qquad \left.\left. -\sigma d_+(x + J_n, r, \sigma)\sqrt{T} - 1\right)\right].
\end{aligned}$$

Using the fact that

$$\partial_\sigma C_{\widetilde{BS}}(t, x, \sigma) = \frac{e^x e^{-d_+^2(\sigma)/2}\sqrt{T - t}}{\sqrt{2\pi}},$$

we can re-write the approximated price as

$$\widehat{V}(0, x, v_0) = \sum_{n=0}^{\infty} p_n(\lambda T)\mathbb{E}_{J_n}\left[C_{\widetilde{BS}}(0, x + J_n, v_0)\right]$$

$$
\begin{aligned}
&+ \quad \partial_\sigma C_{\widetilde{BS}}(v_0) \sum_{n=0}^{\infty} p_n(\lambda T) D_1(x, n, \sigma, T) L[W, M]_0 \\
&+ \quad \partial_\sigma C_{\widetilde{BS}}(v_0) \sum_{n=0}^{\infty} p_n(\lambda T) D_2(x, n, v_0, T) D[M, M]_0.
\end{aligned}
$$

We write $C_{\overline{BS}}(v_0)$ as a shorthand for $C_{\overline{BS}}(0, x, v_0)$. Note that the pricing formula approximation, $\widehat{V}(0, x, v_0)$, has a volatility v_0, meanwhile $I(T, K)$ depends on $\widehat{v}_0$. In order to conciliate one with the other, we consider the Taylor expansion of $C_{\overline{BS}}(0, x, I(T, K))$ around v_0:

$$
\begin{aligned}
C_{\overline{BS}}(0, x, I(T, K)) &= C_{\overline{BS}}(v_0) + \partial_\sigma C_{\overline{BS}}(v_0)(\widehat{v}_0 - v_0 + \rho\nu I_1(T, K) + \nu^2 I_2(T, K) + \cdots) \\
&\quad + \frac{1}{2}\partial_\sigma^2 C_{\overline{BS}}(v_0)(\widehat{v}_0 - v_0 + \rho\nu I_1(T, K) + \nu^2 I_2(T, K) + \cdots)^2 + \cdots \\
&= C_{\overline{BS}}(v_0) + \rho\nu\partial_\sigma C_{\overline{BS}}(v_0) I_1(T, K) + \nu^2 \partial_\sigma C_{\overline{BS}}(v_0) I_2(T, K) \\
&+ \sum_{n=1}^{\infty} \frac{1}{n!}\partial_\sigma C_{\overline{BS}}(v_0)(\widehat{v}_0 - v_0)^n + \cdots .
\end{aligned}
$$

Noticing that

$$
C_{\overline{BS}}(\widehat{v}_0) = C_{\overline{BS}}(v_0) + \sum_{n=1}^{\infty} \frac{1}{n!}\partial_\sigma C_{\overline{BS}}(v_0)(\widehat{v}_0 - v_0)^n
$$

and equating

$$
\widehat{V}(0, x, v_0) = C_{\overline{BS}}(0, x, \widehat{I}(T, K)),
$$

we obtain

$$
\begin{aligned}
\widehat{I}_1(T, K) &:= \rho\nu I_1(T, K) = \frac{\rho}{2} L[W, M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_1(x, n, v_0, T), \\
\widehat{I}_2(T, K) &:= \nu^2 I_2(T, K) = \frac{1}{8} D[M, M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_2(x, n, v_0, T).
\end{aligned}
$$

Hence, we have the following approximation of implied volatility

$$
\begin{aligned}
\widehat{I}(T, K) &= \widehat{v}_0 + \frac{\rho}{2} L[W, M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_1(x, n, v_0, T) \\
&+ \frac{1}{8} D[M, M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_2(x, n, v_0, T).
\end{aligned}
$$

In particular, when we look at the ATM curve, we have that

$$
\widehat{I}^{ATM}(T) = \widehat{v}_0 + \frac{\rho}{2} L[W, M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) \mathbb{E}_{J_n}\left[\frac{e^{J_n + \gamma_n}}{v_0 T}\left(\frac{1}{2} - \frac{J_n}{T v_0^2}\right)\right]
$$

$$- \quad \frac{1}{8}D[M,M]_0 \sum_{n=0}^{\infty} p_n(\lambda T)\mathbb{E}_{J_n}\left[\frac{e^{J_n+\gamma_n}}{v_0 T}\left(\frac{1}{4}+\frac{1}{v^2 T}-\frac{J_n^2}{v_0^4 T^2}\right)\right].$$

Remark 6.3.1. *When T converges to 0, the dynamics of the model is the same as in the Heston model. This is due to the behaviour of the Poisson process when $T \downarrow 0$.*

6.3.2 Deriving an approximated implied volatility surface for Bates model

The Bates model is a particular example of SVJ model of the Heston type. The fact that jumps are also log-normal makes the model more tractable. In this section, we will adapt the generic formulae to this particular case. In this model, after each jump, the drift- and volatility-like parameters will change. We define

$$\widetilde{v}_0^{(n)} = \sqrt{v_0^2 + n\frac{\sigma_J^2}{T}}$$

as the new volatility and

$$\widetilde{r}_n = r - \lambda\left(e^{\mu_J+\frac{1}{2}\sigma_J^2} - 1\right) + n\frac{\mu_J + \frac{1}{2}\sigma_J^2}{T}$$

as the new drift. The parameter n is the number of realized jumps, μ_J and σ_J are the jump-size parameters and λ is the jump intensity. For simplicity, we denote:

$$c_n := -\lambda\left(e^{\mu_J+\frac{1}{2}\sigma_J^2} - 1\right) + n\frac{\mu_J + \frac{1}{2}\sigma_J^2}{T}.$$

As a consequence, we have that

$$d_{\pm}\left(x, \widetilde{r}_n, \widetilde{v}_0^{(n)}\right) = \frac{x - \ln K + \widetilde{r}_n T}{\widetilde{v}_0^{(n)}\sqrt{T}} \pm \frac{\widetilde{v}_0^{(n)}\sqrt{T}}{2}.$$

To simplify the notation, we define

$$\widetilde{\gamma}_n := \frac{d_+^2\left(x,r,\sigma\right) - d_+^2\left(x,\widetilde{r}_n,\widetilde{v}_0^{(n)}\right)}{2}.$$

Following the steps done in the generic formula, we can define the variables

$$D_{B,1}\left(x,\widetilde{r}_n,\widetilde{v}_0^{(n)},T\right) = \frac{e^{\widetilde{\gamma}_n}}{\widetilde{v}_0^{(n)}T}\left(1 - \frac{d_+\left(x,\widetilde{r}_n,\widetilde{v}_0^{(n)}\right)}{\widetilde{v}_0^{(n)}\sqrt{T}}\right),$$

$$D_{B,2}\left(x,\widetilde{r}_n,\widetilde{v}_0^{(n)},T\right) = \frac{e^{\widetilde{\gamma}_n}}{\left(\widetilde{v}_0^{(n)}\right)^3 T^2}\left(d_+^2\left(x,\widetilde{r}_n,\widetilde{v}_0^{(n)}\right)\right.$$

$$-\widetilde{v}_0^{(n)} d_+\left(x, \widetilde{r}_n, \widetilde{v}_0^{(n)}\right)\sqrt{T} - 1\Big).$$

It follows that

$$\begin{aligned}
\widehat{I}_{B,1}(T,K) &= \rho\nu I_{B,1}(T,K) = \frac{\rho}{2} L[W,M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_{B,1}\left(x, \widetilde{r}_n, \widetilde{v}_0^{(n)}, T\right),\\
\widehat{I}_{B,2}(T,K) &= \nu^2 I_{B,2}(T,K) = \frac{1}{8} D[M,M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_{B,2}\left(x, \widetilde{r}_n, \widetilde{v}_0^{(n)}, T\right).
\end{aligned}$$

The approximation of the implied volatility surface has the following shape

$$\begin{aligned}
\widehat{I}_B(T,K) &= \widehat{v}_0 + \frac{\rho}{2} L[W,M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) \frac{e^{\widetilde{\gamma}_n}}{\widetilde{v}_0^{(n)} T}\left(1 - \frac{d_+\left(x, \widetilde{r}_n, \widetilde{v}_0^{(n)}\right)}{\widetilde{v}_0^{(n)}\sqrt{T}}\right)\\
&+ \frac{1}{8} D[M,M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) \frac{e^{\widetilde{\gamma}_n}}{\widetilde{v}_0^{(n)} T}\left(\frac{d_+^2\left(x, \widetilde{r}_n, \widetilde{v}_0^{(n)}\right) - \widetilde{v}_0^{(n)} d_+\left(x, \widetilde{r}_n, \widetilde{v}_0^{(n)}\right)\sqrt{T} - 1}{\left(\widetilde{v}_0^{(n)}\right)^2 T}\right).
\end{aligned}$$

In particular, the ATM implied volatility curve under the studied model takes the form:

$$\begin{aligned}
\widehat{I}_B^{ATM}(T) &= \widehat{v}_0 + \frac{\rho}{2} L[W,M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) \frac{e^{\gamma_n^{ATMBates}}}{\widetilde{v}_0^{(n)} T}\left(\frac{1}{2} - \frac{c_n}{\left(\widetilde{v}_0^{(n)}\right)^2}\right)\\
&- \frac{1}{8} D[M,M]_0 \sum_{n=0}^{\infty} p_n(\lambda T) \frac{e^{\gamma_n^{ATMBates}}}{\widetilde{v}_0^{(n)} T}\left(\frac{1}{4} + \frac{1}{\left(\widetilde{v}_0^{(n)}\right)^2 T} - \frac{c_n^2}{\left(\widetilde{v}_0^{(n)}\right)^4}\right)
\end{aligned}$$

where

$$\gamma_n^{ATMBates} = -\frac{1}{2}\left(c_n T + \frac{c_n^2 T}{\left(\widetilde{v}_0^{(n)}\right)^2}\right).$$

6.3.3 Numerical analysis of the approximation of the implied volatility for the Bates case

We have compared the approximation and semi-closed form formulae for option prices under Bates model. For this model, we also illustrate the approximation quality in terms of implied volatilities.

Because there is no exact closed formula for implied volatilities under the studied model, we take as a reference price the one obtained by means of the complex Fourier transform

(Baustian et al., 2017). Once we have computed the prices, we use a numerical inversion to obtain the desired implied volatilities.

As previously, we start by comparing implied volatilities for well-suited parameter sets. The illustration in Figure 6.4 is obtained by setting $\rho = -0.1$, $\nu = 5\%$ and other parameters as in Section 6.2.2. Typically, for a well-suited parameter set, the absolute approximation errors stay within the range $10^{-5} - 10^{-7}$.

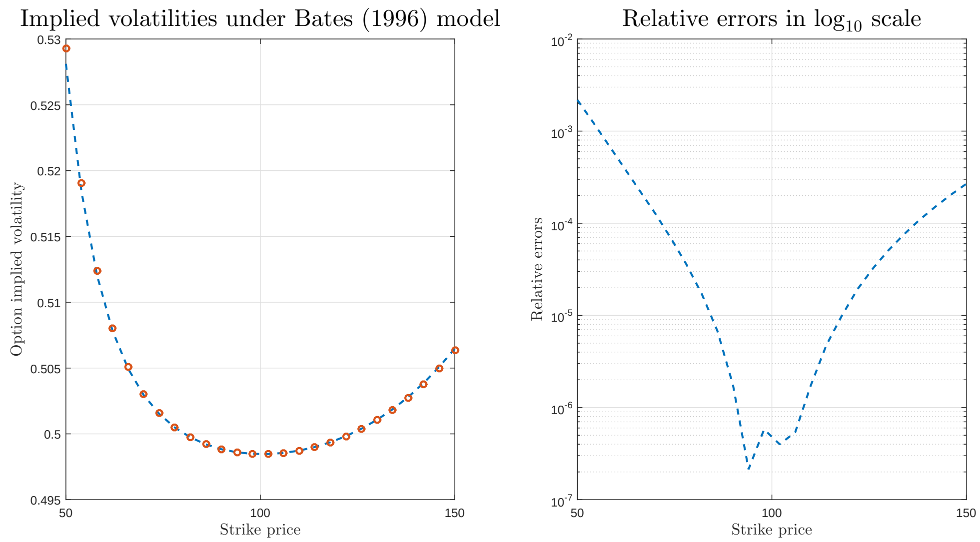


Figure 6.4: Bates model: Short-time implied volatility comparison for low ν and ρ.

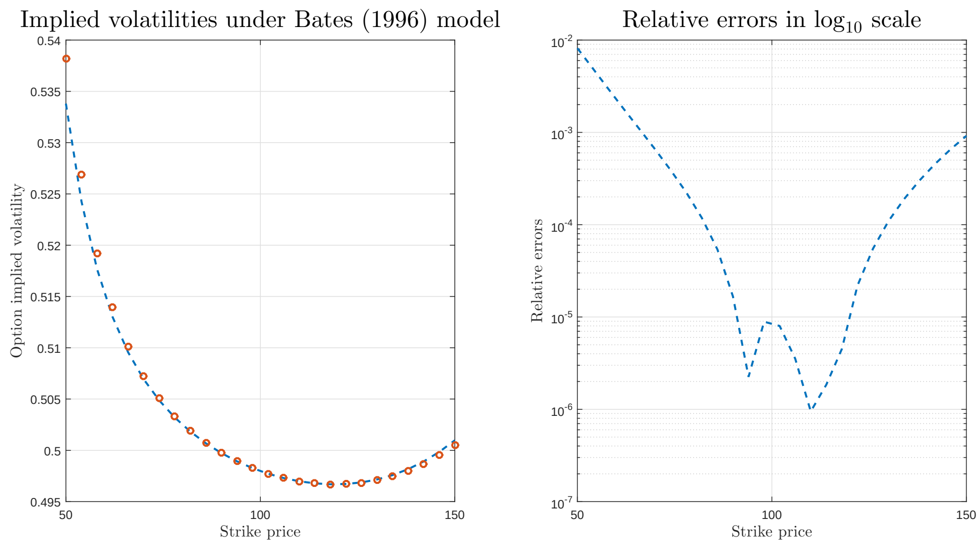


Figure 6.5: Bates model: Short-time implied volatility comparison for low ν and high ρ.

Even for not entirely well-suited parameters, we are able to obtain reasonable errors especially for ATM options, see Figures 6.5 and 6.6. In the mode of high volatility ν of the variance process and high absolute value of the instantaneous correlation ρ, the curvature of the smile is not fully captured. However, the errors are typically well below 10^{-2} even in this adverse setting.

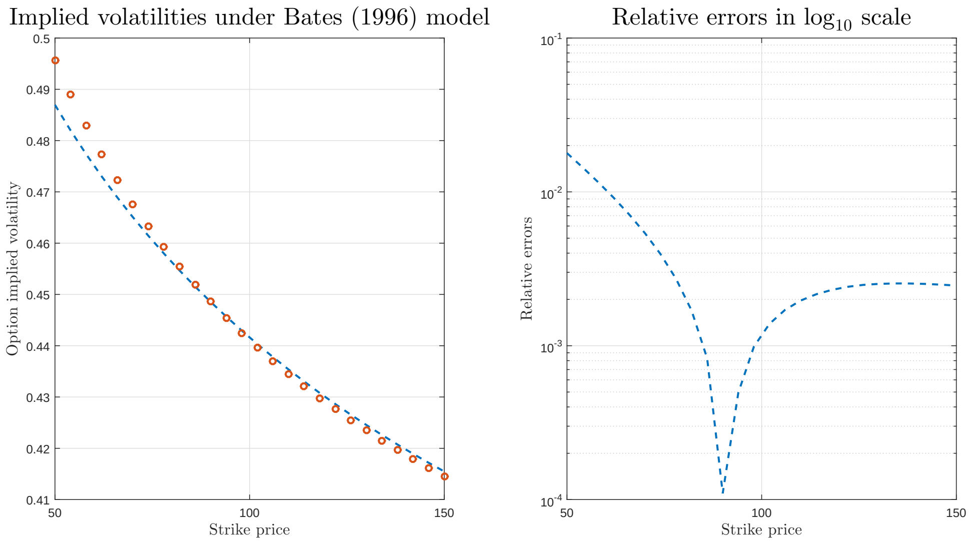


Figure 6.6: Bates model: Implied volatility comparison for high ν and ρ.

CHAPTER 7

Rough Volterra Stochastic Volatility models

In this chapter, following Merino et al. (2020), we develop a decomposition formula for European option prices under general Volterra volatility models. We focus on the particular cases of an exponential Volterra model and exponential fractional volatility model. An approximation formula is obtained for these two models. We present a version of the rBergomi model, introducing a new parameter α, that we will refer as the α rough fractional stochastic volatility model, αRFSV. All the result obtained in this chapter, in contrast with Alòs et al. (2019), have been obtained using the classical Itô calculus. Numerical properties of the approximation for the rBergomi model are studied and we propose a hybrid calibration scheme which combines the approximation formula alongside MC simulations. This scheme can significantly speed up the calibration to financial markets as illustrated in a set of AAPL options.

7.1 Volterra volatility models

7.1.1 General Volterra volatility model

In this section, we apply the generic decomposition formula to model (3.19) with general Volterra volatility process defined in (2.9). Extending the Theorem 3.1 in Sottinen and Viitasaari (2017) enables us to rephrase the adapted projection of the future squared volatility.

Theorem 7.1.1 (Prediction law for Gaussian Volterra processes)**.** *Let* $(Y_t, t \geq 0)$ *be the Gaussian Volterra process* (2.9) *satisfying assumptions* (2.10) *and* (2.11)*. Then, the conditional process* $(Y_u|\mathcal{F}_t, 0 \leq t \leq u)$ *is Gaussian with* $\mathcal{F}_u$*-measurable mean function*

$$\widehat{m}_t(u) := \mathbb{E}_t[Y_u] = \int_0^t K(u,s)\, \mathrm{d}W_s,$$

and deterministic covariance function

$$\widehat{r}(u_1, u_2|t) := \mathbb{E}_t\left[(Y_{u_1} - \widehat{m}_t(u_1))\,(Y_{u_2} - \widehat{m}_t(u_2))\right]$$

$$= r(u_1, u_2) - \int_0^t K(u_1, v) K(u_2, v) \; \mathrm{d}v$$

for $u_1, u_2 \geq t$.

Proof. Let $0 \leq t \leq u$. Then

$$\widehat{m}_t(u) = \mathbb{E}_t[Y_u] = \mathbb{E}\left[\int_0^u K(u,s) \; \mathrm{d}W_s \middle| \mathcal{F}_t^W \right] = \int_0^t K(u,s) \; \mathrm{d}W_s$$

and

$$\begin{aligned}
\widehat{r}(u_1, u_2|t) &= \mathbb{E}\left[\left(Y_{u_1} - \widehat{m}_t(u_1)\right)\left(Y_{u_2} - \widehat{m}_t(u_2)\right) \middle| \mathcal{F}_t^W \right] \\
&= \mathbb{E}\Bigg[\left(\int_0^{u_1} K(u_1, v_1) \, \mathrm{d}W_{v_1} - \int_0^t K(u_1, v_1) \; \mathrm{d}W_{v_1} \right) \\
&\qquad \cdot \left(\int_0^{u_2} K(u_2, v_2) \, \mathrm{d}W_{v_2} - \int_0^t K(u_2, v_2) \; \mathrm{d}W_{v_2} \right) \Bigg| \mathcal{F}_t^W \Bigg] \\
&= \mathbb{E}\left[\int_t^{u_1} K(u_1, v_1) \, \mathrm{d}W_{v_1} \int_t^{u_2} K(u_2, v_2) \, \mathrm{d}W_{v_2} \middle| \mathcal{F}_t^W \right] \\
&= \int_t^{u_1 \wedge u_2} K(u_1, v) K(u_2, v) \; \mathrm{d}v \\
&= r(u_1, u_2) - \int_0^t K(u_1, v) K(u_2, v) \; \mathrm{d}v.
\end{aligned}$$

□

We will denote $\widehat{r}(u|t) := \widehat{r}(u, u|t)$.

Under the general volatility process (2.9), we have

$$v_t^2 = \frac{1}{T-t} \int_t^T \mathbb{E}_t \left[g^2(u, Y_u) \right] \; \mathrm{d}u$$

and the martingale

$$M_t = \int_0^T \mathbb{E}_t \left[g^2(u, Y_u) \right] \; \mathrm{d}u.$$

Let us denote

$$F(t, \widehat{m}_t(u)) := \mathbb{E}_t \left[g^2(u, Y_u) \right].$$

In the upcoming lemma, we express the conditional expectation of the future squared volatility in terms of the mean function $\widehat{m}_t(u)$.

Lemma 7.1.2 (Auxiliary terms in the decomposition formula for the general volatility model)**.** *Let* $0 \leq t \leq u$ *and* $F(t, \widehat{m}_t(u)) = \mathbb{E}_t\left[g^2(u, Y_u)\right]$*, then*

$$\mathrm{d}F(t, \widehat{m}_t(u)) = \left(\partial_1 F(t, \widehat{m}_t(u)) + \frac{1}{2}\partial_{22} F(t, \widehat{m}_t(u))\, K^2(u,t)\right) \mathrm{d}t \\ + \partial_2 F(t, \widehat{m}_t(u))\, \mathrm{d}\widehat{m}_t(u), \tag{7.1}$$

$$\mathrm{d}[M_\cdot, W_\cdot]_t = \int_t^T \partial_2 F(t, \widehat{m}_t(u)) K(u,t)\, \mathrm{d}u\, \mathrm{d}t, \tag{7.2}$$

$$\mathrm{d}[M_\cdot, M_\cdot]_t = \int_t^T \int_t^T \partial_2 F(t, \widehat{m}_t(u_1)) \partial_2 F(t, \widehat{m}_t(u_2)) \cdot \\ \cdot K(u_1, t) K(u_2, t)\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}t. \tag{7.3}$$

Proof. Let $0 \leq t \leq u$ and

$$X_t(u) = \mathbb{E}_t\left[g^2(u, Y_u)\right].$$

Theorem 7.1.1 implies that

$$X_t(u) = \int_{\mathbb{R}} g^2(u,z) \widehat{\varphi}_t(u,z)\ \mathrm{d}z,$$

where

$$\widehat{\varphi}_t(u,z) = \frac{1}{\sqrt{2\pi \widehat{r}(u|t)}} \exp\left\{-\frac{1}{2}\frac{(z - \widehat{m}_t(u))^2}{\widehat{r}(u|t)}\right\} \tag{7.4}$$

is a Gaussian density function with stochastic mean $\widehat{m}_t(u)$ and deterministic variance $\widehat{r}(u|t)$. To calculate the quadratic variation, we note that

$$\widehat{\varphi}_t(u,z) = f(t, \widehat{m}_t(u)),$$

where

$$f(t,m) = \frac{1}{\sqrt{2\pi \widehat{r}(u|t)}} \exp\left\{-\frac{1}{2}\frac{(z-m)^2}{\widehat{r}(u|t)}\right\}.$$

Since

$$\mathrm{d}[\widehat{m}_\cdot(u)]_t = K^2(u,t)\ \mathrm{d}t,$$

we retrieve the following expression by Itô's formula,

$$\mathrm{d}f(t, \widehat{m}_t(u)) = \left(\partial_1 f(t, \widehat{m}_t(u)) + \frac{1}{2}\partial_{22} f(t, \widehat{m}_t(u)) K^2(u,t)\right) \mathrm{d}t \\ + \partial_2 f(t, \widehat{m}_t(u))\, \mathrm{d}\widehat{m}_t(u),$$

and consequently,

$$\mathrm{d}[f(\cdot, \widehat{m}_\cdot(u))]_t = \left(\partial_2 f(t, \widehat{m}_t(u)) K(u,t)\right)^2\ \mathrm{d}t.$$

Due to

$$\partial_2 f(t, \widehat{m}_t(u)) = \frac{z - \widehat{m}_t(u)}{\widehat{r}(u|t)} \widehat{\varphi}_t(u,z),$$

we obtain

$$\mathrm{d}[\widehat{\varphi}_\cdot(u,z)]_t = \left(\frac{z-\widehat{m}_t(u)}{\widehat{r}(u|t)}\widehat{\varphi}_t(u,z)K(u,t)\right)^2 \mathrm{d}t.$$

More generally, we have

$$\mathrm{d}[\widehat{\varphi}_\cdot(u,z_1),\varphi_\cdot(u,z_2)]_t = \frac{(z_1-\widehat{m}_t(u))(z_2-\widehat{m}_t(u))}{\widehat{r}^2(u|t)}\widehat{\varphi}_t(u,z_1)\widehat{\varphi}_t(u,z_2)K^2(u,t)\,\mathrm{d}t,$$

and consequently,

$$\begin{aligned}
\mathrm{d}[X_\cdot(u)]_t &= \mathrm{d}\left[\int_{\mathbb{R}} g^2(u,z_1)\widehat{\varphi}(u,z_1)\,\mathrm{d}z_1, \int_{\mathbb{R}} g^2(u,z_2)\widehat{\varphi}(u,z_2)\,\mathrm{d}z_2\right]_t \\
&= \iint_{\mathbb{R}^2} g^2(u,z_1)g^2(u,z_2)\,\mathrm{d}[\widehat{\varphi}(u,z_1),\widehat{\varphi}(u,z_2)]_t\,\mathrm{d}z_1\,\mathrm{d}z_2 \\
&= \iint_{\mathbb{R}^2} g^2(u,z_1)g^2(u,z_2)(\widehat{m}_t(u)-z_1)(\widehat{m}_t(u)-z_2)\,\cdot \\
&\qquad \cdot\widehat{\varphi}_t(u,z_1)\widehat{\varphi}_t(u,z_2)\left(\frac{K(u,t)}{\widehat{r}(u|t)}\right)^2 \mathrm{d}z_1\,\mathrm{d}z_2\;\mathrm{d}t.
\end{aligned}$$

Let $F=F(t,m)$ be a $\mathcal{C}^{1,2}$-function of time t and ‘spot’ $m=\widehat{m}_t(u)$ of the prediction martingale. Because the the filtrations are the same, we have, in general,

$$\mathrm{d}[F(\cdot,\widehat{m}_\cdot(u)),W_\cdot]_t = \partial_2 F(t,\widehat{m}_t(u))K(u,t)\;\mathrm{d}t.$$

Now we set

$$F(t,\widehat{m}_t(u)) = X_t(u) = \int_{\mathbb{R}} g^2(u,z)\widehat{\varphi}_t(u,z)\;\mathrm{d}z,$$

and applying the Itô formula, we obtain (7.1). Moreover,

$$\begin{aligned}
\mathrm{d}[M_\cdot,W_\cdot]_t &= \mathrm{d}\left[\int_0^T F(\cdot,\widehat{m}_\cdot(u))\,\mathrm{d}u, W_\cdot\right]_t \\
&= \int_0^T \mathrm{d}[F(\cdot,\widehat{m}_\cdot(u)),W_\cdot]_t\;\mathrm{d}u \\
&= \int_0^T \partial_2 F(t,\widehat{m}_t(u))K(u,t)\,\mathrm{d}u\;\;\mathrm{d}t \\
&= \int_t^T \partial_2 F(t,\widehat{m}_t(u))K(u,t)\,\mathrm{d}u\;\;\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}[M_\cdot,M_\cdot]_t &= \mathrm{d}\left[\int_0^T F(\cdot,\widehat{m}_\cdot(u))\,\mathrm{d}u, \int_0^T F(\cdot,\widehat{m}_\cdot(u))\,\mathrm{d}u\right]_t \\
&= \int_0^T\!\!\int_0^T \mathrm{d}[F(\cdot,\widehat{m}_\cdot(u_1)),F(\cdot,\widehat{m}_\cdot(u_2))]_t\;\mathrm{d}u_1\,\mathrm{d}u_2
\end{aligned}$$

$$= \int_0^T \int_0^T \partial_2 F(t, \widehat{m}_t(u_1)) \partial_2 F(t, \widehat{m}_t(u_2)) K(u_1, t) K(u_2, t)\, \mathrm{d}u_1\, \mathrm{d}u_2\ \mathrm{d}t.$$
$$= \int_t^T \int_t^T \partial_2 F(t, \widehat{m}_t(u_1)) \partial_2 F(t, \widehat{m}_t(u_2)) K(u_1, t) K(u_2, t)\, \mathrm{d}u_1\, \mathrm{d}u_2\ \mathrm{d}t.$$

□

7.1.2 Exponential Volterra volatility model

Assume now that X_t is the log-price process (3.19) with σ_t being the *exponential Volterra volatility process* defined by (3.24).

Lemma 7.1.3 (Auxiliary terms in the decomposition formula for the exponential Volterra volatility model)**.** *Let* σ_t *be as in* (3.24) *and* $0 \leq t \leq u$*. Then*

$$\begin{aligned}
F(t, \widehat{m}_t(u)) &= \sigma_0^2 \exp\left\{2\xi \widehat{m}_t(u) + 2\xi^2 \widehat{r}(u|t) - \alpha \xi^2 r(u)\right\}, \\
\partial_2 F(t, \widehat{m}_t(u)) &= 2\xi F(t, \widehat{m}_t(u)), \\
\mathrm{d}[M_\cdot, W_\cdot]_t &= 2\sigma_0^2 \xi \int_t^T \exp\left\{2\xi \widehat{m}_t(u) + 2\xi^2 \widehat{r}(u|t) - \alpha \xi^2 r(u)\right\} K(u,t)\, \mathrm{d}u\ \mathrm{d}t, \\
d[M_\cdot, M_\cdot]_t &= 4\sigma_0^4 \xi^2 \int_t^T \int_t^T \exp\left\{2\xi\left(\widehat{m}_t(u_1) + \widehat{m}_t(u_2)\right)\right\} \cdot \\
&\quad \cdot \exp\left\{2\xi^2\left(\widehat{r}(u_1|t) + \widehat{r}(u_2|t)\right)\right\} \cdot \\
&\quad \cdot \exp\left\{-\alpha\xi^2\left(r(u_1) + r(u_2)\right)\right\} \cdot \\
&\quad \cdot K(u_1, t) K(u_2, t)\, \mathrm{d}u_1\, \mathrm{d}u_2\ \mathrm{d}t.
\end{aligned}$$

Proof. Let $\widehat{\varphi}_t(u, z)$ be given by (7.4). Then

$$\begin{aligned}
F(t, \widehat{m}_t(u)) &= \int_{\mathbb{R}} g^2(u, z) \widehat{\varphi}_t(u, z)\, \mathrm{d}z, \\
&= \sigma_0^2 \mathrm{e}^{-\alpha\xi^2 r(u)} \int_{\mathbb{R}} \mathrm{e}^{2\xi z} \frac{1}{\sqrt{2\pi \widehat{r}(u|t)}} \exp\left(-\frac{1}{2} \frac{(z - \widehat{m}_t(u))^2}{\widehat{r}(u|t)}\right) \mathrm{d}z.
\end{aligned}$$

It is now easy to calculate the partial derivative $\partial_2 F$. We get

$$\partial_2 F(t, \widehat{m}_t(u)) = \sigma_0^2 \mathrm{e}^{-\alpha\xi^2 r(u)} \int_{\mathbb{R}} \mathrm{e}^{2\xi z} \frac{1}{\sqrt{2\pi \widehat{r}(u|t)}} \exp\left(-\frac{1}{2} \frac{(z - \widehat{m}_t(u))^2}{\widehat{r}(u|t)}\right) \frac{z - \widehat{m}_t(u)}{\widehat{r}(u|t)}\, \mathrm{d}z.$$

Changing variables $v = \frac{z - \widehat{m}_t(u)}{\sqrt{\widehat{r}(u|t)}}$ and $\mathrm{d}z = \sqrt{\widehat{r}(u|t)}\, \mathrm{d}v$, we obtain

$$\partial_2 F(t, \widehat{m}_t(u)) = \frac{\sigma_0^2 \mathrm{e}^{-\alpha\xi^2 r(u)}}{\sqrt{\widehat{r}(u|t)}} e^{2\xi \widehat{m}_t(u)} \int_{\mathbb{R}} e^{2\xi \sqrt{\widehat{r}(u|t)} v} v \phi(v)\, \mathrm{d}v$$

$$= \frac{\sigma_0^2 \mathrm{e}^{-\alpha\xi^2 r(u)}}{\sqrt{\widehat{r}(u|t)}} e^{2\xi \widehat{m}_t(u)} \mathbb{E}\left(e^{2\xi\sqrt{\widehat{r}(u|t)}Z} Z \right),$$

where $Z \sim \mathcal{N}(0,1)$. Using formula $\mathbb{E}[Ze^{\alpha Z}] = \alpha e^{\frac{\alpha^2}{2}}$, we get

$$\partial_2 F(t, \widehat{m}_t(u)) = 2\sigma_0^2 \xi \exp\left\{2\xi \widehat{m}_u(u) + 2\xi^2 \widehat{r}(u|t) - \alpha\xi^2 r(u)\right\} = 2\xi F(t, \widehat{m}_t(u)).$$

The remaining formulae follow accordingly. □

Remark 7.1.4. *Using that* $F(t, \widehat{m}_t(u)) = \mathbb{E}_t\left[\sigma_u^2\right]$*, it is straightforward to see that*

$$\mathrm{d}M_t = 2\xi \left(\int_t^T \mathbb{E}_t\left[\sigma_u^2\right] K(u,t)\,\mathrm{d}u \right) \mathrm{d}W_t, \tag{7.5}$$

$$\mathrm{d}[M_\cdot, W_\cdot]_t = 2\xi \int_t^T \mathbb{E}_t\left[\sigma_u^2\right] K(u,t)\,\mathrm{d}u\,\mathrm{d}t, \tag{7.6}$$

$$\mathrm{d}[M_\cdot, M_\cdot]_t = 4\xi^2 \int_t^T \int_t^T \mathbb{E}_t\left[\sigma_{u_1}^2\right] \mathbb{E}_t\left[\sigma_{u_2}^2\right] K(u_1,t) K(u_2,t)\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}t. \tag{7.7}$$

Lemma 7.1.5. *Let* σ_t *be as in* (3.24) *and* $0 \le t \le u$*. Then, we can re-write* $F(t, \widehat{m}_t(u))$ *as*

$$\mathbb{E}_t\left[\sigma_u^2\right] = \sigma_t^2 \exp\Big\{-\alpha\xi^2 (r(u) - r(t)) + 2\xi \int_0^t (K(u,z) - K(t,z))\,dW_z + 2\xi^2 \widehat{r}(u|t)\Big\}. \tag{7.8}$$

Moreover, we also have the following equalities

$$\mathbb{E}_t\left[\sigma_u^3 \exp\left\{2\xi \int_0^u (K(s,z) - K(u,z))\,dW_z\right\}\right]$$
$$= \sigma_t^3 \exp\Big\{-\frac{3}{2}\alpha\xi^2 (r(u) - r(t)) + \xi \int_0^t (2K(s,z) + K(u,z) - 3K(t,z))\,dW_z + \frac{\xi^2}{2} \int_t^u (2K(s,z) + K(u,z))^2\,dz\Big\}$$

and

$$\mathbb{E}_t\left[\sigma_u^4 \exp\left\{2\xi \int_0^u (K(s,z) + K(v,z) - 2K(u,z))\,dW_z\right\}\right]$$
$$= \sigma_t^4 \exp\Big\{-2\alpha\xi^2 (r(u) - r(t)) + 2\xi \int_0^t (K(s,z) + K(v,z) - 2K(t,z))\,dW_z + 2\xi^2 \int_t^u (K(s,z) + K(v,z))^2\,dz\Big\}.$$

Proof. The calculations to obtain these statements are straightforward. □

Proposition 7.1.6 (Terms in the approximation formula for the exponential Volterra volatility model)**.** *Let* σ_t *be as in* (3.24) *and* $0 \leq t \leq u$. *Then*

$$\begin{aligned}\frac{\rho}{2} L[W,M]_t = \rho\xi\sigma_t^3 \int_t^T \int_u^T & \exp\left\{-\frac{3}{2}\alpha\xi^2\left(r(u)-r(t)\right)\right\} \\ &\cdot \exp\left\{\frac{\xi^2}{2}\int_t^u \left(2K(s,z)+K(u,z)\right)^2\,\mathrm{d}z - \alpha\xi^2\left(r(s)-r(u)\right) + 2\xi^2\widehat{r}(s|u)\right\} \\ &\cdot \exp\left\{\xi\int_0^t \left(2K(s,z)+K(u,z)-3K(t,z)\right)\,\mathrm{d}W_z\right\} K(s,u)\,\mathrm{d}s\,\mathrm{d}u \qquad (7.9)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{8} D[M,M]_t = \frac{1}{2}\xi^2\sigma_t^4 \int_t^T\int_u^T\int_u^T & \exp\Big\{-\alpha\xi^2\left(r(s)+r(v)-2r(t)\right) + 2\xi^2\left(\widehat{r}(s|u)+\widehat{r}(v|u)\right) \\ &+ 2\xi\int_0^t \left(K(s,z)+K(v,z)-2K(t,z)\right)\,\mathrm{d}W_z + 2\xi^2\int_t^u \left(K(s,z)+K(v,z)\right)^2\,\mathrm{d}z\Big\} \\ &\cdot K(s,u)K(v,u)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}u. \qquad (7.10)\end{aligned}$$

In particular,

$$\begin{aligned}\frac{\rho}{2} L[W,M]_0 = \rho\xi\sigma_0^3 \int_0^T\int_u^T & \exp\left\{\frac{\xi^2}{2}\int_0^u [2K(s,z)+K(u,z)]^2\,\mathrm{d}z\right\}\cdot \\ &\cdot \exp\left\{2\xi^2\widehat{r}(s|u) - \frac{1}{2}\alpha\xi^2 r(u) - \alpha\xi^2 r(s)\right\} K(s,u)\ \mathrm{d}s\ \mathrm{d}u \qquad (7.11)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{8} D[M,M]_0 = \frac{1}{2}\sigma_0^4\xi^2 \int_0^T\int_u^T\int_u^T & \exp\left\{2\xi^2\int_0^u [K(s,z)+K(v,z)]^2\,\mathrm{d}z\right\}\cdot \\ &\cdot \exp\left\{2\xi^2\left(\widehat{r}(s|u)+\widehat{r}(v|u)\right) - \alpha\xi^2\left(r(s)+r(v)\right)\right\}\cdot \\ &\cdot K(s,u)K(v,u)\,\mathrm{d}s\,\mathrm{d}v\ \mathrm{d}u. \qquad (7.12)\end{aligned}$$

Proof. We have that

$$\begin{aligned}\frac{\rho}{2} L[W,M]_t &= \frac{\rho}{2}\,\mathbb{E}_t\left[\int_t^T \sigma_u\,\mathrm{d}[M_\cdot,W_\cdot]_u\right] \\ &= \rho\xi\,\mathbb{E}_t\left[\int_t^T \sigma_u\left(\int_u^T \mathbb{E}_u\left[\sigma_s^2\right] K(s,u)\,\mathrm{d}s\right)\,\mathrm{d}u\right] \\ &= \rho\xi\int_t^T \mathbb{E}_t\left[\sigma_u\left(\int_u^T \mathbb{E}_u\left[\sigma_s^2\right] K(s,u)\,\mathrm{d}s\right)\right]\,\mathrm{d}u \\ &= \rho\xi\int_t^T\int_u^T \mathbb{E}_t\Big[\sigma_u^3\exp\left\{-\alpha\xi^2\left(r(s)-r(u)\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&\cdot \exp\Big\{2\xi \int_0^u \left(K(s,z) - K(u,z)\right)\, \mathrm{d}W_z + 2\xi^2 \widehat{r}(s|u)\Big\} K(s,u)\Big]\, \mathrm{d}s\, \mathrm{d}u \\
&= \rho\xi \int_t^T \int_u^T \mathbb{E}_t \left[\sigma_u^3 \exp\Big\{2\xi \int_0^u \left(K(s,z) - K(u,z)\right)\, \mathrm{d}W_z\Big\}\right] \\
&\cdot \exp\Big\{-\alpha\xi^2 \left(r(s) - r(u)\right) + 2\xi^2 \widehat{r}(s|u)\Big\} K(s,u)\, \mathrm{d}s\, \mathrm{d}u \\
&= \rho\xi\sigma_t^3 \int_t^T \int_u^T \exp\left\{-\frac{3}{2}\alpha\xi^2 \left(r(u) - r(t)\right)\right\} \\
&\cdot \exp\left\{\xi \int_0^t \left(2K(s,z) + K(u,z) - 3K(t,z)\right)\, \mathrm{d}W_z\right\} \\
&\cdot \exp\Big\{\frac{\xi^2}{2} \int_t^u \left(2K(s,z) + K(u,z)\right)^2\, \mathrm{d}z - \alpha\xi^2 \left(r(s) - r(u)\right) + 2\xi^2 \widehat{r}(s|u)\Big\} K(s,u)\, \mathrm{d}s\, \mathrm{d}u.
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
\frac{1}{8} D[M,M]_t &= \frac{1}{8}\, \mathbb{E}_t \left[\int_t^T \mathrm{d}[M_\cdot, M_\cdot]_u\right] \\
&= \frac{1}{2}\xi^2\, \mathbb{E}_t \left[\int_t^T \left(\int_u^T \mathbb{E}_u\left[\sigma_s^2\right] K(s,u)\, \mathrm{d}s\right)^2 \mathrm{d}u\right] \\
&= \frac{1}{2}\xi^2 \int_t^T \mathbb{E}_t \left[\left(\int_u^T \mathbb{E}_u\left[\sigma_s^2\right] K(s,u)\, \mathrm{d}s\right)^2\right] \mathrm{d}u \\
&= \frac{1}{2}\xi^2 \int_t^T \mathbb{E}_t \left[\left(\int_u^T \int_u^T \mathbb{E}_u\left[\sigma_s^2\right] \mathbb{E}_u\left[\sigma_v^2\right] K(s,u)K(v,u)\, \mathrm{d}s\, \mathrm{d}v\right)\right] \mathrm{d}u \\
&= \frac{1}{2}\xi^2 \int_t^T \mathbb{E}_t \Bigg[\int_u^T \int_u^T \sigma_u^4 K(s,u)K(v,u) \exp\Big\{-\alpha\xi^2 (r(s) + r(v) - 2r(u)) \\
&+\ 2\xi \int_0^u \left(K(s,z) + K(v,z) - 2K(u,z)\right)\, \mathrm{d}W_z + 2\xi^2 \left(\widehat{r}(s|u) + \widehat{r}(v|u)\right)\Big\}\, \mathrm{d}s\, \mathrm{d}v\Bigg]\, \mathrm{d}u \\
&= \frac{1}{2}\xi^2 \int_t^T \int_u^T \int_u^T \mathbb{E}_t \left[\sigma_u^4 \exp\left\{2\xi \int_0^u \left(K(s,z) + K(v,z) - 2K(u,z)\right)\, \mathrm{d}W_z\right\}\right] \\
&\exp\left\{-\alpha\xi^2 (r(s) + r(v) - 2r(u)) + 2\xi^2 \left(\widehat{r}(s|u) + \widehat{r}(v|u)\right)\right\} K(s,u)K(v,u)\, \mathrm{d}s\, \mathrm{d}v\, \mathrm{d}u \\
&= \frac{1}{2}\xi^2\sigma_t^4 \int_t^T \int_u^T \int_u^T \exp\Big\{-\alpha\xi^2 \left(r(s) + r(v) - 2r(t)\right) + 2\xi^2 \left(\widehat{r}(s|u) + \widehat{r}(v|u)\right) \\
&+ 2\xi \int_0^t \left(K(s,z) + K(v,z) - 2K(t,z)\right)\, \mathrm{d}W_z + 2\xi^2 \int_t^u \left(K(s,z) + K(v,z)\right)^2\, \mathrm{d}z\Big\} \\
&\cdot K(s,u)K(v,u)\, \mathrm{d}s\, \mathrm{d}v\, \mathrm{d}u.
\end{aligned}
$$

□

For the exponential Volterra volatility model we can determine an upper error bound for the price approximation in the following way.

Theorem 7.1.7 (Upper error bound for the exponential Volterra volatility model)**.** *Let X_t be a log-price process (3.19) with σ_t being the exponential Volterra volatility process (3.24). Let the processes $D[M,M]_t$ and $L[W,M]_t$ be as in Corollary 4.2.1. Then we can express the call option fair value V_t by*

$$
\begin{aligned}
V_t = & C_{\overline{BS}}(t, X_t, v_t) \\
& + \frac{\rho}{2}\Lambda\Gamma C_{\overline{BS}}(t, X_t, v_t) L[W,M]_t \\
& + \frac{1}{8}\Gamma^2 C_{\overline{BS}}(t, X_t, v_t) D[M,M]_t \\
& + \epsilon_t,
\end{aligned}
$$

where ϵ_t are error terms of order $O\left(\rho\left(\xi^2+\xi^3+\rho\left(\xi+\xi^2\right)\right)+\xi^3+\xi^4\right)$.

Proof. Note that using (7.5) we have that

$$
\mathrm{d}\,[M,M]_t = 4\xi^2 \left(\int_t^T \mathbb{E}_t\left[\sigma_u^2\right] K(u,t)\,\mathrm{d}u\right)^2 \mathrm{d}t, \tag{7.13}
$$

$$
\mathrm{d}\,[M,W]_t = 2\xi \left(\int_t^T \mathbb{E}_t\left[\sigma_u^2\right] K(u,t)\,\mathrm{d}u\right) \mathrm{d}t. \tag{7.14}
$$

Applying the Jensen's inequality to (7.8), we can see that

$$
a_t^2 \geq \sigma_t^2(T-t)\exp\left\{\frac{1}{T-t}\int_t^T \left[-\alpha\xi^2(r(u)-r(t)) + 2\xi\left(\widehat{m}_t(u)-\widehat{m}_t(t)\right) + 2\xi^2 r(u|t)\right]\,\mathrm{d}u\right\}.
$$

Then, it is easy to find that

$$
\begin{aligned}
\frac{T-t}{a_t^2} \leq & \frac{1}{\sigma_0^2}\exp\Big\{-2\xi\widehat{m}_t(t) + \alpha\xi^2 r(t) \\
& - \frac{1}{T-t}\int_t^T \left[-\alpha\xi^2(r(u)-r(t)) + 2\xi\left(\widehat{m}_t(u)-\widehat{m}_t(t)\right) + 2\xi^2 r(u|t)\right]\,\mathrm{d}u\Big\}
\end{aligned}
$$

where the exponent

$$
-2\xi\widehat{m}_t(t) + \alpha\xi^2 r(t) - \frac{1}{T-t}\int_t^T \left[-\alpha\xi^2(r(u)-r(t)) + 2\xi\left(\widehat{m}_t(u)-\widehat{m}_t(t)\right) + 2\xi^2 r(u|t)\right] du.
$$

is a Gaussian process. Therefore $\frac{1}{a_t^2}$ has finite moments of all orders.

Using the error terms specified in (4.8) and (4.9) and Lemma 3.1.10 (ii), we find the following decompositions for each term

$$
\left|\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\overline{BS}}(u, X_u, v_u)\,\mathrm{d}\,[M,M]_u\right] - \frac{1}{8}\Gamma^2 C_{\overline{BS}}(t, X_t, v_t) D[M,M]_t\right|
$$

$$
\begin{aligned}
\leq\ & \frac{C}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^7}+\frac{3}{a_u^6}+\frac{3}{a_u^5}+\frac{1}{a_u^4}\right)\frac{1}{8}D[M,M]_u\,\mathrm{d}\,[M,M]_u\right] \\
+\ & \frac{C\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6}+\frac{2}{a_u^5}+\frac{1}{a_u^4}\right)\frac{1}{8}D[M,M]_u\sigma_u\,\mathrm{d}\,[W,M]_u\right] \\
+\ & C\rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\sigma_u\,\mathrm{d}\left[W,\frac{1}{8}D[M,M]_u\right]_u\right] \\
+\ & \frac{C}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{2}{a_u^4}+\frac{1}{a_u^3}\right)\,\mathrm{d}\left[M,\frac{1}{8}D[M,M]_u\right]_u\right]
\end{aligned}
\tag{7.15}
$$

and

$$
\begin{aligned}
& \left|\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\overline{BS}}(u,X_u,v_u)\sigma_u\,\mathrm{d}\,[W,M]_u\right]-\frac{\rho}{2}\Lambda\Gamma C_{\overline{BS}}(t,X_t,v_t)L[W,M]_t\right| \\
\leq\ & \frac{C}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6}+\frac{2}{a_u^5}+\frac{1}{a_u^4}\right)\frac{\rho}{2}L[W,M]_u\,\mathrm{d}\,[M,M]_u\right] \\
+\ & \frac{C\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{1}{a_u^4}\right)\frac{\rho}{2}L[W,M]_u\sigma_u\,\mathrm{d}\,[W,M]_u\right] \\
+\ & C\rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{1}{a_u^3}\sigma_u\,\mathrm{d}\left[W,\frac{\rho}{2}L[W,M]_u\right]_u\right] \\
+\ & \frac{C}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\,\mathrm{d}\left[M,\frac{\rho}{2}L[W,M]_u\right]_u\right].
\end{aligned}
\tag{7.16}
$$

Since we have a Gaussian driving process and the kernel is square-integrable by assumption (2.10), the integrals are well-defined and the conditional expectations are finite. Substituting the values of $\mathrm{d}\,[W,M]_t$, $\mathrm{d}\,[M,M]_t$, $D[M,M]_t$ and $L[W,M]_t$, equations (7.14), (7.13), (7.9) and (7.10) respectively, an initial estimate of the error is obtained as $O\left(\rho\left(\xi^2+\xi^3+\rho\left(\xi+\xi^2\right)\right)+\xi^3+\xi^4\right)$.

□

Further, we express differentials with respect to the n^{th}-power of the exponential Volterra volatility process when Y_t is a semi-martingale.

Lemma 7.1.8. *Let σ_t be as in (3.24) and Y_t a semi-martingale. Let $n \geq 1$, we have that*

$$
\mathrm{d}\sigma_t^n \;=\; \sigma_t^n K(t,t)\left[n\xi\,\mathrm{d}W_t+\frac{n}{2}\xi^2 K(t,t)\,(n-\alpha)\,\mathrm{d}t\right]. \tag{7.17}
$$

Proof. The formula is an immediate consequence of the Itô formula. □

Lemma 7.1.9. *Let σ_t be as in (3.24) and Y_t is a semi-martingale. We can calculate $\mathrm{d}L[M,W]_t$ and $\mathrm{d}D[M,M]_t$. In order to simplify the notation, we define the two following functions*

$$
\varphi(t,s,x,T) := \exp\Big\{-\frac{3}{2}\alpha\xi^2\Big(r(x)-r(t)\Big)+\xi\int_0^t\Big(2K(s,z)+K(x,z)-3K(t,z)\Big)\,\mathrm{d}W_z\Big\}.
$$

$$\cdot \exp\Big\{\frac{\xi^2}{2}\int_t^x \Big(2K(s,z)+K(x,z)\Big)^2 \mathrm{d}z - \alpha\xi^2\Big(r(s)-r(x)\Big) + 2\xi^2\widehat{r}(s|x)\Big\},$$

$$\psi(t,s,v,x,T) := \exp\Big\{-\alpha\xi^2\Big(r(s)+r(v)-2r(t)\Big) + 2\xi^2\Big(\widehat{r}(s|x)+\widehat{r}(v|x)\Big) + 2\xi\int_0^t \Big(K(s,z)+K(v,z)-2K(t,z)\Big)\mathrm{d}W_z + 2\xi^2\int_t^x \Big(K(s,z)+K(v,z)\Big)^2 \mathrm{d}z\Big\}.$$

Then

$$\begin{aligned}\frac{\rho}{2}\,\mathrm{d}L[W,M]_t &= \rho\xi\sigma_t^3 K(t,t)\Big[3\xi\,\mathrm{d}W_t + \frac{3}{2}\xi^2 K(t,t)\,(3-\alpha)\;\mathrm{d}t\Big]\int_t^T\int_x^T \varphi(t,s,x,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad - \rho\xi\sigma_t^3\int_0^T \varphi(t,s,t,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad + \rho\xi\sigma_t^3\int_t^T\int_x^T \varphi(t,s,x,T)\Big\{\frac{3}{2}\alpha\xi^2\,\mathrm{d}r(t) + \xi\Big(2K(s,t)+K(x,t)-3K(t,t)\Big)\mathrm{d}W_t \\ &\quad + \frac{1}{2}\xi^2\Big(2K(s,t)+K(x,t)-3K(t,t)\Big)^2\mathrm{d}t - \frac{\xi^2}{2}\left(2K(s,t)+K(x,t)\right)^2\,\mathrm{d}t\Big\} \\ &\quad \cdot K(s,x)\,\mathrm{d}s\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\frac{1}{8}\,\mathrm{d}D[M,M]_t &= \frac{1}{2}\xi^2\sigma_t^4 K(t,t)\Big[4\xi\,\mathrm{d}W_t + 2\xi^2 K(t,t)\,(4-\alpha)\;\mathrm{d}t\Big] \\ &\quad \int_t^T\int_x^T\int_x^T \Phi(t,s,v,x,T)\cdot K(s,x)K(v,x)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x \\ &\quad - \frac{1}{2}\xi^2\sigma_t^4\int_0^T\int_0^T \Phi(t,s,v,t,T)K(s,t)K(v,t)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}t \\ &\quad + \frac{1}{2}\xi^2\sigma_t^4\int_t^T\int_x^T\int_x^T \Phi(t,s,v,x,T)K(s,x)K(v,x) \\ &\qquad \Big\{2\alpha\xi^2\,\mathrm{d}r(t) + 2\xi\Big(K(s,t)+K(v,t)-2K(t,t)\Big)\mathrm{d}W_t \\ &\qquad + 2\xi^2\Big(K(s,t)+K(v,t)-2K(t,t)\Big)^2 dt \\ &\qquad - 2\xi^2\Big(K(s,t)+K(v,t)\Big)^2\mathrm{d}t\Big\}\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

We define the following auxiliary function

$$\zeta(t,T) := \int_t^T \mathbb{E}_t\left[\sigma_z^2\right]K(z,t)\,\mathrm{d}z.$$

Then, it is easier to see that the covariations are the following

$$
\begin{aligned}
\mathrm{d}\Big[\frac{\rho}{2}L[W,M]_t, W\Big]_t &= 3\rho\xi^2\sigma_t^3 K(t,t)\int_t^T\int_x^T \varphi(t,s,x,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}x\ \mathrm{d}t\\
&\quad + \rho\xi^2\sigma_t^3\int_t^T\int_x^T \varphi(t,s,x,T)\Big(2K(s,t)+K(x,t)-3K(t,t)\Big)\\
&\quad K(s,x)\,\mathrm{d}s\,\mathrm{d}x\ \mathrm{d}t,\\
\mathrm{d}\Big[\frac{\rho}{2}L[W,M]_t, M\Big]_t &= 6\rho\xi^3\sigma_t^3 K(t,t)\zeta(t,T)\int_t^T\int_x^T \varphi(t,s,x,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}x\ \mathrm{d}t\\
&\quad + 2\rho\xi^3\sigma_t^3\zeta(t,T)\int_t^T\int_x^T \varphi(t,s,x,T)\Big(2K(s,t)+K(x,t)-3K(t,t)\Big)\\
&\quad K(s,x)\,\mathrm{d}s\,\mathrm{d}x\ \mathrm{d}t,\\
\mathrm{d}\Big[\frac{1}{8}D[M,M]_t, W\Big]_t &= 2\xi^3\sigma_t^4 K(t,t)\int_t^T\int_x^T\int_x^T \psi(t,s,v,x,T)\cdot K(s,x)K(v,x)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x\ \mathrm{d}t\\
&\quad + \xi^3\sigma_u^4\int_u^T\int_x^T\int_x^T \psi(t,s,v,x,T)K(s,x)K(v,x)\\
&\quad \Big(K(s,u)+K(v,u)-2K(u,u)\Big)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x\ \mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}\Big[\frac{1}{8}D[M,M]_t, M\Big]_t &= 4\xi^4\sigma_t^4 K(t,t)\zeta(t,T)\int_t^T\int_x^T\int_x^T \psi(t,s,v,x,T)\\
&\quad \cdot K(s,x)K(v,x)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x\ \mathrm{d}t\\
&\quad + 2\xi^4\sigma_t^4\zeta(t,T)\int_t^T\int_x^T\int_x^T \psi(t,s,v,x,T)K(s,x)K(v,x)\\
&\quad \Big(K(s,t)+K(v,t)-2K(t,t)\Big)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x\ \mathrm{d}t.
\end{aligned}
$$

Proof. Now, we can re-write $\frac{\rho}{2}L[W,M]_t$ as

$$
\frac{\rho}{2}L[W,M]_t = \rho\xi\sigma_t^3\int_t^T\int_x^T \varphi(t,s,x,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}x
$$

and $\frac{1}{8}D[M,M]_t$ as

$$
\frac{1}{8}D[M,M]_t = \frac{1}{2}\xi^2\sigma_t^4\int_t^T\int_x^T\int_x^T \psi(t,s,v,x,T)\cdot K(s,x)K(v,x)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x.
$$

We have that

$$
\frac{\rho}{2}\,\mathrm{d}L[W,M]_t = \rho\xi\,\mathrm{d}\sigma_t^3\int_t^T\int_x^T \varphi(t,s,x,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}x
$$

$$
\begin{aligned}
&- \rho\xi\sigma_t^3 \int_0^T \varphi(t,s,t,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}t \\
&+ \rho\xi\sigma_t^3 \int_t^T \int_x^T \varphi(t,s,x,T)\Big\{\frac{3}{2}\alpha\xi^2\,\mathrm{d}r(t) \\
&\quad + \xi\Big(2K(s,t)+K(x,t)-3K(t,t)\Big)\,\mathrm{d}W_t \\
&\quad + \xi^2\Big(2K(s,t)+K(x,t)-3K(t,t)\Big)^2\,\mathrm{d}t \\
&\quad - \frac{\xi^2}{2}\Big(2K(s,t)+K(x,t)\Big)^2\,\mathrm{d}t\Big\}K(s,x)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
$$

Using Lemma 7.1.8, we obtain

$$
\begin{aligned}
\frac{\rho}{2}\,\mathrm{d}L[W,M]_t &= \rho\xi\sigma_t^3K(t,t)\Big[3\xi\,\mathrm{d}W_t + \frac{3}{2}\xi^2K(t,t)\,(3-\alpha)\;\mathrm{d}t\Big]\int_t^T\int_x^T \varphi(t,s,x,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}x \\
&- \rho\xi\sigma_t^3 \int_x^T \varphi(t,s,t,T)K(s,x)\,\mathrm{d}s\,\mathrm{d}t \\
&+ \rho\xi\sigma_t^3 \int_t^T \int_x^T \varphi(t,s,x,T)\Big\{\frac{3}{2}\alpha\xi^2\,\mathrm{d}r(t) + \xi\Big(2K(s,t)+K(x,t)-3K(t,t)\Big)\,\mathrm{d}W_t \\
&\quad + \frac{1}{2}\xi^2\Big(2K(s,t)+K(x,t)-3K(t,t)\Big)^2\,\mathrm{d}t \\
&\quad - \frac{\xi^2}{2}\Big(2K(s,t)+K(x,t)\Big)^2\,\mathrm{d}t\Big\}K(s,x)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
$$

We have that

$$
\begin{aligned}
\frac{1}{8}\,\mathrm{d}D[M,M]_t &= \frac{1}{2}\xi^2\,\mathrm{d}\sigma_t^4 \int_t^T\int_x^T\int_x^T \Phi(t,s,v,x,T)\cdot K(s,x)K(v,x)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x \\
&- \frac{1}{2}\xi^2\sigma_t^4 \int_t^T\int_t^T \Phi(t,s,v,t,T)K(s,t)K(v,t)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}t \\
&+ \frac{1}{2}\xi^2\sigma_t^4 \int_t^T\int_x^T\int_x^T \Phi(t,s,v,x,T)K(s,x)K(v,x)\Big\{2\alpha\xi^2\,\mathrm{d}r(t) \\
&\quad + 2\xi\Big(K(s,t)+K(v,t)-2K(t,t)\Big)\,\mathrm{d}W_t + 2\xi^2\Big(K(s,t)+K(v,t)-2K(t,t)\Big)^2 dt \\
&\quad - 2\xi^2\Big(K(s,t)+K(v,t)\Big)^2\,\mathrm{d}t\Big\}\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
$$

Using Lemma 7.1.8, we obtain

$$
\begin{aligned}
\frac{1}{8}\,\mathrm{d}D[M,M]_t &= \frac{1}{2}\xi^2\sigma_t^4K(t,t)\Big[4\xi\,\mathrm{d}W_t + 2\xi^2K(t,t)\,(4-\alpha)\;\mathrm{d}t\Big] \\
&\int_t^T\int_x^T\int_x^T \Phi(t,s,v,x,T)\cdot K(s,x)K(v,x)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\xi^2\sigma_t^4 \int_t^T \int_t^T \Phi(t,s,v,t,T)K(s,t)K(v,t)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}t \\
&+\frac{1}{2}\xi^2\sigma_t^4 \int_t^T \int_x^T \int_x^T \Phi(t,s,v,x,T)K(s,x)K(v,x) \\
&\quad \Big\{2\alpha\xi^2\,\mathrm{d}r(t) + 2\xi\Big(K(s,t)+K(v,t)-2K(t,t)\Big)\,\mathrm{d}W_t \\
&\quad + 2\xi^2\Big(K(s,t)+K(v,t)-2K(t,t)\Big)^2\,\mathrm{d}t \\
&\quad - 2\xi^2\Big(K(s,t)+K(v,t)\Big)^2\,\mathrm{d}t\Big\}\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
$$

□

Remark 7.1.10. *If Y_t is a semimartingale, terms $L[M,W]_t$ and $D[M,M]_t$ and consequently its covariations can be further specified. Therefore, the error estimation can be improved to $O\Big((\xi^2+\rho\xi)^2\Big)$.*

7.1.3 Exponential fractional volatility model

Let us now focus on a particularly important example of Gaussian Volterra processes, the *fractional Brownian motion* (fBm) defined in (3.25).

Example 7.1.11 (Volatility driven by the approximate fractional Brownian motion)**.** *Let us consider model (3.19) with volatility process*

$$\sigma_t = \sigma_0 \exp\left\{\xi \widetilde{B}_t - \frac{1}{2}\alpha\xi^2 r(t)\right\},$$

where

$$\widetilde{B}_t = \int_0^t \widetilde{K}(t,s)\,\mathrm{d}W_s$$

and

$$\widetilde{K}(t,s) = \sqrt{2H}(t-s+\varepsilon)^{H-1/2}, \quad s \le t, \varepsilon \ge 0, H \in (0,1).$$

Then

$$
\begin{aligned}
r(t,s) &= \int_0^{t\wedge s} \widetilde{K}(t,v)\widetilde{K}(s,v)\,\mathrm{d}v, \\
r(t) &= \int_0^t \widetilde{K}^2(t,v)\,\mathrm{d}v = 2H\int_0^t (t-v+\varepsilon)^{2H-1}\,\mathrm{d}v = (t+\varepsilon)^{2H} - \varepsilon^{2H}.
\end{aligned}
$$

Note that if $\varepsilon = 0$, we get exactly the variance $r(t) = t^{2H}$, that it is the variance of the standard fractional Brownian motion. Further we have

$$\widehat{r}(t|u) = r(t) - \int_0^u \widetilde{K}^2(t,v)\,\mathrm{d}v = r(t) - 2H\int_0^u (t-v+\varepsilon)^{2H-1}\,\mathrm{d}v = (t-u+\varepsilon)^{2H} - \varepsilon^{2H}$$

and thus

$$\begin{aligned}
\frac{\rho}{2}L[W,M]_0 = \rho\sigma_0^3\xi\sqrt{2H}\int_0^T\int_u^T &\exp\Big\{\xi^2 H\int_0^u [(u-v+\varepsilon)^{H-1/2}+2(s-v+\varepsilon)^{H-1/2}]^2\,\mathrm{d}v\Big\}\cdot\\
&\cdot\exp\Big\{2\xi^2[(s-u+\varepsilon)^{2H}-\varepsilon^{2H}]\Big\}\cdot\\
&\cdot\exp\Big\{-\frac{1}{2}\alpha\xi^2[(u+\varepsilon)^{2H}+2(s+\varepsilon)^{2H}-3\varepsilon^{2H}]\Big\}\cdot\\
&\cdot(s-u+\varepsilon)^{H-1/2}\,\mathrm{d}s\,\mathrm{d}u,\\
\frac{1}{8}D[M,M]_0 = \sigma_0^4\xi^2 H\int_0^T\int_u^T\int_u^T &\exp\Big\{4\xi^2 H\int_0^u [(t_1-v+\varepsilon)^{H-1/2}+(t_2-v+\varepsilon)^{H-1/2}]^2\,\mathrm{d}v\Big\}\cdot\\
&\cdot\exp\Big\{2\xi^2[(t_1-u+\varepsilon)^{2H}+(t_2-u+\varepsilon)^{2H}-2\varepsilon^{2H}]\Big\}\cdot\\
&\cdot\exp\Big\{-\alpha\xi^2[(t_1+\varepsilon)^{2H}+(t_2+\varepsilon)^{2H}-2\varepsilon^{2H}]\Big\}\cdot\\
&\cdot(t_1-u+\varepsilon)^{H-1/2}(t_2-u+\varepsilon)^{H-1/2}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}u.
\end{aligned}$$

Example 7.1.12 (Volatility driven by the standard Wiener process)**.** *If in the previous Example 7.1.11 we take $H = 1/2$ and $\varepsilon = 0$, we get model (3.19) with exponential Wiener volatility process*

$$\sigma_t = \sigma_0 \exp\left\{\xi\widetilde{W}_t - \frac{1}{2}\alpha\xi^2 r(t)\right\}, \tag{7.18}$$

where

$$\widetilde{W}_t = \int_0^t \widetilde{K}(t,s)\,\mathrm{d}W_s$$

is the standard Wiener process, i.e. where $\widetilde{K}(t,s) = \mathbf{1}_{\{s\le t\}}$. In this case, we have that

$$\begin{aligned}
v_t^2 &= \frac{\sigma_t^2}{(2-\alpha)\,\xi^2(T-t)}\left[\exp\{(2-\alpha)\,\xi^2(T-t)\}-1\right],\\
r(t,s) &= \int_0^{t\wedge s}\widetilde{K}(t,v)\widetilde{K}(s,v)\,\mathrm{d}v = t\wedge s,\\
r(t) &= \int_0^t \widetilde{K}^2(t,v)\,\mathrm{d}v = t.
\end{aligned}$$

Define

$$\phi(t,T,\alpha) := \int_t^T \exp\left\{(2-\alpha)\xi^2(s-t)\right\}\,\mathrm{d}s. \tag{7.19}$$

It is easy to see that

$$\mathrm{d}M_t = 2\xi\sigma_t^2\,\mathrm{d}W_t\phi(t,T,\alpha),$$

and thus

$$\begin{aligned}
\frac{\rho}{2}L[W,M]_0 &= \rho\sigma_0^3\xi\int_0^T\int_0^T \exp\Big\{\frac{1}{2}\xi^2\int_0^u[\mathbf{1}_{\{v\le u\}}+2\cdot\mathbf{1}_{\{v\le s\}}]^2\,\mathrm{d}v\Big\}\cdot\\
&\qquad\cdot\exp\Big\{2\xi^2(s-u)-\frac{1}{2}\alpha\xi^2u-\alpha\xi^2s\Big\}\mathbf{1}_{\{u\le s\}}\,\mathrm{d}s\,\mathrm{d}u\\
&= \rho\sigma_0^3\xi\int_0^T\int_0^T\exp\Big\{\frac{9}{2}\xi^2u\Big\}\exp\Big\{\frac{1}{2}\xi^2[(4-2\alpha)s-(4+\alpha)u]\Big\}\mathbf{1}_{\{u\le s\}}\,\mathrm{d}s\,\mathrm{d}u\\
&= \frac{2\rho\sigma_0^3}{3(2-\alpha)(3-\alpha)(5-\alpha)\xi^3}\Big[2(2-\alpha)\exp\Big\{\frac{3}{2}\xi^2(3-\alpha)T\Big\}\\
&\quad-3(3-\alpha)\exp\Big\{\xi^2(2-\alpha)T\Big\}+5-\alpha\Big]
\end{aligned}\tag{7.20}$$

and

$$\begin{aligned}
\frac{1}{8}D[M,M]_0 &= \frac{1}{2}\sigma_0^4\xi^2\int_0^T\int_0^T\int_0^T\exp\Big\{2\xi^2\int_0^u[\mathbf{1}_{\{v\le t_1\}}+\mathbf{1}_{\{v\le t_2\}}]^2\,\mathrm{d}v\Big\}\cdot\\
&\qquad\cdot\exp\Big\{2\xi^2[t_1+t_2-2u]-\alpha\xi^2[t_1+t_2]\Big\}\mathbf{1}_{\{u\le t_1\}}\mathbf{1}_{\{u\le t_2\}}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}u\\
&= \frac{1}{2}\sigma_0^4\xi^2\int_0^T\int_0^T\int_0^T\exp\Big\{8\xi^2u\Big\}\exp\Big\{2\xi^2[t_1+t_2-2u]-\alpha\xi^2[t_1+t_2]\Big\}\\
&\quad\mathbf{1}_{\{u\le t_1\}}\mathbf{1}_{\{u\le t_2\}}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}u\\
&= \frac{\sigma_0^4}{8(2-\alpha)^2(4-\alpha)(6-\alpha)\xi^4}\Big[(2-\alpha)^2\exp\Big\{2(4-\alpha)\xi^2T\Big\}\\
&\quad-(4-\alpha)(6-\alpha)\exp\Big\{2(2-\alpha)\xi^2T\Big\}\\
&\quad+8(4-\alpha)\exp\Big\{(2-\alpha)\xi^2T\Big\}-2(6-\alpha)\Big].
\end{aligned}\tag{7.21}$$

For a model without exponential drift ($\alpha=0$) these formulae simplify to

$$\begin{aligned}
\frac{\rho}{2}L[W,M]_0 &= \frac{\rho\sigma_0^3}{45\xi^3}\left[4\exp\Big\{\frac{9}{2}\xi^2T\Big\}-9\exp\Big\{2\xi^2T\Big\}+5\right],\\
\frac{1}{8}D[M,M]_0 &= \frac{\sigma_0^4}{192\xi^4}\left[\exp\Big\{2\xi^2T\Big\}-1\right]^3\left[\exp\Big\{2\xi^2T\Big\}+3\right]
\end{aligned}$$

and for the classical Bergomi model ($\alpha=1$) we get

$$\frac{\rho}{2}L[W,M]_0 = \frac{\rho\sigma_0^3}{6\xi^3}\left[\exp\Big\{3\xi^2T\Big\}-3\exp\Big\{\xi^2T\Big\}+2\right],$$

$$\frac{1}{8}D[M,M]_0 = \frac{\sigma_0^4}{120\xi^4}\left[\exp\Big\{6\xi^2T\Big\} - 15\exp\Big\{2\xi^2T\Big\} + 24\exp\Big\{\xi^2T\Big\} - 10\right].$$

For matter of convenience, we define the functions

$$\psi(t,T,\alpha) = \int_t^T \exp\left\{(8-2\alpha)\xi^2(s-t)\right\}\left[\exp\left\{(2-\alpha)\xi^2(T-s)\right\} - 1\right]^2 \mathrm{d}s \tag{7.22}$$

and

$$\zeta(t,T,\alpha) = \int_t^T \exp\left\{\frac{1}{2}(9-3\alpha)\xi^2(s-t)\right\}\left[\exp\left\{(2-\alpha)\xi^2(T-s)\right\} - 1\right] \mathrm{d}s. \tag{7.23}$$

We can re-write $\frac{\rho}{2}L[W,M]_t$ *and* $\frac{1}{8}D[M,M]_t$ *as*

$$\frac{\rho}{2}L[W,M]_t = \frac{\rho\sigma_t^3}{(2-\alpha)\xi}\zeta(t,T,\alpha) \tag{7.24}$$

and

$$\frac{1}{8}D[M,M]_t = \frac{\sigma_t^4}{2(2-\alpha)^2\xi^2}\psi(t,T,\alpha). \tag{7.25}$$

It is easy to find the $\frac{\rho}{2}\,\mathrm{d}L[W,M]_t$ *and* $\frac{1}{8}\,\mathrm{d}D[M,M]_t$,

$$\begin{aligned}\frac{\rho}{2}\,\mathrm{d}L[W,M]_t &= \frac{\rho\,\mathrm{d}\sigma_t^3}{(2-\alpha)\xi}\zeta(t,T,\alpha) + \frac{\rho\sigma_t^3}{(2-\alpha)\xi}\zeta'(t,T,\alpha)\,\mathrm{d}t \\ &= \frac{\rho\left(3\xi\sigma_t^3\,\mathrm{d}W_t + \frac{1}{2}(18-3\alpha)\xi^2\sigma^3\,\mathrm{d}t\right)}{(2-\alpha)\xi}\zeta(t,T,\alpha) + \frac{\rho\sigma_t^3}{(2-\alpha)\xi}\zeta'(t,T,\alpha)\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\frac{1}{8}\,\mathrm{d}D[M,M]_t &= \frac{\mathrm{d}\sigma_t^4}{2(2-\alpha)^2\xi^2}\psi(t,T,\alpha) + \frac{\sigma_t^4}{2(2-\alpha)^2\xi^2}\psi'(t,T,\alpha)\,\mathrm{d}t \\ &= \frac{4\xi\sigma_t^4\,\mathrm{d}W_t + 2(8-\alpha)\sigma_t^4\,\mathrm{d}t}{2(2-\alpha)^2\xi^2}\psi(t,T,\alpha) + \frac{\sigma_t^4}{2(2-\alpha)^2\xi^2}\psi'(t,T,\alpha)\,\mathrm{d}t.\end{aligned}$$

Remark 7.1.13. *We can do a Taylor expansion of* $\frac{\rho}{2}L[W,M]_0$ *and* $\frac{1}{8}D[M,M]_0$ *to understand their dependencies better. By doing that we obtain*

$$\begin{aligned}\frac{\rho}{2}L[W,M]_0 \sim \rho\xi T^2\sigma_t^3\Bigg(&\frac{1}{2} + \frac{1}{12}(13-5\alpha)\xi^2T + \frac{1}{96}(\alpha(19\alpha-100)+133)\xi^4T^2 \\ &- \frac{1}{960}\left(5\alpha-13\right)\left(\alpha(13\alpha-70)+97\right)\xi^6T^3 + O\left(T^4\right)\Bigg)\end{aligned}$$

and

$$\frac{1}{8}D[M,M]_0 \sim \xi^2 T^3 \sigma_t^4 \left(-\frac{1}{6}(\alpha-2) + \frac{1}{24}(\alpha-2)(5\alpha-14)\xi^2 T \right.$$
$$\left. - \frac{1}{120}(\alpha-2)\left(17\alpha^2 - 96\alpha + 140\right)\xi^4 T^2 + O\left(T^3\right) \right).$$

Theorem 7.1.14 (Decomposition formula for exponential Wiener volatility model)**.** *Let X_t be the log-price process (3.19) with σ_t being the exponential Wiener volatility process defined in (7.18). Assuming without any loss of generality that the options start at time 0, then we can express the call option fair value V_0 using the processes $\frac{\rho}{2}L[W,M]_0$,$\frac{1}{8}D[M,M]_0$ from (7.20) and (7.21) respectively. In particular,*

$$\begin{aligned} V_0 = {} & C_{\overline{BS}}(0,X_0,v_0) \\ & + \frac{\rho}{2}\Lambda\Gamma C_{\overline{BS}}(0,X_0,v_0)L[W,M]_0 \\ & + \frac{1}{8}\Gamma^2 C_{\overline{BS}}(0,X_0,v_0)D[M,M]_0 \\ & + \epsilon \end{aligned}$$

where ϵ denotes error terms and for $\alpha \geq 0$, $|\epsilon|$ is at most of the order $C\xi(\sqrt{T}+\rho\xi^2)T^{3/2}\Pi(\alpha,T,\xi,\rho)$.

Proof. We will find the upper-bound for terms (I) and (II).

Upper-bound for term (I)
For matter of convenience, we define the function

$$\chi_1(t,T,\alpha) := \int_t^T \exp\left\{\frac{1}{2}(9-3\alpha)\xi^2(s-t)\right\}\left[\exp\left\{(2-\alpha)\xi^2(T-s)\right\} - 1\right] \mathrm{d}s.$$

We can rewrite $\frac{\rho}{2}L[W,M]_t$ as

$$\frac{\rho}{2}L[W,M]_t = \frac{\rho\sigma_t^3}{(2-\alpha)\xi}\chi_1(t,T,\alpha).$$

It is easy to find that

$$\begin{aligned} \mathrm{d}\frac{\rho}{2}L[W,M]_t &= \frac{\rho\,\mathrm{d}\sigma_t^3}{(2-\alpha)\xi}\chi_1(t,T,\alpha) + \frac{\rho\sigma_t^3}{(2-\alpha)\xi}\chi_1'(t,T,\alpha)\,\mathrm{d}t \\ &= \frac{\rho\left(3\xi\sigma_t^3\,\mathrm{d}W_t + \frac{1}{2}(18-3\alpha)\xi^2\sigma^3\,\mathrm{d}t\right)}{(2-\alpha)\xi}\chi_1(t,T,\alpha) + \frac{\rho\sigma_t^3}{(2-\alpha)\xi}\chi_1'(t,T,\alpha)\,\mathrm{d}t. \end{aligned}$$

If $\alpha \geq 0$, we can find an upper-bound for $\chi_1(t,T,\alpha)$ which is

$$\chi_1(t,T,\alpha) \quad \leq \quad \int_t^T \exp\left\{\frac{9}{2}\xi^2(s-t)\right\}\left[\exp\left\{2\xi^2(T-s)\right\} - 1\right] \mathrm{d}s$$

$$= \frac{2}{45\xi^2}\left(-9\exp\left\{2\xi^2(T-t)\right\}+4\exp\left\{\frac{9}{2}\xi^2(T-t)\right\}+5\right).$$

We can re-write the decomposition formula as

$$\begin{aligned}
& \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\overline{BS}}(u,X_u,v_u)\sigma_u\,\mathrm{d}\,[W,M]_u\right]-\Lambda\Gamma C_{\overline{BS}}(t,X_t,v_t)\frac{\rho}{2}L[W,M]_t \\
= & \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^5-2\partial_x^4+\partial_x^3\right)\Gamma C_{\overline{BS}}(u,X_u,v_u)\frac{\rho}{2}L[W,M]_u\,\mathrm{d}\,[M,M]_u\right] \\
+ & \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^4-\partial_x^3\right)\Gamma C_{\overline{BS}}(u,X_u,v_u)\frac{\rho}{2}L[W,M]_u\sigma_u\,\mathrm{d}\,[W,M]_u\right] \\
+ & \rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x^2\Gamma C_{\overline{BS}}(u,X_u,v_u)\sigma_u\,\mathrm{d}\left[W,\frac{\rho}{2}L[W,M]_u\right]_u\right] \\
+ & \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^3-\partial_x^2\right)\Gamma C_{\overline{BS}}(u,X_u,v_u)\,\mathrm{d}\left[M,\frac{\rho}{2}L[W,M]_u\right]_u\right].
\end{aligned}$$

Applying Lemma 3.1.10 (ii) and using the definition of a_u, we obtain

$$\begin{aligned}
& \left|\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\overline{BS}}(u,X_u,v_u)\sigma_u\,\mathrm{d}\,[W,M]_u\right]-\Lambda\Gamma C_{\overline{BS}}(t,X_t,v_t)\frac{\rho}{2}L[W,M]_t\right| \\
\leq & \frac{C}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6}+\frac{2}{a_u^5}+\frac{1}{a_u^4}\right)\frac{\rho}{2}L[W,M]_u\,\mathrm{d}\,[M,M]_u\right] \\
+ & \frac{C\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5}+\frac{1}{a_u^4}\right)\frac{\rho}{2}L[W,M]_u\sigma_u\,\mathrm{d}\,[W,M]_u\right] \\
+ & C\rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{1}{a_u^3}\sigma_u\,\mathrm{d}\left[W,\frac{\rho}{2}L[W,M]_u\right]_u\right] \\
+ & \frac{C}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4}+\frac{1}{a_u^3}\right)\mathrm{d}\left[M,\frac{\rho}{2}L[W,M]_u\right]_u\right].
\end{aligned}$$

Noting that $a_u=\sigma_u\phi^{1/2}(u,T,\alpha)$, where ϕ was defined in (7.19),

$$\begin{aligned}
& \left|\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\overline{BS}}(u,X_u,v_u)\sigma_u\,\mathrm{d}\,[W,M]_u\right]-\Lambda\Gamma C_{\overline{BS}}(t,X_t,v_t)\frac{\rho}{2}L[W,M]_t\right| \\
\leq & \frac{C\rho\xi}{2(2-\alpha)}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{\sigma_u}{\phi(u,T,\alpha)}+\frac{2\sigma_u^2}{\phi^{1/2}(u,T,\alpha)}+\sigma_u^3\right)\chi_1(u,T,\alpha)\,\mathrm{d}u\right] \\
+ & \frac{C\rho^2}{(2-\alpha)}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{\sigma_u}{\phi^{3/2}(u,T,\alpha)}+\frac{\sigma_u^2}{\phi(u,T,\alpha)}\right)\chi_1(u,T,\alpha)\,\mathrm{d}u\right] \\
+ & \frac{C\rho^2 3}{(2-\alpha)}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{\sigma_u}{\phi^{3/2}(u,T,\alpha)}\chi_1(u,T,\alpha)\,\mathrm{d}u\right] \\
+ & \frac{3C\rho\xi}{(2-\alpha)}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{\sigma_u}{\phi(u,T,\alpha)}+\frac{\sigma_u^2}{\phi^{1/2}(u,T,\alpha)}\right)\chi_1(u,T,\alpha)\,\mathrm{d}u\right].
\end{aligned}$$

Being σ_u the only stochastic component, we can get the expectation inside. Each power of σ_u has a different forward value, and in this case, we can find the upper-bound $\exp\left\{\frac{9}{2}\xi^2(u-t)\right\}$ for all the terms. We have that

$$\begin{aligned}
&\left|\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\overline{BS}}(u,X_u,v_u)\sigma_u \,\mathrm{d}\,[W,M]_u\right] - \Lambda\Gamma C_{\overline{BS}}(t,X_t,v_t)\frac{\rho}{2}L[W,M]_t\right| \\
\leq\ & \frac{C\rho\xi}{2(2-\alpha)}\int_t^T e^{-r(u-t)}\exp\left\{\frac{9}{2}\xi^2(u-t)\right\}\left(\frac{\sigma_t}{\phi(u,T,\alpha)}+\frac{2\sigma_t^2}{\phi^{1/2}(u,T,\alpha)}+\sigma_t^3\right)\chi_1(u,T,\alpha)\,\mathrm{d}u \\
+\ & \frac{C\rho^2}{(2-\alpha)}\int_t^T e^{-r(u-t)}\exp\left\{\frac{9}{2}\xi^2(u-t)\right\}\left(\frac{\sigma_t}{\phi^{3/2}(u,T,\alpha)}+\frac{\sigma_t^2}{\phi(u,T,\alpha)}\right)\chi_1(u,T,\alpha)\,\mathrm{d}u \\
+\ & \frac{C\rho^2 3}{(2-\alpha)}\int_t^T e^{-r(u-t)}\exp\left\{\frac{9}{2}\xi^2(u-t)\right\}\frac{\sigma_t}{\phi^{3/2}(u,T,\alpha)}\chi_1(u,T,\alpha)\,\mathrm{d}u \\
+\ & \frac{3C\rho\xi}{(2-\alpha)}\int_t^T e^{-r(u-t)}\exp\left\{\frac{9}{2}\xi^2(u-t)\right\}\left(\frac{\sigma_t}{\phi(u,T,\alpha)}+\frac{\sigma_t^2}{\phi^{1/2}(u,T,\alpha)}\right)\chi_1(u,T,\alpha)\,\mathrm{d}u.
\end{aligned}$$

Substituting $\phi(u,T,\alpha)$ and using the upper-bound for $\chi_1(u,T,\alpha)$ when $\alpha\geq 0$, we have

$$\begin{aligned}
&\left|\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\overline{BS}}(u,X_u,v_u)\sigma_u \,\mathrm{d}\,[W,M]_u\right] - \Lambda\Gamma C_{\overline{BS}}(t,X_t,v_t)\frac{\rho}{2}L[W,M]_t\right| \\
\leq\ & \frac{C\rho\xi}{45(2-\alpha)\xi^2}\int_t^T e^{-r(u-t)}\exp\left\{\frac{9}{2}\xi^2(u-t)\right\} \\
& \left(\sigma_t\frac{2\xi^2}{[\exp\{2\xi^2(T-u)\}-1]}+2\sigma_t^2\frac{\sqrt{2}\xi}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]^{\frac{1}{2}}}+\sigma_t^3\right) \\
& \left(-9\exp\left\{2\xi^2(T-u)\right\}+4\exp\left\{\frac{9}{2}\xi^2(T-u)\right\}+5\right)\,\mathrm{d}u \\
+\ & \frac{2C\rho^2}{45(2-\alpha)\xi^2}\int_t^T e^{-r(u-t)}\exp\left\{\frac{9}{2}\xi^2(u-t)\right\} \\
& \left(\sigma_t\frac{2^{\frac{3}{2}}\xi^3}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]^{\frac{3}{2}}}+\sigma_t^2\frac{2\xi^2}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]}\right) \\
& \left(-9\exp\left\{2\xi^2(T-u)\right\}+4\exp\left\{\frac{9}{2}\xi^2(T-u)\right\}+5\right)\,\mathrm{d}u \\
+\ & \frac{6C\rho^2}{45(2-\alpha)\xi^2}\int_t^T e^{-r(u-t)}\exp\left\{\frac{9}{2}\xi^2(u-t)\right\} \\
& \sigma_t\frac{2^{\frac{3}{2}}\xi^3}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]^{\frac{3}{2}}}\left(-9\exp\left\{2\xi^2(T-u)\right\}+4\exp\left\{\frac{9}{2}\xi^2(T-u)\right\}+5\right)\,\mathrm{d}u \\
+\ & \frac{6C\rho\xi}{45(2-\alpha)\xi^2}\int_t^T e^{-r(u-t)}\exp\left\{\frac{9}{2}\xi^2(u-t)\right\} \\
& \left(\sigma_t\frac{2\xi^2}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]}+\sigma_t^2\frac{\sqrt{2}\xi}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]^{\frac{1}{2}}}\right)
\end{aligned}$$

$$\left(-9\exp\left\{2\xi^2(T-u)\right\}+4\exp\left\{\frac{9}{2}\xi^2(T-u)\right\}+5\right)\,\mathrm{d}u.$$

The above upper-bound error is difficult to interpret. In order to do this, we derive a Taylor expansion for one of the terms. Then, the following error behaviour is retrieved:

$$C\frac{\rho\xi^3\sigma_t T^{3/2}}{6(\alpha-2)^2}\left(\frac{32\sqrt{2}\rho}{\sqrt{-(\alpha-2)\xi^2}}+21\sqrt{T}\right).$$

Upper-bound for term (II) For matter of convenience, we define the function

$$\chi_2(t,T,\alpha):=\int_t^T\exp\left\{(8-2\alpha)\xi^2(s-t)\right\}\left[\exp\left\{(2-\alpha)\xi^2(T-s)\right\}-1\right]^2\,\mathrm{d}s.$$

We can re-write R_t as

$$\frac{1}{8}D[M,M]_t=\frac{\sigma_t^4}{2(2-\alpha)^2\xi^2}\chi_2(t,T,\alpha).$$

It is easy to find that

$$\begin{aligned}\mathrm{d}\frac{1}{8}D[M,M]_t&=\frac{\mathrm{d}\sigma_t^4}{2(2-\alpha)^2\xi^2}\chi_2(t,T,\alpha)+\frac{\sigma_t^4}{2(2-\alpha)^2\xi^2}\chi_2'(t,T,\alpha)\,\mathrm{d}t\\&=\frac{4\xi\sigma_t^4\,\mathrm{d}W_t+2(8-\alpha)\sigma_t^4\,\mathrm{d}t}{2(2-\alpha)^2\xi^2}\chi_2(t,T,\alpha)+\frac{\sigma_t^4}{2(2-\alpha)^2\xi^2}\chi_2'(t,T,\alpha)\,\mathrm{d}t.\end{aligned}$$

If $\alpha\geq 0$, we can find an upper-bound for $\chi_2(t,T,\alpha)$ which is

$$\begin{aligned}\chi_2(t,T,\alpha)&\leq\int_t^T\exp\left\{8\xi^2(s-t)\right\}\left[\exp\left\{2\xi^2(T-s)\right\}-1\right]^2\,\mathrm{d}s\\&=\frac{1}{24\xi^2}\left(\exp\left\{2\xi^2(T-t)\right\}-1\right)^3\left(\exp\left\{2\xi^2(T-t)\right\}+3\right).\end{aligned}$$

We can re-write the decomposition formula as

$$\begin{aligned}&\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\overline{BS}}(u,X_u,v_u)\,\mathrm{d}\,[M,M]_u\right]-\Gamma^2 C_{\overline{BS}}(t,X_t,v_t)\frac{1}{8}D[M,M]_t\\=\;&\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^6-3\partial_x^5+3\partial_x^4-\partial_x^3\right)\Gamma C_{\overline{BS}}(u,X_u,v_u)\frac{1}{8}D[M,M]_u\,\mathrm{d}\,[M,M]_u\right]\\+\;&\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^5-2\partial_x^4+\partial_x^3\right)\Gamma C_{\overline{BS}}(u,X_u,v_u)\frac{1}{8}D[M,M]_u\sigma_u\,\mathrm{d}\,[W,M]_u\right]\\+\;&\rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^3-\partial_x^2\right)\Gamma C_{\overline{BS}}(u,X_u,v_u)\sigma_u\,\mathrm{d}\left[W,\frac{1}{8}D[M,M]_u\right]_u\right]\end{aligned}$$

$$+ \quad \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_x^4 - 2\partial_x^3 + \partial_x^2\right)\Gamma C_{\overline{BS}}(u, X_u, v_u)\,\mathrm{d}\left[M, \frac{1}{8}D[M,M]_u\right]_u\right].$$

Applying Lemma 3.1.10 (ii) and using the definition of a_u, we obtain

$$\begin{aligned}
&\left|\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\overline{BS}}(u, X_u, v_u)\,\mathrm{d}\left[M,M\right]_u\right] - \Gamma^2 C_{\overline{BS}}(t, X_t, v_t)\frac{1}{8}D[M,M]_t\right| \\
\le\ & \frac{C}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^7} + \frac{3}{a_u^6} + \frac{3}{a_u^5} + \frac{1}{a_u^4}\right)\frac{1}{8}D[M,M]_u\,\mathrm{d}\left[M,M\right]_u\right] \\
+\ & \frac{C\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^6} + \frac{2}{a_u^5} + \frac{1}{a_u^4}\right)\frac{1}{8}D[M,M]_u\sigma_u\,\mathrm{d}\left[W,M\right]_u\right] \\
+\ & C\rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^4} + \frac{1}{a_u^3}\right)\sigma_u\,\mathrm{d}\left[W, \frac{1}{8}D[M,M]_u\right]_u\right] \\
+\ & \frac{C}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{1}{a_u^5} + \frac{2}{a_u^4} + \frac{1}{a_u^3}\right)\,\mathrm{d}\left[M, \frac{1}{8}D[M,M]_u\right]_u\right].
\end{aligned}$$

Noting that $a_u = \sigma_u\phi^{1/2}(u,T,\alpha)$, we have

$$\begin{aligned}
&\left|\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\overline{BS}}(u, X_u, v_u)\,\mathrm{d}\left[M,M\right]_u\right] - \Gamma^2 C_{\overline{BS}}(t, X_t, v_t)\frac{1}{8}D[M,M]_t\right| \\
\le\ & \frac{C}{4(2-\alpha)^2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{\sigma_u}{\phi^{3/2}(u,T,\alpha)} + \frac{3\sigma_u^2}{\phi(u,T,\alpha)} + \frac{3\sigma_u^3}{\phi^{1/2}(u,T,\alpha)} + \sigma_u^4\right)\chi_2(u,T,\alpha)\,\mathrm{d}u\right] \\
+\ & \frac{C\rho}{2(2-\alpha)^2\xi}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{\sigma_u}{\phi^2(u,T,\alpha)} + \frac{2\sigma_u^2}{\phi^{3/2}(u,T,\alpha)} + \frac{\sigma_u^3}{\phi(u,T,\alpha)}\right)\chi_2(u,T,\alpha)\,\mathrm{d}u\right] \\
+\ & \frac{2C\rho}{(2-\alpha)^2\xi}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{\sigma_u}{\phi^2(u,T,\alpha)} + \frac{\sigma_u^2}{\phi^{3/2}(u,T,\alpha)}\right)\chi_2(u,T,\alpha)\,\mathrm{d}u\right] \\
+\ & \frac{2C}{(2-\alpha)^2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\frac{\sigma_u}{\phi^{3/2}(u,T,\alpha)} + \frac{2\sigma_u^2}{\phi(u,T,\alpha)} + \frac{\sigma_u^3}{\phi^{1/2}(u,T,\alpha)}\right)\chi_2(u,T,\alpha)\,\mathrm{d}u\right].
\end{aligned}$$

Being σ_u the only stochastic component, we can get the expectation inside. Each power of σ_u has a different forward value, in this case, we can bound all terms by $\exp\left\{8\xi^2(u-t)\right\}$. We have that

$$\begin{aligned}
&\left|\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\overline{BS}}(u, X_u, v_u)\,\mathrm{d}\left[M,M\right]_u\right] - \Gamma^2 C_{\overline{BS}}(t, X_t, v_t)\frac{1}{8}D[M,M]_t\right| \\
\le\ & \frac{C}{4(2-\alpha)^2\xi^2}\int_t^T e^{-r(u-t)}\exp\left\{8\xi^2(u-t)\right\} \\
& \left(\frac{\sigma_t}{\phi^{3/2}(u,T,\alpha)} + \frac{3\sigma_t^2}{\phi(u,T,\alpha)} + \frac{3\sigma_t^3}{\phi^{1/2}(u,T,\alpha)} + \sigma_u^4\right)\chi_2(u,T,\alpha)\,\mathrm{d}u \\
+\ & \frac{C\rho}{2(2-\alpha)^2\xi^3}\int_t^T e^{-r(u-t)}\exp\left\{8\xi^2(u-t)\right\}
\end{aligned}$$

$$
\begin{aligned}
& \left(\frac{\sigma_t}{\phi^2(u,T,\alpha)} + \frac{2\sigma_t^2}{\phi^{3/2}(u,T,\alpha)} + \frac{\sigma_t^3}{\phi(u,T,\alpha)}\right) \chi_2(u,T,\alpha)\,\mathrm{d}u \\
+ \quad & \frac{2C\rho}{(2-\alpha)^2\xi^3} \int_t^T e^{-r(u-t)} \exp\left\{8\xi^2(u-t)\right\} \\
& \left(\frac{\sigma_t}{\phi^2(u,T,\alpha)} + \frac{\sigma_t^2}{\phi^{3/2}(u,T,\alpha)}\right) \chi_2(u,T,\alpha)\,\mathrm{d}u \\
+ \quad & \frac{2C}{(2-\alpha)^2\xi^2} \int_t^T e^{-r(u-t)} \exp\left\{8\xi^2(u-t)\right\} \\
& \left(\frac{\sigma_t}{\phi^{3/2}(u,T,\alpha)} + \frac{2\sigma_t^2}{\phi(u,T,\alpha)} + \frac{\sigma_t^3}{\phi^{1/2}(u,T,\alpha)}\right) \chi_2(u,T,\alpha)\,\mathrm{d}u.
\end{aligned}
$$

Substituting $\phi(u,T,\alpha)$ and using the upper-bound for $\chi_2(u,T,\alpha)$ when $\alpha \geq 0$, we have

$$
\begin{aligned}
& \left|\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\overline{BS}}(u,X_u,v_u)\,\mathrm{d}\left[M,M\right]_u\right] - \Gamma^2 C_{\overline{BS}}(t,X_t,v_t)\frac{1}{8}D[M,M]_t\right| \\
\leq \quad & \frac{C}{96(2-\alpha)^2\xi^4} \int_t^T e^{-r(u-t)} \exp\left\{8\xi^2(u-t)\right\} \\
& \Bigg(\sigma_t \frac{(2-\alpha)^{\frac{3}{2}}\xi^3}{\left[\exp\left\{(2-\alpha)\xi^2(T-u)\right\}-1\right]^{\frac{3}{2}}} \\
& + 3\sigma_t^2 \frac{(2-\alpha)\xi^2}{\left[\exp\left\{(2-\alpha)\xi^2(T-u)\right\}-1\right]} + 3\sigma_t^3 \frac{(2-\alpha)^{\frac{1}{2}}\xi}{\left[\exp\left\{(2-\alpha)\xi^2(T-u)\right\}-1\right]^{\frac{1}{2}}} + \sigma_u^4\Bigg) \\
& \left(\exp\left\{2\xi^2(T-u)\right\}-1\right)^3 \left(\exp\left\{2\xi^2(T-u)\right\}+3\right)\,\mathrm{d}u \\
+ \quad & \frac{C\rho}{48(2-\alpha)^2\xi^5} \int_t^T e^{-r(u-t)} \exp\left\{8\xi^2(u-t)\right\} \\
& \Bigg(\sigma_t \frac{(2-\alpha)^2\xi^4}{\left[\exp\left\{(2-\alpha)\xi^2(T-u)\right\}-1\right]^2} + 2\sigma_t^2 \left(\frac{(2-\alpha)^{\frac{3}{2}}\xi^3}{\left[\exp\left\{(2-\alpha)\xi^2(T-u)\right\}-1\right]}\right)^{\frac{3}{2}} \\
& + \sigma_t^3 \frac{(2-\alpha)\xi^2}{\left[\exp\left\{(2-\alpha)\xi^2(T-u)\right\}-1\right]}\Bigg) \\
& \left(\exp\left\{2\xi^2(T-u)\right\}-1\right)^3 \left(\exp\left\{2\xi^2(T-u)\right\}+3\right)\,\mathrm{d}u \\
+ \quad & \frac{C\rho}{12(2-\alpha)^2\xi^5} \int_t^T e^{-r(u-t)} \exp\left\{8\xi^2(u-t)\right\} \\
& \left(\sigma_t \left(\frac{(2-\alpha)^2\xi^4}{\left[\exp\left\{(2-\alpha)\xi^2(T-u)\right\}-1\right]^2}\right) + \sigma_t^2 \left(\frac{(2-\alpha)^{\frac{3}{2}}\xi^3}{\left[\exp\left\{(2-\alpha)\xi^2(T-u)\right\}-1\right]^{\frac{3}{2}}}\right)\right) \\
& \left(\exp\left\{2\xi^2(T-u)\right\}-1\right)^3 \left(\exp\left\{2\xi^2(T-u)\right\}+3\right)\,\mathrm{d}u \\
+ \quad & \frac{C}{12(2-\alpha)^2\xi^4} \int_t^T e^{-r(u-t)} \exp\left\{8\xi^2(u-t)\right\}
\end{aligned}
$$

$$\Big(\sigma_t \left(\frac{(2-\alpha)^{\frac{3}{2}}\xi^3}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]^{\frac{3}{2}}}\right) + 2\sigma_t^2 \frac{(2-\alpha)\xi^2}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]}$$
$$+\sigma_t^3 \left(\frac{(2-\alpha)^{\frac{1}{2}}\xi}{[\exp\{(2-\alpha)\xi^2(T-u)\}-1]^{\frac{1}{2}}}\right)\Big)$$
$$\left(\exp\left\{2\xi^2(T-u)\right\}-1\right)^3 \left(\exp\left\{2\xi^2(T-u)\right\}+3\right) du.$$

The above upper-bound error is difficult to interpret. In order to this, we do a Taylor analysis of one term. Then, the following error behaviour is retrieved

$$C\frac{\xi\sigma_t T^2}{15(\alpha-2)^4}\left(5\left(20\rho+\sqrt{2}\right)+32\sqrt{2}\sqrt{T}\sqrt{-(\alpha-2)\xi^2}\right).$$

□

Remark 7.1.15. *It is worth mentioning that the order of the error bound from Theorem 7.1.14 is better than the general estimate from Theorem 7.1.7, where the time dependency is not considered. To get finer estimates also for the exponential fractional model (case $H \neq 1/2$), a proof similar to the Theorem 7.1.14 would have to be performed with more complicated but still tractable calculations.*

Example 7.1.16 (Volatility driven by the standard fractional Brownian motion)**.** *Let us consider a model with volatility process*

$$\sigma_t = \exp\{\xi B_t^H - \frac{1}{2}\alpha\xi^2 r(t)\},$$

where B_t^H is the standard fractional Brownian motion as defined in (2.17), i.e. with the Molchan-Golosov kernel (2.18) or (2.19). Then, the formulae for $\frac{\rho}{2}L[W,M]_0$ and $\frac{1}{8}D[M,M]_0$ are given in Proposition 7.1.6 with the particular kernel (2.18) or (2.19), autocovariance function (2.15) and $\widehat{r}(t,s|u)$ as in Theorem 7.1.1. In this case, we do not give the formulae for U_0 and R_0 after substituting the Molchan-Golosov kernel, due to these formulae being too long. However, it is worth mentioning that the formulae are explicit and numerical evaluation requires only the computation of some multiple Gaussian integrals.

Remark 7.1.17. *The Molchan-Golosov kernel can be written as*

$$K_H(t,s) \quad = \quad C_H(t-s)^{H-\frac{1}{2}}F\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{s}\right) \tag{7.26}$$

where $F(a,b,c,z)$ is the Gauss hypergeometric function. Then, it is easy to see that

$$K_H(t,s) \leq C\,|t-s|^{H-\frac{1}{2}} \tag{7.27}$$

and $K_H(t,s)$ is square-integrable function.

7.2 Numerical comparison of approximation formula

In this section, we focus on numerical aspects of the introduced approximation formula. We provide details on its numerical implementation and a comparison with the Monte Carlo (MC) simulation framework introduced by Bennedsen et al. (2017) will be made.

In the second part of this section, we also introduce two interesting outcome analysis for rBergomi model. In particular, we show how the model can be efficiently calibrated using the approximation formula to short maturity smiles. We remark that classical SV models (e.g. Heston model) might fail to fit the short-term smiles, unless they exploit high volatility of volatility levels for which they would be typically inconsistent with the long term skew of the volatility surface.

In what follows, we will inspect the approximation quality for rBergomi model and time to maturity / volatility of volatility ξ scenarios. Based on the nature of error terms (see Theorem 7.1.14) those two factors should play prominent roles when it comes to approximation quality.

7.2.1 On implementation of the approximation formula

We note that for the models studied in this chapter, we have obtained either a semi-closed form or analytical formula for standard Wiener case (H=0.5). Moreover, for the class of exponential fractional models, represented by the αRFSV model, we only need to numerically evaluate multiple integrals in $\frac{1}{8}D[M,M]_0$ and $\frac{\varrho}{2}L[W,M]_0$.

In our case, this was done using a trapezoid quadrature routine, not necessarily the most efficient approach, but easy to implement. We used a discretisation[1] of integrands such that the numerical error does not affect the results in a significant way. I.e. to be lower than standard MC errors when compared to simulated prices or lower than the expected approximation error.

For benchmarking, we use a first-order hybrid MC scheme introduced by Bennedsen et al. (2017) alongside 50,000 MC sample paths. Similar to the implementation of the approximation formula, we remark that this scheme could also be improved as described in McCrickerd and Pakkanen (2018).

7.2.2 Sensitivity analysis for rBergomi $\alpha = 1$ approximation w.r.t. increasing ξ and time to maturity τ

In this section, we illustrate the approximation quality for European call options under various model regimes / data set properties as described in Table 7.1. We use option maturities up to 1Y, we are expecting a loss of approximation quality, based on the nature of the approximation formula. As we utilised a first-order approximation arguments with respect to volatility of volatility, we are also expecting more pronounced differences between MC simulations and the formula for large values of ξ.

[1] Typically we used from 1,000 up to 27,000 points for 3D integrals.

Table 7.1: Model / data settings for sensitivity analyses.

Model params	Values
ξ	$\{10\%, 50\%, 100\%\}$ and {from 10% to 50% with 10% step}
σ_0	8%
ρ	-20%
H	0.1
Data set specifics	**Values**
Moneyness S/K	70% – 130% with 5% step
Time to maturity τ	{1M, 3M, 6M, 1Y}
Underlying spot price S_0	100

In Figure 7.1, we illustrate the approximation quality of the rBergomi approximation for low ξ values. We can observe an expected behaviour: a very good match up to 3M expiry and almost linear deterioration of the quality with increasing τ. The approximation formula also provides a similar scale of errors across the tested moneyness.

For different moneyness regimes and 1M time to maturity, we obtained the following discrepancies between the MC trials and the introduced formula, measured in the relative option fair value (FV)[2]:

	Differences in relative FVs		
Spot moneyness	$\xi = 10\%$	$\xi = 50\%$	$\xi = 100\%$
80%	4.5e-04	-8.1e-05	-0.29928
90%	3.9e-04	2.6e-05	-0.02797
100%	2.3e-04	7.2e-04	0.95436
110%	-1.5e-05	-7.7e-05	0.09690
120%	-1.2e-05	-2.7e-04	-0.46417

In the table above, we can see reasonable approximation error measures which fell below standard 1 MC error for $\xi = 10\%$ and $\xi = 50\%$ regimes. Due to the theoretical properties of the approximation formula, we observe a significant deterioration for high volatility of volatility regimes. This also depends on the time to maturity of the approximated option, the shorter maturity we have, the higher ξ is, we can obtain reasonable approximation errors (i.e. of the order 1e-04 and lower in terms of FV).

In order to have a more detailed comparison between the proposed approximation and MC simulations, we have also evaluated pricing differences in terms of implied volatilities within a range from 10% to 50 % for parameter ξ. The differences are provided in Table 7.2.

We note that for in the money options (80% and 90% spot moneyness), the results were significantly affected by Monte Carlo errors even at 150k simulations. However, for

[2]Relative FV is the absolute option fair value divided by the initial spot price.

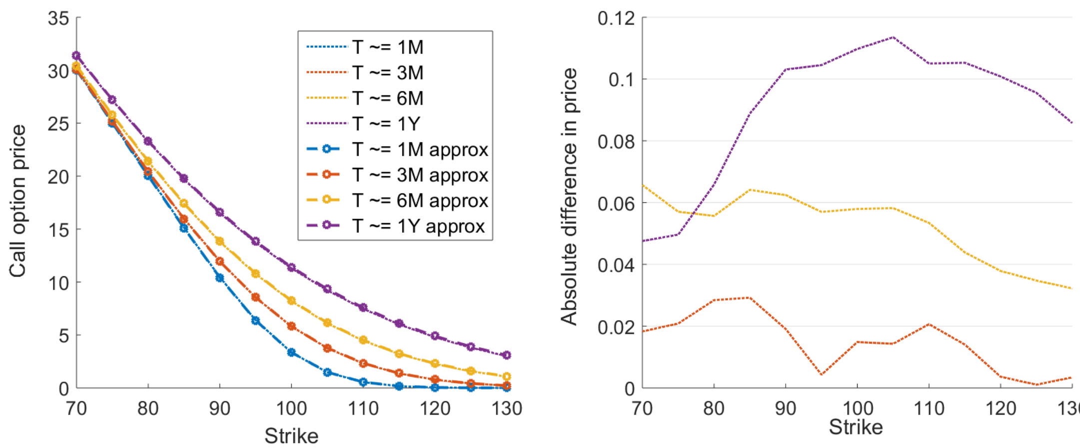


Figure 7.1: rBegomi Model: Comparison of call option fair values calculated by MC simulations and by the approximation formula. (Example 7.1.11 with $\alpha = 1$ and $\varepsilon = 0$. Data and parameter values are: $v_0 = 8\%,\ \xi = 10\%, \rho = -20\%, H = 0.1$)

Table 7.2: Differences in terms of implied volatility

Spot moneyness	$\xi = 10\%$	$\xi = 20\%$	$\xi = 30\%$	$\xi = 40\%$	$\xi = 50\%$
80%	0.0381	0.0143	0.0289	0.0210	0.0062
90%	0.0037	0.0026	0.0043	0.0038	0.0019
100%	0.0008	4e-05	0.0005	0.0014	0.0057
110%	-0.0003	-0.0001	-0.0014	-0.0020	-0.0027
120%	-0.0005	-0.0008	-0.0019	-0.0060	-0.0187

other options we have measured a reasonably good match and only a slight deterioration in approximation quality for increasing ξ.

Although the introduced approximation is typically not suitable for calibrations to the whole volatility surface due to the deterioration of approximation quality when increasing time to maturity, we will illustrate how it can significantly speed up MC calibration to the provided forward At-The-Money (ATMF) backbone.

7.2.3 Short-tenor calibration and a hybrid calibration to ATMF backbone

Unlike previous analyses which were based on artificial data / model parameter values, we inspect an application for the formula on the calibration to real option market data. In particular, we utilise four data sets of AAPL options which were analysed in detail by Pospíšil et al. (2018). Descriptive statistics of the data sets are provided in Table 7.3. The following calibration test trials will be considered.

- Calibration to short maturity smiles:

  This should illustrate how well the model can fit short maturity smiles using the introduced approximation formula without exploiting too high volatility of volatility values (ξ). For each data set, we selected the shortest maturity slice with more than one traded option. The values were not interpolated by any model, i.e. we calibrated to discrete close mid-prices of traded options. We also confirm that both MC simulation and the formula reprice the smile with the final calibrated parameters without significant differences.

- Hybrid calibration to the ATMF backbone:

  In the second trial, we calibrate to the ATMF backbone for each data set. We note that because we have only a discrete set of traded options we might not have for each maturity an option with strike equal to the corresponding forward. Hence, we take an option with the closest strike to the forward value for each expiry. We use the proposed approximation formula only for $\tau < 0.2$, for longer time to maturities we price by MC simulations.

In both cases, the calibration routine was formulated as a standard least-square optimization problem. I.e. to obtain calibrated parameters, we numerically evaluated

$$\widehat{\Theta} = \arg\min f(\Theta) = \arg\min \sum_{i=1}^{N} \left[\text{Mid}_i - \text{rBergomi}_i(\Theta)\right]^2, \tag{7.28}$$

where N is the total number of contracts for the calibration, Mid_i is the mid-price of the i^{th} option and $\text{rBergomi}_i(\Theta)$ represents the corresponding model price based on parameter set Θ. The model price is either obtained by the approximation formula or by means of MC simulations otherwise. The optimization is performed using Matlab's local search trust region optimizer which also needs an initial guess to start with.

	Data # 1	Data # 2	Data # 3	Data # 4
Date (all EOD)	1-Apr-2015	15-Apr-2015	1-May-2015	15-May-2015
Moneyness range	34%–157%	45%–154%	31%–151%	33%–151%
Time to maturity [Yrs]	0.12-1.81	0.08–1.77	0.04–1.73	0.02–1.69
Total nb. of contracts	113	158	201	194

Table 7.3: rBergomi model: Data on AAPL options used in calibration trials

All following results will be quoted in relative FV: e.g. $\text{rBergomi}_i(\Theta)/S_0$ and also differences between market and the calibrated model will be denoted using this measure. For the calibration to the whole surface of European options, errors in FV below 0.5% are typically considered to be acceptable, whereas anything exceeding 1% difference is considered as a significant model inconsistency.

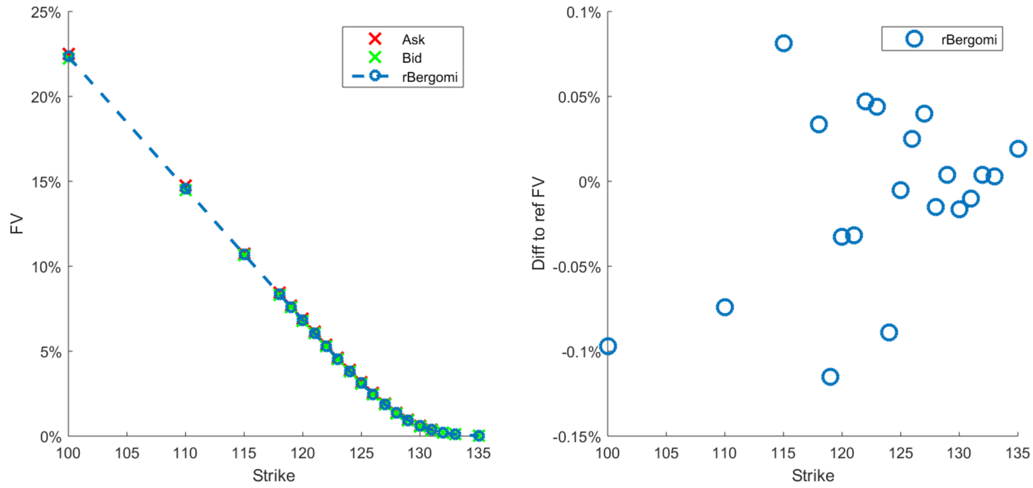
Firstly, we display the results for the short-maturity calibration. In Figure 7.2, we illustrate that even with a not well-suited initial guess for Data # 4, we can obtain satisfactory results (i.e. errors significantly below 0.5% mark). In fact, for all tested data sets, obtained values of the calibrated parameters were not very sensitive to the initial guess (only a number of iterations differed). This is a desired feature which typically is not present under classical SV models, see e.g. Mrázek et al. (2016). For two other sets we have obtained qualitatively similar fitting errors. However for data set # 3, we retrieved 4 errors (out of 22) with absolute FV difference greater than 0.5% and two even greater than 1%, see Figure 7.3. We conclude that this was partly caused due to high ξ values compared to the other three calibration trials and also slightly longer maturity, the data set #3 includes only one option at the shortest maturity, hence the second shortest was used. Still we can conclude that the obtained errors (also verified by using MC simulations) are overall acceptable, although they might not be optimal.

We have observed that at least short maturity smiles ($<$ 1M) can be efficiently calibrated using the proposed approximation formula, while for expiries greater than 1M, one would need to stay within a low volatility of volatility regime, otherwise the discrepancies would lead to a non-optimal solution when recomputed using more precise (but much more costly) MC simulations. To illustrate the efficiency of the approximation, we will show how the approximation can speed up ATMF-backbone calibration.

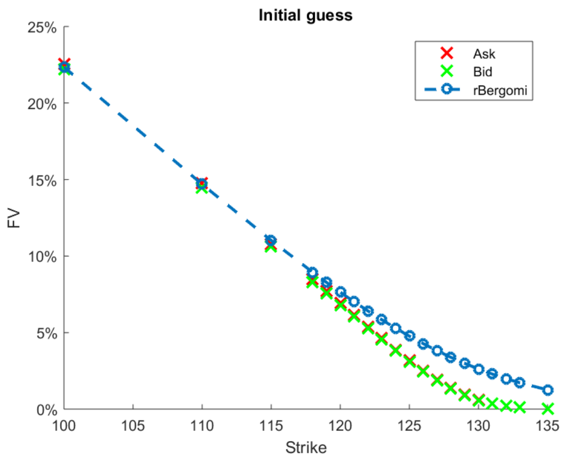
We will now inspect the hybrid calibration where we switch between the approximation and a MC pricer based on properties of options being priced. In particular, we focus on data set from 15^{th} May (Data # 4) and for $\tau < 0.2$ we will use the approximation formula and MC simulations otherwise. For the calibrated parameters, we will also measure the time spent computing FVs by each pricer. For completeness, we remark that MC simulations under the rBergomi model can be performed in a more efficient way using a scheme introduced by McCrickerd and Pakkanen (2018) and similarly numerical integrations within the approximation could be performed by an adaptive quadrature and could be vectorized to improve the computation efficiency.

In Figure 7.4, we illustrate calibration fit to the ATMF backbone of the option price surface. We conclude that we have retrieved similar errors for both the prices computed using the proposed approximation and the longer maturity option prices quantified by MC simulations. The final fit of the calibrated model (recomputed by MC simulations) is very good, especially considering that the studied model has only 4 parameters. Moreover, only a fraction of the time spent by MC pricer was needed to compute all FV using the approximation formula. In particular, 98.43% of the pricing time[3] was spent computing MC simulation estimates of FVs. We also note that 7/9 of total evaluations were computed by the approximation formula.

[3]Excluding any data loading / manipulation routines.



(a) rBergomi model: Comparison of calibrated model and market data (Data # 4)



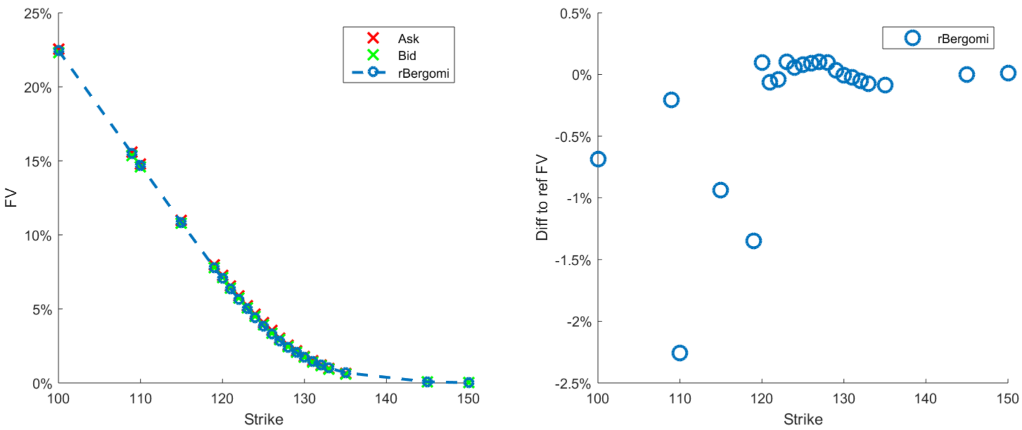
(b) rBergomi model: Errors with respect to the initial guess

Calibrated params:

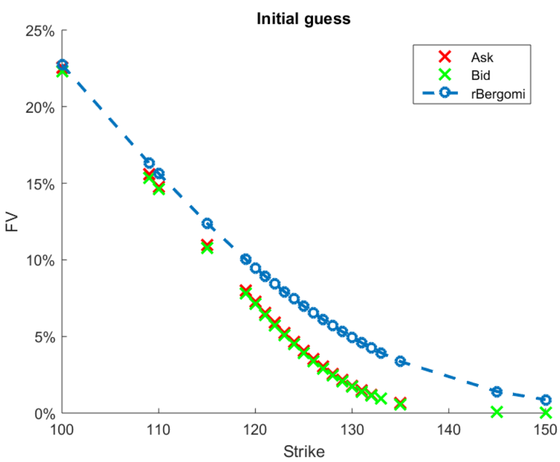
$$\sigma_0 = 3.41\%$$
$$\xi = 39.45\%$$
$$\rho = -98.80\%$$
$$H = 0.3153$$

MSE = 0.0883

Figure 7.2: rBergomi model: Calibration results for short maturity smiles (Data # 4)



(a) rBergomi model: Comparison of calibrated rBergomi and market data (Data # 3)



(b) rBergomi model: Errors with respect to the initial guess

Calibrated params:

$\sigma_0 = 4.57\%$
$\xi = 81.36\%$
$\rho = -98.99\%$
$H = 0.3936$
MSE = 13.9538

Figure 7.3: rBergomi model: Calibration results for short maturity smiles (Data # 3)

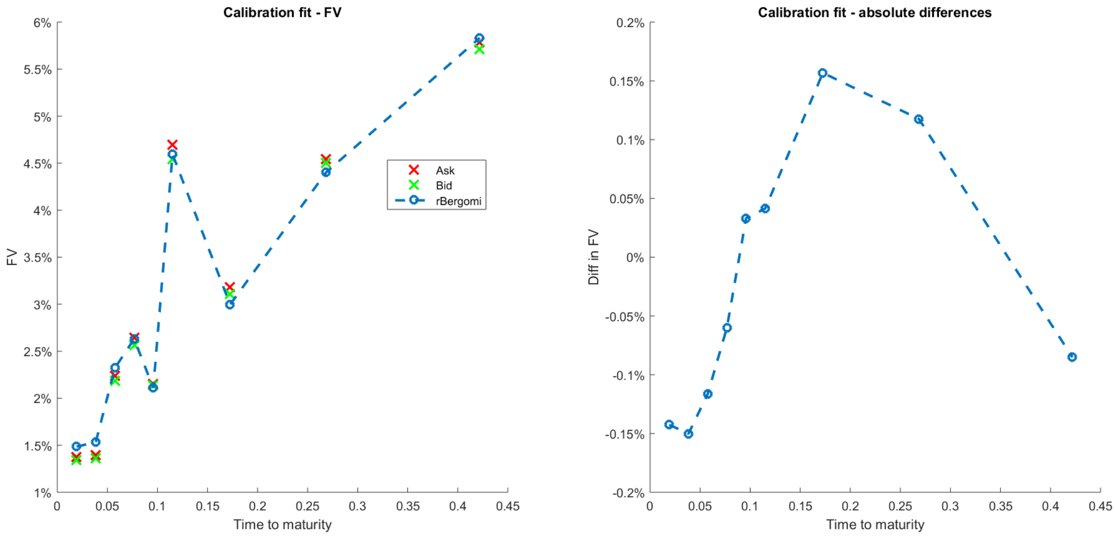


Figure 7.4: rBergomi model: ATMF calibration results when combining the approximation formula ($\tau < 0.2$) and MC simulations.

CHAPTER 8

Decomposition formula for Spot-Dependent Volatility models.

In the previous chapters, we developed a general decomposition formula for stochastic volatility models. Afterwards, we explored different applications. In this chapter, we will use the same type of ideas to develop an approximation of the prices of call options under a spot-dependent volatility model.

The model used assumes that volatility is a deterministic function of the underlying stock price, and therefore, there is only one source of randomness in the model. These models are sometimes called local volatility models in industry and GARCH-type volatility models in financial econometrics. The content of this chapter is based on Merino and Vives (2017)

8.1 Decomposition Formula

The first objective is to find a generic decomposition formula for this type of model. Then, as we did in the case of stochastic volatility models, we will obtain a decomposition formula based on the Black-Scholes-Merton formula.

Remember that in order to generalise the decomposition formula, conditions on Remark 4.1.1 hold.

Theorem 8.1.1 (Decomposition formula for spot-dependent volatility models)**.** *Let S_t be a price process defined in (3.11), let $B(t)$ a function in $\mathcal{C}^2([0,T])$, let $A(t,x,y)$ be a continuous function on the space $[0,T]\times[0,\infty)\times[0,\infty)$ such that $A\in\mathcal{C}^{1,2,2}((0,T)\times(0,\infty)\times(0,\infty))$. Let us also assume that $\mathcal{L}_y A_{\widehat{SD}}(t,x,r,y)=0$. Then, for every $t\in[0,T]$, the following formula holds:*

$$
\begin{aligned}
& \mathbb{E}_t\left[e^{-r(T-t)}A(T,S_T,\upsilon^2(S_T))B(T)\right] \\
= \;& A(t,S_t,\upsilon^2(S_t))B(t) \\
+ \;& \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}A(u,S_u,\upsilon^2(S_u))B'(u)\,\mathrm{d}u\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad r\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{\upsilon^2}A(u,S_u,\upsilon^2(S_u))B(u)(\partial_S\upsilon^2(S_u))S_u\,\mathrm{d}u\right]\\
&+ \quad \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{\upsilon^2}A(u,S_u,\upsilon^2(S_u))B(u)(\partial_S^2\upsilon^2(S_u))\upsilon^2(S_u)S_u^2\,\mathrm{d}u\right]\\
&+ \quad \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{\upsilon^2}^2A(u,S_u,\upsilon^2(S_u))B(u)\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^2(S_u)S_u^2\,\mathrm{d}u\right]\\
&+ \quad \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{S,\upsilon^2}^2A(u,S_u,\upsilon^2(S_u))B(u)(\partial_S\upsilon^2(S_u))\upsilon^2(S_u)S_u^2\,\mathrm{d}u\right].
\end{aligned}
$$

Proof. Applying the Itô formula to process $e^{-rt}A(t,S_t,\upsilon^2(S_t))B(t)$, we obtain:

$$
\begin{aligned}
& e^{-rT}A(T,S_T,\upsilon^2(S_T))B(T)\\
= \quad & e^{-rt}A(t,S_t,\upsilon^2(S_t))B(t)\\
- \quad & r\int_t^T e^{-ru}A(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}u\\
+ \quad & \int_t^T e^{-ru}\partial_uA(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}u\\
+ \quad & \int_t^T e^{-ru}\partial_SA(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}S_u\\
+ \quad & \int_t^T e^{-ru}\partial_{\upsilon^2}A(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}\upsilon^2(S_u)\\
+ \quad & \int_t^T e^{-ru}A(u,S_u,\upsilon^2(S_u))B'(u)\,\mathrm{d}u\\
+ \quad & \frac{1}{2}\int_t^T e^{-ru}\partial_S^2A(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}[S,S]_u\\
+ \quad & \frac{1}{2}\int_t^T e^{-ru}\partial_{\upsilon^2}^2A(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}[\upsilon^2(S),\upsilon^2(S)]_u\\
+ \quad & \int_t^T e^{-ru}\partial_{S,\upsilon^2}^2A(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}[S,\upsilon^2(S)]_u.
\end{aligned}
$$

Substituting the expression of $\mathrm{d}S_u$,

$$
\begin{aligned}
& e^{-rT}A(T,S_T,\upsilon^2(S_T))B(T)\\
= \quad & e^{-rt}A(t,S_t,\upsilon^2(S_t))B(t)\\
- \quad & r\int_t^T e^{-ru}A(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}u\\
+ \quad & \int_t^T e^{-ru}\partial_uA(u,S_u,\upsilon^2(S_u))B(u)\,\mathrm{d}u\\
+ \quad & r\int_t^T e^{-ru}\partial_SA(u,S_u,\upsilon^2(S_u))B(u)S_u\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_t^T e^{-ru}\partial_S A(u,S_u,v^2(S_u))B(u)v(S_u)S_u\,\mathrm{d}W_u \\
&+ \int_t^T e^{-ru}\partial_{v^2} A(u,S_u,v^2(S_u))B(u)\,\mathrm{d}v^2(S_u) \\
&+ \int_t^T e^{-ru} A(u,S_u,v^2(S_u))B'(u)\,\mathrm{d}u \\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_S^2 A(u,S_u,v^2(S_u))B(u)v^2(S_u)S_u^2\,\mathrm{d}u \\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_{v^2}^2 A(u,S_u,v^2(S_u))B(u)\,\mathrm{d}[v^2(S),v^2(S)]_u \\
&+ \int_t^T e^{-ru}\partial_{S,v^2}^2 A(u,S_u,v^2(S_u))B(u)\,\mathrm{d}[S,v^2(S)]_u.
\end{aligned}
$$

Grouping the blue terms, we have the Feynman-Kac formula (3.13), multiplying by e^{rt} and taking conditional expectations, we obtain

$$
\begin{aligned}
& \mathbb{E}_t\left[e^{-r(T-t)}A(T,S_T,v^2(S_T))B(T)\right] \\
=\ & A(t,S_t,v^2(S_t))B(t) \\
+\ & \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}A(u,S_u,v^2(S_u))B'(u)\,\mathrm{d}u\right] \\
+\ & \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{v^2}A(u,S_u,\theta^2(S_u))B(u)dv^2(S_u)\right] \\
+\ & \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{v^2}^2A(u,S_u,v^2(S_u))B(u)\,\mathrm{d}[v^2(S),v^2(S)]_u\right] \\
+\ & \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{S,v^2}^2A(u,S_u,v^2(S_u))B(u)v(S_u)S_u\,\mathrm{d}[W,v^2(S)]_u\right]. \qquad (8.1)
\end{aligned}
$$

On other hand, using Itô calculus rules, it is easy to see that

$$
\mathrm{d}v^2(S_t) \;=\; \partial_S v^2(S_t)rS_t\,\mathrm{d}t + \partial_S v^2(S_t)v(S_t)S_t\,\mathrm{d}W_t + \frac{1}{2}\partial_S^2 v^2(S_t)v^2(S_t)S_t^2\,\mathrm{d}t.
$$

Finally, substituting this expression in (8.1) we finish the proof. □

Now, we can use the decomposition formula to find the exact value of a European call option price under a spot-dependent volatility model. Note that as we have seen before, the price is a sum of terms, being the main term the Black-Scholes-Merton formula.

Corollary 8.1.2 (Price decomposition under a spot-dependent volatility models)**.** *Let S_t be a price process defined in (3.11). Then, for all $t \in [0,T]$, the following formula holds:*

$$
V_t \;=\; C_{BS}(t,S_t,v(S_t))
$$

$$\begin{aligned}
&+ \frac{r}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\upsilon(S_u))(T-u)(\partial_S\upsilon^2(S_u))S_u\,\mathrm{d}u\right]\\
&+ \frac{1}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\upsilon(S_u))(T-u)(\partial_S^2\upsilon^2(S_u))\upsilon^2(S_u)S_u^2\,\mathrm{d}u\right]\\
&+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))(T-u)^2\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^2(S_u)S_u^2\,\mathrm{d}u\right]\\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,\upsilon(S_u))(T-u)(\partial_S\upsilon^2(S_u))\upsilon^2(S_u)S_u\,\mathrm{d}u\right].
\end{aligned}$$

Proof. Substituting $A(t,S_t,\upsilon^2(S_t)) := C_{\widehat{CEV}}(t,S_t,\upsilon(S_t))$ and $B(t)\equiv 1$ in Theorem 8.1.1. Then, apply the following equalities:

$$\partial_{\upsilon^2}C_{\widehat{BS}}(t,S_t,\upsilon(S_t)) = \frac{T-t}{2}S_t^2\partial_S^2 C_{\widehat{BS}}(t,S_t,\upsilon(S_t))$$

and

$$\partial_{\upsilon^2}^2 C_{\widehat{BS}}(t,S_t,\upsilon(S_t)) = \frac{(T-t)^2}{4}S_t^2\partial_S^2\left(S_t^2\partial_S^2 C_{\widehat{BS}}(t,S_t,\upsilon(S_t))\right).$$

Note that $C_{\widehat{BS}}(t,x,y^2) = C_{BS}(t,x,y)$ and this equality holds deriving with respect to x. □

Remark 8.1.3. *For simplicity, we will refer to each of the terms of the price decomposition above as*

$$V_t = C_{BS}(t,S_t,\upsilon(S_t)) + (\mathcal{I}) + (\mathcal{II}) + (\mathcal{III}) + (\mathcal{IV}).$$

Looking carefully at this decomposition, one may notice that it is an alternative decomposition to the example 4.1.7. In the example, the base function was $C_{BS}(t,S_t,v_t)$, whereas now it is $C_{BS}(t,S_t,\upsilon(S_t))$. The second formula performs better than the first one because it captures the non-linearity of the price formula. Note that the price of being more accurate is using, in the deterministic volatility case, two terms more.

8.2 Approximation formula

As we have noted in previous chapters, the decomposition formula for European call options is not numerically manageable. For this reason, we have to find an approximation. The main idea is to use again Theorem 8.1.1 to approximate the price of a call option and to obtain an estimation of the errors.

Corollary 8.2.1 (Approximation price decomposition under a spot-dependent volatility models)**.** *Let S_t be a price process defined in (3.11). Then, for all $t\in[0,T]$, we can express the call option price V_t in the following way:*

$$V_t = C_{BS}(t,S_t,\upsilon(S_t))$$

$$
\begin{aligned}
&+ \quad \frac{1}{4}r(\partial_S v^2(S_t))S_t\Gamma C_{\widehat{BS}}(t,S_t,v(S_t))(T-t)^2 \\
&+ \quad \frac{1}{8}(\partial_S^2 v^2(S_t))v^2(S_t)S_t^2\Gamma C_{\widehat{BS}}(t,S_t,v(S_t))(T-t)^2 \\
&+ \quad \frac{1}{24}\left(\partial_S v^2(S_t)\right)^2 v^2(S_t)S_t^2\Gamma^2 C_{\widehat{BS}}(t,S_t,v(S_t))(T-t)^3 \\
&+ \quad \frac{1}{4}(\partial_S v^2(S_t))v^2(S_t)S_t\Lambda\Gamma C_{\widehat{BS}}(t,S_t,v(S_t))(T-t)^2 \\
&+ \quad \epsilon_t
\end{aligned}
$$

where ϵ_t is an error.

Proof. The main idea is to use iteratively the Theorem 8.1.1 to be able to find an approximation formula. The new terms arising will be considered the error of the approximation.

The term $(\mathcal{I})$ is decomposed using Theorem 8.1.1 with

$$A(t,S_t,v^2(S_t)) = (\partial_S v^2(S_t))S_t\Gamma C_{SD}(t,S_t,v(S_t)) \text{ and } B_t = \frac{r}{2}\int_t^T (T-u)\,\mathrm{d}u.$$

This gives

$$
\begin{aligned}
& \quad \frac{r}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u(\partial_S v^2(S_u))S_u\,\mathrm{d}u\right] \\
&- \quad \frac{r}{4}(\partial_S v^2(S_t))S_t\Gamma C_{\widehat{BS}}(t,S_t,v(S_t))(T-t)^2 \\
&= \quad \frac{r^2}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^3(\partial_S v^2(S_u))^2 S_u^2\,\mathrm{d}u\right] \\
&+ \quad \frac{r}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}(\partial_S v^2(S_u))\Gamma^2 C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^3(\partial_S^2 v^2(S_u))v^2(S_u)S_u^3\,\mathrm{d}u\right] \\
&+ \quad \frac{r}{32}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^3 C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^4\left(\partial_S v^2(S_u)\right)^3 v^2(S_u)S_u^3\,\mathrm{d}u\right] \\
&+ \quad \frac{r}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left((\partial_S v^2(S_u))S_u\Gamma^2 C_{\widehat{BS}}(u,S_u,v(S_u))\right)\tau_u^3(\partial_S v^2(S_u))v^2(S_u)S_u\,\mathrm{d}u\right].
\end{aligned}
$$

The term $(\mathcal{II})$ is decomposed using Theorem 8.1.1 with

$$A(t,S_t,v^2(S_t)) = (\partial_S^2 v^2(S_t))v^2(S_t)S_t^2\Gamma C_{\widehat{BS}}(t,S_t,v(S_t)) \text{ and } B_t = \frac{1}{4}\int_t^T (T-u)du.$$

We obtain

$$
\begin{aligned}
& \quad \frac{1}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u(\partial_S^2 v^2(S_u))v^2(S_u)S_u^2\,\mathrm{d}u\right] \\
&- \quad \frac{1}{8}(\partial_S^2 v^2(S_t))v^2(S_t)S_t^2\Gamma C_{\widehat{BS}}(t,S_t,v(S_t))(T-t)^2
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{r}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_S^2\upsilon^2(S_u)\Gamma C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^2(\partial_S\upsilon^2(S_u))S_u^3\,\mathrm{d}u\right]\\
+ \; & \frac{r}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}(\partial_S^2\upsilon^2(S_u))\upsilon^2(S_u)S_u^3\Gamma^2 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^3(\partial_S\upsilon^2(S_u))\,\mathrm{d}u\right]\\
+ \; & \frac{1}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^2(\partial_S^2\upsilon^2(S_u))^2\upsilon^2(S_u)S_u^4\,\mathrm{d}u\right]\\
+ \; & \frac{1}{32}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^3(\partial_S^2\upsilon^2(S_u))^2\upsilon^4(S_u)S_u^4\,\mathrm{d}u\right]\\
+ \; & \frac{1}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^3(\partial_S^2\upsilon^2(S_u))\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^2(S_u)S_u^4\,\mathrm{d}u\right]\\
+ \; & \frac{1}{64}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}(\partial_S^2\upsilon^2(S_u))\Gamma^3 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^4\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^4(S_u)S_u^4\,\mathrm{d}u\right]\\
+ \; & \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\partial_S^2\upsilon^2(S_u)S_u^2\Gamma C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\right)\tau_u^2(\partial_S\upsilon^2(S_u))\upsilon^2(S_u)S_u\,\mathrm{d}u\right]\\
+ \; & \frac{1}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left((\partial_S^2\upsilon^2(S_u))\upsilon^2(S_u)S_u^2\Gamma^2 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\right)\tau_u^3(\partial_S\upsilon^2(S_u))\upsilon^2(S_u)S_u\,\mathrm{d}u\right].
\end{aligned}
$$

The term $(\mathcal{III})$ is decomposed using Theorem 8.1.1 with

$$
A(t,S_t,\upsilon^2(S_t)) = \left(\partial_S\upsilon^2(S_t)\right)^2\upsilon^2(S_t)S_t^2\Gamma^2 C_{\widehat{BS}}(t,S_t,\upsilon(S_t)) \text{ and } B_t = \frac{1}{8}\int_t^T (T-u)^2\,\mathrm{d}u.
$$

We have

$$
\begin{aligned}
& \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^2\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^2(S_u)S_u^2\,\mathrm{d}u\right]\\
- \; & \frac{1}{24}\left(\partial_S\upsilon^2(S_t)\right)^2\upsilon^2(S_t)S_t^2\Gamma^2 C_{\widehat{BS}}(t,S_t,\upsilon(S_t))(T-t)^3\\
= \; & \frac{r}{24}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_S\upsilon^2(S_u)\right)^2\Gamma^2 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^3(\partial_S\upsilon^2(S_u))S_u^3\,\mathrm{d}u\right]\\
+ \; & \frac{r}{48}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^2(S_u)\Gamma^3 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^4(\partial_S\upsilon^2(S_u))S_u^3\,\mathrm{d}u\right]\\
+ \; & \frac{1}{48}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_S\upsilon^2(S_u)\right)^2\Gamma^2 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^3(\partial_S^2\upsilon^2(S_u))\upsilon^2(S_u)S_u^4\,\mathrm{d}u\right]\\
+ \; & \frac{1}{96}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^2(S_u)\Gamma^3 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^4(\partial_S^2\upsilon^2(S_u))\upsilon^2(S_u)S_u^4\,\mathrm{d}u\right]\\
+ \; & \frac{1}{48}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^3 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^4\left(\partial_S\upsilon^2(S_u)\right)^4\upsilon^2(S_u)S_u^4\,\mathrm{d}u\right]\\
+ \; & \frac{1}{192}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^2(S_u)\Gamma^4 C_{\widehat{BS}}(u,S_u,\upsilon(S_u))\tau_u^5\left(\partial_S\upsilon^2(S_u)\right)^2\upsilon^2(S_u)S_u^4\,\mathrm{d}u\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{24}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\left(\partial_S v^2(S_u)\right)^2\Gamma^2 C_{\widehat{BS}}(u,S_u,v(S_u))\right)\tau_u^3(\partial_S v^2(S_u))v^2(S_u)S_u^3\,\mathrm{d}u\right]\\
&+ \frac{1}{48}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\left(\partial_S v^2(S_u)\right)^2 v^2(S_u)\Gamma^3 C_{\widehat{BS}}(u,S_u,v(S_u))\right)\tau_u^4(\partial_S v^2(S_u))v^2(S_u)S_u^3\,\mathrm{d}u\right].
\end{aligned}
$$

The term $(\mathcal{IV})$ is decomposed using Theorem 8.1.1 with

$$
A(t,S_t,v^2(S_t)) = (\partial_S v^2(S_t))v^2(S_t)S_t\Lambda\Gamma C_{\widehat{BS}}(t,S_t,v(S_t)) \text{ and } B_t = \frac{1}{2}\int_t^T (T-u)\,\mathrm{d}u.
$$

This gives

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u(\partial_S v^2(S_u))v^2(S_u)S_u\,\mathrm{d}u\right]\\
&- \frac{1}{4}(\partial_S v^2(S_t))v^2(S_t)S_t\Lambda\Gamma C_{\widehat{BS}}(t,S_t,v(S_t))(T-t)^2\\
&= \frac{r}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_S v^2(S_u)\right)\Lambda\Gamma C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^2(\partial_S v^2(S_u))S_u^2\,\mathrm{d}u\right]\\
&+ \frac{r}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}(\partial_S v^2(S_u))v^2(S_u)\Lambda\Gamma^2 C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^3(\partial_S v^2(S_u))S_u^2\,\mathrm{d}u\right]\\
&+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(\partial_S v^2(S_u)\right)\Lambda\Gamma C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^2(\partial_S^2 v^2(S_u))v^2(S_u)S_u^3\,\mathrm{d}u\right]\\
&+ \frac{1}{16}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}(\partial_S v^2(S_u))v^2(S_u)\Lambda\Gamma^2 C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^3(\partial_S^2 v^2(S_u))v^2(S_u)S_u^3\,\mathrm{d}u\right]\\
&+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma^2 C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^3\left(\partial_S v^2(S_u)\right)^3 v^2(S_u)S_u^3\,\mathrm{d}u\right]\\
&+ \frac{1}{32}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}(\partial_S v^2(S_u))v^2(S_u)\Lambda\Gamma^3 C_{\widehat{BS}}(u,S_u,v(S_u))\tau_u^4\left(\partial_S v^2(S_u)\right)^2 v^2(S_u)S_u^3\,\mathrm{d}u\right]\\
&+ \frac{1}{4}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\left(\partial_S v^2(S_u)\right)S_u\Lambda\Gamma C_{\widehat{BS}}(u,S_u,v(S_u))\right)\tau_u^2(\partial_S v^2(S_u))v^2(S_u)S_u\,\mathrm{d}u\right]\\
&+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left((\partial_S v^2(S_u))v^2(S_u)S_u\Lambda\Gamma^2 C_{\widehat{BS}}(u,S_u,v(S_u))\right)\tau_u^3(\partial_S v^2(S_u))v^2(S_u)S_u\,\mathrm{d}u\right].
\end{aligned}
$$

□

CHAPTER 9

CEV model

In this chapter, following the development of a decomposition formula for spot-dependent volatility models, we will find the decomposition of the price as well as an approximation, of a European call option under the CEV model. As an application of the approximation price formula, we will obtain two approximations of the implied volatility surface: an approximation of the at-the-money implied volatility curve as a function of time, and an approximation of the implied volatility smile as a function of log-moneyness near the expiration date. We will see that we can use these two volatility approximations to retrieve the model parameters from the CEV model. The content of this chapter is based on Merino and Vives (2017).

9.1 Approximation of the CEV model.

Applying Corollary 8.2.1 to CEV model, we obtain an exact decomposition formula.

Corollary 9.1.1 (CEV Exact Formula)**.** *Let S_t be a price process defined in (3.14). Then, for all $t \in [0,T]$, we can express the call option fair value V_t by*

$$\begin{aligned} V_t &= C_{BS}(t, S_t, \sigma S_t^{\beta-1}) \\ &+ r(\beta-1)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u, S_u, \sigma S_u^{\beta-1})(T-u)\sigma^2 S_t^{2(\beta-1)}\,\mathrm{d}u\right] \\ &+ \frac{(\beta-1)(2\beta-3)}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u, S_u, \sigma S_u^{\beta-1})(T-u)\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right] \\ &+ \frac{(\beta-1)^2}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u, S_u, \sigma S_u^{\beta-1})(T-u)^2\sigma^6 S_u^{6(\beta-1)}\,\mathrm{d}u\right] \\ &+ (\beta-1)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u, S_u, \sigma S_u^{\beta-1})(T-u)\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right]. \end{aligned}$$

We will write

$$\begin{aligned}
&\mathbb{E}_t\left[e^{-r(T-t)}C_{\widehat{BS}}(T,S_T,\sigma S_T^{\beta-1})\right] \\
= \;& C_{BS}(t,S_t,\sigma S_t^{\beta-1}) + (I_{CEV}) + (II_{CEV}) + (III_{CEV}) + (IV_{CEV}).
\end{aligned}$$

The exact formula can be difficult to use in practice, so we will need an approximation.

Corollary 9.1.2 (CEV Approximation Formula)**.** *Let S_t be a price process defined in* (3.14)*. Then, for all $t \in [0,T]$, we can approximate the call option fair value V_t by*

$$\begin{aligned}
V_t \;=\; & C_{BS}(t,S_t,\sigma S_t^{\beta-1}) \\
+ \;& \frac{1}{2}(\beta-1)r\sigma^2 S_t^{2(\beta-1)}\Gamma C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^2 \\
+ \;& \frac{1}{4}(\beta-1)(2\beta-3)\sigma^4 S_t^{4(\beta-1)}\Gamma C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^2 \\
+ \;& \frac{1}{6}(\beta-1)^2\sigma^6 S_u^{6(\beta-1)}\Gamma^2 C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^3 \\
+ \;& \frac{1}{2}(\beta-1)\sigma^4 S_t^{4(\beta-1)}\Lambda\Gamma C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^2 \\
+ \;& \epsilon_t
\end{aligned}$$

where ϵ_t is an error. We have that $\epsilon_t \le (\beta-1)^2\Pi(t,T,r,\sigma,\beta)$ and Π is an increasing function on every parameter.

Proof. The proof is a direct consequence of applying Lemma 3.1.10 (i) to $(I_{CEV})-(IV_{CEV})$.

Decomposition of the term I_{CEV}:

$$\begin{aligned}
& r(\beta-1)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)\,\sigma^2 S_u^{2(\beta-1)}\,\mathrm{d}u\right] \\
- \;& \frac{1}{2}(\beta-1)r\sigma^2 S_t^{2(\beta-1)}\Gamma C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^2 \\
= \;& \frac{r^2}{2}(\beta-1)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^3\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right] \\
+ \;& \frac{r}{4}(\beta-1)^2(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^3\,\sigma^6 S_u^{6(\beta-1)}\,\mathrm{d}u\right] \\
+ \;& \frac{r}{4}(\beta-1)^3\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^3 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^4\,\sigma^8 S_u^{8(\beta-1)}\,\mathrm{d}u\right] \\
+ \;& \frac{r}{2}(\beta-1)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\sigma^2 S_u^{2(\beta-1)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\right)(T-u)^3\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right].
\end{aligned}$$

Upper-Bound of the term I_{CEV}:

$$\left| r(\beta-1)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\sigma^2 S_u^{2(\beta-1)}\,(T-u)\,\mathrm{d}u\right]\right.$$

$$
\begin{aligned}
&- \frac{1}{2}(\beta-1)r\sigma^2 S_t^{2(\beta-1)}\Gamma C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^2\Big| \\
&\leq \frac{r^2}{2}C_1(\beta-1)^2\sigma\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
&+ \frac{r}{4}C_2(\beta-1)^2(2\beta-3)\sigma^3\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
&+ \frac{r}{4}C_3(\beta-1)^3\sigma^3\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
&+ C_4 r(\beta-1)^3\sigma^3\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
&+ \frac{r}{2}C_5(\beta-1)^2\sigma^2\int_t^T e^{-r(u-t)}\left(T-u\right)\,\mathrm{d}u \\
&\leq C(\beta-1)^2\Pi_1(t,r,\sigma,\beta)
\end{aligned}
$$

where $\Pi_1(t,T,r,\sigma,\beta)$ is an increasing function for every parameter, C_i $(i=1,\ldots,5)$ are some constants and $C=\max(C_i)$.

Decomposition of the term II_{CEV}*:*

$$
\begin{aligned}
& \frac{1}{2}(\beta-1)(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_T^{\beta-1})\,(T-u)\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right] \\
-\ & \frac{1}{4}(\beta-1)(2\beta-3)\sigma^4 S_t^{4(\beta-1)}\Gamma C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^2 \\
=\ & \frac{r}{2}(\beta-1)^2(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^2\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right] \\
+\ & \frac{r}{4}(\beta-1)^2(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\sigma^6 S_u^{6(\beta-1)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^3\,\mathrm{d}u\right] \\
+\ & \frac{1}{4}(\beta-1)^2(2\beta-3)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^2\,\sigma^6 S_u^{6(\beta-1)}\,\mathrm{d}u\right] \\
+\ & \frac{1}{8}(\beta-1)^2(2\beta-3)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^3\,\sigma^8 S_u^{8(\beta-1)}\,\mathrm{d}u\right] \\
+\ & \frac{1}{2}(\beta-1)^3(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\theta(S_u))\,(T-u)^3\,\sigma^8 S_u^{8(\beta-1)}\,\mathrm{d}u\right] \\
+\ & \frac{1}{8}(2\beta-3)(\beta-1)^3\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^3 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^4\,\sigma^{10} S_u^{10(\beta-1)}\,\mathrm{d}u\right] \\
+\ & \frac{1}{2}(\beta-1)^2(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\sigma^2 S_u^{2(\beta-1)}\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\right)(T-u)^2\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right] \\
+\ & \frac{1}{4}(\beta-1)^2(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\sigma^4 S_u^{4(\beta-1)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\right)(T-u)^3\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right].
\end{aligned}
$$

Upper-Bound of the term II_{CEV}:

$$\begin{aligned}
& \Big|\frac{1}{2}(\beta-1)(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_T^{\beta-1})\,(T-u)\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right] \\
- & \frac{1}{4}(\beta-1)(2\beta-3)\sigma^4 S_t^{4(\beta-1)}\Gamma BS(t,S_t,\sigma S_t^{\beta-1})(T-t)^2\Big| \\
\leq & \frac{r}{2}C_1(\beta-1)^2(2\beta-3)\sigma^3\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
+ & \frac{r}{4}C_2(\beta-1)^2(2\beta-3)\sigma^3\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
+ & \frac{1}{4}C_3(\beta-1)^2(2\beta-3)^2\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
+ & \frac{1}{8}C_4(\beta-1)^2(2\beta-3)^2\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
+ & \frac{1}{2}C_5(\beta-1)^3(2\beta-3)\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
+ & \frac{1}{8}C_6(2\beta-3)(\beta-1)^3\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
+ & C_7(\beta-1)^3(2\beta-3)\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
+ & \frac{1}{2}C_8(\beta-1)^2(2\beta-3)\sigma^4\int_t^T e^{-r(u-t)}\,(T-u)\,\mathrm{d}u \\
+ & C_9(\beta-1)^3(2\beta-3)\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3\,\mathrm{d}u \\
+ & \frac{1}{4}C_{10}(\beta-1)^2(2\beta-3)\sigma^4\int_t^T e^{-r(u-t)}\,(T-u)\,\mathrm{d}u \\
\leq & C(\beta-1)^2\Pi_2(t,r,\sigma,\beta)
\end{aligned}$$

where $\Pi_2(t,T,r,\sigma,\beta)$ is an increasing function for every parameter, C_i $(i=1,\ldots,10)$ are some constants and $C=\max(C_i)$.

Decomposition of the term III_{CEV}:

$$\begin{aligned}
& \frac{1}{2}(\beta-1)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_T^{\beta-1})\,(T-u)^2\,\sigma^6 S_u^{6(\beta-1)}du\right] \\
- & \frac{1}{6}(\beta-1)^2\sigma^6 S_u^{6(\beta-1)}\Gamma^2 C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^3 \\
= & \frac{r}{3}(\beta-1)^3\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^3\,\sigma^6 S_u^{6(\beta-1)}\,\mathrm{d}u\right] \\
+ & \frac{r}{6}(\beta-1)^3\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^3 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^4\,\sigma^8 S_u^{8(\beta-1)}\,\mathrm{d}u\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{6}(\beta-1)^3(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^3\,\sigma^8 S_u^{8(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{1}{12}(\beta-1)^3(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^3 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^4\,\sigma^{10} S_u^{10(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{1}{3}(\beta-1)^4\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^3 C_{\widehat{BS}}(u,S_u,\theta(S_u))\,(T-u)^4\,\sigma^{10} S_u^{10(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{1}{12}(\beta-1)^4\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^4 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^5\,\sigma^{12} S_u^{12(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{1}{3}(\beta-1)^3\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} S_u^2\Lambda\left(\sigma^4 S_u^{4\beta-6}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\right)(T-u)^3\left(\sigma^4 S_u^{4(\beta-1)}\right)\mathrm{d}u\right]\\
&+ \frac{1}{6}(\beta-1)^3\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} S_u^2\Lambda\left(\sigma^6 S_u^{6\beta-8}\Gamma^3 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\right)(T-u)^4\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right].
\end{aligned}
$$

Decomposition of the term IV_{CEV}:

$$
\begin{aligned}
& (\beta-1)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\sigma^4 S_u^{4(\beta-1)}\,(T-u)\,\mathrm{d}u\right]\\
&- \frac{1}{2}(\beta-1)\sigma^4 S_t^{4(\beta-1)}\Lambda\Gamma C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^2\\
&= r(\beta-1)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^2\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{r}{2}(\beta-1)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^3\,\sigma^6 S_u^{6(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{1}{2}(\beta-1)^2(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^2\,\sigma^6 S_u^{6(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{1}{4}(\beta-1)^2(2\beta-3)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^3\,\sigma^8 S_u^{8(\beta-1)}\,\mathrm{d}u\right]\\
&+ (\beta-1)^3\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma^2 C_{\widehat{BS}}(u,S_u,\theta(S_u))\,(T-u)^3\,\sigma^8 S_u^{8(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{1}{4}(\beta-1)^3\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma^3 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\,(T-u)^4\,\sigma^{10} S_u^{10(\beta-1)}\,\mathrm{d}u\right]\\
&+ (\beta-1)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\sigma^2 S_u^{2(\beta-1)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\right)(T-u)^2\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right]\\
&+ \frac{1}{2}(\beta-1)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\left(\sigma^4 S_u^{4(\beta-1)}\Lambda\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\right)(T-u)^3\,\sigma^4 S_u^{4(\beta-1)}\,\mathrm{d}u\right].
\end{aligned}
$$

Upper-Bound of the term III_{CEV}:

$$
\left|\frac{1}{2}(\beta-1)^2\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,\sigma S_T^{\beta-1})\sigma^6 S_u^{6(\beta-1)}\,(T-u)^2\,\mathrm{d}u\right]\right.
$$

$$
\begin{aligned}
&- \frac{1}{6}(\beta-1)^2\sigma^6 S_u^{6(\beta-1)}\Gamma^2 C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^3\Big| \\
&\leq \frac{r}{3}C_1(\beta-1)^3\sigma^3\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3 \mathrm{d}u \\
&+ \frac{r}{6}C_2(\beta-1)^3\sigma^3\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3 \mathrm{d}u \\
&+ \frac{1}{6}C_3(\beta-1)^3(2\beta-3)\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3 \mathrm{d}u \\
&+ \frac{1}{12}C_4(\beta-1)^3(2\beta-3)\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3 \mathrm{d}u \\
&+ \frac{1}{3}C_5(\beta-1)^4\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3 \mathrm{d}u \\
&+ \frac{1}{12}C_6(\beta-1)^4\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3 \mathrm{d}u \\
&+ \frac{2}{3}C_7(\beta-1)^3(2\beta-3)\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3 \mathrm{d}u \\
&+ \frac{1}{3}C_8(\beta-1)^3\sigma^4\int_t^T e^{-r(u-t)}(T-u)\,\mathrm{d}u \\
&+ \frac{1}{3}C_9(\beta-1)^3(3\beta-4)\sigma^5\int_t^T e^{-r(u-t)}\left(\sqrt{T-u}\right)^3 \mathrm{d}u \\
&+ \frac{1}{6}C_{10}(\beta-1)^3\sigma^4\int_t^T e^{-r(u-t)}(T-u)\,\mathrm{d}u \\
&\leq C(\beta-1)^3\Pi_3(t,r,\sigma,\beta)
\end{aligned}
$$

where $\Pi_3(t,T,r,\sigma,\beta)$ is an increasing function for every parameter, C_i $(i=1,\ldots,10)$ are some constants and $C=\max(C_i)$.

Decomposition of the term IV_{CEV}:

$$
\begin{aligned}
&\left|(\beta-1)\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,\sigma S_u^{\beta-1})\sigma^4 S_u^{4(\beta-1)}(T-u)\,du\right]\right. \\
&- \frac{1}{2}(\beta-1)\sigma^4 S_t^{4(\beta-1)}\Lambda\Gamma C_{\widehat{BS}}(t,S_t,\sigma S_t^{\beta-1})(T-t)^2\Big| \\
&\leq C_1 r(\beta-1)^2\sigma^2\int_t^T e^{-r(u-t)}(T-u)\,\mathrm{d}u \\
&+ \frac{r}{2}C_2(\beta-1)^2\sigma^2\int_t^T e^{-r(u-t)}(T-u)\,\mathrm{d}u \\
&+ \frac{1}{2}C_3(\beta-1)^2(2\beta-3)\sigma^4\int_t^T e^{-r(u-t)}(T-u)\,\mathrm{d}u \\
&+ \frac{1}{4}C_4(\beta-1)^2(2\beta-3)\sigma^4\int_t^T e^{-r(u-t)}(T-u)\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad C_5(\beta-1)^3\sigma^4 \int_t^T e^{-r(u-t)}\,(T-u)\,\mathrm{d}u \\
&+ \quad \frac{1}{4}C_6(\beta-1)^3\sigma^4 \int_t^T e^{-r(u-t)}\,(T-u)\,\mathrm{d}u \\
&+ \quad 2C_7(\beta-1)^3\sigma^4 \int_t^T e^{-r(u-t)}\,(T-u)\,\mathrm{d}u \\
&+ \quad C_8(\beta-1)^2\sigma^3 \int_t^T e^{-r(u-t)}\sqrt{T-u}\,\mathrm{d}u \\
&+ \quad 2C_9(\beta-1)^3\sigma^4 \int_t^T e^{-r(u-t)}\,(T-u)\,\mathrm{d}u \\
&+ \quad \frac{1}{2}C_{10}(\beta-1)^2\sigma^3 \int_t^T e^{-r(u-t)}\sqrt{T-u}\,\mathrm{d}u \\
&\leq \quad C(\beta-1)^2\Pi_4(t,r,\sigma,\beta)
\end{aligned}
$$

where $\Pi_4(t,T,r,\sigma,\beta)$ is an increasing function for every parameter, C_i $(i=1,\ldots,10)$ are some constants and $C=\max(C_i)$. □

9.2 Numerical analysis of the approximation for the CEV case

In this section, we compare our numerically approximated price of a European call option under CEV model with the following different pricing methods:

- The exact formula, see Cox (1975), Emanuel and MacBeth (1982) and Schroder (1989). The Matlab code is available in Kienitz and Wetterau (2012).
- The Singular Perturbation Technique, see Hagan and Woodward (1999).

The results of a call option with parameters $S_0 = 100$, $K = 100$, $\sigma = 20\%$ and $r = 1\%$ are as follows.

$T-t$	β	Exact Formula	Approximation Error	HW Error
0.25	0.25	0.2882882	-1.92 E-07	8.64 E-05
1		1.0103060	-9.78 E-07	2.68 E-04
2.5		2.4709883	-1.04 E-06	1.57 E-04
5		4.8771276	-2.22 E-07	1.77 E-05
0.25	0.5	0.5356736	-2.89 E-06	2.41 E-04
1		1.3886303	-2.26 E-05	1.75E-03
2.5		2.8506826	-8.42 E-05	5.68E-03
5		5.1658348	-2.09 E-04	1.15E-02
0.25	0.75	1.3887209	-2.30 E-05	3.92 E-04
1		3.0389972	-1.83 E-04	3.10 E-03
2.5		5.2954739	-7.13 E-04	1.19 E-02
5		8.2781049	-1.98 E-03	3.22 E-02
0.25	0.90	2.6404164	-2.92 E-05	3.14 E-04
1		5.5191736	-2.32 E-04	2.49 E-03
2.5		9.1446125	-9.03 E-04	9.70 E-03
5		13.5553379	-2.50 E-03	2.67 E-02

Table 9.1: CEV Model: Comparison between different price approximations

Note that the new approximation is more accurate than the approximation obtained in Hagan and Woodward (1999).

In Figure 9.1, we plot the surface of errors between the exact formula and our approximation.

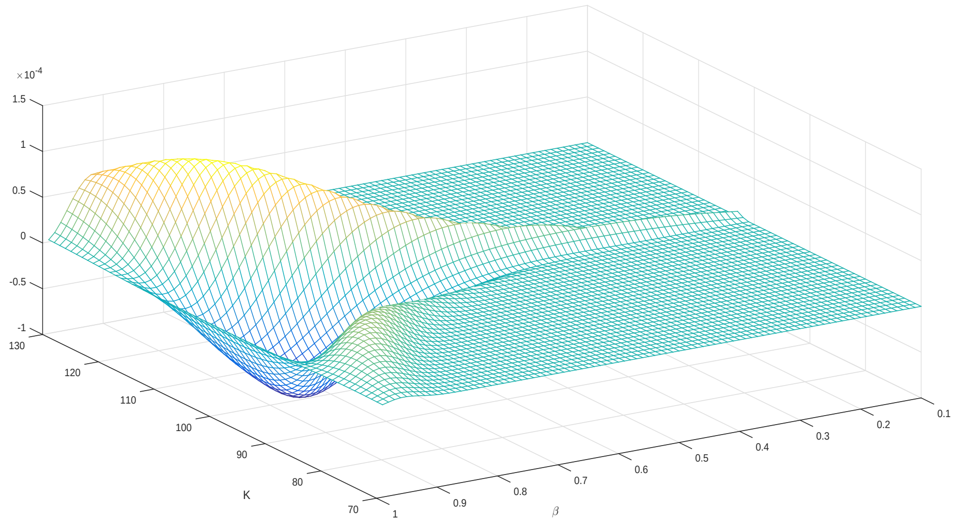


Figure 9.1: CEV Model: Error surface between the exact formula and our approximation for $S_0 = 100$, $\sigma = 20\%$ and $r = 5\%$.

We calculate also the speed time of execution (in seconds) of every method running the function timeit of Matlab 1.000 times. The computer used is an Intel Core i7 CPU Q740 @1.73 GHz 1.73GHz with 4GB of RAM with a Windows 10 (x64). The results are:

Measure	Exact formula	Approximation	HW
Average	2.56 E-02	1.73 E-04	1.67 E-04
Standard Deviation	3.03 E-03	2.86 E-05	2.52 E-05
Max	4.68 E-02	3.65 E-04	3.67 E-04
Min	2.42 E-02	1.64 E-04	1.59 E-04

Table 9.2: CEV Model: Statistical analysis of the price approximations

We observe that the singular perturbation method is the fastest method to calculate the price of CEV call option. The method developed in this work is a little more expensive in computation time. But to compute the exact price is much more costly than any of the other two methods. Note that in our method, we are also able to calculate the price and the Gamma of the log-normal price simultaneously.

9.3 The approximated implied volatility surface under CEV model.

In the above section, we have computed a bound for the error between the exact and the approximated pricing formulae for the CEV model. Now, we are going to derive an approximated implied volatility surface of second order in the log-moneyness. This approximated implied volatility surface can help us to understand better the volatility dynamics. Moreover, we obtain an approximation of the ATM implied volatility dynamics.

9.3.1 Deriving an approximated implied volatility surface for the CEV model.

For simplicity and without losing generality, we assume $t = 0$.

Expanding the implied volatility function $I(T, K)$ with respect to $(\beta - 1)$, as we did in Subsection 6.3.1, we have

$$I(T, K) = v_0 + (\beta - 1)I_1(T, K) + (\beta - 1)^2 I_2(T, K) + O((\beta - 1)^3)$$

and the approximation

$$\widehat{I}(T, K) = v_0 + (\beta - 1)I_1(T, K) + (\beta - 1)^2 I_2(T, K).$$

Let be $v_0 := \sigma S_0^{\beta - 1}$. Write $C_{BS}(v_0)$ as a shorthand for $C_{BS}(0, S_0, v_0)$. We can re-write Corollary 9.1.2 as

$$\begin{aligned} \widehat{V}(0, S_0, v_0) &= C_{BS}(v_0) \\ &+ \frac{1}{4}(\beta - 1)T v_0 \left(2r + v_0^2 \left[1 - \frac{2d_+}{v_0\sqrt{T}}\right]\right) \partial_\sigma C_{BS}(v_0) \\ &+ \frac{1}{6}(\beta - 1)^2 v_0^3 T \left(\left[d_+^2 - v_0\sqrt{T} d_+ + 2\right]\right) \partial_\sigma C_{BS}(v_0). \end{aligned}$$

On other hand we can consider the Taylor expansion of $C_{BS}(0, S_0, I(T, K))$ around v_0. We have that

$$\begin{aligned} \widehat{V}_0 &= C_{BS}(v_0) \\ &+ \partial_\sigma C_{BS}(v_0) \left((\beta - 1)I_1(T, K) + (\beta - 1)^2 I_2(T, K) + \cdots\right) \\ &+ \frac{1}{2}\partial_\sigma^2 C_{BS}(v_0) \left((\beta - 1)I_1(T, K) + (\beta - 1)^2 I_2(T, K) + \cdots\right)^2 \\ &+ \cdots \end{aligned}$$

and this expression can be rewritten as

$$\begin{aligned} C_{BS}(I(T, K)) &= C_{BS}(v_0) \\ &+ (\beta - 1)\partial_\sigma C_{BS}(v_0) I_1(T, K) \\ &+ (\beta - 1)^2 \partial_\sigma C_{BS}(v_0) I_2(T, K) \\ &+ O((\beta - 1)^2). \end{aligned}$$

Then, equating this expression to $\widehat{V}_0$ we have

$$I_1(T, K) = \frac{T v_0}{4}\left(2r + v_0^2\left[1 - \frac{2d_+}{v_0\sqrt{T}}\right]\right)$$

and

$$I_2(T,K) = \frac{Tv_0^3}{6}(d_+^2 - v_0\sqrt{T}d_+ + 2).$$

Note that $I_1(T,K)$ is linear with respect to the log-moneyness, while $I_2(T,K)$ is quadratic.

Remark 9.3.1. *Note that the pricing formula has an error of $O((\beta-1)^2)$ as we have proved in Corollary 9.1.2, and this is translated into an error of $O((\beta-1)^2)$ into our approximation of the implied volatility. The quadratic term of the volatility shape is not accurate.*

We calculate now the short time behaviour of the approximated implied volatility $\widehat{I}(T,K)$. We write the approximated equations in terms of $1-\beta$, because the case $\beta < 1$ is the most interesting, and in terms of the log-moneyness $\ln K - \ln S_0$.

Lemma 9.3.2. *For T close to 0 we have*

$$\hat{I}(T,K) \approx v_0 - \frac{v_0}{2}(1-\beta)(\ln K - \ln S_0) + \frac{v_0}{6}(1-\beta)^2 (\ln K - \ln S_0)^2 \tag{9.1}$$

Proof. Note that

$$\lim_{T\to 0} I_1(T,K) = \frac{v_0}{2}(\ln K - \ln S_0)$$

and

$$\begin{aligned}\lim_{T\to 0} I_2(T,K) &= \lim_{T\to 0} \frac{v_0^3 T}{6}\left(d_+^2 - v_0\sqrt{T}d_+ + 2\right) \\ &= \frac{v_0}{6}(\ln K - \ln S_0)^2 .\end{aligned}$$

□

Remark 9.3.3. *Note that equation (9.1) is a parabolic equation in the log-moneyness. Also, from the above expression it is easy to see that the slope with respect to $\ln K$ is negative when $K < S_0 exp\left(\frac{3}{2(1-\beta)}\right)$ and positive when $K > S_0 exp\left(\frac{3}{2(1-\beta)}\right)$, showing that the implied volatility from short time to maturity is smile-shaped. This is consistent with the result in Renault and Touzi (1996). Furthermore, there is a minimum of the implied volatility with respect to $\ln K$ attained at $K = S_0 exp\left(\frac{3}{2(1-\beta)}\right)$.*

Remark 9.3.4. *Note that, in stochastic volatility models, the implied volatility depends homogeneously on the pair (S,K), and in fact, it is a function of the log-moneyness $\ln(S_0/K)$. As extensively discussed in Renault (1997) and exemplified for GARCH option pricing in Garcia and Renault (1998), this homogeneity property is at odds with any type of GARCH option pricing. We also found this phenomenon in the quadratic expansion (9.1).*

The behaviour of the approximated implied volatility when the option is ATM is easy to obtain:

$$\hat{I}(T,K) = v_0 + \left(\frac{v_0 r(\beta-1)}{2} + \frac{v_0^3(\beta-1)^2}{3}\right)T - \frac{(\beta-1)^2 v_0^6}{24}T^2. \tag{9.2}$$

9.3.2 Numerical analysis of the approximation of the implied volatility for the CEV case.

Here, we compare numerically our approximated implied volatilities with the implied volatilities computed from call option prices calculated with the exact formula and with the ones obtained using the following formula obtained from Hagan and Woodward (1999):

$$\widehat{I}(T,K) = \frac{\sigma}{f_{av}^{1-\beta}}\left[1 + \frac{(1-\beta)(2+\beta)}{24}\left(\frac{F_0 - K}{f_{av}}\right)^2 + \frac{(1-\beta)^2}{24}\frac{\sigma^2 T}{f_{av}^{2-2\beta}}\right]$$

where $f_{av} = \frac{1}{2}(F_0 - K)$ and F_0 the forward price.

In Figure 9.2, we can see that the implied volatility dynamics behaves well for long dated maturities and short dated maturities when β is close to 1. When this is not the case, the formula behaves well at-the-money but the error increases far from the ATM value. This behaviour is a consequence of the quadratic error in our approximation.

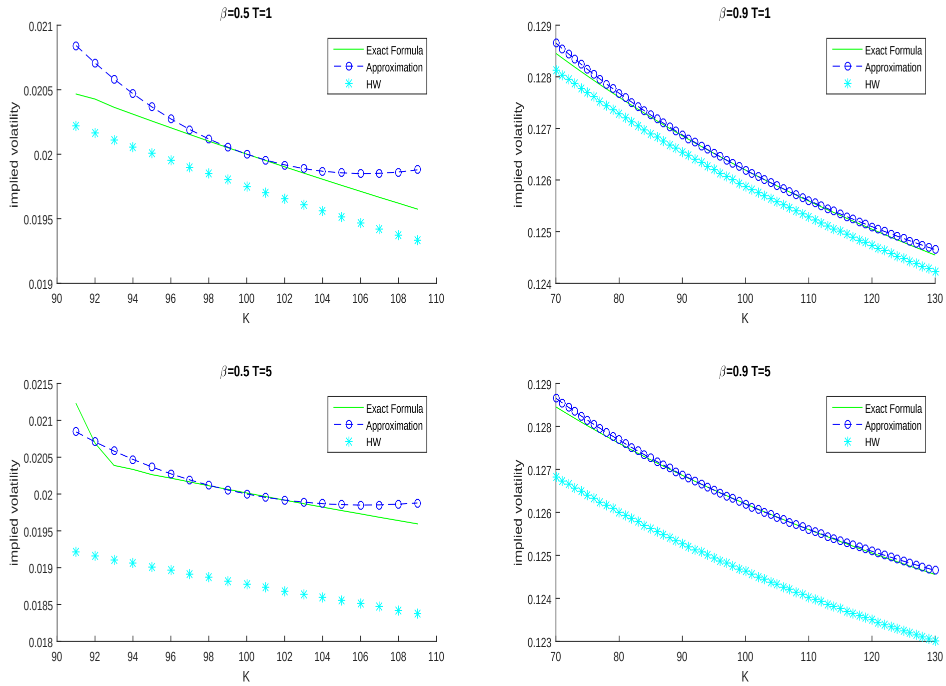


Figure 9.2: CEV Model: Comparison of implied volatility approximations for $S_0 = 100$, $\sigma = 20\%$ and $r = 5\%$.

In Figure 9.3, we observe that for ATM options, the approximated implied volatility surface fits really well the real implied volatility structure.

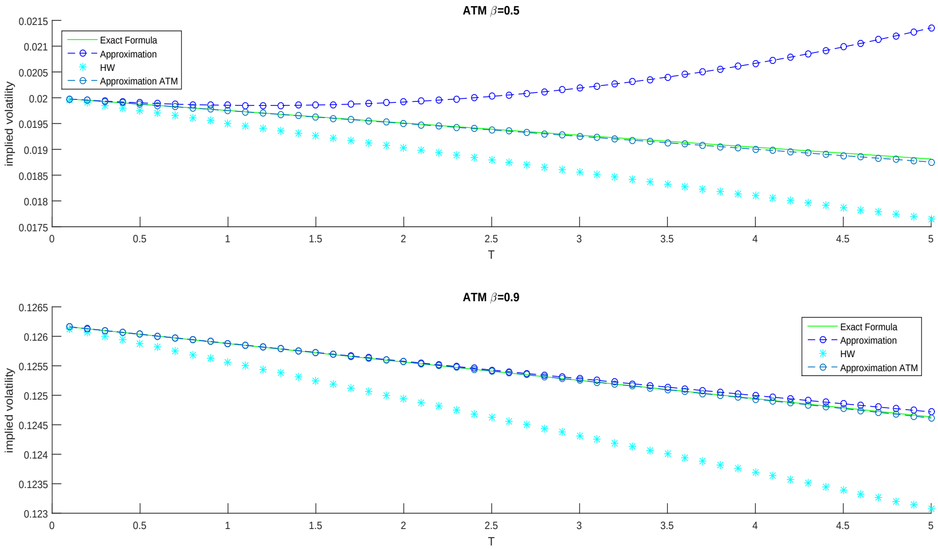


Figure 9.3: CEV Model: Comparison of ATM implied volatility approximations for $S_0 = 100$, $\sigma = 20\%$ and $r = 5\%$.

Now, we put the implied volatility approximation found in (9.1) into Black-Scholes-Merton formula and compare the obtained results with Hagan and Woodward results. The results for a call option with parameters $\beta = 0.25$, $S_0 = 100$, $K = 100$, $\sigma = 20\%$ and $r = 1\%$ are in the following table.

$T-t$	β	Exact Formula	BS with IV (9.1) error	HW error
0.25	0.25	0.2882882	3.51 E-08	8.64 E-05
1		1.0103060	2.36 E-07	2.68 E-04
2.5		2.4709883	2.94 E-07	1.57 E-04
5		4.8771276	6.72 E-08	1.77 E-05
0.25	0.5	0.5356736	4.27 E-07	2.41 E-04
1		1.3886303	3.65 E-06	1.75 E-03
2.5		2.8506826	1.54 E-05	5.68 E-03
5		5.1658348	4.36 E-05	1.15 E-02
0.25	0.75	1.3887209	3.29 E-06	3.92 E-04
1		3.0389972	2.64 E-05	3.10 E-03
2.5		5.2954739	1.05 E-04	1.19 E-02
5		8.2781049	3.01 E-04	3.22 E-02
0.25	0.9	2.6404164	4.17 E-06	3.14 E-04
1		5.5191736	3.31 E-05	2.49 E-03
2.5		9.1446125	1.29 E-04	9.70 E-03
5		13.5553379	3.59 E-04	2.67 E-02

Table 9.3: CEV Model: Comparison between prices using different implied volatility approximations

Our approximation is better than the Hagan and Woodward one.

We also compare the execution times.

Measure	HW	BS with IV (9.1)
Average	1.67 E-04	1.66 E-04
Standard Deviation	2.52 E-05	2.37 E-05
Max	3.67 E-04	3.48 E-04
Min	1.59 E-04	1.58 E-04

Table 9.4: CEV Model: Statistical analysis of prices using different implied volatility approximations

We can observe that both formulae are similar in computation time with the new approximation formula being a bit faster.

9.4 Calibration of the model.

Following the ideas of Alòs et al. (2015), we propose a method to calibrate the model. This method will allow us to find σ and β using quadratic linear regression. We can recover the parameters with a set of options of the same maturity with (9.1) or with ATM options of different maturities (9.2).

9.4.1 Calibration using the smile of volatility.

Using a set of options with the same maturity and the parameters $S_0 = 100$, $\sigma = 20\%$, $r = 5\%$, $K = 98 \cdots 102$. We calculate the price and their implied volatilities with the exact formula. We do a quadratic regression adjusting a parabola $a+bc+cx^2$ with $x = \ln K - \ln S_0$ to the implied volatilities. Using (9.1), it is easy to see that $\beta = \frac{2b}{a} + 1$ and $\sigma = \frac{a}{S^{\beta-1}}$. We do the following cases.

- For $T = 1$ and $\beta = 0.5$, , we find that

  $$0.000200446x^2 - 0.00497683x + 0.020000611$$

  from which we obtain $\beta = 0.50233$ and $\sigma = 19.787\%$.

- For $T = 5$ and $\beta = 0.5$, we find that

  $$-0.001234308x^2 - 0.004881387x + 0.020013633$$

  from which we obtain $\beta = 0.51219$ and $\sigma = 18.921\%$.

- For $T = 1$ and $\beta = 0.9$, we find that

  $$0.000382876x^2 - 0.006311173x + 0.126192162$$

  from which we obtain $\beta = 0.89997$ and $\sigma = 20.002\%$.

- For $T = 5$ and $\beta = 0.9$, we find that

  $$0.00010393x^2 - 0.00628411x + 0.126198861$$

  from which we obtain $\beta = 0.90041$ and $\sigma = 19.963\%$.

9.4.2 Calibration using ATM implied volatilities.

Using a set of ATM options with the same maturity and parameters $S_0 = 100$, $\sigma = 20\%$, $r = 5\%$, we calculate the price and their implied volatilities with the exact formula. Then we do a quadratic regression adjusting a parabola $a + bc + cx^2$ with $x = T$ to the implied volatilities. Using (9.2), it is easy to see that $\beta = 1 + \frac{-3r \pm \sqrt{9r^2+16ab}}{4a^2}$ and $\sigma = \frac{a}{S^{\beta-1}}$. We do the following cases.

- For $T = 0.3, 0.5, 0.8, 0.9, 1$ and $\beta = 0.5$

  $$0.0000086x^2 - 0.0002577x + 0.0200020$$

  from which we obtain $\beta = 0.48324$ (or $\beta = -185.94$ which we can discard) and $\sigma = 21.607\%$.

- For $T = 1, 2, 3, 4, 5$ and $\beta = 0.5$, we find that

$$0.0000024x^2 - 0.0002495x + 0.0199997$$

from which we obtain $\beta = 0.49970$ (or $\beta = -186.0055$ which we can discard) and $\sigma = 20.028\%$.

- For $T = 0.3, 0.5, 0.8, 0.9, 1$ and $\beta = 0.9$, we find that

$$-0.0000054x^2 - 0.0003076x + 0.1261899$$

from which we obtain $\beta = 0.90040$ (or $\beta = -3.6103$ which we can discard) and $\sigma = 19.963\%$.

- For $T = 1, 2, 3, 4, 5$ and $\beta = 0.9$, we find that

$$0, 0000006x^2 - 0.0003141x + 0.1261907$$

from which we obtain $\beta = 0.89822$ (or $\beta = -3.6081$ which we can discard) and $\sigma = 20.164\%$.

We have seen that to do a quadratic regression is enough to recover a good approximation of the parameters.

CHAPTER 10

Conclusion

10.1 Conclusion

Albert Einstein once said:

As far as laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Ever since people have been able to make promises, there have been financial derivatives. Naturally, from its inception, the question has arisen of how to find the right price for these contracts. Over the last century, a lot of research has been conducted to solve this problem. For example, sometimes studying the behaviour of the market, others the dynamics of prices and some others in ways of calculating prices numerically, especially quickly and accurately. Financial mathematics is the constant exercise of modelling reality finding the correct model, the one able to assign a probability to rare events. But sadly, there is no perfect model yet.

This can seem discouraging. In the words of George E.P. Box:

All models are wrong, but some are useful.

For that reason, we are still using a wide variety of pricing models today. Each of them is capable of explaining a part of reality. For decades, different models have been proposed, each with different properties and capable of explaining different behaviours observed in the market. With them, pricing methods have emerged and evolved at the same time as computers improved. Some of them are very effective and accurate, but most of the time, they are not intuitive to the researcher or practitioner.

This thesis is dedicated to the decomposition of the European Plain Vanilla options under different models. The most extensive research on the decomposition formula has been done using Malliavin calculus. However, this thesis is focused on the alternative approach of using Itô calculus. The methodology introduced by Alòs (2012) allows us to decompose the price of an option such as the Black-Scholes-Merton formula plus other terms that correct the effects of having a more complex model. Moreover, the corrections depend on the derivatives, also named sensitivities or Greeks, of the Black-Scholes-Merton formula. In some cases, a small correction in the Black-Scholes-Merton formula has to be applied as we have seen in Chapter 9. Therefore, one of the advantages of this technique is that in order to price an option in a complex model, it can be expressed by objects well-known to researchers and practitioners.

In Alòs (2012) and Alòs et al. (2015), the decomposition method is applied to the Heston model. A decomposition formula and an approximation are found for the Heston model, and subsequently an implicit volatility approximation. From this research, the idea to extend the decomposition formula to other models arose. For this, the decomposition formula is generalised under a general stochastic volatility model for a generic functional. Furthermore, it can be applied to stochastic volatility jump diffusion models with finite activity by conditioning the jump process. As an application, on the one hand, we apply the decomposition formula and its approximation to the rough volatility model. In particular, as an alternative to the rBergomi model. On the other hand, we apply the decomposition formula and its approximation to the Bates model. An approximation of the implied volatility is found. Additionally, a numerical comparison of the effectiveness is performed, by first comparing the option price approximation and then the implied volatility approximation.

In a different direction, we have seen that it is possible to obtain a decomposition formula for a local volatility model, particularly when the volatility model depends on the spot. It is interesting to note that in this situation, the decomposition formula seems more complex than the stochastic volatility case as it has more terms to approximate. As an application, the decomposition formula for the CEV model is studied, obtaining a price approximation formula and an approximation of the implicit volatility dynamics when an option is ATM or close to expiration.

When studying the previous cases, we realised that by applying the decomposition formula recursively, the approximation formula can be improved. In particular, we realised that the decomposition formula can be transformed into a Taylor type formula containing an infinite series with stochastic terms. As an application, we expanded the approximation formula in the Heston model case by adding several terms. We developed new approximations, even when there was no correlation between asset and volatility. The new approximation formulae have an error of order $O(\nu^3(|\rho| + \nu))$ and $O(\nu^4(1 + |\rho|))$. In the particular case of zero correlation, we derived an approximation formula with an error of order $O(\nu^6)$. In addition, for each approximation given, an upper-bound error was given and a numerical comparison was performed with a different benchmark prices.

10.2 Future research

In this dissertation, we have studied the decomposition formula for a wide variety of models. In particular, the results on stochastic volatility jump diffusion models only worked when the jumps were of finite activity. One possible line of research is extending the decomposition formula for a general Levy process. See, for example, the preprints of Arai (2020) and Lagunas and Ortiz-Latorre (2020).

Another topic of research is the improvement of the decomposition formula for exponential models. We noticed that for large-time maturities as well as high vol-vol, the approximation formula is not accurate enough. A possible way to improve this behaviour is by expanding the approximation formula, but in the rough volatility case it can be hard as it cannot be calculated explicitly. It can also be applied to the SABR model, where the CEV is mixed with a stochastic volatility model.

An interesting topic as well is understanding how the Taylor type formula works. This can help us to develop a more powerful way to calculate the approximations or to develop a better implied volatility approximation.

Appendices

APPENDIX A

Decomposition formula using Malliavin calculus.

In this thesis, we have focused on developing the decomposition formula using the classical Itô calculus for different models. But, in the first approaches, the methodology used was the Malliavin calculation. The anticipating stochastic calculus is a powerful extension of the Itô calculus which allows us to work with non-adapted processes. We have been able to use the Itô calculus because we have changed a non-adapted process like the future average volatility

$$\bar{\sigma}_t^2 := \frac{1}{T-t}\int_t^T \sigma_s^2 \, \mathrm{d}s$$

into an adapted process

$$\mathbb{E}_t(\bar{\sigma}_t^2) = \frac{1}{T-t}\int_t^T \mathbb{E}_t[\sigma_s^2] \, \mathrm{d}s.$$

In this appendix, we will present a basic exposition of definitions and propositions of Malliavin calculus, necessary to find a decomposition formula for a general stochastic volatility model.

A.1 Basic elements of Malliavin Calculus.

In this section, we present a brief introduction to the basic facts of Malliavin calculus. For more information, see Nualart (1995).

Let $W = \{W(t), t \in [0,T]\}$ a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $H = L^2([0,T])$ and denote by

$$W(h) := \int_0^T h(s) \, \mathrm{d}W_s,$$

the Itô integral of a deterministic function $h \in H$, also known as Wiener integral. Consider $\mathcal{S}$ to be the set of smooth and cylindrical random variables of the form

$$F = f\left(W(h_1, \ldots, W(h_n)\right)$$

where $n \geq 1$, $h_1, \ldots, h_n \in H$ and f is differentiable bounded function.

Definition A.1.1. *Given a random variable F, we define its Malliavin derivative, $D_t^W F$, as the stochastic process given by*

$$D_t^W F = \sum_{i=1}^{n} \partial_{x_i} f\left(W(h_1), \ldots, W(h_n)\right) h_i(x), \quad t \in [0,T].$$

The derivative operator D^W is a closable and unbounded operator defined as:

$$\begin{aligned} D^W : L^2(\Omega) \quad &\longrightarrow \quad L^2([0,T]^n \times \Omega) \\ F = f\left(W(h_1), \ldots, W(h_n)\right) \quad &\longrightarrow \quad D_{\cdot}^W F = \sum_{i=1}^{n} \partial_{x_i} f\left(W(h_1), \ldots, W(h_n)\right) h_i(\cdot). \end{aligned}$$

We can define the iterated derivative operator $D^{W,n}$ as:

$$D_{t_1,\ldots,t_n}^{W,n} = D_{t_1}^W \cdots D_{t_n}^W F.$$

The iterated derivative operator $D^{W,n}$ is closable and unbounded from $L^2(\Omega)$ into $L^2([0,T]^n \times \Omega)$, for all $n \geq 1$.

We denote by $\mathbb{D}^{n,2}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|_{n,2}^2 := \|F\|_{L^2(\Omega)}^2 + \sum_{k=1}^{n} \left\|D^{W,k} F\right\|_{L^2([0,T]^k \times \Omega)}^2.$$

We define δ^W as the adjoint of derivative operator D^W, also referred to as the Skorohod integral. The domain of δ^W, $Dom\delta^W$, is the set of elements $u \in L^2([0,T] \times \Omega)$ such that exist a constant $c > 0$ satisfying

$$\left| E\left[\int_0^T D_t^W F u_t \, \mathrm{d}t\right] \right| \leq c \|F\|_{L^2(\Omega)},$$

for all $F \in \mathcal{S}$.

If $u \in Dom\delta^W$ and $F \in \mathbb{D}^{1,2}$, $\delta^W(u)$ is an element of $L^2(\Omega)$ characterized by

$$E\left[\delta^W(u) F\right] = E\left[\int_0^T D_t^W F u_t \, \mathrm{d}t\right].$$

The operator δ is an extension of the Itô integral in the sense that the set $L_a^2([0,T] \times \Omega)$ of square integrable and adapted processes is included in $Dom\delta$ and the operator δ restricted to $L_a^2([0,T] \times \Omega)$ coincides with the Itô stochastic integral.

For any $u \in Dom\delta^W$, we will use the following notation

$$\delta^W(u) = \int_0^T u_t \, \mathrm{d}W_t.$$

The space of functions $\mathbb{L}_W^{n,2} := L^2\left([0,T];\mathbb{D}_W^{n,2}\right)$ is contained in the domain of δ for all $n \geq 1$. The variance of the Skorohod integral can be calculated for any process $u \in \mathbb{L}^{1,2}$ as follows:

$$E\left[\delta^W(u)\right]^2 = E\left[\int_0^T u_t^2 \, \mathrm{d}t\right] + E\left[\int_0^T \int_0^T D_s^W u_t D_t^W u_s \, \mathrm{d}t \, \mathrm{d}s\right].$$

We will need the following result on the Skorohod integral.

Proposition A.1.2. *Let $u \in Dom\delta^W$ and consider a random variable $F \in \mathbb{D}^{1,2}$ such that*

$$\mathbb{E}\left[F^2 \int_0^T u_t^2 \, \mathrm{d}t\right] < \infty.$$

Then

$$\int_0^T F u_t \, \mathrm{d}W_t = F \int_0^T u_t \, \mathrm{d}W_t - \int_0^T (D_t F) u_t \, \mathrm{d}t,$$

in the sense that $Fu \in Dom\delta$ if and only if the right-hand side of the above equation is square integrable.

Proof. See Nualart (1995), Subsection 1.3.1, (4). □

Proposition A.1.3. *Let $u \in L_a^2([0,T] \times \Omega)$. Then, for all $0 \leq t < s \leq T$, we have that*

$$D_s u_t = 0.$$

Proof. See Nualart (1995), Proposition 1.3.3, (4). □

Now, we will present the Itô formula for anticipative processes. This Theorem is the cornerstone to develop a decomposition formula by means of Malliavin calculus.

Theorem A.1.4. *Consider a stochastic process as*

$$X_t = X_0 + \int_0^t u_s \, \mathrm{d}W_s + \int_0^t v_s \, \mathrm{d}s,$$

where X_0 is a $\mathcal{F}_0$-measurable random variable and $u, v \in L_a^2([0,T] \times \Omega)$. Moreover, consider a process

$$Y_t = \int_t^T \theta_s \, \mathrm{d}s,$$

for some $\theta \in \mathbb{L}^{1,2}$.

Let $F : \mathbb{R}^3 \to \mathbb{R}$ *be a twice continuously differentiable function such that, for all* $t \in [0,T]$, F *and its derivatives evaluated in* (t, X_t, Y_t) *are bound by a positive constant* C. *Then it follows that*

$$\begin{aligned} F(t, X_t, Y_t) &= F(0, X_0, Y_0) + \int_0^t \partial_s F(s, X_s, Y_s)\, \mathrm{d}s \\ &+ \int_0^t \partial_x F(s, X_s, Y_s) dX_s \\ &+ \int_0^t \partial_y F(s, X_s, Y_s) dY_s \\ &+ \int_0^t \partial^2_{x,y} F(s, X_s, Y_s)(D^- Y)_s u_s\, \mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \partial^2_x F(s, X_s, Y_s) u_s^2\, \mathrm{d}s, \end{aligned}$$

where $(D^- Y)_s := \int_s^T D_s^W Y_r\, \mathrm{d}r$.

Proof. See Alòs (2006), Theorem 3. □

The next proposition is useful when we want to calculate the Malliavin derivative.

Proposition A.1.5. *Consider an Itô process*

$$X_t = X_0 + \int_0^t H_s\, \mathrm{d}W_s + \int_0^t K_s\, \mathrm{d}s.$$

Then, we have

$$D_s X_t = H_s \exp\left(\int_s^t \partial_s H_s\, \mathrm{d}W_s + \int_s^t \lambda_s\, \mathrm{d}s \right) \mathbb{1}_{[0,t]}(s).$$

where $\lambda_s = [\partial_s K - \frac{1}{2} (\partial_s H)^2]_s$.

Proof. See Nualart (1995), Section 2.2. □

A.2 Decomposition formula.

In this section, we use the Malliavin calculus to extend the call option price decomposition in an anticipative framework. The idea is to give a general decomposition formula.

Theorem A.2.1 (Functional decomposition under a SV model using Malliavin Calculus)**.** *Let* S_t *be a price process defined in* (3.26)*, let* $A(t, x, y)$ *be a continuous function on the space* $[0,T] \times [0,\infty) \times [0,\infty)$ *such that* $A \in \mathcal{C}^{1,2,2}((0,T) \times (0,\infty) \times (0,\infty))$. *Let us also* *assume that* $\mathcal{L}_y A_{\widehat{GSV}}(t,x,r,y) = 0$ *and* $\bar{\sigma}_t^2$ *as defined in* (4.2)*. Then, for every* $t \in [0,T]$*, the following formula holds:*

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}_t\left[A(T,S_T,\bar{\sigma}_T^2)\right] &= A(t,S_t,\bar{\sigma}_t^2) \\ &+ \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y A(u,S_u,\bar{\sigma}_u^2)\frac{v_u^2-\sigma_u^2}{T-u}\,\mathrm{d}u\right] \\ &+ \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{x,y}^2 A(u,S_u,\bar{\sigma}_u^2)(D^-\sigma^2)_u\frac{\theta(u,S_u,\sigma_u)}{T-u}\,\mathrm{d}u\right] \\ &+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x^2 A(u,S_u,\bar{\sigma}_u^2)\left(\theta^2(u,S_u,\sigma_u)-S_u^2v_u^2\right)\,\mathrm{d}u\right]. \end{aligned}$$

Proof. Notice that $e^{-rT}A(T,S(T),\bar{\sigma}^2(T)) = e^{-rT}V_T$. As $e^{-rt}V(t)$ is a martingale we can write

$$e^{-rt}V(t) = \mathbb{E}_t\left(e^{-rT}V(T)\right) = \mathbb{E}_t\left(e^{-rT}A(T,S(T),\bar{\sigma}^2(T))\right).$$

Now we will apply the previous anticipative Itô formula to the process $e^{-rt}A(t,S_t,v_t^2)$. It is obtained

$$\begin{aligned} e^{-rT}A(T,S_T,\bar{\sigma}_T^2) &= e^{-rt}A(t,S_t,\bar{\sigma}_t^2) \\ &+ r\int_t^T e^{-ru}A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}u \\ &+ \int_t^T e^{-ru}\partial_u A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}u \\ &+ \int_t^T e^{-ru}\partial_x A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}S_u \\ &+ \int_t^T e^{-ru}\partial_y A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}v_u^2 \\ &+ \rho\int_t^T e^{-ru}\partial_{x,y}^2 A(u,S_u,\bar{\sigma}_u^2)(D^-v^2)_u\theta(u,S_u,\sigma_u)\,\mathrm{d}u \\ &+ \frac{1}{2}\int_t^T e^{-ru}\partial_x^2 A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}[S,S]_u. \end{aligned}$$

Developing the terms, it can be found

$$\begin{aligned} e^{-rT}A(T,S_T,\bar{\sigma}_T^2) &= e^{-rt}A(t,S_t,\bar{\sigma}_t^2) \\ &+ r\int_t^T e^{-ru}A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}u \\ &+ \int_t^T e^{-ru}\partial_u A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}u \\ &+ \int_t^T e^{-ru}\partial_x A(u,S_u,\bar{\sigma}_u^2)\mu(u,S_u)\,\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&+ \int_t^T e^{-ru}\partial_x A(u,S_u,\bar{\sigma}_u^2)\theta(u,S_u,\sigma_u)\left(\rho\,\mathrm{d}W_u + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_u\right)\\
&- \int_t^T e^{-ru}\partial_y A(u,S_u,\bar{\sigma}_u^2)\frac{v_u^2-\sigma_u^2}{T-u}\\
&+ \rho\int_t^T e^{-ru}\partial_{x,y}^2 A(u,S_u,\bar{\sigma}_u^2)(D^-v^2)_u\theta(u,S_u,\sigma_u)\,\mathrm{d}u\\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_x^2 A(u,S_u,\bar{\sigma}_u^2)\theta^2(u,S_u,\sigma_u)\,\mathrm{d}u.
\end{aligned}
$$

We add and subtract the term

$$
\frac{1}{2}\int_t^T e^{-ru}S_u^2\partial_x^2 A(u,S_u,\bar{\sigma}_u^2)v_u^2\,\mathrm{d}u,
$$

having

$$
\begin{aligned}
e^{-rT}A(T,S_T,\bar{\sigma}_T^2) &= e^{-rt}A(t,S_t,\bar{\sigma}_t^2)\\
&- r\int_t^T e^{-ru}A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}u\\
&+ \int_t^T e^{-ru}\partial_u A(u,S_u,\bar{\sigma}_u^2)\,\mathrm{d}u\\
&+ \int_t^T e^{-ru}\partial_x A(u,S_u,\bar{\sigma}_u^2)\mu(u,S_u)\,\mathrm{d}u\\
&+ \int_t^T e^{-ru}\partial_x A(u,S_u,\bar{\sigma}_u^2)\theta(u,S_u,\sigma_u)\left(\rho\,\mathrm{d}W_u + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_u\right)\\
&+ \int_t^T e^{-ru}\partial_y A(u,S_u,\bar{\sigma}_u^2)\frac{v_u^2-\sigma_u^2}{T-u}\,\mathrm{d}u\\
&+ \rho\int_t^T e^{-ru}\partial_{x,y}^2 A(u,S_u,\bar{\sigma}_u^2)(D^-v^2)_u\theta(u,S_u,\sigma_u)\,\mathrm{d}u\\
&+ \frac{1}{2}\int_t^T e^{-ru}\partial_x^2 A(u,S_u,\bar{\sigma}_u^2)\theta^2(u,S_u,\sigma_u)\,\mathrm{d}u\\
&+ \frac{1}{2}\int_t^T e^{-ru}S_u^2\partial_x^2 A(u,S_u,\bar{\sigma}_u^2)v_u^2\,\mathrm{d}u\\
&- \frac{1}{2}\int_t^T e^{-ru}S_u^2\partial_x^2 A(u,S_u,\bar{\sigma}_u^2)v_u^2\,\mathrm{d}u.
\end{aligned}
$$

Grouping the blue terms, the corresponding Feynman-Kac formula $\mathcal{L}_y A_{\widehat{GSV}}(t,x,r,y)$ is obtained, so those terms vanish. Multiplying by e^{-rt} and using conditional expectations, we see that

$$
e^{-r(T-t)}\mathbb{E}_t\left[A(T,S_T,\bar{\sigma}_T^2)\right] = A(t,S_t,\bar{\sigma}_t^2)
$$

$$
\begin{aligned}
&+ \quad \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_y A(u,S_u,\bar{\sigma}_u^2)\frac{v_u^2-\sigma_u^2}{T-u}\,\mathrm{du}\right]\\
&+ \quad \rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_{x,y}^2 A(u,S_u,\bar{\sigma}_u^2)(D^-\sigma^2)_u\frac{\theta(u,S_u,\sigma_u)}{T-u}\,\mathrm{du}\right]\\
&+ \quad \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\partial_x^2 A(u,S_u,\bar{\sigma}_u^2)\left(\theta^2(u,S_u,\sigma_u)-S_u^2v_u^2\right)\,\mathrm{du}\right].
\end{aligned}
$$

□

The following statement can be derived from Theorem A.2.1.

Corollary A.2.2. *Let function A as in Theorem A.2.1. Suppose that the function A satisfies*

$$\partial_y A(t,x,y) = \frac{1}{2}x^2\partial_x^2 A(t,x,y)(T-t). \tag{A.1}$$

Let $A_t := A(t,S_t,\bar{\sigma}_t^2)$ $\forall t\in[0,T]$. Then, for every $t\in[0,T]$, the following formula holds:

$$
\begin{aligned}
e^{-r(T-t)}\mathbb{E}_t\left[A(T,S_T,\bar{\sigma}_T^2)\right] &= A(t,S_t,\bar{\sigma}_t^2)\\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma A(u,S_u,\bar{\sigma}_u^2)\left(\theta^2(u,S_u,\sigma_u)-\sigma_u^2\right)\,\mathrm{du}\right]\\
&+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma A(u,S_u,\bar{\sigma}_u^2)(D^-\sigma^2)_u\frac{\theta(u,S_u,\sigma_u)}{S_u}\,\mathrm{du}\right].
\end{aligned}
$$

Proof. Substituting (A.1) in Theorem A.2.1 and using the definitions of Λ and Γ the proof is straightforward. □

Remark A.2.3. *Note that $C_{\widehat{SV}}$ satisfies all the conditions of Corollary A.2.2.*

Remark A.2.4. *Being $C_{SV}(t,S_t,\sigma_t)$ the price of a call option under the model (3.26), notice that*

$$V_T = C_{SV}(T,S_T,\sigma_T) = C_{\widehat{BS}}(T,S_T,\bar{\sigma}_T).$$

Then,

$$V_t = C_{SV}(t,S_t,\sigma) = e^{-r(T-t)}\mathbb{E}_t\left[C_{\widehat{BS}}(T,S_T,\bar{\sigma}_T)\right].$$

Assuming that $A = C_{\widehat{BS}}$. Therefore, the price under the model (3.26) can be obtained as the following decomposition

$$
\begin{aligned}
V_t &= C_{BS}(t,S_t,\bar{\sigma}_t)\\
&+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{BS}(u,S_u,\bar{\sigma}_u^2)\left(\theta^2(u,S_u,\sigma_u)-\sigma_u^2\right)\,\mathrm{du}\right]\\
&+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{BS}(u,S_u,\bar{\sigma}_u^2)(D^-\sigma^2)_u\frac{\theta(u,S_u,\sigma_u)}{S_u}\,\mathrm{du}\right].
\end{aligned}
$$

and it is equal to $C_{SV}(t,S_t,\sigma)$. Note that $C_{BS}(t,S_t,\bar{\sigma}_t) = C_{\widehat{BS}}(t,S_t,\bar{\sigma}_t)$ and this equivalence is maintained for any derivative with respects to the price.

Remark A.2.5. *When we have a lognormal model, i.e.* $\theta(t, S_t, \sigma_t) = \sigma_t S_t$*. Then, we find*

$$\begin{aligned} V_t &= C_{BS}(t, S_t, \bar{\sigma}_t) \\ &+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{BS}(u, S_u, \bar{\sigma}_u)(D^-\sigma^2)_u \sigma_u \, \mathrm{d}u\right]. \end{aligned}$$

As it was proven in Alòs (2006).

Notice that decomposition formula obtained using the Itô formula, see Example 4.1.6, has an extra term. This extra term explains the vol-vol impact. In the case of the Malliavin calculus, this term is embedded in the decomposition.

Example A.2.6 (The Heston model)**.** *Assuming that the asset price follows the Heston model, 3.22, we have the following decomposition formula*

$$\begin{aligned} V(t) &= C_{BS}(t, S_t, \bar{\sigma}_t) \\ &+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{BS}(u, S_u, \bar{\sigma}_u)\left(\int_u^T D_u^W \sigma_r^2 dr\right)\sigma_u \, \mathrm{d}u\right]. \end{aligned}$$

where

$$D_u^W \sigma_r^2 = \nu\sigma_u \exp\left(\frac{\nu}{2}\int_u^r \frac{1}{\sigma_s}\mathrm{d}W_s + \int_u^r \left[-k - \frac{\nu^2}{8\sigma_s^2}\right]\mathrm{d}s\right).$$

For more information, see Alòs and Ewald (2008).

Example A.2.7 (SABR Model)**.** *Assuming that the asset prices follows the SABR model, 3.23, we have the following decomposition formula*

$$\begin{aligned} V(t) &= \mathbb{E}_t\left[C_{BS}(t, S_t, \bar{\sigma}_t)\right] \\ &+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{BS}(u, S_u, \bar{\sigma}_u)\sigma_u^2\left(S_u^{2(\beta-1)} - 1\right)\mathrm{d}u\right] \\ &+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{BS}(u, S_u, \bar{\sigma}_u)\left(\int_u^T D_u^W \sigma_r^2 \, \mathrm{d}r\right)\sigma_u \, \mathrm{d}u\right]. \end{aligned}$$

where

$$D_u^W \sigma_r^2 = 2\alpha\sigma_u^2 1_{[0,r]}(u).$$

APPENDIX B

An expression of the derivative of the implied volatility

In Alòs et al. (2007), an expression of the derivative of the implied derivative ATM is found. Following on from those ideas, and combining them with the alternative decomposition formulae for the Black-Scholes-Merton formula presented in Remark 4.1.5 and A.2.4, we can obtain a more generic formula.

Let $I(S_t)$ denote the implied volatility process, which satisfies by definition $V_t = C_{\widehat{BS}}(t, S_t, I(S_t))$. We calculate the derivative of the implied volatility.

Proposition B.0.1. *Under (3.26), for every fixed $t \in [0, T)$ and assuming that $(v_t)^{-1} < \infty$ a.s., we notice that*

$$\begin{aligned}\partial_S I(S_t^*) &= \frac{\mathbb{E}_t\left[\int_t^T \partial_S F_2(u, S_u^*, v_u)\,\mathrm{d}u\right]}{\partial_\sigma C_{\widehat{BS}}(t, S_t^*, I(S_t^*))} \\ &- \frac{\mathbb{E}_t\left[\int_t^T (F_1(u, S_u^*, v_u) + \partial_S F_3(u, S_u^*, v_u))\,\mathrm{d}u\right]}{2S\partial_\sigma C_{\widehat{BS}}(t, S_t^*, I(S_t^*))}.\end{aligned}$$

where

$$\begin{aligned}&\mathbb{E}_t\left[\int_t^T F_1(u, S_u, v_u)\,\mathrm{d}u\right] \\ =\ & \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u, S_u, v_u)\left(\frac{\theta^2(u, S_u, \sigma_u)}{S_u^2} - \sigma_u^2\right)\mathrm{d}u\right] \\ +\ & \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u, S_u, v_u)\,\mathrm{d}\left[M, M\right]_u\right] \\ +\ & \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\frac{\theta(u, S_u, \sigma_u)}{S_u}\Lambda\Gamma C_{\widehat{BS}}(u, S_u, v_u)\,\mathrm{d}\left[W, M\right]_u\right],\end{aligned}$$

$$\mathbb{E}_t\left[\int_t^T F_2(u, S_u, v_u)\,\mathrm{d}u\right]$$

$$= \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,v_u)\frac{\theta(u,S_u,\sigma_u)}{S_u}\,\mathrm{d}\,[W,M]_u\right]$$

and

$$\begin{aligned} &\mathbb{E}_t\left[\int_t^T F_3(u,S_u,v_u)\,\mathrm{d}u\right] \\ = &\frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,v_u)\left(\frac{\theta^2(u,S_u,\sigma_u)}{S_u^2}-\sigma_u^2\right)\mathrm{d}u\right] \\ + &\frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma^2 C_{\widehat{BS}}(u,S_u,v_u)\,\mathrm{d}\,[M,M]_u\right]. \end{aligned}$$

Proof. Taking partial derivatives with respect to S_t in the expression $V(t) = C_{\widehat{BS}}(t,S_t,I(S_t))$, we obtain

$$\partial_S V_t = \partial_S C_{\widehat{BS}}(t,S_t,I(S_t)) + \partial_\sigma C_{\widehat{BS}}(t,S_t,I(S_t))\partial_S I(S_t). \tag{B.1}$$

On the other hand, from Remark 4.1.5, we deduce that

$$V_t = C_{\widehat{BS}}(t,S_t,v_t) + \mathbb{E}_t\left[\int_t^T F_1(u,S_u,v_u)\,\mathrm{d}u\right], \tag{B.2}$$

which implies that

$$\partial_S V_t = \partial_S BS(t,S_t,v_t) + \mathbb{E}_t\left[\int_t^T \partial_S F_1(u,S_u,v_u)\,\mathrm{d}u\right]. \tag{B.3}$$

Using that $(v_t)^{-1} < \infty$, we can check that $\partial_S V_t$ is well-defined and finite a.s. Thus, using that $S_t^* = K\exp(r(T-t))$, (B.1) and (B.3), we obtain

$$\begin{aligned} \partial_S I(S_t^*) &= \frac{\partial_S C_{\widehat{BS}}(t,S_t^*,v_t) - \partial_S C_{\widehat{BS}}(t,S_t^*,I(S_t))}{\partial_\sigma C_{\widehat{BS}}(t,S_t^*,I(S_t))} \\ &+ \frac{\mathbb{E}_t\left[\int_t^T \partial_S F_1(u,S_u^*,v_t)\,\mathrm{d}u\right]}{\partial_\sigma C_{\widehat{BS}}(t,S_t^*,I(S_t))}. \end{aligned}$$

From Renault and Touzi (1996), we know that $\partial_S I_t^0 = 0$ at ATM moneyness, where I_t^0 is the implied volatility in the case $\rho = 0$, so

$$\partial_S C_{\widehat{BS}}(t,S_t^*,v_t) = \partial_S C_{\widehat{BS}}(t,S_t^*,I^0(S_t)) - \mathbb{E}_t\left[\int_t^T \partial_S F_3(u,S_u^*,v_u)\,\mathrm{d}u\right].$$

Therefore, we note that

$$\partial_S I(S_t^*) = \frac{\partial_S C_{\widehat{BS}}(t,S_t^*,I_t^0) - \partial_S C_{\widehat{BS}}(t,S_t^*,I(S_t^*))}{\partial_\sigma C_{\widehat{BS}}(t,S_t^*,I(S_t^*))}$$

$$+ \quad \frac{\mathbb{E}_t\left[\int_t^T \partial_S F_2(u, S_u^*, v_u)\,\mathrm{d}u\right]}{\partial_\sigma C_{\widehat{BS}}(t, S_t^*, I(S_t^*))}.$$

On the other hand, we note that

$$\partial_S C_{\widehat{BS}}(t, S_t^*, v_t) = \phi(d)$$

and

$$C_{\widehat{BS}}(t, S_t^*, v_t) = S\left(\phi(d) - \phi(-d)\right)$$

where ϕ is the standard Gaussian density. Then

$$\partial_S C_{\widehat{BS}}(t, S_t^*, v_t) = \frac{C_{\widehat{BS}}(t, S_t^*, v_t) + S}{2S}$$

and

$$\begin{aligned}
& \partial_S C_{\widehat{BS}}(t, S_t^*, I_t^0) - \partial_S C_{\widehat{BS}}(t, S_t^*, I(S_t^*)) \\
= \; & \frac{1}{2S}\left(C_{\widehat{BS}}(t, S_t^*, I_t^0) - C_{\widehat{BS}}(t, S_t^*, I(S_t^*))\right) \\
= \; & -\frac{1}{2S}\mathbb{E}_t\left[\int_t^T \left(F_1(u, S_u^*, v_u) + \partial_S F_3(u, S_u^*, v_u)\right)\,\mathrm{d}u\right].
\end{aligned}$$

□

We can do it analogously using Mallivin calculus and Remark A.2.4. In Alòs et al. (2007), we find a proof when $\theta(t, S_t, \sigma_t) = \sigma_t S_t$.

Proposition B.0.2. *Under (3.26), for every fixed $t \in [0, T)$, $(\tilde{v}(t))^{-1} < \infty$ a.s. Then it follows*

$$\begin{aligned}
\partial_S I(S_t^*) \; &= \; \frac{\mathbb{E}_t\left[\int_t^T \partial_S F_2(u, S_u^*, \tilde{\sigma}_u)\,\mathrm{d}u\right]}{\partial_\sigma C_{\widehat{BS}}(t, S_t^*, I(S_t^*))} \\
&- \; \frac{\mathbb{E}_t\left[\int_t^T F_1(u, S_u^*, \tilde{\sigma}_u) + \partial_S F_3(u, S_u^*, \tilde{\sigma}_u)\,\mathrm{d}u\right]}{2S\partial_\sigma C_{\widehat{BS}}(t, S_t^*, I(S_t^*))}.
\end{aligned}$$

where

$$\begin{aligned}
& \mathbb{E}_t\left[\int_t^T F_1(u, S_u, \tilde{\sigma}_u)\,\mathrm{d}u\right] \\
= \; & \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u, S_u, \tilde{\sigma}_u)\left(\left(\frac{\theta(u, S_u, \sigma_u)}{S_u}\right)^2 - \sigma_u^2\right)\,\mathrm{d}u\right]
\end{aligned}$$

$$+ \quad \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,\tilde{\sigma}_u)\left(\int_s^T D_s^W\sigma_r^2\,\mathrm{d}r\right)d\left[W,M\right]_u\right],$$

$$\mathbb{E}_t\left[\int_t^T F_2(u,S_u,\tilde{\sigma}_u)\,\mathrm{d}u\right] = \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Lambda\Gamma C_{\widehat{BS}}(u,S_u,\tilde{\sigma}_u)\left(\int_s^T D_s^W\sigma_r^2\,\mathrm{d}r\right)d\left[W,M\right]_u\right]$$

and

$$\begin{aligned}&\mathbb{E}_t\left[\int_t^T F_3(u,S_u,\tilde{\sigma}_u)\,\mathrm{d}u\right]\\ = \; &\frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\Gamma C_{\widehat{BS}}(u,S_u,\tilde{\sigma}_u)\left(\left(\frac{\theta(u,S_u,\sigma_u)}{S_u}\right)^2-\sigma_u^2\right)\,\mathrm{d}u\right].\end{aligned}$$

Proof. It is similar to the previous proof. □

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