Spin-dependent Effects on the Hyperfine Structure in Heavy Hybrid Mesons

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Abstract: We study the hyperfine structure of a heavy hybrid meson, interesting because of the non-trivial contribution of its gluon field on the heavy quak-antiquark pair. We will be able to work on a non-relativistic frame, thanks to the meson being much heavier than the energy of the gluonic field, and so use the Schrödinger equation to define the system. With the results from the hyperfine structure — which come attenuated by a $\frac{1}{m_Q}$ factor — we will be able to draw the relationships between the different possible masses of the system.

I. INTRODUCTION

A meson is a quark and an antiquark $(q - \overline{q})$ system bound by a gluon field. It is described by the orbital angular momentum L, the coupled spin S and a time-independent potential in line with the Strong Force symmetries — C, T, P and total angular momentum. The states of such a system can be properly outlined with J^{PC}, where J, P and C are, respectively, the total angular momentum, the parity and the charge conjugation. This system can, however, be complicated if one takes into account the contribution of the gluon field as it has been theorised from the early days of Quantum Chromodynamics (see [1]). To take this into account we need to also define the state of the gluon field as J_g^{PC}. Furthermore, given that the intrinsic parity of gluons is -1 and looking at the first non-trivial gluonic contribution we can get the following compound states.

J_g^{PC}	L	S	J^{PC}
1^{+-}	0	0	1^{--}
1^{+-}	1	1	$(0,1,2,3)^{+-}$

TABLE I: Possible states of the hybrid system when the gluon field has an angular momentum equal to one.

To properly define the hybrid system, then, new variables will be needed, as we'll be working with both L and J_g. Therefore, it is interesting to define $J = L + J_g$ and $\mathcal{J} = J + S$, where S is the spin of the quark-antiquark system.

Normally the description of these states would involve using QCD and QFT, but as the mass of the heavy quarks here studied m_Q is much bigger than the gluonic energy contribution, we can safely work in a non-relativistic frame, where the gluon field will react immediately to the motion of the meson. In short, we will not need a propagation speed or a reaction time. Moreover, as we will use the Born-Oppenheimer approximation, we will be able to use the Schrödinger equation to describe the movement of $q - \overline{q}$. This will be achieved by associating a stationary potential $V(r)$ to each stationary state of the gluon field. Finally, this work will only look at the lowest non-trivial energy contributions from the gluon field. Namely $J_g = 1$. The demonstration behind why such a value represents the lowest energy can be seen in [2].

The first part of the work will consist on developing a never-studied-before potential, so that its hyperfine matrix can be found, i.e.: the matrix of values that tells us the structure of the hyperfine energy levels. After that, we will use perturbation theory on this matrix to find the relations and differences between the masses of the different meson states — basically those in the same spin multiplet.

II. HYPERFINE SPLITTING

The main body of this project is to study the hyperfine structures that arise in the system described in the introduction. These structures have of course been studied before, and so we know that they come attenuated by a factor of $\frac{1}{m_Q}$ as shown in [3].

To properly define the system that we are working with, we will need a mathematical object able to represent the spin of the quarks and their relative momentum and the spin of the gluon field all at the same time. This can be achieved by using the tensorial spherical harmonics (defined in Appendix V A). Moreover we will need to define the wave function, which will be:

$$H^j = \frac{1}{\sqrt{2}}\left(H_0^j \mathbb{I} + \sigma^i H_1^{ji}\right)$$
$$H_1^{ij} = \sum_{L,J,\mathcal{J},\mathcal{M}} P_{1\mathcal{J}\mathcal{M}}^{LJ}(r)\, Y_{\mathcal{J}\mathcal{M}}^{ijLJ}(\hat{r}) \tag{1}$$

Where the only new index is $\mathcal{M}$, which refers to the third component of the total angular momentum. From this expression, H_0^i corresponds to zero-spin $Q\bar{Q}$ and H_1^{ij} corresponds to spin-one $Q\bar{Q}$.

Given the discrete symmetries of P, C and T, the only possible energy contributions to the Lagrangian density

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(again, attenuated by $\frac{1}{m_Q}$) are (see [4]):

$$
\begin{gathered}
i\varepsilon^{ijk}V^S(r)\,tr\left(H^{i\dagger}\left[\sigma^k,H^j\right]\right)\\
ir^i\varepsilon^{ijk}\tilde{V}^S(r)\,tr\left(\vec{r}\vec{H}^\dagger\left[\sigma^k,H^j\right]\right)+H.c.
\end{gathered}
\tag{2}
$$

It should be noted that, at leading order, the potential is independent from m_Q or the spin of the quarks, which is the reason why we can write the Lagrangian in terms of expression (1). Moreover, $H.c.$ represents the Hermitic conjugate from the second expression.

From the two expressions in (2), the first one has already been discussed in [3], so our job will be to work on the second one.

By developing the potential using (1), and given that by definition $rH^\dagger = r^p\frac{1}{\sqrt{2}}\left(H_0^{p\dagger}\mathbb{I}+H_1^{pq\dagger}\sigma^q\right)$, we get:

$$
\begin{aligned}
&ir^i\varepsilon^{ijk}\tilde{V}^S(r)\,tr\left(\vec{r}\vec{H}^\dagger\left[\sigma^k,H^j\right]\right)=\\
&=2\tilde{V}^S(r)\,r^ir^p\left(H_1^{pi\dagger}H_1^{jj}-H_1^{pj\dagger}H_1^{ji}\right)
\end{aligned}
\tag{3}
$$

Where the identity (4) for the Levi-Civita symbols has also been used in an intermidiate step.

$$
\varepsilon^{ijk}\varepsilon^{klr}=-\left(\delta_r^i\delta_l^j-\delta_r^j\delta_l^i\right)
\tag{4}
$$

The next step is to contract r^ir^p such as:

$$
r^ir^p=r^ir^p-\frac{\delta^{ip}}{3}+\frac{\delta^{ip}}{3}=(\mathrm{T}_2)^{ip}+\frac{\delta^{ip}}{3}
\tag{5}
$$

From equation (5) it is apparent that our potential has two separable contributions. The term $\frac{\delta^{ip}}{3}$ will be identical (with a factor of $\frac{1}{3}$) to what has already been found in previously cited works, so we will disregard it. On the other hand, the term $(\mathrm{T}_2)^{ip}$ has not yet been looked into, and so we will focus on it.

The part with which we will work is, then, outlined in (6)

$$
2\tilde{V}^S(\mathrm{T}_2)^{ip}\left(H_1^{pi\dagger}H_1^{jj}-H_1^{pj\dagger}H_1^{ji}\right)
\tag{6}
$$

These two elements turn into the following expressions by reworking the wave functions H_1^{ij} into the eigenbasis $\mathcal{J}$, $\mathcal{M}$, J^2, L^2:

$$
\begin{aligned}
&(\mathrm{T}_2)^{ip}H_1^{pi\dagger}H_1^{jj}=\\
&=\sum_{\substack{L,J,\mathcal{J},\mathcal{M}\\L',J',\mathcal{J}',\mathcal{M}'\\\mu,\nu,\mu',\nu'}}C\left(J,1,\mathcal{J};\mathcal{M}-\nu,\nu\right)C\left(L,1,J;\mathcal{M}-\mu-\nu,\mu\right)\cdot\\
&\cdot C\left(J',1,\mathcal{J}';\mathcal{M}'-\nu',\nu'\right)C\left(L',1,J';\mathcal{M}'-\mu'-\nu',\mu'\right)\cdot\\
&\cdot \mathrm{P}^{*L'J'}_{1\mathcal{J}'\mathcal{M}'}\mathrm{P}^{LJ}_{1\mathcal{J}\mathcal{M}}\mathrm{Y}^{*\mathcal{M}'-\mu'-\nu'}_{L'}\mathrm{Y}^{\mathcal{M}-\mu-\nu}_{L}(\mathrm{T}_2)^{ip}\chi^{*i}_{\nu'}\chi^{j}_{\nu}\chi^{*p}_{\mu'}\chi^{j}_{\mu}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&(\mathrm{T}_2)^{ip}H_1^{pj\dagger}H_1^{ji}=\\
&=\sum_{\substack{L,J,\mathcal{J},\mathcal{M}\\L',J',\mathcal{J}',\mathcal{M}'\\\mu,\nu,\mu',\nu'}}C\left(J,1,\mathcal{J};\mathcal{M}-\nu,\nu\right)C\left(L,1,J;\mathcal{M}-\mu-\nu,\mu\right)\cdot\\
&\cdot C\left(J',1,\mathcal{J}';\mathcal{M}'-\nu',\nu'\right)C\left(L',1,J';\mathcal{M}'-\mu'-\nu',\mu'\right)\cdot\\
&\cdot \mathrm{P}^{*L'J'}_{1\mathcal{J}'\mathcal{M}'}\mathrm{P}^{LJ}_{1\mathcal{J}\mathcal{M}}\mathrm{Y}^{*\mathcal{M}'-\mu'-\nu'}_{L'}\mathrm{Y}^{\mathcal{M}-\mu-\nu}_{L}(\mathrm{T}_2)^{ip}\chi^{*j}_{\nu'}\chi^{i}_{\nu}\chi^{*p}_{\mu'}\chi^{j}_{\mu}
\end{aligned}
\tag{8}
$$

As we can see, both these expression must still be worked on before we can integrate them. So, given the properties of $(\mathrm{T}_2)^{ip}$ (outlined in Appendix V B) and the tensorial spherical harmonics we finally get:

$$
\begin{aligned}
&\int(\mathrm{T}_2)^{ip}H_1^{pi\dagger}H_1^{jj}d^3\vec{r}=\int\sum_{\substack{L,J,\mathcal{J},\mathcal{M},\mu\\L',J',\nu',\mu'}}(-1)^{-\mu+\mu'+\nu'}\sqrt{\frac{2\pi}{15}}\cdot\\
&\cdot 2^{\frac{|\mu'+\nu'|}{2}}\left(\frac{2}{3}\right)^{\delta_{-\mu'\nu'}}2^{\delta_{-\mu'0}\delta_{\nu'0}}\sqrt{\frac{5\left(2L+1\right)}{4\pi\left(2L'+1\right)}}\cdot\\
&\cdot C\left(L,2,L';\mathcal{M},-\mu'-\nu'\right)C\left(L,2,L';0,0\right)\cdot\\
&\cdot C\left(J,1,\mathcal{J};\mathcal{M}+\mu,-\mu\right)C\left(L,1,J;\mathcal{M},\mu\right)\cdot\\
&\cdot C\left(J',1,\mathcal{J};\mathcal{M}-\nu',\nu'\right)C\left(L',1,J';\mathcal{M}-\mu'-\nu',\mu'\right)\\
&\cdot \mathrm{P}^{*L'J'}_{1\mathcal{J}\mathcal{M}}\mathrm{P}^{LJ}_{1\mathcal{J}\mathcal{M}}r^2dr
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\int(\mathrm{T}_2)^{ip}H_1^{pj\dagger}H_1^{ji}d^3\vec{r}=\int\sum_{\substack{L,J,\mathcal{J},\mathcal{M}\\L',J',\nu,\nu',\mu'}}(-1)^{\mu'}\sqrt{\frac{2\pi}{15}}\cdot\\
&\cdot 2^{\frac{|-\mu'+\nu|}{2}}\left(\frac{2}{3}\right)^{\delta_{-\mu'-\nu}}2^{\delta_{-\mu'0}\delta_{-\nu0}}\sqrt{\frac{5\left(2L+1\right)}{4\pi\left(2L'+1\right)}}\cdot\\
&\cdot C\left(L,2,L';\mathcal{M}-\nu-\nu',-\mu'+\nu\right)C\left(L,2,L';0,0\right)\cdot\\
&\cdot C\left(J,1,\mathcal{J};\mathcal{M}-\nu,\nu\right)C\left(L,1,J;\mathcal{M}-\nu-\nu',\nu'\right)\cdot\\
&\cdot C\left(J',1,\mathcal{J};\mathcal{M}-\nu',\nu'\right)C\left(L',1,J';\mathcal{M}-\mu'-\nu',\mu'\right)\\
&\cdot \mathrm{P}^{*L'J'}_{1\mathcal{J}\mathcal{M}}\mathrm{P}^{LJ}_{1\mathcal{J}\mathcal{M}}r^2dr
\end{aligned}
\tag{10}
$$

The calculation of these two terms is an extremely complex one, so to get the result we had to compute them with Mathematica.

The results of these calculations are presented in Table II and III. Table II outlines the special cases of $\mathcal{J}=0$ and $\mathcal{J}=1$ (Tables IIa and IIb, respectively), as in such cases there are some states that don't exist (namely, those states that would reach negative values in any of the variables J, J', L or L'). The null symbol $\varnothing$ signifies those cells which represent nonphysical states. On the other hand, Table III shows the values for any greater-than-one arbitrary $\mathcal{J}$.

It should be noted that the terms in the aforementioned tables are all multiplied by the constant $2\tilde{V}^S$, which was left out to ease the reading.

J = -	L = -	∅	∅	∅	∅	∅	∅	∅	∅	∅
J = -	L = 0	∅	∅	∅	∅	∅	∅	∅	∅	∅
J = -	L = +	∅	∅	∅	∅	∅	∅	∅	∅	∅
J = 0	L = -	∅	∅	∅	∅	∅	∅	∅	∅	∅
J = 0	L = 0	∅	∅	∅	∅	0	0	0	0	0
J = 0	L = +	∅	∅	∅	∅	0	0	0	0	0
J = +	L = -	∅	∅	∅	∅	0	0	0	0	$-\frac{2\sqrt{2}}{3}$
J = +	L = 0	∅	∅	∅	∅	0	0	0	$-\frac{1}{3}$	0
J = +	L = +	∅	∅	∅	∅	0	0	$\frac{\sqrt{2}}{3}$	0	$-\frac{1}{3}$
		J' = -	J' = -	J' = -	J' = 0	J' = 0	J' = 0	J' = +	J' = +	J' = +
		L' = -	L' = 0	L' = +	L' = -	L' = 0	L' = +	L' = -	L' = 0	L' = +

(a) Values for $\mathcal{J} = 0$

∅	∅	∅	∅	∅	∅	∅	∅	∅
∅	0	0	0	0	0	0	0	0
∅	0	0	0	$\frac{2}{3\sqrt{3}}$	0	0	0	0
∅	0	0	0	0	$-\frac{\sqrt{2}}{3}$	0	0	0
∅	0	$-\frac{4}{3\sqrt{3}}$	0	$-\frac{1}{6}$	0	$-\frac{1}{6\sqrt{15}}$	0	$\frac{1}{\sqrt{10}}$
∅	0	0	$\frac{1}{3\sqrt{2}}$	0	$-\frac{1}{6}$	0	$\frac{1}{2\sqrt{3}}$	0
∅	0	0	0	$\frac{\sqrt{\frac{5}{3}}}{6}$	0	$\frac{1}{10}$	0	$-\frac{3\sqrt{\frac{3}{2}}}{5}$
∅	0	0	$\frac{1}{\sqrt{6}}$	0	$-\frac{1}{2\sqrt{3}}$	0	$-\frac{1}{6}$	0
∅	0	0	0	0	0	$\frac{\sqrt{6}}{5}$	0	$-\frac{4}{15}$
J' = -	J' = -	J' = -	J' = 0	J' = 0	J' = 0	J' = +	J' = +	J' = +
L' = -	L' = 0	L' = +	L' = -	L' = 0	L' = +	L' = -	L' = 0	L' = +

(b) Values for $\mathcal{J} = 1$

TABLE II: Table of special discrete $\mathcal{J}$ values

. The two left-most tag columns are only written once, to save up space.

J = -	L = -	$\frac{2-\mathcal{J}}{-3+6\mathcal{J}}$	0	$\frac{\sqrt{(\mathcal{J}-1)\mathcal{J}}}{2\mathcal{J}-1}$	0	0	0	0	0	0
J = -	L = 0	0	$\frac{1}{3\mathcal{J}}$	0	$-\frac{\sqrt{\frac{\mathcal{J}-1}{\mathcal{J}+1}}(\mathcal{J}^2-1)}{3\mathcal{J}(1+2\mathcal{J})}$	0	$\frac{\sqrt{\frac{\mathcal{J}-1}{\mathcal{J}}}(1+\mathcal{J})}{1+2\mathcal{J}}$	0	0	0
J = -	L = +	$-\frac{(\mathcal{J}-1)^{\frac{3}{2}}}{\sqrt{\mathcal{J}}(2\mathcal{J}-1)}$	0	$\frac{1-\mathcal{J}^2}{3\mathcal{J}-6\mathcal{J}^2}$	0	$\frac{(1+\mathcal{J})\sqrt{1-\frac{2}{1+2\mathcal{J}}}}{3\mathcal{J}}$	0	0	0	0
J = 0	L = -	0	$\frac{\sqrt{\mathcal{J}^2-1}}{3\mathcal{J}}$	0	$-\frac{1}{3\mathcal{J}}+\frac{1}{1+2\mathcal{J}}$	0	$-\frac{\sqrt{1+\frac{1}{\mathcal{J}}}}{1+2\mathcal{J}}$	0	0	0
J = 0	L = 0	$\frac{\mathcal{J}^2-1}{\sqrt{\mathcal{J}(\mathcal{J}-1)(2\mathcal{J}-1)(2\mathcal{J}+1)}}$	0	$-\frac{(1+\mathcal{J})^2}{3\mathcal{J}\sqrt{4\mathcal{J}^2-1}}$	0	$-\frac{1}{3\mathcal{J}(1+\mathcal{J})}$	0	$-\frac{\mathcal{J}^2}{3(1+\mathcal{J})\sqrt{3+4\mathcal{J}(2+\mathcal{J})}}$	0	$\frac{\mathcal{J}(2+\mathcal{J})}{\sqrt{(1+\mathcal{J})(2+\mathcal{J})(1+2\mathcal{J})(3+2\mathcal{J})}}$
J = 0	L = +	0	0	0	$\frac{\sqrt{\frac{\mathcal{J}}{1+\mathcal{J}}}}{1+2\mathcal{J}}$	0	$-\frac{2+\mathcal{J}}{3+9\mathcal{J}+6\mathcal{J}^2}$	0	$\frac{\sqrt{\mathcal{J}(2+\mathcal{J})}}{3+3\mathcal{J}}$	0
J = +	L = -	0	0	0	0	$\frac{1}{3}\frac{\mathcal{J}}{\mathcal{J}+1}\sqrt{\frac{2\mathcal{J}+3}{2\mathcal{J}+1}}$	0	$\frac{\mathcal{J}(2+\mathcal{J})}{9+15\mathcal{J}+6\mathcal{J}^2}$	0	$-\frac{(2+\mathcal{J})^{\frac{3}{2}}}{\sqrt{1+\mathcal{J}}(3+2\mathcal{J})}$
J = +	L = 0	0	0	0	$\frac{\mathcal{J}\sqrt{1+\frac{1}{1+\mathcal{J}}}}{1+2\mathcal{J}}$	0	$-\frac{\sqrt{\mathcal{J}}(2+\mathcal{J})^{\frac{3}{2}}}{3(1+\mathcal{J})(1+2\mathcal{J})}$	0	$-\frac{1}{3+3\mathcal{J}}$	0
J = +	L = +	0	0	0	0	0	0	$\frac{\sqrt{(1+\mathcal{J})(2+\mathcal{J})}}{3+2\mathcal{J}}$	0	$-\frac{3+\mathcal{J}}{9+6\mathcal{J}}$
		L' = -	L' = 0	L' = +	L' = -	L' = 0	L' = +	L' = -	L' = 0	L' = +
		J' = -	J' = -	J' = -	J' = 0	J' = 0	J' = 0	J' = +	J' = +	J' = +

TABLE III: Values for arbitrary $\mathcal{J} > 1$

III. MASS PREDICTIONS

From the results derived from the hyperfine splitting we can get still more information about our system. That is because in particle physics, if one were to take two mesons, they could still have different masses even if they were constituted by the same particles. This is a result of the strong force dependence on spin, as different states on the same multiplet will present different masses. It should also be noted that, as the coupling of the quark spin is on J, the expressions will use this variable instead of $\mathcal{J}$. Thanks to [5] we know that $L = J$ does not mix with $L = J \pm 1$. To find the relationships between these masses we will apply perturbation theory on our potential — so, practically speaking, on our hyperfine matrix. To do so, we would left-multiply the hyperfine matrix by a row vector of all possible P^{ij} states and right-multiply it by a column vector of all possible $P^{i'j'}$ states. Doing such an operation on (7) and (8) results in:

$$\begin{aligned} M^{0i}_{1J-i} - M_{0J} &= V^{0i}_{0i}\left(J-i\right)a + \tilde{V}^{0i}_{0i}\left(J-i\right)b \\ M^{\pm i}_{1J-i} - M_{0J} &= \left(V^{+i}_{+i} + V^{-i}_{-i}\right)\left(J-i\right)\tilde{a} + \tilde{V}^{+i}_{+i}\left(J-i\right)\tilde{b} + \\ &+ \left(\tilde{V}^{+i}_{-i}\left(J-i\right) + \tilde{V}^{-i}_{+i}\left(J-i\right)\right)\tilde{c} + \\ &+ \tilde{V}^{-i}_{-i}\left(J-i\right)\tilde{d} \end{aligned} \tag{11}$$

Where a, b, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ are the structure constants and $V^{LJ}_{L'J'}(J)$, $\tilde{V}^{LJ}_{L'J'}(J)$ are the elements from the hyperfine splitting matrix (the first being from the matrix found in [5] and [6] and the second being the one developed in this work). The expression $J - i$ refers to $J \pm 1$, J for $\mp$, 0, respectively. In our case, the expressions we get for the case $L = J$ and $L = J \pm 1$ are shown in (12) and (13), respectively.

$$\begin{aligned} M^{0-}_{1J+1} - M_{0J} &= \frac{1}{J+1}a + \frac{1}{3}\frac{1}{J}b \\ M^{00}_{1J} - M_{0J} &= -\frac{1}{J\left(J+1\right)}a - \frac{1}{3}\frac{1}{J}\frac{1}{J+1}b \\ M^{0+}_{1J-1} - M_{0J} &= -\frac{1}{J}a - \frac{1}{3}\frac{1}{J+1}b \end{aligned} \tag{12}$$

$$\begin{aligned} M^{\pm+}_{1J-1} - M_{0J} &= -\frac{1}{J}\tilde{a} - \frac{1}{3}\frac{2+J}{1+2J}\tilde{b} - \\ &- \frac{\sqrt{J+1}}{\sqrt{J}\left(2J+1\right)}\tilde{c} + \frac{1}{3}\frac{\left(J-1\right)\left(J+1\right)}{J\left(1+2J\right)}\tilde{d} \\ M^{\pm-}_{1J+1} - M_{0J} &= \frac{1}{J+1}\tilde{a} + \frac{1}{3}\frac{J\left(J+2\right)}{\left(J+1\right)\left(1+2J\right)}\tilde{b} + \\ &+ \frac{\sqrt{J}}{\sqrt{J+1}\left(2J+1\right)}\tilde{c} + \frac{1}{3}\frac{1-J}{1+2J}\tilde{d} \\ M^{\pm 0}_{1J} - M_{0J} &= -\frac{1}{J\left(J+1\right)}\tilde{a} - \frac{1}{3}\frac{2+J}{\left(J+1\right)\left(2J+1\right)}\tilde{b} - \\ &- \frac{1}{\sqrt{J\left(1+J\right)}\left(1+2J\right)}\tilde{c} + \frac{1}{3}\frac{J-1}{J\left(2J+1\right)}\tilde{d} \end{aligned} \tag{13}$$

The specific expressions from the different coefficients a, b, $\tilde{a}$, etc are shown in Appendix V C.

If these expressions are adequately manipulated then, we can get the following relations, respectively:

$$\frac{M^{0+}_{1J-1} - M_{0J}}{M^{00}_{1J} - M_{0J}} = J+1 \tag{14}$$

$$\frac{M^{0-}_{1J+1} - M_{0J}}{M^{00}_{1J} - M_{0J}} = -J \tag{15}$$

$$\begin{aligned} \frac{M^{\pm+}_{1J-1} - M_{0J}}{M^{\pm 0}_{1J} - M_{0J}} &= J+1 \\ \frac{M^{\pm-}_{1J+1} - M_{0J}}{M^{\pm 0}_{1J} - M_{0J}} &= -J \end{aligned} \tag{16}$$

It should be noted that result (14) is specially interesting as it reaffirms previous findings (in [5]). (16), on the other hand, gives us a new insight on mass relations with an analogous result.

IV. CONCLUSIONS

Firstly, we can conclude, from the results of the matrix in III, that we have two groups, composed of different parity terms. From this we can affirm that there is no mixing between such parity-differing results. Another conclusion is that, while the results obtained from [6] presented more structure, our work has resulted in more mixing between the terms. On the other hand, however, as we will discuss in a moment, such mixing does not contradict the previous findings insofar as mass relations are regarded.

Secondly, related to the mass predictions, we can conclude that this new potential under scrutiny is consistent with previous works (such as [6]) even though it's quite different and more complex, which is quite surprising. In the case of $L = J$ we have found that the results of the mass predictions are exactly the same, and in the case of $L = J \pm 1$ we have found an analogous result for the differing terms, which bodes really well for further investigations.

V. APPENDIX

A. Tensor Spherical Harmonics

The main properties of the tensorial spherical harmonics used throughout this report are as follows:

$$\begin{aligned} Y^{ijLJ}_{\mathcal{J}\mathcal{M}} &= \sum_{\nu=-1}^{1} C\left(J,1,\mathcal{J};\mathcal{M}-\nu,\nu\right)\mathcal{Y}^{Li}_{J\mathcal{M}-\nu}\chi^{j}_{\nu} \\ \mathcal{Y}^{Li}_{JM} &= \sum_{\mu=-1}^{1} C\left(L,1,J;\mathcal{M}-\mu,\mu\right)\mathrm{Y}^{M-\mu}_{L}\chi^{i}_{\mu} \end{aligned} \tag{17}$$

Where the coefficients C are the Clebsch-Gordan Coefficients $C\left(J_1,J_2,J_3;M_1,M_2\right)$.

B. Properties of $(\mathbf{T}_2)^{ip}$

$$r^i r^j = \left(r^i r^j - \frac{\delta^{ij}}{3}\right) + \frac{\delta^{ij}}{3} = \left(T_2\right)^{ij} + \frac{\delta^{ij}}{3} \tag{18}$$

$$\begin{aligned} \chi^i_\nu * \left(T_2\right)^{ij}\chi^j_\mu &= r_\nu r_\mu - \frac{\delta^{ij}}{3} = \left(T_2\right)_{\nu\mu} \\ \chi^i_\nu \left(T_2\right)^{ij}\chi^j_\mu &= (-1)^\nu \chi^i_{-\nu} * \left(T_2\right)^{ij}\chi^j_\mu = (-1)^\nu \left(T_2\right)_{-\nu\mu} \\ \chi^i_\nu * \left(T_2\right)^{ij}\chi^j_\mu * &= (-1)^\mu \chi^i_\nu * \left(T_2\right)^{ij}\chi^j_{-\mu} = (-1)^\mu \left(T_2\right)_{\nu-\mu} \end{aligned} \tag{19}$$

$$\left(T_2\right)_{\mu\nu} = \sqrt{\frac{2\pi}{15}}2^{\frac{|\mu-\nu|}{2}}\left(\frac{2}{3}\right)^{\delta_{\mu\nu}} 2^{\frac{\delta_{\mu 0}\delta_{\nu 0}}{2}}(-1)^\nu Y_{2\mu-\nu} \tag{20}$$

Where Y_{LM} is the spherical harmonic for $L=2$, $M=\mu-\nu$.

$$\int d\Omega Y_{L'M'} * Y_{2\mu-\nu}Y_{LM} = \sqrt{\frac{5\left(2L+1\right)}{4\pi\left(2L'+1\right)}}\delta_{M'M+\mu-\nu}\cdot \\ \cdot C\left(L,2,L';0,0\right)C\left(L,2,L';M,\mu-\nu\right) \tag{21}$$

C. Structure Constants

The expressions for the constants shown in (12) and (13) are as follows:

$$\begin{aligned} a &= \int drr^2\left(P^0_J\right)^2\left(-2V^S + \frac{1}{3}\tilde{V}^S\right) \\ b &= \int drr^2\left(P^0_J\right)^2\left(2\tilde{V}^S\right) \end{aligned} \tag{22}$$

$$\begin{aligned} \tilde{a} &= \int drr^2\left(\left(P^+_J\right)^2 + \left(P^-_J\right)^2\right)\left(-2V^S + \frac{1}{3}\tilde{V}^S\right) \\ \tilde{b} &= \int drr^2\left(P^+_J\right)^2\left(2\tilde{V}^S\right) \\ \tilde{c} &= \int drr^2\left(P^+_J\right)\left(P^-_J\right)\left(2\tilde{V}^S\right) \\ \tilde{d} &= \int drr^2\left(P^-_J\right)^2\left(2\tilde{V}^S\right) \end{aligned} \tag{23}$$

The $\frac{1}{3}\tilde{V}^S$ contribution in a and $\tilde{a}$ comes from the $\frac{\delta^{ip}}{3}$ and it's added to keep the potential numerically in line with [6].

Acknowledgments

I would like to thank Joan Soto, my advisor, for the unconditional support and care that he poured onto this project, helping me understand and solve any doubt which I've encountered during these past few months.

I would also like to thank my family and classmates for the last-minute check-up they did on this document, so as to find any mistakes that could have been missed. Your help is valued.

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