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THE DENJOY-WOLFF THEOREM, EXTENSIONS AND APPLICATIONS

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Abstract

The aim of this project is to prove the *Denjoy-Wolff Theorem*, which deals with iteration of holomorphic self-maps of the unit disk $\mathbb{D}$. It claims that either the map is conjugate to a rotation about the origin or all the points converge to a unique point in $\overline{\mathbb{D}}$ under iteration. We will also prove that there always exists a *fundamental set*, an invariant subset reached by all the compact sets in a finite number of iterations and where the map is one-to-one. Fundamental sets can be classified in four different types, up to conformal conjugation.

Finally, we will use this results to classify the periodic Fatou components of entire maps. For each of them, we can find a fundamental set. In the case of attracting or parabolic components or Siegel disks, the dynamics in the fundamental set is determined up to conformal conjugation. However, in the case of Baker domains three different types can occur and we will present some examples of them.

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Contents

INTRODUCTION.

Throughout this project, we study the iteration of holomorphic functions in the unit disk and apply the results in a more general context of complex dynamics.

The iteration of holomorphic functions in the unit disk started with the Schwarz lemma, a result about analytic self-maps of $\mathbb{D}$ that fix the origin. Specifically, it asserts that, given a self-map of $\mathbb{D}$ that fixes the origin, the disks centred in the origin of any radius remain invariant. This result was stated by H. Schwarz (1880) only for one-to-one maps, and generalised by C. Carathéorody (1912) for every self-map of $\mathbb{D}$.

Few was discussed on the topic until 1926, when J. Wolff published an article at the *Comptes rendus hebdomadaires des séances de l'Academie des sciences* [24]. He proved that, given an analytic self-map of $\mathbb{D}$, not a Möbius Transformation, without fixed points in $\mathbb{D}$, all orbits converge to the same point in the boundary. However, he assumed that the map extended continuously to the boundary. Two weeks later, Wolff published another article where he showed that the hypothesis of continuity at the boundary was superfluous [25]. One week later, A. Denjoy published an alternative proof for the theorem [11]. Because of this independent, but nevertheless collaboration, the theorem was named after the two mathematicians.

Denjoy-Wolff Theorem. Let f be an holomorphic self-map of $\mathbb{D}$, that is not an elliptic automorphism. Then, there is $a \in \overline{\mathbb{D}}$, such that $\forall z \in \mathbb{D}$, $f^n(z) \to a$, as $n \to \infty$.

The point a is called the **Denjoy-Wolff point** of f.

Months later, Wolff published *Sur une géneralisation d'un théorème de Schwarz* [26], where he gave an alternative proof for the theorem based on Schwarz lemma. He proved that given a self-map f with Denjoy-Wolff point a, any disk tangent to $\partial\mathbb{D}$ at a remains invariant under f. This result, known as the Wolff lemma, was useful to compute the limit and derivative at the Denjoy-Wolff point, in some generalised sense.

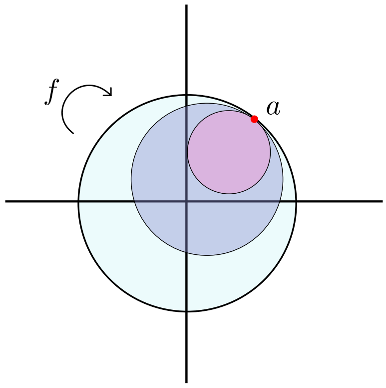


Figure 1: Wolff lemma asserts that tangent disks to $\partial\mathbb{D}$ at the Denjoy-Wolff point a are mapped into itselves.

In this project, we present a modern proof of these results due to A. Beardon (1990) that can be found in [9] and [21]. The novelty relies on using the hyperbolic metric of $\mathbb{D}$, instead of the Euclidean one. The Schwarz-Pick lemma, that asserts that holomorphic maps are always contractive with respect to the hyperbolic metric, is the key element of this proof. Chapters 1, 2 and 3 are devoted to preliminary results and finally, in chapter 4 we prove the Denjoy-Wolff Theorem, as well as the Wolff lemma.

Once we know that all the orbits converge to the same point, it is natural to ask *how* they converge. If we assume that the self-map of $\mathbb{D}$ is also analytic in a neighbourhood of the Denjoy-Wolff point, then this question is closely related to the linearization around the fixed point, i.e.

to whether the function is locally conjugate to its linear part, a classical problem widely studied by many mathematicians going back to Schröder (1871), Koenigs (1884) or Leau (1897).

On the other hand, the general problem of convergence when the extension to the Denjow-Wolff point is not assumed, remained open until 1981, when Cowen [10] proved the existence of a subset of $\mathbb{D}$ where the dynamics are essentially linear. He called it a *fundamental set*.

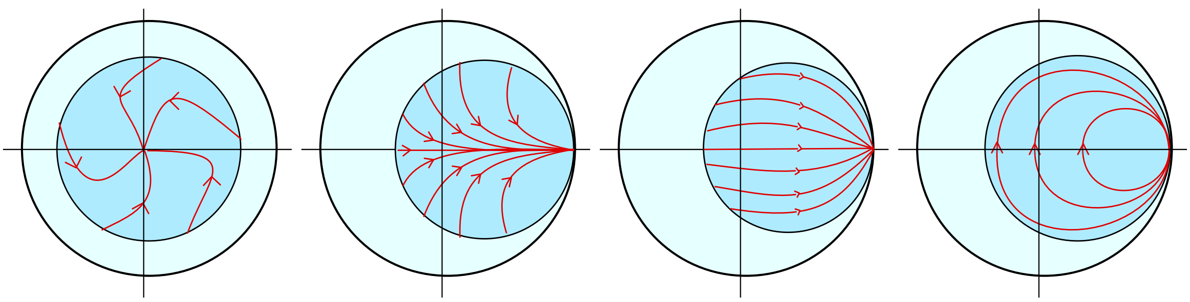
Definition. Let f be a map of a domain $\Delta \subset \mathbb{C}$ into itself, we say V is a **fundamental set** for f on Δ if V is an open, connected, simply connected subset of Δ such that: $f(V) \subset V$ and for every compact set $K \subset \Delta$, there is a positive integer n so that $f^n(K) \subset V$.

Cowen's Theorem. Let $f\colon \mathbb{D} \to \mathbb{D}$ be analytic, nonconstant and not conformal, with Denjoy-Wolff point a. Suppose that the angular derivative $f'(a)$ is nonzero. Then there is a fundamental set V for f on $\mathbb{D}$, a domain Ω ($\mathbb{C}$ or $\mathbb{D}$), a Möbius transformation $\phi\colon \Omega \to \Omega$ and an analytic map $\sigma\colon \mathbb{D} \to \Omega$, such that σ and f are univalent in V, $\sigma(V)$ is a fundamental set for ϕ on Ω and the following diagram is commutative:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "f"] \arrow[d, "\sigma"] & \mathbb{D} \arrow[d, "\sigma"] \\ \Omega \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

Cowen's results go actually further, proving that Ω and ϕ fall into four essentially different cases.

1. $\Omega = \mathbb{C}$, $\phi(z) = sz$, $0 < |s| < 1$.
2. $\Omega = \mathbb{C}$, $\phi(z) = z + 1$.
3. $\Omega = \mathbb{D}$, $\phi(z) = \dfrac{(1+s)z + (1-s)}{(1-s)z + (1+s)}$ with $0 < s < 1$.
4. $\Omega = \mathbb{D}$, $\phi(z) = \dfrac{(1 \pm 2i)z - 1}{z - 1 \pm 2i}$.



(a) *CASE 1* (b) *CASE 2* (c) *CASE 3* (d) *CASE 4*

Figure 2: The different types of convergence to the Denjoy-Wolff point.

The results of Denjoy, Wolff and Cowen form together a fairly complete description of the possible dynamics of holomorphic self-maps of $\mathbb{D}$, arguably a quite restricted class of functions. It turns out however that their results can be applied in a much more general setting, namely the iteration of holomorphic maps in all of $\mathbb{C}$, which had started some years earlier. Indeed, the French Academy of Sciences announced that it would award its *1918 Grand Prix des Sciences Mathématiques* for the study of complex iteration in the Riemann sphere $\widehat{\mathbb{C}}$. At that time, P. Fatou and G. Julia developed independently a groundbreaking theory, based on the results on normal families of P. Montel (1912). They reached the same results with different approaches, but their most important idea was to split $\widehat{\mathbb{C}}$ into two different sets, one where the iterates are well-behaved and the other with chaotic dynamics. These two sets were named after them: the Fatou set and the Julia set.

In chapter 6 we analyse the behaviour of the iterates in the Fatou set, in particular in each connected component of it, which are called Fatou components. In the case of entire functions, Fatou components are simply connected which connects, via the Riemann map, to the work done in the previous sections. The main theorem is as follows.

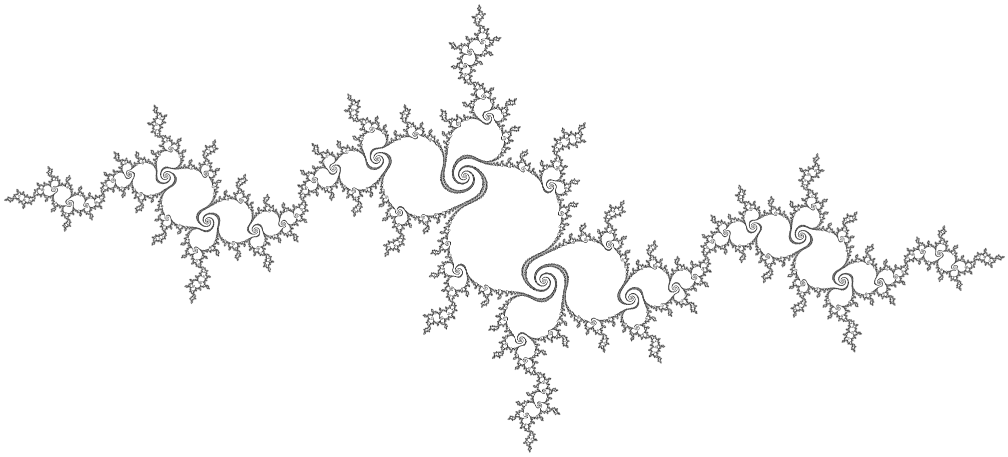


Figure 3: The Julia set of the polynomial $P(z) = z^2 - 1.12 + 0.222i$. [23]

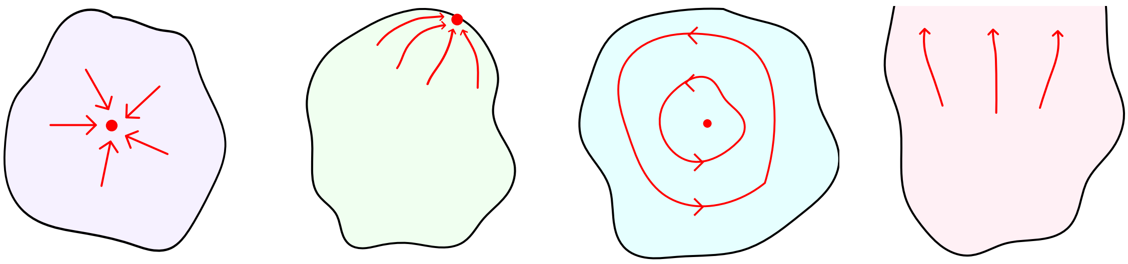
Theorem. (Classification of Fatou components) *Let f be a entire function, but not a linear polynomial, and U be a periodic Fatou component of period k. Then exactly one of the following holds:*

1. *U contains an attractive periodic point z_0 and $f^{nk} \to z_0$ uniformly on compact subsets of U. Then U is a component of the* ***basin of attraction*** *of z_0.*

2. *∂U contains a parabolic periodic point z_0 and $f^{nk} \to z_0$ uniformly on compact subsets of U. Then U is a component of the* ***parabolic basin of attraction*** *of z_0.*

3. *There exists $z_0 \in U$, an irrationally neutral periodic point and $f^k_{|U}$ is conformally conjugate to an irrational rotation. Then, U is a* ***Siegel disk****.*

4. *If f is transcendental, U can also be a* ***Baker domain****. That is $f^{nk}(z) \to \infty$ uniformly on compact subsets of U.*



(a) Basin of attraction. (b) *Parabolic basin.* (c) *Siegel disk.* (d) *Baker domain.*

Figure 4: Schematic representation of the different types of Fatou components.

The classification theorem is due essentially to Fatou (1919) and Cremer (1932). In 1926, Fatou extend some of his results in the case of transcendental entire functions [15], i.e. entire maps with an essential singularity at ∞, although he realised that *"the already complex phenomena that occur in the iteration of polynomials acquire here an even bigger complexity"*. He admitted to feeling overcome and only proved few general theorems. However, he realised the presence of Baker domains and gave some examples.

At that time, Baker domains were called *infinite Fatou components*, *essentially parabolic domains* or *domains at* ∞. In the above form the theorem was stated first by Baker, Kötus and Lü (1991).

We shall give a proof of the classification theorem using the Denjoy-Wolff theory developed in the previous sections.

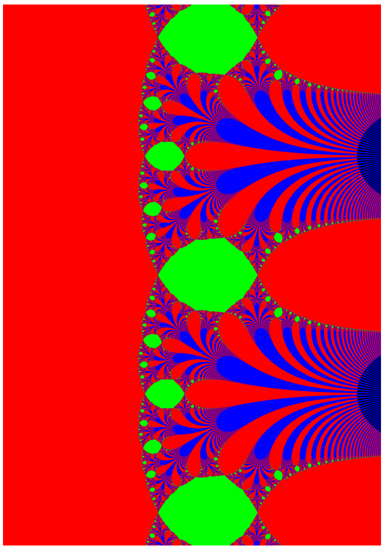


Figure 5: A Julia set with a Baker domain of the left.[7]

Our last goal is to use Cowen's classification to describe the dynamics in the different types of Fatou components. Around attracting and parabolic fixed points and in Siegel disks the dynamics are determinate up to conjugacy. The situation is different for Baker domains, where several cases can occur, as it is shown in the next and final theorem.

Theorem. **(Classification of Baker domains)** *Let B be a Baker domain of f and $V \subset B$ a fundamental set for f in B. Then, taking $\Omega = \mathbb{C}$ or $\Omega = \mathbb{H}$, there exists a map $\psi\colon B \to \Omega$, which is one-to-one in V, and a Möbius transformation $\phi\colon \Omega \to \Omega$, such that $\psi \circ f = \phi \circ \psi$. Moreover, Ω is unambiguously determined and ϕ is unique up to conjugation, and they can be chosen among the following:*

*(a) $\Omega = \mathbb{C}$ and $\phi(z) = z + 1$. In this case, we say that B is **doubly-parabolic**.*

*(b) $\Omega = \mathbb{H}$ and $\phi(z) = sz$, with $0 < s < 1$. In this case, we say that B is **hyperbolic**.*

*(c) $\Omega = \mathbb{H}$ and $\phi(z) = z \pm 1$. In this case, we say that B is **simply-parabolic**.*

We follow the proof of J. König [18] and we provide examples of the different types of Baker domains.

1 BACKGROUND ON COMPLEX ANALYSIS.

Here we give the main results in complex analysis that we use throughout this project. First we start by defining the concept of *holomorphic function*.

Definition 1.1. Let S be $\mathbb{C}$ or $\widehat{\mathbb{C}}$ and V an open subset of S. Let us consider $f\colon V \to S$. We say that f is **holomorphic** at $z \in V$ if the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists. In that case we denote it by $f'(z)$.

We say that f is holomorphic in V if it is holomorphic at all points of V. We define $H(V)$ to be the collection of analytic functions in V.

We remark that, in the previous definition, if $S = \widehat{\mathbb{C}}$ then the limit is taken with respect to the spherical metric.

We refer to [19] for basic background on holomorphic maps. Nevertheless, we recall that any holomorphic map $f\colon V \to S$ is open and analytic, i.e. it has a power expansion about any $z_0 \in V$. As a consequence, f is $\mathcal{C}^\infty$ and all zeros of f are isolated. Holomorphic maps of the Riemann sphere $\widehat{\mathbb{C}}$ are precisely the rational maps.

Given any holomorphic function f, we shall be interested in the sequence $\mathcal{F} = \{f^n\}_n$ of iterates of f.

In general, when considering a family $\mathcal{F}$ of holomorphic functions, we use *uniform convergence on compact subsets*, which is stronger than pointwise convergence. Let Ω be any open subset of $\widehat{\mathbb{C}}$.

Definition 1.2. A sequence of functions $\{f_n\}$ on Ω **converges uniformly** to a funtion f on Ω if $\forall \varepsilon > 0$ $\exists n_0$ such that $\forall n \geq n_0$, $\sigma(f_n(z) - f(z)) < \varepsilon$ $\forall z \in \Omega$.

A sequence of functions f_n on Ω **converges uniformly on compact subsets (u.c.c)** if for every compact subset K in Ω, the restrictions $f_{n\restriction_K}$ converge uniformly.

The concept of *equicontinuity* is specially relevant. Intuitively, a family of functions is *equicontinuous* at one point if the image of z_0 is close to the image of z_1, when z_0 and z_1 are close enough, for all members of the family. We will also define the concept of *normal family* because, by Arzelà-Ascoli theorem, it is equivalent to *equicontinuous*.

Definition 1.3. Let (X_1, d_1), (X_2, d_2) be metric spaces. A family $\mathcal{F}$ of maps of (X_1, d_1) into (X_2, d_2) is **equicontinuous** at x_0, if for every $\varepsilon > 0$, there is $\delta > 0$ such that, for all $x \in X_1$ and $f \in \mathcal{F}$: if $d_1(x_0, x) < \delta$, then $d_2(f(x_0), f(x)) < \varepsilon$.

We say $\mathcal{F} \subset H(\Omega)$ is a **normal** family if every sequence $\{f_n\}$ in $\mathcal{F}$ contains a subsequence that converges u.c.c. in Ω.

Theorem 1.4. (Arzelà-Ascoli) *Let $\mathcal{F}$ be a family of holomorphic functions is normal in Ω if and only if it is equicontinuous on every compact subset of Ω, with respect to the spherical metric.*

By Weierstrass Theorem, if $\mathcal{F} \subset H(\Omega)$, then every limit function is also holomorphic in Ω.

Since we are considering the spherical metric, sequences can converge uniformly to ∞. Then, in the definitions of equicontinuity and normality, we allow $f_n \to \infty$. This is because we want to have the concept of proximity of the iterates also for points that are close to ∞.

Montel's Theorem gives us an easier characterisation for normal families. In particular, bounded families are normal.

Theorem 1.5. (Montel) *Let $\mathcal{F}$ be a family of holomorphic functions on Ω. If there exist three distinct points of $\widehat{\mathbb{C}}$ which are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal.*

In sections 4 and 5, we work with holomorphic self-maps of $\mathbb{D}$, which therefore are bounded. The next corollary tells us that, in this case, pointwise convergence is equivalent to convergence *u.c.c.* So, when working with holomorphic self-maps of $\mathbb{D}$ we will only care about pointwise convergence.

Corollary 1.6. *If $\{f_n\}_n$ is a sequence of bounded holomorphic functions on Ω converging pointwise to f, then f is holomorphic and the convergence is u.c.c.*

Proof. The family $\mathcal{F} = \{f_n\}_n$ is bounded so, by Montel's Theorem, every subsequence of f_n has a convergent sub-subsequence. This limit must be f and it is analytic. Suppose now that the whole sequence $\{f_n\}_n$ does not converge to f u.c.c. Then there exists $K \subset \Omega$ compact and $\varepsilon > 0$ such that for all n_0 there exists $n \geq n_0$ and some $z \in K$ such that $|f_n(z) - f(z)| \geq \varepsilon$. So we can take a subsequence $\left\{f_{n_j}\right\}_j$ and points z_j in K such that $\left|f_{n_j}(z_j) - f(z_j)\right| \geq \varepsilon$, $\forall j \geq 1$. Therefore, the sequence $\left\{f_{n_j}\right\}_j$ does not admit any uniformly convergent subsequence in K, which is a contradiction. $\square$

In section 6, we will work with entire functions, that is, functions that are holomorphic in the whole plane $\mathbb{C}$. As a consequence of Liouville's theorem, entire functions must have a singularity at ∞, either a pole or an essential singularity. The ones that have a pole at ∞ are polynomials and the ones that have an essential singularity are transcendental entire functions. Polynomials can be extended to holomorphic functions of $\widehat{\mathbb{C}}$ by defining ∞ to be the image of itself, whereas transcendental entire functions cannot be extended to $\widehat{\mathbb{C}}$. The two following theorems will be used when working with entire functions.

Theorem 1.7. (Picard) *Let f be an entire non-constant function. Then, $f(\mathbb{C}) = \mathbb{C}$ or $f(\mathbb{C}) = \mathbb{C} \setminus \{z_0\}$, for some z_0 in $\mathbb{C}$.*

Moreover, if f is transcendental, then for any neighbourhood U of ∞, $f(U) = \mathbb{C}$ or $f(U) = \mathbb{C} \setminus \{z_0\}$, for some z_0 in $\mathbb{C}$.

Theorem 1.8. *Let f be a transcendental entire function. Then, $f^2(z) - z$ has infinitely many zeros in $\mathbb{C}$.*

Finally, we define harmonic function and state a result that we will use in section 6.4.

Definition 1.9. If G is an open subset of $\mathbb{C}$, then a function $u \colon G \to \mathbb{R}$ is **harmonic** if u has continuous second partial derivatives and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Theorem 1.10. (Harnack's inequality) *Let V be an open subset of $\mathbb{C}$ and K a compact subset of V. Suppose that h is a harmonic positive function in V. Then, there exists some C depending only on V and K such that*

$$\sup_{x \in K} h(x) \leq C \inf_{x \in K} h(x).$$

2 CONFORMAL SELF-MAPS OF THE UNIT DISK.

Consider $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and let Ω_1, Ω_2 be open subsets of $\widehat{\mathbb{C}}$. We say f is a **conformal map** between Ω_1 and Ω_2 if $f : \Omega_1 \to \Omega_2$ is holomorphic and bijective.

The **Möbius Transformations** $T(z) = \dfrac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, are the only conformal maps of $\widehat{\mathbb{C}}$. It is easy to see that the composition and the inverse of Möbius Transformations are Möbius Transformations. Therefore, they form a group with respect to composition. One important property is that they map circles in $\widehat{\mathbb{C}}$ onto circles in $\widehat{\mathbb{C}}$, where a straight line in $\mathbb{C}$ is thought as a circle through ∞.

The conformal maps of $\mathbb{C}$ are the Möbius Transformations that fix ∞. Hence, c must be 0 and $T(z) = az + b$, with $a, b \in \mathbb{C}$ and $a \neq 0$. T can be thought as the composition of a rotation and a translation.

We are going to focus on the conformal self-maps of $\mathbb{D}$, that is:

Definition 2.1. $f \colon \mathbb{D} \to \mathbb{D}$ is a **conformal self-map of** $\mathbb{D}$ if it is holomorphic and bijective.

Notice that the conformal self-maps of $\mathbb{D}$ form a group with respect to the composition.

Now we are going to prove the Schwarz Lemma, a powerful tool to work with self-maps of $\mathbb{D}$ that fix the origin. As a corollary, we will obtain that the only conformal self-maps that fix the origin are rotations.

Theorem 2.2. (Schwarz Lemma) *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that $f(0) = 0$. Then:*

$$|f(z)| \leq |z| \quad \forall z \in \mathbb{D} \quad \textit{and} \quad \left|f'(0)\right| \leq 1.$$

The equality holds if and only if $f(z) = e^{i\theta} z$, for some θ.

Proof. Consider the function $g(z) = \dfrac{f(z)}{z}$ defined in $\mathbb{D} \setminus \{0\}$. Since $\lim_{z \to 0} g(z) = f'(0)$, g has a removable singularity in $z = 0$ and it can be extended to an holomorphic map $\widetilde{g}$ defined in $\mathbb{D}$.

Now we notice that:

$$|\widetilde{g}(z)| = |g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r} \quad \forall\, |z| = r,\ r \in (0,1)$$

By the Maximum Modulus Principe, this inequality holds for all z with $|z| \leq r$. Making $r \to 1$, $|\widetilde{g}(z)| \leq 1 \ \ \forall z \in \mathbb{D}$. Therefore, $|f(z)| \leq |z| \ \ \forall z \in \mathbb{D}$. As $f'(0) = \widetilde{g}(0)$ and $|\widetilde{g}(0)| \leq 1$, then $|f'(0)| \leq 1$.

Suppose now that $|f(z)| = |z|$ for some $z \in \mathbb{D}$. Then, $|\widetilde{g}(z)| = 1$ and by the Maximum Modulus Principe, $\widetilde{g}(z) = e^{i\theta}$, for some θ. The reciprocal is obvious.

□

Corollary 2.3. *Let $f \colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that $f(0) = 0$. Suppose f is not a rotation. Let K be a compact subset of $\mathbb{D}$. Then, there exists $k = k(K) \in (0,1)$ such that $|f(z)| \leq k\,|z|\,,\ \forall z \in K$.*

Proof. Consider the function $\widetilde{g}$ as defined in the proof of Schwarz lemma. $|\widetilde{g}|$ is a continuous function which takes positive real values so, by Weierstrass Theorem, $|\widetilde{g}|$ has a maximum, call it k. Clearly, $0 < k$. And $k < 1$ because, by Schwarz lemma, $|f(z)| < |z|$, so $|\widetilde{g}(z)| < 1$ for all z in K. □

Corollary 2.4. *The only conformal self-maps of $\mathbb{D}$ that fix the origin are the rotations, $f(z) = e^{i\theta}z$, with $0 \leq \theta \leq \pi$.*

Proof. Applying Schwarz lemma to both f and f^{-1}, we obtain $|f(z)| \leq |z|$ and $|z| \leq |f(z)|$, so $|f(z)| = |z|$. Therefore, $f(z) = e^{i\theta}z$, for some θ. $\square$

Now we are going to prove that there is an explicit formula for all the conformal self-maps of $\mathbb{D}$. With this formula, computing the fixed points of f and its dynamics will be quite easy.

Theorem 2.5. *The conformal self-maps of $\mathbb{D}$ are of the form:*

$$f(z) = e^{i\varphi} f_a(z) = e^{i\varphi} \frac{z-a}{1-\overline{a}z}$$

with $a \in \mathbb{D}$ and $0 \leq \varphi \leq 2\pi$.

Proof. Observe that f_a is a Möbius Transformation, so it sends circles to circles in $\mathbb{C}^*$. Let us prove that the unit circle is invariant under f_a:

$$\left|e^{i\theta} - a\right| = \left|\overline{e^{i\theta} - a}\right| = \left|e^{-i\theta} - \overline{a}\right| = \left|1 - \overline{a}e^{i\theta}\right|$$

So $|f_a(z)| = 1$, if $z = e^{i\theta}$. Therefore, $f_a(z)$ maps the unit circle onto itself and, since $f_a(a) = 0$, it maps $\mathbb{D}$ onto $\mathbb{D}$. Therefore, f_a is a conformal self-map of $\mathbb{D}$.

Now consider h any self-map of $\mathbb{D}$ and let $a \in \mathbb{D}$ be such that $h(a) = 0$. Then, consider $h \circ f_a^{-1}$. Since $h(f_a^{-1}(0)) = 0$, corollary 2.4 applies and $h(f_a^{-1}(w)) = e^{i\theta}w$, for some θ. Writing $w = f_a(z)$, h has the required form. $\square$

2.1 Iteration of conformal self-maps of $\mathbb{D}$.

With an explicit formula for the self-maps of $\mathbb{D}$, we are able to compute its fixed points. Notice that the conformal self-maps of $\mathbb{D}$ are well-defined in $\partial\mathbb{D}$, so it makes sense to refer to a fixed point of f in $\partial\mathbb{D}$.

Suppose $f \neq id$. The fixed points of f satisfy the equation $f(z) = z$, that is:

$$\overline{a}z^2 + (e^{i\varphi} - 1)z - ae^{i\varphi} = 0$$

If $a = 0$, the equation is $(e^{i\varphi} - 1)z = 0$, whose only solution is $z = 0$, provided we have supposed $f \neq id$, so $e^{i\varphi} \neq 1$. If $a \neq 0$, there are two solutions z_1, z_2 to the previous equation. They verify $z_1 z_2 = -\frac{a}{\overline{a}}e^{i\varphi} \in \partial\mathbb{D}$ (so $|z_1|\,|z_2| = 1$) and $\overline{a}z_1 + \overline{a}z_2 = 1 - e^{i\varphi}$. In addition,

$$f'(z_1)f'(z_2) = e^{2i\varphi}\frac{1-|a|^2}{(1-\overline{a}z_1)^2}\frac{1-|a|^2}{(1-\overline{a}z_2)^2} = e^{2i\varphi}\frac{(1-|a|^2)^2}{(1-\overline{a}z_1-\overline{a}z_2+\overline{a}\overline{a}z_1z_2)^2} = e^{2i\varphi}\frac{(1-|a|^2)^2}{(1-1+e^{i\varphi}-e^{i\varphi}\overline{a}a)^2} = 1$$

Therefore, one of the derivatives has modulus less or equal to 1.

This basic analysis shows us that there are three different situations:

1. f **is elliptic,** that is f has a unique fixed point z_0 in $\mathbb{D}$.
   If $z_0 = 0$, applying the Schwarz lemma, we obtain that f is a rotation with centre 0. So, $f(z) = e^{i\theta}\ \ \forall z \in \mathbb{D}$, for some θ. Then, the iterates of f are $f^n(z) = e^{i\theta n}z\ \ \forall z \in \mathbb{D}$.

   If $z_0 \neq 0$, let us conjugate f by the conformal map $f_{z_0}(z) = \dfrac{z-z_0}{1-\bar{z}_0 z}$. $\widetilde{f} = f_{z_0} \circ f \circ f_{z_0}^{-1}$ is a conformal self-map of $\mathbb{D}$ which maps 0 to 0. Then, $\widetilde{f}$ is a rotation with centre 0: $\widetilde{f}(z) = e^{i\theta}z$. The iterates of $\tilde{f}$ verify $\widetilde{f^n}(z) = e^{i\theta n}z$ and, since $f = f_{z_0}^{-1} \circ \widetilde{f} \circ f_{z_0}$, we obtain the iterates of f: $f = f_{z_0}^{-1}(e^{i\theta n} \cdot f_{z_0}(z))\ \ \forall z \in \mathbb{D}$.

2. f **is parabolic**, that is f has a unique fixed point $z_0 \in \mathbb{D}$. Using the conformal map $h(z) = \dfrac{z_0 + z}{z_0 - z}$ we can go over the right half-plane $\mathbb{H}$ as follows: $\widetilde{f} = h^{-1} \circ f \circ h$.

   Notice that h cuts $\partial\mathbb{D}$ at z_0 and maps $\partial\mathbb{D}$ to the axis $\{Im(w) = 0\}$, so that the opposed point of z_0 in $\partial\mathbb{D}$, $-z_0$, is mapped to 0. The points of $\mathbb{D}$ are mapped into $\mathbb{H}$ (see Figure 6).

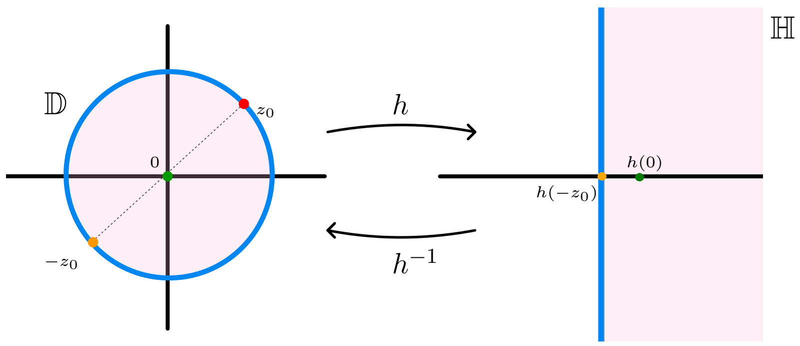


   Figure 6: How the conformal map h from $\mathbb{D}$ to $\mathbb{H}$.

   The map $\widetilde{f}$ is a conformal self-map of $\mathbb{H}$ which only fixes ∞, so it has the form $\widetilde{f}(z) = z + i\beta$, for some $\beta \in \mathbb{R}$. Notice that $\beta \neq 0$, because we are assuming $f \neq Id$. Iterating:

$$\widetilde{f}^n(z) = z + ni\beta \ \ \forall z \in \mathbb{H}, \quad \lim_{n\to\infty} \widetilde{f}(z) = \infty$$

   So, ∞ is the only fixed point and it attracts every point in $\mathbb{H}$. Returning to f, z_0 is the only fixed point of f in $\overline{\mathbb{D}}$ and, for all point in $\mathbb{D}$, their orbit will converge to z_0.

3. f **is hyperbolic,** that is f has two fixed points $z_1, z_2 \in \partial\mathbb{D}$. Using the conformal map $h(z) = \dfrac{z - z_1}{z - z_2}$, we can go over some half-plane $\mathbb{H}^*$: $\widetilde{f} = h^{-1} \circ f \circ h$.

   We see that h cuts $\partial\mathbb{D}$ at z_2 and extend to a straight line. This line will pass through 0, as it is the image of z_1.

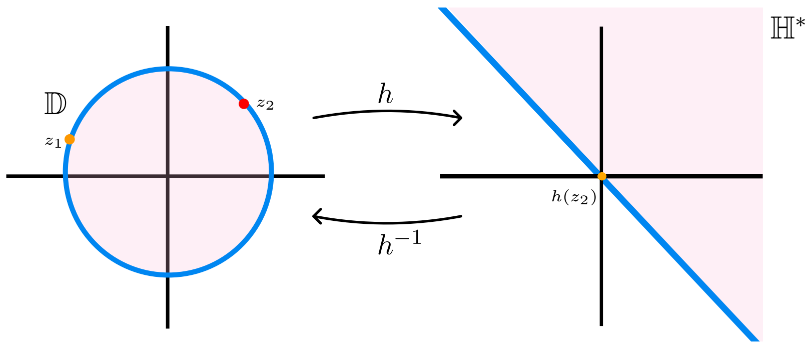


   Figure 7: How the conformal map h from $\mathbb{D}$ to $\mathbb{H}$.

   Hence, $\widetilde{f}$ is a conformal self-map of $\mathbb{H}^*$ which fixes 0 and ∞, so it must be $\widetilde{f}(z) = \alpha z$, for some $\alpha \in \mathbb{R}_+$. Moreover, $\alpha \neq 1$ because otherwise f would be the identity. If $\alpha < 1$, the point 0 attracts every point in $\mathbb{H}^*$, and if $\alpha > 1$, ∞ is the global attractor. Therefore, z_1 or z_2 is the global attractor for f in $\mathbb{D}$, depending on the case.

In each of the cases above, there is one point, call it a, such that $a \in \overline{\mathbb{D}}$ and $f^n(z) \to a$, for all z in $\mathbb{D}$. We will see that we have the same situation for any map $f\colon \mathbb{D} \to \mathbb{D}$, not only for the conformal ones. This is actually the central result in this project.

3 HYPERBOLIC GEOMETRY.

We are going now to define a new metric based on the condition that isometries must be precisely the conformal self-maps of $\mathbb{D}$. We call this new metric hyperbolic metric and we will see that it has many advantages with respect to the euclidean one when working in $\mathbb{D}$. For example, all the holomorphic self-maps of $\mathbb{D}$ will be contractive with respect to this new metric.

The contents of this section can be found in the book [17].

We define the hyperbolic length of a path as the line integral of some density $\rho(t)$ along this path. Then, we force ρ be such that the conformal self-maps are the isometries of this metric.

Definition 3.1. Let γ be a path in the unit disk $\mathbb{D}$ joining the points p and q. We define the **hyperbolic length** (or **ρ-length**) of this path as: $\rho(\gamma) = \int_\gamma \rho(t)\,|d(t)|$.

In the same way, we define the **hyperbolic distance** (or **ρ-distance**) from p to q as the infimum of $\rho(\gamma)$, taken over all possible path γ from p to q.

From now on, to avoid confusions, we will use $D(z_0, r)$ to refer to an euclidean disk of center z_0 and radius r, and $D_\rho(z_0, r)$ for the hyperbolic disk.

To get an explicit formula for the hyperbolic density ρ we are going to impose the conformal mappings to be isometries with this metric. If f is a conformal self-map of $\mathbb{D}$, then

$$\rho(f(t))\left|f'(t)\right| = \rho(t) \tag{3.1}$$

Now, as $f\colon \mathbb{D} \to \mathbb{D}$ is conformal, it must be $f(z) = e^{i\varphi}\dfrac{z-a}{1-\overline{a}z}$ with $a \in \mathbb{D}$ and $0 \le \varphi \le 2\pi$. Then, $f'(z) = e^{i\varphi}\dfrac{1-|a|^2}{(1-\overline{a}z)^2}$. Taking $z = 0$, we have $f(0) = e^{i\varphi}a$ and $|f'(0)| = 1-|a|^2$.

Applying (3.1) with $t = 0$, we obtain $\rho(e^{i\varphi}a) = \dfrac{\rho(0)}{1-|a|^2}$. To simplify the notation, we will take $\rho(0) = 1$. Notice that it only depend on the modulus of the point we are considering:

$$\rho(a) = \frac{1}{1-|a|^2} \tag{3.2}$$

From the formula of the $\rho-$density and the definition of the $\rho-$distance, we can get a formula to compute the hyperbolic distance between two points in $\mathbb{D}$. First, suppose that one of this points is the origin and the other is $p \neq 0$. Consider the straight segment γ_0 that joins 0 and p, $\gamma_0 = \{tp, 0 \le t \le 1\}$. Then, the $\rho-$length of γ_0 is:

$$\rho(\gamma_0) = \int_{\gamma_0} \rho(t)\,|dt| = \int_{\gamma_0} \frac{1}{1-t^2\,|p|^2}\,|p|\,dt = \frac{1}{2}\log\frac{1+|p|}{1-|p|}$$

Proposition 3.2. *$\gamma_0 = \{tp, 0 \le t \le 1\}$ is the shortest path between 0 and p and, therefore:*

$$\rho(0,p) = \frac{1}{2}\log\frac{1+|p|}{1-|p|} \tag{3.3}$$

Proof. Suppose that γ is any path joining 0 and p. Consider any partition $0 = t_1 < t_2 < ... < t_n = 1$ of the unit interval. Let P be the integral estimate of γ by the partition $\{t_i\}$:

$$P = \sum_{i=1}^{n-1} \rho(\gamma(t_i))\,|\gamma(t_{i+1}) - \gamma(t_i)|$$

Let us project radially the points $\gamma(t_i)$, so they have the same modulus but lie on γ_0, that is $\gamma_0^i = |\gamma(t_i)| \dfrac{p}{|p|}$. Observe that $\rho(\gamma(t_i)) = \rho(\gamma_0^i)$.

Joining γ_0^i to γ_0^{i+1} with straight segments, we obtain a path between 0 and p which may go back and forward over itself, but remains on γ_0. Then:

$$\rho(\gamma_0) \leq \sum_{i=1}^{n-1} \rho(\gamma_0^i) \left|\gamma_0^{i+1} - \gamma_0^i\right| \leq \sum_{i=1}^{n-1} \rho(\gamma(t_i)) \left|\gamma(t_{i+1}) - \gamma(t_i)\right| = P$$

Since the partition was arbitrary, $\rho(\gamma_0) \leq \rho(\gamma)$. $\square$

Now we are interested in getting a formula to compute the hyperbolic distance between any two points in $\mathbb{D}$. We are going to move them with an isometry, so that one of them is the origin. Then, we can compute the distance between them, which must be the same as between the original points.

Consider $a, w \in \mathbb{D}$ and $f_w(z) = \dfrac{z - w}{1 - \overline{w}z}$. Let s be such that $f_w(a) = \dfrac{a - w}{1 - \overline{w}a} = s$. Then,

$$\rho(a, w) = \rho(f_w(a), f_w(w)) = \rho(s, 0) = \frac{1}{2} \log \frac{1 + |s|}{1 - |s|} = \frac{1}{2} \log \frac{1 + \left|\frac{w-a}{1-\overline{w}a}\right|}{1 - \left|\frac{w-a}{1-\overline{w}a}\right|}.$$

Therefore, for any two points $a, w \in \mathbb{D}$ we have:

$$\rho(a, w) = \frac{1}{2} \log \frac{|1 - \overline{a}w| + |w - a|}{|1 - \overline{a}w| - |w - a|}. \tag{3.4}$$

3.1 Properties of the hyperbolic metric.

Our next goal is to show that the distance defined is, in fact, a distance. Moreover, we prove that $\mathbb{D}$ with this metric is a complete metric space. Notice that $\mathbb{D}$ with the euclidean metric is not a complete metric space.

Proposition 3.3. *$\mathbb{D}$ with the hyperbolic metric ρ is a complete metric space.*

Proof. First we must check that $\rho\colon \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ defines a distance in $\mathbb{D}$, which is straightforward from the definition that ρ is symmetric.

Indeed, by formula (3.4), ρ is non-negative and it is zero only when the inside of the logarithm is 1, that is, when $w = a$.

The triangular inequality comes from: $\rho(p, q) = \inf_\gamma \leq \inf_{\gamma_r} = \rho(p, r) + \rho(r, q)$, where γ denotes any curve joining p and q and γ_r denotes any curve joining p and q through r.

It is left to see that any Cauchy sequence is covergent. Let $\{z_n\}_n \subset \mathbb{D}$ be a Cauchy sequence. As $\overline{\mathbb{D}}$ is compact, there exists a convergent subsequence with respect to the euclidean metric: $z_{n_k} \to z$, for some $z \in \overline{\mathbb{D}}$. If $z \in \partial\mathbb{D}$, then $\rho(0, z_{n_k}) \to \infty$. But, since $\{z_n\}$ is a Cauchy sequence, $\rho(0, z_{n_k})$ must be bounded. Therefore, $z \in \mathbb{D}$.

It remains to see that $\{z_{n_k}\}_k$ converges to z also with the hyperbolic metric. Indeed, using that $|z_{n_k} - z| \to 0$,

$$\rho(z_{n_k}, z) = \frac{1}{2} \log \frac{|1 - \overline{z}z_{n_k}| + |z - z_{n_k}|}{|1 - \overline{z}z_{n_k}| - |z - z_{n_k}|} \longrightarrow 0$$

Since any Cauchy sequence with a convergent subsequence is convergent, $\{z_n\}_n$ converges to p with respect to the hyperbolic metric. $\square$

Proposition 3.4. *The set of hyperbolic circles coincides with the set of euclidean circles in* $\mathbb{D}$.

Proof. First, consider the hyperbolic circle with center 0 and radius r:

$$C_\rho(0,r) = \{z \in \mathbb{D} : \rho(0,z) = r\}.$$

This is, by formula (3.3), an euclidean disk $C(0,r')$ with center 0 and euclidean radius $r' = \dfrac{e^{2r}-1}{e^{2r}+1}$.

Recall that the isometries with respect to the hyperbolic metric are the Möbius transformations of $\mathbb{D}$, so they send euclidean circles to euclidean circles.

So, if C is any hyperbolic circle $C_\rho(w,r)$, consider an isometry f with $f(w) = 0$. Then $f(C_\rho(w,r)) = C_\rho(0,r)$, so it is an euclidean circle $C(0,r')$ for some r'. Hence, $C_\rho(w,r) = f^{-1}(C(0,r'))$ is an euclidean circle.

Conversely, suppose that $C(w,r)$ is an euclidean circle. Consider a hyperbolic isometry f such that $f(w) = 0$. Then, $f(C(w,r)) = C(0,r')$ for another radius r'. Hence, $C(0,r') = C_\rho(0,r'')$, and $C(w,r) = f^{-1}(C_\rho(0,r''))$ is a hyperbolic circle. □

As the last proposition is true for open disks (provided that their closure lies in $\mathbb{D}$), we have that the set of open hyperbolic disks coincides with the set of euclidean disks in $\mathbb{D}$. Therefore, the topology induced by the hyperbolic metric coincides with the one induced by the euclidean metric.

So far we have built a metric in $\mathbb{D}$, whose isometries are the conformal self-maps of $\mathbb{D}$. Our goal is to transfer this metric to any simply connected open subset of $\mathbb{C}$. Indeed, let U be a simply connected open subset of $\mathbb{C}$, different from $\mathbb{C}$. By the Riemann Mapping Theorem, there is a conformal map $\varphi\colon U \to \mathbb{D}$. Hence, we define:

$$\rho_U(z,w) = \rho(\varphi(z),\varphi(w)),\ \ z,w \in U.$$

Clearly, it is invariant under conformal self-maps of U. Indeed, f is a conformal self-map of U if and only if $g = \varphi \circ f \circ \varphi^{-1}$ is a conformal self-map of $\mathbb{D}$ and,

$$\rho_U(f(z),f(w)) = \rho(\varphi(f(z)),\varphi(f(w))) = \rho(g(\varphi(z)),g(\varphi(w))) = \rho(\varphi(z),\varphi(w)) = \rho_U(z,w).$$

Lemma 3.5. *Let* $\{z_n\}_n \subset U$ *such that* $z_n \to \partial U$*, with respect to the euclidean metric. Then, fixed* $r > 0$*, the euclidean diameter of* $D_{\rho_U}(z_n,r)$ *tends to* 0.

Proof. Suppose first that $U = \mathbb{D}$. Since $\{z_n\}_n$ escapes from all K compact subset of $\mathbb{D}$, we can find n such that $\rho(K,z_n) \geq r$. Then, $D_\rho(z_n,r) \not\subset K$. Repeating the same argument for $K = \overline{D(0,r)}$, with $r \to 1$, we get that the euclidean diameter of $D(z_n,r)$ must tend to 0 when n tends to ∞.

Let us consider $U \neq \mathbb{D}$ and consider the Riemann Mapping $\varphi\colon U \to \mathbb{D}$. Then, $\{\varphi(z_n)\}_n$ is a sequence that escapes from all compact subset of $\mathbb{D}$, so the euclidean diameter of $D_\rho(\varphi(z_n),r)$ must tend to 0 when n tends to ∞. Therefore, the euclidean diameter of $D_{\rho_U}(z_n,r)$ also tends to 0. □

Finally notice that, the formula (3.4) is an increasing function of

$$d_{\mathbb{D}}(z,w) := \left|\frac{w-z}{1-\overline{w}z}\right|$$

We call $d_{\mathbb{D}}$ the **pseudo-hyperbolic distance** in $\mathbb{D}$. It is easier to work with this formula and we don not lose the information of which points are closer or further. We do lose properties, for example, it does not satify the triangular inequality, so it is not actually a distance. However, it is symmetric and, for all $z, w \in \mathbb{D}$, $0 \leq d_{\mathbb{D}}(z, w) < 1$. Moreover, it is equal to zero if and only if $z = w$ and it is equal to one if one point lies in $\partial\mathbb{D}$.

By the process described above, we can define the pseudo-hyperbolic distance Since the right half-plane $\mathbb{H}$ and $\mathbb{D}$ are conformally equivalent, we can think in the pseudo-hyperbolic distance of two points in $\mathbb{H}$ as the distance between their corresponding points in $\mathbb{D}$. Then,

$$d_{\mathbb{H}}(z, w) := d_{\mathbb{D}}\left(\frac{z-1}{z+1}, \frac{w-1}{w+1}\right) = \frac{|z-w|}{|\overline{z}+w|}$$

is the formula for the pseudo-hyperbolic distance in $\mathbb{H}$. We will use $D_{\mathbb{D}}(z_0, r)$ and $D_{\mathbb{H}}(z_0, r)$ to refer to disks with the pseudo-hyperbolic distance in $\mathbb{D}$ and $\mathbb{H}$ respectively.

3.2 The Schwarz-Pick Lemma.

Now we are going to prove one of the most powerful tools when working with holomorphic self-maps of $\mathbb{D}$: the Schwarz-Pick lemma. It says that all holomorphic self-maps of $\mathbb{D}$ are contractions with respect to the hyperbolic metric. Moreover, they are strictly contractive if they are not conformal.

Theorem 3.6. (Schwarz-Pick Lemma) *Let $f : \mathbb{D} \to \mathbb{D}$ be an holomorphic function. Then, f is both an infinitesimal and a global contraction with respect to the hyperbolic metric. That is,*

$$\rho(f(t)) \cdot |f'(t)| \leq \rho(t) \quad \forall t \in \mathbb{D}, \quad \textit{and} \quad \rho(f(z), f(w)) \leq \rho(z, w) \quad \forall z, w \in \mathbb{D}.$$

Proof. Consider $g(z) = h(f(h_0(z)))$, where $h(z) = \dfrac{z - f(t)}{1 - \overline{f(t)}z}$ and $h_0(z) = \dfrac{z+t}{1+\overline{t}z}$. Notice that $h(f(t)) = 0$ and $h_0(0) = t$. Hence $g(0) = 0$, then Schwarz lemma applies and $|g'(0)| < 1$.

By the chain rule $g'(0) = h'(f(t) \cdot f'(t) \cdot h_0'(0)$, where $h'(f(t)) = \dfrac{1}{1 - |f(t)|^2}$ and $h_0'(0) = 1 - |t|^2$.

Thus, $|g'(0)| = \left| f'(t) \cdot \dfrac{1-|t|^2}{1-|f(t)|^2} \right| = \left| f'(t) \cdot \dfrac{\rho(f(t))}{\rho(t)} \right| \leq 1$ and, therefore, $\rho(f(t)) \cdot |f'(t)| \leq \rho(t)$.

It remains to see that f is a global contraction. We take $z, w \in \mathbb{D}$ and γ the shortest path between z and w. Then $f(\gamma)$ is a path joining $f(z)$ and $f(w)$, probably not the shortest one. So,

$$\rho(f(z), f(w)) \leq \rho(f(\gamma)) = \int_{f(\gamma)} \rho(t)\,|dt| = \int_{\gamma} \rho(f(t))\,|f'(t)|\,|dt| \leq \int_{\gamma} \rho(t)\,|dt| = \rho(\gamma) = \rho(z, w)$$

where the last inequality is due to the infinitesimal bound. □

Theorem 3.7. (Isometries) *Let $z, w \in \mathbb{D}$ be such that $\rho(f(z), f(w)) = \rho(z, w)$. Then, f is a conformal self-map of $\mathbb{D}$ and equality holds for all $z, w \in \mathbb{D}$.*

Proof. Consider $g(t) = h_1(f(h_0(t)))$, where $h_0(t) = \dfrac{z+t}{1+\overline{z}t}$ and $h_1(t) = \dfrac{t - f(z)}{1 - \overline{f(z)}t}$. Notice that $g(0) = 0$. Let us set $w_0 = \dfrac{w-z}{1-\overline{z}w}$. Then $h_0(w_0) = w$ and $g(w_0) = \dfrac{f(w) - f(z)}{1 - \overline{f(z)}f(w)}$.

By formula (3.4), we obtain that $\rho(f(z), f(w)) = \rho(z, w)$ is equival to $\left| \dfrac{w-z}{1-\overline{z}w} \right| = \left| \dfrac{f(w)-f(z)}{1-\overline{f(z)}f(w)} \right|$. Hence, $|w_0| = |g(w_0)|$. By Schwarz lemma, g is a rotation, and therefore, f is conformal. □

Hence, we have proved that $f\colon \mathbb{D} \to \mathbb{D}$ is a hyperbolic isometry if and only if it is a conformal self-map of $\mathbb{D}$.

Corollary 3.8. (Strict contractions) *Let K be a compact subset of $\mathbb{D}$ and $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic but not conformal. Then, there exists $k = k(K)$, $k \in (0,1)$ such that:* $\rho(f(z), f(w)) \leq k\rho(z,w) \quad \forall z, w \in K$.

Proof. It is enough to prove it for compact sets of the form $K = \overline{D(0,r)}$, $r \in (0,1)$, because any compact K' can be put inside a closed ball.

Let us fix $z_1, z_2, z_3 \in K$. We can take $\varepsilon > 0$ such that $D(z_i, \varepsilon)$ are pairwise disjoint and consider, for $i = 1,2,3$, the functions $f_i\colon K \smallsetminus D(z_i,\varepsilon) \to \mathbb{R}$, defined by $f_i(z) = \dfrac{\rho(f(z_i), f(z))}{\rho(z_i, z)}$.

For each i, the function is well-defined and continuous, since the denominator is never zero. Moreover, it is positive and, by Schwarz-Pick lemma, $f_i(z) < 1 \quad \forall z \in K \smallsetminus D(z_i,\varepsilon)$. As $K \smallsetminus D(z_i,\varepsilon)$ is a compact subset of $\mathbb{D}$, by Weierstrass Theorem, there exists $k_i < 1$ the maximum of f_i.

Now take $k = \max k_i < 1$. We claim that $\forall z, w \in K$, $\rho(f(z), f(w)) \leq k\rho(z,w)$. If $z, w \notin D(z_i,\varepsilon)$ for any i, it is clear. Suppose now that $z \in D(z_1,\varepsilon)$ and $w \in D(z_2,\varepsilon)$. As $D(z_i,\varepsilon)$ are pairwise disjoint, then $z, w \notin D(z_3,\varepsilon)$ and $\dfrac{\rho(f(z_i), f(z))}{\rho(z_i,z)} < k_3 < k$. □

Note that at this point we can already prove convergence of the iterates to any fixed point in $\mathbb{D}$ if this exists.

Corollary 3.9. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic but not conformal and suppose there exists $a \in \mathbb{D}$, a fixed point of f. Then, for all $z \in \mathbb{D}$, $f^n(z) \to a$.*

Proof. We can suppose that $a = 0$, otherwise we can conjugate f by a Möbius transformation that brings a to 0. Fixed any point $z \in \mathbb{D}$, consider $K = \overline{D(0,r)}$ compact subset of $\mathbb{D}$ such that $z \in K$. Then, by Corollary 2.3 to Schwarz lemma, $f(K) \subset K$ and there exist $k \in (0,1)$ such that $|f(z)| \leq k\,|z|$. Iterating f, we get that $|f^n(z)| \leq k^n\,|z|$. Therefore, $f^n(z) \to 0$, when $n \to \infty$, as desired. □

4 THE DENJOY-WOLFF THEOREM.

In previous sections, we proved that if f is a conformal self-map of $\mathbb{D}$, but not elliptic, all orbits converge to the same fixed point. We also proved this result for self-maps not necessarily conformal with a fixed point in $\mathbb{D}$. Our goal in this section is to prove the Denjoy-Wolff Theorem, which asserts this is true for every holomorphic self-map of $\mathbb{D}$.

Theorem 4.1. (Denjoy-Wolff Theorem) *Let f be an holomorphic self-map of $\mathbb{D}$, that is not an elliptic automorphism. Then, there is $a \in \overline{\mathbb{D}}$, such that $\forall z \in \mathbb{D}$, $f^n(z) \to a$.*

*The point a is called the **Denjoy-Wolff point** of f.*

The original proofs due to A. Denjoy and J. Wolff can be found in [11] and [25]. However, we follow the ideas of a modern proof given by A. Beardon, for which we refer to [9] and [21].

Recall that, by Corollary 1.6 to Montel's Theorem, pointwise convergence implies uniform convergence on compact subsets. Therefore, the sequence of iterates f^n converges to a uniformly on compact subsets of $\mathbb{D}$.

As mentioned before, we know this result for conformal maps and for those which have a fixed point in $\mathbb{D}$. Now we are going to focus on those which are not conformal and have no fixed point in $\mathbb{D}$. We remark that we are not assuming that f has a continuous extension to $\partial\mathbb{D}$.

We need two preliminary lemmas.

Lemma 4.2. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function, not conformal. Let $w \in \mathbb{D}$ be a point such that $f^n(w) \to a \in \overline{\mathbb{D}}$. Then, for all z in $\mathbb{D}$, $f^n(z) \to a$.*

In other words, if the orbit of one point converges, all orbits have the same limit.

Proof. If $a \in \mathbb{D}$, it is an immediate consequence of Corollary 3.9. Hence, suppose $a \in \partial\mathbb{D}$. Set $\varepsilon > 0$. Since $\lim_n f^n(w) = a$, there exists n_1 such that $\forall n \geq n_1$, $|f^n(w) - a| \leq \frac{\varepsilon}{2}$.

On the other hand, for $z \in \mathbb{D}$ arbitrary there exists some r such that $z \in D_\rho(w, r)$. By Schwarz-Pick lemma, for all $n \geq 1$, $f^n(z) \in D_\rho(f^n(w), r)$. Since $f^n(w) \to a \in \partial\mathbb{D}$, by lemma 3.5 $|f^n(z) - f^n(w)| \to 0$. Therefore, there exists some n_2 such that $\forall n \geq n_2$, $|f^n(z) - f^n(w)| \leq \frac{\varepsilon}{2}$.

Finally, if $n_0 = \max\{n_1, n_2\}$, for all $n \geq n_0$

$$|f^n(z) - a| \leq |f^n(z) - f^n(w)| + |f^n(w) - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $f^n(z) \to a$, as desired. □

We remark that this lemma is also true for any simply connected open subset, since lemma 3.5 was proven for any simply connected open subset.

Lemma 4.3. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be analytic, but not conformal. Then, either $f^n(z) \to \partial\mathbb{D}$ $\forall z \in \mathbb{D}$ or there is a fixed point $z_0 \in \mathbb{D}$.*

Proof. Suppose there is a point $w \in \mathbb{D}$ such that $f^n(w) \nrightarrow \partial\mathbb{D}$. Then there is a compact subset $L \subset \mathbb{D}$ such that $f^n(a) \in L$ for infinitely many n. Now we take a compact subset K with $L \cup f(L) \subset K \subset \mathbb{D}$. As K is compact, there will be a constant $k = k(K) < 1$ such that $\rho(f(z_1), f(z_2)) \leq k\rho(z_1, z_2)$, for all $z_1, z_2 \in K$. As $L \cup f(L) \subset K$ and $f^{n+1} = f \circ f^n$, we have:

$$0 \leq \rho(f^{n+2}(w), f^{n+1}(w)) \leq k\rho(f^{n+1}(w), f^n(w)), \text{ for all } n \text{ such that } f^n(w) \in L.$$

As $\rho(f(z), f(w)) < \rho(z,w)$ for all $z, w \in \mathbb{D}$, we have that

$$0 \leq \rho(f^{n+1}(w), f^n(w)) \leq k^m \rho(f(w), w),$$

where $m \leq n$ is the number of times that the iteration lies in L. Making n tend to ∞, m tends to ∞ too, so $\rho(f^{n+1}(w), f^n(w)) \to 0$. Therefore, $f^n(w)$ and $f^{n+1}(w)$ will have the same limit when n tends to ∞.

As $f^n(w)$ lies in K for infinitely many $n's$, we can take a subsequence $\{f^{n_k}(w)\}_k \subset K$. Since K is compact we can suppose it convergent (if not, we can take a sub-subsequence), so $\lim\limits_{k\to\infty} f^{n_k}(w) = a$, for some $a \in K$.

Then, as f is continuous and f^{n_k} and f^{n_k+1} have the same limit:

$$a = \lim_{k\to\infty} f^{n_k+1}(w) = f(\lim_{k\to\infty} f^{n_k}(w)) = a$$

So, $a \in K \subset \mathbb{D}$ is a fixed point and the lemma is proved. $\square$

4.1 The proof of the Denjoy-Wolff Theorem.

As f is not conformal and has no fixed points in $\mathbb{D}$ then, by lemma 4.2, $\forall z \ f^n(z) \to \partial\mathbb{D}$. It remains to be shown that the orbit of 0 accumulates in one unique limit point in $\partial\mathbb{D}$.

For $\varepsilon \in (0,1)$, let us consider:

$$\begin{aligned} f_\varepsilon \colon \mathbb{D} &\to \mathbb{D} \\ z &\mapsto (1-\varepsilon)\cdot f(z) \end{aligned}$$

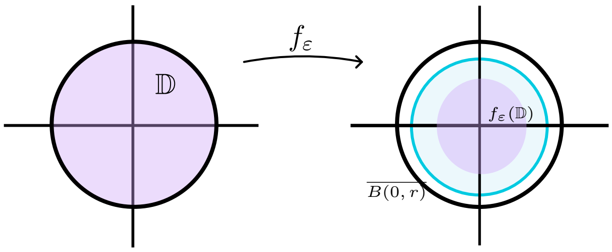


Figure 8: How f_ε maps $\mathbb{D}$ into a compact disk $\overline{B(0,r)}$.

This map is holomorphic but not conformal. Therefore, f_ε is a contraction with respect to the hyperbolic metric: $\rho(f_\varepsilon(z), f_\varepsilon(w)) < \rho(z,w)$, $\forall z, w \in \mathbb{D}$.

We can suppose $f_\varepsilon(\mathbb{D}) \subset \overline{D(0,r)}$, for some $r \in (0,1)$, where $\overline{D(0,r)}$ is the euclidean closed disk with centre 0 and radius r. Then $f_\varepsilon(\overline{D(0,r)}) \subset \overline{D(0,r)}$. So, we can consider the function restricted to $\overline{D(0,r)}$, which is compact. The Brouwer fixed-point theorem[2] applies and f_ε has some fixed point z_ε in $\overline{D(0,r)}$, but it must be in $f_\varepsilon(\mathbb{D})$.

Remark. *The fixed point of f_ε is unique.*
Proof. Suppose that f_ε has two fixed points $z_\varepsilon, z'_\varepsilon \in \mathbb{D}$, then $\rho(z_\varepsilon, z'_\varepsilon) = \rho(f_\varepsilon(z_\varepsilon), f_\varepsilon(z'_\varepsilon)) < \rho(z_\varepsilon, z'_\varepsilon)$, a contradiction. $\square$

Let us consider a sequence $\{\varepsilon_k\}_k$ such that $\varepsilon_k \to 0$ when $k \to \infty$. Let $\{z_{\varepsilon_k}\}_k$ be the associated sequence of fixed points and $a \in \overline{\mathbb{D}}$ any limit point of this sequence. There exists at least one of them, because $\overline{\mathbb{D}}$ is compact.

[2]See [8]

Remark. *If $a \in \mathbb{D}$, then a is a fixed point of f.*

Proof. As f is continuous in $\mathbb{D}$ and z_{ε_k} is a fixed point of f_{ε_k}:

$$a = \lim_{k\to\infty} z_{\varepsilon_k} = \lim_{k\to\infty} f_{\varepsilon_k}(z_{\varepsilon_k}) = \lim_{k\to\infty}(1-\varepsilon_k)f(z_{\varepsilon_k}) = f(\lim_{k\to\infty} z_{\varepsilon_k}) = f(a)$$

Therefore, $a \in \partial\mathbb{D}$, as we assumed that f had no fixed points in $\mathbb{D}$. □

For each ε, let us define: $D_\varepsilon = \{w \in \mathbb{D} \colon \rho(z_\varepsilon, w) < \rho(0, z_\varepsilon)\}$, the hyperbolic disk with centre z_ε and radius $\rho(0, z_\varepsilon)$. We observe that $0 \in \partial D_\varepsilon$. By proposition 3.4, D_ε is also an euclidean disk, with probably another centre and radius.

Remark. *The disks D_ε are invariant under f_ε, that is, $f_\varepsilon(D_\varepsilon) \subset D_\varepsilon$.*

Proof. As f is contractive with respect to the hyperbolic metric, $\rho(f_\varepsilon(w), z_\varepsilon) = \rho(f_\varepsilon(w), f_\varepsilon(z_\varepsilon)) < \rho(w, z_\varepsilon)$, for all $w \in \mathbb{D}$. □

Taking the previous sequence $\{\varepsilon_k\}_k$, we can build an associated sequence of euclidean disks $\{D_{\varepsilon_k}\}_k$. Any limit disk D of this sequence will be an euclidean disk, with $0 \in \partial D$, invariant under f an tangent to $\partial\mathbb{D}$. Observe that D has only one tangent point to $\partial\mathbb{D}$, or otherwise, $D = \mathbb{D}$ and $0 \notin \partial D$.

Remark. *D is unique.*

Proof. Suppose that this is not the case, so there exist D_1 and D_2 such that they are distinct, tangent to $\partial\mathbb{D}$ (at diferent points), $0 \in \partial D_1 \cap \partial D_2$, $f(D_1) \subset D_1$ and $f(D_2) \subset D_2$.

There are two cases.

a) If $\overline{D_1} \cap \overline{D_2} = \{0\}$. As $0 \in \overline{D_1} \cap \overline{D_2}$, there exist $\{w_n\} \subset D_1$ such that $w_n \to 0$ and $\{z_n\} \subset D_2$ such that $z_n \to 0$. Using that f is continuous in $\mathbb{D}$, $f(w_n) \to f(0)$ and $f(z_n) \to f(0)$, so $f(0) \in \overline{D_1} \cap \overline{D_2} = \{0\}$. Thus, 0 is a fixed point of f in $\mathbb{D}$, a contradiction.

b) If $\overline{D_1} \cap \overline{D_2} \neq \{0\}$. Then $\overline{D_1} \cap \overline{D_2} \subset \mathbb{D}$ is compact, connected and invariant under f. By the Brouwer fixed-point theorem, f has a fixed point in $\overline{D_1} \cap \overline{D_2}$, a contradiction.

□

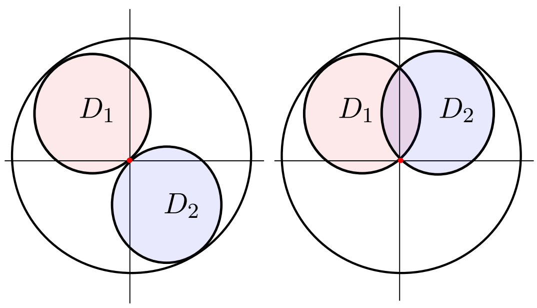


Figure 9: The two cases that appear supposing D is not unique.

Now, let a be the point such that $\{a\} = \partial\mathbb{D} \cap \overline{D}$. Since $0 \in \overline{D}$ and $f^n(D) \subset D$, $f^n(0) \in \overline{D}$, for all n. We know that the orbit of 0 accumulates in $\partial\mathbb{D}$, but $\{a\} = \partial\mathbb{D} \cap \overline{D}$. Therefore, $f^n(0) \to a$, when $n \to \infty$, and by lemma 4.2 the proof is finished.

4.2 Limit at the Denjoy-Wolff point.

If the Denjoy-Wolff point a is in $\mathbb{D}$, f is well-defined and holomorphic at a. Suppose now that $a \in \partial\mathbb{D}$. We are interested in computing the limit (and later, the derivative) in some sense of f at a. We cannot consider the usual definition of limit, because the function may not be defined in $\partial\mathbb{D}$. Therefore, we are going to consider the radial limit (which is taken approaching a from the radial line) and the angular limit (which is taken approaching a from a so-called Stolz angle).

Definition 4.4. A **Stolz angle** at $a \in \partial\mathbb{D}$ is a region:

$$\Delta = \{z \in \mathbb{D} \colon \ |Arg(1 - \overline{a}z)| < \alpha, \quad |z - a| < \rho\}$$

with $0 < \alpha < \frac{\pi}{2}$ and $\rho < \cos 2\alpha$.

The exact shape of this region is irrelevant. The important fact is that all the points in Δ are at a finite hyperbolic distance of the radial segment $[0, a]$.

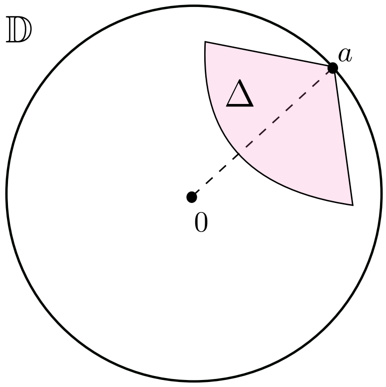


Figure 10: Example of a Stolz angle at a.

Definition 4.5. Let $f \colon \mathbb{D} \to \widehat{\mathbb{C}}$. Then, f has **radial limit** $b \in \widehat{\mathbb{C}}$ at $a \in \mathbb{D}$ if $\lim\limits_{r \to 1^-} f(ar) = b$. Moreover, if $\lim\limits_{z \to a, \ z \in \Delta} f(z) = b$, for all Δ Stolz angle at a, we say that f has **angular limit** $b \in \widehat{\mathbb{C}}$ at $a \in \mathbb{D}$. Then we write $f(a) = b$.

The following result shows that, for bounded functions, if the radial limit exists, the angular limit also exists and they are equal.

Theorem 4.6. *Let f be a bounded analytic function in $\mathbb{D}$. Then, if $\lim\limits_{r \to 1^-} f(ar) = b$, then f has angular limit b at a.*

Proof. The proof can be found in [22]. □

The following lemma will be useful to compute the radial limit in the Denjoy-Wolff point. The original proof can be found in [26].

Theorem 4.7. (Wolff Lemma) *Let $f \colon \mathbb{D} \to \mathbb{D}$ be holomorphic without fixed points in $\mathbb{D}$. Let $a \in \partial\mathbb{D}$ be the Denjoy-Wolff point of f.*

Then, if $D \subset \mathbb{D}$ is a disk tangent to $\partial\mathbb{D}$ at a, $f(D) \subset D$. That is,

$$\frac{1 - |z|^2}{|a - z|^2} \leq \frac{1 - |f(z)|^2}{|a - f(z)|^2}, \quad \forall z \in \mathbb{D}. \tag{4.1}$$

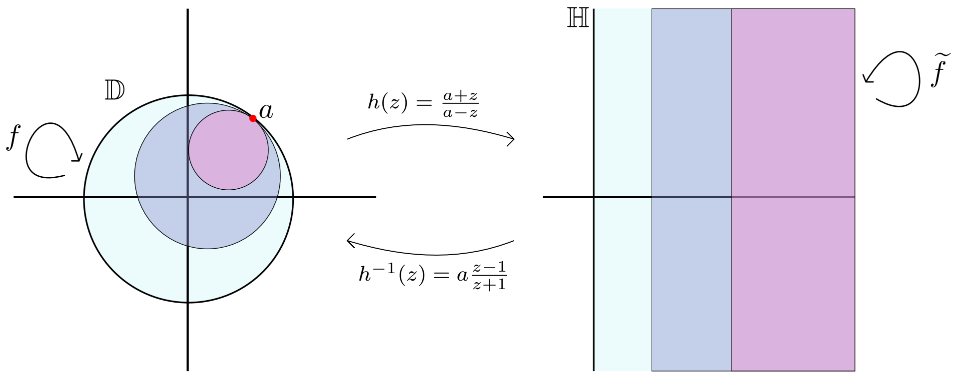


Figure 11: Tangent disks at a in $\mathbb{D}$ are equivalent to vertical straight lines in $\mathbb{H}$.

Before starting the proof of Theorem 4.7, we are going to see that (4.1) is equivalent to $f(D) \subset D$, for all disks $D \in \mathbb{D}$ tangent to $\partial\mathbb{D}$ at a. It is easier to deal with this problem in the right half-plane $\mathbb{H}$. Consider the conformal map $h(z) = \dfrac{a+z}{a-z}$ from $\mathbb{D}$ to $\mathbb{H}$, mapping the Denjoy-Wolff point a to ∞. Let $\widetilde{f} = h \circ f \circ h^{-1}$ be the corresponding self-map of $\mathbb{H}$. It is clear that a tangent disk to $\partial\mathbb{D}$ at a goes to a vertical straight line in $\mathbb{H}$. Therefore, $f(D) \subset D$, for all disks $D \in \mathbb{D}$ tangent to $\partial\mathbb{D}$ at a is equivalent to $Re(w) \leq Re(\widetilde{f}(w))$, for all $w \in \mathbb{H}$.

Let $w = h(z) \in \mathbb{H}$, for some $z \in \mathbb{D}$. Then, $Re(w) = \frac{1-|z|^2}{|a-z|^2}$ and

$$Re(\widetilde{f}(w)) = Re(\widetilde{f}(h(z))) = Re(h(f(z))) = \frac{1-|f(z)|^2}{|a-f(z)|^2}.$$

Hence the equivalence is proven.

Proof of Theorem 4.7. As in the proof of the Denjoy-Wolff Theorem, let $f_\varepsilon(z) = (1-\varepsilon)f(z)$ and $z_\varepsilon \in \mathbb{D}$ be the fixed point of f_ε. Consider some euclidean disk D tangent to $\partial\mathbb{D}$ at the Denjoy-Wolff point a. Hence, there is some $\delta \in (0,1)$ such that $D = D(\delta a, 1-\delta)$, where we have used that $\delta\,|a| = 1$. Taking $z \in \mathbb{D} \cup \partial D$, we have $|z - \delta a| = 1 - \delta$, so we are able to compute explicitly the value of δ: $\delta = \dfrac{1-|z|^2}{2 - 2Re(\overline{a}z)}$.

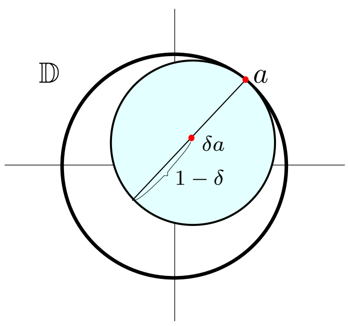


Figure 12: Schematic representation of D in terms of δ.

Now, for each z_ε, consider $h_\varepsilon(z) = \dfrac{z - z_\varepsilon}{1 - \overline{z_\varepsilon}z}$. Let $\tilde{f}_\varepsilon = h_\varepsilon \circ f_\varepsilon \circ h_\varepsilon^{-1}$. Since $\tilde{f}_\varepsilon$ is a self-map of $\mathbb{D}$ and $\tilde{f}_\varepsilon(0) = 0$, Schwarz lemma applies and $\left|\tilde{f}_\varepsilon(z)\right| \leq |z| \quad \forall z \in \mathbb{D}$. Applying this inequality to the point $h_\varepsilon(z) \in \mathbb{D}$, we have $|h_\varepsilon(f_\varepsilon(z))| \leq \left|\dfrac{z - z_\varepsilon}{1 - \overline{z_\varepsilon}z}\right|$.

Hence, $h_\varepsilon(f_\varepsilon(z))$ lies in the euclidean disk of radius $\rho_\varepsilon = \left|\frac{z - z_\varepsilon}{1 - \overline{z_\varepsilon} z}\right|$ and center the origin. Therefore, $f_\varepsilon(z)$ is in the disk $h_\varepsilon(\{w \in \mathbb{D} \colon \ |w| \le \rho_\varepsilon\})$. An easy calculation shows that this is the disk of euclidean center $c(\varepsilon) = \frac{1-\rho_\varepsilon^2}{1-\rho_\varepsilon^2 |z_\varepsilon|^2} z_\varepsilon$ and radius $R(\varepsilon) = \frac{1-|z_\varepsilon|^2}{1-\rho_\varepsilon^2|z_\varepsilon|^2}\rho_\varepsilon$.

Computing explicitly the center and the radius and taking limits when $\varepsilon \to 0$:

$$1 - \rho_\varepsilon^2 |z_\varepsilon|^2 = \frac{(1-|z_\varepsilon|^2)(1+|z_\varepsilon|^2 - 2Re(\overline{z_\varepsilon} z))}{|1-\overline{z_\varepsilon} z|^2}$$

$$c(\varepsilon) = \frac{1-\rho_\varepsilon^2}{1-\rho_\varepsilon^2|z_\varepsilon|^2} z_\varepsilon = \frac{1-|z|^2}{1+|z_\varepsilon|^2 - 2Re(\overline{z_\varepsilon} z)} \xrightarrow[\varepsilon\to 0]{} \frac{1-|z|^2}{2-2Re(\overline{a} z)} a = \delta a$$

$$R(\varepsilon) = \frac{1-|z_\varepsilon|^2}{1-\rho_\varepsilon^2|z_\varepsilon|^2}\rho_\varepsilon = \frac{|1-\overline{z_\varepsilon} z|^2}{1+|z_\varepsilon|^2 - 2Re(\overline{z_\varepsilon} z)}\left|\frac{z-z_\varepsilon}{1-\overline{z_\varepsilon} z}\right| \xrightarrow[\varepsilon\to 0]{} \frac{|z-a|^2}{2-2Re(\overline{a} z)} = 1 - \frac{1-|z|^2}{2-2Re(\overline{a} z)} = 1-\delta$$

Therefore, $|f(z) - \delta a| \le 1 - \delta$, for all $z \in \partial D \cap \mathbb{D}$. Thus, $f(\partial D) \subset \overline{D}$. Since this is verified for all disks tangent to $\partial\mathbb{D}$ in a, we have $f(D) \subset \overline{D}$. Finally, since f is holomorphic and non constant, by the Open Mapping Theorem, $f(D) \subset D$. □

Corollary 4.8. (Existence of radial limit at the Denjoy-Wolff point) *Let $f \colon \mathbb{D} \to \mathbb{D}$ be holomorphic with Denjoy-Wolff point $a \in \partial\mathbb{D}$. Then, $\lim_{r\to 1^-} f(ra) = a$.*

Proof. By the previous theorem, each tangent disk to $\partial\mathbb{D}$ at a is mapped into itself. So, making the radius infinitely smaller, we get that $\lim_{r\to 1^-} f(ra) = a$. □

4.3 Derivative at the Denjoy-Wolff point.

We are going to define the angular derivative in a similar way that we have defined the angular limit and comptuting it at the Denjoy-Wolff point a, when it lies in $\partial\mathbb{D}$.

Definition 4.9. Let f be analytic in $\mathbb{D}$. We say that f has **angular derivative** $b \in \widehat{\mathbb{C}}$ at $a \in \partial\mathbb{D}$ if the angular limit $f(a) \neq \infty$ exists and $\lim_{z\to a,\ z\in\Delta} \frac{f(z)-f(a)}{z-a}$ also exists, for each Stolz angle Δ at a. Then we write $b = f'(a)$.

With the following theorem we are able to compute the angular derivative at $a \in \partial\mathbb{D}$.

Theorem 4.10. (Julia-Wolff lemma) *Let f be holomorphic in $\mathbb{D}$ with angular limit $f(a)$ at $a \in \partial\mathbb{D}$. If $f(\mathbb{D}) \subset \mathbb{D}$ and $f(a) \in \partial\mathbb{D}$, then $f'(a)$ exists and*

$$0 < a\frac{f'(a)}{f(a)} = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{|a-z|^2} \frac{|f(a)-f(z)|^2}{1-|f(z)|^2} \le \infty$$

Moreover, $\frac{1}{f'(a)} = \inf_{x>0} \frac{Re(f(x+iy))}{x}$.

Proof. The proof can be found in [22]. □

Corollary 4.11. (Angular derivative at the Denjoy-Wolff point) *Let $f \colon \mathbb{D} \to \mathbb{D}$ be holomorphic with Denjoy-Wolff point $a \in \mathbb{D}$. Then, $0 < f'(a) \le 1$.*

Proof. Since a is the Denjoy-Wolff point of f, by 4.8, $f(a) = a$. Applying the previous theorem and Wolff lemma (4.7), we have: $0 < f'(a) \leq 1$. $\square$

Notice that the Denjoy-Wolff theorem does not discard the existence of others fixed point in the border $\partial\mathbb{D}$. Nevertheless, if they exists, their derivative is larger than 1, as we show in the following lemma.

Lemma 4.12. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic, not a Möbius Transformation, with Denjoy-Wolff point a. If $b \neq a$, $b \in \partial\mathbb{D}$ and $\lim_{r\to 1^-} f(rb) = b$, then $\lim_{r\to 1^-} f'(rb) > 1$.*

Proof. First, we note that, by theorem 4.10, $f'(b)$ exists and $0 < f'(b) \leq \infty$.

Suppose that a is in $\mathbb{D}$. We can assume, without loss of generality, that $a = 0$ and $b = 1$. Let $\tilde{f}(z) = \dfrac{f(z)}{z^k}$, where $k \geq 1$ is the multiplicity of 0 as a zero of f. Then, $\tilde{f}$ is an analytic map of $\mathbb{D}$ into $\mathbb{D}$ and $\lim_{r\to 1^-} \tilde{f}(r) = 1$. Therefore, $\lim_{r\to 1^-} f'(rb) = \lim_{r\to 1^-} kr^{k-1}\tilde{f}(r) + r^k\tilde{f}'(r) = k + \lim_{r\to 1^-} f'(rb) > 1$.

Now, suppose $a \in \partial\mathbb{D}$. We can assume, without loss of generality, that $a = -1$ and $b = 1$. Let $\tilde{f}(z) = \frac{1}{2}(f(z) - f(-z))$. $\tilde{f}$ is an analytic map of $\mathbb{D}$ into $\mathbb{D}$ with $\tilde{f}(0) = 0$ and $\lim_{r\to 1^-} \tilde{f}(r) = 1$. So we are in the previous case and, therefore, $\lim_{r\to 1^-} \tilde{f}'(r) > 1$. Then, by the definition of $\tilde{f}$, $\lim_{r\to 1^-} f(ra) + \lim_{r\to 1^-} \tilde{f}(rb) > 2$. But since a is the Denjoy-Wolff point of f, $f'(a) \leq 1$, and therefore $\lim_{r\to 1^-} f'(rb) > 1$. $\square$

5 FUNDAMENTAL SETS AND CONJUGACIES.

Up to now, we proved the Denjoy-Wolff Theorem, which asserts that, given a holomorphic self-map of $\mathbb{D}$, either it is conjugate to a rotation or all orbits converge to the same point, namely to the Denjoy-Wolff point. In the second case, now the question is *how* the orbits converge to the Denjoy-Wolff point.

In general, given a dynamical system we are not able to compute a explicit solution for the orbit of any initial condition. Our aim is to give a description of the orbits, although we cannot find an explicit formula for them. We want to describe the dynamics *qualitatively*.

The tool that we have to describe the dynamics qualitatively are the *conjugacies*. We say that a map $f\colon U \to U$ is **conformally conjugate** to a map $g\colon V \to V$ if there is a conformal map $\varphi\colon U \to V$ such that $g = \varphi \circ f \circ \varphi^{-1}$. We call the map φ a **conjugacy** between f and g. The maps f and g can be thought as the same map seen in different coordinate systems. It comes straightforward from the definition that f^n and g^n are also conjugate. Qualitatively, the dynamics of both maps f and g is the same.

Given a dynamical system in $\mathbb{D}$ defined by f, we want to find a conformal conjugacy between f and another map, g, that is expected to be as simple as possible. The simplest maps are the Möbius Transformations, which are studied in detail in section 2. Obviously, in general we cannot expect to find a conformal conjugacy between the given map f and a Möbius Transformation, because it would imply that f is conformal. However, we will prove that such conjugacy exists in an appropriate neighbourghood of the Denjoy-Wolff point.

Therefore, our goal in this section is to study the behaviour of a self-map of $\mathbb{D}$, f, in some neighbourhood of its Denjoy-Wolff point a. This neighbourhood should be small enough so that f is conjugate to a Möbius Transformation, but large enough that $f^n(K)$ belongs to V with a finite number of iterations, for each compact subset K. We call it *fundamental set* and the formal definition follows:

Definition 5.1. Let f be a map of a domain $\Delta \subset \mathbb{C}$ into itself, we say V is a **fundamental set** for f on Δ if V is an open, connected, simply connected subset of Δ such that: $f(V) \subset V$ and for every compact set $K \subset \Delta$, there is a positive integer n so that $f^n(K) \subset V$.

Fundamental sets are often also called **absorbing domains**.

In the following, we use the notations $f(a)$ and $f'(a)$, where a is the Denjoy-Wolff point of f. When a lies in $\mathbb{D}$, the image and the derivative at a are well-defined. However, if $a \in \partial\mathbb{D}$, $f(a)$ and $f'(a)$ refers to the angular limit and the angular derivative at a.

The following theorems claim the existence of fundamental sets and explain how the self-map of $\mathbb{D}$ is conjugate to a Möbius Transformation of the complex plane or the unit disk.

Theorem 5.2. [Cowen] (Existence of fundamental sets) *Let $f\colon \mathbb{D} \to \mathbb{D}$ be analytic, nonconstant and not conformal, with Denjoy-Wolff point a. If $f'(a) \neq 0$, there is a fundamental set V for f in $\mathbb{D}$ such that f is univalent on V.*

Theorem 5.3. [Cowen] (Nature of fundamental sets) *Let $f\colon \mathbb{D} \to \mathbb{D}$ be analytic, nonconstant and not conformal, with Denjoy-Wolff point a. Suppose $f'(a) \neq 0$. Then there exists:*

1. *a fundamental set V for f on $\mathbb{D}$,*
2. *a domain Ω, that can be either the complex plane or the unit disk,*
3. *a Möbius transformation ϕ mapping Ω onto Ω*
4. *and an analytic map σ from $\mathbb{D}$ to Ω*

such that:

a) σ and f are univalent in V,

b) $\sigma(V)$ is a fundamental set for ϕ on Ω

c) and the following diagram is commutative:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "f"] \arrow[d, "\sigma"] & \mathbb{D} \arrow[d, "\sigma"] \\ \Omega \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

Moreover, ϕ is unique up to conjugation by a Möbius transformation mapping Ω onto Ω, and ϕ and σ depend only on f, not on the particular fundamental set V.

In addition, we will see that σ can be defined in a so that σ is continuous in $V \cup \{a\}$. Then, Ω and ϕ can be chosen from the following:

CASE 1. $\Omega = \mathbb{C}$, $\sigma(a) = 0$ and $\phi(z) = sz$ with $0 < |s| < 1$.

CASE 2. $\Omega = \mathbb{C}$, $\sigma(a) = \infty$ and $\phi(z) = z + 1$.

CASE 3. $\Omega = \mathbb{D}$, $\sigma(a) = 1$ and $\phi(z) = \dfrac{(1+s)z + (1-s)}{(1-s)z + (1+s)}$ with $0 < s < 1$.

CASE 4. $\Omega = \mathbb{D}$, $\sigma(a) = 1$ and $\phi(z) = \dfrac{(1 \pm 2i)z - 1}{z - 1 \pm 2i}$.

We dedicate the next sections to prove Theorems 5.2 and 5.3, after some preliminary results.

A more delicate issue is to decide which of the cases correspond to a given map f and it is discussed in section 5.4. We will see that *Case 1* takes places if and only if the Denjoy-Wolff point lies in $\mathbb{D}$. Hence, the others occur when $a \in \partial\mathbb{D}$. To distinguish among them, we use the derivative at a and the concept of *nontangential convergence*, which we define next.

The contents of this section can be found in the paper [10]. For more background in nontangential convergence we refer to [22].

5.1 Nontangential convergence.

Up to this point we have discussed the convergence to the Denjoy-Wolff point. In what follows, we discuss the different types of convergence.

Definition 5.4. Let $\{z_n\}_n \subset \mathbb{D}$ and $\lim\limits_n z_n = a$, where $a \in \mathbb{D}$. We say that the sequence z_n **converges nontangentially** to a if $\sup\limits_n |Arg(1 - \overline{a}z_n)| < \frac{\pi}{2}$.

Otherwise, we say that the sequence **converges tangentially** to a.

Notice that a sequence converges nontangentially if and only if there exists some Stolz angle Δ at a such that $z_n \in \Delta$, for all n.

Sometimes we prefer to work in the right half-plane $\mathbb{H}$, so it is useful to translate the definition of nontangential convergence to $\mathbb{H}$. We may suppose that $a = 1$, if not conjugate with an appropiate rotation. Consider $h(z) = \frac{1+z}{1-z}$, a conformal self-map of $\mathbb{D}$ onto $\mathbb{H}$ mapping 1 to ∞. The straight lines that define the Stolz angle intersect at 1 (and at ∞, because they are straight lines), so are mapped to cirles intersecting in $h(1) = \infty$ (so they are, in fact, straight lines) and in $h(\infty) = -1$. Call $w_n = x_n + iy_n = h(z_n)$ the points of the sequence in $\mathbb{H}$. Therefore, nontangentially convergence to ∞ in $\mathbb{H}$ means $\sup\limits_n \left|\frac{y_n}{x_n}\right| < \infty$.

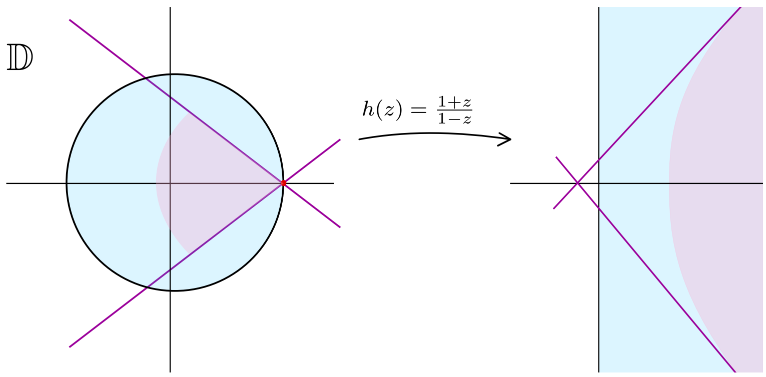


Figure 13: A Stolz angle in $\mathbb{D}$ and its corresponding region in $\mathbb{H}$.

The following result is a technical lemma. However, it is important to remark that it gives a bound for the quotient $\left|\dfrac{y}{x}\right|$ for all points in $D_{\mathbb{H}}(w, r)$.

Lemma 5.5. *Let $z = x + iy,\ w = u + iv \in \mathbb{H}$.*
If $d_{\mathbb{H}}(z, w) \leq r < 1$, then $x \geq u\dfrac{1-r}{1+r}$ and $|y| \leq |v| + \dfrac{2ru}{1-r^2}$.

Proof. Taking into account that $d_{\mathbb{H}}(z, w) = \left|\dfrac{z-w}{\overline{z}+w}\right|$, the proof is straightforward. □

Lemma 5.6. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic with Denjoy-Wolff point $a \in \partial\mathbb{D}$. If $\lim_{r\to 1^-} f'(ra) = s < 1$, then $\forall z \in \mathbb{D}$ the sequence $f^n(z)$ converges nontangentially to a.*

Proof. Consider φ the corresponding self-map of $\mathbb{H}$. Given $z_0 \in \mathbb{H}$, let $\varphi(z) = z_n = x_n + iy_n$. Since $\lim_{x\to\infty} \varphi(x) = \infty$. Note that the radial limit in $\mathbb{D}$ is equivalent to taking the limit in $\mathbb{H}$ through the real axis, so we have that $\lim_{x\to\infty} \varphi(x) = \infty$ and $c = \lim_{x\to\infty} \varphi(x) = \dfrac{1}{s}$. By theorem 4.10, $x_{n+1} = cx_n$, for all n.

Let us set $r = d_{\mathbb{H}}(z_1, z_0) < 1$. By Schwarz-Pick lemma, $d_{\mathbb{H}}(z_{n+1}, z_n) \leq d_{\mathbb{H}}(z_n, z_{n-1}) \leq \ldots \leq r$. By lemma 5.5, $\dfrac{|y_{n+1} - y_n|}{x_{n+1} - x_n} \leq M$, for all n. Then, $\dfrac{|y_n - y_0|}{x_n - x_0} \leq M$ (it is easily seen geometrically, with M thought as the slope of the straight line joining z_n and z_0 and taking into account that z_n is always in the left of z_{n+1}). Since $x_n \to \infty$, we have for all n: $\dfrac{|y_n|}{x_n} \leq \dfrac{|y_n|}{x_n} + \dfrac{|y_0|}{x_n - x_0} \leq M'$, for some M'.

Therefore, the sequence $\varphi^n(z_0)$ converges nontangentially to ∞. □

Lemma 5.7. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic with Denjoy-Wolff point $a \in \partial\mathbb{D}$. Suppose that for some $z_0 \in \mathbb{D}$ the sequence $f^n(z_0)$ converges nontangentially to a. Then, for all K compact subset of $\mathbb{D}$, the sequence $f^n(z)$ converges nontangentially to a, for all $z \in K$.*

Proof. Taking φ the corresponding self-map of $\mathbb{H}$, we have $z_0 \in \mathbb{H}$ such that φ^n converges to infinity nontangentially. Calling $\varphi^n(z_0) = z_n = x_n + iy_n$, we have $\left|\dfrac{y_n}{x_n}\right| \leq M < \infty$.

On the other hand, given K compact subset of $\mathbb{H}$, there is $r < 1$ such that $d_{\mathbb{H}}(z, z_0) \leq r$ for all $z \in K$. By Schwarz-Pick lemma, $d_{\mathbb{H}}(z, z_n) \leq r$ for all $z \in \varphi^n(K)$. By lemma 5.5, writing $z = x + iy$, $\left|\dfrac{y}{x}\right| \leq \dfrac{|y_n|}{x_n}\dfrac{1+r}{1-r} + \dfrac{2r}{1-r} = M'$, being M' a bound that depend on M and r, but it is uniform for all the points in K and does not depend on n. Therefore, φ^n converges nontangentially to ∞. $\square$

This last result tells us that if we have nontangential convergence for some point $z_0 \in \mathbb{D}$, we have it for all the points in $\mathbb{D}$. Conversely, to have tangential convergence in z_0 we need to have it for all the points in $\mathbb{D}$. Therefore, checking which type of convergence we have in one point is enough to determine it for all $\mathbb{D}$.

5.2 The proof of Theorem 5.2.

We are going to distinguish two cases, depending on if a lies in $\mathbb{D}$ or not.

First suppose that $a \in \mathbb{D}$. If $a = 0$, by Schwarz lemma and since $f'(a) \neq 0$, we have that $0 < |f'(a)| < 1$. By Schwarz lemma we also know that each euclidean disk with centre 0 is invariant by f. The non-vanishing of the derivative at 0 guarantees that we can find $\varepsilon > 0$ small enough that f is one-to-one in $D(0, \varepsilon)$.

$D(0, \varepsilon)$ is open, connected and simply connected. By Schwarz lemma, $f(D(0, \varepsilon)) \subset D(0, \varepsilon)$ and f is one-to-one there. It is left to see that for each $K \subset \mathbb{D}$ compact, there exists n such that $f^n(K) \subset D(0, \varepsilon)$.

By 2.3, there is $k \in (0, 1)$ such that $|f(z)| \leq k\,|z|$. Hence, $|f^n(z)| \leq k^n\,|z|$ and, since $k^n \to 0$, there exists n_0 such that $|f^{n_0}(z)| < \varepsilon$ and, therefore, $f^{n_0}(z) \in D(0, \varepsilon)$.

Thus, taking $V = D(0, \varepsilon)$, V is a fundamental set for f in $\mathbb{D}$.

Suppose now that $a \neq 0$. Conjugating by $f_a(z) = \dfrac{z - a}{1 - \overline{a}z}$:

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "f"] & \mathbb{D} \\
a \arrow[r, mapsto] \arrow[d, "f_a"] & a \arrow[d, "f_a"'] \\
\mathbb{D} \arrow[r, "\widetilde{f}"] & \mathbb{D} \\
0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

Then, $\widetilde{f}(0) = 0$ and $0 < \left|\widetilde{f}'(0)\right| < 1$, so we can take $V = f_a^{-1}(D(0, \varepsilon))$.

Suppose now that $a \in \partial\mathbb{D}$. Without loss of generality we can assume $a = 1$. By Julia Wolff Theorem (4.10), $s = \lim\limits_{r \to 1^-} f'(ra)$, with $0 < s \leq 1$. We are going to divide the proof into three steps. First we will see that for any $K \subset \mathbb{D}$ compact we can find N such that f is one-to-one in $\bigcup\limits_{n=N}^{\infty} f^n(K)$. Note that this set is invariant by f. Second, taking $\{K_k\}_k$ exhaustion for compacts of $\mathbb{D}$, we are going to construct a sequence of nested open subsets U_k such that $f^n(K_k) \subset U_k$ for all $n \geq N$, $f_{|U_k}$ is one-to-one and $f(U_k) \subset U_k$. Third, we will use these open subsets to construct the fundamental set V.

1. **For each compact subset, we find the required N.**
   We are going to use this result due to Pommerenke [22].

   Theorem 5.8. *Let φ be a self-map of $\mathbb{H}$, with angular derivative $c \geq 1$ at ∞. Let S be a Stolz angle at ∞. Set $0 < \delta < 1$ and $0 < \lambda < \infty$. Let $\varphi^n(1) = x_n + iy_n$ and $S_n = \{x + iy\colon\ \delta x_n \leq x < \infty \text{ and } |y - y_n| \leq \lambda x_n\}$. If $G = S \cup \bigcup\limits_{n=1}^{\infty} S_n$, then there is $\rho > 0$ so that φ is univalent in $G \cap \{w\colon\ |w| > \rho\}$.*

Using the conformal map $h(z) = \dfrac{1+z}{1-z}$, we can move our problem to the right half-plane:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "f"] \arrow[d, "h"] & \mathbb{D} \arrow[d, "h"] \\ \mathbb{H} \arrow[r, "\varphi"] & \mathbb{H} \end{tikzcd}$$

Let K be any compact subset of $\mathbb{D}$, there exists some $r < 1$ such that $|z| < r$, for all $z \in K$. Consider $D_r = h(\{z\colon\ |z| < r\})$. Notice that D_r are euclidean disks in $\mathbb{H}$ with different radius and centers, but $1 \in D_r$, $\forall r$. With respect to the pseudo-hyperbolic metric in $\mathbb{H}$ ($d_{\mathbb{H}}$), they are disks with center 1 and radius r. Then, by Schwarz-Pick lemma, $\varphi^n(D_r) \subset D_{\mathbb{H}}(w_n, r)$, where $\varphi^n(1) = w_n = x_n + iy_n$.

It is an easy calculation to check the following inclusion:

$$D_{\mathbb{H}}(w_n, r) = \left\{ w\colon\ \frac{|w - w_n|}{|w + \overline{w}_n|} < r \right\} \subset \{w = x + iy \in \mathbb{H}\colon\ x > \delta \text{ and } |y - y_n| < \lambda x_n\} =: S_n$$

where $\delta = \dfrac{1-r}{1+r}$ and $\lambda = \dfrac{2r}{1+r}$.

Let $G = S \cup \bigcup\limits_{n=1}^{\infty} S_n$, where S is any Stolz angle at ∞. Since φ is a self-map of $\mathbb{H}$ and it has angular derivative $\frac{1}{s} \geq 1$ at infinity, by Pommerenke's Theorem 5.8, there is $\rho > 0$ such that φ is one-to-one in $G \cap \{w\colon\ |w| > \rho\}$.

Since $\varphi \to \infty$ uniformly on compact subsets of $\mathbb{H}$, there is N such that $\bigcup\limits_{n=N}^{\infty} \varphi^n(h(K)) \subset G \cap \{w\colon\ |w| > \rho\}$. Thus, φ is one-to-one in $\bigcup\limits_{n=N}^{\infty} \varphi^n(h(K))$, so f is one-to-one in $\bigcup\limits_{n=N}^{\infty} \varphi^n(K)$.

2. **Construction of the sequence of open subsets $\{U_k\}_k$.**
For $k \geq 2$, let $K_k = \left\{z :\ |z| \leq 1 - \frac{1}{k}\right\}$ and $\mathring{K}_k = \left\{z :\ |z| < 1 - \frac{1}{k}\right\}$. Let $m \geq 2$ be the smallest integer such that $f(0) \in \mathring{K}_m$. We will build, by induction in k, a sequence of subsets of the unit disk: $U_m \subset U_{m+1} \subset ... \subset U_k$, such that U_k is open, connected, $\exists N_k$ such that for all $n \geq N_k$, $f^n(\mathring{K}_k) \subset U_k$, $f_{|U_k}$ is one-to-one and $f(U_k) \subset U_k$.

   - **Base case ($k = m$):** Applying the previous reasoning we can find a positive integer N_m such that f restricted to $\bigcup\limits_{n=N_m}^{\infty} f^n(K_m)$ is one-to-one. Take $U_m = \bigcup\limits_{n=N_m}^{\infty} f^n(\mathring{K}_m)$.
   Clearly, U_m is open, $f_{|U_m}$ is one-to-one and $f(U_m) \subset U_m$. By construction, $\forall n \geq N_m$, $f^n(\mathring{K}_m) \subset U_m$. We must see that U_m is connected.
   Since f is continuous, $f^n(\mathring{K}_m)$ is connected for all n. By the choice of m, $f(0) \in \mathring{K}_m$ and $f(0) \in f(\mathring{K}_m)$. Applying successively f, we have that $f^n(0) \in f^{n-1}(\mathring{K}_m)$ and $f^n(0) \in f^n(\mathring{K}_m)$. Thus, U_m is connected.
   - **Inductive step:** Suppose we have $U_m \subset U_{m+1} \subset ... \subset U_{k-1}$ that satisfy the required properties and we are going to construct U_k.
   Taking K_k and applying the previous reasoning we can find $N'_k \geq N_{k-1}$ such that f restricted to $\bigcup\limits_{n=N'_k}^{\infty} f^n(K_k)$ is one-to-one. Take $U'_k = \bigcup\limits_{n=N'_k}^{\infty} f^n(\mathring{K}_k)$.
   Note that U'_k does not fulfill the required conditions because we do not know if $U_{k-1} \subset U'_k$. Taking $U_k = U'_k \cup U_{k-1}$ does not solve our problem because we do not know if the function is one-to-one here.

Consider $L = \overline{U_{k-1} \setminus U'_k} \subset \mathbb{D}$. L is compact and, therefore, $f(L)$ is also compact. Since f^n converges uniformly on compact subsets to $a \in \partial\mathbb{D}$, so we can find $N_k \geq N'_k$ such that:

$$\bigcup_{n=N_{k+1}}^{\infty} f^n(\mathring{K}_k) \cap f(L) = \varnothing$$

Take $U_k = U_{k-1} \cup \bigcup_{n=N_k}^{\infty} f^n(\mathring{K}_k)$. Then, U_k is open and $U_{k-1} \subset U_k$. By the construction of U_k, $f^{N_k}(\mathring{K}_k) \in U_k$ and $f(U_k) \subset U_k$.

We have to see that U_k is connected. U_{k-1} is connected (by induction hypotesis) and $\bigcup_{n=N_k}^{\infty} f^n(\mathring{K}_k)$ also. Since $f^{N_k}(0) \in \bigcup_{n=N_{k-1}}^{\infty} f^n(\mathring{K}_{k-1}) \subset U_{k-1}$ and $f^{N_k}(0) \in \bigcup_{n=N_{k-1}}^{\infty} f^n(\mathring{K}_{k-1})$, U_k is connected.

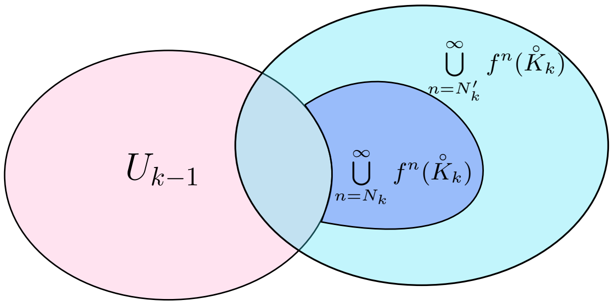


Figure 14: Schematic represetation of the choice of N'_k and N_k.

It is left to see that $f_{|U_k}$ is one-to-one. Suppose we have $\alpha, \beta \in U_k$ with $\alpha \neq \beta$. If $\alpha, \beta \in U_k$ or $\alpha, \beta \in \bigcup_{n=N'_k}^{\infty} f^n(\mathring{K}_k)$, then $f(\alpha) \neq f(\beta)$. Suppose $\alpha \in \bigcup_{n=N_k}^{\infty} f^n(\mathring{K}_k) \subset \bigcup_{n=N'_k}^{\infty} f^n(\mathring{K}_k)$ and $\beta \notin \bigcup_{n=N'_k}^{\infty} f^n(\mathring{K}_k)$, so $\beta \in L$. Then, $f(\alpha) \in \bigcup_{n=N_{k+1}}^{\infty} f^n(\mathring{K}_k)$ and $f(\beta) \subset L$. Since we have chosen N_k so that the previous sets are disjoint, $f(\alpha) \neq f(\beta)$. Thus, f is one-to-one in U_k.

3. **Construction of the fundamental set V.**

   Having built the sets U_k, for $k = m, m+1, \ldots$, we can define $V' = \bigcup_{n=m}^{\infty} U_k$. V' is an open connected subset of $\mathbb{D}$, with $f(V') \subset V'$ and in which f is one-to-one. Moreover, for each $K \subset \mathbb{D}$ compact, since $K \subset \mathring{K}_n$ for some n, then $f^{N_n}(K) \subset U_n \subset V'$.

   Our problem is that V' may not be simply connected. We take $V = V' \cup$(holes of V'). Clearly, V' is open, connected, simply connected and for all K, there is n such that $f^n(K) \subset V$. We have to see that f is one-to-one in V and that $f(V) \subset V$.

   As a corollary to the Argument Principle,[3] we have that to see if an analytic function is one-to-one on a region, it is enough to check it on its boundary. Given a hole of V', take a simple curve γ in V' surrounding the hole. Since f is one-to-one in γ, it is one-to-one in the points surrounded by it, in particular in the hole. Applying the same argument to all the holes of V', we get that f is one-to-one in V.

   Finally, $f(V) \subset V$ is a direct consequence of the Mean-Value Property.

And this finishes the proof.

[3] See, for example [19].

5.3 The proof of Theorem 5.3.

Since $f'(a) \neq 0$, by Theorem 5.2, there exists V fundamental set for f on $\mathbb{D}$ such that f is univalent on V. Once we saw the existence of a fundamental set $V \subset \mathbb{D}$ we are now interested in proving that the dynamics on V are conjugate to the dynamics of a Möbius Transformation. This is precisely the content of Theorem 5.3.

To do so, we are going to focus on the points in V, where we know that f is univalent. Our problem is that some point in V may not be in the image of f, others in the image of f^2... So we are going to build an abstract surface S that contains the (abstract) preimages of all the points in V so every point of S is in the image of f^n.

We will introduce S as a set of points and give it topological and analytic structure, so it will be a Riemann surface. Moreover, it will be simply connected and compact. Then, by the Uniformization Theorem, S will be conformally isomorphic to either $\mathbb{D}$ or $\mathbb{C}$ (that will be our domain Ω) by a conformal map ρ. We will then have

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\pi"] \arrow[dd, bend right, "\sigma"'] & V \arrow[d, "\pi"'] \arrow[dd, bend left, "\sigma"] \\
S \arrow[r, "\psi"] \arrow[d, "\rho"] & S \arrow[d, "\rho"'] \\
\Omega \arrow[r, "\phi"] & \Omega
\end{tikzcd}
$$

Finally, we will prove that we can extend σ to all points in $\mathbb{D}$ and prove the uniqueness up to conformal conjugation.

1. **Construction of the Riemann surface S.**

   (a) **Construction of S as a set of points.**
   Our goal is to build an abstract surface S in which f is bijective. To do so, we are going to add the preimages of all point of V, so that each point in S is in the image of f^n. However, we have to do it respecting the previous structure in V, so if $z = f^n(w)$, for some z and w, the same has to happen to the analogous points in S.
   If $n, m \in \mathbb{Z}$ and $z, w \in V$ we say that $(z, n) \sim (w, m)$ if there exists some k such that $f^{n+k}(z) = f^{m+k}(w)$. Necessarily, $k \geq \max\{-n, -m\}$, because only the positive iterations of f are defined. Due to the univalence of f, if there exists $k \geq \max\{-n, -m\}$ such that $f^{n+k}(z) = f^{m+k}(w)$, then it happens for all $k \geq \max\{-n, -m\}$.
   We use the notation $[(z, n)]$ to denote the equivalence class containing (z, n). Now, we define $S\colon = \{[(z, n)]\colon z \in V, n \in \mathbb{Z}\}$.

   (b) **Introduction of a topology.**
   If U is an open subset of V and $n \in \mathbb{Z}$, let $\mathcal{B} = \left\{\mathcal{R}_n^U\right\}$ be a basis for a topology in S, where $\mathcal{R}_n^U = \{[(z, n)]\,|z \in U\}$. We must check that $\mathcal{B}$ defines indeed a topology in S.
   First we observe that S can be written as a union of elements of $\mathcal{B}$. Taking $U = V$, which is open, $S = \bigcup\limits_{n \in \mathbb{Z}} \mathcal{R}_n^V$.
   We also must check that $\forall w_* \in \mathcal{R}_{m_1}^{W_1} \cap \mathcal{R}_{m_2}^{W_2}$, there exists one element of $\mathcal{B}$ such that it is in $\mathcal{R}_{m_1}^{W_1} \cap \mathcal{R}_{m_2}^{W_2}$ and w_* belongs to it. Suppose $w_* \in \mathcal{R}_{m_1}^{W_1} \cap \mathcal{R}_{m_2}^{W_2}$, then $w_* = [(w_1, m_1)] = [(w_2, m_2)]$, with $w_1 \in W_1$ and $w_2 \in W_2$. We have $p = f^{k+m_1}(w_1) = f^{k+m_2}(w_2)$, where $k = \max\{-m_1, -m_2\}$. So $Y = f^{k+m_1}(W_1) \cap f^{k+m_2}(W_2)$ is a neighbourhood of p. Then $\mathcal{R}_0^Y = \mathcal{R}_{k+m_1}^{W_1} \cap \mathcal{R}_{k+m_2}^{W_2}$ and $\mathcal{R}_{-k}^Y = \mathcal{R}_{m_1}^{W_1} \cap \mathcal{R}_{m_2}^{W_2}$, with $w_* \in \mathcal{R}_{-k}^Y$.

We note that S with this topology verifies the second axiom of countability, that is, there exists a countable basis. Since V is an open subset of $\mathbb{C}$, the euclidean topology on V admits a countable basis $\{U_m\}_{m\in\mathbb{N}}$. Then, $\{\mathcal{R}_n^{U_m}\}_{m\in\mathbb{N},n\in\mathbb{Z}}$ is a basis of our topological space and it is countable.

We also note that S with this topology is a Hausdorff space. Let $z_{1*} = [(z_1, n_1)]$ and $z_{2*} = [(z_2, n_2)]$ with $z_{1*} \neq z_{2*}$. Let $k = \max\{-n_1, -n_2\}$, then $(z_1, n_1) \sim (f^{k+n_1}(z), -k)$ and $(z_2, n_2) \sim (f^{k+n_2}(z), -k)$. Therefore we can choose U_1 and U_2 disjoint open subsets of V such that $f^{k+n_1}(z_1) \in U_1$ and $f^{k+n_2}(z_2) \in U_2$. Then $\mathcal{R}_{-k}^{U_1}$ and $\mathcal{R}_{-k}^{U_2}$ are disjoint open neighbourhoods of z_{1*} and z_2* respectively.

(c) **Introduction of an analytic structure.**

Let us introduce an analytic structure to S by defining the coordinate charts $c_n\colon V \to S$, with $c_n(z) = [(z, n)]$. We note that c_n is one-to-one. In fact, if $[(z, n)] = [(w, n)]$, then $f^n(z) = f^n(w)$ and $z = w$, because f is univalent in V.

We have to prove that c_n is continuous with respect to the topology defined in S. We have to see that, if $\mathcal{R}_n^U$ is an open subset of S, then $c_m^{-1}(\mathcal{R}_n^U)$ is an open subset of V. If $n = m$ is quite obvious, so suppose $n \neq m$. Consider $\mathcal{R}_n^U \cap \mathcal{R}_m^U = \big\{[(z, n)] \in \mathcal{R}_n^U\colon\ \exists w \in U \quad (z, n) \sim (w, m)\big\}$. Since we are dealing with a topology, $\mathcal{R}_n^U \cap \mathcal{R}_m^U$ is open and it can be written as the union of open subsets of the basis $\mathcal{B}$: $\bigcup\limits_{i\in I} \mathcal{R}_m^{W_i} = \mathcal{R}_m^{\cup W_i} = \mathcal{R}_m^W$, where $W = \bigcup\limits_{i\in I} W_i$ is an open subset of V. Then, $c_m^{-1}(\mathcal{R}_n^U) = W$ and, as it is open, we have proved that c_m is continuous.

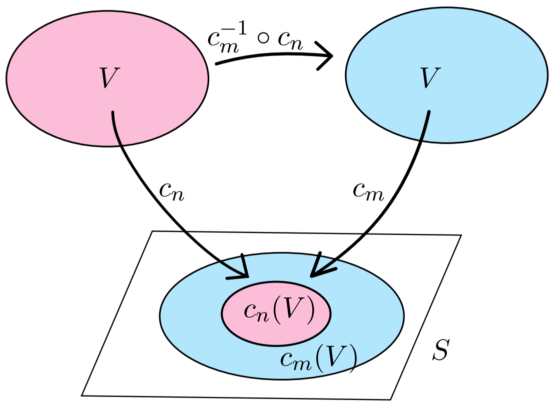


Figure 15: How the coordinate charts and the transition maps act on V and S.

Since V is locally compact and S is Hausdorff, c_n is an homeomorphism between V and $c_n(V)$. We notice that the union of all the charts is S.

It is left to see that the transition maps are holomorphic, that is, we have to see that $c_m^{-1} \circ c_n$ and $c_m^{-1} \circ c_n$ are holomorphic on their domain of definition. Suppose $n = m + l$, with $l \geq 0$. Then, $c_m^{-1} \circ c_n$ is well defined in V, because all the points of $f^n(V)$ belong to $f^m(V)$. Now: $c_m^{-1}(c_n(z)) = c_m^{-1}([(z, n)]) = c_m^{-1}([(z, m + l)]) = c_m^{-1}(\big[(f^l(z), m)\big]) = f^l(z)$, which is holomorphic in V.

Finally, as f^l is one-to-one in V, $c_n^{-1} \circ c_m$ is holomorphic in $f^l(V)$, which is the domain of definition of $c_n^{-1} \circ c_m$.

(d) S **is simply connected.**

Suppose $\gamma\colon\ [0, 1] \to S$ is a loop on S. Since $\gamma([0, 1])$ is compact, then there is $n \in \mathbb{Z}$ such that $\gamma([0, 1]) \subset c_n(V)$. But $c_n(V) \cong V$, which is simply connected (by definition of fundamental set). Then, $\gamma \sim 0$ in $c_n(V)$, so $\gamma \sim 0$ in S.

(e) S **is not compact.**

We must prove that there exists a sequence of points in S that does not have any convergent subsequence.

Consider $z_k^* = [(z,-k)]$, with $k \in \mathbb{N}$ and $z \neq a$. Set $n \in \mathbb{Z}$ and U open subset of V with compact closure. Suppose $z_k^* \in \mathcal{R}_n^U$ for all $k > -n$, that is $f^{-k}(z) \in f^n(U)$. Therefore, for all $k > -n$, $z \in f^{n+k}(U)$, contrary to the fact that f^n converges uniformly on U to a.

The contradiction comes from supposing that $z_k^* \in \mathcal{R}_n^U$ for all $k > -n$. Thus the sequence z_k^* can have at most finitely many terms in any open subset with compact closure of S, so it cannot have a convergent subsequence n S.

We have already proved that S is a Riemann surface, that it is simply connected and not compact. Therefore, by the Uniformization Theorem, S is conformally isomorphic to either the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$.

2. **The maps ψ and π.**
Let us define:

$$\begin{aligned} \pi\colon V &\longrightarrow S \\ z &\longmapsto [(z,0)] \end{aligned} \qquad\qquad \begin{aligned} \psi\colon S &\longrightarrow S \\ [(z,n)] &\mapsto [(f(z),n)] \end{aligned}$$

We observe that both π and ψ are holomorphic. Moreover, ψ is well defined, as it does not depend on the representative we choose. In fact, if $(z,n) \sim (w,n)$, then $f^n(z) = f^m(z)$ and $f^{n+1}(z) = f^{m+1}(z)$, so $\psi([(z,n)]) = [(f(z),n)] = [(f(w),m)] = \psi([(w,m)])$.

Both π and ψ are one-to-one, due to the univalence of f in V. In addition, ψ is onto on S because $[(z,n)] = [(f(z),n-1)] = \psi([(z,n-1)])$. Clearly, $\psi \circ \pi = \pi \circ f$.

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & V \\ z & \longmapsto & f(z) \\ \big\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi}\big\downarrow \\ S & \xrightarrow{\quad \psi \quad} & S \\ [(z,0)] & \longmapsto & [(f(z),0)] \end{array}$$

Moreover, we observe that $\pi(V)$ is a fundamental set for ψ on S. Indeed, if $K \subset S$ is compact, we can find $n \in \mathbb{Z}$ such that $K \subset c_n(V)$, then $\psi^n(K) \subset \pi(V)$.

3. **The maps ϕ and σ, and the domain Ω.**
We already know that S is conformally isomorphic to either $\mathbb{D}$ or $\mathbb{C}$. We call Ω this domain. Then there exists $\rho\colon S \to \Omega$ conformal. We define $\sigma\colon V \to \Omega$ as $\sigma = \rho \circ \pi$ and $\phi\colon \Omega \to \Omega$ as $\phi = \rho \circ \psi \circ \rho^{-1}$.
Clearly, ϕ is conformal and, therefore, it is a Möbius transformation. The map σ is one-to-one and $\phi \circ \sigma = \sigma \circ f$. Moreover, $\sigma(V)$ is a fundamental set for ϕ on Ω.

4. **Extension of the functions from V to $\mathbb{D}$.**
Since V is a fundamental set for f on $\mathbb{D}$, there exists n large enough that $f^n(z) \in V$, where σ is defined. Once we are in Ω, we know that ϕ is bijective, so we can compose with ϕ^{-n}.

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\quad f^n \quad} & \mathbb{D} \\ z & \longmapsto & f^n(z) \in V \\ \big\downarrow{\scriptstyle \sigma} & & {\scriptstyle \sigma}\big\downarrow \\ S & \xleftarrow{\quad \phi^{-n} \quad} & \Omega \\ \phi^{-n}(\sigma(f^n(z))) & \longleftarrow & \sigma(f^n(z)) \end{array}$$

We have to check if the extension is well defined. We already know that, for $z \in V$, $\phi^{-k}(\sigma(f^k(z))) = \sigma(z)$. But, if we have $z \in \mathbb{D}$, not necessarily in V, with $f^n(z) \in V$ and $f^m(z) \in V$, $n \neq m$, we obtain the same $\sigma(z)$. Indeed, suppose $n = m + k$, then $\phi^{-n}(\sigma(f^n(z))) = \phi^{-m}(\phi_{-k}(\sigma(f^k(f^m(z))))) = \phi^{-m}(\sigma(f^m(z)))$.
The extension of σ is holomorphic because it is the composition of holomorphic functions.

5. **Uniqueness up to conformal conjugation.**
Suppose that $\widetilde{V}$ is another fundamental set for f on $\mathbb{D}$ and consider the corresponding domain $\widetilde{\Omega}$ and the maps $\widetilde{\phi}$ and $\widetilde{\sigma}$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\sigma"] & V \arrow[d, "\sigma"] \\ \Omega \arrow[r, "\phi"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} \widetilde{V} \arrow[r, "f"] \arrow[d, "\widetilde{\sigma}"] & \widetilde{V} \arrow[d, "\widetilde{\sigma}"] \\ \Omega \arrow[r, "\widetilde{\phi}"] & \Omega \end{tikzcd}$$

Consider $K = \{t \cdot f(0) \colon 0 \leq t \leq 1\}$. K is compact and connected. Moreover, $\forall N$, $\bigcup\limits_{n=N}^{\infty} f^n(K)$ is connected. Indeed, $f^n(K)$ is connected and $f^n(f(0)) = f^{n+1}(0 \cdot f(0)) \in f^n(K) \cap f^{n+1}(K)$. Since V and $\widetilde{V}$ are fundamental sets for f, there exists N such that $\bigcup\limits_{n=N}^{\infty} f^n(K) \in V \cap \widetilde{V}$. Let W be the connected component of $V \cap \widetilde{V}$ that contains $\bigcup\limits_{n=N}^{\infty} f^n(K) \in V \cap \widetilde{V}$. Clearly W is a fundamental set for f on $\mathbb{D}$. Hence, $\sigma(W)$ and $\widetilde{\sigma}(W)$ are fundamental sets for ϕ and $\widetilde{\phi}$ on Ω and $\widetilde{\Omega}$.

Let us define

$$\begin{aligned} \tau \colon \Omega &\longrightarrow \widetilde{\Omega} \\ z &\longmapsto \phi^{-n}(\widetilde{\sigma}(\sigma^{-1}(\phi^n(z)))) \end{aligned}$$

where n is large enough that $\phi^n(z) \in \sigma(W)$. It is well-defined because it does not depend on which n we choose.

Since τ is a conformal map between Ω and $\tilde{\Omega}$, then $\Omega = \tilde{\Omega}$. Therefore, τ is a Möbius transformation, $\tilde{\phi} = \tau \circ \phi \circ \tau^{-1}$ and $\tilde{\sigma} = \tau \circ \sigma$ in V.

And this finishes the proof.

5.4 Classification of the dynamics in the fundamental set.

First we claim that ϕ cannot have any fixed point apart from the ones of f. This is not obvious because we could have a fixed point in $\Omega \setminus \sigma(\mathbb{D})$, so it would not be a fixed point of f, but we claim this cannot happen. We will use the setup and notation introduced in section 5.3.

Proposition 5.9. *ϕ has a fixed point in Ω if and only if f has a fixed point in $\mathbb{D}$ and, in this case, the fixed point of ϕ is $\sigma(a)$, where a is the Denjoy-Wolff point of f.*

Proof. If p is a fixed point of ϕ in Ω, since ρ is a conjugacy, $p = \rho([(z,n)])$, where $[(z,n)]$ is a fixed point of ψ in S. Hence, $(z,n) \sim (f(z),n)$, so z is a fixed point of f in $V \subset \mathbb{D}$.

Conversely, if f has any fixed point in $\mathbb{D}$, it must be the Denjoy-Wolff point a. Then, $a \in V$ and $(a,n) \sim (a,m)$, for all n, m. $[(a,n)]$ is a fixed point of ψ in S, so $p = \rho([(a,n)])$ is a fixed point of ϕ in Ω. $\square$

Notice that ϕ cannot be a rotation. Otherwise, for each $w \in \Omega$, the sequence $\phi^n(w)$ has w as a limit point, contrary to the fact that $f^n(z) \to a$. Moreover, if we define $\sigma(a) = \lim_n \phi^n(\sigma(z))$, for some $z \in \mathbb{D}$, σ is continuous in $\mathbb{D} \cup \{a\}$. Notice that the limit is well-defined, because by the Denjoy-Wolff theorem, all $z \in \mathbb{D}$ converge to the same point under iteration of f.

Considering the facts above, **which Möbius Transformations $\phi\colon \Omega \to \Omega$ can we have?** First, consider $\Omega = \mathbb{C}$. The conformal mappings of $\mathbb{C}$ onto $\mathbb{C}$ are the polynomials of degree one: $\widetilde{\phi}(z) = \alpha z + \beta$, with $\alpha \neq 0$. We distinguish two cases:

CASE 1. If $\alpha \neq 1$, $\widetilde{\phi}$ has one fixed point $z_* = \dfrac{\beta}{1-\alpha} \in \mathbb{C}$. Conjugating by $h(z) = 1 - \dfrac{\beta}{1-\alpha}$, we get that $\widetilde{\phi}$ is conjugated to $\phi(z) = \alpha z$. Then, $\sigma(a) = 0$ and $0 < |\alpha| < 1$, because a must be mapped to the fixed point and it must be attracting.

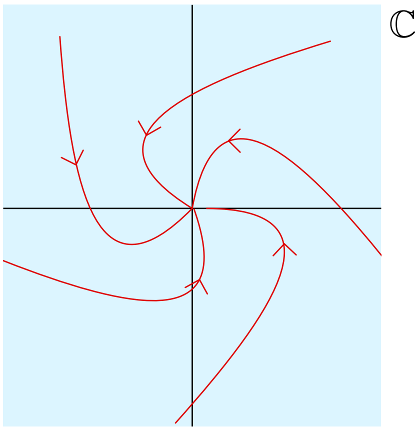


Figure 16: *CASE 1.* Schematic representation of the function $\phi(z) = \alpha z$.

It is easy to prove that $\alpha_1 z$ and $\alpha_2 z$, with $0 < |\alpha_i| < 1$ and $\alpha_1 \neq \alpha_2$, cannot be conformally conjugated.

CASE 2. If $\alpha = 1$, then $\widetilde{\phi}$ has no fixed points in $\mathbb{C}$, so $\sigma(a) = \infty$. Moreover, $\widetilde{\phi}(z) = z + \beta$, with $\beta \neq 0$, is conjugate to $\phi(z) = z + 1$ by $h(z) = \dfrac{z}{\beta}$. We note that $\phi'(\infty) = 1$.[4]

Note that αz and $z + 1$ are not conformally conjugate because they have a different number of fixed points. Therefore, *Case 1* and *Case 2* are actually different cases.

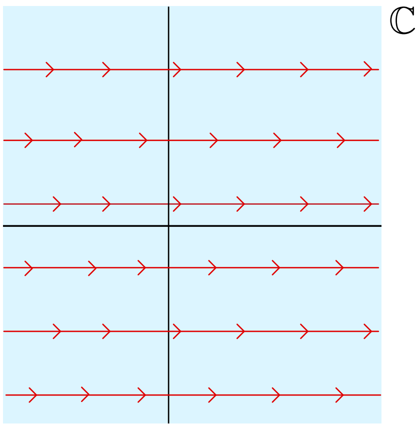


Figure 17: *CASE 2.* Schematic representation of the function $\phi(z) = z + 1$.

[4]The procedure to calculate the derivative at ∞ will be explained in the next chapter.

Consider now $\Omega = \mathbb{D}$. First notice that ϕ cannot have any fixed point in $\mathbb{D}$. Otherwise it will be conjugate to a rotation, contradicting our assumptions. Since $\mathbb{D}$ is conformally equivalent to the right half-plane $\mathbb{H}$, we will work with the Möbius Transformations of $\mathbb{H}$ onto $\mathbb{H}$ without fixed points in $\mathbb{H}$.

CASE 3. If there are two fixed points, we can suppose $\phi(z) = sz$ up-to conjugation, with $s > 1$ and $\sigma(a) = \infty$.

Thinking it as a self-map of $\mathbb{D}$ it is $\phi(z) = \dfrac{(1+s)z + (1-s)}{(1-s)z + (1+s)}$, and $\sigma(a) = 1$.

We note that $\phi'(1) = s$.

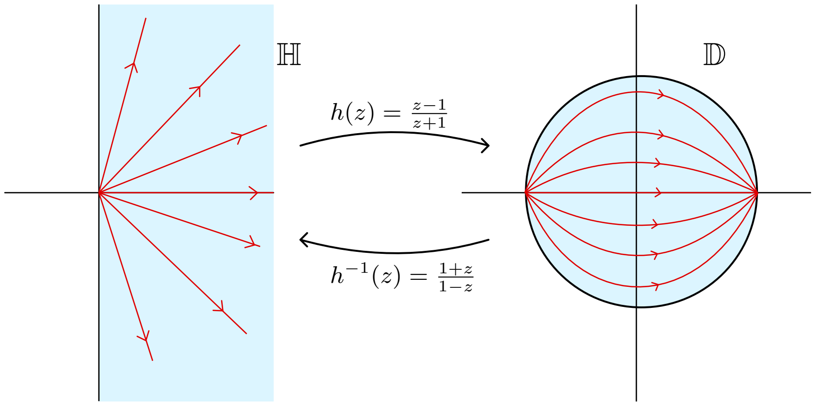


Figure 18: *CASE 3.* Schematic representation of the function $\phi(z) = sz$ in $\mathbb{H}$ and its transformation to a function in $\mathbb{D}$. Thus we see the dynamics in $\mathbb{D}$, what was difficult to see from the formula.

CASE 4. If the only fixed point of $\mathbb{H}$ is ∞, then ϕ is a translation and it is conjugated either to $\phi_1(z) = z + i$ or to $\phi_2(z) = z - i$. In both cases, since the fixed point is ∞, $\sigma(a) = \infty$.

It is easy to prove than ϕ_1 and ϕ_2 are not conformally conjugated.

Thinking them as self-maps of $\mathbb{D}$, they are $\phi_1(z) = \dfrac{(1-2i)z - 1}{z - 1 - 2i}$ and $\phi_2(z) = \dfrac{(1+2i)z - 1}{z - 1 + 2i}$. In both cases, $\sigma(a) = 1$.

We note that $\phi'(1) = 1$.

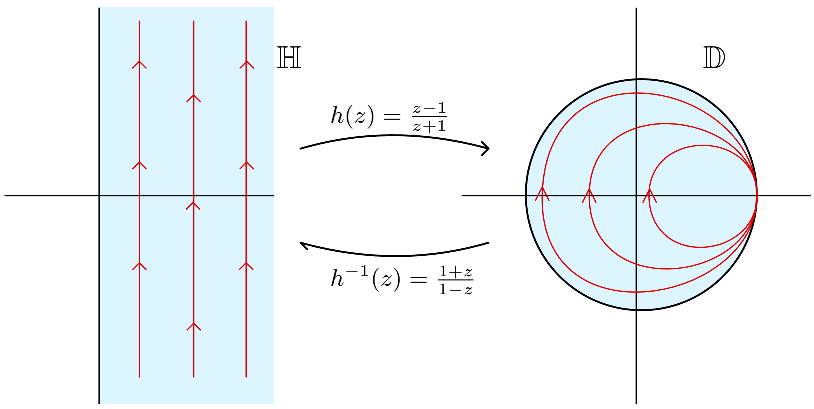


Figure 19: *CASE 4.* Schematic representation of the function $\phi_1(z) = z + i$ in $\mathbb{H}$ and its transformation to a function in $\mathbb{D}$.

Observe that the four cases can actually occur. For Cases 1, 3 and 4, take $f = \phi$ and $\sigma = Id$. For Case 2 take $f(z) = \sigma^{-1}(\sigma(z) + 1)$, where σ is the conformal map of $\mathbb{D}$ onto the right half-plane $\mathbb{H}$. Namely $\sigma(z) = \dfrac{1+z}{1-z}$ and $f(z) = \dfrac{1+z}{3-z}$.

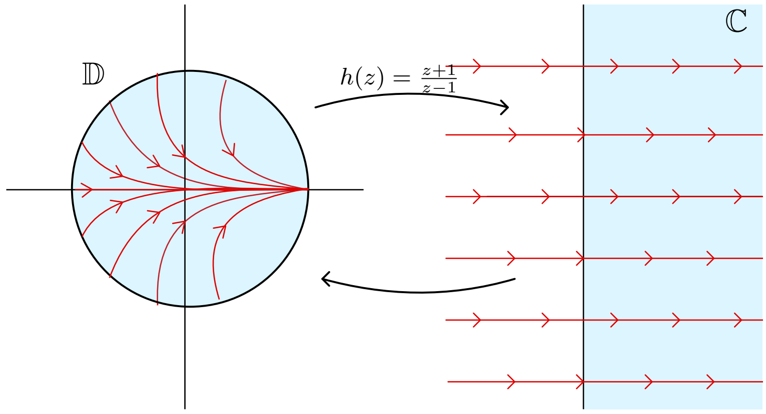


Figure 20: Schematic representation for the function f and how σ transfers it to $\mathbb{C}$. We see that in $\mathbb{C}$ it is equivalent to the translation $z + 1$.

Now that we know the possible cases, the logical question to ask is: **Given a self-map of $\mathbb{D}$, do we know in which case we are?** The easiest case is the first one: the case in which we have an inner fixed point in Ω which correspond to an inner fixed point in $\mathbb{D}$.

Proposition 5.10. *Let f, ϕ and Ω as in Theorem 5.3. Then, $\Omega = \mathbb{C}$ and $\phi(z) = sz$ (Case 1) if and only if the Denjoy-Wolff point a is in $\mathbb{D}$ and $f'(a) = s$.*

Proof. Since in *Case 1* there is a fixed point in Ω, it must correspond to a fixed point in $\mathbb{D}$. Conversely, if we have a fixed point in $\mathbb{D}$ we must be in *Case 1*, because it is the only one with an inner fixed point.

Noting that $\phi'(0) = s$ and $\sigma'(a) \neq 0$ (because it is one-to-one in a neighbourhood of a), differentiating $\sigma(\varphi(a)) = \phi(\sigma(a))$, we obtain $f'(a) = s$. ☐

To distinguish the cases in which $a \in \partial\mathbb{D}$, we need the following theorem. It asserts that, with stronger conditions of regularity, $\phi'(a)$ behaves as expected.

Theorem 5.11. *Let f, ϕ and Ω as in Theorem 5.3. Let $a \in \partial\mathbb{D}$ be the Denjoy-Wolff point of f. If f' can be defined at a so that it is continuous in $\mathbb{D} \cup \{a\}$, then:*

$$\phi'(\sigma(a)) = \lim_{r \to 1^-} f'(ra)$$

Proof. The proof can be found in [10], page 81. ☐

As a corollary, we can distinguish *Case 3* from the others, because is the only one with $a \in \partial\mathbb{D}$ and derivative $\phi'(\sigma(a)) = s < 1$.

Corollary 5.12. *Let f be an holomorphic self-map of $\mathbb{D}$ with Denjoy-Wolff point $a \in \partial\mathbb{D}$. If . If $f'(a) = s < 1$, we may take $\Omega = \mathbb{D}$ and $\phi(z) = \dfrac{(1+s)z + (1-s)}{(1-s)z + (1+s)}$ (Case 3).*

Finally, the following theorem allows as to decide between *Case 2* and *Case 4* in some cases. We will see that sometimes it is inconclusive.

Theorem 5.13. *Let f be an holomorphic self-map of $\mathbb{D}$ with Denjoy-Wolff point $a \in \partial\mathbb{D}$. Suppose f' can be defined at a so that it is continuous in $\mathbb{D} \cup \{a\}$ and $f'(a) = 1$. Then, if for some $z_0 \in \mathbb{D}$, $f^n(z_0) \to a$ nontangentially, then $\Omega = \mathbb{C}$ (Case 2).*

Proof. The proof can be found in [10], page 84. $\square$

We note that the converse is not true. We can be in *Case 2* and have tangential convergence. However, if we are in *Case 4* we have tangential convergence. The maps ϕ_1 and ϕ_2 defined in *Case 4* are examples. We saw that $f(z) = \dfrac{1+z}{3-z}$ is an example of *Case 2* and it has nontangential convergence. For an example of *Case 2* with tangential convergence we may take $f(z) = \sigma^{-1}(\sigma(z)+1)$, where σ is a conformal map of $\mathbb{D}$ onto $\{z = x+iy\colon x > 0, -x < y < \sqrt{x}\}$. Note that by the Riemann Mapping Theorem, σ exists.

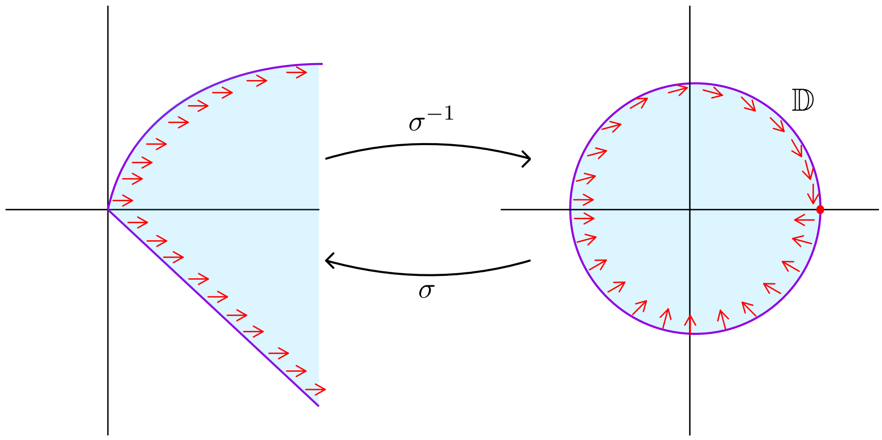


Figure 21: Schematic representation of $f(z) = \sigma^{-1}(\sigma(z)+1)$, where σ is a conformal map of $\mathbb{D}$ onto $\{z = x+iy\colon x > 0, -x < y < \sqrt{x}\}$. It does not try to be accurate, but to give an idea of the tangential convergence in $\mathbb{D}$. Since σ is conformal, it preserves angles, so we have tangential convergence for the points near the upper branch. Recall that tangential convergence for one point implies that all the points converge tangentially.

6 CLASSIFICATION OF THE FATOU COMPONENTS OF ENTIRE FUNCTIONS.

The aim of this section is to use the Denjoy-Wolff Theorem and the Cowen's classification of fundamental domains in a more general setting.

Up to now, we worked with holomorphic self-maps of the unit disk $\mathbb{D}$, which can be thought as a very restrictive kind of maps. However, by the Riemann Mapping Theorem, any self-map of any simply connected region of the complex plane is conjugate to a self-map of the unit disk, where the results we proved apply.

In this chapter, we deal with entire functions, that is holomorphic maps defined in the whole complex plane $\mathbb{C}$. Given an entire map f, we are interested in the dynamical system defined by f. Our goal is to study the long-term behaviour of the points of the complex plane under the iteration of f. A special case are the fixed points, that is the points that remain motionless under f. We study them in section 6.1. This will give us some ideas about how the points that are close to fixed points move under the iteration of f, but we will not be able to describe the dynamics globally yet.

To study globally the dynamical systems, we need the concepts of Fatou and Julia sets, that are introduced in section 6.2. The idea is that we will split the plane into two different invariant sets, namely the Fatou and Julia sets. The Fatou set contains the points where the function is well-behaved and the Julia set, the points where it is chaotic.

We are interested in the connected components of the Fatou set, that we will prove to be simply connected. In particular, for those which are invariant under f, we can conjugate f to a self-map of the unit disk. Applying the Denjoy-Wolff Theorem, we will be able to describe the dynamics on the Fatou components. The classification of the Fatou components depending on its dynamics (Theorem 6.11) is the main result of this chapter.

Finally, we use Cowen's classification of fundamental sets to describe the dynamics inside each Fatou component. As we will see, the interesting dynamics happen in the so-called *Baker domains*, where all the types of convergence can occur. This is studied in detail in section 6.6.

We refer to [4], [9] or [21] for general background in complex dynamics. The content specifically related to entire functions can be found in [6] and [16]. The classification of Baker domains can be found in [18].

6.1 Local Theory.

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic mapping. We write $f^0 = id$ and $f^n = f \circ f^{n-1}$. Given any point $z \in \mathbb{C}$ we define its **orbit** as the set of points $\{z, z_1 = f(z), z_2 = f^2(z), ...\}$.

We say that z_0 is a **fixed point** of f if $f(z_0) = z_0$. The number $\lambda = m(f, z_0) = f'(z_0)$ is called the **multiplier** of f at the fixed point z_0. It allows us to classify the fixed points into three types: attracting ($|\lambda| < 1$), repelling ($|\lambda| > 1$) or neutral ($|\lambda| = 1$). Among the neutral fixed points, we distinguish the rationally neutral (if λ is a root of the unity) and the irrationally neutral (otherwise).

A point z_0 is called **periodic** if $f^n(z_0) = z_0$, for some n. The minimal $n > 0$ such that this equality is true is called the **period** of z_0. The fixed points of f are periodic points of period one. More generally, z_0 has period n if and only if z_0 is a fixed point of f^n but not of any lower-order iterate. The orbit $\{z_0, z_1, ..., z_{n-1}\}$ is called a **cycle**. As we have done with the fixed points, we can classify the cycles into attracting, repelling or neutral, according to their multiplier as fixed points of f^n. By the chain rule, $\lambda = m(f^n, z_0) = (f^n)'(z_0) = f'(z_0) \cdot ... \cdot f'(z_{n-1})$. Therefore, the derivative $(f^n)'$ has the same value at each point of the cycle, and we can speak of attracting, repelling or neutral cycles.

A point z_0 is **pre-periodic** if it is not periodic under f but some image, say $f^m(z_0)$ is. That is to say that $f^k(z_0) \neq z_0$ for all k, but there exist m, n such that $f^{m+n}(z_0) = f^m(z_0)$.

In what follows we study a little more in detail the different types of fixed points. We want to see that what we have defined as attracting or repelling fixed points actually attract or repel close points. Since the periodic points of f are fixed points of f^n for some n, the following description extends to periodic points.

6.1.1 Attracting fixed points.

If z_0 is an attracting point under f, then there exists ρ such that $|f'(z_0)| < \rho < 1$. Hence, taking z in a small enough neighbourhood of z_0, we have:

$$\frac{|f(z) - z_0|}{|z - z_0|} = \frac{|f(z) - f(z_0)|}{|z - z_0|} < \rho$$

That is to say, $|f(z) - z_0| < \rho\,|z - z_0|$. So $f(z)$ is closer to z_0 than z. Repeating the argument inductively, we get that $|f^n(z) - z_0| < \rho^n\,|z - z_0|$ and, therefore, the iterates f^n restricted to some neighbourhood of z_0 converge uniformly to z_0.

We define the **basin of attraction** of an attracting fixed point z_0, denoted by $A(z_0)$, as the set of points that converge to z_0 under f. That is:

$$A(z_0) = \left\{ z\colon\ f^n(z) \underset{n}{\longrightarrow} z_0 \right\}$$

We observe that $A(z_0)$ is open. In fact, since z_0 is attracting , there exists U open neighbourhood of z_0 where f^n converge uniformly to z_0 and $A(z_0) = \bigcup\limits_n f^{-n}(U)$, where $f^{-n}(U)$ is to be understood as $f^{-n}(U) = \{z \in \mathbb{C}\colon f^n(z) \in U\}$. The **immediate basin of attraction** of z_0, denoted by $A^*(z_0)$, is defined as the connected component of $A(z_0)$ that contains z_0.

In the case of an attracting cycle $\{z_0, z_1, ..., z_{n-1}\}$, its basin of attraction is defined as the set consisting of all the points z such that the successive iterates $f^n(z)$, $f^{2n}(z)$, ... converge towards a point of the cycle. That is:

$$A(\{z_0, .., z_{n-1}\}) = \left\{ z\colon\ f^{nk}(z) \underset{k}{\longrightarrow} z_i, \text{ for some } i \right\}$$

We remark that, if $f^{nk}(z) \underset{k}{\longrightarrow} z_0$, then $f^{nk+i}(z) \underset{k}{\longrightarrow} z_i$, for $0 < i < n$. Therefore, although the definition of the basin of attraction of a cycle only considers the convergence of one subsequence (namely, the iterates which are multiple of n), the convergence of the other subsequences is unequivocally determined.

A special case is the **super-attracting** fixed points, that is the fixed points that have multiplier $\lambda = 0$. The dynamics around them is the same as if they were attractive. The main difference is that f has local inverse near z_0 if it is attracting, but not if it is super-attracting, since the map has local degree 2 or larger.

6.1.2 Repelling fixed points.

Suppose now that z_0 is a repelling fixed point of f. Then, $|f'(z_0)| > 1$. So, in a small enough neighbourhood of z_0, $|f(z) - f(z_0)| > |z - z_0|$. Therefore, points close to z_0 get away from it, at least during the first iterations. This does not mean that the orbit cannot return, but it is true that we can build some neighbourhood U of z_0 such that the orbits of the points of U escape from U after some iterates.

Since f is one-to-one in a neighbourhood of z_0, one can consider the branch g of f^{-1} which fixes z_0, and then z_0 is an attracting fixed point of g with multiplier $\dfrac{1}{f'(z_0)}$.

6.1.3 Rationally neutral fixed points.

We defined a fixed point z_0 as rationally neutral if its multiplier λ is a root of the unity. First suppose that $\lambda = 1$. Conjugating by a proper map, we can suppose that $z_0 = 0$ and write:

$$f(z) = z + az^{n+1} + (\text{higher terms}), \quad a \neq 0$$

The integer $n + 1 \geq 2$ is called the **multiplicity** of the fixed point. By definition, the fixed points with multiplier $\lambda \neq 1$ have multiplicity 1.

By the Inverse Function Theorem, since $f'(0) = 1$, we can find a neighbourhood N of 0 so that $f\colon N \to N' = f(N)$ is a diffeomorphism. Since N and N' are open neighbourghoods of the origin, its intersection is also an open neighbourhood of the origin.

Definition 6.1. Let U be a connected open subset such that $\overline{U} \subset N \cap N'$. U is an **attracting petal** for U at the origin if:

$$f(\overline{U}) \subset U \cup \{0\} \quad \text{and} \quad \forall z \in U,\ f^n(z) \to 0$$

U' is called a **repelling petal** for f if U' is an attracting petal for f^{-1}.

Theorem 6.2. (Leau-Fatou Flower Theorem) *Suppose that the origin is a fixed point with multiplicity $n+1 \geq 2$. Then, there exist n disjoint attracting petals U_i and n disjoint repelling petals U_i' such that the union of these $2n$ petals and the origin form a neighbourhood N_0 of the origin.*

Moreover, these petals alternate with each other such that U_i only intersects U_i' and U_{i-1}'.

Proof. We say that $v \in \mathbb{C}$ is an *attracting direction* if av^n is real and positive. Then, ignoring the higher order terms, $f(v) = v(1+av^n)$, so $|f(v)| < |v|$ and $arg(f(v)) = arg(v)$. On the other hand, v is a *repelling direction* if av^n is real and positive. Since $av^n = 1$ and $av^n = -1$ have n different solutions, there exists n equally spaced attracting directions which are separed by n equally spaced repelling directions.

Now, consider the change of coordinates $z \mapsto w = \frac{-1}{naz^n}$. Then, the sector between two repelling directions in the z-plane will correspond to the whole w-plane minus the negative real axis. In particular, the part of a neighbourhood of the origin between two consecutive repelling directions correspond to a neighbourhood of ∞ in the slit w-plane.

The corresponding map in the w-plane is $g(w) = w\left(1 + \frac{1}{w} + o\left(\frac{1}{w}\right)\right) = w + 1 + o(1)$. So, for all $\varepsilon > 0$, there exists $r > 0$ such that $|f(w) - (w+1)| < \varepsilon$ for all w, $|w| > r$. Taking ε small enough, we can take $M > r$ such that $U = \{w = u + iv,\ u > M\}$ is an attracting petal. Indeed, $f(\overline{U}) \subset U$ and, for all $z \in U$, $f(z) \to \infty$. □

As a consequence of this theorem, we have that in each attracting petal U_i, $f^n_{|U_i}$ converges uniformly towards the origin. On the other hand, all the orbits that start in a repelling petal U_i' must escape from it. This tells us that there cannot be any periodic orbit totally contained in N_0 (apart from the origin, that is a fixed point).

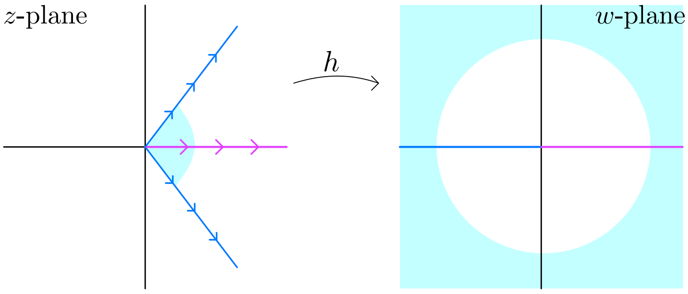


Figure 22: The correspondance, by h, of a neighbourhood of the origin to a neighbourhood of ∞.

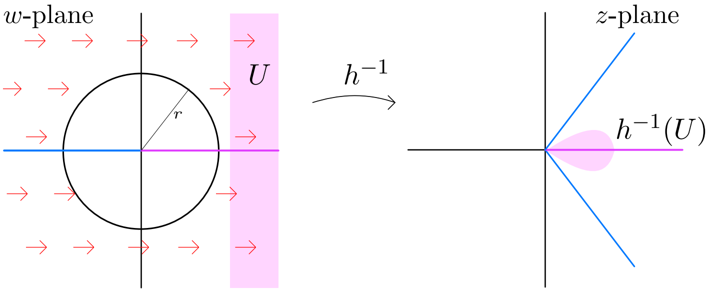


Figure 23: How we choose the petal U and its preimage by h.

Each attracting petal U_i determines a **parabolic attracting basin** Ω_i that consists in all the points that each orbit eventually falls into U_i and, therefore, converge towards the fixed point through U_i.

Suppose now that the multiplier λ is different from 1. Therefore, λ is a root of the unity and we can write $\lambda = e^{2\pi i \frac{p}{q}}$, where $\frac{p}{q}$ is a fraction in lowest terms. As above, supposing that the fixed point is the origin, we can write:

$$f(z) = \lambda z + ...$$

If we compute the q-fold iterate:

$$f^q(z) = z + az^{m+1} + ... \quad \text{where } a \neq 0$$

Then, $m(f^q, 0) = 1$ and we are in the previous case. Therefore, f^q has m attracting petals and m repelling petals. The set of attracting petals U_i is forward invariant under f, but not each petal, which are invariant under f^q. Therefore, f permute the petals in cycles of length q.

6.1.4 Irrationally neutral fixed points.

Now we are interested in irrationally neutral fixed points, that is the ones with multiplier $\lambda = e^{2\pi i \xi}$, where $\xi \notin \mathbb{Q}$. We will classify them depending on if f is locally linearizable around them or not.

We say that f is **locally linearizable** around a neutral fixed point with multiplier λ, if there exists a conjugacy h from f to $g(z) = \lambda z$, in a small enough neighbourhood of z_0. That is

$$h \circ f(z) = \lambda h(z),$$

for z in a neighbourhood of z_0.

Theorem 6.3. *Let z_0 be a fixed point of f with multiplier λ, $|\lambda| = 1$. Then, f is locally linearizable around z_0 if and only if the family of iterates $\{f^n\}_n$ is normal in some neighbourhood of z_0.*

Proof. We may assume $z_0 = 0$. If there exists such conjugation, $\{f^n\}_n$ must be bounded in some neighbourhood of the origin, so the family of iterates restricted to this neighbourhood is normal.

Conversely, suppose $\{f^n\}_n$ is normal in some neighbourhood of the origin. Therefore, it is equicontinuous near 0. So, for each $\varepsilon > 0$, there is some $\delta > 0$ such that: if $|z| < \delta$, $|f^n(z)| < \varepsilon$, for all n. For each $n \geq 1$, consider:

$$\phi_n(z) = \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} f^j(z).$$

Observe that, for all n and for $|z| < \delta$, $|\phi_n(z)| < \varepsilon$. So the family $\{\phi_n\}_n$ is normal and contains a convergent subsequence that converges uniformly to a function ϕ. Since $\phi_n(f(z)) = \frac{n+1}{n}\lambda\phi_{n+1}(z) - \frac{\lambda z}{n}$, we obtain $\phi \circ f = \lambda\phi$. □

Definition 6.4. We say that an irrationally neutral fixed point z_0 is a **Siegel point** if it is locally conjugate to an irrational rotation $g(z) = \lambda z$. The maximal neighbourhood of z_0 on which f is conjugate to the irrational rotation is called the **Siegel disk**.

If such conjugation is not possible, we say that z_0 is a **Cremer point**.

6.2 Global Theory. Fatou and Julia sets.

Once we analysed the local behaviour of the dynamics around the fixed points, we want to study the dynamics in a more global sense.

Given a entire map f, we use it to split the complex plane into two disjoint invariant sets, namely the Fatou set and the Julia set. The Fatou set is the set on which the function is well-behaved, in the sense that each point behaves similarly to the points around it. On the other hand, the Julia set is the set of points where the behaviour of the function is chaotic and the behaviour of one point cannot be predicted by the behaviour of the points around it.

Now we give a formal definition of the Fatou and Julia sets.

Definition 6.5. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic map.

Take $z_0 \in \mathbb{C}$. If there is a neighbourhood U of z_0 so that the sequence of iterates $\{f^n\}_n$ restricted to U forms a normal family, we say that z_0 belongs to the **Fatou set** $F(f)$.

Otherwise, if no such neighbourhood exists, we say that z_0 belongs to the **Julia set** $J(f)$.

It is straightforward from the definition that the Fatou set $F(f)$ is open and, therefore, the Julia set $J(f)$ is closed.

Theorem 6.6. *For any integer $p \geq 1$, the Fatou set and the Julia set for f and for f^p coincide. That is, $F(f^p) = F(f)$ and $J(f^p) = J(f)$.*

Proof. We write $g = f^p$. First, we observe that $F(g) \subset F(f)$, because $\{g^n,\ n \geq 1\}$ is a subfamily of $\{f^n,\ n \geq 1\}$, so it is going to be normal wherever $\{f^n,\ n \geq 1\}$ is.

To see the other inclusion, for any $k \geq 0$, we consider the family $\mathcal{F}_k = \left\{f^k \cdot g^n,\ n \geq 1\right\}$. $\mathcal{F}_k$ is normal wherever $\{g^n,\ n \geq 1\}$ is. Therefore, $\mathcal{F}_k$ is normal in $F(g)$ and $\mathcal{F}_0 \cup ... \cup \mathcal{F}_{p-1}$ too. But $\mathcal{F}_0 \cup ... \cup \mathcal{F}_{p-1} = \bigcup_{k=0}^{p-1} \left\{f^k \cdot g^n,\ n \geq 1\right\} = \{f^n,\ n \geq 1\}$. Thus, $\{f^n,\ n \geq 1\}$ is normal in $F(g)$, so $F(f) \subset F(g)$. □

Now we are going to see the property of invariance of the Julia and Fatou sets. We begin by defining the different types of invariance. If f is a map of $\mathbb{C}$ into itself and E is a subset of $\mathbb{C}$, E is *forward invariant* if $f(E) \subset E$ and *backward invariant* if $f^{-1}(E) \subset E$. We remark that $f^{-1}(E)$ means the set of the preimages of all the points in E. We say that E is **completely invariant** if it is both forward and backward invariant.

Theorem 6.7. *Given a holomorphic map f, its Fatou set is completely invariant under f. Consequently, the Julia set is also completely invariant under f.*

Proof. Let $z \in F(f)$ and we consider a neighbourhood U of z such that $U \subset F(f)$. Therefore, the sequence of iterates $\{f^n\}_n$ restricted to U forms a normal family, so any subsequence admits a convergent sub-subsequence in U. By the Open Mapping Theorem, $f(U)$ is open. We consider any subsequence $\{f^{n_k}\}_k$ in $f(U)$. Then, $\left\{f^{n_k+1}\right\}_k$ is a subsequence in U, that admits a convergent sub-subsequence $\left\{f^{n_{k_j}+1}\right\}_j$. Then, $\left\{f^{n_{k_j}}\right\}_j$ is a convergent subsequence of $\{f^{n_k}\}_k$. Therefore, $f(z)$ admits a neighbourhood $f(U)$ where $\{f^n\}_n$ is normal, so $f(z) \in F(f)$.

Applying the same argument to $\left\{f^{n_k-1}\right\}_k$ and taking into account that $f^{-1}(U)$ is open, we get that all the preimages $f^{-1}(z)$ admit a neighbourhood $f^{-1}(U)$ where $\{f^n\}_n$ is normal. $\square$

In the following proposition we classify the periodic points depending on if they belong to the Fatou or to the Julia set.

Theorem 6.8. *Let f be holomorphic and z_0 a periodic point of f. Then, if z_0 is an attracting periodic point or a Siegel periodic point, it belongs to the Fatou set of f. Moreover, the entire basin of attraction or the Siegel disk is contained in the Fatou set.*

On the other hand, if z_0 is repelling, parabolic or a Cremer periodic point, it belongs to the Julia set.

Proof. We may assume that z_0 is a fixed point.

First let z_0 be an attractive fixed point. For any z in the basin of attraction $A(z_0)$, we can find a neighbourhood of z, with its closure contained in $A(z_0)$. In such neighbourhood, $\{f^n\}_n$ converge uniformly to z_0. So, for all $z \in A(z_0)$, z belongs to the Fatou set.

Now, let z_0 be a repelling fixed point, so $f'(z_0) = a > 1$. Thus, $(f^n(z_0))' = a^n \to \infty$. If we suppose that $\{f^n\}_n$ form a normal family in some neighbourhood of z_0, there would exist a subsequence that converge to an homormorphic function g. It satisfies: $g(z_0) = z_0$ and $g'(z_0) = \lim_{n\to\infty} (f^n(z_0))' = \infty$, what is a contradiction. Therefore, there does not exists any neighbourhood of z_0 where the family of iterates is normal, so z_0 belongs to the Julia set.

Let us suppose that z_0 is a parabolic fixed point. Without loss of generality, we can suppose $z_0 = 0$ and $f'(0) = 1$. We can write f as: $f(z) = z + az^p + \cdots$, with $a \neq 0$. By induction, $f(z) = z + naz^p + \cdots$ and $(f^n)^{(p)}(0) = p!na \underset{n}{\to} \infty$. Hence, $\{f^n\}_n$ cannot be normal in any neighbourhood of the origin. Otherwise, some subsequence would converge to an holomorphic function g with $g(0) = 0$ and $g'(0) = \infty$. Therefore, 0 belongs to the Julia set.

The case of irrationally neutral fixed points is straighforward from Theorem 6.3. $\square$

We notice that $\partial A(z_0)$ belongs to the Julia set of f, for any attracting point z_0. In fact, since the points of $A(z_0)$ converge to z_0 but not in its complementary, any neighbourhood of a point in $\partial A(z_0)$ contains points which converge to z_0 and points that not.

Theorem 6.9. *$J(f)$ is the closure of the repelling periodic points of f.*

Proof. It is easy to see that periodic points of f are dense in $J(f)$, the difficulty comes in proving that there are infinitely many repelling periodic points.

We prove that periodic points of f are dense in $J(f)$. Suppose that U is an open subset such that $U \cap J(f) \neq \emptyset$ and U does not contain any periodic point of f. We may assume that U does not contain no Picard's exceptional value (in the case of transcendental entire functions) nor zeros of the derivative.

If φ_1, φ_2 are two different branches of f^{-1} in U, since there are no periodic points in U,

$$g_n = \frac{f^n - \varphi_1}{f^n - \varphi_2}$$

omits the values $0, 1, \infty$ in U. By Montel's Theorem, $\{g_n\}_n$ is normal in U, and hence so is $\{f_n\}_n$, a contradiction. SO periodic points must be dense in U.

For the proof of the existence of infinitelly many repelling fixed points, we refer to [4] or [9] for the rational functions, and to [16] for transcedental entire functions. □

By a **Fatou component** of a holomorphic function f we mean any connected component of the Fatou set of f.

Proposition 6.10. *If U is a Fatou component of f, then $f(U)$ is also a Fatou component.*

Proof. If U is a Fatou component, then U is open and $\partial U \subset J(f)$. Since $F(f)$ and $J(f)$ are completely invariant under f, $f(U) \subset F(f)$ and $f(\partial U) \subset J(f)$. Moreover $\partial f(U) \subset f(\partial U)$. Hence $f(U)$ must be a Fatou component. □

Therefore, we can think that f defines a dynamical system on the Fatou components. If U is a Fatou component of f, there are several possibilities for its orbit under f:

1. If $f^n(U) = U$ for some $n \geq 1$, then we call U a **periodic component** of f.
2. If $f^m(U)$ is periodic for some $m \geq 1$ but U is not periodic, we say that U is a **pre-periodic component** of f.
3. Otherwise, $f^n(U) \cap f^m(U) = \emptyset$, for all $n \neq m$. In this case, we call U a **wandering domain**.

Our goal is to prove the following theorem that classifies the periodic Fatou components of f, where f is an entire function. We recall that, among the entire functions, we distinguish the polynomials from the transcendental entire functions, as we saw in section 1. We remark that case 4 in the classification can only occur when the function is transcendental.

Theorem 6.11. (Classification of Fatou components) *Let f be a entire function, but not a linear polynomial, and U be a periodic Fatou component of period k. Then exactly one of the following holds:*

1. *U contains an attractive periodic point z_0 and $f^{nk} \to z_0$ uniformly on compact subsets of U. Then U is a component of the **basin of attraction** of z_0.*
2. *∂U contains a parabolic periodic point z_0 and $f^{nk} \to z_0$ uniformly on compact subsets of U. Then U is a component of the **parabolic basin of attraction** of z_0.*
3. *There is an $z_0 \in U$ irrationally neutral periodic point and $f^k_{|U}$ is conformally conjugate to an irrational rotation. Then, U is a **Siegel disk**.*

4. If f is transcendental, U can also be a ***Baker domain****. That is $f^{nk}(z) \to \infty$ uniformly on compact subsets of U.*

One important step of the proof is to show that all periodic components are simply connected and therefore, by the Riemann Mapping Theorem, conformally equivalent to $\mathbb{D}$, where the Denjoy-Wolff Theorem applies. Proving the simply connecteness requires different arguments depending on if the function is a polynomial or transcendental but, after that, the proof of the theorem finishes equal in both cases.

We remark that, if $\{U_1, \ldots, U_n\}$ is a cycle of Fatou components, they would be all of the same type. Indeed, if none of them is a Baker domain, we will have a cycle $\{p_1, \ldots, p_n\}$ of periodic points with the same multiplier. On the other hand, if U is a Baker domain, $f(U)$ must be a Fatou component without any periodic point in $\overline{f(U)}$, so $f(U)$ is a Baker domain.

6.3 Polynomials.

The aim now is to prove the following proposition, which asserts that all bounded Fatou components of a polynomial are simply connected.

Proposition 6.12. *Let f be a polynomial of degree $d \geq 2$, and U be a bounded Fatou component of f. Then U is simply connected.*

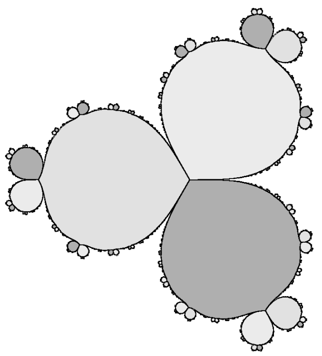


Figure 24: The Julia set of the polynomial $P(z) = z^4 + z$. [20]

First we show that polynomials can be extended to holomorphic maps of $\widehat{\mathbb{C}}$. Indeed, holomorphic maps of $\widehat{\mathbb{C}}$ are the rational maps, and polynomials are rational maps. Hence, given f polynomial, it can be extended to $\widehat{\mathbb{C}}$ by defining $f(\infty) = \lim_{|z| \to \infty} f(z) = \infty$

Therefore, ∞ is a fixed point of f. We are interested in computing its multiplier, but we do not know how to differentiate the function at ∞.

To do so, we consider the change of coordinates $h(z) = 1/z$. It moves ∞ to 0. Consider $g = h^{-1} \circ f \circ h$, so $g(z) = \frac{1}{f(\frac{1}{z})}$. Then, ∞ is a fixed point of f if and only if 0 is a fixed point of g. In this case, we define $m(f, \infty) = g'(0)$. We remark that we can find a neighbourhood of $g(0) = 0$ that does not contain ∞, so $g'(0)$ can be computed without problems.

Proposition 6.13. *Let f be a polynomial of degree $d \geq 2$. Then, ∞ is a super-attracting fixed point of f. That is $m(f, \infty) = 0$.*

Proof. We have already seen that ∞ is a fixed point of any polynomial. It remains to compute its multiplier.

As we have seen before, we may consider $g(z) = \frac{1}{f\left(\frac{1}{z}\right)}$. In our case, if f is a polynomial of degree $d \geq 2$: $f(z) = a_0 + a_1 z + ... + a_d z^d$. Therefore:

$$g(z) = \frac{1}{f\left(\frac{1}{z}\right)} = \frac{z^d}{a_d + ... + a_0 z^d}$$

Taking into account that $d \geq 2$, it is easy to check that $m(f, \infty) = g'(0) = 0$. □

Therefore, it makes sense to consider the basin of attraction of ∞, $A(\infty)$, which belongs to the Fatou set of f. Moreover, we deduce that the Julia set of a polynomial is bounded.

Recall that if a point z belongs to the Julia set of f, in each neighbourhood U of z the family $\{f^n\}_n$ is not normal. As a direct consequence to Montel's Theorem, there cannot exist three diferent values omitted by all the $f^n_{|U}$ in the family, otherwise it would be normal. Therefore, there exists $a, b \in \widehat{\mathbb{C}}$ such that

$$\widehat{\mathbb{C}} \setminus \{a, b\} \subset \bigcup_n f^n(U).$$

This phenomenon is known as the *blow-up property* of the Julia set and it is the key to prove the following propositions.

Proposition 6.14. *The Julia set $J(f)$ of any rational function f is non-empty.*

Proof. Suppose f of degree $d \geq 2$. Assume that the Julia set $J(f)$ is empty, so $\{f^n\}_n$ is normal in $\widehat{\mathbb{C}}$. Therefore, there exists a convergent subsequence $\{f^{n_j}\}_j$. Its limit function g must be holomorphic in $\widehat{\mathbb{C}}$, so it must be a rational function. If g is constant, then the image of f^{n_j} is eventually contained in a small neighbourhood of the constant value, which contradicts Liouville Theorem. If g is not constant then, by Hurwitz Theorem, there exists n_0 so that, for all $n \geq n_0$, f^n has the same number of zeros than f. But that is impossible because f^n has degree d^n. □

Proposition 6.15. *Let f be a rational function of degree $d \geq 2$. If z_0 belongs to the Julia set $J(f)$, then the set of all the bakward iterates of z_0 is dense in $J(f)$.*

Proof. We take any point $w \in J(f)$ and U some neighbourhood of it. We have to see that U contains some preimage of z_0. By the *blow-up property*, there exist some $a, b \in \widehat{\mathbb{C}}$ such that:

$$\widehat{\mathbb{C}} \setminus \{a, b\} \subset \bigcup_n f^n(U).$$

Therefore, if z_0 is neither a nor b, there exist some $N > 0$, such that $z_0 \in f^N(U)$ and for some $z \in U$, $f^{-N}(z_0) = z$.

We should consider the case of z_0 being a or b. Since $\{a, b\}$ is completely invariant under f and f is surjective, there are two options: they are both fixed points or it is an orbit of period 2. If a is a fixed point of f, then a is a zero of multiplicity d of $f - a$. So $f(z) - a = (z - a)^d + \cdots$ and $f'(a) = 0$. Then a belongs to the Fatou set, so $z_0 \neq a$. Similarly, we argue that $z_0 \neq b$. For the case that $\{a, b\}$ is a periodic orbit, a is a zero of multiplicity d of $f - a$, so $f(z) - a = (z - b)^d + \cdots$ and $f'(b) = 0$. Similarly, $f'(a) = 0$, so the multiplier $m(f, \{a, b\}) = f'(a)f'(b) = 0$. Therefore, $\{a, b\}$ belongs to the Fatou set, so z_0 can be neither a nor b. □

Proposition 6.16. *Any completely invariant subset of $J(f)$ is dense in $J(f)$.*

Moreover, if D is a union of Fatou components that is completely invariant, then $J(f) = \partial D$.

Proof. The second statement is a direct consequence of the first. Indeed, since D is completely invariant, ∂D is also completely invariant, which is closed and a subset of $J(f)$. Applying the first statement, $J = \overline{\partial D} = \partial D$.

For the first statement, we take J_1 subset of $J(f)$, which is completely invariant. By 6.15, for all $z \in J_1$, $\bigcup_n f^{-n}(z)$ is dense in J, but J_1 is completely invariant, so $\bigcup_n f^{-n}(z) \subset J_1$. Then, J_1 is dense in J. □

As a consequence of theorem 6.8 and proposition 6.16, we get that, for any attracting fixed point z_0, $J = \partial A(z_0)$. Clearly, $A(z_0)$ is completely invariant under f, so $\partial A(z_0)$ is completely invariant too. It is contained in the Julia set because any neighbourhood of any point in $\partial A(z_0)$ contains points that converge to z_0 and points that not, so the family of iterates cannot be normal. In particular, we have $J = \partial A(\infty)$.

Finally, we are ready to prove the main result of this section, Proposition 6.13, which asserts that all bounded Fatou components of a polynomial are simply connected.

Proof of Proposition 6.13. First, we are going to prove that $U \not\subset A(\infty)$. Suppose, on the contrary, $U \subset A(\infty)$. Since in ∂U the iterates do not tend to ∞, there exists M such that $|f^n(z)| < M$, for all $z \in \partial U$. By the Maximum Modulus Principle, f^n cannot tend to ∞ in U, so $U \subset A(\infty)$.

Now, since U is not simply connected, there is a non-contractible curve γ in U. We call V the bounded connected component of $\mathbb{C} \setminus \gamma$. Since $\gamma \not\subset A(\infty)$, there exists M such that $|f^n(z)| < M$, for all $z \in \gamma$. By the Maximum Modulus Principle, $|f^n(z)| < M$, for all $z \in V$. As we have seen before, $J(f) = \partial A(\infty)$, so V contains points of $A(\infty)$ and that is a contradiction. Therefore, bounded Fatou components of a polynomial must be simply connected. □

6.4 Transcendental Entire Functions.

Our goal now is to prove the following theorem, that claims that the periodic Fatou components of a transcendental map are simply connected.

Theorem 6.17. *Let f be a transcendental entire function and U a multiply connected Fatou component of f. Then, U is a wandering domain.*

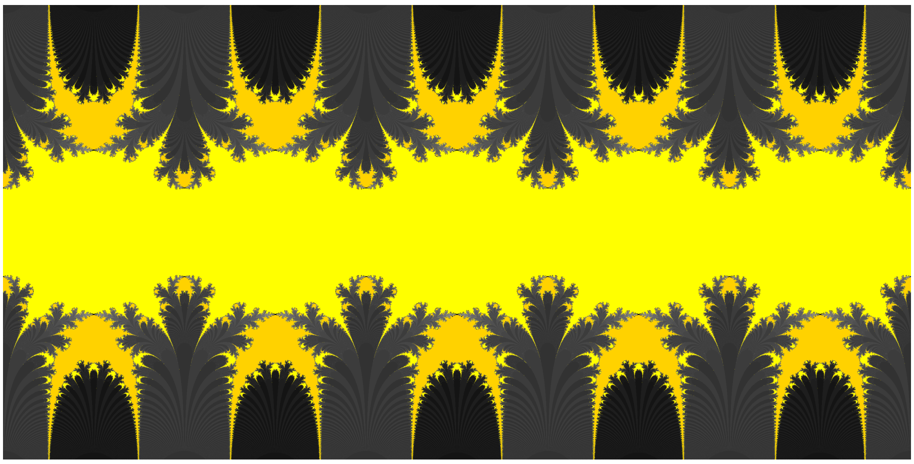


Figure 25: The Julia set of an entire function, with a Baker domain. [7]

We observe that we cannot follow the same strategy that in polynomials, because the functions cannot be extended continuously to $\widehat{\mathbb{C}}$. Indeed, transcendental entire maps have a essential singularity at ∞ and, by Picard's Theorem (1.7), the image of any neighbourhood of ∞ is the whole complex plane excluding, at most, one point. Hence, the function has a chaotical behaviour around ∞.

The idea of the proof of Theorem 6.17 is the following. First, we prove that the Julia set of a transcendetal entire function is non-empty and, in fact, unbounded (Theorem 6.18). Second, we show that in multiply connected Fatou components, $f^n \to \infty$ (Proposition 6.19). Third, we prove that multiply connected Fatou components are bounded (Proposition 6.22). With all these preliminary results, we will be able to prove the theorem.

The proof of Theorem 6.17, as well as the previous results, can be found are due to A. Baker an can be found in its papers [2] and [3], and also in the book [16].

Theorem 6.18. *Let f be a transcendental entire function. Then $J(f)$ contains an infinite number of points and it is unbounded.*

Proof. Consider $g = f^2$. Since $J(g) = J(f)$, we have to prove that, given $R > 0$, $J(g) \cap D \neq \emptyset$, where $D = \{z\colon\ |z| > R\}$. By Theorem 1.8, $g(z) - z$ has infinitely many zeros in $\mathbb{C}$. They cannot accumulate in the finite part of the plane, so there are infinitely many zeros in D, and hence we can take distinct $a_1, a_2 \in D$ such that $g(a_1) = a_1$ and $g(a_2) = a_2$. Applying Picard's Theorem (1.7) to $\{z\colon\ |z| < |a_1|\}$ we can find $b \in D$ such that $g(b) = a_1$ and $b \neq a_1$.

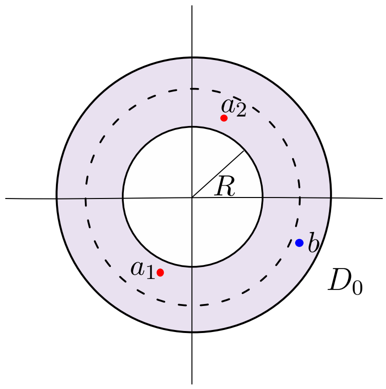


Figure 26: Schematic representation of the subset D_0.

Set $D_0 = \{z\colon R \leq |z| \leq |a_1| + |a_2| + |b| + 1\}$, so $a_1, a_2, b \in D_0$. We suppose $J(g) \cap D_0 = \emptyset$ and we are going to find a contradiction. Since D_0 does not contain points of the Julia set, $\{g_n\}_n$ is normal in D_0, so a_1 and a_2 are either attracting fixed points or Siegel points.

Suppose a_1 is attracting, then there exists $C = \{z\colon\ |z - a_1| < r\}$, with $C \subset D_0$ and $g^n \to a_1$ uniformly in C. On the other hand, since $\{g_n\}_n$ is normal in D_0, there exists some convergent subsequence g^{n_k} to some function h. As $g^n \to a_1$ in C, it must be $h \equiv a_1$ in C and, by analytic continuation, $h \equiv a_1$ in D_0, in particular $h(a_2) = a_1$. But this is impossible because $g^n(a_2) = a_2$ for all n. Therefore, a_1 must be a Siegel point, and a_2 too, by the same argument.

Now, suppose a_1 is a Siegel point. There exists $C = \{z\colon\ |z - a_1| < r\} \subset D_0$ and we can find a subsequence $\{g^{n_k}\}_k$ so that $g^{n_k}(z) \to z$ uniformly in C. By normality, there exists a convergent subsequence of $\{g^{n_k}\}_k$ to a function h. We have $h(z) = z$ in C and, by analytic continuation, $h(z) = z$ in D_0. In particular, $h(b) = b$, but $g^n(b) = a_1$ for all n. We have reached a contradiction, so it must be $J(g) \cap D_0 \neq \emptyset$ and the theorem is proved. $\square$

Proposition 6.19. *Let f be a transcendental entire function and U a multiply connected Fatou component of f. Then, $f^n \to \infty$ in U.*

Proof. Suppose that $f^n \nrightarrow \infty$ in U, that is, there exists some subsequence $\{f^{n_k}\}_k$ that remains bounded. So there exists M such that $|f^{n_k}(z)| < M$, for all $z \in U$ and for all k.

Consider a non-contractible curve γ in U and denote by V the bounded component of $\mathbb{C} \setminus \gamma$. Since f is holomorphic in V, the Maximum Modulus Principle applies and $|f^{n_k}(z)| < M$, for all $z \in V$. Therefore, the derivatives $(f^{n_k})'$ are bounded in V.

On the other hand, by Theorem 6.9, there must be repelling points in V. As we have seen in the proof of Theorem 6.8, $(f^n)'(z)$ tends to ∞ if z is repelling. And that is a contradiction with the fact that $(f^{n_k})'$ are bounded in V. □

Lemma 6.20. *Let f be a transcendental entire function and U a multiply-connected Fatou component of f. Suppose γ is a non-contractible curve in U, then there exists n_0 such that $Ind(f^n(\gamma), 0) > 0$ for all $n \geq n_0$.*

Proof. First, we remark that, since $f^n \to \infty$ in γ, for n big enough $0 \notin f^n(\gamma)$, it makes sense to consider $Ind(f^n(\gamma), 0)$.

Now, suppose that there exists some sequence $\{n_k\}_k$ such that $Ind(f^{n_k}(\gamma), 0) = 0$ for all n_k. By the Argument Principle, f^{n_k} has not any zero surrounded by γ. Then, by the Minimum Modulus Principle, $f^{n_k} \to \infty$ inside γ. And this is a contradiction because there are points of the Julia set inside γ. □

Lemma 6.21. *Let f be an entire function. Suppose that $\{\gamma_n\}_n$ is a collection of closed curves such that $\gamma_n \to \infty$ and $Ind(\gamma_n, 0) > 0$. Then, if $|f(z)| < |z|^C$ for all $z \in \gamma_n$, f is a polynomial.*

Proof. From each curve γ_n we can extract a Jordan curve which surrounds the origin. Therefore, we can think we have a collection of Jordan curves $\{\widetilde{\gamma}_n\}_n$ such that $Ind(\widetilde{\gamma}_n, 0) = 1$, $\widetilde{\gamma}_n \to \infty$ and $\widetilde{\gamma}_n \cap \widetilde{\gamma}_{n+1} = \emptyset$. We consider the space between two consecutive curves and we call it V.

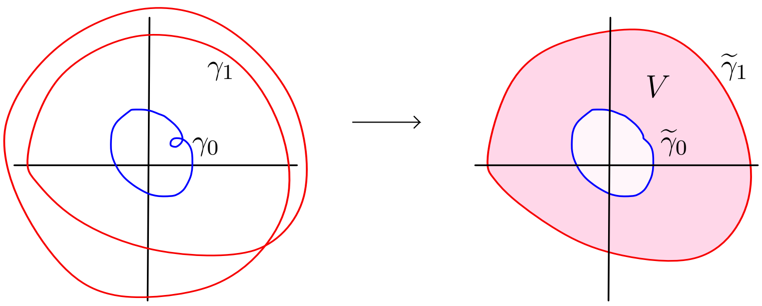


Figure 27: From the non simple curves on the left, we can extract the ones on the right that are simple and surround the origin. V is the spaces between any consecutive curves.

Consider the function

$$g(z) = \frac{f(z)}{z^C}.$$

It is holomorphic in V and $|g(z)| < 1$, for $z \in \partial V$. By the Maximum Modulus Principle, $|g| < 1$ in V so, for all $z \in V$, $|f(z)| < |z|^C$. Applying the same argument to all the curves $\widetilde{\gamma}_n$, we have the previous inequality for all the points in the unbounded connected component of $\mathbb{C} \setminus \widetilde{\gamma}_0$. So there exists $R > 0$ and a positive integer m such that:

$$|f(z)| < |z|^m, \quad \forall z \in \{w \colon |w| > R\}.$$

So, for all $r > R$, by Cauchy's inequalities, we have:

$$\left| f^{(m)}(z) \right| \leq \frac{m! \sup_{|z|=r} |f(z)|}{r^m} = \frac{m! \, |z|^m}{r^m} = m!, \quad \forall z \in D(0, r).$$

Making $r \to \infty$, we get that $\left|f^{(m)}\right|$ is bounded in $\mathbb{C}$. By Liouville's Theorem, $f^{(m)}$ must be constant. Hence, f is a polynomial. $\square$

Proposition 6.22. *Let f be a transcendental entire function and U an unbounded Fatou component of f. Then, U is simply connected.*

Proof. Suppose that U is unbounded and multiply connected, so there is a Jordan curve γ non-contractible in U. For n large enough, $Ind(f^n(\gamma), 0) > 0$ and $f^n_{|\gamma} \to \infty$, hence $f^n(\gamma)$ intersects U. As $f^n(\gamma) \subset F(f)$, $f^n(\gamma) \subset U$ for all n big enough. Therefore, U is invariant under f.

Take V a connected bounded open subset of U such that γ and $f(\gamma)$ are in V and $\overline{V} \subset U$. By Harnack's inequality (1.10) applied to $h_n(z) = \ln|f^n(z)|$, we get that for $z \in \gamma$ and $w \in f(\gamma)$,

$$|f^n(w)| \leq |f^n(z)|^C, \quad \forall n \geq n_0.$$

On the other hand, by lemma 6.21, for all $n \geq n_0$ there is $a \in \gamma_n$ such that $|f(a)| > |a|^C$. Setting $a = f^n(z_0)$, for some $z_0 \in \gamma$ and $f(z_0) = z_1 \in f(\gamma)$, we have:

$$|f^n(z_1)| > |f^n(z_0)|^C.$$

So we reached a contradiction and U must be simply connected. $\square$

Finally, we finish the proof of Theorem 6.17.

Proof of Theorem 6.17. It is straightforward from propositions 6.19 and 6.22. Indeed, if there exists k such that $f^k(U) = U$ and $f^n \to \infty$ in U, it must be $\infty \in \overline{U}$, but this contradicts the fact that U is bounded. $\square$

6.5 The proof of Thereom 6.11.

We can suppose $f(U) = U$. In the case of polynomials, if $U = A(\infty)$, then $f^n(z) \to \infty$, for all $z \in U$, so it corresponds to the first case. Hence, we can suppose that U is simply connected. As $U \neq \mathbb{C}$, by the Riemann Mapping Theorem, there exists $\varphi\colon U \to \mathbb{D}$. Therefore, f is conformly conjugate to $g := \varphi \circ f \circ \varphi^{-1}$. Then, g is a self-map of $\mathbb{D}$ so, by Denjoy-Wolff theorem, we have the following cases:

a) There is a unique fixed point $p \in \mathbb{D}$ of g and, for all $z \in \mathbb{D}$, $g^n(z) \to p$. Going back to U, that implies that $\varphi^{-1}(p) \in U$ is the only fixed point of f and, for all $z \in U$, $f^n(z) \to \varphi^{-1}(p)$. It corresponds to the first case.

b) There exists a point $p \in \partial\mathbb{D}$ such that, for all $z \in \mathbb{D}$, $g^n(z) \to p$.

c) There is a conjugation between g and a rational rotation. Then, there is some n such that $g^n = id_{\mathbb{C}}$. By analytic continuation g^n, and therefore f^n, would be the identity in $\mathbb{D}$, and that is not possible because we are avoiding the trivial case of f being a linear polynomial. Hence, this case never occurs.

d) There is a conjugacy between g and an irrational rotation. Then, U is a Siegel disk and it corresponds to the third case.

Our goal is to see that case (b) corresponds to the second or the fourth case in the theorem. We remark that the function φ given by the Riemann Mapping theorem may not be extendible to the boundary and $\varphi^{-1}(p)$ may not exist.

First we notice that, if we are in case (b), then $f^n(z) \to \partial U$, for all $z \in U$. Indeed, let z be any point in U and K any compact subset of U. Then $\varphi(z)$ is a point in $\mathbb{D}$ and $\varphi(K)$ a compact subset of $\mathbb{D}$. As we are supposing that case (b) applies, there exists some n such that $\varphi(f^n(z)) = g^n(\varphi(z)) \notin \varphi(K)$. Then, $f^n(z) \notin K$ and, as it happens for any compact subset of U, $f^n(z) \to \partial U$.

Second, we are going to proof that, in fact, all the orbits converge to a unique point in the boundary. That is that there exists some $p \in \partial U$ such that, for all $z \in U$, $f^n(z) \to p$. By lemma 4.2, it is enough to prove that, for some $w \in U$, its orbit converges to a single boundary point. By Schwarz-Pick lemma, if $\rho_U(w, f(w)) < r$, then $\rho_U(f^n(w), f^{n+1}(w)) < r$ for all n. Since $f^n(w) \to \partial U$, by lemma 3.5, $\left|f^n(w) - f^{n+1}(w)\right| \to 0$. We know that f is continuous in $\overline{U}$ except at most in one point, ∞, which can only occur in the case of U being an unbounded component of a transcendental entire function. Since $\overline{U}$ is compact (as a subset of $\widehat{\mathbb{C}}$), there exists some limit point of $\{f^n(w)\}_n$ in $\overline{U}$. If there exists some limit point p where f is defined, it must be a fixed point of f:

$$p = \lim_n f^n(w) = \lim_n f^{n+1}(w) = f(\lim_n f^n(w)) = f(p).$$

But fixed points are isolated and limit sets are connected, hence $f^n(w)$ must converge to p. On the other hand, if there is no limit point of $\{f^n(w)\}_n$ where f is defined, the only possibility is $f^n(w) \to \infty$. In this case, we have that U is a Baker domain (case 4 in the theorem). We remark that this can only happen with transcendental entire functions in unbounded Fatou components.

Observe that p cannot be attracting, since $p \in J(f)$, so $|f'(p)| \geq 1$. On the other hand, p cannot be repelling, since it attracts all the points in U, so $|f'(p)| \leq 1$. Therefore, $|f'(p)| = 1$.

Finally, we are going to proof that $f'(p) = 1$. Intuitively, we expect $f'(p) = 1$ to hold because, since $|f'(p)| = 1$, f acts like a rotation about p on a small neighbourhood of p and, as $f(U) = U$, this rotation must be trivial. Now we prove it rigorously.

We can assume that $p = 0$ and $\infty \in \partial U$. We can take a connected subset W of U, forward invariant by f and with $f_{|W}$ one-to-one. Indeed, since $|f'(0)| = 1$, f is one-to-one in some neighbourhood of 0 and $f^n \to 0$ uniformly on compact subsets. Then, we take V any open subset of U such that for some $z \in V$, $f(z) \in V$. Then, for some N,

$$W = \bigcup_{n=N}^{\infty} f^n(V)$$

has the required properties.

Then, consider $z_0 \in W$ and, for all $n \geq 1$, we define:

$$\begin{aligned} \varphi_n \colon W &\longrightarrow \widehat{\mathbb{C}} \\ z &\longmapsto \varphi_n(z) = \frac{f^n(z)}{f^n(z_0)} \end{aligned}$$

We remark that, for all n, $\varphi_n(z_0) = 1$.

Remark. *The family $\{\varphi_n\}_n$ is normal in W.*

Proof. First, φ_n does not take the values 0 and ∞ in W, because f^n does not. Since φ_n is one-to-one in W, because f is, and $\varphi_n(z_0) = 1$ for all n, then φ_n does not take any of the values $0, 1, \infty$ in $W \setminus \{z_0\}$. Then, $\{\varphi_n\}_n$ is normal in $W \setminus z_0$.

To proof that $\{\varphi_n\}_n$ is normal in W it is enough to see that it is normal in some neighbourhood of z_0. Let $D \coloneqq D(z_0, r)$, for some $r > 0$ such that $D \subset W$ and let $C \coloneqq \partial D$. By normality, we can find a subsequence $\left\{\varphi_{n_j}\right\}_j$, which converges uniformly on compact subsets to some function φ on $W \setminus \{z_0\}$. Then we have

$$\varphi_{n_j}(z) = \frac{1}{2\pi i}\int_C \frac{\varphi_{n_j}(w)}{w-z}dw \longrightarrow \frac{1}{2\pi i}\int_C \frac{\varphi(w)}{w-z}dw,$$

and the convergence is uniform. Defining $\varphi(z_0) = 1$, it follows that φ_{n_j} converges uniformly on compact subsets to φ on W. $\square$

Having proved the normality of $\{\varphi_n\}_n$, we can assume that there exists some subsequence φ_{n_j} converges uniformly on compact subsets to φ on W. Then,

$$\varphi_n(f(z)) = \frac{f^n(f(z))}{f^n(z_0)} = \frac{f^n(f(z))}{f^n(z)}\frac{f^n(z)}{f^n(z_0)} = \varphi_n(z)\frac{f^n(f(z))}{f^n(z)} = \varphi_n(z)\frac{f^n(f(z)) - f(0)}{f^n(z) - 0}.$$

So, when $n \to \infty$, $\varphi(f(z)) = f'(0)\varphi(z)$. Since φ_n are one-to-one on W, by Hurwitz Theorem, φ must be constant or one-to-one in W. If φ is constant in W, then it must be $\varphi \equiv 1$, as $\varphi(z_0) = 1$. Therefore, $f'(0) = 1$, as we wanted to proof.

Now assume that φ is one-to-one in W. Setting $\lambda = f'(0)$, we have $\varphi(f(z)) = \lambda\varphi(z)$ and $\varphi(f^n(z)) = \lambda^n\varphi(z)$. Evaluating in z_0, it gives us $\varphi(f^n(z_0)) = \lambda^n\varphi(z_0) = \lambda^n$. Since $|\lambda| = 1$, there exists an increasing sequence of integers $\{m_j\}_j$ such that $\lambda^{m_j} \to 1$ (because either $\lambda^m = 1$ for some m or $\{\lambda^n\}_n$ is dense in S^1). Then, $\varphi(f^{m_j}(z_0)) = \lambda^{m_j} \to 1$. Since $1 \in \varphi(W)$ and $\varphi(f_j^m(z_0)) \in \varphi(W)$ and φ is invertible in W (because it is one-to-one), $f^{m_j}(z_0) \to z_0 \in W$. This is a contraction with the fact that $f^n(z) \to 0 \in \partial U$, for all $z \in U$. So φ must be constant and the theorem is proved.

6.6 Application of Cowen's Theorem: Classification of Baker domains.

Finally, we classify the Baker domains according on the dynamics inside them. Since they are simply connected, all the theory developed in section 5 applies. We will also see that for the other types of Fatou components the dynamics near the fixed point are not subject to classification since they do not depend on the map.

We follow the ideas of the paper [18]. For more examples on Baker domains, see [5] and [12].

Indeed, if we take any Fatou component of f, different from $A(\infty)$ in the case of polynomials, it need to be simply connected. By the Riemann Mapping Theorem, there exists a conformal map $\varphi\colon U \to \mathbb{D}$, so f is conformally conjugate to $g = \varphi \circ f \circ \varphi^{-1}$. Since g is a self-map of $\mathbb{D}$, Cowen's Theorem 5.3 applies, so there exists a fundamental set for g in $\mathbb{D}$ where g is one-to-one. We take $V \subset U$ so that $\varphi(V)$ is such a fundamental set. Moreover, we know that $g_{|\varphi(V)}$ is conjugate by σ to some Möbius Transformation of Ω, with $\Omega = \mathbb{C}$ or $\Omega = \mathbb{D}$. Writing $\psi = \sigma \circ \varphi$, we have:

$$\begin{tikzcd}
V \subset U \arrow[r, "f"] \arrow[d, "\varphi"] \arrow[dd, bend right=60, "\psi"'] & V \subset U \arrow[d, "\varphi"'] \arrow[dd, bend left=60, "\psi"] \\
\varphi(V) \subset \mathbb{D} \arrow[r, "g"] \arrow[d, "\sigma"] & \varphi(V) \subset \mathbb{D} \arrow[d, "\sigma"'] \\
\psi(V) \subset \Omega \arrow[r, "\phi"] & \psi(V) \subset \Omega
\end{tikzcd}$$

Since σ is one-to-one in the fundamental domain, ψ is one-to-one in V, so ψ conjugates f and ϕ in a neighbourhood of the fixed point.

When the fixed point lies in U is easy to describe the dynamics around it. If we take U to be a Siegel disk, we know that it is conjugate to an irrational rotation, so we must take $\Omega = \mathbb{D}$ and ϕ an irrational rotation. Moreover, the only possible fundamental set V is $V = U$.

On the other hand, if the fixed point is attracting, it corresponds to a fixed point of a non-conformal self-map of $\mathbb{D}$, so we must take $\Omega = \mathbb{C}$ and $\phi(z) = \lambda z$, where λ is the multiplier of the fixed point.

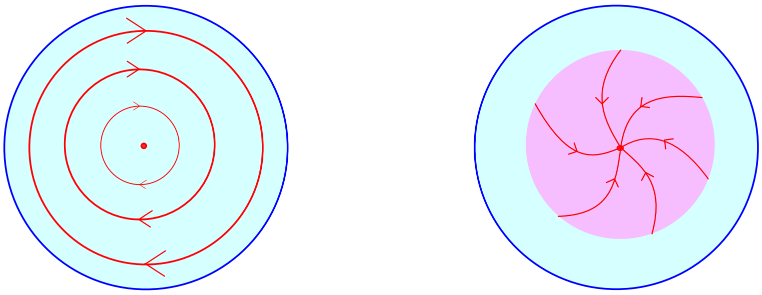


Figure 28: In the left, the dynamics in a Siegel disk. In the right, the dynamics in some neighbourhood of an attracting fixed point.

Suppose now that the orbits in U converge to a parabolic fixed point p of multiplicity $n \geq 1$. Then, $f'(p) = 1$ and, by Theorem 6.2, there are n equally spaced attracting directions separed by n equally spaced directions. Therefore, U is placed between two repelling directions, and it contains a petal. The petal is a fundamental set, so its dynamics must be conjugate to some Möbius transformation of either $\mathbb{C}$ or $\mathbb{D}$.

On the other hand, $f'(p) = 1$. Assuming that the Denjoy-Wolff point of g is $1 \in \mathbb{D}$, so φ extends continuously to $U \cup \{p\}$ with $\varphi(p) = 1$, then:

$$\lim_{r \to 1^-} g'(r) = \lim_{r \to 1^-} \left(\varphi \circ f \circ \varphi^{-1}\right)'(r) = \lim_{r \to 1^-} \varphi'(f(\varphi^{-1}(r)))f'(\varphi^{-1}(r))\frac{1}{\varphi'(r)} = \varphi'(p)f'(p)\frac{1}{\varphi'(p)} = 1.$$

Therefore, it must be *Case 2* or *Case 4* in Cowen's classification. However, since the points of the attracting direction converge nontangentially, it is *Case 2*. So we must take $\Omega = \mathbb{C}$ and $\phi(z) = z + 1$.

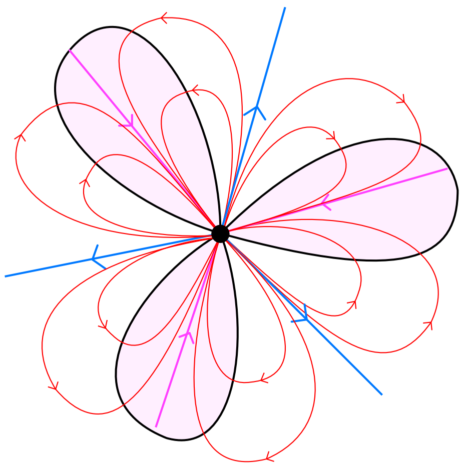


Figure 29: The dynamics around a parabolic fixed point.

The case of U being a Baker domain is different because f is not defined at ∞. Beforehand, it can correspond to *Cases 2, 3* or *4* in Cowen's classification and, actually, all occur.

We denote by $\mathbb{H}$ the upper half-plane.

Theorem 6.23. (Classification of Baker domains) *Let B be a Baker domain of f and $V \subset B$ a fundamental set for f in B. Then, taking $\Omega = \mathbb{C}$ or $\Omega = \mathbb{H}$, there exists a map $\psi\colon B \to \Omega$, which is one-to-one in V, and a Möbius transformation $\phi\colon \Omega \to \Omega$, such that $\psi \circ f = \phi \circ \psi$. Moreover, Ω is unambiguously determined and ϕ is unique up to conjugation, and they can be chosen among the following:*

(a) $\Omega = \mathbb{C}$ *and* $\phi(z) = z + 1$. *In this case, we say that B is* ***doubly-parabolic***.

(b) $\Omega = \mathbb{H}$ *and* $\phi(z) = sz$, *with* $0 < s < 1$. *In this case, we say that B is* ***hyperbolic***.

(c) $\Omega = \mathbb{H}$ *and* $\phi(z) = z \pm 1$. *In this case, we say that B is* ***simply-parabolic***.

As we show in the following theorem, if the function is one-to-one in a Baker domain, then it cannot be doubly-parabolic.

Theorem 6.24. (Classification of univalent Baker domains) *Let B be a Baker domain of f and suppose that $f_{|B}$ is one-to-one. Then B must be hyperbolic or simply-parabolic.*

Proof. Since $f_{|B}$ is one-to-one and $f(B) = B$, we have that $f\colon B \to B$ is conformal. Taking φ a conformal map between B and $\mathbb{H}$, f is conjugate to $\psi = \varphi \circ f \circ \varphi^{-1}$, which is a Möbius transformation of $\mathbb{H}$ without fixed points in $\mathbb{H}$.

Then, by the unicity in Theorem 5.3, $\Omega = \mathbb{H}$. So it must be *Case 3* or *Case 4*, which correspond to hyperbolic and simply-parabolic, respectively. $\square$

We remark that, applying the same argument that in this last theorem, we can deduce that f cannot be one-to-one in U, when U is a fixed Fatou component with an attracting or parabolic fixed point. Therefore, there is a point in U when f' vanishes.

Finally, we present examples of the different types of Baker domains.

Example 6.25. (Doubly-parabolic) Let us consider the map:

$$f(z) = z + e^{-z}.$$

It is semi-conjugate to $g(w) = we^{-w}$ by the map $h(z) = e^{-z} = w$, as follows:

$$\begin{tikzcd} z \arrow[r, "f"] \arrow[d, "h"] & z + e^{-z} \arrow[d, "h"'] \\ w \arrow[r, "g"] & we^{-w} \end{tikzcd}$$

First, we analyse the map g. Its only fixed point is the origin, and it has multiplier $g'(0) = 1$. Therefore, it is parabolic and, since $g(w) = w - w^2 + O(w^3)$ near the origin, there is one attracting and one repelling direction and a parabolic basin of attraction A. The real axis is invariant under g and, for $x < 0$, $g(x) < x$ and, for $x > 0$, $0 < g(x) < x$. Therefore, the positive real axis is contained in A.

Returning to the map f, we observe that the preimages of $\mathbb{R}^-$ under e^{-z} are the horitzontal lines $\{\mathrm{Im}(z) = (2k+1)\pi,\ k \in \mathbb{Z}\}$. They are invariant under f and their points converge to ∞ to the left. Each strip $\mathbb{R} \times ((2k-1)\pi, (2k+1)\pi)$, $k \in \mathbb{Z}$, contains a preimage of A and therefore a preimage of the attracting direction, that is a straight line whose points converge to ∞ to the right. The following figure shows an outline of the dynamics, and we deduce that it is a doubly-parabolic Baker domain.

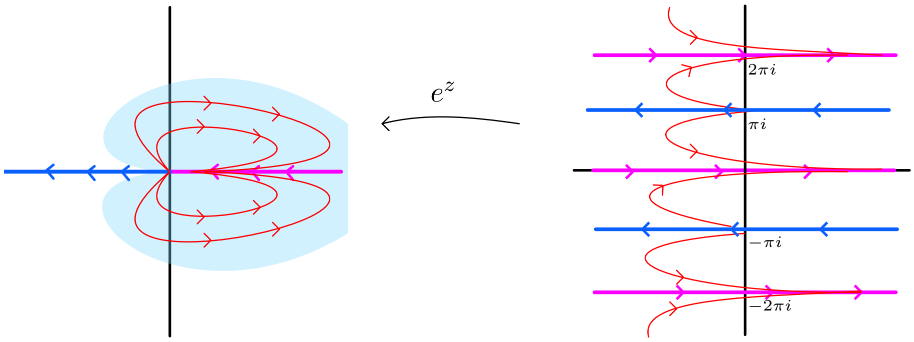


Figure 30: On the left, an outline dynamic plane of g, with the parabolic basin of the origin and the atracting and repelling directions. On the right, an outline dynamic plane of f, with a Baker domains on each horitzontal strip.

Example 6.26. (Hyperbolic) Let us consider the map:

$$f(z) = 2 - \log 2 + 2z - e^z.$$

Note that f is semi-conjugate to $g(w) = \frac{1}{2}w^2e^{2-w}$ by the map $h(z) = e^z = w$, as follows:

$$\begin{tikzcd} z \arrow[r, "f"] \arrow[d, "h"] & 2 - \log 2 + 2z - e^z \arrow[d, "h"] \\ w \arrow[r, "g"] & \frac{1}{2}w^2e^{2-w} \end{tikzcd}$$

The map g has two fixed points: 0 and 2, which is super-attracting. Consider its immediate basin of attraction $A(0)$, which its preimage by e^z consists on a Baker domain B. It contains some left half-plane $\{\mathrm{Re}(z) < z_0\}$, for some z_0.

Observe that, for z with $|z|$ big enough, the map f acts like $\widetilde{f}(z) = 2z$. Therefore, B must be hyperbolic.

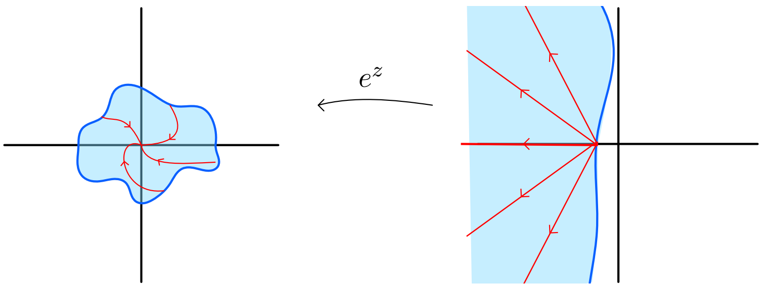


Figure 31: On the left, an outline dynamic plane of g, with the immediate attracting basin $A(0)$. On the right, an outline dynamic plane of f, with the hyperbolic Baker domain on the left.

Example 6.27. (Simply-parabolic) Let us consider the map:

$$f(z) = \alpha + z + e^z, \quad \alpha \in \mathbb{C}.$$

It is semi-conjugate to $g(w) = e^\alpha we^w$ by the map $h(z) = e^z = w$, as follows:

$$\begin{tikzcd} z \arrow[r, "f"] \arrow[d, "h"] & \alpha + z + e^z \arrow[d, "h"] \\ w \arrow[r, "g"] & e^\alpha we^w \end{tikzcd}$$

The map g has an only fixed point at the origin, with mutiplier e^{α}. It can be taken $\alpha = i\theta$, $\theta \in \mathbb{R}$, so that the origin is a Siegel point. Therefore, there is a neighbourhood of the origin where the orbits rotate around it.

Coming back to f, we see that the preimages of this Siegel disk under e^z correspond to an invariant domain U that contains a left half-plane $\{\mathrm{Re}(z < x_0)\}$, for some x_0. The orbits inside the Siegel disk correspond to almost vertical lines in U, whose points converge vertically to ∞. Thus, U is a simply-parabolic Baker domain.

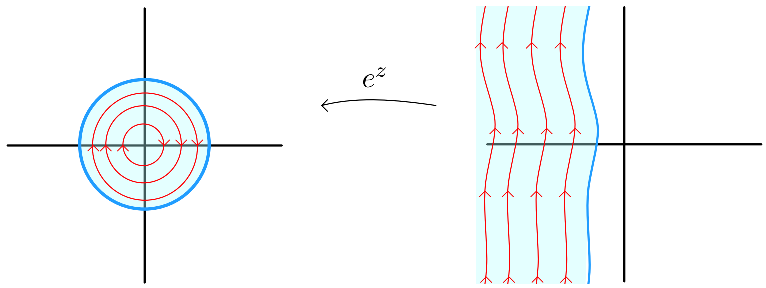


Figure 32: On the left, an outline dynamic plane of g, with the Siegel disk around 0. On the right, an outline dynamic plane of f, with a Baker domain on the left.

CONCLUSIONS.

Finally, I proceed to provide an overview of the project. I comment some of the results in a more personal and informal way than when they are discussed in the dissertation.

First of all, since the study of dynamical systems is a branch of mathematics characterised by its multidisciplinary, in this project we use material studied in many different subjects of the degree. Apart from background in Complex Analysis and Dynamical Systems, we also need some tools developed in Differential Geometry, such as the Hyperbolic metric, or in Topology, such as all the properties of topological spaces used in Cowen's proof. With that in mind, it was interesting to see a practical usage for some of these results and examples of some topological concepts that I had not seen before appear out of their context.

Our main goal was to study the proof of the Denjoy-Wolff theorem. Provided that most of the necessary knowledge had not been studied in the degree, I had to devote some sections to preliminary results. Therefore, I studied some properties of holomorphic functions that are useful when dealing with iteration, such as the Schwarz-Pick lemma. It is almost magic that changing the metric implies that holomorphic functions are contractions. In general, when working with holomorphic functions, there appear many surprising properties, different to the ones of real analytic functions. Such properties are precisely the ones that allow us to prove many of the theorems presented in this project.

In the last chapter of the dissertation, we dealt with the Fatou and Julia set of entire functions, that is complex dynamics in a more global sense. We saw that, as Fatou said [15], the degree of complexity is even bigger when dealing with transcendental entire functions, due to the essential singularity at ∞. However, it is also more exciting because there appear interesting phenomena such as Baker domains or wandering domains, which bring wealth to the dynamics.

We focused on what can be seen as a drop of water in the middle of the ocean: during this project I realised the large extent of this branch of the mathematics. Without going too far, the theorem of classification can be extended to rational functions, with the appearance of Herman rings, or it can be proved that rational functions have no wandering domains. What seemed to me as a side note in the course of Mathematical Models, is actually a huge active field of research.

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