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Facultat de Matemàtiques i Informàtica

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HOMOTOPICAL REALIZATIONS OF INFINITY GROUPOIDS

Autor: Jan McGarry Furriol

Director: Dr. Carles Casacuberta Vergés

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Abstract

Grothendieck's homotopy hypothesis asserts that the study of homotopy types of topological spaces is equivalent to the study of ∞-groupoids, illustrating how important ideas in higher category theory stem from basic homotopical concepts. In practice there are distinct models for ∞-groupoids, and providing a proof of the homotopy hypothesis is a test for the suitability of any such model. In this thesis, we give a proof of the homotopy hypothesis using topological categories (i.e., categories enriched over topological spaces) as models for ∞-groupoids. In the same context, we propose a manageable model for the fundamental ∞-groupoid of a topological space.

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Introduction

This thesis is devoted to the study of a homotopy theoretic equivalence between topological spaces and ∞-groupoids. We begin by reviewing a result that culminated a thorough study of the relationship between simplicial sets and topological spaces, developed in the 1950s and 1960s. Its origins go back to Kan, on his work introducing adjoint functors, and reached a great milestone with the work of Quillen on model categories. After this, we move forward into the context of higher category theory, which is the framework of ∞-groupoids, in order to prove the homotopy hypothesis.

Spaces as infinity groupoids

How such an equivalence may be realized can be intuitively foreseen by associating to a topological space X its fundamental ∞-groupoid, which generalizes the standard construction of the fundamental groupoid by considering "higher paths" (homotopies) in X. Let us develop this idea.

At the very beginning of the use of algebraic tools in the study of topology, Poincaré introduced, in the 1890s, the fundamental group $\pi_1(X, x)$ of a topological space X with basepoint x. This is an algebraic object containing information about the topological space. Later on, it was seen that the fundamental group of a space, for all choices of base points, could be assembled in a very natural object which we can now describe in the following way. The *fundamental groupoid* of X, denoted $\Pi_1(X)$, is the category that has the points of X as objects, and for every pair of objects x and y, the morphisms from x to y are the homotopy classes of paths in X starting at x and ending at y. Then, the fundamental group of X with basepoint $x \in X$ can be recovered by considering the morphisms from x to itself in $\Pi_1(X)$.

Furthermore, one might consider the nth homotopy group, denoted $\pi_n(X, x)$, and regard the family of these groups $\{\pi_n(X, x)\}_{n \geq 1}$ as algebraic data of the topological space; we may add the set $\pi_0(X)$ of path-connected components of X. It turns out that this data cannot tell apart two (weakly) homotopy equivalent topological spaces. Now, we may try to assemble all this data together using the language of category theory, analogously to the case of the fundamental groupoid. Let us first do a single step.

Let X be a topological space. We want to extract a category from it, as we did before, but instead of considering homotopy classes of paths between points, we consider paths as morphisms and encode the information about the homotopies between these paths separately. A way to do this is by considering a 2-*category*, in which we do not only have a collection of objects and morphisms between these objects but also morphisms between morphisms, which are called 2-*morphisms*.

More generally, there exists a notion of *n-category*; these are objects of study in higher category theory. Back to our construction, we may consider a 2-category $\Pi_2(X)$ as follows. The objects of $\Pi_2(X)$ are the points of X. If $x, y \in X$, then the morphisms from x to y are given by continuous paths $[0,1] \to X$ starting at x and ending at y. The 2-morphisms are given by homotopies of paths, considered up to homotopy. That is, two homotopies are equal if and only if they are homotopic. Following an analogous construction, we can extract an n-category $\Pi_n(X)$. Note that if $n = 1$, the category obtained is the fundamental groupoid of X. We therefore call $\Pi_n(X)$ the *fundamental n-groupoid*.

As Grothendieck envisioned, there should exist a good notion of *n-groupoid* for which the study of n-truncated homotopy types of topological spaces is essentially equivalent to the study of n-groupoids, by associating to a topological space its fundamental n-groupoid. Furthermore, the same should happen by somehow letting n go to infinity and considering the *fundamental* ∞*-groupoid* $\Pi_\infty(X)$, where one has morphisms of all orders. Grothendieck's vision happens to be veracious, and the object of this thesis is to make precise this equivalence, known as the homotopy hypothesis. The homotopy hypothesis ought to express that the homotopy theory of topological spaces and that of ∞-groupoids are essentially equivalent. In particular, the fundamental ∞-groupoid cannot tell apart two (weakly) homotopy equivalent topological spaces.

Homotopy Theories

We now give a brief overview of what we mean by "the homotopy theory of topological spaces is essentially equivalent to the homotopy theory of ∞-groupoids". There are many well known contexts in which homotopy theory arises, such as topological spaces, chain complexes, or simplicial sets. In all of these contexts, the usual notion of "sameness" given by an isomorphism was dropped for a weaker notion, in view of the first being too restrictive. Namely, the notion of homotopy equivalence or other invariants such as weak homotopy equivalence of topological spaces or simplicial sets, and quasi-isomorphisms of chain complexes.

In the 1960s, Quillen characterized the common behavior of various examples of homotopy theories and presented a set of axioms for a homotopy theory. The structure that satisfies these axioms is called a *model category*. So if one can set such a structure in a category, one can carry out homotopy theory in it. For example, the category of topological spaces is endowed with a model category structure such that the resulting homotopy theory is the classical homotopy theory of spaces.

Grothendieck expressed his first thoughts on the program he envisioned in a letter to Quillen [10], dated 1983, where he proposed the following:

> *"One comment is that presumably, the category of ∞-groupoids (which is still to be defined) is a "model category" for the usual homotopy category; this would be at any rate one plausible way to make explicit the intuition referred to before, that a homotopy type is "essentially the same" as an ∞-groupoid up to ∞-equivalence."*

Unfortunately, there is no record of a response from Quillen. But we will see that the category of ∞-groupoids can be realized within a larger model category. In such a structure, we have a notion of *equivalences* and of its *homotopy category*, which are the essential information of the homotopy theory. What we will then show is that the *homotopy categories* of topological spaces and of ∞-groupoids are equivalent, and this will prove the homotopy hypothesis.

The fundamental ∞-groupoid

The fundamental ∞-groupoid $\Pi_\infty(X)$ presented before is a genuine model of what an ∞-groupoid should be, but needs to be suited with a formal definition of such an object. First of all, notice that $\Pi_\infty(X)$ does not have a strictly associative composition; however, this is something we want it to have, similarly to a category. For example, defining composition as concatenation of paths by traveling along one path in half the time of the interval, and traveling along the second path during the second half of the interval, is not strictly associative: if α, β and γ are three composable paths, then $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ carry different parametrizations. This composition is associative only up to homotopy.

In the literature, there does not appear to be an obvious transparent model of the fundamental ∞-groupoid for our definition of ∞-groupoid, nor a direct equivalence from topological spaces to ∞-groupoids. This has brought us to a detour through the theory of simplicial sets and simplicial categories in order to prove the homotopy hypothesis, with which we conclude Chapter 2. As a consequence, we will obtain an ∞-groupoid $\mathcal{G}(X)$ associated to a topological space X, which fits with our idea of what the fundamental ∞-groupoid should be.

Nevertheless, we believe we can go a step further in realizing our guiding hope to find a model for ∞-groupoids in which the fundamental ∞-groupoid is realized in a manageable and accessible way. This motivates the third and last chapter, in which we propose such a model, based on Moore paths. We believe that it accomplishes our goal.

The homotopy hypothesis after Grothendieck

After writing to Quillen, Grothendieck proceeded to further develop his ideas in his manuscript *À la poursuite des champs*, which also extends ideas he explored in letters to Breen in the 1970s. A deep study of Grothendieck's approach has been carried on by Maltsiniotis [17]. The homotopy hypothesis opened many other paths in the search for a theory of higher categories, being regarded as a principle for such a theory.

One of these paths has been the development of the theory of quasicategories, a special kind of simplicial sets which have been thoroughly studied as a model for higher categories. This theory goes back to Boardman and Vogt in the 1970s, and subsequent work of Joyal and Lurie has pushed forward this model. This approach has provided the most complete theory of higher categories until now, established in [15] in 2009. Two other approaches are those using simplicial categories or topological categories, which have advanced parallelly to the work of Cordier, Dwyer and Kan, and Porter in the 1980s. Remarkably, all such approaches are equivalent, as we will see.

With Quillen's axiomatization as a formal basis, the homotopy hypothesis has set homotopy theory in a larger framework, enhancing a far-reaching interplay of this field of study with many areas of research in mathematics. On the one hand, the homotopy hypothesis gives algebraic and combinatorial insights in the homotopy theory of topological spaces; we will see the way in which it connects simplicial sets and ∞-groupoids to spaces. On the other hand, the homotopy hypothesis lies at the very heart of higher category theory and expresses how many ideas in this subject arise from homotopy theory; for example, ∞-groupoids are an object of study of higher category theory. Moreover, such structures emerge in many parts of mathematics. In the case of the ∞-groupoid structure of a topological space, it arises by keeping the information about homotopies of all orders; in more general words, we are keeping information about a collection of identifications. In such a fashion, we can see higher categorical structures arise in homotopy type theory, showing how higher category theory meets the very foundations of mathematics. This allows for the previous point of view to be turned around to provide spatial insights in many other areas in mathematics; such is the case in Martin–Löf type theory, giving rise to homotopy type theory.

Chapter 1

Foundations

This chapter is meant to introduce and summarize standard mathematical concepts which will be needed for the development of the main chapter. The first two sections are dedicated to present two fundamental notions of category theory, which are essential to the ideas in this thesis: in the first section, we define the concept of adjunction and present some of its characterizations; in the second section, we talk about equivalence of categories. In the third section, simplicial sets are introduced, the theory of which is very well established and will play a very important role when transitioning to a higher categorical context. Finally, in the fourth section, we present the necessary theory of model categories in order to formalize our work.

1.1 Adjoint Functors

Definition 1.1. An *adjunction* consists of a pair of functors

$$F\colon \mathcal{C} \to \mathcal{D} \quad \text{and} \quad G\colon \mathcal{D} \to \mathcal{C}$$

together with a bijection $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$ for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, that is natural in both variables. Here, F is *left adjoint* to G and G is *right adjoint* to F.

Example 1.2. Denote by **Set** the category of sets and by **Grp** the category of groups. There is a *forgetful* functor $U\colon \mathbf{Grp} \to \mathbf{Set}$ sending each group G to its underlying set, which is right adjoint to the functor $F\colon \mathbf{Set} \to \mathbf{Grp}$ sending each set X to the *free group* over X. For each set X and each group G there is a bijection $\mathbf{Grp}(FX, G) \cong \mathbf{Set}(X, UG)$ sending a map $f\colon FX \to G$ to the map $g\colon X \to UG$ that makes the same assignments as f on the generators of FX (which are the elements of X). One can check naturality with the help of the discussion below. This is one of many examples of the "free-forgetful" kind; another such example is given by *free category* on a *directed graph*.

In this section, we will write $f^\sharp \colon Fc \to d$ and $f^\flat \colon c \to Gd$ for the morphisms that correspond to each other under the bijection in the definition above. We say one is the *transpose* (or the *adjunct*) of the other.

Naturality in $\mathcal{D}$ says that for any $k \colon d \to d'$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(Fc, d) & \xrightarrow{\cong} & \mathcal{C}(c, Gd) \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle Gk_*} \\ \mathcal{D}(Fc, d') & \xrightarrow[\cong]{} & \mathcal{C}(c, Gd'), \end{array}$$

where k_* denotes the map that postcomposes with k. The above says that for all $f^\sharp \colon Fc \to d$ and $k \colon d \to d'$, the transpose of $k \circ f^\sharp \colon Fc \to d'$ is equal to the following composition:

$$\begin{array}{ccc} c & \xrightarrow{f^\flat} & Gd \\ & {\scriptstyle (k\circ f^\sharp)^\flat}\searrow & \downarrow{\scriptstyle Gk} \\ & & Gd'. \end{array}$$

Naturality in $\mathcal{C}$ says that for any morphism $h \colon c \to c'$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(Fc', d') & \xrightarrow{\cong} & \mathcal{C}(c', Gd') \\ {\scriptstyle Fh^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ \mathcal{D}(Fc, d') & \xrightarrow[\cong]{} & \mathcal{C}(c, Gd'), \end{array}$$

where h^* denotes the map that precomposes with h. The above says that for all $g^\sharp \colon Fc' \to d'$ and $h \colon c \to c'$, the transpose of $g^\sharp \circ Fh \colon Fc \to d'$ is given by

$$\begin{array}{ccc} c & & \\ {\scriptstyle h}\downarrow & \searrow{\scriptstyle (g^\sharp\circ Fh)^\flat} & \\ c' & \xrightarrow[g^\flat]{} & Gd'. \end{array}$$

Let (F, G) be an adjunction. With this notation we indicate that F is the left adjoint. We will also write $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, with the left-to-right arrow on top.

Lemma 1.3. *Consider a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$. Then, for any morphisms with domains and codomains as displayed below,*

$$\begin{array}{ccc} Fc & \xrightarrow{f^\sharp} & d \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle k} \\ Fc' & \xrightarrow[g^\sharp]{} & d' \end{array} \qquad\qquad \begin{array}{ccc} c & \xrightarrow{f^\flat} & Gd \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Gk} \\ c' & \xrightarrow[g^\flat]{} & Gd', \end{array}$$

the left-hand square commutes in $\mathcal{D}$ if and only if the right-hand square commutes in $\mathcal{C}$.

Proof. By naturality in $\mathcal{C}$ and $\mathcal{D}$, we have the commutative diagrams

$$\begin{tikzcd} c \arrow[d, "h"'] \arrow[dr, "(g^\sharp \circ Fh)^\flat"] & \\ c' \arrow[r, "g^\flat"'] & Gd' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "f^\flat"] \arrow[dr, "(k \circ f^\sharp)^\flat"'] & Gd \arrow[d, "Gk"] \\ & Gd' \end{tikzcd}$$

respectively. Therefore, the right square in the statement commutes if and only if $(g^\sharp \circ Fh)^\flat = (k \circ f^\sharp)^\flat$. By the bijection $\mathcal{D}(Fc, d') \cong \mathcal{C}(c, Gd')$, the latter is equivalent to $g^\sharp \circ Fh = k \circ f^\sharp$, which corresponds to the commutativity of the left square in the statement. □

For every object $c \in \mathcal{C}$, we can consider the transpose of id_{Fc}, which is a morphism $\eta_c \colon c \to GFc$. We now show that the moprhisms η_c are the components of a natural transformation $\eta \colon \mathrm{Id}_\mathcal{C} \Rightarrow GF$.

Lemma 1.4. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. There is a natural transformation $\eta \colon \mathrm{Id}_\mathcal{C} \Rightarrow GF$, whose component at c is $\eta_c \colon c \to GFc$.*

Proof. We have to show that the following square at the left commutes for every $f \colon c \to c'$ in $\mathcal{C}$:

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] & GFc \arrow[d, "GFf"] \\ c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd} \qquad \begin{tikzcd} Fc \arrow[r, "\mathrm{id}_{Fc}"] \arrow[d, "Ff"'] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "\mathrm{id}_{Fc'}"'] & Fc'. \end{tikzcd}$$

This holds as a consequence of Lemma 1.3, because the square at the right commutes. □

The natural transformation $\eta \colon \mathrm{Id}_\mathcal{C} \Rightarrow GF$ is called the *unit* of the adjunction (F, G). Dually, for every object $d \in \mathcal{D}$ we consider the transpose of id_{Gd}, which is a map $\epsilon_d \colon FGd \to d$. The dual of Lemma 1.4 is:

Lemma 1.5. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. There is a natural transformation $\epsilon \colon FG \Rightarrow \mathrm{Id}_\mathcal{D}$, whose component at d is $\epsilon_d \colon FGd \to d$.*

The natural transformation $\epsilon \colon FG \Rightarrow \mathrm{Id}_\mathcal{D}$ is called the *counit* of the adjunction. In fact, the unit and counit of the adjunction permit a computation of the transpose of a morphism. For example, by naturality of the isomorphism between sets of maps we can compute the transpose $f^\sharp$ of a morphism $f^\flat \colon c \to Gd$ as the composition

$$Fc \xrightarrow{Ff^\flat} FGd \xrightarrow{\epsilon_d} d.$$

Lemma 1.6. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Then, the natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ satisfy the following triangle identity: for every $c \in \mathcal{C}$, there is a commutative diagram*

$$\begin{array}{ccc} Fc & \xrightarrow{F\eta_c} & FGFc \\ & {\scriptstyle \mathrm{id}_{Fc}}\searrow & \downarrow{\scriptstyle \epsilon_{Fc}} \\ & & Fc. \end{array}$$

Proof. We must show that the following square at the left commutes for every $c \in \mathcal{C}$:

$$\begin{array}{ccc} Fc & \xrightarrow{\mathrm{id}_{Fc}} & Fc \\ {\scriptstyle F\eta_c}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{Fc}} \\ FGFc & \xrightarrow[\epsilon_{Fc}]{} & Fc \end{array} \qquad\qquad \begin{array}{ccc} c & \xrightarrow{\eta_c} & GFc \\ {\scriptstyle \eta_c}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{GFc}} \\ GFc & \xrightarrow[\mathrm{id}_{GFc}]{} & GFc. \end{array}$$

This follows from Lemma 1.3, because the square at the right commutes. □

To end this section, we show that adjoint functors have a especially good relationship with respect to the dual notions of *limit* and *colimit* (see [16] or [23] for details). In particular, it holds for *products* and *coproducts*.

Proposition 1.7. *Every left adjoint functor preserves colimits and every right adjoint functor preserves limits.*

Proof. We prove the first part of the statement; the second part is dual and the proof is analogous. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Let $D\colon I \to \mathcal{C}$ be a *diagram* in $\mathcal{C}$; i.e., a functor where the objects of the category I form a set. Let $\operatorname{colim} D$ be the colimit of D and recall that the universal property of the colimit is characterized by natural bijections

$$\mathcal{C}(\operatorname{colim} D, X) \cong \lim \mathcal{C}(D, X) \quad \text{for all } X \in \mathcal{C}.$$

To prove that F preserves colimits, we need to show that the natural morphism $\operatorname{colim} FD \to F \operatorname{colim} D$ is an isomorphism. Thus, we will show that $F \operatorname{colim} D$ satisfies the universal property of $\operatorname{colim} FD$. For every object X of $\mathcal{C}$ we have natural bijections

$$\begin{aligned} \mathcal{C}(F \operatorname{colim} D, X) &\cong \mathcal{D}(\operatorname{colim} D, GX) \cong \lim \mathcal{D}(D, GX) \\ &\cong \lim \mathcal{D}(FD, X) \cong \mathcal{C}(\operatorname{colim} FD, X). \end{aligned}$$

Finally, since the bijections $\mathcal{C}(F \operatorname{colim} D, X) \cong \mathcal{C}(\operatorname{colim} FD, X)$ are natural for all $X \in \mathcal{C}$, the Yoneda Lemma ([23, Proposition 2.3.1]) yields the desired isomorphism $\operatorname{colim} FD \cong F \operatorname{colim} D$. □

1.2 Equivalence of Categories

Definition 1.8. An *equivalence of categories* consists of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ together with natural isomorphisms $\mathrm{Id}_{\mathcal{C}} \cong GF$ and $FG \cong \mathrm{Id}_{\mathcal{D}}$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists an equivalence between them.

As expected, equivalence of categories defines an equivalence relation. First of all, let us present the following definitions in order to shed light on the notion of equivalence of categories.

Definition 1.9. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is

- *full* is for each $c, c' \in \mathcal{C}$, the map $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is surjective;
- *faithful* if for each $c, c' \in \mathcal{C}$, the map $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is injective;
- and *essentially surjective* if for every object $d \in \mathcal{D}$ there is some $c \in \mathcal{C}$ such that d is isomorphic to Fc.

The above definitions are *local* characterizations of a functor. For example, fullness or faithfulness do not assure that F is injective or surjective *globally* on morphisms of $\mathcal{C}$. However, this is true if F is injective on objects; in such case, F is a *full embedding* of the domain category into the codomain category. Nevertheless, if a full and faithful functor is in addition essentially surjective, then it contains all the data of the codomain category up to isomorphism. If a functor is full and faithful, we will say it is *fully faithful*.

Consequently, it is not surprising that functors F and G giving an equivalence of categories are fully faithful and essentially surjective; in fact, this will be a very useful characterization.

Theorem 1.10. *If a functor $F \colon \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories, then it is fully faithful and essentially surjective. Assuming the axiom of choice holds, the converse is also true.*

Proof. Here we only sketch a proof; we refer the reader to [23, Theorem 1.5.9] for further details. For the first part of the statement, observe that the natural isomorphism $FG \cong \mathrm{Id}_{\mathcal{D}}$ provides for every object $d \in \mathcal{D}$ an object Gd satisfying that $FGd \cong d$, and this shows that F is essentially surjective. Moreover, the natural isomorphism $\mathrm{Id}_{\mathcal{C}} \cong GF$ assures that F is fully faithful.

Conversely, any fully faithful and essentially surjective functor F yields an equivalence of categories. Using essential surjectiveness and the axiom of choice, one can choose an object Gd for every object d of $\mathcal{D}$ together with an isomorphism $FGd \cong d$. The fact that F is fully faithful permits to assign to each arrow in $\mathcal{D}$

an arrow in $\mathcal{C}$ in a functorial way, and such that the isomorphisms $FGd \cong d$ are natural. Full faithfulness also allows to define a natural isomorphism $\mathrm{Id}_{\mathcal{C}} \cong GF$. In summary, one obtains a functor $G\colon \mathcal{D} \to \mathcal{C}$ witnessing that F yields an equivalence of categories. □

Before presenting an example, recall that a *groupoid* is a category in which every morphism is an isomorphism. Also, we say that a category is *connected* if any pair of objects can be linked by a finite *zig-zag* of morphisms, i.e., a "path" of not necessarily composable arrows. Finally, notice that if G is a group, we can view it as a category with one object $*$, with an arrow from $*$ to itself for each element of G; composition of two arrows is defined as the product of the two elements they represent in G.

Proposition 1.11. *Any connected groupoid is equivalent, as a category, to the automorphism group of any of its objects.*

Proof. Let $\mathcal{G}$ be a connected groupoid and x be one of its objects. We may view $\mathrm{Hom}_{\mathcal{G}}(x,x)$ as a category with one object as explained above; observe that all its morphisms are isomorphisms, so it is indeed a group. Then the inclusion functor $\mathrm{Hom}_{\mathcal{G}}(x,x) \hookrightarrow \mathcal{G}$ is fully faithful, by definition, and essentially surjective, because $\mathcal{G}$ is a connected groupoid. □

In other words, any groupoid "collapses", up to equivalence, into one of its objects and its automorphism group.

Example 1.12. Let X be a path-connected topological space and $\Pi_1(X)$ its fundamental groupoid. Then, $\Pi_1(X)$ is a connected groupoid, and applying the previous lemma transitively shows that any choice of basepoint $x \in X$ gives an isomorphic fundamental group $\pi_1(X,x)$. Indeed, consider two basepoints x and x', and recall that their automorphism groups, as objects of $\Pi_1(X)$, are the fundamental groups $\pi_1(X,x)$ and $\pi_1(X,x')$ respectively. Then, by the previous lemma, both are equivalent to $\Pi_1(X)$, and hence to each other. Finally, notice that a functor between these two one-object categories induces a group homomorphism (in fact, an isomorphism) between the groups $\pi_1(X,x)$ and $\pi_1(X,x')$.

1.3 Simplicial Sets

In the Introduction we mentioned that we would describe a result that culminated a thorough study of the relationship between simplicial sets and topological spaces. Precisely, we will see that the homotopy theory of simplicial sets and that of topological spaces are equivalent in a very strong sense (Section 2.1). First, we need to introduce these objects.

We denote by Δ the category whose objects are the sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and whose morphisms are order-preserving functions $[n] \to [m]$.

Definition 1.13. A *simplicial set* is a functor $X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$, where **Set** denotes the category of sets.

A *simplicial map* $f\colon X \to Y$ between simplicial sets is a natural transformation. The category of simplicial sets with simplicial maps is denoted by $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ or, more concisely, as **sSet**.

For a simplicial set X, we normally write X_n instead of $X[n]$, and call it the set of *n-simplices* of X. There are injections $\delta_i^n\colon [n-1] \to [n]$ forgetting i and surjections $\sigma_i^n\colon [n+1] \to [n]$ repeating i for $0 \leq i \leq n$ that give rise to functions

$$d_i^n\colon X_n \to X_{n-1}\,, \qquad s_i^n\colon X_n \to X_{n+1}\,,$$

called *faces* and *degeneracies* respectively. Since every order-preserving function $[n] \to [m]$ is a composite of a surjection followed by an injection, the sets $\{X_n\}_{n\geq 0}$ together with the faces d_i^k and degeneracies s_j^l determine uniquely a simplicial set X. Faces and degeneracies satisfy the *simplicial identities*:

$$\begin{aligned} d_i^{n-1} \circ d_j^n &= d_{j-1}^{n-1} \circ d_i^n \quad \text{if } i < j; \\ d_i^{n+1} \circ s_j^n &= \begin{cases} s_{j-1}^{n-1} \circ d_i^n & \text{if } i < j; \\ \mathrm{id}_{X_n} & \text{if } i = j \text{ or } i = j+1; \\ s_j^{n-1} \circ d_{i-1}^n & \text{if } i > j+1; \end{cases} \\ s_i^{n+1} \circ s_j^n &= s_{j+1}^{n+1} \circ s_i^n \quad \text{if } i \leq j. \end{aligned}$$

For $n \geq 0$, the *standard n-simplex* is the simplicial set $\Delta[n] = \Delta(-, [n])$, that is,

$$\Delta[n]_m = \Delta([m], [n])$$

for all $m \geq 0$. Then the Yoneda Lemma ([23, Theorem 2.2.4]) implies that

$$X_n = X[n] \cong \mathbf{sSet}(\Delta(-, [n]), X) = \mathbf{sSet}(\Delta[n], X)$$

for each simplicial set X and $n \geq 0$. Moreover, every simplicial set X is a colimit of standard n-simplices, as we now explain. The *category of simplices* of X is the category $(\Delta \downarrow X)$ that has as objects the morphisms $\Delta[n] \to X$ in **sSet**, and as morphisms natural transformations. Then the Density Theorem ([16, Corollary 3, Section 6, Chapter X]) asserts that X is isomorphic to the colimit of the diagram

$$(\Delta \downarrow X) \xrightarrow{U} \Delta \xrightarrow{Y} \mathbf{Set}^{\Delta^{\mathrm{op}}}$$

where U sends $\Delta[n]$ to $[n]$, followed by the Yoneda embedding from Δ to simplicial sets, which sends $[n]$ to $\Delta[n]$. We write

$$X \cong \operatorname*{colim}_{(\Delta \downarrow X)} \Delta[n] \,.$$

The *geometric realization* of a simplicial set X is defined as

$$|X| = \operatorname*{colim}_{(\Delta \downarrow X)} \Delta^n \,,$$

where $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} |\, 0 \le t_i \le 1, \sum t_i = 1\}$ with the Euclidean topology. In other words, $| - |$ is the left Kan extension of the functor from Δ to the category of topological spaces sending $[n]$ to Δ^n, along the Yoneda embedding from Δ to the category of simplicial sets sending $[n]$ to $\Delta[n]$. Consequently, we may think of $|X|$ as constructed by means of copies of Δ^n for all n in the same way as X is constructed by means copies of $\Delta[n]$ for all n.

For $k \ge 0$, the *k-skeleton* of a simplicial set X is the smallest sub-simplicial-set of X containing $X_0, \ldots, X_k$. We now define the *boundary* of $\Delta[n]$, denoted $\partial\Delta[n]$, as the $(n-1)$-skeleton of $\Delta[n]$. The *kth horn* $\Lambda^k[n]$ is the sub-simplicial-set of $\Delta[n]$ resulting from removing the kth face. The horns with $0 < k < n$ are called *inner horns* and those with $k = 0$ and $k = n$ are *outer horns*.

What follows is the definition of a kind of simplicial sets which share a particularly rich structure, and can serve as models for the *homotopy types* of simplicial sets. But since we will show an equivalence with the homotopy theory of topological spaces, they are also models for the *homotopy types* of spaces.

Definition 1.14. A simplicial set X is called a *Kan complex* if for any $0 \le k \le n$ and $n \ge 1$, every map $\Lambda^k[n] \to X$ admits an extension $\Delta[n] \to X$ (called a *filler*). That is, the map $\Delta[n] \to X$ fits into a commutative diagram

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

where $\Lambda^k[n] \to \Delta[n]$ is the canonical inclusion.

In order to relate simplicial sets to topological spaces we will need functors between their categories, where the category of topological spaces and continuous maps is denoted **Top**. These functors will be the geometric realization and the one defined next.

Definition 1.15. Let X be a topological space. The *singular complex* $\operatorname{Sing} X$ is the simplicial set defined by

$$(\operatorname{Sing} X)_n = \mathbf{Top}(\Delta^n, X)\,,$$

with face and degeneracy maps induced by the usual face and degeneracy maps between the standard n-simplices Δ^n.

The assignation $X \mapsto \operatorname{Sing} X$ fits into a functor $\operatorname{Sing}\colon \mathbf{Top} \to \mathbf{sSet}$, which sends a map of topological spaces $f\colon X \to Y$ to the map of simplicial sets

$$\operatorname{Sing} f\colon \operatorname{Sing} X \to \operatorname{Sing} Y$$

that postcomposes with f.

Lemma 1.16. *The functor* $\operatorname{Sing}\colon \mathbf{Top} \to \mathbf{sSet}$ *is right adjoint to the geometric realization.*

Proof. Recall $\Delta^n = |\Delta[n]|$ and that the Yoneda Lemma gives a natural bijection $(\operatorname{Sing} X)_n \cong \mathbf{sSet}(\Delta[n], X)$. Then, for any topological space X and simplicial set Y, we have natural bijections

$$\begin{aligned}\mathbf{Top}(|Y|, X) &\cong \lim_{(\Delta\downarrow Y)} \mathbf{Top}(\Delta^n, X)\\ &\cong \lim_{(\Delta\downarrow Y)} \mathbf{sSet}(\Delta[n], \operatorname{Sing} X)\\ &\cong \mathbf{sSet}(Y, \operatorname{Sing} X).\end{aligned}$$

The first and third bijection follow directly from the universal property of the colimit $|Y| = \operatorname*{colim}_{(\Delta\downarrow Y)} \Delta^n$, and the second bijection follows from the definition of $\operatorname{Sing} X$. □

Example 1.17. The singular complex of a topological space X is a Kan complex. Indeed, the extension problem of a map $\Lambda^k[n] \to \operatorname{Sing} X$ to a map $\Delta[n] \to X$, corresponds, by adjointness (Lemma 1.3), to finding an extension of a map $|\Lambda^k[n]| \to X$ to a map $|\Delta[n]| \to X$. The later follows because $|\Lambda^k[n]|$ is a retract of $|\Delta[n]|$.

Definition 1.18. Let X be a simplicial set. We say that X is a *quasicategory* if for any $0 < k < n$, every map $\Lambda^k[n] \to X$ admits a filler.

Notice that every Kan complex is a quasicategory. These two notions are very important ones in the study of higher categories, as explained in the Introduction. Quasicategories are also known as *weak Kan complexes*, as Boardman and Vogt called them, or ∞-*categories*, the term used by Lurie; it was Joyal that introduced the term quasicategories. As the name suggests, quasicategories can be thought in a similar way as categories, and in fact serve as a model for higher categories. There is a very complete theory following this approach which involves the work of the aforementioned mathematicians; the main reference is [15].

In a quasicategory X, the filler of a map $\Lambda^1[2] \to X$ can be thought as a *homotopy* witnessing the 1-face as a composite for the two edges, but this composition is not unique as in a category: there may exist many different fillers. Additionally, if X is also a Kan complex, fillers for outer horns give an "inverse" for every "morphism". The problem of not having composition defined as a map has actually been solved by Nikolaus, with his introduction of *algebraic Kan complexes* in [20]. However, we will pursue an approach to higher categories considering ∞-*categories* as topological categories, which we believe to be more intuitive. This is also an approach developed in [15] and seen to be equivalent to the one using quasicategories. In later sections, we will see the how and why of this equivalence, which is formalized in the language of model categories.

1.4 Model Categories

Definition 1.19. A *model category* is a category $\mathcal{C}$ with three distinguished classes of maps: *weak equivalences* ($\xrightarrow{\sim}$), *fibrations* ($\twoheadrightarrow$), and *cofibrations* ($\hookrightarrow$), each closed under composition and containing all isomorphisms and such that the following axioms hold:

(MC1) All (set-indexed) limits and colimits exist in $\mathcal{C}$.

(MC2) The class of weak equivalences satisfy the *two-out-of-three property*: if $f\colon X \to Y$ and $g\colon Y \to Z$ are maps in $\mathcal{C}$ such that two of f, g and $g \circ f$ are weak equivalences, then so is the third.

(MC3) All three classes of maps are closed under *retracts*.

(MC4) Consider any commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y. \end{array}$$

If either i or p is a weak equivalence, then a dotted arrow exists rendering the diagram commutative.

(MC5) Any map $f\colon X \to Y$ can be factored in two ways:

(i) $X \overset{\sim}{\hookrightarrow} X' \twoheadrightarrow Y$, and

(ii) $X \hookrightarrow Y' \overset{\sim}{\twoheadrightarrow} Y$.

From $\mathcal{C}$ having limits and colimits we deduce that $\mathcal{C}$ has an *initial object* (an object $\varnothing$ with a unique map $\varnothing \to X$ for any object X of $\mathcal{C}$), and a *terminal object* (an object $*$ with a unique map $X \to *$ for any object X of $\mathcal{C}$).

Definition 1.20. Let $\mathcal{C}$ be a model category. An object X of $\mathcal{C}$ is said to be *cofibrant* if the unique map $\varnothing \to X$ is a cofibration. Dually, we say that X is *fibrant* if the unique map $X \to *$ is a fibration.

Consider an object X of a model category $\mathcal{C}$. Applying axiom MC5 to the map $\varnothing \to X$, we obtain an object X^c and a factorization $\varnothing \hookrightarrow X^c \overset{\sim}{\twoheadrightarrow} X$. Since $\varnothing \hookrightarrow X^c$ is a cofibration, the object X^c is cofibrant; moreover, as X^c and X are weakly equivalent, we say that X^c is a *cofibrant replacement* of X. Dually, we apply MC5 to the map $X \to *$, and obtain a weakly equivalent fibrant object X^f and a factorization $X \overset{\sim}{\hookrightarrow} X^f \twoheadrightarrow *$; we then say that X^f is a *fibrant replacement* of X.

In modern texts about model categories, such as [12], one assumes the existence of *functorial* factorizations in MC5. Thus, we may choose a cofibrant replacement functor and a fibrant replacement functor. Precisely, on the one hand we get a functor $Q\colon \mathcal{C} \to \mathcal{C}$ taking cofibrant values, together with a natural transformation $QX \overset{\sim}{\twoheadrightarrow} X$; we call Q a *cofibrant replacement functor* on $\mathcal{C}$. On the other hand, we get a *fibrant replacement functor* $R\colon \mathcal{C} \to \mathcal{C}$ together with a natural transformation $X \overset{\sim}{\hookrightarrow} RX$. Since the mentioned natural transformations are weak equivalences, by the two-out-of-three property, both Q and R preserve weak equivalences.

Remark 1.21. Let X be an object of $\mathcal{C}$. Consider a fibrant replacement obtained as above. Then, a cofibrant replacement of X^f, denoted X^{cf}, is still fibrant. Indeed, we obtained X^f via a factorization

$$X \overset{\sim}{\hookrightarrow} X^f \twoheadrightarrow *,$$

and then X^{cf} with

$$\varnothing \hookrightarrow X^{cf} \overset{\sim}{\twoheadrightarrow} X^f.$$

By MC2, the composition $X^{cf} \overset{\sim}{\twoheadrightarrow} X^f \twoheadrightarrow *$ is a fibration.

Given a model category $\mathcal{C}$, we will denote by $\mathcal{C}_{cf}$ the full subcategory of $\mathcal{C}$ consisting of those objects that are both cofibrant and fibrant. We may think of this subclass of objects as the "nice" objects of the model category. For example, we will see that for these objects we have a suitable notion of homotopy and that the notions of homotopy equivalence and weak equivalence agree on these objects. Furthermore, MC5 assures that every object of the model category is weakly equivalent to at least one of these "nice" objects.

Definition 1.22. A map $f\colon X \to Y$ in a model category $\mathcal{C}$ that is both a fibration and a weak equivalence is called a *trivial fibration*. If f is both a cofibration and a weak equivalence, we say that f is a *trivial cofibration*.

There is a notion of *homotopy* in every model category $\mathcal{C}$. First of all, given an object X of $\mathcal{C}$, we can consider the coproduct $X \amalg X$, and then, for the identity map $\mathrm{Id}\colon X \to X$, the universal property of the coproduct yields a unique map $X \amalg X \to X$ called the *fold map*. In the case $\mathcal{C} = \mathbf{Top}$, the coproduct is the disjoint union.

Let $f, g\colon X \to Y$ be two maps in $\mathcal{C}$. A *cylinder object* $X \times I$ for X is a factorization of the fold map $X \amalg X \to X$ into a cofibration $i_0 + i_1\colon X \amalg X \to X \times I$ followed by a weak equivalence $X \times I \to X$. Then, a *left homotopy* from f to g is a map $H\colon X \times I \to Y$ for some cylinder object $X \times I$ for X such that $Hi_0 = f$ and $Hi_1 = g$. When X is a fibrant object, we can assume that the cylinder object in this definition is the object obtained by applying the factorization of MC5 to the map $X \amalg X \to X$, yielding a cylinder object $X \times I$ together with a trivial fibration $X \times I \overset{\sim}{\twoheadrightarrow} X$. The notation $X \times I$ is meant to suggest the product of X with an interval, but we should warn that a cylinder object is not necessarily any kind of product. However, in the usual model structure on topological spaces (Section 2.1) a choice of cylinder object for a space X is the product $X \times [0, 1]$.

Dually, one may define the notion of *right homotopy* using *path objects*, and when Y is cofibrant the factorization yields a path object Y^I together with a trivial cofibration $Y \overset{\sim}{\hookrightarrow} Y^I$. We say that f and g are *homotopic* if there exists both a left homotopy and a right homotopy from f to g. When X is cofibrant and Y is fibrant, the left homotopy and right homotopy relations coincide in $\mathcal{C}(X, Y)$ and are equivalence relations. Consequently, the homotopy relation on the morphisms of $\mathcal{C}_{cf}$ is an equivalence relation and it follows from the above definitions that it is compatible with composition. Hence, the following is well defined.

Definition 1.23. Let $\mathcal{C}$ be a model category. We define the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ as follows:

- The objects of $\mathrm{Ho}(\mathcal{C})$ are the objects of $\mathcal{C}$ that are both fibrant and cofibrant.
- If X, Y are two objects, we define $\mathrm{Ho}(\mathcal{C})(X, Y) = [X, Y]$, where the brackets denote homotopy classes of maps.
- If $[f] \in \mathcal{C}(X, Y)$ and $[g] \in \mathcal{C}(Y, Z)$, then their composition is defined as $[g] \circ [f] := [g \circ f] \in \mathcal{C}(X, Z)$.

The machinery of model categories allows us to show that the category $\mathrm{Ho}(\mathcal{C})$ we have just defined is equivalent to the category obtained by formally inverting all weak equivalences in $\mathcal{C}$; the second is known as a *localization* of $\mathcal{C}$ with respect to the class W of weak equivalences, and is denoted by $\mathcal{C}[W^{-1}]$. In the literature, $\mathrm{Ho}(\mathcal{C})$ is sometimes defined as such a localization, although this definition entails set-theoretical difficulties because there could be a proper class of morphisms between two objects of $\mathcal{C}[W^{-1}]$. However, the localization $\mathcal{C}[W^{-1}]$ is shown to exist by proving that it is equivalent to the category given in Definition 1.23 (see [12] for details). Specifically, there are equivalences of categories

$$\mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{C}_{cf}[W_0^{-1}] \longrightarrow \mathcal{C}[W^{-1}]$$

where we have denoted by W_0 the class of those weak equivalences W that are also maps in $\mathcal{C}_{cf}$. The second equivalence is induced by the inclusion $\mathcal{C}_{cf} \to \mathcal{C}$, and its inverse by the cofibrant and fibrant replacement functors. The first equivalence is in fact an isomorphism because of the universal property of the localization $\mathcal{C}_{cf}[W_0^{-1}]$ and the fact that a map of $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence ([12, Proposition 1.2.8]), hence an isomorphism in $\mathrm{Ho}(\mathcal{C})$. Observe, however, that the definition of $\mathcal{C}[W^{-1}]$ solely depends on the class of weak equivalences. This indicates that the weak equivalences carry the essential information about the "homotopy theory" in $\mathcal{C}$.

Definition 1.24. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction between model categories. We say that the pair (F, G) is a *Quillen adjunction* between $\mathcal{C}$ and $\mathcal{D}$ if any of the following equivalent conditions is satisfied:

(1) The functor F preserves cofibrations and trivial cofibrations.

(2) The functor G preserves fibrations and trivial fibrations.

(3) The functor F preserves cofibrations, and the functor G preserves fibrations.

(4) The functor F preserves trivial cofibrations, and the functor G preserves trivial fibrations.

The equivalences follow from adjointness and lifting properties as in MC4 (see [7, Remark 9.8]). Here, we say that F is a *right Quillen functor* and G is a *left Quillen functor*.

Remark 1.25. A property of Quillen functors that we will need at a particular moment is that every right Quillen functor preserves weak equivalences between fibrant objects, by Ken Brown's Lemma ([12, Lemma 1.1.12]), which also assures the dual result for left Quillen functors.

Lemma 1.26. *Every left Quillen functor preserves cofibrant objects and every right Quillen functor adjoint preserves fibrant objects.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a left Quillen functor. Consider an initial object $\emptyset$ of $\mathcal{C}$. Since F is left adjoint and $\emptyset$ is a colimit, we deduce that $F\emptyset$ is an initial object of $\mathcal{D}$. On the other hand, given a cofibrant object X of $\mathcal{C}$ we have that the map $\emptyset \to X$ is a cofibration, and by assumption F preserves cofibrations. Therefore the induced map $F\emptyset \to FX$ is a cofibration, i.e., the object FX is cofibrant in $\mathcal{D}$. The second part of the statement is dual and the proof is analogous. □

Theorem 1.27. *Every right or left Quillen functor induces a functor between homotopy categories.*

Proof. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. We start by defining a functor G_* induced by G, with domain $\mathrm{Ho}(\mathcal{D})$ and codomain $\mathrm{Ho}(\mathcal{C})$. Observe that, by the previous lemma, the restriction $G\colon \mathcal{D}_{cf} \to \mathcal{C}$ takes fibrant values. Thus, we have a functor

$$QG\colon \mathcal{D}_{cf} \to \mathcal{C}_{cf}$$

where Q is a cofibrant replacement functor for $\mathcal{C}$. We define G_* by sending each object $X \in \mathrm{Ho}(\mathcal{D})$ to the object $QFX \in \mathrm{Ho}(\mathcal{C})$, and the homotopy class of a map $f\colon X \to Y$ in $\mathrm{Ho}(\mathcal{D})$ to the class of $QFf\colon QFX \to QFY$ in $\mathrm{Ho}(\mathcal{C})$. To check that this assignation is well defined, we must show that G_* sends two homotopic maps in $\mathcal{D}_{cf}$ to the same map in $\mathrm{Ho}(\mathcal{C})$.

Let $f, g\colon X \to Y$ be two homotopic maps in $\mathcal{D}_{cf}$. That is, there exists a homotopy $H\colon X \times I \to Y$ where $X \times I$ is the cylinder object given by the factorization

$$X \amalg X \overset{i_0+i_1}{\hookrightarrow} X \times I \underset{\sim}{\overset{w}{\twoheadrightarrow}} X$$

of MC5. Since $wi_0 = \mathrm{Id}_X = wi_1$, we have $QGwQGi_0 = QGwQGi_1$. Notice that G preserves trivial fibrations because it is a rigth Quillen functor, and R preserves weak equivalences. Hence, since w is a trivial fibration we have that QGw is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. From this and the previous equality we deduce that $QGi_0 = QGi_1$. On the other hand $QGf = QGHQGi_0 = QGHQGi_1 = QGg$ and we conclude that $QGf = QGg$, as we wanted.

Similarly F induces a functor $F_*\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ which sends an object X of $\mathrm{Ho}(\mathcal{C})$ to RFX, and the homotopy class of a map f to the class of RFf, where R is a fibrant replacement functor for $\mathcal{D}$. □

Corollary 1.28. *Every Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ satisfying $F(\mathcal{C}_{cf}) \subset \mathcal{D}_{cf}$ and $G(\mathcal{D}_{cf}) \subset \mathcal{C}_{cf}$ induces an adjunction between homotopy categories. If in addition the components of both the unit and counit of the adjunction (F, G) are all weak equivalences, then the induced adjunction is also an equivalence of categories.*

Proof. If $F(\mathcal{C}_{cf}) \subset \mathcal{D}_{cf}$, then the fibrant replacement functor R is not needed in the construction of F_* and we have the functor

$$F_*\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D}), \quad X \mapsto FX.$$

Similarly when $G(\mathcal{D}_{cf}) \subset \mathcal{C}_{cf}$. The fact that (F_*, G_*) is an adjunction then follows directly because (F, G) is an adjunction and the definition of composition in the homotopy category.

Let η and ϵ be the unit and counit of (F, G) respectively. The homotopy classes of the components η_X for every $X \in \mathcal{C}$ assemble into the unit of (F_*, G_*), and they are all isomorphisms if and only if η_X is a weak equivalence for every $X \in \mathcal{C}$. Similarly, the homotopy classes of the components of ϵ_Y for every $Y \in \mathcal{D}$ assemble into the counit of (F_*, G_*), and they are all isomorphisms if and only if ϵ_Y is a weak equivalence for every $Y \in \mathcal{D}$. This shows the second part of the statement. □

Remark 1.29. The hypotheses of Corollary 1.28 might seem restrictive because the result is true for an arbitrary Quillen adjunction. However, these hypotheses will be satisfied for the Quillen adjunctions we will work with, and will permit an appreciated simplification. When the moment to use Corollary 1.28 comes, in order to check if the conditions in the statement hold it will suffice to show that $F\colon \mathcal{C}_{cf} \to \mathcal{D}$ takes fibrant values and that $G\colon \mathcal{D}_{cf} \to \mathcal{C}$ takes cofibrant values. Indeed, see Lemma 1.26.

Definition 1.30. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction between model categories. We say that the pair (F, G) is a *Quillen equivalence* if for every cofibrant object $X \in \mathcal{C}$ and every fibrant object $Y \in \mathcal{D}$, a map $X \to GY$ is a weak equivalence in $\mathcal{C}$ if and only if the adjunct map $FX \to Y$ is a weak equivalence in $\mathcal{D}$.

A Quillen equivalence is a very strong notion of equivalence between model categories, to the point that we may regard the homotopy theories of the Quillen equivalent model categories as coincident. In particular, a Quillen equivalence induces an equivalence between homotopy categories ([12, Proposition 1.3.13]).

Theorem 1.31. *Given Quillen equivalences $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ and $F' : \mathcal{D} \rightleftarrows \mathcal{E} : G'$, the compositions $(F'F, GG')$ form a Quillen equivalence.*

Proof. First of all, we see that $F'F$ and GG' are a pair of adjoint functos. Indeed, for every $X \in \mathcal{C}$ and every $Y \in \mathcal{E}$ we have natural isomorphisms

$$\mathcal{E}(F'FX, Y) \cong \mathcal{D}(FX, G'Y) \cong \mathcal{C}(X, GG'Y).$$

The composition $F'F$ clearly preserves cofibrations because F and F' do, and GG' preserves fibrations because G' and G do. Hence the pair $(F'F, GG')$ is a Quillen adjunction. Finally, let X be a cofibrant object of $\mathcal{C}$ and Y a fibrant object of $\mathcal{E}$; notice that then FX is a cofibrant object of $\mathcal{D}$ and $G'Y$ is a fibrant object of $\mathcal{D}$. We have that $X \xrightarrow{\sim} GG'Y$ is a weak equivalence if, and only if, $FX \xrightarrow{\sim} G'Y$ if, and only if, $F'FX \xrightarrow{\sim} Y$. We conclude that $(F'F, GG')$ is a Quillen equivalence. $\square$

Chapter 2

The Homotopy Hypothesis

In the Introduction we have seen that a topological space hides an ∞-groupoid structure by considering its fundamental ∞-groupoid, but we have not yet given a rigorous definition of this object; it will be defined in Section 2.2. We will see that the precise formulation of the homotopy hypothesis expresses how ∞-groupoids are equivalent to topological spaces.

The purpose of the first section is to prove that the homotopy theory of simplicial sets is equivalent to the homotopy theory of topological spaces by giving a Quillen equivalence. In the second section we define ∞-groupoids as a special family of topological categories. We next introduce simplicial categories in the third section, and proceed to present the functors that form the bridge between topological spaces and ∞-groupoids. In Section 2.5, we put together two important results to conclude that there exists a Quillen equivalence between simplicial sets and topological categories. We end in Section 2.7 by proving that ∞-groupoids satisfy the homotopy hypothesis.

2.1 Topological Spaces and Simplicial Sets

The goal of this section is to prove that the homotopy theories of topological spaces and of simplicial sets are equivalent. This will be accomplished by considering appropriate model structures and giving a Quillen equivalence.

Before going into the more technical concepts of this section, let us introduce a couple of notions from classical homotopy theory. Consider X a topological space with basepoint $x \in X$, and fix $n \geq 1$. Consider also the n-dimensional sphere S^n with basepoint $(1, 0, \overset{n}{\ldots}, 0)$. Then, the set of homotopy classes of basepoint-preserving maps from S^n to X has a group structure. This group is called the nth homotopy group and is denoted by $\pi_n(X, x)$. The second notion is that of a CW-complex. Roughly, a CW-complex is a topological space constructed inductively by attaching n-dimensional disks D^n (also called *cells*) along their boundary (see [11, Appendix] for details).

Definition 2.1. A map of topological spaces $f\colon X \to Y$ is said to be a *weak homotopy equivalence* if it induces a bijection between sets of path components and isomorphisms $\pi_n(X, x) \cong \pi_n(Y, f(x))$ for all $n \geq 1$ and all $x \in X$.

Definition 2.2. A map of topological spaces $p\colon X \to Y$ is called a *Serre fibration* if for each CW-complex A and every commutative square of solid arrows

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ A \times [0,1] & \longrightarrow & Y \end{array}$$

where $A \times \{0\} \to A \times [0,1]$ is the canonical inclusion, there exists a dotted arrow rendering the diagram commutative. The map p is said to have the *right lifting property* with respect to the map $A \times \{0\} \to A \times [0,1]$.

The category of topological spaces admits a model structure in which the weak equivalences are weak homotopy equivalences, fibrations are Serre fibrations, and cofibrations are *retracts* of inclusions given by *cell attachments* ([7, Section 8]). A proof goes back to Quillen in [21, Chapter II, Section 3]. Even more details can be found in [12, Section 2.4]. Henceforth, we will consider **Top** as a model category, with the structure we have just described. All objects are fibrant in this model category, and the cofibrant objects are *retracts* of CW-complexes. In particular, CW-complexes are fibrant and cofibrant.

Remark 2.3. The category of topological spaces admits other model structures, such as the Hurewicz model structure. In this structure the weak equivalences are the homotopy equivalences and its homotopy category is the category of all topological spaces and homotopy classes of maps. The later is what is sometimes known as the *homotopy category* of topological spaces, and we now see that it is actually a homotopy category in the sense of model categories. However, it is not the homotopy category of **Top** with the model structure described above, but they are equivalent because every topological space X is weakly equivalent to a CW-complex, by the CW approximation theorem ([11, Proposition 4.13]).

Definition 2.4. A map of simplicial sets $p\colon X \to Y$ is called a *Kan fibration* if it has the right lifting property with respect to the horn inclusions $\Lambda^k[n] \to \Delta[n]$ for all $0 \leq k \leq n$ and all $n \geq 1$.

Theorem 2.5. *The geometric realization of a Kan fibration is a Serre fibration.*

Proof. See [9, Theorem 10.10] or [22]. □

The above is a technical result due to Quillen, which involves the theory of *minimal* Kan fibrations.

Definition 2.6. A map of simplicial sets $f\colon X \to Y$ is a *weak homotopy equivalence* if $|f|\colon |X| \to |Y|$ is a weak homotopy equivalence of topological spaces.

The category of simplicial sets admits a model structure in which the weak equivalences are weak homotopy equivalences, fibrations are Kan fibrations, and cofibrations are monomorphisms ([9, Theorem 11.3]). We will refer to this structure as the *Quillen model structure*. In this section, we will consider **sSet** as a model category with the Quillen model structure.

We now show two quick results, with which we will deduce that the adjunction $|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$ is a Quillen adjunction.

Lemma 2.7. *Let $p\colon X \to Y$ be a Serre fibration. Then, the simplicial map*

$$\mathrm{Sing}\, p\colon \mathrm{Sing}\, X \to \mathrm{Sing}\, Y$$

is a Kan fibration.

Proof. We need to show that for every diagram in **sSet** as the one bellow at the right

$$\begin{array}{ccc} |\Lambda^k[n]| & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ |\Delta[n]| & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \mathrm{Sing}\, X \\ \big\downarrow & \nearrow & \big\downarrow \mathrm{Sing}\, p \\ \Delta[n] & \longrightarrow & \mathrm{Sing}\, Y, \end{array}$$

where the outer square commutes, there exists a dotted arrow rendering the diagram commutative. By adjointness (Lemma 1.3), this is equivalent to finding a dotted arrow in the left hand square, which exists, by definition, because the geometric realization of a simplicial set carries a canonical CW-structure ([9, Proposition 2.3]), and p is a Serre fibration. □

Lemma 2.8. *Let $f\colon X \to Y$ be a weak homotopy equivalence of topological spaces. Then, the simplicial map* $\mathrm{Sing}\, f\colon \mathrm{Sing}\, X \to \mathrm{Sing}\, Y$ *is a weak homotopy equivalence of simplicial sets.*

Proof. It suffices to show that, for any topological space X and every basepoint $x \in X$, we have $\pi_n(\mathrm{Sing}\, X, x) \cong \pi_n(X, x)$ for all $n \geq 1$; the sets of path components are clearly in bijective correspondence. We denote by S^n_{simp} the simplicial n-sphere $\Delta[n]/\partial\Delta[n]$, which we consider with the vertex 1 as a base point. We have

$$\pi_n(\mathrm{Sing}\ X, x) = [S^n_{\mathrm{simp}}, \mathrm{Sing}\ X]$$

where the right expression denotes the set of homotopy classes of base point preserving simplicial maps from S^n_{simp} to Sing X. By adjointness, we have a natural isomorphism $\mathbf{sSet}_*(S^n_{\text{simp}}, \text{Sing } X) \cong \mathbf{Top}_*(|S^n_{\text{simp}}|, X)$, and also, that a simplicial homotopy $\Delta[n] \times \Delta[1] \to \text{Sing } X$ corresponds to a topological homotopy $I^n \times I \to X$, since $|\Delta[n]| \simeq I^n$. Hence,

$$[|S^n_{\text{simp}}|, X] \cong [S^n, X] = \pi_n(X, x)\,.$$

□

Corollary 2.9. *The adjunction* $|-| : \textit{\textbf{sSet}} \rightleftarrows \textit{\textbf{Top}} : \text{Sing}$ *is a Quillen adjunction.*

Proof. The right adjoint functor Sing preserves fibrations, by Lemma 2.7. It also preserves weak equivalences, by Lemma 2.8. □

We now proceed to prove that the components of the counit ϵ of the adjunction $(|-|, \text{Sing})$ are weak homotopy equivalences of topological spaces. Before the proof, we recall a result from classical homotopy theory. Let $p \colon E \to B$ be a fibration, where E and B are topological spaces. Consider a point $b \in B$, and $F = p^{-1}(b)$, which we call the *fiber* of p over b. Let $i \colon F \to E$ be the natural inclusion. We will say we have a fibration

$$F \xrightarrow{i} E \xrightarrow{p} B.$$

Then for every $x \in F$ there are induced morphisms $p_* \colon \pi_n(E, x) \to \pi_n(B, b)$ and $i_* \colon \pi_n(F, x) \to \pi_n(E, x)$ for all $n \geq 0$, and a morphism $\partial \colon \pi_{n+1}(B, b) \to \pi_n(F, x)$ for all $n \geq 0$, such that we have the following long exact sequence([11, Theorem 4.41]):

$$\cdots \pi_{n+1}(E, x) \xrightarrow{p_*} \pi_{n+1}(B, b) \xrightarrow{\partial} \pi_n(F, x) \xrightarrow{i_*} \pi_n(E, x) \to \cdots\,.$$

Example 2.10. Let (X, x) be a pointed topological space. We define the *space of paths* of X based at x as

$$PX = \{f \colon [0,1] \to X \mid f \text{ continuous, } f(0) = x\}.$$

The path space PX is a topological space equipped with the compact-open topology ([11, p. 529]). We consider (PX, c_x) pointed, where we denote by c_x the constant path. Then the map $p \colon PX \to X$ which sends a path f to $f(1)$ is a fibration. The fiber of p over x is the *loop space*

$$\Omega X = \{f \colon [0,1] \to X \mid f \text{ continuous, } f(0) = f(1) = x\}.$$

Furthermore, the path space is contractible, hence $\pi_n(PX, c_x) = 0$ for all $n \geq 0$. Thus the above exact sequence gives

$$\pi_{n+1}(X, x) \underset{\partial}{\xrightarrow{\cong}} \pi_n(\Omega X, c_x)$$

for all $n \geq 0$.

Proposition 2.11. *Let X be a topological space. Then the counit map $\epsilon_X \colon |\operatorname{Sing} X| \to X$ is a weak homotopy equivalence of topological spaces.*

Proof. We will prove that for any topological space X, ϵ_X induces isomorphisms

$$\pi_n(|\text{Sing } X|, x) \xrightarrow[\epsilon_X]{\cong} \pi_n(X, \epsilon x)$$

for every basepoint $x \in X$ and for all $n \geq 1$; it is clear that ϵ_X induces a bijection between the sets of path components.

Consider a topological space X. We will now prove that

$$\pi_{n+1}(|\text{Sing } X|, x) \xrightarrow[\epsilon_X]{\cong} \pi_{n+1}(X, \epsilon x)$$

assuming that ϵ_Y induces isomorphisms between $\pi_i(|\operatorname{Sing} Y|, y)$ and $\pi_i(Y, \epsilon y)$ for every topological space Y and all $i \leq n$. Since ϵ is a natural transformation, we have a commutative diagram

$$\begin{array}{ccc} \pi_{n+1}(|\operatorname{Sing} X|, x) & \xrightarrow{\epsilon_X} & \pi_{n+1}(X, \epsilon x) \\ \partial \downarrow & & \partial \downarrow \cong \\ \pi_n(|\operatorname{Sing} \Omega X|, c_x) & \xrightarrow[\epsilon_{\Omega X}]{\cong} & \pi_n(\Omega X, c_{\epsilon x}). \end{array}$$

The bottom arrow is an isomorphism because of the induction hypothesis. To show that the top arrow is an isomorphism, it suffices to show that the left vertical arrow is an isomorphism.

Now, we observe that we have a fibration $|\operatorname{Sing} \Omega X| \to |\operatorname{Sing} PX| \to |\operatorname{Sing} X|$, since the functor Sing sends Serre fibrations to Kan fibrations, by Lemma 2.7, and the geometric realization sends Kan fibrations to Serre fibrations, by Theorem 2.5. Thus we get a long exact sequence of homotopy groups. Since PX retracts to its basepoint, so does $|\operatorname{Sing} PX|$. Therefore, we have

$$\pi_{n+1}(|\text{Sing } X|, x) \xrightarrow[\partial]{\cong} \pi_n(|\text{Sing } \Omega X|, c_x)\,.$$

□

As a consequence of the result above, we can now show that the unit map $\eta_S \colon S \to \operatorname{Sing}|S|$ is a weak equivalence for every simplicial set S. Indeed, the map $\eta_S \colon S \to \operatorname{Sing}|S|$ is a weak equivalence of simplicial sets if, and only if, $|\eta_S| \colon |S| \to |\operatorname{Sing}|S||$ is a weak equivalence of topological spaces. The triangle identity of the adjunction (Lemma 1.6) gives a commutative diagram

$$\begin{array}{ccc} |S| & \xrightarrow{|\eta_S|} & |\operatorname{Sing}|S|| \\ & \searrow^{\operatorname{Id}_{|S|}} & \downarrow \epsilon_{|S|} \\ & & |S| \end{array}$$

where Id_S is obviously a weak equivalence; the counit $\epsilon_{|S|}$ is a weak equivalence, by Proposition 2.11. Since weak equivalences satisfy the two-out-of-three property, it follows that the unit $\eta_S \colon S \to \mathrm{Sing}\,|S|$ is a weak equivalence of simplicial sets.

Finally, we reach a great milestone in the comparison between simplicial sets and topological spaces, which proves, as we promised, that the homotopy theory of topological spaces is equivalent to the homotopy theory of simplicial sets.

Theorem 2.12. *The adjunction* $|-| : \textbf{sSet} \rightleftarrows \textbf{Top} : \mathrm{Sing}$ *is a Quillen equivalence.*

Proof. We must show that for every simplicial set X and every topological space Y, a map $f\colon X \to \mathrm{Sing}\,Y$ is a weak equivalence in **sSet** if and only if the adjoint map $g\colon |X| \to Y$ is a weak equivalence in **Top**. Observe that g factors as a composition

$$|X| \xrightarrow{|f|} |\mathrm{Sing}\,Y| \xrightarrow{\epsilon_Y} Y$$

(see discussion after Lemma 1.5), where ϵ_Y is a weak equivalence (Proposition 2.11). By definition, f is a weak equivalence in **sSet** if and only if $|f|$ is a weak equivalence in **Top**. By the two-out-of-three property, it follows that f is a weak equivalence if and only if g is a weak equivalence. □

Consequently, the Quillen equivalence $(|-|, \mathrm{Sing})$ induces an equivalence between the homotopy categories $\mathrm{Ho}(\textbf{sSet})$ and $\mathrm{Ho}(\textbf{Top})$. This is a result that was known in different ways since the late 1950s. Its origins go back to Kan, on his work introducing adjoint functors. This comparison between simplicial sets and topological spaces appears in work of Gabriel and Zisman [8], May [18], and Quillen [21], all published in 1967.

In the next section, we enter the context of higher category theory in order to present our definition of ∞-groupoid. Nevertheless, the results we have obtained in this section are crucial in what follows; using this passage to simplicial sets and back, we will be able to reach topological categories and prove that ∞-groupoids satisfy the homotopy hypothesis.

2.2 Topological Categories

We now consider especially "rich" categories, in which we have a notion of higher order morphisms. Two such families of categories are simplicial categories and topological categories; both can serve as a definition of ∞-*categories* and allow to develop a whole theory of higher categories. As we mentioned when introducing simplicial sets, quasicategories also provide an approach to higher category theory, and we will see in the following sections that these different approaches are actually equivalent (in an appropriate sense) and, in fact, give rise to the most complete theory of higher categories until now. However, our focus will be centered on the relationship between topological spaces and ∞-groupoids under this equivalence.

If $\mathcal{C}$ is a *monoidal* category with product $\times$ and unit I (see [15, Appendix A.1.3 and A.1.4] for details of the present discussion), a *$\mathcal{C}$-enriched category $\mathcal{D}$* consists of:

(1) a collection of objects;

(2) for every pair of objects $X, Y \in \mathcal{D}$, a mapping object $\mathrm{Map}_{\mathcal{D}}(X, Y)$ of $\mathcal{C}$;

(3) for every triple of objects $X, Y, Z \in \mathcal{D}$, an associative composition map

$$\mathrm{Map}_{\mathcal{D}}(X, Y) \times \mathrm{Map}_{\mathcal{D}}(Y, Z) \to \mathrm{Map}_{\mathcal{D}}(X, Z).$$

(4) for every object $X \in \mathcal{D}$, a morphism $I \to \mathrm{Map}_{\mathcal{D}}(X, X)$ acting as an identity element.

We also say that $\mathcal{D}$ is a category *enriched* over $\mathcal{C}$. If $\mathcal{D}$ and $\mathcal{D}'$ are two categories enriched over a category $\mathcal{C}$, then a *$\mathcal{C}$-enriched functor* $F\colon \mathcal{D} \to \mathcal{D}'$ consists of a map from the objects of $\mathcal{D}$ to the objects of $\mathcal{D}'$ and a collection of morphisms

$$\mathrm{Map}_{\mathcal{D}}(X, Y) \to \mathrm{Map}_{\mathcal{D}'}(FX, FY)$$

in $\mathcal{C}$ preserving composition and identity elements. In general, an enriched category is not a category, since mapping objects need not have underlying sets of any kind. However, every enriched category has an underlying category, defined as follows. The *underlying* category $\mathcal{D}_0$ of a $\mathcal{C}$-enriched category $\mathcal{D}$ has the same objects as $\mathcal{D}$ and

$$\mathcal{D}_0(X, Y) = \mathcal{C}(I, \mathrm{Map}_{\mathcal{D}}(X, Y)).$$

Definition 2.13. A *topological category $\mathcal{C}$* is a category enriched over the category of (compactly generated and Hausdorff [11, p.523]) topological spaces.

Remark 2.14. The case of topological categories is a special one because mapping objects are topological spaces, hence sets with a topology. Thus, the underlying sets can be viewed as sets of morphisms in the underlying category admitting a topology. This does not happen in the case of simplicial categories.

Remark 2.15. We consider the enrichment over the category of compactly generated Hausdorff spaces (denoted **CGHaus**), rather than arbitrary topological spaces, so that the geometric realization commutes with finite products; that is, $|X \times Y| \cong |X| \times |Y|$, where we take the product in **CGHaus** ([8, Section 3]). This will allow us to define a functor from simplicial categories to topological categories, as we will see in the next section. We do not lose any homotopic information by restricting ourselves to these "nice" spaces, because every topological space is weakly equivalent to one of them; namely, to a CW-complex, by the CW approximation theorem [11, Proposition 4.13].

The category of topological categories and topologically enriched functors will be denoted by **tCat**. If we say $F\colon \mathcal{D} \to \mathcal{D}'$ is a functor between topological categories, we assume it is a topologically enriched functor. The next thing we want is to find a good notion of equivalence between topological categories for our purpose. One may at first consider an enriched version of the notion of equivalence of categories by demanding the existence of homeomorphisms between mapping spaces. However, this is too restrictive for us because we want the notion of "sameness" between mapping spaces to be that of homotopy equivalence. With this in mind, we give the following two definitions.

Definition 2.16. Let $\mathcal{C}$ be a topological category. The *homotopy category* $\mathrm{h}\mathcal{C}$ is defined as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- If $X, Y \in \mathcal{C}$, we define $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y) = \pi_0 \operatorname{Map}_{\mathcal{C}}(X,Y)$.
- Composition in $\mathrm{h}\mathcal{C}$ is induced from the composition of morphisms in $\mathcal{C}$ by applying the functor π_0.

The following should be reminiscent of the characterization of a functor giving an equivalence of categories as a fully faithful and essentially surjective functor. However, our goal requires to introduce a weakened (and topologically enriched) version of this notion.

Definition 2.17. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between topological categories. We say that F is a *weak equivalence* if the following hold:

- For every pair of objects $X, Y \in \mathcal{C}$, the induced map
$$\operatorname{Map}_{\mathcal{C}}(X,Y) \to \operatorname{Map}_{\mathcal{D}}(FX,FY)$$
is a weak homotopy equivalence of topological spaces.
- Every object of $\mathcal{D}$ is isomorphic in $\mathrm{h}\mathcal{D}$ to FX, for some $X \in \mathcal{C}$.

Topological categories admit a model structure [2] in which weak equivalences are the ones defined in 2.17. All objects are fibrant in this model structure, and the cofibrant objects are the topological categories in which each mapping space is a *retract* of a CW-complex (a cofibrant object in **Top**).

Finally, we give the expected definition of an ∞-groupoid. The definition we now present captures the idea of an algebraic object with a strictly associative composition, in which every morphism is invertible up to a higher order morphism.

Definition 2.18. Let $\mathcal{C}$ be a topological category. We say that $\mathcal{C}$ is an ∞-*groupoid* if h$\mathcal{C}$ is a groupoid.

Let us help to illustrate this definition by characterizing the condition that h$\mathcal{C}$ is a groupoid. Consider an ∞-groupoid $\mathcal{G}$ and $f\colon X \to Y$ a morphism in $\mathcal{G}$. The assumption that h$\mathcal{G}$ is a groupoid garanties that there exists a morphism $g\colon Y \to X$ in $\mathcal{G}$ such that gf and Id_X lie in the same path-connected component of $\mathrm{Map}_{\mathcal{G}}(X, X)$, and, in the same way, that fg equals Id_Y in h$\mathcal{G}$. In other words, f and g are mutual inverses in $\mathcal{G}$ up to *homotopy*.

The category of ∞-groupoids and topologically enriched functors will be denoted by **∞-Grpd**. Its *homotopy category* is the full subcategory of Ho(**tCat**) formed by all ∞-groupoids, and will be denoted by Ho(**∞-Grpd**).

The next step is to study the functors permitting passage between topological spaces and ∞-groupoids, and that will give the desired equivalence. In particular, we will obtain a functor that associates an ∞-groupoid to each topological space. We can think of this assignation as "strictifying" the composition of the fundamental ∞-groupoid.

2.3 Simplicial Categories

The way in which we will prove that the homotopy categories of topological spaces and ∞-groupoids are equivalent will be through an equivalence with simplicial sets, then simplicial categories, and finally, we will reach topological categories, and in particular, ∞-groupoids.

Definition 2.19. A *simplicial category* is a category which is enriched over the category of simplicial sets.

Remark 2.20. This definition of simplicial category should not be confused with the more general notion of a *simplicial object* in the category of categories. The second notion does correspond to the first if one imposes that faces and degeneracies induce identities on objects, but we will not use this fact.

The category of simplicial categories and simplicially enriched functors will be denoted by **sCat**. A diagram of the plan outlined in the introductory paragraph above is:

$$\textbf{Top} \leftrightarrows \textbf{sSet} \rightleftarrows \textbf{sCat} \rightleftarrows \textbf{tCat}.$$

A Quillen equivalence between topological spaces and simplicial sets has been studied in Section 2.1; a Quillen equivalence between simplicial sets and simplicial categories is presented in this section; and a Quillen equivalence between simplicial categories and topological categories will be given in Section 2.5. The two latter equivalences will be composable in order to give a Quillen equivalence between simplicial sets and topological categories. To prove the homotopy hypothesis, we will have to argue accordingly in order to put together the above equivalences when taking into consideration the model structures. First, let us briefly compare simplicial categories with topological categories.

Let $\mathcal{C}$ be a simplicial category. We define the topological category $|\mathcal{C}|$ as follows:

- The objects of $|\mathcal{C}|$ are the objects of $\mathcal{C}$.
- If $X, Y \in \mathcal{C}$, then $\mathrm{Map}_{|\mathcal{C}|}(X, Y) = |\mathrm{Map}_{\mathcal{C}}(X, Y)|$.
- The composition law for morphisms in $|\mathcal{C}|$ is induced from the composition law on $\mathcal{C}$ by applying the geometric realization.

Observe that the composition law is well defined because of Remark 2.15. The functor Sing commutes with products because it is a right adjoint functor. Hence, given a topological category $\mathcal{D}$, we define a simplicial category $\mathrm{Sing}\,\mathcal{D}$ analogously. After the results of Section 2.1 it is not surprising that the functors we have just defined give an equivalence between simplicial categories and topological categories. This equivalence is precisely a Quillen equivalence, as we will see later on. However, we will now be able to see the first manifestations of it. First, we give two definitions which are analogous to the case of topological categories.

Definition 2.21. Let $\mathcal{C}$ be a simplicial category. We define the *homotopy category* $\mathrm{h}\mathcal{C}$ as $\mathrm{h}|\mathcal{C}|$.

Definition 2.22. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between simplicial categories. We say that F is a *weak equivalence* if $|F|\colon |\mathcal{C}| \to |\mathcal{D}|$ is a weak equivalence of topological categories.

Observe how the above definition reminds of that of a weak homotopy equivalence of simplicial sets and of an equivalence of categories. Indeed, a map of simplicial sets is a weak homotopy equivalence if its geometric realization is a weak homotopy equivalence of spaces. Thus, a simplicially enriched functor $F\colon \mathcal{C} \to \mathcal{D}$ is a weak equivalence if and only if the following hold:

- For every pair of objects $X, Y \in \mathcal{C}$, the induced map

  $$\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{D}}(FX,FY)$$

  is a weak homotopy equivalence of simplicial sets.

- Every object of $\mathcal{D}$ is isomorphic in $\mathrm{h}\mathcal{D}$ to FX, for some $X \in \mathcal{C}$.

A weak equivalence as we have just defined it is also known as a *Dwyer-Kan equivalence*. Like in the analogous case of topological categories, this is the good notion of equivalence between simplicial categories for our purpose. For this reason, Dwyer-Kan equivalences are the weak equivalences of the model structure on simplicial categories known as the Bergner model structure, and which we will soon describe with more detail.

It follows that $|-|$ and Sing determine an adjunction between **sCat** and **tCat**. In fact, the unit and counit maps

$$\mathcal{C} \to \mathrm{Sing}\,|\mathcal{C}|, \qquad |\,\mathrm{Sing}\,\mathcal{D}| \to \mathcal{D}$$

are weak equivalences, because they are the identity on objects and the induced maps between morphism spaces are weak equivalences, by Proposition 2.11 and the discussion following it. Thus, they induce bijections between the sets of morphisms in the homotopy categories, and, consequently, induce isomorphisms of categories

$$\mathrm{h}\mathcal{C} \xrightarrow{\cong} \mathrm{h}\,\mathrm{Sing}\,|\mathcal{C}|, \qquad \mathrm{h}|\,\mathrm{Sing}\,\mathcal{D}| \xrightarrow{\cong} \mathrm{h}\mathcal{D}.$$

2.3.1 From Simplicial Sets to Simplicial Categories

In this section, a functor from the category of simplicial sets to the category of simplicial categories is introduced. This functor will happen to be left adjoint to the homotopy coherent nerve (Section 2.3.2); together, they yield a Quillen equivalence between simplicial sets and simplicial categories.

The homotopy coherent nerve was introduced by Cordier [6], and is a variant of the *nerve* of an ordinary category. The latter is a functor which assigns to every category $\mathcal{C}$ a simplicial set $N\mathcal{C}$ given by the formula

$$(N\mathcal{C})_n = \mathbf{Cat}([n], \mathcal{C}).$$

More precisely, $(N\mathcal{C})_n$ is the set of all composable sequences of morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

of length n in $\mathcal{C}$. The face map d_i^n sends such a sequence to the one obtained by composing the ith and $(i+1)$th morphisms; the degeneracy map s_i^n sends it to the sequence obtained by including the identity as the $(i+1)$th morphism in the sequence.

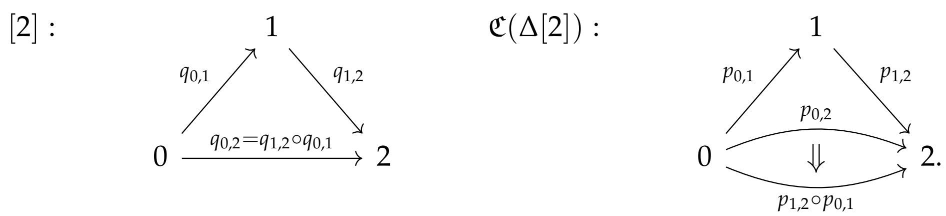
But by only considering the ordinary nerve functor we would not obtain the desired equivalence, because we would not be making use of the additional richness of simplicial categories. Therefore, we next consider a functor which will give an appropriate substitute for $[n]$ in the formula defining the nerve of a category. However, absorbing its definition will require some work.

Definition 2.23. The simplicial category $\mathfrak{C}(\Delta[n])$ is defined as follows:

- The objects of $\mathfrak{C}(\Delta[n])$ are the elements of $[n]$.
- If $i, j \in [n]$, then
$$\mathrm{Map}_{\mathfrak{C}(\Delta[n])}(i,j) = \begin{cases} \varnothing & \text{if } j < i \\ NP_{i,j} & \text{if } i \leq j \end{cases}$$
where $P_{i,j}$ is the partially ordered set $\{I \subseteq J \mid (i,j \in I),\ \forall k \in I\ (i \leq k \leq j)\}$.
- If $i_0 \leq i_1 \leq \cdots \leq i_n$, then the composition
$$\mathrm{Map}_{\mathfrak{C}(\Delta[n])}(i_0,i_1) \times \cdots \times \mathrm{Map}_{\mathfrak{C}(\Delta[n])}(i_{n-1},i_n) \to \mathrm{Map}_{\mathfrak{C}(\Delta[n])}(i_0,i_n)$$
is induced by the map of partially ordered sets
$$P_{i_0,i_1} \times \cdots \times P_{i_{n-1},i_n} \to P_{i_0,i_n}$$
$$(I_1, \ldots, I_n) \mapsto I_1 \cup \ldots \cup I_n.$$

If $i \leq j$, we have a vertex in $\mathrm{Map}_{\mathfrak{C}(\Delta[n])}(i,j)$ given by $\{i,j\} \in P_{i,j}$. We will denote this vertex by $p_{i,j}$. All the vertices in each mapping space of $\mathfrak{C}(\Delta[n])$ are a composition of vertices of the kind $p_{i,j}$.

Example 2.24. The simplicial category $\mathfrak{C}(\Delta[2])$ can be described as follows: the objects are 0, 1 and 2; the mapping spaces $\mathrm{Map}_{\mathfrak{C}(\Delta[2])}(0,1)$ and $\mathrm{Map}_{\mathfrak{C}(\Delta[2])}(1,2)$ each have, as non-degenerate simplices, the vertices $p_{0,1}$ and $p_{1,2}$ respectively; the non-degenerate simplices of $\mathrm{Map}_{\mathfrak{C}(\Delta[2])}(0,2)$ are the two vertices $p_{0,2}$ and $p_{1,2} \circ p_{0,1}$, and one edge from the first to the second. See the figure at the right:



Let us continue with some descriptions of the simplicial categories $\mathfrak{C}(\Delta[n])$. If $0 \leq i \leq j \leq n$, then

$$|\operatorname{Map}_{\mathfrak{C}(\Delta[n])}(i,j)| = I^{j-i-1}$$

the $(j-i-1)$-dimensional cube, where I is the unit interval. We now compare the categories $\mathfrak{C}(\Delta[n])$ and $[n]$. Both categories have the same objects. The totally ordered set $[n]$, viewed as a category in the usual way, has a unique morphism $q_{i,j}\colon i \to j$ for each pair $i \leq j$. In particular, these satisfy $q_{j,k} \circ q_{i,j} = q_{i,k}$, whenever $i \leq j \leq k$. In $\mathfrak{C}(\Delta[n])$, in contrast, $p_{j,k} \circ p_{i,j} \neq p_{i,k}$ whenever $i < j < k$, and instead, we have an edge from $p_{i,k}$ to $p_{j,k} \circ p_{i,j}$ in the simplicial set $\operatorname{Map}_{\mathfrak{C}(\Delta[n])}(i,k)$. In other words, the vertices $p_{i,k}$ and $p_{j,k} \circ p_{i,j}$ are equal up to homotopy. We may paraphrase this comparison as follows: $\mathfrak{C}(\Delta[n])$ is an "inflated" version of $[n]$. (Compare the two figures above.)

Definition 2.25. Let $f\colon [n] \to [m]$ be a map in Δ. We define the simplicial functor $\mathfrak{C}(f)\colon \mathfrak{C}(\Delta[n]) \to \mathfrak{C}(\Delta[m])$ as follows:

- For each object $i \in \mathfrak{C}(\Delta[n])$, $\mathfrak{C}(f)(i) = f(i) \in \mathfrak{C}(\Delta[m])$.
- If $i \leq j$ in $[n]$, then the map $\operatorname{Map}_{\mathfrak{C}(\Delta[n])}(i,j) \to \operatorname{Map}_{\mathfrak{C}(\Delta[m])}(f(i), f(j))$ induced by f is the nerve of the map

$$P_{i,j} \to P_{f(i),f(j)}$$
$$I \mapsto f(I).$$

It follows that $\mathfrak{C}$ is a functor

$$\Delta \to \mathbf{sCat}$$
$$[n] \mapsto \mathfrak{C}(\Delta[n]).$$

For the same reasons as the geometric realization, the functor $\mathfrak{C}\colon \Delta \to \mathbf{sCat}$ extends to a functor $\mathbf{sSet} \to \mathbf{sCat}$. We will also denote this extension by $\mathfrak{C}$. Then, for a simplicial set X,

$$\mathfrak{C}(X) = \operatorname*{colim}_{(\Delta \downarrow X)} \mathfrak{C}(\Delta[n])\,.$$

Consequently, we may think of $\mathfrak{C}(X)$ as the simplicial category "freely generated" by X. Consider the simple case in which X has no simplices of order $n \geq 2$. Then $\mathfrak{C}(X)$ is the free simplicial category on the graph with X_0 as the set of vertices, and the non-degenerate 1-simplices of X as the set of edges.

Remark 2.26. In general, a vertex f in a mapping space of $\mathfrak{C}(X)$ is a composition

$$X = Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} \cdots Z_n = Y$$

where each f_i is a vertex of a copy of $\mathfrak{C}(\Delta[1])$ corresponding to a 1-simplex in X. In particular, each f_i corresponds to an edge in S. In other words, each vertex in a mapping space of $\mathfrak{C}(X)$ corresponds to a path in X.

Remark 2.27. If we have a 2-simplex σ in X, with faces f, g and h as shown in the diagram

$$\begin{array}{ccc} & y & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ x & \xrightarrow{\quad h \quad} & z, \end{array}$$

then, in $\mathfrak{C}(X)$, we have two vertices $g \circ f$ and h, and an edge from h to $g \circ f$ in $\mathrm{Map}(x, z)$, given by the 2-simplex σ.

Taking the two remarks above into account, we see that if X is a quasicategory, then the simplices of orders 0, 1 and 2 of X determine $\mathrm{h}\mathfrak{C}(X)$. This is an immediate consequence of the extension property of quasicategories for 2-inner-horns $\Lambda^1[2] \to X$. In fact, when X is a quasicategory, the category $\mathrm{h}\mathfrak{C}(X)$ has a nice description, which we next present.

For a quasicategory X, we will define a category $\pi(X)$, which we will call the *homotopy category* of the quasicategory X. This category will actually be isomorphic to the category $\mathrm{h}\mathfrak{C}(X)$. Therefore, we will refer to both $\mathrm{h}\mathfrak{C}(X)$ and $\pi(X)$ as the homotopy category of X. (We refer the reader to [15, Section 1.2.3] for the omitted details on this construction.)

If X is a quasicategory, we call *objects* of X its vertices, and *morphisms* of X the 1-simplices. The face $d_1^1 \colon X_1 \to X_0$ is called *source map* and $d_0^0 \colon X_1 \to X_0$ is the *target map*. We denote by $f \colon x \to y$ a morphism with source x and target y. For each object $x \in X_0$, we let id_x denote the degenerate 1-simplex $s_0^0 x$.

If $f, g \colon x \to y$ are morphisms in X, we say that f is *homotopic* to g if there exists a 2-simplex $\sigma \in X_2$ (called a *homotopy* between f and g) which has, as its faces, the following 1-simplices:

$$\begin{array}{ccc} & y & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle \mathrm{id}_y} \\ x & \xrightarrow{\quad g \quad} & y. \end{array}$$

The homotopy relation is an equivalence relation. The homotopy class of f is denoted by $[f]$. The homotopy category $\pi(X)$ has X_0 as its set of objects and $\mathrm{Hom}_{\pi(X)}(x, y) = \{[f] \mid f \colon x \to y\}$. Composition $[g] \circ [f]$ is well defined and unique, namely $[g] \circ [f] = [h]$ where h is any filler for

$$x \xrightarrow{f} y \xrightarrow{g} z.$$

The extension property for $\Lambda^1[3] \hookrightarrow \Delta[3]$ yields uniqueness of h up to homotopy.

If K is a Kan complex, then $\pi(K) \cong \mathrm{h}\mathfrak{C}(K)$ is the fundamental groupoid of K. Equivalently, given a topological space X, the category $\mathrm{h}\mathfrak{C}(\mathrm{Sing}\, X)$ is the fundamental groupoid of the space X. In particular, we are saying that $|\mathfrak{C}(K)|$ is an ∞-groupoid. By now, we should be convinced about the veracity of the previous assertion. Nevertheless, we will give a formal proof in terms of $\mathfrak{C}$ in Section 2.6 (Proposition 2.37).

2.3.2 The Homotopy Coherent Nerve

Definition 2.28. Let $\mathcal{C}$ be a simplicial category. The *homotopy coherent nerve* (or *coherent nerve*) $N\mathcal{C}$ is the simplicial set defined by

$$(N\mathcal{C})_n = \mathbf{sCat}(\mathfrak{C}(\Delta[n]), \mathcal{C}).$$

From now on, N will always symbolize the coherent nerve. It should not bring confusion because we will not be making use of the ordinary nerve, unless otherwise stated.

Remark 2.29. The coherent nerve of a simplicial category differs, in general, from the ordinary nerve of a category. However, both notions coincide if the mapping spaces of the simplicial category are discrete. The reason is that, in such case, there are no non-constant edges in the mapping spaces. Thus, giving a functor $\mathfrak{C}(\Delta[n]) \to \mathcal{C}$ is equivalent to giving a functor $[n] \to \mathcal{C}$ (cf. Example 2.24).

Lemma 2.30. *The coherent nerve is right adjoint to* $\mathfrak{C}$.

Proof. The proof is analogous to Lemma 1.16. □

We are finally prepared to be specific about the equivalence between simplicial sets and simplicial categories. This equivalence is better understood in the context of the comparison between the approach to higher category theory using quasicategories and the approach using simplicial categories, which can be formalized in the language of model categories.

The category of simplicial sets admits a model strucutre in which the weak equivalences are a subclass of the weak homotopy equivalences (definitions and more details will be given in Section 2.4). The model structure to which we refer is known as the *Joyal model structure* [14, Theorem 6.12]. What we would like to remark for now is that the objects that are both fibrant and cofibrant in Joyal's structure are the quasicategories.

On the other hand, the category of simplicial categories can be equipped with a model structure in which the weak equivalences are the Dwyer-Kan equivalences.

The objects that are both fibrant and cofibrant in this structure are simplicial categories in which all the mapping spaces are Kan complexes. The alluded structure is called the *Bergner model structure* [3]. It is worth remarking that the fibrant simplicial categories are precisely the categories in which every morphism of order higher than two is invertible, because the mapping spaces are Kan complexes (recall the discussion after Definition 1.18). Thus, this is what is known as an $(\infty, 1)$-*category* or ∞-*category* in the modern language of higher category theory.

Remark 2.31. The coherent nerve carries fibrant simplicial categories to quasicategories. A direct proof can be found in [15, Theorem 1.1.5.10]. However, we will later be able to deduce this because N is in fact a right Quillen adjoint.

We reach the result that formalizes the equivalence between these two approaches, and is a pilar of the foundations of higher category theory:

Theorem 2.32. [15, Theorem 2.2.5.1]. *The adjoint functors $(\mathcal{C}, N)$ determine a Quillen equivalence between* **sSet** *(with the Joyal model structure) and* **sCat** *(with Bergner's model structure).*

The reference given is for a proof by Lurie; a different proof was given by Joyal in [13]. A result that plays a very important role in the proof of the equivalence above is the one coming up next, and which we present because it will be specifically useful for our goal; it shows that the counit of the adjunction $(\mathfrak{C}, N)$ is a weak equivalence of simplicial categories, because it induces weak homotopy equivalences between the mapping spaces, and the categories $\mathfrak{C}(N\mathcal{C})$ and $\mathcal{C}$ have the same objects.

Theorem 2.33. *Let $\mathcal{C}$ be a fibrant simplicial category (that is, a simplicial category in which each mapping space $\mathrm{Map}_{\mathfrak{C}(N\mathcal{C})}(x, y)$ is a Kan complex) and let $x, y \in \mathcal{C}$ be a pair of objects. The counit map*

$$\mathrm{Map}_{\mathfrak{C}(N\mathcal{C})}(x, y) \to \mathrm{Map}_{\mathcal{C}}(x, y)$$

is a weak homotopy equivalence of simplicial sets.

Proof. See [15, Theorem 2.2.0.1]. □

2.4 Comparing Model Structures on Simplicial Sets

At this point, we have shown that **Top** and **sSet** are Quillen equivalent, where we consider **sSet** with Quillen's model structure; and that **sSet** and **sCat** are Quillen equivalent, with **sSet** considered with Joyal's model structure. To assemble these equivalences together we should be careful, because Quillen's and

Joyal's model categories are not Quillen equivalent. However, our goal is to show that, for Kan complexes, they are sufficiently similar. The weak equivalences in the Joyal model structure are the following:

Definition 2.34. Let $f\colon X \to Y$ be a map of simplicial sets. We say f is a *weak categorical equivalence* if $\mathfrak{C}(f)\colon \mathfrak{C}(X) \to \mathfrak{C}(Y)$ is a weak equivalence of simplicial categories.

Remark 2.35. Recall that $\mathfrak{C}(f)$ is a weak equivalence of simplicial categories if $|\mathfrak{C}(f)|$ is a weak equivalence of topological categories.

Every weak categorical equivalence is a weak homotopy equivalence of simplicial sets, and the converse is true for maps between Kan complexes ([14, Corollary 6.16]). Furthermore, the cofibrations in Joyal's model structure are the same as in Quillen's model structure on simplicial sets. Consequently, the fibrant objects in Joyal's structure (quasicategories) include the fibrant objects in Quillen's structure (Kan complexes); all simplicial sets are cofibrant with both structures. From now on, the category of simplicial sets with Joyal's model structure will be denoted by $\mathbf{sSet}_J$, and when equipped with Quillen's model structure, it will be written as $\mathbf{sSet}_Q$. Let us describe the relationship between their homotopy categories.

The identity functor $\mathrm{Id}\colon \mathbf{sSet}_J \to \mathbf{sSet}_Q$ preserves cofibrations and trivial cofibrations, by the description above. Hence, the functors

$$\mathrm{Id}\colon \mathbf{sSet}_J \rightleftarrows \mathbf{sSet}_Q : \mathrm{Id}$$

form a Quillen adjunction. Since every Kan complex is a quasicategory, the right Quillen functor induces a functor

$$\mathrm{Id}_*\colon \mathrm{Ho}(\mathbf{sSet}_Q) \to \mathrm{Ho}(\mathbf{sSet}_J)$$

by Corollary 1.28. The left Quillen functor $\mathrm{Id}\colon \mathbf{sSet}_J \to \mathbf{sSet}_Q$ takes fibrant values when restricted to Kan complexes. Therefore, if we denote by $\mathrm{Ho}(\mathbf{Kan}_J)$ the full subcategory of $\mathrm{Ho}(\mathbf{sSet}_J)$ formed by Kan complexes, again by Corollary 1.28, we have an induced functor

$$\mathrm{Id}_*\colon \mathrm{Ho}(\mathbf{Kan}_J) \to \mathrm{Ho}(\mathbf{sSet}_Q).$$

All in all, we have functors $\mathrm{Id}_*\colon \mathrm{Ho}(\mathbf{Kan}_J) \rightleftarrows \mathrm{Ho}(\mathbf{sSet}_Q) : \mathrm{Id}_*$ which clearly induce bijections between the sets of maps. Since both categories have the same objects, the above is an isomorphism of categories. Consequently, from now on, we will write $\mathrm{Ho}(\mathbf{Kan})$ to denote either of these two isomorphic categories, and call it the *homotopy category* of Kan complexes.

2.5 Simplicial Sets and Topological Categories

At the beginning of Section 2.3, we outlined an equivalence between simplicial sets and topological categories. After all the technicalities introducing the functors $\mathfrak{C}$ and N in the previous sections, we can formalize and shape this equivalence. Precisely, that the adjoint pair $(|\mathfrak{C}|, N\,\mathrm{Sing})$ yields a Quillen equivalence between $\mathbf{sSet}_J$ and $\mathbf{tCat}$. The remaining ingredient is the result that follows, which was proved by Amrani in [2].

Theorem 2.36. [2]. *The adjoint functors* $(|-|, \mathrm{Sing})$ *determine a Quillen equivalence between* $\mathbf{sCat}$ *and* $\mathbf{tCat}$*.*

We recall that $(\mathfrak{C}, N)$ is a Quillen equivalence between the model categories $\mathbf{sSet}_J$ and $\mathbf{sCat}$. Hence, the composition

$$|\mathfrak{C}| : \mathbf{sSet}_J \rightleftarrows \mathbf{tCat} : N\,\mathrm{Sing}$$

is a Quillen equivalence. These functors induce an equivalence between homotopy categories as in 1.28. Indeed, $|\mathfrak{C}|$ takes fibrant values because every topological category is fibrant, and $N\,\mathrm{Sing}$ takes cofibrant values because every simplicial set is cofibrant.

To focus on the relationship between topological spaces and ∞-groupoids under this equivalence, we need to show that it restricts to an equivalence between the homotopy categories of Kan complexes and of ∞-groupoids; the equivalence between the homotopy categories of Kan complexes and of topological spaces is assured by the first section of this chapter.

2.6 Kan Complexes and ∞-Groupoids, back and forth

The goal of this section is to show that the pair of functors $(|\mathfrak{C}|, N\,\mathrm{Sing})$ between simplicial sets and topological categories restrict to functors between Kan complexes and ∞-groupoids. After this, we will be ready to prove the homotopy hypothesis.

Proposition 2.37. *If K is a Kan complex, then $|\mathfrak{C}(K)|$ is an ∞-groupoid.*

Proof. We have to prove that $\mathrm{h}|\mathfrak{C}(K)|$ is a groupoid. Since morphisms in the same path-component of $|\mathfrak{C}(K)|$ are identified in $\mathrm{h}|C(K)|$, it suffices to prove that every vertex in every mapping space of $\mathfrak{C}(K)$ becomes invertible in $\mathrm{h}|\mathfrak{C}(K)|$. Recall that a vertex in $\mathfrak{C}(K)$ corresponds to a path in K, in the sense of Remark 2.26. Hence, we will be done if we show that every 1-simplex in X yields an invertible morphism (up to homotopy) in $|\mathfrak{C}(K)|$. We consider a 1-simplex $\alpha\colon \Delta[1] \to K$ of K, which determines a functor

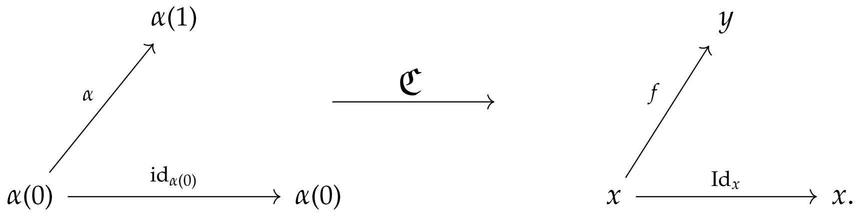
$$F\colon \mathfrak{C}(\Delta[1]) \to \mathfrak{C}(K)$$

that sends the objects 0 and 1, to objects x and y, and the vertex $p_{0,1}$ to a vertex $f \in \mathrm{Map}_{\mathfrak{C}(K)}(x,y)$. In other words, $F(\mathfrak{C}(\Delta[1]))$ is the copy of $\mathfrak{C}(\Delta[1])$ in $\mathfrak{C}(K)$ which corresponds to the 1-simplex α. We prove that f becomes invertible in $\mathrm{h}|\mathfrak{C}(K)|$.

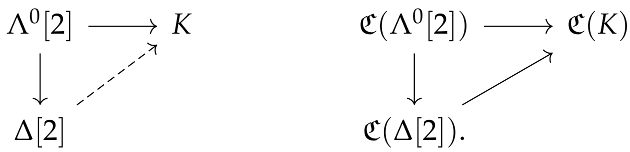
Consider the degenerate 1-simplex $s_0^0\alpha(0) = \mathrm{id}_{\alpha(0)}$ of K, which determines another functor

$$\mathfrak{C}(\Delta[1]) \to \mathfrak{C}(K)$$

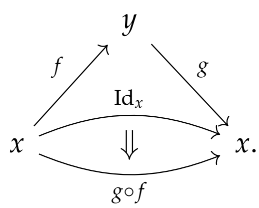
that sends both 0 and 1 to x, and the vertex $p_{0,1}$ to Id_x. Note that α and $\mathrm{id}_{\alpha(0)}$, joined at their initial vertex $\alpha(0)$, form a 2-horn $\Lambda^0[2] \to K$:



Consider the following extension problem in **sSet** at the left:



Since, by assumption, K is a Kan complex, there exists a map of simplicial sets $\Delta[2] \to K$ rendering the above diagram commutative. By applying $\mathfrak{C}$, we obtain the commutative diagram above at the right. The extension $\mathfrak{C}(\Delta[2]) \to \mathfrak{C}(K)$ gives the following in $\mathfrak{C}(K)$:



That is, $g \circ f$ and Id_x lie in the same path-component of $\pi_0 \mathrm{Map}_{\mathfrak{C}(K)}(x,x)$. Similarly, by considering a degenerate 1-simplex at the end vertex of α, we get that there exists a vertex h of $\mathrm{Map}_{\mathfrak{C}(K)}(y,x)$ such that $f \circ h$ and Id_y lie in the same path-component of $\pi_0 \mathrm{Map}_{\mathfrak{C}(K)}(y,y)$. This means that f has both a right and left inverse in $\mathrm{h}|\mathfrak{C}(K)|$, and it follows that f becomes an isomorphism in the homotopy category $\mathrm{h}|\mathfrak{C}(K)|$. We conclude that $\mathrm{h}|\mathfrak{C}(K)|$ is a groupoid. $\square$

Theorem 2.38. *A quasicategory X is a Kan complex if, and only if, the homotopy category of X is a groupoid.*

Proof. We have proven the left to right implication in 2.37. The right to left implication is due to Joyal ([14, Theorem 4.14]). □

Proposition 2.39. *Let $\mathcal{C}$ be a fibrant simplicial category. If $\mathrm{h}\mathcal{C}$ is a groupoid, then $N\mathcal{C}$ is a Kan complex.*

Proof. We know that $N\mathcal{C}$ is a quasicategory (Remark 2.31). We will prove that $\mathrm{h}\mathfrak{C}(N\mathcal{C})$ is a groupoid, and then, by Theorem 2.38, we will be able to conclude that $N\mathcal{C}$ is a Kan complex.

In order to prove that the assumptions in the statement imply that $\mathrm{h}\mathfrak{C}(N\mathcal{C})$ is a groupoid, we will prove that this category is isomorphic to $\mathrm{h}\mathcal{C}$; this will suffice.

By Theorem 2.33, the counit $\epsilon_{\mathcal{C}}\colon \mathfrak{C}(N\mathcal{C}) \to \mathcal{C}$ is a weak equivalence of simplicial categories. In particular, for every pair of objects X and Y of $\mathfrak{C}(N\mathcal{C})$, the following induced map is a weak homotopy equivalence of simplicial sets

$$\mathrm{Map}_{\mathfrak{C}(N\mathcal{C})}(X,Y) \to \mathrm{Map}_{\mathcal{C}}(X,Y).$$

Thus, it induces a bijection between their sets of path components. That is,

$$\mathrm{Map}_{\mathrm{h}\mathfrak{C}(N\mathcal{C})}(X,Y) \cong \mathrm{Map}_{\mathrm{h}\mathcal{C}}(X,Y).$$

Both categories have the same objects. We conclude that $\mathrm{h}\mathcal{C}$ and $\mathrm{h}\mathfrak{C}(N\mathcal{C})$ are isomorphic categories. □

Corollary 2.40. *Let $\mathcal{C}$ be an ∞-groupoid. The simplicial set $N\,\mathrm{Sing}\,\mathcal{C}$ is a Kan complex.*

Proof. The mapping spaces of the simplicial category $\mathrm{Sing}\,\mathcal{C}$ are Kan complexes (Example 1.17). Also, $\mathrm{h}\mathcal{C}$ is a groupoid if and only if $\mathrm{h}\,\mathrm{Sing}\,\mathcal{C} = \mathrm{h}|\,\mathrm{Sing}\,\mathcal{C}|$ is a groupoid, because these two categories are isomorphic (see the discussion before Section 2.3.1). The result follows from the previous proposition. □

2.7 Proof of the Homotopy Hypothesis

We end this chapter with a proof that the study of topological spaces, up to weak homotopy equivalence, is equivalent to the study of ∞-groupoids, up to weak equivalence. Precisely, we will give an equivalence between the homotopy categories of topological spaces and of ∞-groupoids. This will prove the homotopy hypothesis.

In the previous section, we saw that the adjunction $|\mathfrak{C}| : \mathbf{sSet} \rightleftarrows \mathbf{tCat} : N\,\mathrm{Sing}$ restricts to an adjunction

$$|\mathfrak{C}| : \mathbf{Kan} \rightleftarrows \infty\text{-}\mathbf{Grpd} : N\,\mathrm{Sing}\,.$$

Consequently, the induced adjunction $|\mathfrak{C}|_* : \mathrm{Ho}(\mathbf{sSet}_J) \rightleftarrows \mathrm{Ho}(\mathbf{tCat}) : N\,\mathrm{Sing}_*$ from Section 2.5 also restricts to an adjunction

$$|\mathfrak{C}|_* : \mathrm{Ho}(\mathbf{Kan}) \rightleftarrows \mathrm{Ho}(\infty\text{-}\mathbf{Grpd}) : N\,\mathrm{Sing}_*\,.$$

In order to conclude that the later is an equivalence of categories, we will show that the components of both the unit and counit of the adjunction $(|\mathfrak{C}|, N\,\mathrm{Sing})$ are weak equivalences, and thus, isomorphisms in the homotopy categories (Corollary 1.28).

Proposition 2.41. *Let $\mathcal{G}$ be an ∞-groupoid. The counit map $\epsilon_{\mathcal{G}} \colon |\mathfrak{C}(N\,\mathrm{Sing}\,\mathcal{G})| \to \mathcal{G}$ is a weak equivalence of topological categories.*

Proof. The map $\epsilon_{\mathcal{G}}$ is the composition

$$|\mathfrak{C}(N\,\mathrm{Sing}\,\mathcal{G})| \to |\,\mathrm{Sing}\,\mathcal{G}| \to \mathcal{G}$$

where the first is given by the counit of $(\mathfrak{C}, N)$ and the second is given by the counit of $(|-|, \mathrm{Sing})$. Note that $\mathrm{Sing}\,\mathcal{G}$ is a simplicial category in which each mapping space is a Kan complex. Hence, by Theorem 2.33, the first map is a weak equivalence. We have also seen, before starting Section 2.3.1, that the second map is a weak equivalence. We conclude that the counit map $|\mathfrak{C}(N\,\mathrm{Sing}\,\mathcal{G})| \to \mathcal{G}$ is a weak equivalence. □

Now, we can deduce that the components of the unit of the adjunction are weak categorical equivalences. By definition, the unit map $X \to N\,\mathrm{Sing}\,|\mathfrak{C}(X)|$ is a weak categorical equivalence if $|\mathfrak{C}(X)| \to |\mathfrak{C}(N\,\mathrm{Sing}\,|\mathfrak{C}(X)|)|$ is a weak equivalence of topological categories. From the triangle identity of the adjunction $(|\mathfrak{C}|, N\,\mathrm{Sing})$ (Lemma 1.6), we have a commutative diagram

$$\begin{array}{ccc} |\mathfrak{C}(X)| & \xrightarrow{|\mathfrak{C}\eta_X|} & |\mathfrak{C}(N\,\mathrm{Sing}\,|\mathfrak{C}(X)|)| \\ & {\scriptstyle \mathrm{Id}_{|\mathfrak{C}(X)|}} \searrow & \downarrow {\scriptstyle \epsilon_{|\mathfrak{C}(X)|}} \\ & & |\mathfrak{C}(X)| \end{array}$$

where $\mathrm{Id}_{|\mathfrak{C}(X)|}$ is clearly a weak equivalence; the counit map $\epsilon_{|\mathfrak{C}(X)|}$ is also a weak equivalence, as we have shown above. By the two-out-of-three property, it follows that the top arrow is a weak equivalence, like we wanted to show.

Proposition 2.42. *Let K be a Kan complex. The unit map $K \to N\,\mathrm{Sing}\,|\mathfrak{C}(K)|$ is a weak homotopy equivalence of simplicial sets.*

Proof. We have seen above that the unit map $K \to N\,\mathrm{Sing}\,|\mathfrak{C}(K)|$ is a weak categorical equivalence. Recall, from Section 2.4, that every weak categorical equivalence is a weak homotopy equivalence. $\square$

Joining this with the results of Section 2.1, we have functors

$$\mathbf{Top} \underset{\mathrm{Sing}}{\overset{|-|}{\leftrightarrows}} \mathbf{Kan} \underset{N\,\mathrm{Sing}}{\overset{|\mathfrak{C}|}{\rightleftarrows}} \infty\text{-}\mathbf{Grpd}$$

which induce equivalences between the corresponding homotopy categories. This shows that ∞-groupoids satisfy the homotopy hypothesis.

Chapter 3

The Fundamental ∞-Groupoid as a Topological Category

In the previous section, we described a way to associate to a topological space X an ∞-groupoid $|\mathfrak{C}(\operatorname{Sing} X)|$ that gets very close to our idea of what $\Pi_\infty(X)$ should be: it is an algebraic object with a strictly associative composition, and it encodes the homotopy type of the space X. Furthermore, $\mathrm{h}|\mathfrak{C}(\operatorname{Sing} X)|$ is the fundamental groupoid of X. Consequently, we could define the fundamental ∞-groupoid of X as $|\mathfrak{C}(\operatorname{Sing} X)|$. However, the topological category $|\mathfrak{C}(\operatorname{Sing} X)|$ is definitely not an easy object to handle.

3.1 Moore Paths

The fundamental ∞-groupoid of a topological space X is a genuine example of an ∞-groupoid, which should admit a more manageable and accessible model. In this last chapter we propose such a model, which is a topological category that we denote by $\Pi_\infty(X)$.

Definition 3.1. Let X be a topological space and $x \in X$ a point. We define the space of *Moore paths* based at x as the topological space

$$P'_x X = \{(f, r) \in X^{\mathbb{R}_+} \times \mathbb{R}_+ \mid f(0) = x,\ f(s) = f(r) \text{ for } s \geq r\}$$

with the product topology, where $\mathbb{R}_+$ has the Euclidean topology and $X^{\mathbb{R}_+}$ has the compact-open topology ([11, p. 529]). Here, $\mathbb{R}_+ = [0, \infty)$ and $X^{\mathbb{R}_+}$ is the space of continuous maps from $\mathbb{R}_+$ to X.

We call the real number r the *length* of the path f if it is the smallest such that $f(s) = f(r)$ for all $s \geq r$. Observe that the ordinary space of paths $P_x X$ based at x

embeds into $P'_x X$ as the subspace of paths of length 1. The space of Moore paths $P'_x X$ is homotopy equivalent to $P_x X$, since $P_x X$ is a deformation retract of $P'_x X$ ([5, Proposition 5.1.1]).

Definition 3.2. Let X be a topological space. We define a topological category $\Pi_\infty(X)$ in the following way:

- The objects of $\Pi_\infty(X)$ are the points of X.
- If x and y are two objects, we define $\mathrm{Map}_{\Pi_\infty(X)}(x,y)$ as

  $$\{(f,r) \in X^{\mathbb{R}_+} \times \mathbb{R}_+ \mid f(0) = x,\ f(r) = y,\ \ f(s) = f(r) \text{ for } s \geq r\}$$

  viewed as a subspace of $P'_x X$.
- Composition is given by the continuous map

  $$\mathrm{Map}_{\Pi_\infty(X)}(x,y) \times \mathrm{Map}_{\Pi_\infty(X)}(y,z) \longrightarrow \mathrm{Map}_{\Pi_\infty(X)}(x,z)$$

  sending $((f,r),(g,s))$ to $(f * g, r+s)$, where $f * g$ denotes concatenation of paths, that is,

  $$(f * g)(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq r \\ g(t-r) & \text{if } t \geq r. \end{cases}$$
- For every object $x \in \Pi_\infty(X)$, the constant path $(c_x, 0)$ of length 0 acting as an identity element.

Our choice of Moore paths instead of ordinary paths (i.e., paths parametrized by $0 \leq t \leq 1$) makes composition strictly associative, and therefore $\Pi_\infty(X)$ is indeed a topological category.

Proposition 3.3. *The topological category $\Pi_\infty(X)$ is an ∞-groupoid for every space X.*

Proof. Let $(f,r) \in \mathrm{Map}_{\Pi_\infty(X)}(x,y)$ for some objects x and y of $\Pi_\infty(X)$. We claim that (f,r) becomes invertible in $\mathrm{h}\Pi_\infty(X)$. Indeed, any path that undoes the travel of f provides a homotopy inverse for (f,r). To be specific, consider the morphism $(g,r) \in \mathrm{Map}_{\Pi_\infty(X)}(y,x)$ where g is defined by

$$g(t) = \begin{cases} f(r-t) & \text{if } 0 \leq t \leq r \\ x & \text{if } t \geq r. \end{cases}$$

We have that $(f * g, r+r)$ lies in the same path component of $\mathrm{Map}_{\Pi_\infty(X)}(y,y)$ as the constant path $(c_y, 0)$. Similarly for $(g * f, r+r)$ and $(c_x, 0)$. We conclude that (f,r) and (g,r) become mutual inverses in $\mathrm{h}\Pi_\infty(X)$. □

The above definition yields a functor from topological spaces to ∞-groupoids: given a continuous map $f\colon X \to Y$, we let $\Pi_\infty(f)\colon \Pi_\infty(X) \to \Pi_\infty(Y)$ send each object x to $f(x)$, and for every pair of objects x, y of $\Pi_\infty(X)$ we consider the map

$$\mathrm{Map}_{\Pi_\infty(X)}(x,y) \longrightarrow \mathrm{Map}_{\Pi_\infty}(f(x), f(y))$$

sending a morphism (g, r) to $(f \circ g, r)$.

3.2 Realizing the Fundamental ∞-Groupoid

Finally, we achieve our goal of finding a genuine and manageable model of the fundamental ∞-groupoid. Our candidate is the ∞-groupoid $\Pi_\infty(X)$ defined in the previous section; in this section we prove that $\Pi_\infty(X)$ encodes the whole homotopy type of X, i.e., it is a model for its homotopy type. We start with a characteristic property of ∞-groupoids. Namely, every connected ∞-groupoid "collapses", up to weak equivalence, into the space of endomorphisms of any one of its objects. Observe the resemblance with Proposition 1.11.

Proposition 3.4. *Let $\mathcal{G}$ be a connected ∞-groupoid. Let x be an arbitrary object of $\mathcal{G}$, and consider $\mathrm{Map}_{\mathcal{G}}(x,x)$ viewed as a topological category with one object. Then the inclusion functor $\mathrm{Map}_{\mathcal{G}}(x,x) \hookrightarrow \mathcal{G}$ is a weak equivalence of topological categories.*

Proof. The inclusion functor is fully faithful because it induces the identity on the single mapping space. It is essentially surjective because $\mathrm{h}\mathcal{G}$ is a connected groupoid. □

Example 3.5. For every space X with a base point x_0, the space $\Omega' X$ of Moore loops based at x_0 is the topological monoid of endomorphisms of x_0 within the topological category $\Pi_\infty(X)$. Thus, viewing $\Omega' X$ as a topological category with one object, the ∞-groupoids $\Pi_\infty(X)$ and $\Omega' X$ are weakly equivalent.

Now we can prove that our model $\Pi_\infty(X)$ of the fundamental ∞-groupoid has the right homotopy type.

Theorem 3.6. *For every topological space X, the ∞-groupoids $|\mathfrak{C}(\mathrm{Sing}(X)|$ and $\Pi_\infty(X)$ are weakly equivalent.*

Proof. The topological categories $\Pi_\infty(X)$ and $|\operatorname{Sing}\Pi_\infty(X)|$ are weakly equivalent by the discussion before Section 2.3.1. Suppose first that X is path-connected, and choose a point $x \in X$. If we collapse $\Pi_\infty(X)$ onto $\Omega' X$ for the chosen point x, we have a weak equivalence

$$|\operatorname{Sing}\Pi_\infty(X)| \simeq |\operatorname{Sing}\Omega' X|,$$

because $|-|$ preserves weak equivalences, by definition, and also does Sing, by Lemma 2.8. On the other hand, the simplicial category $\operatorname{Sing}\Omega' X$ is weakly equivalent to $\mathfrak{C}(\operatorname{Sing} X)$ by the argument given in [24, Proposition 7.2]. All in all,

$$\Pi_\infty(X) \simeq |\operatorname{Sing}\Pi_\infty(X)| \simeq |\operatorname{Sing}\Omega' X| \simeq |\mathfrak{C}(\operatorname{Sing} X)|.$$

Since the path-connected components of X correspond to the connected components of $\Pi_\infty(X)$, the argument holds componentwise for arbitrary (not necessarily path-connected) topological spaces. □

Corollary 3.7. *The ∞-groupoid $\Pi_\infty(X)$ is a model for the homotopy type of a space X.*

Proof. This result is assured by the previous theorem and the proof of the homotopy hypothesis, as we now show. By the previous theorem, $\Pi_\infty(X)$ and $|\mathfrak{C}(\operatorname{Sing} X)|$ are weakly equivalent. Thus, applying the functor $|N\operatorname{Sing}|$ yields a weak homotopy equivalence of topological spaces

$$|N\operatorname{Sing}\Pi_\infty(X)| \simeq |N\operatorname{Sing}|\mathfrak{C}(\operatorname{Sing} X)||.$$

Indeed, $|-|$ and Sing preserve weak equivalences; since Sing takes fibrant values and N is a right Quillen functor, the composition $N\operatorname{Sing}$ also preserves weak equivalences (see Remark 1.25).

Finally, by Proposition 2.42, the unit map $\operatorname{Sing}(X) \to N\operatorname{Sing}|\mathfrak{C}(\operatorname{Sing} X)|$ is a weak homotopy equivalence of simplicial sets, and, consequently,

$$|N\operatorname{Sing}|\mathfrak{C}(\operatorname{Sing} X)|| \simeq |\operatorname{Sing} X| \simeq X,$$

where the second equivalence is assured by Proposition 2.11. In summary, we have shown that $|N\operatorname{Sing}\Pi_\infty(X)| \simeq X$, proving that $\Pi_\infty(X)$ encodes the whole homotopy type of the topological space X. □

Conclusions

The goal of this thesis has been to prove a homotopy theoretic equivalence between topological spaces and ∞-groupoids: the homotopy hypothesis. This brought us to learn about different models for ∞-groupoids, finally choosing that of topological categories, which we believe to be the most intuitive.

In order to prove the homotopy hypothesis, we have followed reflections of the classical homotopy theory of topological spaces in other contexts of mathematics, a road that has brought us from the comparison with simplicial sets, back in the mid 20th century, to ∞-groupoids and the lively and far-reaching field of higher category theory. We have started out by reviewing the equivalence between the homotopy theory of topological spaces and that of simplicial sets, which is by itself a very important result and gives combinatorial insights in the homotopy theory of spaces. Furthermore, it has been crucial for our goal because from simplicial sets we have been able to reach simplicial categories, and it has given us passage back into a topological context where ∞-groupoids lie.

By transitioning from spaces to simplicial sets, and then ∞-groupoids, we have been able to give a wide view of the foundations of higher category theory establishing an equivalence between some of its different approaches, and to formalize a proof of the homotopy hypothesis with this theory as a base.

Since the beginning we have pursued the aspiration of finding a model for ∞-groupoids in which the fundamental ∞-groupoid is realized in a manageable and transparent way, and which would make ∞-groupoids more accessible. We believe we have accomplished this goal in Chapter 3.

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