AN ECONOMIC REVIEW OF VALUE: FINANCIAL VALUE.

Its application to the Financial Analysis of Investment*

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ABSTRACT

Conventional financial analysis does not meet the requirements of the *complex operations*, mainly because of their different times of *financial immobilization*. The present study overcomes this limitation by means of a *financial reduction* which, respecting the financial equivalence of the market, defines an *Average Financial Time* (AFT) which provides a rigorous description of complex operations, both *financing* and *investment*.

To do this, it introduces an initial formalization of the financial operation, with a strictly financial definition of the *income* and *profitability* magnitudes based on the measure of financial imbalance pursued by investment operations. The study also includes a critical, rigorous study of the IRR which shows it to be imperfect, introducing at the same time an original complete calculation of its solutions based on the definition of the AFT. This conceptual framework can be immediately applied to the financial selection of investments.

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CONTENT

0. INTRODUCTION

The Department of Economic, Financial and Actuarial Mathematics of the University of Barcelona has carried out a research programme, previously explained in several publications*, which has made it possible to incorporate the subject known as **Investment Mathematics** within its Academic Syllabus. This is a continuation of the **Mathematics of Finance**, and both subjects can be included within the field known as **Financial Mathematics**.

This paper aims to offer a synthesis of the conceptual and methodological foundations that make it possible to extend conventional financial analysis to complex financial operations. By means of **financial reduction** we will present magnitudes that describe the yield of an investment as *yield or return rates*, which replace conventional IRR. These enable there to be greater perfection in the financial selection of investment projects. This study also includes a thorough critical study of the strictly financial meaning of IRR, possible solutions and the implication of the paradoxes involved. An original method for studying, calculating and representing financial operations will also be put forward, by introducing the magnitude AFT (Average Financial Time). Finally, a computer application that describes the financial operation of investment in any *financial ambience* (measured by market interest rate structure), mainly based on the financial functions AFT and DUR (Duration), and the auxiliary functions HYP (Hyperbola) and DEV (Deviation) will be presented.

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* Rodríguez, A. (ed.1974,1984,1994) *Matemática de la Financiación* [Mathematics of financing operations]. University of Barcelona.
Rodríguez, A. (1983, 1987) *Matemática de la Inversión* [Mathematics of investment operations]. Romargraf: Barcelona.
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1. ECONOMIC INSUFFICIENCY OF MONETARY VALUE: FINANCIAL VALUATION

The monetary value of goods, a service or a productive element is expressed by means of an amount of money in the currency used as a unit of reference. Monetary valuation governs all economic transactions, both as regards consumption and in production. The market carries out an *objective* valuation, which, as a consequence of being generally accepted in a certain financial ambience, justifies the monetary equiibrium of the transactions. Another parallel valuation, a *subjective* one, which economic agents carry out in the course of decision-making, also exists. The difference between the latter and objective valuations is the driving force and the reason behind economic transactions, in which both parties seek to gain a profit. Objective valuation, because of its general nature, is what is described by the science of Economics, and the subject of interest in this study.

When assetss or patrimonial elements endowed with different *degrees of liquidity,* or alternatively liabilities affected by different *degrees of requirability or claim*, are involved, monetary valuation is insufficient to describe holdings and patrimonial transactions. The *waiting time* of the monetary unit that expresses the monetary value, until it is converted into available liquid resources, is not unimportant for *economic value*. The same is true, if it is a matter of a liability, of the *waiting time* until it can be required. *Preference for liquidity*, as far as assetss are concerned, or an *aversion to require*, in the case of liabilities, is based on the greater appreciation and economic value of *immediate* money in

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Rodríguez, A. (1998) *Fundamentos de la Matemática Financiera* [Foundations of Financial Mathematics]. Gráficas Rey: Barcelona.

comparison with *distant* or future money. This principle is what lies at the heart of the *financial phenomenon.*

The *liquidity* of an amount of money is measured by the *deferment* time until its *expiry date*. While the *amount* measures the *monetary value* of a patrimonial element (whether an assets or a liabilities for the person concerned), the *deferment* time measures its *degree of liquidity* or *requirability*. The **financial value** summarises both measurements of a patrimonial holding. It can be expressed by means of a *binary complex* (C,T) called **financial capital**, the two components of which, (C) **amount**, and (T) **deferment time**, are real non-negative numbers, which measure the monetary value and liquidity respectively,

$$(C,T); \quad C,T \in \mathbf{R}^{+}$$

It should be observed that in the case of non-financial assetss, without a defined expiry date, liquidity is also present and is demonstrated in the process of *depreciation* of immobilised assetss, and the *realization* of movable assetss.

In simple financial transactions, the different degree of liquidity is often corrected by means of adding *interest*, which reestablishes the equilibrium between the different financial values of the monetary unit. Nevertheless, such a *financial corrector* does not resolve the valuation in the case of other assetss or more complex holdings, which are made up by non-financial elements, and which exhibit varying degrees of liquidity or *requirability*. A paradigmatic example can be found in a business balance sheet, which expresses an accounting balance for the company's complex accumulation of patrimonial assetss and liabilities in purely *monetary* terms.

2. THE COMPANY BALANCE SHEET: MONETARY EQUILIBRIUM

Let us consider the following, highly schematic, company balance sheet:

ASSETS		**LIABILITIES**	
IMMOBILISED	2,000,000	CAPITAL	1,500,000
MOVABLE	500,000	LONG TERM CREDITORS	1,000,000
DEBTORS	150,000	SHORT TERM CREDITORS	175,000
LIQUID ASSETSS	50,000	DIVIDENDS (sharehlders) ...	25,000
TOTAL	2,700,000	TOTAL	2,700,000

The sum total of the Assets expresses the monetary value of a collection of patrimonial amounts of widely varying liquidity. In practice, *long-term* immobilised assetss, such as land and buildings, are added to the liquid assetss, whose liquidity is immediate. They coexist with other intermediate accounts, in turn forming part of other elements of widely varying liquidity. Thus, it is apparent that the sum of the Assets does not respect the *principle of financial homogeneity* of the amounts added together. The sum of the Assets shows the *monetary value* of the collective patrimony that it incorporates, but it *does not explain its financial value.*

As far as the Liabilities are concerned, the company's own funds, which are not requirable, coexist alongside expired loans, which are immediately demandable. In intermediate accounts, liabilities of varying degrees of requirability are also added. Hence, the sum total of the Liabilities does not describe the financial valuation, but rather only the *monetary value*.

The equilibrium that this balance sheet reflects is an *accounting or monetary balance* of the sum total of the Assets and the Liabilities. In its patrimonial representation, the balance sheet ignores financial analysis, which considers the *degree of liquidity* and *demandability* of the cumulative patrimonial assetss. For this reason, the entries that the accounts and the balance sheet aggregate in monetary terms lack *financial homogeneity*.

3. FINANCIAL AGGREGATION

Both business and private holdings and patrimonial transactions are made up by aggregates of assetss and liabilities. The *financial value* of such aggregates formally requires the definition of a very special algebra in the set of financial capitals (*financial algebra*), since the **financial sum** contemplates the aggregation of financial capitals that are complex binary elements. As the financial sum is an *internal composition law* in the set of financial capitals, the result has to be a **financial capital** that represents the financial value of the aggregate, with its two components, the monetary *amount* and the *deferment* time, clearly defined.

The **amount** of the financial sum is defined, with a precise *accounting sense,* as the arithmetic addition of the amounts that make up the aggregate. The definition of the second component, the **deferment** of the financial sum, has to deal with the inevitable dispersion of deferment terms or degrees of liquidity in the aggregate of patrimonial elements. For this temporal component, the arithmetic sum total is not significant. This is the difficulty that, with a greater or lesser degree of awareness, prevents conventional financial analysis from developing further, and hinders greater precision in the description of complex financial operations, as will be seen below.

In aggregates of fixed interest financial assetss, *financial updating or adjustment* (the financial discount) provides a solution, based on the replacement of the *face* value of assetss by their *current value,* with the consequent zero *equideferment* in all cases. Once a *financial law* has been defined (an interest rate) to update the amounts, the arithmetic sum total of their current values is the current value of the aggregate of the assetss. This monetary amount, with zero *deferment* (absolute liquidity), has the same financial value as the financial sum total of the aggregate.

To the lack of realism that *financial updating* implies (aggregate assetss are never immediately realised), a conversion or replacement of the other component, the sum total capital that is *replaced* by the other amount, is added, with disregard for the monetary and accounting meaning that this has. In addition, financial updating requires a financial law that defines the *replacement relationship* between the two components of the financial capital. This law (an interest rate) totally conditions the monetary amount that is *updated*. It seriously infringes the nature of the amount, independent of the *financial ambience* and of the market interest rates. An *updated* financial capital that is endowed with a smaller amount and less deferment (zero) than its antecedent may have (depending on the financial law) the *same financial value*, but can never replace it without distorting its monetary and temporal meaning. Financial updating, even if it only refers to fixed income financial assetss, *artificially* homogenises assets liquidity, at the expense of *destroying its monetary representation* and its *real nature.*

The *measurement* of financial value is financial capital, formally a *complex binary vector*, not a bidimensional one (the components have different monetary and temporal units), the aggregation of which lacks the operative simplicity of scale or numerical algebra, characteristic of magnitudes whose measurement is a real number. The algebra of financial values must necessarily be complex vectorial algebra.

4. FINANCIAL REDUCTION

The methodological challenge that vectorial aggregation presents is resolved by the **financial reduction of** the aggregate.

The *deferments* of the elements composing an aggregate exhibit dispersion, the description of which corresponds, methodologically, to statistical reduction, and, in the case in hand, to *financial-statistical reduction*. This provides a definition of an **average deferment time** (**T**)[*] for the aggregate, as the deferment that, together with the arithmetic sum of its amounts ($\mathbf{C}=\Sigma C_r$), determines a financial capital (**C**,**T**) that represents the financial value of the complex aggregate,

$$\{(C_r, T_r)\};\ r=1,2,..n$$

Such a simple assets (**C**,**T**), of a single *amount* and deferment time, financially replaces the complex aggregate $\{(C_r, T_r)\}$, of multiple amounts and varying deferment times, without altering the monetary value of the aggregate,

[*] Calculation of the *average deferment* requires the *financial ambience* described by the ETTI (market interest rate structures) to be considered. The analysis carried out here takes advantage of algorithms and computer applications that enable it to be defined in any financial ambience whether stationary or dynamic.

or the financial equilibrium of the transactions in which it takes part, due to its definition being conditioned by the financial equivalence. It is defined as the **financial sum** of the complex aggregate, and it provides the operation of *aggregation* to the binary vector algebra,

$$\Sigma\{(C_r, T_r)\} = (\mathbf{C},\mathbf{T}) \;/\; \{(C_r, T_r)\} \sim (\mathbf{C},\mathbf{T})$$

Returning to the business balance sheet considered above, the sum totals of the Assets ($\mathbf{C}_A$) and the Liabilities ($\mathbf{C}_P$) are the monetary components of the respective aggregates, which are completed with the respective *average deferment times*, ($\mathbf{T}_A$) y ($\mathbf{T}_P$), their second time components. Therefore, the *monetary equilibrium* of the balance sheet, as demonstrated by the sum totals of the Assets and the Liabilities,

$$\mathbf{C}_A = \mathbf{C}_P$$

is complemented by the possible *financial disequilibrium,*

$$\mathbf{T}_A \neq \mathbf{T}_P$$

an exponent of the deviation in liquidity between the Assets and the Liabilities. By means of its sign and amount, the difference, $\mathbf{T}_A - \mathbf{T}_P$ indicates the direction and the degree of liquidity existing in the company's balance sheet.

5. FINANCIAL REDUCTION OF OPERATIONS

In complex transactions many patrimonial elements with varying degrees of liquidity are exchanged. A **complex operation** can be formalised by means of an *input-output* scheme of emergent and salient sets of financial capital in the following way[*]

$$input: \{(C_r,T_r)\}; \quad r = 1,2,..n$$

$$output: \{(C'_s,T'_s)\}; \quad s = 1,2,..m$$

The financial reduction of the aggregates forming part of the operation,

$$\{(C_r,T_r)\} \sim (\mathbf{C},\mathbf{T})$$

$$\{(C'_s,T'_s)\} \sim (\mathbf{C}',\mathbf{T}')$$

makes it possible to **reduce a *complex* operation to an elementary** or simple one, without weakening its properties when faced with the financial equilibrium of the market (a consequence of the transitive nature of the relationship of equivalence). This *reduced equivalent elementary operation* is, thus,

$$input: (\mathbf{C},\mathbf{T})$$

$$output: (\mathbf{C}',\mathbf{T}')$$

The operational power that the financial reduction of operations brings to financial analysis will immediately be appreciated.

6. BASIC MAGNITUDES OF INVESTMENT: AMOUNT, IMMOBILISATION AND YIELD

An **investment operation** supposes a provision of assetss (*inputs*), with the restitution of others (*outputs*), in a financial ambience of certainty or uncertainty (which is unlikely to be stochastic). If the assetss are of a financial

[*] Their financial equivalence or non-equivalence in the money market makes it possible to differentiate them as **financing operations** (FFO) or **investment operations** (OFI).

nature, the amounts and the liquidity of the capitals participating in the investment operation will be well defined. However, if the assetss are not financial, it is also possible to describe them by means of the financial capitals that represent them.

When describing an investment operation (OFI), the **immobilised amount** is the arithmetic sum total of the *input* amounts. It thus coincides with the amount (**C**) of the financial total of the *input*. The **immobilisation period** is not a single one in complex operations, different periods of immobilisation coexisting among the monetary units taking part, depending on their starting and recovery dates (a difficulty that has not been solved by conventional financial analysis).

The **absolute yield** is the difference between the aggregate amounts of the *output* and the *input,*

$$\mathbf{R} = \mathbf{C}' - \mathbf{C}.$$

It is independent of the immobilisation period and fails to fulfil the principle of *financial homogeneity* in the terms, as the sum amounts, (**C**') and (**C**), have different liquidities, (**T')** and (**T**), respectively.

The **relative yield** (profitability) relates the *absolute yield* to the *amount* and the *immobilisation time*. In *elementary operations* (simple ones) both magnitudes, amount and time, are well defined. Therefore, relative yield is also well defined. In contrast, in *complex operations,* the indefinition of the immobilisation time makes it impossible to define profitability directly. Conventional financial analysis, faced with need to overcome the indefinition of the time, falls back on an indirect measurement of profitability, of a highly debatable nature, known as the IRR (Internal Rate of Return). The latter, apart from eliminating the immobilisation time from its calculations, does not take into account the current financial ambience (current interest rates). This is rather paradoxical for a unit of measurement that seeks to claim a role in finance.

Leaving aside deeper and more critical analysis of the IRR to a later moment, we should now consider, without taking it into account, the correct way to measure relative yield (profitability) in complex investment operations. This is made possible by the application of financial reduction to such operations, which supplies a simple solution to the indefinition of the immobilisation period in complex operations.

7. THE AVERAGE FINANCIAL TIME (AFT)

Financial reduction makes it possible to define an **average financial time** (AFT) for the immobilisation of the monetary units invested in a complex investment operation, with the capacity to replace the effectively immobilised times without altering their financial properties in transactions.

In the simple or elementary operation,

input: (C,T)

output: (C',T')

the immobilisation time of the amount (C) is well defined by the difference between the deferments of the *output* and the *input*,

$$t = T' - T$$

This time is *effective* and *common* to all the (C) monetary units invested in (T), and which will be repaid in (T'). Thus, similarly in the reduced simple operation, equivalent to the complex one, the monetary units that make up the immobilised amount ($\mathbf{C}=\Sigma C_r$), the arithmetic sum of the amounts in the *input* of the operation, have a well-defined immobilisation time through the difference of the *average deferments* of the complex operation, of the *output* (**T'**) and the *input* (**T**). That is to say,

$$\mathbf{t} = \mathbf{T'} - \mathbf{T}$$

or, in order to show its dependence on the current interest rate (ρ^o),

$$\mathbf{t}(\rho^o) = \mathbf{T'}(\rho^o) - \mathbf{T}(\rho^o)$$

since the *average deferments* of the *input* and *output* sets are functions of ρ^o.

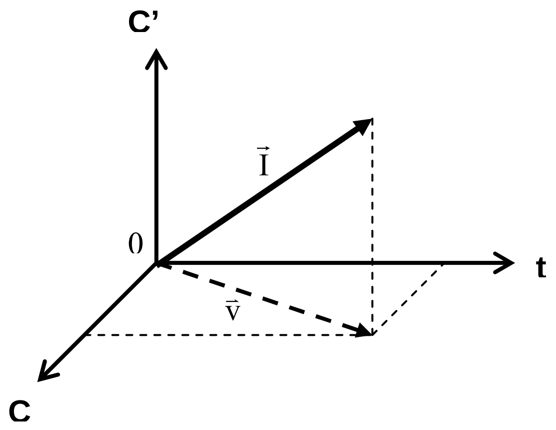
The **average financial time** behaves statistically as an *average time* of the set of various times that take part in the immobilisation. Because of its financial definition, it is financially the same (according to the law governing the calculation) to consider the diversity of the real immobilisation times existing for each of the monetary units taking part as to attribute the AFT to all of them. The absence of a single immobilisatiion time in complex operations, which makes it impossible to define profitability directly, can be overcome by introducing the AFT. In other words, the difficulty is avoided by replacing the complex operation by its reduced elementary equivalent. This makes it possible to study profitability in complex operations correctly and there is no need to resort to the deficient IRR as at present.

8. VECTORIAL REPRESENTATION OF THE FINANCIAL OPERATION (OF)

In any financial operation there are three basic magnitudes that define it in financial terms. They are the ones that define the equivalent elementary operation: the aggregate amount of the *input* (**C**), the AFT (**t**) and the aggregate amount of the *output* (**C**'). From these, the remaining derived magnitudes that describe the operation can be deduced. They make it possible to formalise the financial operation through a financial vector, in a three-dimensional space,

$$\vec{I} \equiv (\mathbf{C}, \mathbf{t}, \mathbf{C'})$$

The first two components of the vector, (**C**) and (**t**), define the **financial immobilisation** in amount and in AFT, these being the components of the vector projection of $\vec{I}$ on the plan, $\vec{v} \equiv (\mathbf{C}, \mathbf{t})$. The remaining component (**C**'), together with the previous components, makes it possible to determine the **yield** in its different meanings.



9. "PRODUCTIVITY" PROFITABILITY

In its most common meaning, the profitability of invested capital is identified with the economic and technical sense of *productivity.* In practice, the immobilised invested capital is the *productive factor* in the investment, the *product* of which is the yield. The *productivity* of a factor is the relationship between the physical units *of the product* obtained and the physical units of the *factor* consumed. In financial investment, the product-yield is expressed in monetary units and the immobilisation factor is a monetary-temporal magnitude.

The relationship between them

$$r = \frac{\mathbf{R}}{\mathbf{C.t}}$$

defines the *financial productivity* of the investment, as monetary yield per monetary unit/year of investment. The financial productivity is thus considered as the **profitability** of the investment.

Within this concept of *productivity-profitability*, it is also possible to propose a further, more precise definition, if we consider the gradual reinvestment of the yield obtained, and not withdrawn, within the same investment operation. This always arises in the simple or elementary operation, in which the yields gradually increase the initial immobilisation during the deposit time, since they are not withdrawn until it comes to an end. Taking into account the reinvestment of the yield (increasing variable immobilisation), **strict profitability** in the elementary operation is*

$$\rho = \frac{1}{t}\ln\frac{C'}{C}$$

which can also be extended to a complex operation as strict profitability, provided it is calculated in the equivalent elementary reduced operation.

Neither the *absolute yield* nor the *productivity-profitability*, whether strict or not, deduct the *financing cost* of the immobilised capital from the result. At the most, they deduct the opportunity cost if the investment is financed by the investor himself. If (**I)** is the current interest rate, corresponding to the amount

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* *Strict profitability* is obtained from the equation $C' = C.e^{\rho.t}$.

and the immobilisation time, this is the *financial cost of the investment.* If we consider the current *interest rates* to be (i^o) nominal and (ρ^o) continuous, the expressions

$$\hat{\mathbf{R}} = \mathbf{R} - \mathbf{I}$$
$$\hat{r} = r - i^o$$
$$\hat{\rho} = \rho - \rho^o$$

define the **net yield** and the two **net profitabilities** as derived magnitudes, **non strict and strict,** respectively

10. "ECONOMIC" PROFITABILITY

Economic Theory defines the *profitability* of the productive factor, according to a more economic-financial sense, as the relationship between the *monetary value* of the product obtained and the *monetary cost* of the factor. Owing to the monetary nature of the financial investment, the *monetary value* of the product coincides with the yield (**R**). The *monetary cost* of the investment is (**I**), the *financial cost* of immobilisation. Thus,

$$\delta = \frac{\mathbf{R}}{\mathbf{I}} \quad and \quad \hat{\delta} = \frac{\hat{\mathbf{R}}}{\mathbf{I}}$$

which are respectively the **gross profitability** and **net profitability** magnitudes derived in this economic sense. Between them, there exists the relationship

$$\delta = \hat{\delta} + 1$$

which justifies that they are indifferent in order to select the optimum investment when faced with investment alternatives according to economic-financial criteria.

11. THE FINANCIAL SIGNIFICANCE OF THE IRR

Conventional financial analysis does not know the *financial reduction* of the operation or the possible definition of the AFT to describe the immobilisation time in complex operations. The lack of existence of a single time in complex operations, together with considerable *conceptual confusion* between yield and interest, has given rise to the IRR as a measurement of profitability in complex operations, in the sense described as *productivity-profitability.*

The IRR is an implicit interest rate. As a rate, it has a *strict* nature, since its definition includes the accumulation of interest generated by the operation itself (compound interest). At this juncture, it is important to demonstrate the serious conceptual error that the IRR introduces by confusing interest with yield.

- **Interest** is a money market *price* rewarding the liquidity transferred. Its definition is *external* and *exogenous* to the investment operation. As a price, interest *defines a market balance*, in this case of the money market. Like any price, interest always has a *positive* value.
- **Yield**, in contrast, is an *internal* magnitude of the investment operation. It is a *result* of the operation. It is derived from the *disequilibrium* between the financial values of the *input* and the *ouput.* Its nature is *endogenous* and *marginal*, like that of any business result. The value of the yield accepts the two signs, *positive or negative*, indicating profits or a loss.

In spite of these fundamental differences, financial practice confuses *interest* with *yield*, totally ignoring their nature and economic functionality, which are quite dissimilar.

Gross yield includes both *net yield* and *interest*. If the investor invests his own capital**,** he receives the two together, as a gross yield of a mixed nature. If the investor finances the operation externally, he must deduct the interest paid for the financing process in order to determine the yield, the part remaining as *net yield*.

The interest (**I**), added to the aggregate amount of the *input* (**C**), (**C+I**), makes it possible to compare it *in homogeneous financial terms* with the aggregate amount of the *output,* (**C**'), since both have the same deferment, (**T**'). The difference between them is the *net yield*. In contrast, the *gross yield* is a difference between sums of money that are not financially homogeneous, (**C**') and (**C**), since their deferment times are different, (**T**') and (**T**) respectively**.**

The IRR **is not a rate of yield**; it is an implicit interest rate. When a financial financing operation (FFO) is carried out in the money market (financially equilibrated), and the interest rate is not made explicit, the implicit rate is deduced from the *financial equilibrium* between the *input* and the *output*. This is what the IRR calculates, as a solution of a polynomial exponential equation*. This equation, because of its analytical structure, may have several solutions, just one or none at all. Positive solutions may even coexist alongside negative ones. Nevertheless, profitability can only be measured in one way, with a single solution. The explanation for this paradox can be found in the confusion existing between yield and interest.

While there is but one measurement of profitability, several interest rates may satisfy the *financial equilibrium* in a complex operation. It may also occur that it is impossible to achieve equilibrium between the *input* and the *output* with any interest rate. In this case, the operation is impossible in an equilibrated market. The peculiar financial structure of a complex operation makes all these scenarios feasible*.

Financial reduction, apart from defining the AFT, and facilitating a correct interpretation of profitability in complex operations, also introduces a methodology that makes it possible to define an **algorithm*** for calculating the IRR, which recognises the parameters that anticipate the sign and the number of solutions (contributions that clearly go far beyond conventional analysis). Research enables one to state that the *maximum number of solutions* for IRR is **three**, one of which is of the opposite sign to the other two. Thus, if there can only be two positive solutions, the financial equilibrium of the operation can only be achieved by means of two interest rates at the most.

The interpretation of the IRR as a rate of yield can lead to **aberrant results.** In addition to the abovementioned absurd multiplicity of solutions, and the contradictory meaning of the two signs in them, the IRR updates the capital participating in the operation with the *internal interest rate* that it defines itself. Thus, a negative IRR, which describes an investment project as ruinous, is based on updating the participating capital with a negative interest rate. This goes against elementary financial logic. However, if the investment project is

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* The equation that conditions and determines the IRR ρ is $\sum_r C_r.e^{\rho.T_r} = \sum_s C'_s.e^{\rho.T'_s}$

* In the work entitled "Fundamentos de la Matemática Financiera", *op.cit.*, pp. 59 and 60, three numerical cases of financial operations in such circumstances are explained.

* The description of the algorithm can be found in "Matemática de la Financiación", *op.cit.*, p. 211 *et seq.*

analysed by readjusting the capital with the current market interest rate (which is always positive), the investment project can become exceptionally profitable. It is also possible that there may be no solution for the IRR in an investment project that, as such, has a clearly defined profitability*.

12. REASONS THAT JUSTIFY THE PRESENT USE OF THE IRR

In spite of such criticism (which in no way affects the significance of IRR as an *implicit interest rate*), IRR clearly continue to be used to analyze investment projects in financial terms. This is for the following reasons:

1) Ignorance of other alternative magnitudes that are capable of measuring profitability in complex operations (the *financial rates of yield* that will be defined below).
2) In *elementary* operations the IRR always exists, and it has a single solution. It coincides with the *gross financial rate of yield*, it being possible to deduce the *net rate of yield* as a result of the difference with the current market interest rate (such properties in the *elementary operation* are incorrectly and unconsciously extended to *complex operations*).
3) In *quasi-elementary* operations (a single *input* and multiple *output*: e.g. a bond acquired for cash) the IRR exists, and there is also but a single solution. However, no longer does it coincide with the *gross rate of yield* and it involves an erroneous calculation of the profitability of the operation (it is incorrect to deduce from it the *net rate of yield* as the difference with the current market interest rate).

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* In “Matemática de la Financiación”, *op.cit*., p. 217 *et seq*. numerical cases are presented.

4) In the remaining *complex* operations (with multiple *inputs* and *outputs*), the solutions can be multiple in number or non-existent, but only in exceptional circumstances. In the multiple case, one of the two possible positive solutions can be rejected as an interest rate because of its abnormal value. On the other hand, if the IRR is calculated in conventional terms, only one of the solutions (the smallest one) is detected, while all the remaining possibilities are ignored. Be that as it may, none of these corresponds to the *gross rate of yield* (neither is it acceptable to obtain the *net yield* as a result of the difference with the current market interest rate).
5) The IRR fails to take into account the study of the *financial ambience* (market interest rates). Although this is incorrect in financial analysis, it simplifies the process and makes it easier.
6) Finally, if the IRR is close to the current market interest rate (which supposes a minimum net yield in the operation, close to equilibrium), the error that the IRR introduces is much reduced.

13. THE GROSS RATE OF RETURN (GRR)

The strict gross rate of return (GRR), which was repeatedly referred to in the previous section, corresponds to the expression

$$\rho = \frac{1}{\mathbf{t}(\rho^{o})} \ln \frac{\mathbf{C'}}{\mathbf{C}}$$

Its value is a function of the current market interest rate (ρº) (continuous interest rate). It differes from the IRR, whose expression is obtained from the reduced operation, as a solution of the equation [*]

$$\rho = \frac{1}{\mathbf{t}(\rho)} \ln \frac{\mathbf{C'}}{\mathbf{C}}$$

Comparison of the two preceding equations allows us to make the following observations:

a) In the latter, it can be seen that the current market interest rate (ρ°) is replaced by the IRR itself (ρ), thus confirming its use in the updating of the participating capital.

b) The former only has one value, whereas the IRR is the result of an equation with multiple solutions, a single one, or even none at all.

c) In *elementary operations*, the AFT, the denominator in both expressions, is a constant that does not depend on the interest rate, which is the reason why both rates, GRR and IRR coincide. In contrast, in *complex operations*, the GRR is a function of the interest rate (ρ°), whereas the IRR is a function of itself (ρ). Therefore, the two rates no longer coincide (ρ°≠ρ), and only approach each other when (ρ) is close to (ρ°).

d) The extraordinary sensitivity of rates of yield to the volatility of interest rates means that interpreting them by means of the IRR, in order to .analyze the possibilities of an investment alternative, may decisively modify the choice of the best option[*].

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[*] ρ is obtained from the equation $C' = C.e^{\rho.t(\rho)}$.

[*] Several numerical examples in which this alteration occurs are developed in "Matemática de la Inversión", *op.cit*., p. 84 *et seq*.

e) The *net rate of yield* is obtained by subtracting the current market interest rate from the gross rate of return GRR. The IRR does not include the current market interest rate in its calculation, and has the nature of a rate of gross yield. The *rate of net yield* can only be deduced from the IRR in elementary operations, in which the IRR coincides with the GRR, but not in complex operations.

f) If the IRR is very distant from the current market interest rate, the error introduced in analysis is quite considerable, and the results are unreliable for investment project selection.

Whatever the case, the **most significant magnitude** for investment decisions is not a *productivity-profitability* rate (IRR or GRR), but rather the *"economic"* yield rate of the investment (δ), both in its *gross* and in its *net* form.

14. CHOOSING THE OPTIMUM INVESTMENT IN AN ALTERNATIVE

An **investment alternative** is, in formal terms, a series of ν options or investment vectors,

$$\{\vec{I}_j\}; \text{ when } \vec{I}_j(\mathbf{C'}_j,\mathbf{t}_j,\mathbf{C}_j); \quad j = 1,2..\nu$$

The *optimum investment option* can be chosen by any economic-financial criteria.

Two financial criteria are in simultaneous use when ordering the alternative, the *absolute yield* and *relative yield*. In conventional selection processes, such criteria correspond to the *capital value* and to the IRR respectively.

In the case of investment possibilities of *flexible immobilisation*, which make it possible to exhaust all the financing available, as regards both the amount and the investment time, *the two criteria always coincide*. It is not the case in investment alternatives with *rigid immobilisation*, in which each one determines the exact amount to be invested, the time of immobilisation, or both of them.

So as to order alternatives according to **absolute yield criteria**, the most representative magnitude is the *current value of the net yield* ($\hat{\mathbf{R}}_0{}^j$), updated by the current market interest rate. This coincides with the *capital value of the investment* and it is justified on the grounds of the *equal deferment*, which makes it possible to compare and order all the options. The unupdated net return ($\hat{\mathbf{R}}^j$) has, for each option, the liquidity of the average deferment of its *output* ($\mathbf{T}'_j$), with the result that it cannot be compared with the different investment options in the alternative.

So as to order alternatives according to **relative yield criteria**, the most representative magnitude is *net economic profitability* ($\hat{\delta}$), rather than the IRR, for the reasons expounded above. This magnitude is *adimensional* and thus does not respond to changes in either monetary or temporal units. It therefore allows different possible options in the alternative to be compared immediately.

It is now easy to see the indifference between the two selection criteria in the case of **flexible immobilisation** options. Since all the options have the same immobilisation, as regards amount and time (total investment of the capital), all the options bear the same financial cost, the interest ($\mathbf{I}$). Moreover, it is also the updated value of the financial cost ($\mathbf{I}_0$). Thus, the denominators in the expression of ($\hat{\delta}^j$) coincide for all the options,

$$\hat{\delta}^j = \frac{\hat{\mathbf{R}}^j}{\mathbf{I}} = \frac{\hat{\mathbf{R}}_0{}^j}{\mathbf{I}_0}$$

it being indifferent which of the two, whether the absolute return ($\hat{R}_0{}^j$) or the relative return ($\hat{\delta}^j$), is the criterion chosen for the purpose of selecting.

In contrast, in the case of **rigid immobilisation** options, the financial costs of the options (I^j) are different, the two criteria not being indifferent. This clarifies a well-known controversy concerning the identification of conventional criteria, that of capital value (*absolute return*) and that of the IRR *(relative return*).

In the case of investment possibilities with rigid options, preference for one of the criteria is *subjective*. High profitability does not justify this preference if the immobilisation is very limited in comparison with another option, with lower profitability, but offering a higher absolute yield because of its far higher immobilisation time. The boundary between the two criteria, relative and absolute, is subjective. Choosing one of them does not correspond to objective analysis.

The investor can be further enlightened, by progressing in the description of the alternatives. On a scale of preferences, 0 - 1, a parameter λ can be defined so that 0 corresponds to the *objective preference* for relative yield, and 1 an objective preference for absolute returns. The remaining values, between 0 and 1, will correspond to *degrees of subjective preference* situated between the two criteria. The *preference index*

$$\gamma = \hat{\mathbf{R}}_0{}^{\lambda} . \hat{\delta}^{1-\lambda}$$

that combines the two criteria makes it possible to determine a **critical value** λ_{rs} for each pair of options in the alternative, $(\vec{I}_r, \vec{I}_s)$, which indicates the boundary of the change in criteria, and the ordering of the pair. Such critical values, which are *objective*, make it possible to formulate an *objective ordering* in the alternative, by dividing the scale 0 - 1 into subintervals defined by the critical values of the pairs existing in the alternative. Such an ordering process makes simultaneous use of both absolute and relative criteria*.

This description of the investment choice brings to a close the *objective information* that can be deduced from the financial character of the options. All this is independent of the criteria finally chosen by the investor, whose decision in this case is always *subjective*. Once the latter has been taken, any investor will know the interval in which their unknown parameter λ is placed, which will inform them about the relative level of their decision in front of the two financial criteria.

15. A COMPUTER APPLICATION DESCRIBING THE OPERATIONS

The financial analysis presented above makes it possible to describe the financial investment operation by means of a computer application. This is prorammed on the basis of the financial data of the *input* and the *output* and it shows the characteristic properties of the operation.

This **application** quantifies the basic magnitudes according to the current market interest rate (*immobilisation, average financial time, absolute and relative yield, gross and net yield, capital value, IRR, gross and net*

* The complete ordering of the investment possibilities, taking into account both criteria and the parameter λ, which defines the relative position between the two on the scale 0 - 1, as well as the determination of the critical values of each pair of options in which an alteration in their ordering takes place, have been programmed in a computer application which only needs the financial data that define the options to be introduced, together with the market interest rate structure (ETTI). Numerical examples illustrating this application can be found in "Matemática de la Inversión", *op.cit*. p. 91 *et seq*.

profitability). It provides information about the *critical parameters*, about *the immunisation rates* and the *degeneration intervals.*

The operation is represented graphically by means of the *characteristic financial functions* AFT and DUR and their auxiliaries HYP and DEV.*

We do not intend to develop the analysis presented here, which is explained in more detail in the publications mentioned in the notes, at greater length. In order to provide an example for the purpose of illustration, we limit ourselves to presenting the results obtained in a complex operation chosen for its wealth of financial properties.

16. STUDY OF A COMPLEX INVESTMENT FINANCIAL OPERATION (OFI)

An investment project is subsidised at the point of origin with 100 m.u. The financial contributions and withdrawals detailed on the following time scale are made during its period of development. Finally, closure of the project requires expenses to the sum of 50 m.u. The interest rate for financing them is 4.5%.

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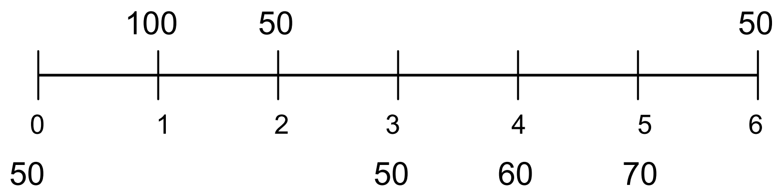
* These are the characteristic auxiliary functions:

AFT. $t(\rho) = \frac{1}{\rho}\left(\ln\frac{\mathbf{C'}}{\mathbf{C}} - \ln\frac{\sum C'_s .e^{-\rho.T'_s}}{\sum C_r .e^{-\rho.T_r}} \right)$; DUR. $d(\rho) = \frac{1}{\rho}\left(\ln\frac{\mathbf{C'}}{\mathbf{C}} - \ln\frac{\sum C'_s .T'_s .e^{-\rho.T'_s}}{\sum C_r .T_r .e^{-\rho.T_r}} \right)$

HIP. $t(\rho) = \frac{1}{\rho}\ln\frac{\mathbf{C'}}{\mathbf{C}}$; DES. $\Delta(\rho) = t(\rho) - d(\rho)$

The function DUR supposes an extension to the *duratio* (Macauley) of the OFI only as regards its elasticity (Hicks). Another interpretation, as an average financial time, does not fulfil the financial equivalence.



inputs: (100;1), (50;2)

outputs: (50;0), (50;3), (60;4), (70;5)

Basic parameters: **C**=200; **C**'=230; **t**=0.7317

$$\vec{I} \equiv (200;0.7317;230)$$

Results:

- Gross yield, R = 30
- Financial costs, I = 6.5454
- Net yield, $\hat{R}$ = 23.4536
- Capital value, $\hat{R}_0$ = 20.4250
- Gross rate of yield, GRR = 21.0467%[*]
- Net rate of yield, GRRN = 15.8242%
- Gross rate of profitability, ∂ = 4.5827
- Net rate of profitability, $\hat{\partial}$ = 3.5827
- Results of the IRR:

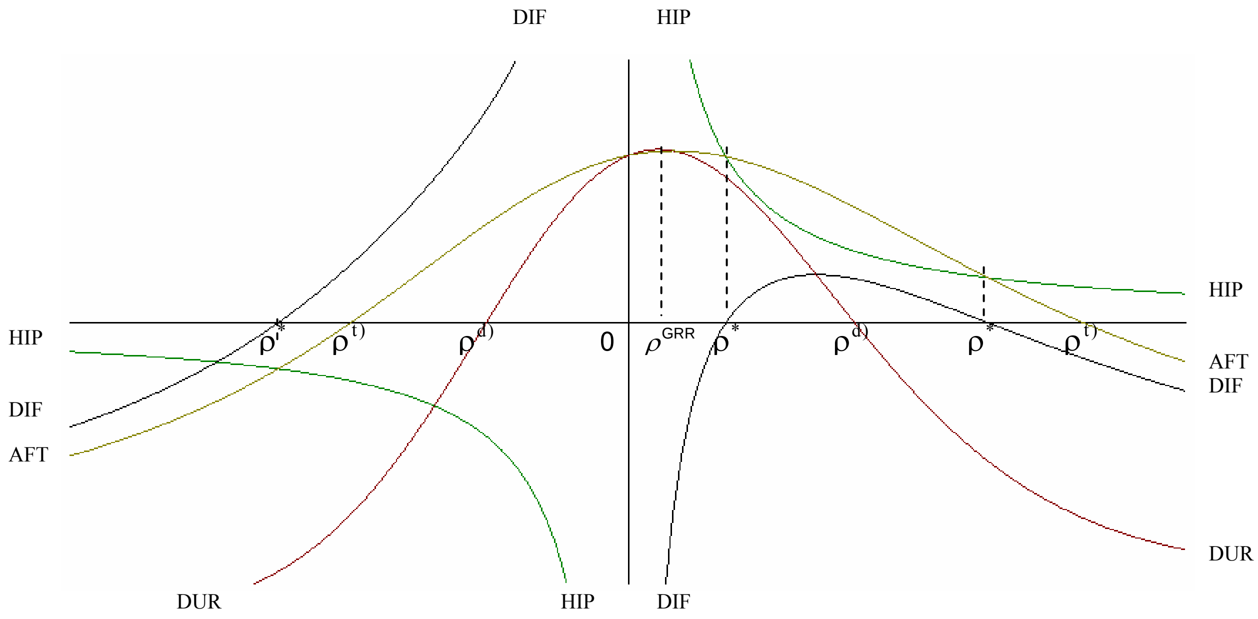
  1) –50.6471%

  2) 21.6916%

  3) 106.2396%

[*] The GRR benefits from *financial immunity* for an interest rate for financing purposes of 9.5%.

4) Graphic representation of the financial investment operation (OFI),



ρ^{*} ≡ IRR

$\rho^{t)}$ ≡ *degeneration rates*

$\rho^{d)}$ ≡ *immunity rates (profitability)*

ρ^{GRR} ≡ *immunity rates (productivity)*

17. CONCLUSIONS

1) The *monetary* valuation of a patrimonial amount (asset or liability) does not provide an adequate explanation of its *economic* value, since it lacks the evaluation of the other *financial* component, its *liquidity* or *demandability*. The *financial* valuation combines both the monetary *amount* and *deferment time*, which can be formalized in a binary complex called the *financial capital* (C,T).

   Except in individual financial applications, the *present value* of the monetary amount (C'_0) should not be substituted by the financial capital, because it requires previous submission to a certain interest rate (financial equivalence), admitting a substitutive *amount/deferment time* relation which integrates the latter component inside the former, thus the distorting the explanatory value of both components of the liquidity or accountability of the asset.

2) The *aggregation* of patrimonial amounts requires the definition of an internal operation for the set of financial capitals. This definition is immediate and provides a sufficient account of the first component, amount (**C**), as the addition of the amounts of the aggregated capitals. This is not the case for the second component*, average deferment time* (**T**), which requires the definition of an *average deferral* to provide a sufficient explanation. This is defined and determined as the deferral that maintains and respects the financial equivalence of the market, and where the aggregation of financial capitals $\{(C_r,T_r)\}$ can be replaced with just one (**C**,**T**), which is their financial sum total, such that (**C**,**T**) $\sim \{(C_r,T_r)\}$.

3) The operation described as the *financial aggregation* of a set of capitals makes it easy to apply the financial analysis and the description of simple operations to *complex operations*, where more than one financial capital participates in the *input* or *output*. This is done through the *financial reduction* of the complex operation to a simple or elementary operation, without losing any of its financial properties in the market (financial ambient) as defined by financial equivalence. Through financial reduction, complex operations can be described,

input, $\{(C_r,T_r)\}$

output, $\{(C'_s,T'_s)\}$

through its reduced elementary operation,

input, (**C**,**T**)

output, (**C**',**T**')

Its descriptive power makes it possible to apply to complex operations all the primary and derivative financial magnitudes which describe simple operations. Among these, it allows for the determination of an *average financial time* for financial immobilization (AFT), which is the well-defined immobilization term in the equivalent simple operation, t=**T**'-**T**. In complex operations, however, there is no single immobilization term.

This extension of financial analysis avoids the use of the controversial *internal return rate* (IRR), whose paradoxes and contradictions are described in the study, and also provides a powerful calculation of its possible solutions based on the magnitude introduced, the AFT. This is important because the IRR has been mistakenly defined as an implicit return rate, when it is in fact an *implicit interest rate.* The error arises from the confusion between return and interest.

4) There are basically three magnitudes which financially describe an investment operation (IFO): the *amount*, the *immobilization* and the *yield. Risk* is not an independent magnitude. It can affect the three basic magnitudes, and its intervention is related to the description of a possible uncertainty in them. These magnitudes are formally described by the *investment vector*, $\vec{I} \equiv$ (**C**',**t**,**C**).

Yield is an endogenous differential derivative magnitude, arising from the imbalance of the IFO in the face of the financial equivalence of the money market (financial ambient), which shows its deviation from the exogenous magnitude, interest rate or price of money. The *gross yield,* **R** = **C'**-**C**, integrates the *interest* and *net yield*, **R**= **I** + $\hat{\mathbf{R}}$.

Immobilization, a binary magnitude, is described by the (**C**,**t**) complex, which is the $\vec{I}$ projection on the plane.

Profitability relates the (gross or net) yield with the immobilization (the productivity of this factor) or, more specifically, with the cost of the *productive* factor, equivalent to the absolute interest (profitability in the economic sense). When the reinvestment of non-withdrawn returns is considered, derivative magnitudes of *strict profitability* δ arise (gross or net).

The *present value* of the investment is the present value of net return, $\hat{R}_0$.

5) After deriving the most relevant basic and derivative magnitudes for the financial analysis of the IFO, we approach their application to the *selection of the optimal investment* when faced with an investing alternative with several options, by *ordering* the options with a financial criterion. Two criteria are possible, not always opposed but independent: absolute return (present value, $\hat{R}_0$) and relative return (profitability, δ). Their selection is subjective, and both can be combined in an index, $\gamma(\lambda) = \hat{\mathbf{R}}_0{}^{\lambda}.\hat{\delta}^{1-\lambda}$, whose λ parameter shows the investor's subjective choice for these criteria.

6) The study ends by mentioning computer applications designed for the financial analysis proposed, and provides a partial presentation of their results in a numerically defined operation.