Connecting Quantum Algorithms with Quantum Games:
Grover's algorithm and the Battle of the Sexes

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(Dated: February 2, 2021)

Abstract: A new game is created based on the Battle of the Sexes with a quantum algorithm taking the central stage. The way it works and the impact it has on the development of the game is shown. Different versions are studied in which quantum algorithms take on different relevance and are compared in an ordinary situation.

I. INTRODUCTION

Every day we face situations where there are different paths to be chosen that will lead us to different rewards and, usually, we will take the strategy that will guarantee us a higher profit. These situations can be reproduced by games, specifically, there is a discipline, Game theory, which is dedicated to describe the evolution of possible scenarios in which players can take different strategies.

Since J. V. Neuman and O. Morgenstern published the first paper about the theory in 1944, many simple situations and games have been studied by scientists to understand the theory, such as the popular Prisoner's Dilemma [1] or the Battle of the Sexes [2], the game that will serve as the basis for the paper. Many ways of interpreting the games were introduced in order to find the optimal strategy for the players, such as mixed strategies, which involve some probability mechanism in the choice of a move. Following these different interpretations, in 1999 Meyer [3] introduced a way of understanding the theory from the quantum point of view, resulting in Quantum Game theory.

Parallel to the emergence of this new theory, the so-called quantum computing began to appear. It is based on the use of qubits for the development of quantum algorithms. A qubit is the quantum version of the familiar bit, which corresponds to a two-state quantum-mechanical system, and can therefore be in both states simultaneously, unlike the classical bit. For the creation of these algorithms, combinations of qubits are used that allow to handle probabilities associated with the quantum states of the qubits, creating a superposition of states.

In this paper we will start with the well-known game Battle of the Sexes and introduce a quantum algorithm, the Grover Search Algorithm [4], in order to study different strategies for the players and see if any of them can be more favorable. This will allow us to observe the close relationship between quantum algorithms and quantum game theory and to learn about both fields, which are relatively new and constantly developing.

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II. CONSTRUCTION OF THE GAME

A. Battle of the Sexes

This game is not really a battle, it is a love fest with conflicting values. Alice and Bob want to spend an evening together. Alice prefers going to the Art Gallery (A), while Bob prefers spending the evening at the Bar (B) and both would like to be together. Alice and Bob are both at their respective jobs and they are not able to communicate. They show up at the place they decide, hoping to meet the other there. Depending on their decisions they receive different payoffs, displayed in Tab. I. The term payoff refers to the reward they get for doing one thing or another. In this game it refers to the happiness they will get depending on the situation they end up in.

	Bob A	Bob B
Alice A	(α, β)	(γ, γ)
Alice B	(δ, δ)	(β, α)

TABLE I: Payoff matrix for Battle of the Sexes. Shown: Alice first, then Bob.

It is assumed that $\alpha > \beta > \gamma \geq \delta$, since activity as a couple takes precedence over personal satisfaction. It can be seen that this game lacks optimal play, it does not have the well-known in game theory Nash's Equilibrium [7].

B. Grover's Search Algorithm

Alice and Bob will have to choose between 4 options, so let us look for a 4-item search algorithm. As Grover's search algorithm is one of the fundamental techniques of quantum computation it will be used in the context of the problem.

To each of the 4 options, a quantum state can be associated depending on the state of two qubits. The first one determines where she goes and the second one where he goes. The different states will be:

$$AA : |00\rangle \equiv |0\rangle \ ; \ AB : |01\rangle \equiv |1\rangle \ ; \qquad (1)$$
$$BA : |10\rangle \equiv |2\rangle \ ; \ BB : |11\rangle \equiv |3\rangle \ ;$$

determining an orthonormal basis of a 4-dimensional space.

Before explaining how the algorithm works, it is necessary to introduce the Walsh-Hadamard Transformation (WHT), which is a unitary transformation defined recursively the following way for n > 1:

$$W_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ ; \ W_{2^n} = \frac{1}{\sqrt{2^n}}\begin{pmatrix} W_{2^{n-1}} & W_{2^{n-1}} \\ W_{2^{n-1}} & -W_{2^{n-1}} \end{pmatrix}.$$

In the problem, since it is in a 4-dimensional space, $n = 2$.

Grover's algorithm is a four-step quantum calculation algorithm that attempts to rotate an initial $|\varphi\rangle$ superposition on the origin by the plane spanned by $|\varphi\rangle$ and a target $|a\rangle$, until it is as close as possible to the target. Usually, it takes about $\frac{\pi}{4}\sqrt{n}$ searches to find the target with a probability close to 100%. To achieve it the algorithm uses an oracle and the transformation defined above. An oracle is a black box, an operation that has some property that you do not know and that you are trying to find out about.

In the game, Alice creates an unitary initial target $|\varphi_a\rangle$ as a superposition of the states in Eq. (1) and builds the oracle as:

$$R_a = I - 2\,|\varphi_a\rangle\,\langle\varphi_a|\,. \qquad (2)$$

Bob prepares an initial two-qubit register in 0, $|00\rangle$. Then he applies the WHT to get an equally weighted superposition of the standard basis states, obtaining the initial state for the algorithm, which will be called $|\varphi_s\rangle$.

Now, Bob has to decide how many searches is he going to do, denoted as k. Each of these searches is equivalent to applying the algorithm once, which consists of the following product:

$$|\varphi_i\rangle = W_{2^2} \cdot R_s \cdot W_{2^2}^{-1} \cdot R_a \cdot |\varphi_s\rangle\,, \qquad (3)$$

where $R_s = -(I - 2\,|0\rangle\,\langle 0|)$ and each of these products consists of one step. It can be seen that it corresponds to a rotation since:

- $\det(W_{2^2} \cdot R_s \cdot W_{2^2}^{-1} \cdot R_a) = 1$.
- $(W_{2^2} \cdot R_s \cdot W_{2^2}^{-1} \cdot R_a)^T = (W_{2^2} \cdot R_s \cdot W_{2^2}^{-1} \cdot R_a)^{-1}$.

This algorithm has great capabilities and allows to find a target from 2^n options with a very high probability of success and in a number of steps of the order of $\sqrt{n}$, whereas classically it would require approximately $\frac{1}{2} \cdot 2^n$ tries to find the target with a probability of 50%.

C. Final Quantum Game

Alice will choose what she wants to do as a couple, creating the target $|\varphi_a\rangle$, that will not be known by Bob. He will set the initial state

$$|\varphi_s\rangle = W_2^2\,|0\rangle = \frac{1}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle) \qquad (4)$$

and decide how many times is he playing Grover's algorithm (k), a thing that will not be revealed to Alice.

After making the different calculations, they will make a measurement of the final quantum state and it will show the probabilities of finishing in each of the four options. It is important to note that Alice's initial choices to determine $|\varphi_a\rangle$ do not limit the final state to just a few elements of the base.

The final payoff for each of them will be:

$$\pi_k^C = \sum_{j=0}^{3} |a_j|^2 \cdot \pi_C(|j\rangle) \ ; \ C = A \text{ or } B. \qquad (5)$$

In Eq. (5), a_j refers to the probability amplitude for each of the states that form the final overlay. The payoff matrix in Tab. I will be modified a little bit to make some calculations less cumbersome. As they both want to do something together, the payoff will be set to zero for the case in which neither of them is doing either what they prefer or is accompanied. Also $\alpha = \beta + \gamma$ is imposed. Getting the payoffs matrix displayed in Tab. II.

	Bob A	Bob B
Alice A	(α, β)	(γ, γ)
Alice B	(0, 0)	(β, α)

TABLE II: Payoff matrix for the Quantum Game. Shown: Alice first, then Bob.

In the next sections, the different ways the problem can be faced are going to be studied.

III. SIMPLE TARGET

Alice chooses only one of the basis state as the target. Let us assume she choose $|\varphi_a\rangle = |0\rangle$. She can now set the oracle:

$$R_a = I - 2 \cdot |0\rangle\,\langle 0| = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (6)$$

Bob, not knowing what she has played, can now choose the k he will play.

If he plays $k = 1$, they will end up with a probability of 100% of doing whatever Alice has chosen, as $k = 1$ is optimal in this case. If Bob plays $k = 2$, the final state will be:

$$|\varphi_f\rangle = \frac{1}{2} \cdot (|0\rangle - |1\rangle - |2\rangle - |3\rangle). \qquad (7)$$

So they will end with a 25% of probability of doing each option. If he plays $k = 3$ they will obtain the same chances, and for $k = 4$ the same one as $k = 1$. This happens because, as was explained before, the algorithm rotates an initial state, the equiprobable state created by Bob. In this version of the problem the angle between $|a^{\perp}\rangle$ and $|\varphi_s\rangle$ is $\phi = \frac{\pi}{6}$. Every time Bob applies the algorithm is rotating the initial state by 2ϕ, so at $k = 4$ a new cycle will start, repeating the same results as for $k = 1, 2$ and 3. It can be seen graphically in Figure 1.

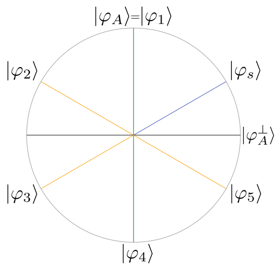

FIG. 1: Rotation of the initial state. $|\varphi_k\rangle$ denotes the state after applying the algorithm k times.

As it can be noticed, to play k higher than 4 is to repeat the cycle, so from now on the k value chosen by Bob will be limited to 4, although in the following sections the cycle does not have to start again at this k value. It will be a way to limit the game.

The payoff if Bob plays $k = 1$ or similar will be maximum, α for Alice and β for Bob. In case Bob plays $k = 2$ or alike, both will get $\pi_2^{A,B} = \frac{\alpha}{2}$. With some calculation and the conditions imposed above, $\frac{\alpha}{2} < \beta$. So, the best option for Bob is to play $k = 1$.

The results would be the same regardless of $|\varphi_a\rangle$, adjusting the reward to $k = 1$. But in the end, she wants to go with him to the Art Gallery, so the state chose, $|0\rangle$, makes the most sense.

IV. OVERLAPPING TARGET

Alice will be able to choose two basis states to create her initial state, the target. In this version, the following types of Alice will be considered:

1. Social: She wants to be with Bob either at the Art Gallery ($|0\rangle$) or at the Bar ($|3\rangle$).
2. Selfish: She wants to go to the Art Gallery with Bob ($|0\rangle$) or alone ($|1\rangle$).
3. Evil: Either they go to the art gallery together ($|0\rangle$), or she will try to make sure that neither of them is rewarded ($|2\rangle$).

All this information is listed in the Tab. III. This is part of Alice's idiosyncrasy, her personality, so it will be known by Bob.

	$\|i\rangle$	$\|j\rangle$
Social	$\|0\rangle$	$\|3\rangle$
Selfish	$\|0\rangle$	$\|1\rangle$
Evil	$\|0\rangle$	$\|2\rangle$

TABLE III: Types of Alice with the states she is choosing.

Now they are facing two different games: consider that Alice simply chooses a probability to play one option or another, called Non-Quantum Alice, or consider that she establishes her initial state as a superposition of two of the four states, creating a whole new oracle with the probability amplitudes, named after Quantum Alice.

A. Non-Quantum Alice

Alice will play mixed strategies. She is going to put a probability $\rho \in \{0, 1\}$ to do one thing or another, to play one oracle or another. The resultant payoff will correspond, numerically, to make the product of probabilities between the actual probabilities of going one place or another obtained in Sec. III and the probability she has chosen. For $k = 1$:

$$\pi_1^{A,B} = \rho \cdot \pi_{A,B}(|i\rangle) + (1 - \rho) \cdot \pi_{A,B}(|j\rangle). \qquad (8)$$

For $k = 2$ the payoff for both will be $\pi_2^{A,B} = \frac{\alpha}{2}$, as in the previous section it has been seen that the reward obtained for k does not depend on what Alice has chosen.

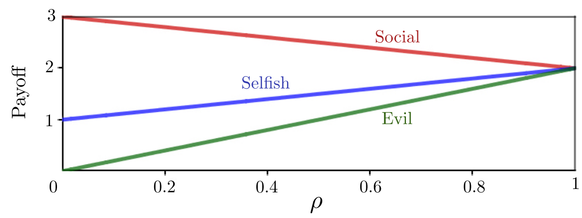

FIG. 2: Bob's payoff for different kinds of Alice at $k = 1$. It has been assumed $\alpha = 3$, $\beta = 2$ and $\gamma = 1$.

It can be seen in Fig. 2 that if Alice is social it is clear that the best option for Bob is to play $k = 1$ since for any ρ value he gets a higher reward than the one obtained with $k = 2$. On the other hand, if Alice is evil Bob must ensure a minimum payoff and play $k = 2$, since for $k = 1$ he gets a lower reward for almost all ρ. The case that is not so clear is what to do if Alice is selfish. If Bob's reasoning is not to risk it and guarantee himself a minimum payoff he should play $k = 2$. Also, by choosing this k, he gets the same probability for all base states, as it can be concluded from Eq. (7), so he could get a payoff of α sometime.

B. Quantum Alice

Alice will choose a degree $\theta \in [0, \pi]$ and set an initial state [8]:

$$|\varphi_A\rangle = \cos(\theta) \cdot |i\rangle + \sin(\theta) \cdot |j\rangle \,. \tag{9}$$

Using this she will be able to construct the oracle as explained earlier in Eq. (2). Now, the different types of Alice defined above will be studied separately. For each of them, the best options for both will be examined. Bob's reasoning will always be to ensure a minimum of payoff. From now on, the values of the parameters shown in the caption of Fig. 2 will be considered in all representations or expressions.

If Alice is social the oracle will be:

$$R_a = \begin{pmatrix} -\cos(2\theta) & 0 & 0 & -\sin(2\theta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(2\theta) & 0 & 0 & \cos(2\theta) \end{pmatrix} . \tag{10}$$

Depending on the number of k Bob will get different rewards. For $k = 2$ and $k = 4$ he can obtain a payoff lower than 1, which will be dramatic for him compared to the ones obtained with $k = 1$ or $k = 3$, represented in Fig. 3.

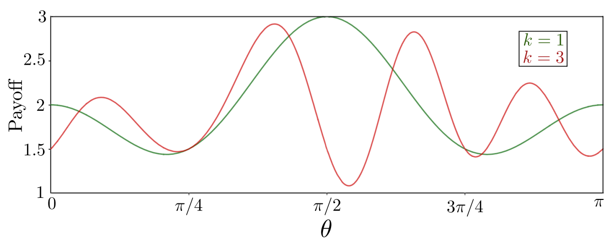

FIG. 3: Bob rewards for $k = 1$ and $k = 3$ based on the θ value played by Alice.

Using Bob's rationale, the best option for him is to play $k = 1$, since he maximises his minimum and is also able to get the highest payout, 3. For this k the probabilities obtained for the different states are showed in Fig. 4.

As it can be seen in this figure, for $\theta = 0$ and π Alice gets full certainty of going to the Art Gallery with Bob, something that had been seen in the Sec. III as she would be playing a simple target. For $\theta = \frac{\pi}{4}$ and $\frac{3\pi}{4}$ equiprobability for all states is obtained, which translates into a

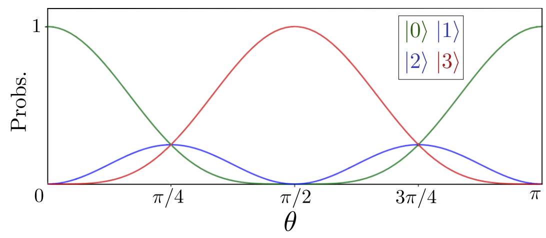

FIG. 4: Probabilities for each base state for $k = 1$ based on the θ value played by Alice. Note that $|1\rangle$ and $|2\rangle$ collapse.

payoff of $\frac{3}{2}$ for both. Finally, one can see that the maximum reward for Bob matches with the case in which Alice chooses $\theta = \frac{\pi}{2}$, since $\sin(\frac{\pi}{2}) = 1$. In particular, Alice's payoff will be:

$$\pi_1^A = 3 \cdot \cos^4(x) + \frac{1}{4} \cdot \sin^2(2x) + 2 \cdot \sin^4(x). \tag{11}$$

Depending on her preferences, she will choose one θ or another, being assured of a reward of at least $\frac{3}{2}$.

If Alice is selfish, the best option for Bob is to play $k = 3$, as this will ensure him a maximum minimum reward of approximately 1.1, shown in Fig. 5. In this case the maximum peaks for Bob are around the ones for Alice, which could benefit him in case she is looking for a maximum payoff.

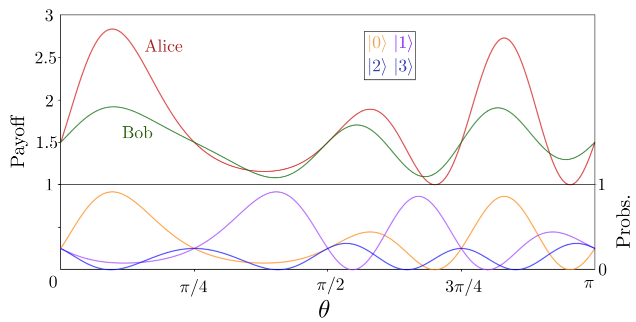

FIG. 5: Above and with the y-axis on the left: Alice and Bob rewards for $k = 3$. Down and with the y-axis on the right: the probabilities for each base state for $k = 3$.

It can be seen that Alice is now not getting her maximum reward at the ends of the range, for as seen in the Sec. III they would be getting the final state shown in Eq. (7). Taking into account that Alice prioritises being with Bob, her best option will be to play $\theta \simeq 0.3$ or 2.6, as in all other cases she has a very low probability of doing something with Bob, which can be seen in the lower part of Fig. 5.

In the case that Alice is evil, Bob is in front of two options where he maximises his minimum. In the first of them, $k = 2$, there is only one point for which he would obtain that minimum. Moreover, in the interval $(0, \frac{\pi}{2})$ he would guarantee a probability equal to or greater than 25% of going to the Bar with Alice. On the other hand, the maximum he could get is less than 2 and he would only get it for a value of θ. The second option, $k = 3$, has two possible extremes where he could get the minimum value, but otherwise there are two points with a reward equal to 2, as it can be seen in Fig. 6. To make the right choice, Bob should stop and think about what he thinks Alice will do: At the θ values where Bob gets the minimum, she also gets very low payoffs, but for all values of k, around $\theta \simeq 2.6$ she has a maximum payoff, so it would be rational for her to play a value similar to that to guarantee herself a maximum (or close to it) whatever Bob's reasoning. So following this thinking the best thing for Bob would be to play $k = 3$ as he gets the maximum maximum in that area. However, he is the one who knows his wife, and he might choose to play $k = 2$ if he thinks she will play a θ in the interval discussed above and could get to a payoff of 3. Furthermore, for $k = 2$ he also has a maximum in $\theta \simeq 2.5$, although smaller than the one in $k = 3$.

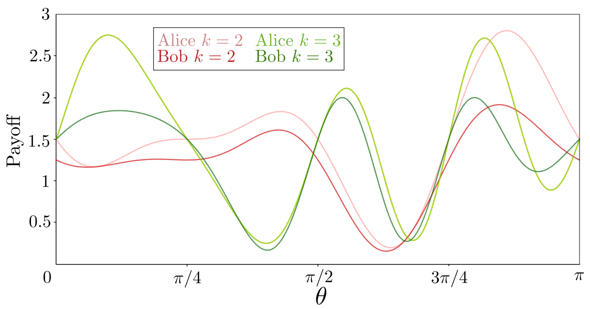

FIG. 6: Rewards for Bob and Alice for $k = 2$ and $k = 3$.

V. CONCLUSIONS

So what do quantum algorithms have to do with quantum games? The game shown in the text represent it: a quantum algorithm for a game can be understood as a quantum strategy for a player.

Different perspectives have been shown from a famous classical problem that has been studied from many quantum versions in recent years. It has been possible to see how it develops and how quantum algorithms affect it. Starting from a simple version, in Sec. III, as an example of how the problem works, and progressing, in Secs. IV A and IV B, to a complete version of the problem, where all those situations that, based on common sense, were logical were studied. In order to reduce the number of results and fit the dimensions of the paper, certain restrictions had been made, such as limiting $k = 4$ or not allowing superpositions of more than 2 states.

In the game created, it has been observed that neither of the two can get a clear advantage or always get away with it. Alice usually gets a higher payoff, and therefore a higher probability of doing what she wants to do, to go with Bob to the Art Gallery. Whatever their character, one has to think that Alice and Bob are a couple and want no harm for each other. Moreover, no one would try to get a very low reward, which translates into happiness, when she could try to get much higher happiness.

During the course of the project, other reasoning behind Bob's method of choice, such as social benefit, were studied. In other words, Bob seeks the best for both, the maximum of the sum of rewards. Finally, it has not been introduced in the work due to lack of space and because it was considered more interesting the reasoning in which both try to achieve what they want with their partner. So this game offers more research than presented and performed. It also could be interesting to compare results using different quantum algorithms, such as the Bernstein-Vazirani algorithm [9].

Acknowledgments

I deeply thank Dr. Miquel Montero for advising and providing me his guidance and feedback throughout this project. I also would like to thank his careful revision of all my work. Finally, last but by no means least; I wish to express my love and my gratitude to my family for their constant support and encouragement during all this research.

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[6] C. Zalka. "Grover's quantum searching algorithm is optimal". Phys. Rev. A **60**, 2746-2751 (1999).
[7] Nash Equilibrium tries to find a strategy that maximises the profit given the decisions of the others. Ergo no player has any incentive to individually modify his strategy.
[8] No consideration is given to the relative phase. The mathematical complexity is reduced while keeping the diversity of solutions. It also allows the comparison $\rho = \cos^2(\theta)$.
[9] Similar to Grover's algorithm, using WTH but with a different oracle and capabilities.