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AUTOMORPHISMS OF $\mathbb{C}^n$:

A survey of the Andersén-Lempert theory

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Abstract

In this project, we study the automorphisms of $\mathbb{C}^n$. We give a proof of the Global Andersén Lempert Theorem (Theorem 3.1) as well as the Local Andersén Lempert Theorem (Theorem 3.11). We illustrate the importance of these results by giving three geometric applications (Theorems 4.1, 4.10, and 4.18).

1 Introduction

It has always been of interest to classify maps that preserve a structure. For example, the linear automorphisms of a vector space, the affine bijections on affine spaces, the isometries of $\mathbb{R}^n$ ...In this project, we will focus our attention on the space $\mathrm{Aut}(\mathbb{C}^n)$, that is, the space of holomorphic bijective maps with holomorphic inverse. We call the maps of $\mathrm{Aut}(\mathbb{C}^n)$ *automorphisms of* $\mathbb{C}^n$.

It would be ideal to end up giving a list of maps defining the whole $\mathrm{Aut}(\mathbb{C}^n)$. Such a list exists for $\mathrm{Aut}(\mathbb{C})$ as the following theorem shows:

Theorem 1.1.

$$\mathrm{Aut}(\mathbb{C}) = \{f \in \mathcal{H}(\mathbb{C}) \mid f(z) = az + b \textit{ where } a, b \in \mathbb{C} \textit{ and } a \neq 0\}.$$

Proof. Let us see that for $f \in \mathrm{Aut}(\mathbb{C})$ the singularity at infinity can only be a pole. Indeed, if it is removable then f is bounded, and by Liouville's Theorem [5] f is constant, and thus, not injective. If the singularity at infinity is essential, then by the Great Picard's Theorem [5] the function f fails to be injective. Thus f must have a pole at infinity. This implies that f is a polynomial, but by the Fundamental Theorem of Algebra, the only polynomials that are bijective are those of degree 1. That is, $f(z) = az + b$ for some $a, b \in \mathbb{C}$ and $a \neq 0$. □

No similar result holds in $\mathbb{C}^n$. In fact $\mathrm{Aut}(\mathbb{C}^n)$ is a very complicated space. To illustrate this, we present the following example:

Consider $h\colon \mathbb{C}^{n-1} \to \mathbb{C}$ a holomorphic function, and define the map $f_m\colon \mathbb{C}^n \to \mathbb{C}^n$ as

$$f_m(z) = z + h(z_1, \ldots, \widehat{z_m}, \ldots, z_m)e_m, \tag{1}$$

where $1 \leq m \leq n$ is an integer, and e_m is the m-th vector of the canonical basis in $\mathbb{C}^n$. It is easy to see that this map is an automorphism of $\mathbb{C}^n$.

The interesting thing about this map f_m is that it shows that there are at least as many automorphisms of $\mathbb{C}^n$, as holomorphic functions in $\mathcal{H}(\mathbb{C}^{n-1})$. Thus, if $n > 1$, then $\mathrm{Aut}(\mathbb{C}^n)$ is infinite-dimensional!

A similar example can be constructed if we take $h\colon \mathbb{C}^{n-1} \to \mathbb{C}$ to be a nowhere zero holomorphic function. Then an automorphism associated to h, say $g_m\colon \mathbb{C}^n \to \mathbb{C}^n$, can be defined as

$$g_m(z) = \big(z_1, \ldots, z_m h(z_1, \ldots, \widehat{z_m}, \ldots, z_n), \ldots, z_n\big). \tag{2}$$

We will see that these maps will be useful for our study.

Definition 1.2. We will call $f\colon \mathbb{C}^n \to \mathbb{C}^n$ a *shear* map if after a linear change of variables, f is given by equation (1). We will say that $g\colon \mathbb{C}^n \to \mathbb{C}^n$ is an *overshear* map, if after a linear change of variables g satifies equation (2).

These two particular types of maps were very important to the study of $\mathrm{Aut}(\mathbb{C}^n)$. In fact, for some time, mathematicians studied the space $\mathrm{Aut}(\mathbb{C}^n)$ through shears and overshears. Such studies ([22] for example) gave, for example, some information on how $\mathrm{Aut}(\mathbb{C}^n)$ acted on countable sets. In 1988, Walter Rudin and Jean Pierre Rosay proved that there were some sequences that could not be mapped to $\mathbb{N} \times \{0\} \times \cdots \times \{0\}$ by automorphisms of $\mathbb{C}^n$, while many others could be. This and some other results showed that $\mathrm{Aut}(\mathbb{C}^n)$ had a very delicate structure.

Of course, automorphisms of $\mathbb{C}^n$ were not only studied from the point of view of shears and overshears. Before the work of Rudin and Rosay, Masakazu Suzuki in 1974, gave some results for polynomial automorphisms of $\mathbb{C}^n$ i.e. automorphisms where every component is a polynomial (see [20] for more information). But it was not until 1990 that a remarkable result concerning all types of automorphisms was proven. In [2], Erik Andersén showed that every volume-preserving automorphism of $\mathbb{C}^n$ is a uniform limit, on compact sets, of compositions of shears (in fact, Erik Andersén was motivated by a question asked in the paper by Rudin and Rosay). Shortly after, in 1992, Erik Andersén and Lázló Lempert proved two groundbreaking results similar to the one proved by Erik Andersén. They proved that every automorphism of $\mathbb{C}^n$ is a uniform limit, on compact sets, of compositions of overshears. They also proved that any biholomorphism on an open set is a uniform limit, on compact sets, of compositions of overshears.

These three results allowed the birth of approximation-type theorems for automorphisms of $\mathbb{C}^n$.

This will be the goal of our project. To prove the three theorems proved by Erik Andersén and Lázló Lempert, and to give some applications to illustrate how powerful the theorems from Andersén and Lempert are.

1.1 Main theorems

Here we state the main theorems that we will prove in this project. We begin with the two versions of the Andersén-Lempert Theorem.

Theorem 1.3 (Andersén-Lempert). *Let $f \in \operatorname{Aut}(\mathbb{C}^n)$. Then there exists a sequence $(\psi_k)_k$ of finite composition of overshears such that ψ_k tends to f uniformly on compact sets of $\mathbb{C}^n$.*

Moreover if f is volume-preserving (that is, $\det Df(z) = 1$ for all z), then the sequence $(\psi_k)_k$ can be chosen so that each ψ_k is a finite composition of shears.

Theorem 1.4 (Local Andersén-Lempert Theorem). *Let $\Omega \subset \mathbb{C}^n$ be an open set, and $H\colon \big[0,1\big] \times \Omega \to \mathbb{C}^n$ be an isotopy of biholomorphisms such that each each Ω_t is Runge in $\mathbb{C}^n$. Then if $H(0,\cdot)$ can be approximated by automorphisms of $\mathbb{C}^n$ uniformly on compact sets of Ω, then for every $t \in \big[0,1\big]$ the map $H(t,\cdot)$ can also be approximated by automorphisms of $\mathbb{C}^n$ uniformly on compact sets of Ω.*

The three following geometric applications, follow from the Local Andersén-Lempert Theorem.

Theorem 1.5. *Let $K_1, \dots, K_m$ be pairwise disjoint compact star-shaped domains of $\mathbb{C}^n$. Let for each $1 \leq k \leq m$ $\phi_k \in \operatorname{Aut}(\mathbb{C}^n)$. If the sets $\phi_k(K_k)$ are pairwise disjoint, and $K = \bigcup_{k=1}^m K_k$ and $K' = \bigcup_{k=1}^m \phi_k(K_k)$ are polynomially convex, then for each $1 \leq k \leq m$ there exists U_k a neighborhood of K_k and a sequence $(\psi_j)_j \subset \operatorname{Aut}(\mathbb{C}^n)$ such that ψ_j converges to ϕ_k uniformly on U_k, for each $1 \leq k \leq m$.*

Theorem 1.6. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex Runge domain and $Z = (z_k)_{k\geq 0} \subset \Omega$ a discrete sequence (that is with no accumulation points). Then there exists a proper holomorphic embedding $f\colon \mathbb{D} \subset \mathbb{C} \to \Omega$ satisfying $Z \subset f(\mathbb{D})$.*

Theorem 1.7. *Let $\gamma\colon [0,1] \to \mathbb{C}^n$ be an injective real analytic arc with $\gamma' \neq 0$. Then there is a sequence $(\psi_k)_k$ of automorphisms of $\mathbb{C}^n$ such that $\psi_k \circ \gamma$ converges uniformly on $[0,1]$ to the map $\nu(t) = (t, 0, \dots, 0)$ $(t \in [0,1])$.*

1.2 Structure of the project

We now explain how the project is structured.

Section 1 gives a brief introduction to the topic of automorphisms of $\mathbb{C}^n$. In Section 2 we give some background results on the theory of ODEs, and explain how automorphisms are tied to ODEs. Next, in Section 3 we give the proofs of the Global and Local versions of the Andersén-Lempert Theorem, and we comment on the existence of automorphisms that are not compositions of shears nor overshears. Section 4 is divided into three subsections one for Theorem 1.5 and some results concerning polynomial convexity. One for Theorem 1.6, where we give a proof of such theorem and comment on the existence of Fatou-Bieberbach domains. And finally, the third subsection is devoted to Theorem 1.7. The last section, Section 5, gives some conclusions of the project.

These four sections are the main ones. We have also included an appendix (Appendix A) introducing some concepts from the theory of several complex variables.

2 Automorphisms through ODEs

This section is devoted to explaining one of the key ingredients on the proof of the Andersén-Lempert Theorem, that is to show how are the solutions to ODEs related to automorphisms of $\mathbb{C}^n$. We only have included those proofs that we thought were important, or not well-known. Nevertheless, we encourage the reader to see [18] and [4] if he or she is not familiar with some of the concepts or results we mention. We begin by recalling some notions of the theory of ODEs.

2.1 Complete holomorphic vector fields

We will work with continuous time-dependent vector field $X\colon \mathbb{R}\times\mathbb{C}^n \to \mathbb{C}^n$, if X is locally Lipschitz on the second variable uniformly with respect to the first one. Remember that for such vector fields, the *flow of* X at $(t_0, z_0) \in \mathbb{R}\times\mathbb{C}^n$ is the map $\varphi(\cdot; t_0, z_0)$ defined on a neighborhood I_{t_0} of t_0 which satisfies the following Cauchy problem

$$\begin{cases} \dfrac{d\varphi}{dt}(t; t_0, z_0) &= X\big(t, \varphi(t; t_0, z_0)\big) \\ \varphi(t_0; t_0, z_0) &= z_0, \end{cases}$$

with $t \in I_{t_0}$. We will also refer to φ as the solution to the vector field X.

Definition 2.1. Let $X\colon \mathbb{R}\times\mathbb{C}^n \to \mathbb{C}^n$ be a continuous time-dependent vector field, locally Lipschitz on the second variable uniformly with respect to the first one. We say that X is *complete* if the flow φ of X at any $(t_0, z_0) \in \mathbb{R}\times\mathbb{C}^n$ is defined for all $t \in \mathbb{R}$. In other words, X is *complete* if for any $(t_0, z_0) \in \mathbb{R}\times\mathbb{C}^n$ the solution to the Cauchy problem

$$\begin{cases} \dfrac{d\varphi}{dt}(t; t_0, z_0) &= X\big(t, \varphi(t; t_0, z_0)\big) \\ \varphi(t_0; t_0, z_0) &= z_0, \end{cases}$$

is defined for all $t \in \mathbb{R}$.

On an autonomous vector field X (those that only depend on z), the choice of t_0 is irrelevant to define the flow of X. Thus we will always assume that $t_0 = 0$, and we will write $\varphi(t; z)$ instead of $\varphi(t; 0, z)$. We will refer to φ as *the solution generated by* X, or a *solution to* X.

What will be important to us is that given a complete holomorphic autonomous vector field $X\colon \mathbb{C}^n \to \mathbb{C}^n$, the flow of X defines a 1-parameter group of automorphisms. More concretely, for any fixed $t \in \mathbb{R}$ the map $\varphi(t; \cdot)\colon \mathbb{C}^n \to \mathbb{C}^n$ (referred to as the *time-t map*) will be an automorphism of $\mathbb{C}^n$ with inverse $\varphi(-t; \cdot)$. A converse of this result is also true, as we will see in Section 3.

Let us recall how the divergence of a vector field X is tied to the determinant of the differential of its solution.

Proposition 2.2. Let $\Omega \subset \mathbb{C}^n$ be open, $X\colon \Omega \to \mathbb{C}^n$ be a $\mathcal{C}^1$ vector field, and φ be its flow. Then

$$\frac{d}{dt} \det D_z\varphi(t; z) = \operatorname{div} X\big(\varphi(t; z)\big) \cdot \det D_z\varphi(t; z)$$

where $t \in \mathbb{R}$ and $z \in \mathbb{C}^n$ are so that $\varphi(t; z)$ is defined. In particular, $\operatorname{div} X = 0$ if and only if $\det D_z\varphi(t; z)$ is constant in t.

Proof. This is a straightforward computation. Let us put $D_z\varphi(t;z)$ in vector notation, i.e.

$$D_z\varphi(t;z) = \left(\frac{\partial \varphi}{\partial z_1}(t;z), \dots, \frac{\partial \varphi}{\partial z_n}(t;z)\right).$$

Then the derivative of the determinant is

$$\begin{aligned}
\frac{d}{dt}\det D_z\varphi(t;z) &= \sum_{j=1}^{n} \det\left(\frac{\partial \varphi}{\partial z_1}(t;z), \dots, \frac{d}{dt}\Big(\frac{\partial \varphi}{\partial z_j}\Big)(t;z), \dots, \frac{\partial \varphi}{\partial z_n}(t;z)\right) = \\
&= \sum_{j=1}^{n} \det\left(\frac{\partial \varphi}{\partial z_1}(t;z), \dots, \frac{\partial}{\partial z_j}\Big(\frac{d\varphi}{dt}\Big)(t;z), \dots, \frac{\partial \varphi}{\partial z_n}(t;z)\right) = \\
&= \sum_{j=1}^{n} \det\left(\frac{\partial \varphi}{\partial z_1}(t;z), \dots, \frac{\partial}{\partial z_j}\Big(X\big(\varphi(s;w)\big)\Big)(t;z), \dots, \frac{\partial \varphi}{\partial z_n}(t;z)\right) = \\
&= \sum_{j=1}^{n} \det\left(\frac{\partial \varphi}{\partial z_1}(t;z), \dots, \frac{\partial X}{\partial z_j}\big(\varphi(t;z)\big)\cdot\frac{\partial \varphi}{\partial z_j}(t;z), \dots, \frac{\partial \varphi}{\partial z_n}(t;z)\right) = \\
&= \sum_{j=1}^{n} \frac{\partial X}{\partial z_j}\big(\varphi(t;z)\big)\cdot \det\left(\frac{\partial \varphi}{\partial z_1}(t;z), \dots, \frac{\partial \varphi}{\partial z_j}(t;z), \dots, \frac{\partial \varphi}{\partial z_n}(t;z)\right) = \\
&= \operatorname{div} X\Big(\varphi(t;z)\Big)\cdot \det D_z\varphi(t;z)
\end{aligned}$$

where we have used that φ is the flow generated by X, and φ is $\mathcal{C}^1$ in (t,z). □

Remark 2.3. Observe that if $\operatorname{div} X = 0$, and $\det D_z\varphi_z(0;z) = 1$, then we have that $\det D_z\varphi(t;z) = 1$. That is $\varphi(t;\cdot)$ is a volume-preserving automorphism. This fact will prove crucial in Section 3.

2.2 Approximation of solutions to ODEs

Another useful result for us will be how the solution to an ODE changes when we make a small change in the vector field defining the equation. To give a full proof of that we first need the following lemma.

Lemma 2.4 (Gronwall inequality)**.** Let $T > 0$ and $\psi\colon [0,T] \to \mathbb{R}$. Suppose there exist $a, b, c \in \mathbb{R}$ with $b \geq 0$ such that

$$\psi(t) \leq a + \int_0^t b\psi(s) + c ds, \quad t \in [0,T].$$

Then

$$\psi(t) \leq ae^{bt} + \frac{c}{b}(e^{bt} - 1).$$

Proof. Define $\phi(t) = e^{-bt}$, and for all $t \in [0,T]$ consider the derivative

$$\frac{d}{dt}\Big(\phi(t)\int_0^t b\psi(s) + cds\Big) = b\phi(t)\Big(\frac{c}{b} + \psi(t) - \int_0^t b\psi(s) + cds\Big)$$

Now using the hypothesis on ψ we have

$$\frac{d}{dt}\Big(\phi(t)\int_0^t b\psi(s) + cds\Big) \leq b\phi(t)\Big(a + \frac{c}{b}\Big),$$

and integrating we get

$$\phi(t)\int_0^t b\psi(s)+cds \le \int_0^t b\phi(s)\Big(a+\frac{c}{b}\Big)ds = (1-e^{-bt})\Big(a+\frac{c}{b}\Big).$$

Multiplying by $\phi(t)^{-1}$ and adding a to both sides yields

$$a+\int_0^t b\psi(s)+cds \le ae^{bt}+\frac{c}{b}(e^{bt}-1), \quad t\in[0,T].$$

Applying the inequality of the hypothesis to the left-hand side gets us the desired result. □

As a consequence of the Gronwall inequality, we have the following theorem.

Theorem 2.5. *Let $\Omega\subset\mathbb{C}^n$ be open, $I\subset\mathbb{R}$ an open interval, and $f,g\colon I\times\mathbb{C}^n\to\mathbb{C}^n$ be time-dependent continuous vector fields. Suppose f is locally Lipschitz on the second variable, uniformly with respect to the first one. And let ψ and φ be the solutions to the following Cauchy problems*

$$\begin{cases}\dfrac{d\varphi}{dt}(t;t_0,z_0) & = f\big(t,\varphi(t;t_0,z_0)\big)\\ \varphi(t_0;t_0,z_0) & = z_0,\end{cases} \qquad \begin{cases}\dfrac{d\psi}{dt}(t;t_0,w_0) & = g\big(t,\psi(t;t_0,w_0)\big)\\ \psi(t_0;t_0,w_0) & = w_0,\end{cases}$$

then we have that

$$||\varphi(t;t_0,z_0)-\psi(t;t_0,w_0)|| \le ||z_0-w_0||e^{L|t-t_0|}+\frac{M}{L}\Big(e^{L|t-t_0|}-1\Big),$$

where

$$L=\sup_{(t,z)\neq(t,w)\in U}\frac{||f(t,z)-f(t,w)||}{||z-w||},$$

and

$$M=\sup_{(t,z)\in U}||f(t,z)-g(t,z)||$$

being U a set containing both graphs of $\varphi(t;t_0,z_0)$ and $\psi(t;t_0,w_0)$.

Proof. To prove this we will only need to see that the function $||\varphi(t;t_0,z_0)-\psi(t;t_0,w_0)||$ satisfies the hypothesis of Gronwall's inequality lemma. This is a direct result from the definition of φ, ψ, L, and M. Indeed,

$$\begin{aligned}||\varphi(t;t_0,z_0)-\psi(t;t_0,w_0)|| &\le ||z_0-w_0||+||\varphi(t;t_0,z_0)-z_0-\psi(t;t_0,w_0)+w_0|| \le\\ &\le ||z_0-w_0||+\int_{t_0}^t ||f(s,\varphi(s,t_0,z_0))-g(s,\psi(s;t_0,w_0))||ds \le\\ &\le ||z_0-w_0||+\int_{t_0}^t ||f(s,\varphi(s,t_0,z_0))-f(s,\psi(s;t_0,w_0))||ds+\\ &+\int_{t_0}^t ||f(s,\psi(s;t_0,z_0))-g(s,\psi(s;t_0,w_0))||ds \le\\ &\le ||z_0-w_0||+\int_{t_0}^t L||\varphi(s;t_0,z_0)-\psi(s;t_0,w_0)||+Mds.\end{aligned}$$

Now applying the Gronwall inequality with $a=||z_0-w_0||$, $b=L$, $c=M$ and $\psi(t)=||\varphi(t;t_0,z_0)-\psi(t;t_0,w_0)||$, we get the result we wanted. □

In a sense, this theorem tells us that if the vector fields f and g are close enough, then the solutions defined by φ and ψ will also be close (provided that z_0 and w_0 are also close enough). This will be a crucial fact that we will use in the following section.

Finally, we present another result on approximating solutions to ODEs which will be very important on the proof of the Andersén-Lempert Theorem.

Theorem 2.6. *Let X be a holomorphic vector field, and let φ be its flow (which we assume it is defined in $\big[0,1\big] \times \mathbb{C}^n$). Suppose there exist complete holomorphic vector fields $X_1, \dots, X_m$ such that $X = \sum_{k=1}^N X_k$, and let ϕ_k denote the flow of X_k. And for $k \geq m$ define $\phi^k(t;z)$ by the following recurrence*

$$\begin{cases} \phi^{k+1}\Big(t;z\Big) & = \phi^k\Big(t;\phi^1\big(t;z\big)\Big) \\ \phi^1(t;z) & = \phi_m\Big(t;\phi_{m-1}\big(t;\dots;\phi_1(t;z)\big)\dots\big)\Big) \end{cases}$$

Then, $\phi^j\left(\frac{t}{j};z\right)$ converges uniformly on compact sets of $\big[0,1\big] \times \mathbb{C}^n$ to $\varphi(t;z)$.

This theorem tells us that given the decomposition $X = \sum_{k=1}^N X_k$, composing the flows of the X_k's in cyclically (first ϕ_1, then ϕ_2 up to ϕ_m, then again ϕ_1 and repeat) enough times with small steps in the variable t, gives an approximation to the flow of X.

Proof. Observe first that since each X_k is complete, $\phi^j(t;z)$ is defined for all $(t,z) \in \mathbb{R} \times \mathbb{C}^n$. Because we are interested in uniform convergence on compact sets, it is enough that we prove that we have local uniform convergence. Let us then prove that we have local uniform convergence.

Observe that

$$\begin{aligned} &\frac{d}{dt}\bigg(\phi^1\Big(\frac{t}{j};z\Big)\bigg)\bigg|_{t=0} = X_m\bigg(\phi_{m-1}\Big(\frac{t}{j};\dots;\phi_1\Big(\frac{t}{j};z\Big)\dots\Big)\bigg)\frac{1}{j}\bigg|_{t=0} + \\ &+ D_z\phi_m\bigg(\frac{t}{j};\phi_{m-1}\Big(\frac{t}{j};\dots;\phi_1\Big(\frac{t}{j};z\Big)\dots\Big)\bigg)\cdot\frac{d}{dt}\bigg(\phi_{m-1}\Big(\frac{t}{j};\dots;\phi_1\Big(\frac{t}{j};z\Big)\dots\Big)\bigg)\bigg|_{t=0} = \\ &= \frac{1}{j}X_m(z) + D_z\phi_m(0;z)X_{m-1}(z)\frac{1}{j} + D_z\phi_m(0;z)D_z\phi_{m-1}(0;z)\frac{d}{dt}\bigg(\phi_{m-2}\Big(\frac{t}{j};\dots;\phi_1\Big(\frac{t}{j};z\Big)\dots\Big)\bigg)\bigg|_{t=0} = \\ &= \frac{1}{j}X_m(z) + \frac{1}{j}D_z\phi_m(0;z)X_{m-1}(z) + \dots + \frac{1}{j}D_z\phi_m(0;z)\cdots D_z\phi_2(0;z)X_1(z). \end{aligned}$$

Since each ϕ_k is of class $\mathcal{C}^1$ and $\phi_k(0;z) = z$, we have that $D_z\phi_k(0;z) = \mathrm{Id}$. Then our last formula tells us that

$$\frac{d}{dt}\bigg(\phi^1\Big(\frac{t}{j};z\Big)\bigg)\bigg|_{t=0} = \frac{1}{j}\sum_{k=1}^m X_k(z) = \frac{1}{j}X(z) = \frac{d}{dt}\bigg(\varphi\Big(\frac{t}{j};z\Big)\bigg)\bigg|_{t=0}$$

which implies that

$$\frac{d}{dt}(\phi^1 - \varphi)\Big(\frac{t}{j};z\Big)\bigg|_{t=0} = 0.$$

In particular, for a fixed $(t_0,z_0) \in \big[0,1\big] \times \mathbb{C}^n$, there exists a neighborhood U_0 of (t_0,z_0) satisfying $(\phi^1 - \varphi)\Big(t/j;z\Big) = o\Big(\big|t/j\big|\Big)$ uniformly on U_0. Let now L be a Lipschitz constant for X. By Gronwall's inequality (Lemma 2.4) we have

$$||\varphi(t;z) - \varphi(t;w)|| \leq e^{L|t|}||z-w||. \tag{3}$$

Now let us rewrite $\varphi(t;z) - \phi^j(t/j;z)$ as follows:

$$\begin{aligned}\varphi(t;z) - \phi^j\Big(\frac{t}{j};z\Big) &= \varphi^j\Big(\frac{t}{j};z\Big) - \phi^j\Big(\frac{t}{j};z\Big) = \\ &= \varphi^{j-1}\Big(\frac{t}{j};\varphi\Big(\frac{t}{j};z\Big)\Big) - \varphi^{j-1}\Big(\frac{t}{j};\phi^1\Big(\frac{t}{j};z\Big)\Big) + \\ &+ \varphi^{j-2}\Big(\frac{t}{j};\varphi\Big(\frac{t}{j};\phi^1\Big(\frac{t}{j};z\Big)\Big)\Big) - \varphi^{j-2}\Big(\frac{t}{j};\phi^2\Big(\frac{t}{j};z\Big)\Big) + \ldots + \\ &+ \varphi^{j-k}\Big(\frac{t}{j};\varphi\Big(\frac{t}{j};\phi^{k-1}\Big(\frac{t}{j};z\Big)\Big)\Big) - \varphi^{j-k}\Big(\frac{t}{j};\phi^k\Big(\frac{t}{j};z\Big)\Big) + \ldots + \\ &+ \varphi\Big(\frac{t}{j};\phi^{n-1}\Big(\frac{t}{j};z\Big)\Big) - \phi^1\Big(\frac{t}{j};\varphi^{n-1}\Big(\frac{t}{j};z\Big)\Big).\end{aligned}$$

Therefore using (3) we have

$$\begin{aligned}||\varphi(t;z) - \phi^j\Big(\frac{t}{j};z\Big)|| &\leq \sum_{k=1}^{j} ||\varphi^{j-k}\Big(\frac{t}{j};\varphi\Big(\frac{t}{j};\phi^{k-1}\Big(\frac{t}{j};z\Big)\Big)\Big) - \varphi^{j-k}\Big(\frac{t}{j};\phi^k\Big(\frac{t}{j};z\Big)\Big)|| \leq \\ &\leq \sum_{k=1}^{j} e^{L|t/j|(j-k)}||\varphi\Big(\frac{t}{j};\phi^{k-1}\Big(\frac{t}{j};z\Big)\Big) - \phi^1\Big(\frac{t}{j};\phi^{k-1}\Big(\frac{t}{j};z\Big)\Big)||.\end{aligned}$$

Finally, for all $(t,z) \in U_0$ we have

$$\begin{aligned}||\varphi(t;z) - \phi^j\Big(\frac{t}{j};z\Big)|| \leq \sum_{k=1}^{j} e^{L|t/j|(j-k)} o(t/j) &= \frac{e^{L|t/j|} - e^{L|t|}}{1 - e^{L|t/j|}} o(t/j) = \\ &= \frac{e^{L|t/j|} - e^{L|t|}}{j/t - j/te^{L|t/j|}} \frac{o(t/j)}{t/j} \underset{j\to\infty}{\longrightarrow} 0\end{aligned}$$

because $(e^{L|t/j|} - e^{L|t|})/(j/t - j/te^{L|t/j|}) \underset{j\to\infty}{\longrightarrow} -1$ uniformly on U_0. This finishes the proof. □

3 The Andersén-Lempert Theorem

This section is devoted to the proof of the Andersén-Lempert Theorem (the global and local versions). The Andersén-Lempert Theorem as we present it here is actually two theorems, one by Erik Andersén proved in 1990 and the other one by Erik Andersén and László Lempert proved in 1992. Instead of following their original proofs (which can be found in [2], and [3]) we follow an argument presented by J-P.Rosay in [11], a paper published in 1999.

3.1 The Global Andersén-Lempert Theorem

Let us state exactly what we want to prove.

Theorem 3.1 (Andersén-Lempert)**.** *Let $f \in \operatorname{Aut}(\mathbb{C}^n)$. Then there exists a sequence $(\psi_k)_k$ of finite composition of overshears such that ψ_k tends to f uniformly on compact sets of $\mathbb{C}^n$.*

Moreover, if f is volume-preserving (that is, $\det Df(z) = 1$ for all z), then the sequence $(\psi_k)_k$ can be chosen so that each ψ_k is a finite composition of shears.

The outline of the proof will be divided into several parts:

1st part:
First, we will connect f to the identity via a map $\varphi(t; z)$ in a way that $\varphi(0; z) = z$, $\varphi(1; z) = f(z)$, and φ is the solution to some vector field X (which will depend on f, and which will be time-dependent). So f will be the time-1 map of X. At this point, the idea is first to approximate in intervals of the form $\left[\frac{k}{m}, \frac{k+1}{m}\right]$ the vector field X with time-independent vector fields $Y_{k,m}$. This will imply that a solution of $Y_{k,m}$ will approximate the solution of X, φ, in $\left[\frac{k}{m}, \frac{k+1}{m}\right]$. Thus concatenating the solutions of $Y_{k,m}$ will yield an approximation to φ.

2nd part:
Secondly, we will approximate each vector field $Y_{k,m}$ with a vector field whose solutions are overshears. For that, we will first need to approximate $Y_{k,m}$ by a polynomial vector field and then decompose the polynomial vector field into a sum of complete vector fields Z_j satisfying that the time$-t$ maps of Z_j are overshears.

3rd part:
The third and final part is to compose in the variable t, the time$-t$ maps of the Z_j's, which will give us an approximation (this, will be a composition of overshears) to the solution of $Y_{k,m}$, thus an approximation to φ, and in particular setting $t = 1$ we will get an approximation to f.

Following the outline of the proof, we first give some lemmas that will serve us for the first part of the proof.

Lemma 3.2. Let $f \in \operatorname{Aut}(\mathbb{C}^n)$ with $f(0) = 0$ and $Df(0) = \mathrm{Id}$. Then $\varphi\colon \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\begin{cases} \varphi(t; z) = \dfrac{f(tz)}{t} & , \text{ if } t \neq 0 \\ \varphi(t; z) = z & , \text{ if } t = 0 \end{cases}$$

is a 1-parameter group of automorphisms of $\mathbb{C}^n$, analytic in t and z. Moreover φ satisfies the Cauchy problem

$$\begin{cases} \dfrac{d\varphi}{dt}(t; z) & = X\big(t, \varphi(t; z)\big) \\ \varphi(0; z) & = z, \end{cases} \tag{4}$$

where

$$X(t,z) = \frac{d\varphi}{dt}(t, \varphi(-t,z)),$$

thus in particular X is complete. If in addition $\det Df(z)$ is constant then $\mathrm{div}_z X = 0$.

Proof. To prove that $\varphi(t;z)$ is analytic on both t and z we will see that φ can be expressed as a power series.

Because f is an automorphism, we have that for every compact $K \subset \mathbb{C}^n$

$$f(z) = \Big(\sum_{|\alpha|\geq 0} c_\alpha^{(1)} z^\alpha, \dots, \sum_{|\alpha|\geq 0} c_\alpha^{(n)} z^\alpha \Big), \quad z \in K.$$

And since $f(0) = 0$ we have that $c_0^{(1)} = \ldots = c_0^{(n)} = 0$, therefore

$$f(z) = \Big(\sum_{|\alpha|\geq 1} c_\alpha^{(1)} z^\alpha, \dots, \sum_{|\alpha|\geq 1} c_\alpha^{(n)} z^\alpha \Big), \quad z \in K.$$

Now for $t \neq 0$

$$\frac{f(tz)}{t} = \Big(\sum_{|\alpha|\geq 1} c_\alpha^{(1)} t^{|\alpha|-1} z^\alpha, \dots, \sum_{|\alpha|\geq 1} c_\alpha^{(n)} t^{|\alpha|-1} z^\alpha \Big).$$

This series defines an analytic map on both t and z. Moreover since $Df(0) = \mathrm{Id}$, we have

$$\lim_{t\to 0} \frac{f(tz)}{t} = \Big(\sum_{|\alpha|=1} c_\alpha^{(1)} z^\alpha, \dots, \sum_{|\alpha|=1} c_\alpha^{(n)} z^\alpha \Big) = z.$$

Thus in the end, we have that φ is analytic on both t and z.

The fact that φ is bijective comes from f being bijective. And because f^{-1} is also entire, we get that the inverse of φ is also analytic in both t and z (for this it is enough to repeat the previous argument substituting f by f^{-1}). Therefore φ is a 1-parameter path of automorphisms. What makes φ into a 1-parameter group of automorphisms is the fact that it satisfies the Cauchy problem (4), which follows directly from the definition of X.

Finally observe that $D_z\varphi(t;z) = Df(z)$. Thus if $\det Df(z)$ is constant, then $\det D_z\varphi(t;z)$ is also constant. Which by Proposition 2.2 implies that $\mathrm{div}_z X = 0$. □

We now approximate the vector field X by time-independent vector fields.

Lemma 3.3. Let f and X be as in the previous lemma. For every integer $m \geq 1$, consider the vector field $Y_m \colon [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$Y_m(t,z) = \begin{cases} X(0,z) & \text{if } \ 0 \leq t < \frac{1}{m} \\ X\left(\frac{1}{m}, z\right) & \text{if } \ \frac{1}{m} \leq t < \frac{2}{m} \\ \vdots & \\ X\left(\frac{m-1}{m}, z\right) & \text{if } \ \frac{m-1}{m} \leq t \leq 1 \end{cases}$$

and let $y_m(t;z)$ be the so that

$$\begin{cases} \dfrac{dy_m}{dt}(t;z) &= Y_m\Big(t, y_m(t;z)\Big), \\ y_m(0;z) &= z \end{cases}$$

Then $(y_m)_m$ converges uniformly on compact sets of $[0,1]\times\mathbb{C}^n$ to φ the solution generated by X.
If in addition $\mathrm{div}_z X = 0$, then $\mathrm{div}_z Y_m = 0$.

Proof. We want to use Theorem 2.5 to the vector fields X and Y_m. But Y_m is not continuous, so we will have to restrict ourselves to intervals of the form $\left[\frac{k}{m}, \frac{k+1}{m}\right]$ with $0 \le k < m$.

First, fix $K \subset \mathbb{C}^n$ any compact set, and let $\varepsilon > 0$. Let

$$L = \sup_{(t,z)\neq(t,w)\in[0,1]\times K} \frac{||X(t,z)-X(t,w)||}{||z-w||},$$

and

$$M_n = \sup_{(t,z)\in[0,1]\times K} ||X(t,z)-Y_m(t,z)||.$$

Put $C = \max\left(e^L, \frac{e^L-1}{L}\right)$, observe that $C \ge 1$. Because X is continuous, it is absolutely continuous in $[0,1]\times K$. Then there exists $m_0 \in \mathbb{N}$ such that for every $m \ge m_0$, $M_m < \frac{\varepsilon}{2C}$. Let us now see that for each $0 \le k < m$ we have that $||\varphi(t;z) - y_m(t;z)|| < \varepsilon$ in $[\frac{k}{m}, \frac{k+1}{m}] \times K$. To that end we use induction on k.

For $k = 0$, because Y_m is continuous in $[0, \frac{1}{m}]$ and X is continuous and locally Lipschitz on z uniformly with respect to t (every $\mathcal{C}^1$ map is locally Lipschitz), we can use Theroem 2.5 to get that for every $(t,z) \in [0, \frac{1}{m}] \times K$

$$\begin{aligned} ||\varphi(t;z) - y_m(t;z)|| \le ||z-z||e^{Lt} + \frac{M_m}{L}\Big(e^{Lt}-1\Big) \le \frac{M_m}{L}\Big(e^{L}-1\Big) < \\ < \frac{e^L-1}{L}\frac{\varepsilon}{2C} \le \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Now suppose the result is true for k and let us prove it for $k+1$. By the previous argument, we can apply Theorem 2.5 to get that for every $(t,z) \in [\frac{k+1}{m}, \frac{k+2}{m}] \times K$ we have

$$\begin{aligned} ||\varphi(t;z) - y_m(t;z)|| &\le \left|\left|\varphi\Big(\frac{k+1}{m};z\Big) - y_m\Big(\frac{k+1}{m};z\Big)\right|\right| e^{Lt} + \frac{M_m}{L}\Big(e^{Lt}-1\Big) \le \\ &\le \left|\left|\varphi\Big(\frac{k+1}{m};z\Big) - y_m\Big(\frac{k+1}{m};z\Big)\right|\right| e^{L} + \frac{M_m}{L}\Big(e^{L}-1\Big) < \\ &< \frac{\varepsilon}{2C}e^L + \frac{e^L-1}{L}\frac{\varepsilon}{2C} \le \varepsilon. \end{aligned}$$

In the end, for any $m \ge m_0$ and any $0 \le k < m$ we have

$$||\varphi(t;z) - y_m(t;z)||_{[\frac{k}{m},\frac{k+1}{m}]\times K} < \varepsilon$$

thus in the end we have

$$||\varphi(t;z) - y_m(t;z)||_{[0,1]\times K} < \varepsilon.$$

That is, $(y_m)_m$ converges uniformly on compact sets of $[0,1]\times\mathbb{C}^n$ to φ.
The fact that Y_m has divergence 0 with respect to z when X does, comes directly from the definition of Y_m. ☐

Now we present the necessary results to get the second part of the proof of the Andersén-Lempert Theorem. Let us focus on an interval of the form $[\frac{k}{m}, \frac{k+1}{m}]$ so that Y_m is time-independent. For simplicity, we perform a dilation sending $[\frac{k}{m}, \frac{k+1}{m}]$ to $[0,1]$, and let us refer to Y_m simply by Y.

Observe that since Y is holomorphic, it can be approximated by a polynomial vector field uniformly on compact sets (because of Theorem A.8). Let us first see that the divergence of this second vector field is also zero.

Lemma 3.4. Let $Y\colon \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic vector field with $\mathrm{div} Y = 0$. Then there exists a polynomial vector field $Z\colon \mathbb{C}^n \to \mathbb{C}^n$ approximating Y uniformly on compact sets such that $\mathrm{div} Z = 0$.

Proof. Fix $K \subset \mathbb{C}^n$ compact and $\varepsilon > 0$. To the vector field $Y(z) = (y_1(z), \ldots, y_n(z))$ assign the holomorphic $(n-1,0)-$form

$$\omega(z) = \sum_{k=1}^{n} (-1)^{k-1} Y_k(z) dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n.$$

Observe that the exterior derivative of ω is

$$\begin{aligned} dw &= \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial Y_k}{\partial z_k} dz_k \wedge dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n = \\ &= \sum_{k=1}^{n} \frac{\partial Y_k}{\partial z_k} dz_1 \wedge \ldots \wedge dz_n = \mathrm{div} Y dz_1 \wedge \ldots \wedge dz_n = 0, \end{aligned}$$

because $\mathrm{div} Y = 0$. Thus because $\mathbb{C}^n$ is simply connected there exists a holomorphic $(n-2,0)-$form τ such that $\omega = d\tau$. Now because τ is holomorphic we can find a polynomial $(n-2,0)-$form σ approximating τ uniformly on compact sets. That is, if τ and σ are given by

$$\tau = \sum_{1 \le i < j \le n} \tau_{ij} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n$$

and

$$\sigma = \sum_{1 \le i < j \le n} \sigma_{ij} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n,$$

then there exists $\delta > 0$ such that for every $z, z' \in K$, if $||z - z'|| < \delta$, we have that $|\tau_{ij}(z) - \sigma_{ij}(z)| < \varepsilon$. Calculating the exterior derivative of both τ and σ yields

$$\begin{aligned} \omega = d\tau = &\sum_{1 \le i < j \le n} (-1)^{i-1} \frac{\partial \tau_{ij}}{\partial z_i} dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n + \\ &+ \sum_{1 \le i < j \le n} (-1)^{j-2} \frac{\partial \tau_{ij}}{\partial z_j} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n \end{aligned}$$

and

$$\begin{aligned} d\sigma = &\sum_{1 \le i < j \le n} (-1)^{i-1} \frac{\partial \sigma_{ij}}{\partial z_i} dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n + \\ &+ \sum_{1 \le i < j \le n} (-1)^{j-2} \frac{\partial \sigma_{ij}}{\partial z_j} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n. \end{aligned}$$

Let us now see that $d\sigma$ approximates $d\tau$ uniformly on K. For that, we use the Cauchy inequalities. Since K is compact, it can be covered with finitely many polydisks with radius $r < \delta$ (that is $r_1, \ldots, r_n < \delta$). Then it is enough to see that $d\sigma$ approximates $d\tau$ uniformly on a polydisk $\Delta^n(w, r) \subset K$, with $r < \delta$. By the Cauchy inequalities we have

$$\left| \frac{\partial \tau_{ij}}{\partial z_k}(z) - \frac{\partial \sigma_{ij}}{\partial z_k}(z) \right| \leq \frac{1}{r} \sup_{z \in \overline{\Delta^n(w,r)}} |\tau_{ij}(z) - \sigma_{ij}(z)| < \frac{\varepsilon}{r}, \quad \forall z \in \overline{\Delta^n(w, r)}.$$

Therefore we have that $d\sigma$ approximates $d\tau$ uniformly on K. Finally, rewrite $d\sigma$ as

$$d\sigma = \sum_{k=1}^{n} (-1)^{k-1} Z_k dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n,$$

and let $Z(z) = \big(Z_1(z), \ldots, Z_n(z)\big)$ be a holomorphic polynomial vector field. Because Z is assigned to $d\sigma$ the same way that Y is assigned to $w = d\tau$, and $d\sigma$ approximates $d\tau$, we have that Z approximates Y uniformly on K. Moreover

$$0 = d^2\sigma = \sum_{k=1}^{n} \frac{\partial Z_k}{\partial z_k} dz_1 \wedge \ldots \wedge dz_n = \operatorname{div} Z dz_1 \wedge \ldots \wedge dz_n.$$

In the end Z is a holomorphic polynomial vector field approximating Y uniformly on compact sets with $\operatorname{div} Z = 0$, just as we wanted. □

The next step is to prove that any polynomial holomorphic vector field Z can be decomposed into a sum of complete holomorphic vector fields, whose solutions are overshears, or shears if $\operatorname{div} Z = 0$. To prove that fact, we need the three following lemmas.

Lemma 3.5. Let $P \in \mathbb{C}[z]$ be a polynomial of degree $d \geq 0$. Then $\operatorname{span}\big(P(z), P(z-1), \ldots, P(z-d)\big) = \mathbb{C}_{\leq d}[z]$. That is, $P(z), P(z-1), \ldots, P(z-d)$ span the space of polynomials of one variable of degree at most d with complex coefficients.

Proof. We use induction on d. The case $d = 0$ it is clear, because P is a constant, and thus $\operatorname{span}\big(P(z)\big) = \mathbb{C} = \mathbb{C}_0[z]$.
Suppose now the result is true for d, and let us prove it for $d+1$. Let then $P \in \mathbb{C}_{\leq d+1}[z]$, and define $Q(z) = P(z) - P(z-1)$. Because Q has degree d, by the induction hypothesis $\operatorname{span}\big(Q(z), \ldots, Q(z-d)\big) = \mathbb{C}_{\leq d}[z]$. Therefore $\operatorname{span}\big(P(z), Q(z), \ldots, Q(z-d)\big) = \operatorname{span}\big(P(z), P(z-1), \ldots, P(z-d-1)\big) = \mathbb{C}_{\leq d+1}[z]$. □

Lemma 3.6. Let $P \colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Then there exist $l_1, \ldots, l_r$ linear forms on $\mathbb{C}^n$ and $P_1, \ldots, P_r \colon \mathbb{C} \to \mathbb{C}$ polynomials such that

$$P(z) = \sum_{k=1}^{r} P_k\big(l_k(z)\big).$$

Proof. Let $P \colon \mathbb{C}^n \to \mathbb{C}$ be the polynomial given by

$$P(z) = \sum_{|\alpha| \leq N} c_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $c_\alpha = c_{\alpha_1, \ldots, \alpha_n}$, and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Thus we can write P as

$$P(z) = \sum_{\alpha_1 + \ldots + \alpha_n \leq N} c_{\alpha_1, \ldots, \alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

thus it is enough to prove the lemma for expressions of the form $z_1^{\alpha_1}\cdots z_n^{\alpha_n}$. Let us prove it by induction on n. First consider the case $n=2$, that is we want to prove it for $z_1^{\alpha_1}z_2^{\alpha_2}$. Put $m\geq \alpha_1,\alpha_2$, which we will fix later. By Lemma 3.5 applied to $z_1^{\alpha_1}$ and $d=m$ we know there exist some numbers c_j with $0\leq j\leq m$ such that

$$z_1^{\alpha_1}=\sum_{j=0}^{m} c_j(z_1-j)^m. \tag{5}$$

Replacing in (5) z_1 by z_1/z_2 yields

$$z_1^{\alpha_1}z_2^{m-\alpha_1}=\sum_{j=0}^{m} c_j(z_1-jz_2)^m.$$

Finally, by putting $m=\alpha_1+\alpha_2$ we get what we want.

Now suppose we have the result for $n\geq 2$ and let us prove it for $n+1$. By the induction hypothesis we have

$$\begin{aligned} z_1^{\alpha_1}\cdots z_{n+1}^{\alpha_{n+1}} = (z_1^{\alpha_1}\cdots z_n^{\alpha_n})z_{n+1}^{\alpha_{n+1}} = \sum_{k=1}^{m} P_k\big(l_k(z_1,\ldots,z_n)\big)z_{n+1}^{\alpha_{n+1}} = \\ = \sum_{k=1}^{m}\sum_{j=0}^{N_k} c_{k,j}\big(l_k(z_1,\ldots,z_n)\big)^j z_{n+1}^{\alpha_{n+1}}, \end{aligned}$$

where $P_k(x)=\sum_{j=0}^{N_k} c_{k,j}x^j$. Applying again the case $n=2$ now to each $\big(l_k(z_1,\ldots,z_n)\big)^j z_{n+1}^{\alpha_{n+1}}$ we get

$$\big(l_k(z_1,\ldots,z_n)\big)^j z_{n+1}^{\alpha_{n+1}} = \sum_{r=1}^{\widetilde{N}_j} Q_r\big(s_r(l_k(z_1,\ldots,z_n),z_{n+1})\big) = \sum_{r=1}^{\widetilde{N}_j} Q_r\big(\widetilde{s}_{r,k}(z_1,\ldots,z_{n+1})\big).$$

Putting this together with our last formula yields

$$z_1^{\alpha_1}\cdots z_{n+1}^{\alpha_{n+1}} = \sum_{k=1}^{m}\sum_{j=0}^{N_k}\sum_{r=1}^{\widetilde{N}_j} c_{k,j}Q_r\big(\widetilde{s}_{r,k}(z_1m\ldots,z_{n+1})\big)$$

which is an expression of the form that we wanted. □

Lemma 3.7. For every polynomial $P\colon \mathbb{C}^n\to\mathbb{C}$, there exist complete polynomial holomorphic vector fields $X_1,\ldots,X_m$ such that

$$\operatorname{div}\Bigg(\sum_{k=1}^{m} X_k(z)\Bigg) = P(z)$$

Proof. By Lemma 3.6 it is enough to prove it when $P(z)=P_1\big(l(z)\big)$, where P_1 is a polynomial in one variable. Indeed, assume that we have proved it for polynomials of the form $P_1\big(l(z)\big)$. Let P be any polynomial in n variables. Then by Lemma 3.6 we can express P as

$$P(z)=\sum_{k=1}^{m} P_k\big(l_k(z)\big).$$

Now, because we are assuming the result to hold for $P_k\big(l_k(z)\big)$, for each $1 \leq k \leq m$ there exist $X_{k1}, \ldots, X_{km_k}$ complete polynomial holomorphic vector fields satisfying

$$\operatorname{div}\Big(\sum_{j=1}^{m_k} X_{kj}(z)\Big) = P_k\big(l_k(z)\big).$$

Then

$$\operatorname{div}\Big(\sum_{k=1}^{m}\sum_{j=1}^{m_k} X_{kj}(z)\Big) = \sum_{k=1}^{m}\operatorname{div}\Big(\sum_{j=1}^{m_k} X_{kj}(z)\Big) = \sum_{k=1}^{m} P_k\big(l_k(z)\big) = P(z).$$

Thus if we are able to prove the lemma for a polynomial of the form $P_1\big(l(z)\big)$ we will have completed the proof. Let us then focus on proving the result for $P_1\big(l(z)\big)$. Rearranging the variables if necessary, we can suppose that $l(1,0,\ldots,0) \neq 0$. Then performing the change of variables

$$\begin{cases} w_1 = l(z) \\ w_j = z_j \quad \text{for } 2 \leq j \leq n \end{cases}$$

we have that $P_1\big(l(z)\big) = P_1(w_1)$. Define the vector field $X(w) = (0, w_2P_1(w_1), 0, \ldots, 0)$ which satisfies

$$\operatorname{div}X(w) = P_1(w_1) = P(z)$$

and is complete because it has flow

$$\varphi(t; w) = (w_1, w_2e^{tP_1(w_1)}, w_3, \ldots, w_n)$$

which clearly is defined for all $t \in \mathbb{R}$ and all $w \in \mathbb{C}^n$. Because our change of variables is linear, we have that $X(z)$ will be a complete polynomial vector field with $\operatorname{div}X = 0$. □

Remark 3.8. Observe that the time$-t$ maps of X in the previous lemma, are overshears. This will be useful when proving the Andersén-Lempert Theorem.

We finally give a proof that every polynomial vector field can be decomposed into a sum of complete polynomial vector fields.

Theorem 3.9 (Decomposition of polynomial vector fields)**.** *Let X be a polynomial holomorphic vector field. Then there exist $X_1, \ldots, X_k$ complete holomorphic vector fields such that*

$$X = \sum_{j=1}^{k} X_j.$$

and the time $t-$maps of each X_j are overshears. If in addition $\operatorname{div}X = 0$*, then every X_j can be chosen so that* $\operatorname{div}X_j = 0$*, and the time$-t$ maps of each X_j are shears.*

To prove this theorem, first, we will see that with the help of the previous lemmas it is enough to consider the case where $\operatorname{div}X = 0$. Then to X we will attatch a $(n-1,0)-$form ω, as we did in Lemma 3.4. Then we will consider τ a $(n-2,0)-$form such that $d\tau = \omega$. Using Lemma 3.6 we will get a decomposition of τ, which will induce a decomposition of ω, $\omega = \sum_{j=1}^{m}\omega_j$. Finally our vector fields X_j will be those attached to each ω_j.

Proof. Let us first see that it is enough to see the case when $\mathrm{div}X = 0$.

Indeed, supose we have proved the theorem when $\mathrm{div}X = 0$, and let X be any polynomial holomorphic vector field. Then $\mathrm{div}X$ is also a polynomial, and by Lemma 3.7 there exist $X_1, \ldots, X_m$ complete polynomial holomorphic vector fields satisfying

$$\mathrm{div}\Big(\sum_{k=1}^{m} X_k\Big) = \mathrm{div}(X).$$

Thus the vector field $X - \sum_{k=1}^{m} X_k$ has divergence zero. By our assumption there exist complete holomorphic vector fields $X_{m+1}, \ldots, X_{m'}$ satisfying

$$X - \sum_{k=1}^{m} X_k = \sum_{k=m+1}^{m'} X_k$$

which implies

$$X = \sum_{k=1}^{m'} X_k.$$

That is, X is a sum of complete holomorphic vector fields.

We now prove the case where $\mathrm{div}X = 0$. Denote $X = (X_1, \ldots, X_n)$, and consider the holomorphic $(n-1,0)-$form ω defined by

$$\omega = \sum_{k=1}^{n} (-1)^{k+1} X_k dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n.$$

Since $d\omega = \mathrm{div}X dz_1 \wedge \ldots \wedge dz_n = 0$, and $\mathbb{C}^n$ is simply-connected there exists a holomorphic $(n-2,0)-$form τ with $d\tau = \omega$. Because ω is polynomial we can also take τ to be polynomial. Put

$$\tau(z) = \sum_{1 \le i < j \le n} \tau_{ij}(z) dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n,$$

where τ_{ij} are polynomials. Now to each τ_{ij} apply Lemma 3.6 to get

$$\tau_{ij}(z) = \sum_{r=1}^{m_{ij}} P_{ijr}\big(l_{ijr}(z)\big)$$

with P_{ijr} polynomials in one variable and l_{ijr} linear forms. Now let ω_{ijr} be the $(n-1,0)-$form defined by

$$\omega_{ijr} = d\Big(P_{ijr}\big(l_{ijr}(z)\big) dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n\Big).$$

More precisely put

$$\begin{aligned}\omega_{ijr} =& (-1)^{i-1} P'_{ijr}\big(l_{ijr}(z)\big) c_{ijr}^{(i)} dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n + \\ &+ (-1)^{j} P'_{ijr}\big(l_{ijr}(z)\big) c_{ijr}^{(j)} dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n.\end{aligned}$$

To ω_{ijr} associate the vector field Y_{ijr} defined as

$$Y_{ijr}(z) = \Big(0, \ldots, 0, \overbrace{(-1)^{i+j} Q'_{ijr}\big(l_{ijr}(z)\big) c_{ijr}^{(j)}}^{i}, 0, \ldots, 0, \overbrace{(-1)^{i+j-1} Q'_{ijr}\big(l_{ijr}(z)\big) c_{ijr}^{(i)}}^{j}, 0, \ldots, 0\Big).$$

As we have seen in other proofs we have $\operatorname{div}Y_{ijr}dz_1\wedge\ldots\wedge dz_n = d\omega_{ijr} = 0$. And because ω_{ijr} is a decomposition of ω we have

$$X(z) = \sum_{1\leq i<j\leq n}\sum_{r=1}^{m_{ij}} Y_{ijr}.$$

If we see that each Y_{ijr} is complete and their time$-t$ maps are shears we will be done. To that end, consider the linear change of variables

$$w = Az,$$

where $w_1 = l_{ijr}(z)$ and A is a unitary matrix, that is, we complete A so that the rows form an orthonormal system (if needed modify P_{ijr} so that the vector defining l_{ijr} has norm 1). In these new coordinates we have

$$\begin{aligned}&P_{ijr}\big(l_{ijr}(z)\big)dz_1\wedge\ldots\wedge\widehat{dz_i}\wedge\ldots\wedge\widehat{dz_j}\wedge\ldots\wedge dz_n =\\ &=\sum_{1\leq l<k\leq n}\lambda_{lk}P_{ijr}(w_1)dw_1\wedge\ldots\wedge\widehat{dw_l}\wedge\ldots\wedge\widehat{dw_k}\wedge\ldots\wedge dw_n\end{aligned}$$

which implies that

$$\omega_{ijr} = P'_{ijr}(w_1)\left(\sum_{k=2}^{n}\lambda_{1k}dw_1\wedge\ldots\wedge\widehat{dw_k}\wedge\ldots\wedge dw_n\right).$$

Therefore the vector field Y_{ijr} in the w coordinates is

$$Y_{ijr}(z) = \Big(0, -P'_{ijr}(w_1)\lambda_{12},\ldots,(-1)^{n-1}P'_{ijr}(w_1)\lambda_{1n}\Big)$$

and has as flow

$$\varphi_{ijr}(t;w) = \big(w_1, w_2 - tP'_{ijr}(w_1)\lambda_{12},\ldots,w_n + (-1)^{n-1}P'_{ijr}(w_1)\lambda_{1n}\big),$$

which for every t, it is a shear. Since completeness is invariant under a change of variables, we get that each Y_{ijr} is complete, and thus X is a sum of complete holomorphic vector fields with $\operatorname{div}Y_{ijr} = 0$. □

Finally, we have all the necessary ingredients to give a proof of the Andersén-Lempert Theorem.

3.1.1 Proof of the global Andersén-Lempert Theorem

Let $f\colon\mathbb{C}^n\to\mathbb{C}^n$ be an automorphism of $\mathbb{C}^n$. We first prove the general case.

Fix any compact $K\subset\mathbb{C}^n$ and a $\varepsilon>0$. Observe that any linear map (and translation) can be written as a composition of overshears. Then we can suppose that $f(0)=0$ and $Df(0)=\mathrm{Id}$ (if needed we rewrite f as $f(z) = Df(0)^{-1}\big(f(z)-f(0)\big)$. Then by Lemma 3.2 $\varphi(t;z) = \frac{f(tz)}{t}$ if $t\neq 0$ and $\varphi(0;z) = z$ is a 1-parameter group of automorphisms of $\mathbb{C}^n$. Moreover φ can be viewed as the flow of the time-dependent vector field X. Let now Y_m be the vector field described in Lemma 3.3 and y_m its flow. Because y_m approximates φ uniformly on compact sets we have that there exists $m_0\in\mathbb{N}$ such that for all $m\geq m_0$

$$||\varphi - y_m||_{[0,1]\times K} < \frac{\varepsilon}{3}.$$

Because Y_m is a time-independent vector field in the intervals of the form $\big[k/m,(k+1)/m\big]$, we can approximate Y_m uniformly on $\big[k/m,(k+1)/m\big]\times K$ by a polynomial vector field $Z^{(k)}$ whose flow $\phi_{(k)}$ satisfies

$$||y_m-\phi_{(k)}||_{[k/m,(k+1)/m]\times K}<\frac{\varepsilon}{3}.$$

Lastly, by the Decomposition Theorem (Theorem 3.9) there exist $Z_1^{(k)},\dots,Z_M^{(k)}$ complete holomorphic vector fields satisfying

$$Z^{(k)}=\sum_{j=1}^{m}Z_j^{(k)}.$$

Let $\psi_{(k)}^j$ represent the composition of the flows of each $Z_j^{(k)}$, as in Theorem 2.6. Then by Theorem 2.6 we have that there exists $j_k\in\mathbb{N}$ such that for all $j\geq j_k$ we have that for all $(t,z)\in\big[\frac{k}{m},\frac{k+1}{m}\big]\times K$ we have

$$\Big|\Big|\phi_{(k)}(t;z)-\psi_{(k)}^j\Big(\frac{t}{j};z\Big)\Big|\Big|<\frac{\varepsilon}{3}.$$

Therefore we have that

$$\begin{aligned}\Big|\Big|\varphi-\phi_{(k)}^j\Big|\Big|_{[k/m,(k+1)/m]\times K}&\leq||\varphi-y_m||_{[k/m,(k+1)/m]\times K}+||y_m-\phi_{(k)}||_{[k/m,(k+1)/m]\times K}+\\&+\Big|\Big|\phi_{(k)}-\psi_{(k)}^j\Big|\Big|_{[k/m,(k+1)/m]\times K}<\varepsilon.\end{aligned}$$

Putting it all together, for any fixed $t\in\big[0,1\big]$, $\psi_{(k)}^j$ (whith k depending on t) will be a uniform approximation in K of $\varphi(t;\cdot)$. Finally, because of the Decomposition Theorem, we know that the flows of each $Z_j^{(k)}$ are overshears, we have that for a fixed t, $\psi_{(k)}^j(t/j;z)$ is a composition of overshears. Putting $t=1$ gives us that $\varphi(1;z)=f(z)$ is approximated by a composition of overshears, just as we wanted.

For the case where f is a volume-preserving automorphism, we have (as stated by each lemma and theorem), that the approximations can be carried out so that all the vector fields have zero divergence. By the Decomposition Theorem (Theorem 3.9), this implies that the time t-maps of each $Z_j^{(k)}$ are shears. Thus $\psi_{(k)}^j$ is a composition of shears, and therefore f can be approximated uniformly on compact sets by a composition of shears. This concludes the proof. □

3.1.2 Not all automorphisms are compositions of overshears/shears

At this point, having seen that automorphisms of $\mathbb{C}^n$ are approximable by overshears it is natural to ask ourselves if all automorphisms are compositions of overshears (or compositions of shears). The answer (as the title of this section hints) is no. There are some automorphisms that are not compositions of shears nor overshears. The first known example was provided by Erik Andersén in his paper [2], and it is the following map

$$f(z_1,z_2)=(z_1e^{z_1z_2},z_2e^{-z_1z_2}).$$

This example only covers the case of $n=2$ and where we are considering approximation by shears. It was later proved that any map of the form

$$f(z_1,z_2)=(z_1e^{\phi(z_1z_2)},z_2e^{\phi(z_1z_2)})$$

where $\phi\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and non-constant, is not a composition of shears.

Later Erik Andersén and Lázló Lempert proved in [3], that there exists a volume-preserving map $f \in \mathrm{Aut}(\mathbb{C}^n)$ that is not a composition of shears, and there also exists a map $f \in \mathrm{Aut}(\mathbb{C}^n)$ that is not a composition of overshears.
The proofs of these two results were not constructive. That is the authors were not able to find an explicit automorphism which is not a composition of shears/overshears. To this day (that we know of) there have not been found explicit automorphisms not being a finite composition of shears/overshears.

3.2 The Local Andersén-Lempert Theorem

Having seen the Global Andersén-Lempert Theorem, we now move to state and prove the local version. To do that it is useful to consider the following definition.

Definition 3.10. Let $\Omega \subset \mathbb{C}^n$ be open. We say that $H\colon \big[0,1\big] \times \Omega \to \mathbb{C}^n$ is an *isotopy of biholomorphisms of* Ω (we will also refer to H just as an isotopy) if

i) H is of class $\mathcal{C}^2$ on $\big[0,1\big] \times \Omega$

ii) For each $t \in \big[0,1\big]$ the map $H(t,\cdot)\colon \Omega \to \mathbb{C}^n$ is a biholomorphism from Ω into $\mathbb{C}^n$.

For each $t \in \big[0,1\big]$, we will put Ω_t to indicate the range of $H(t,\cdot)$.

The concept of isotopy will replace the role of φ on the proof of the Global Andersén-Lempert Theorem.
It will be important to refer to the inverse of H for a fixed t, i.e. the inverse of the map $H(t,\cdot)$. We will denote this inverse as $H^{-1}(t,\cdot)$. So $H^{-1}(t,z)$ will be the map $H^{-1}(t,\cdot)$ evaluated at z.
We now state the Local Andersén-Lempert Theorem.

Theorem 3.11 (Local Andersén-Lempert Theorem)**.** *Let $\Omega \subset \mathbb{C}^n$ be an open set, and $H\colon \big[0,1\big] \times \Omega \to \mathbb{C}^n$ be an isotopy of biholomorphisms such that each Ω_t is Runge in $\mathbb{C}^n$. Then if $H(0,\cdot)$ can be approximated by automorphisms of $\mathbb{C}^n$ uniformly on compact sets of Ω, then for every $t \in \big[0,1\big]$ the map $H(t,\cdot)$ can also be approximated by automorphisms of $\mathbb{C}^n$ uniformly on compact sets of Ω.*

Observe that one of the main differences between this theorem and the Global version is that we start with a biholomorphism $H(0,\cdot)$ which is defined only in an open set Ω instead of the whole $\mathbb{C}^n$. That is why it is used the term "Local". The second main difference is that we require the sets Ω_t to be Runge in $\mathbb{C}^n$, the reason for doing this will be clear when discussing the outline of the proof.

The outline of the proof is very similar to the one of the Global Andersén-Lempert Theorem. We mainly focus on the parts that are different.

1st part: We will begin by noting that we can reduce ourselves to the case where $H(0,\cdot) = \mathrm{Id}_\Omega$. Then, using H we will construct a time-dependent vector field X with domain $\big[0,1\big] \times \Omega_t$, whose flow will be H. The rest of this part is to approximate X with a vector field Y_m that is time-independent on intervals of the form $[k/m,(k+1)/m]$.

2nd part: Focusing on an interval $[k/m,(k+1)/m]$, Y_m will be time-independent. Then using that each Ω_t is Runge, we will be able to approximate Y_m by a polynomial vector field Z.

Using the Decomposition Theorem we will decompose Z into a sum of complete holomorphic vector fields whose solutions are automorphisms of $\mathbb{C}^n$.

3rd part: Finally, we will compose the different time-t maps resulting from the previous decomposition (which will yield a composition of automorphisms of $\mathbb{C}^n$). This composition will end up approximating $H(t,\cdot)$.

Proof. It is enough to see the result when $H(0,\cdot) = \mathrm{Id}_\Omega$.

Indeed, suppose we have proved the theorem when $H(0,\cdot) = \mathrm{Id}_\Omega$, and let H be an isotopy which at time $t = 0$ is not the identity. Let $K \subset \Omega$ be compact and $\varepsilon > 0$.

We have that the map $\widetilde{H}(t,z) \colon \big[0,1\big] \times \Omega_0$ given by

$$\widetilde{H}(t,z) = H\big(t, H^{-1}(0,z)\big)$$

is an isotopy of biholomorphisms of Ω_0 such that $\widetilde{H}(0,z) = z$. Observe also that the range of $\widetilde{H}(t,\cdot)$ is also Ω_t. Then by our assumption, for any $t \in \big[0,1\big]$, since $K_0 = H(0,K)$ is compact, there exists $\psi \in \mathrm{Aut}(\mathbb{C}^n)$ satisfying that for every $z \in K_0$

$$||\widetilde{H}(t,z) - \psi(z)|| < \frac{\varepsilon}{2},$$

which implies that for every $z \in K$

$$||H(t,z) - \psi\big(H(0,z)\big)|| < \frac{\varepsilon}{2}.$$

Now by hypothesis, for every $\delta > 0$ there exists $\Psi \in \mathrm{Aut}(\mathbb{C}^n)$ such that for every $z \in K$

$$||H(0,z) - \Psi(z)|| < \delta,$$

and because ψ is uniformly continuous on K, we can choose δ small enough so that for all $z \in K$

$$||\psi\big(H(0,z)\big) - \psi\big(\Psi(z)\big)|| < \frac{\varepsilon}{2}.$$

Putting it all together yields that for every $z \in K$

$$||H(t,z) - \psi\big(\Psi(z)\big)|| \leq ||H(t,z) - \psi\big(H(0,z)\big)|| + ||\psi\big(H(0,z)\big) - \psi(\Psi(z))|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ψ and Ψ are both automorphisms $\psi \circ \Psi$ is also an automorphism, thus $H(t,z)$ can be approximated uniformly on compact sets of Ω by automorphisms.

Let us now prove the case when $H(0,\cdot) = \mathrm{Id}_\Omega$. Let $K \subset \Omega$ be compact and $\varepsilon > 0$. Define the time-dependent vector field $X \colon [0,1] \times \Omega_t$ as

$$X(t,z) = \frac{dH}{dt}\big(t, H^{-1}(t,z)\big).$$

Observe that its flow is given by H because

$$X\big(t, H(t,z)\big) = \frac{dH}{dt}\Big(t, H^{-1}\big(t, H(t,z)\big)\Big) = \frac{dH}{dt}(t,z),$$

and $H(0,z) = z$. Moreover, X is holomorphic in z, and it is of class $\mathcal{C}^1$ on both t and z because H is of class $\mathcal{C}^2$. Thus X is locally Lipschitz on the second variable uniformly with respect to the first one. Let Y_m denote the vector field defined on the proof of the Global Andersén-Lempert Theorem (replacing φ with H). We already know that y_m, the flow of Y_m, for a large enough m satisfies that

$$||H - y_m||_{[0,1]\times K} < \frac{\varepsilon}{3}.$$

Let us focus now on an interval of the form $[k/m, (k+1)/m]$ (where Y_m is time-independent).

Because each Ω_t is Runge, and Y_m is holomorphic, we can approximate Y_m by a polynomial holomorphic vector field Z, whose flow $\phi_{(k)}$ satisfies

$$||y_m - \phi_{(k)}||_{[k/m,(k+1)/m]\times K} < \frac{\varepsilon}{3}.$$

At this point the proof carries out exactly as in the Global Andersén-Lempert Theorem. Thus (keeping the same notation) we can find $\psi^j_{(k)}$ such that for all $(t, z) \in \left[\frac{k}{m}, \frac{k+1}{m}\right] \times K$ we have

$$\left|\left|\phi_{(k)}(t;z) - \psi^j_{(k)}\left(\frac{t}{j};z\right)\right|\right| < \frac{\varepsilon}{3}.$$

Moreover for a fixed t, $\psi^j_{(k)}(t/j;\cdot)$ is a composition of automorphisms of $\mathbb{C}^n$. In the end we have that

$$\left|\left|H - \phi^j_{(k)}\right|\right|_{[k/m,(k+1)/m]\times K} \leq ||H - y_m||_{[k/m,(k+1)/m]\times K} + ||y_m - \phi_{(k)}||_{[k/m,(k+1)/m]\times K} + \\ + \left|\left|\phi_{(k)} - \psi^j_{(k)}\right|\right|_{[k/m,(k+1)/m]\times K} < \varepsilon.$$

Therefore, for any fixed $t \in \left[0, 1\right]$, $H(t, \cdot)$ can be uniformly approximated in K by an automorphism of $\mathbb{C}^n$. This ends the proof of the theorem. □

Remark 3.12. In practice, if we have $f : \Omega \to \mathbb{C}^n$ a biholomorphism with $f(\Omega)$ Runge. It will be enough to see that Ω is star-shaped to deduce that f can be approximated by automorphisms of $\mathbb{C}^n$ uniformly on compact sets of Ω.
Indeed, suppose Ω is star-shaped with respect to z_0. Then the map $F : \left(\Omega - z_0\right) \to \mathbb{C}^n$ defined by

$$F(z) = Df(z_0)^{-1}\big(f(z + z_0) - f(z_0)\big)$$

is a biholomorphism satifying $F(0) = 0$ and $DF(0) = \mathrm{Id}$. Thus $H : \left[0, 1\right] \times \left(\Omega - z_0\right) \to \mathbb{C}^n$ defined by

$$\begin{cases} H(t,z) = \dfrac{F(tz)}{t} & \text{, if } t \neq 0 \\ H(t,z) = z & \text{, if } t = 0 \end{cases}$$

is well defined and also, because of Lemma 3.2, an isotopy. Moreover, since linear maps and translations preserve the Runge property, we have that $\left(\Omega - z_0\right)_t$ is Runge for all $t \in \left[0, 1\right]$. Therefore we can apply the Local Andersén-Lempert Theorem to see there is a sequence $(\psi_k)_k \subset \mathrm{Aut}(\mathbb{C}^n)$ such that ψ_k converges uniformly on compact sets to $H(1, z) = F(z)$. Thus the sequence defined by $Df(z_0)\psi_k(z - z_0) + f(z_0)$ will converge uniformly on compact sets of Ω to f.

4 Applications of the Andersén-Lempert Theorem

In this section, we give, as an illustration, three major theorems that follow from the Andersén-Lempert Theorem.

4.1 Approximation near polynomially convex sets

What we will prove in this section, in loose terms, is that given a finite collection of compact star-shaped sets $K_1, \dots, K_m$, and an automorphism ϕ_k for each K_k. Then, under some conditions there exists a sequence $(\psi_j)_j \subset \mathrm{Aut}(\mathbb{C}^n)$ such that ψ_j approximates simultaneously each ϕ_k uniformly on a neighborhood of K_k. Notice that we are approximating different maps by just one map!

In more concrete terms, we will prove the following:

Theorem 4.1. *Let $K_1, \dots, K_m$ be pairwise disjoint compact star-shaped domains of $\mathbb{C}^n$. Let for each $1 \leq k \leq m$ $\phi_k \in \mathrm{Aut}(\mathbb{C}^n)$. If the sets $\phi_k(K_k)$ are pairwise disjoint, and $K = \bigcup_{k=1}^m K_k$ and $K' = \bigcup_{k=1}^m \phi_k(K_k)$ are polynomially convex, then for each $1 \leq k \leq m$ there exists U_k a neighborhood of K_k and a sequence $(\psi_j)_j \subset \mathrm{Aut}(\mathbb{C}^n)$ such that ψ_j converges to ϕ_k uniformly on U_k, for each $1 \leq k \leq m$.*

The idea of the proof is first to see an auxiliary result saying that if we have an isotopy of biholomorphisms H of $\Omega \subset \mathbb{C}^n$, and K a polynomially convex set, then approximating H (for a fixed t) on a neighborhood of K is equivalent to checking that for each t, $K_t = H(t, K)$ is polynomially convex.
Therefore to prove Theorem 4.1, we will only need to construct an isotopy H where every $H(t, K)$ is polynomially convex.

But all of this will be pointless unless we have a condition to check whether the union of the compact polynomially convex sets $K_1, \dots, K_m$ is polynomially convex. That is exactly what Eva Kallin did in her revolutionary paper [12] in 1964. Before proving Theorem 4.1 we present her findings.

4.1.1 The Separation Lemma and the three-sphere problem

As we have already mentioned, Eva Kallin gave in her paper [12] a condition on two compact polynomially convex sets to ensure that their union is polynomially convex. For the proofs of the following results, we have followed her paper [12].

Theorem 4.2 (Separation Lemma)**.** *Let K_1, K_2 be two compact sets in $\mathbb{C}^n$. If f is a polynomial satisfying $\widehat{f(K_1)} \cap \widehat{f(K_2)} = \emptyset$, then $(\widehat{K_1 \cup K_2}) = \widehat{K}_1 \cup \widehat{K}_2$.*

Observe that because both $f(K_1)$ and $f(K_2)$ are compact sets in $\mathbb{C}$, the condition of f being such that $\widehat{f(K_1)} \cap \widehat{f(K_2)} = \emptyset$ just translates to $f(K_1)$ not surrounding $f(K_2)$ and vice versa.

The idea of the proof is to use Runge's approximation theorem to construct polynomials satisfying $||h||_{\widehat{K}_1 \cup \widehat{K}_2} < |h(a)|$ for some $a \in \mathbb{C}^n \setminus (\widehat{K_1 \cup K_2})$, thus implying that $(\widehat{K_1 \cup K_2}) \subset \widehat{K}_1 \cup \widehat{K}_2$.

Proof. First, let us prove that $\widehat{K}_1 \cup \widehat{K}_2 \subset (\widehat{K_1 \cup K_2})$ (which always holds). Let $a \in \widehat{K}_1 \cup \widehat{K}_2$. By definition we have that for every holomorphic polynomial P we have $|P(a)| \leq ||P||_{K_1}$ or $|P(a)| < ||P||_{K_2}$. Without loss of generality we may assume that the former holds. Then we also have that for every holomorphic polynomial P

$$|P(a)| \leq \max(||P||_{K_1}, ||P||_{K_2}) = ||P||_{K_1 \cup K_2}.$$

Thus $a \in \widehat{(K_1 \cup K_2)}$, which proves the inclusion.

For the other inclusion, suppose $a \in \mathbb{C}^n \setminus (\widehat{K}_1 \cup \widehat{K}_2)$ and let us prove that $a \notin \widehat{(K_1 \cup K_2)}$. We now have two cases, either $f(a) \notin \widehat{f(K_1)} \cup \widehat{f(K_2)}$ or $f(a) \in \widehat{f(K_1)} \cup \widehat{f(K_2)}$.

On the first case, choose g a holomorphic function in one variable satisfying $g(f(a)) = 1$ and $||g \circ f||_{K_1 \cup K_2} < \delta$, with $\delta > 0$ (here g depends on δ). Now applying the Runge approximation theorem to g on the compact $f(a) \cup f(K_1) \cup f(K_2)$ (by hypothesis $\mathbb{C} \setminus \big(f(K_1) \cup f(K_2)\big)$ is connected, thus $\mathbb{C} \setminus \big(f(K_1) \cup f(K_2) \cup f(a)\big)$ is also connected), we get that for every $\varepsilon > 0$ ther exists a polynomial P_ε such that $|P_\varepsilon(f(a)) - 1| < \varepsilon$ and $||P \circ g||_{K_1 \cup K_2} < \varepsilon$. Choose $\varepsilon < 1/2$. Then we have

$$|P(f(a))| - ||P \circ f||_{K_1 \cup K_2} > 1 - 2\varepsilon > 0.$$

Since $P \circ f$ is a polynomial, we get that $a \notin \widehat{(K_1 \cup K_2)}$.

On the second case, if $f(a) \in \widehat{f(K_1)} \cup \widehat{f(K_2)}$, since $\widehat{f(K_1)} \cap \widehat{f(K_2)} = \emptyset$, either $f(a) \in \widehat{f(K_1)}$ or $f(a) \in \widehat{f(K_2)}$. Without loss of generality assume that $f(a) \in \widehat{f(K_1)}$. Since $a \notin K_1$ (because $K_1 \subset \widehat{K}_1$), we can find q a polynomial (in several variables) such that $q(a) = 1$ and $||q||_{K_1} < 1/2$. Put $M = ||g||_{K_2}$. Again using Runge's approximation theorem we can find a polynomial p satisfying

$$||p - 1||_{f(K_1)} < \frac{1}{3}$$

and

$$||p||_{f(K_2)} < \frac{1}{2M}.$$

Then the polynomial $h(z) = q(z) \cdot p\big(f(z)\big)$ satisfies

$$|h(a) - 1| < \frac{1}{3}$$

and

$$||h||_{K_1 \cup K_2} < \frac{2}{3}.$$

Indeed,

$$|h(a) - 1| = |q(a)p\big(f(a)\big) - 1| = |p\big(f(a)\big) - 1| \leq ||p - 1||_{\widehat{f(K_1)}} < \frac{1}{3}$$

where in the first inequality we have used that $f(a) \in \widehat{f(K_1)}$. The other inequality, namely $||h||_{K_1 \cup K_2} < \frac{2}{3}$, comes from $||h||_{K_1 \cup K_2} = \max(||h||_{K_1}, ||h||_{K_2})$ and

$$||h||_{K_1} = ||p||_{f(K_1)} ||q||_{K_1} < \frac{4}{3}\frac{1}{2} = \frac{2}{3}, ||h||_{K_2} \qquad = ||p||_{f(K_2)} ||q||_{K_2} < \frac{1}{2M} M = \frac{1}{2}.$$

In the end we have that

$$||h||_{K_1 \cup K_2} < \frac{2}{3} < |h(a)|,$$

which because h is a polynomial, it implies that $a \notin \widehat{(K_1 \cup K_2)}$.

In both cases we ended up with $a \notin \widehat{(K_1 \cup K_2)}$, and since $a \in \mathbb{C}^n \setminus (\widehat{K}_1 \cup \widehat{K}_2)$ is arbitrary we have that $\widehat{(K_1 \cup K_2)} \subset \widehat{K}_1 \cup \widehat{K}_2$. This finishes the proof. □

From the previous theorem, we can deduce two important corollaries.

Corollary 4.3. If K_1 and K_2 are two disjoint convex compact sets, then $K_1 \cup K_2$ is polynomially convex

Proof. First observe that since K_1 and K_2 are convex, they are polynomially convex. Since K_1 and K_2 are disjoint convex sets, there is a hyperplane H separating K_1 from K_2. Now the equation of such hyperplane is a polynomial p of degree one. Then $p(K_1)$ and $p(K_2)$ are also disjoint convex sets, thus $\widehat{p(K_1)} \cap \widehat{p(K_2)} = \emptyset$. Therefore by the Separation Lemma, we have that $K_1 \cup K_2$ is polynomially convex. □

Corollary 4.4. Let $a_1, \dots, a_k \in \mathbb{C}^n$ be different points. Then there exist $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, the set $\bigcup_{j=1}^k \overline{B(a_j, \delta)}$ is polynomially convex.

The proof relies on finding a complex line L satisfying that for every pair of points a_j, and a_m we have $\pi(a_j) \neq \pi(a_m)$, where π is the orthogonal projection onto L. Having found this line, we then can send this line L to $\mathbb{C} \times \{0\} \times \cdots \times \{0\}$ via a translation and rotations. The resulting map will be a polynomial separating the points a_j's, thus separating some balls with a small radius.

Proof. First, we want to choose a line L satisfying that for any two points a_j, a_m we have $\pi(a_j) \neq \pi(a_m)$ (where π is the orthogonal projection onto L). Observe that $\pi(a_j) = \pi(a_m)$ if and only if a_j and a_m belong to a common hyperplane H_{a_j,a_m} which is perpendicular to L. We would like to choose a line L whose defining unitary vector v is not any perpendicular vector to the hyperplanes H_{a_j,a_m} containing a_j and a_m. We claim that we can always find such a vector.

Indeed, with $1 \leq j < m \leq k$ fixed, let $S_{a_j,a_m} \subset \mathbb{S}^n_{\mathbb{C}}$ be the set of unit normal vectors defining each hyperplane containing both a_j and a_m (here $\mathbb{S}^n_{\mathbb{C}}$ is the unit sphere of complex coordinates of complex dimension n). Observe that $\dim S_{a_j,a_m} = n-1$ as a complex manifold. Then the set of all the possible normal unitary vectors defining the hyperplanes passing through each pair a_j, a_m is the set $\bigcup_{1 \leq j < m \leq k} S_{a_j,a_m} \subset \mathbb{S}^n_{\mathbb{C}}$. Observe that this is a finite union of sets of dimension $n-1$, therefore $\bigcup_{1 \leq j < m \leq k} S_{a_j,a_m} \neq \mathbb{S}^n_{\mathbb{C}}$. This implies that there is a vector $v \in \mathbb{S}^n_{\mathbb{C}} \setminus \bigcup_{1 \leq j < m \leq k} S_{a_j,a_m}$, i.e. v is a vector that is not perpendicular to any hyperplane H_{a_j,a_k}. Thus the line L defined by v satisfies what we wanted.

Let then π denote the orthogonal projection of $\mathbb{C}^n$ onto L. Then for every a_j, a_m we have that $\pi(a_j) \neq \pi(a_m)$. By means of a translation and rotations (which are polynomials of degree one in each coordinate) send L to the set $\mathbb{C} \times \{0\} \times \cdots \times \{0\}$ (denote g the composition of such a translation and rotations). Therefore the first coordinate of $g \circ \pi$ is a polynomial f such that $f(a_j) \neq f(a_k)$.

Put $\varepsilon_0 = 1/2 \min_{1 \leq j < k \leq m}(|f(a_j) - f(a_k)|)$, and choose $\varepsilon < \varepsilon_0$. Because f is continuous, for each $1 \leq j \leq m$ there is $\delta_j > 0$ such that for every $z \in \mathbb{C}^n$, if $||z - a_j|| < \delta_j$ then $|f(z) - f(a_j)| < \varepsilon$. Take $\delta < 1/2 \min(\delta_1, \dots, \delta_m)$ and $\delta < 1/2 \min_{1 \leq j < k \leq m}(||a_j - a_k||)$. Then for all j, k different we have that $\overline{B(a_j, \delta)} \cap \overline{B(a_k, \delta)} = \emptyset$, and because $f\big(\overline{B(a_j, \delta)}\big) \subset B(f(a_j), \varepsilon)$ we also have that $f\big(\overline{B(a_j, \delta)}\big) \cap f\big(\overline{B(a_k, \delta)}\big) = \emptyset$.

In the end we have found a polynomial f separating the sets $\overline{B(a_j, \delta)}$ whenever $\delta < \delta_0$. Then by the Separation Lemma, we have that $\bigcup_{j=1}^k \overline{B(a_j, \delta)}$ is polynomially convex. □

The work of Eva Kallin was important because it answered a question mathematicians had not been able to answer for some time. Kallin was able to prove that the polynomial convex hull of three spheres (of any radius) is the union of the closed balls whose boundaries are such spheres. It is still a mystery to this day if the same holds for more spheres.

Theorem 4.5 (Three Sphere Theorem)**.** *The polynomial hull of the three spheres S_1, S_2, S_3 in $\mathbb{C}^n$ is $\widehat{S}_1 \cup \widehat{S}_2 \cup \widehat{S}_3$, where $\widehat{S}_j$ is the closed ball whose surface is S_j.*

The proof is a direct consequence of the Separation Lemma. The strategy is then to construct a polynomial separating the three closed balls.

Proof. We may begin assuming that the balls $\widehat{S}_j$ are all disjoint. If not, since the spheres are disjoint by hypothesis, we have the following two cases:

i) Two of the balls are included in the third one.

ii) Only one ball is included in another.

On case 1), by renaming the balls we can suppose that $\widehat{S}_1, \widehat{S}_2 \subset \widehat{S}_3$. Then because $S_1 \cup S_2 \cup S_3 \subset \widehat{S}_3$ we have $\widehat{(S_1 \cup S_2 \cup S_3)} = \widehat{S}_3 = \widehat{S}_1 \cup \widehat{S}_2 \cup \widehat{S}_3$. This, in turn, implies $\widehat{(S_1 \cup S_2 \cup S_3)} = \widehat{S}_3 = \widehat{S}_1 \cup \widehat{S}_2 \cup \widehat{S}_3$.

On case 2), without loss of generality, we can assume $\widehat{S}_1 \subset \widehat{S}_2$ and $\widehat{S}_2 \cap \widehat{S}_3 = \emptyset$. Then by Corollary 4.3 we have that $\widehat{S}_1 \cup \widehat{S}_2 \cup \widehat{S}_3 = \widehat{S}_2 \cup \widehat{S}_3$ is polynomially convex.

Having discussed these two small cases we turn our attention to the case where all three balls are disjoint. By Corollary 4.3 and the Separation Lemma, it is enough to find a polynomial separating one ball from the other two, to deduce $\widehat{S}_1 \cup \widehat{S}_2 \cup \widehat{S}_3$ is polynomially convex. Let us find such a polynomial.

By means of renaming the balls and performing a dilation, we can take S_1 to be the ball with the biggest radius, and to have its radius equal to 1. Let $r_2, r_3 \leq 1$ be the radii of S_2 and S_3 respectively. Now perform a translation so the center of S_1 is 0. In the space spanned by the center of the spheres take coordinates so that the center S_1 is in $(0, 0)$ and the center of S_2 lies on the real z_1-axis. Next, perform a rotation in z_1 and then in z_2 so that the center of S_3 is of the form (α, β) with $\alpha, \beta \in \mathbb{R}$ (essentially we are just multiplying the first and second coordinates by complex numbers of modulus 1). This leaves the center of S_1 at $(0, 0)$ and the center of S_2 at $(\gamma, 0)$ with $\gamma \in \mathbb{C}$. We claim that the polynomial f given by

$$f(z_1, z_2) = z_1^2 + z_2^2$$

separates $\widehat{S}_1$ from $\widehat{S}_2 \cup \widehat{S}_3$. Let us see this.

Clearly $|f(z)| \leq 1$ for all $z \in \widehat{S}_1$. We also have that $\mathrm{Re}(f(z)) > 1$ in $\widehat{S}_2$. To see this, write for $j = 1, 2$, $z_j = x_j + iy_j$, and remember that $\widehat{S}_1 \cap \widehat{S}_2 = \emptyset$, thus $\alpha^2 + \beta^2 > (1 + r_2)^2$. Therefore we have that for $(z_1, z_2) \in \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1 - \alpha|^2 + |z_2 - \beta|^2 \leq r_2^2\}$ the following inequality holds

$$(x_1 - \alpha)^2 + y_1^2 + (x_2 - \beta)^2 + y_2^2 \leq r_2^2.$$

Put $\eta = \sqrt{\alpha^2 + \beta^2} - 1 - r_2 > 0$, and $\varepsilon = r_2 - \sqrt{(x_1 - \alpha)^2 + (x_2 - \beta)^2} \geq 0$. Then we have that

$$\begin{aligned} \mathrm{Re}(z_1^2 + z_2^2) &= x_1^2 + x_2^2 - (y_1^2 + y_2^2) \geq x_1^2 + x_2^2 + (x_1 - \alpha)^2 + (x_2 - \beta)^2 - r_2^2 \geq \\ &\geq (1 + \eta + \varepsilon)^2 + (r_2 - \varepsilon)^2 - r_2^2 = 1 + 2\eta + 2\varepsilon(1 - r) + (\eta + \varepsilon)^2 + \varepsilon^2 > 1 \end{aligned}$$

because $\eta > 0$, $\varepsilon \geq 0$, and $0 < r_2 \leq 1$.

Now if we replace z_1, z_2, α, β, and r_2 by $\frac{|\gamma|}{\gamma} z_1$, $\frac{|\gamma|}{\gamma} z_2$, $|\gamma|, 0$ and r_3 we get

$$\mathrm{Re}\left(\frac{|\gamma|^2}{\gamma^2}(z_1^2 + z_2^2)\right) > 1$$

for all $(z_1, z_2) \in \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1 - \gamma|^2 + |z_2|^2 \leq r_3\}$, because $|\gamma| > 1 + r_3$ and $r_3 \leq 1$.

In the end $|f(z)| > 1$ for all $z \in \widehat{S}_2 \cup \widehat{S}_3$. Thus f separates $\widehat{S}_1$ from $\widehat{S}_2 \cup \widehat{S}_3$ which is what we wanted. ∎

4.1.2 Auxiliary results

As we have mentioned at the beginning of Section 4.1, to prove Theorem 4.1 we first need an auxiliary result saying that having an isotopy of biholomorphisms preserving polynomial convexity is equivalent to being able to approximate each biholomorphism by automorphisms of $\mathbb{C}^n$.

To prove the auxiliary result we first need two previous results.

Lemma 4.6. Let $f\colon \Omega \to \Omega'$ be a biholomorphism from a domain $\Omega \subset \mathbb{C}^n$ onto the domain $\Omega' \subset \mathbb{C}^n$ that can be approximated uniformly on compact sets by automorphisms of $\mathrm{Aut}(\mathbb{C}^n)$. Then:

i) $K \Subset \Omega$ is polynomially convex if and only if $f(K) \Subset \Omega'$ is polynomially convex.

ii) Ω is Runge if and only if Ω' is Runge.

Proof. a)
We will prove that if $K' = f(K)$ is polynomially convex then K is also polynomially convex. For that choose $a \in \Omega \setminus K$ and let us prove that $a \notin \widehat{K}$. Since $f(a) \notin K'$ and K' is polynomially convex (by hypothesis) we can find a holomorphic polynomial p satisfying

$$||P\big(f(a)\big)|| > ||P||_{K'} = ||P \circ f||_K.$$

Put $d = |P\big(f(a)\big)| - ||P \circ f||_K > 0$. By hypothesis we can find $\psi \in \mathrm{Aut}(\mathbb{C}^n)$ satisfying

$$||P \circ \psi - P \circ f||_{K \cup \{a\}} < \frac{d}{2}.$$

Therefore we have that

$$||P \circ \psi||_K < ||P \circ f||_K + \frac{d}{2} < |P\big(f(a)\big)| - \frac{d}{2} < |P\big(\psi(a)\big)|.$$

Now because $P \circ \psi \in \mathcal{H}(\mathbb{C}^n)$, we can approximate $P \circ \psi$ uniformly on $K \cup \{a\}$ by a polynomial Q satisfying $||Q||_K < |Q(a)|$ (we just need to repeat the same argument replacing $P \circ \psi$ with Q and $P \circ f$ with $P \circ \psi$). In the end, we have that $a \notin \widehat{K}$. Because $a \in \Omega \setminus K$ is arbitrary, we have proved $\widehat{K} \subset K$ which implies that $K = \widehat{K}$. In short, K is polynomially convex.

For the other implication, the same argument works replacing f by f^{-1} (because f is a biholomorphism).
b)
Suppose that Ω' is Runge, and let us show that Ω is also Runge. Let $K \Subset \Omega$ and $g \in \mathcal{H}(\Omega)$. To see that Ω is Runge it is enough to see that g can be uniformly approximated by polynomials. Choose U an open set satisfying $f(K) \subset U \Subset \Omega'$. Now because Ω' is Runge, $g \circ f^{-1}$ can be uniformly approximated on $\overline{U}$ by a polynomial P. Therefore $P \circ f$ approximates g uniformly on $f(\overline{U})$. Using the hypothesis on f, we can choose $\psi \in \mathrm{Aut}(\mathbb{C}^n)$ so that it approximates f in $\overline{U}$ and $\psi(K) \subset U$. Finally $P \circ \psi$ approximates g uniformly on K, and since $P \circ \psi \in \mathcal{H}(\mathbb{C}^n)$, g can be approximated uniformly on K by polynomials. In the end, Ω is Runge.

Repeating the same argument for f^{-1} instead of f yields the other implication. □

Lemma 4.7. Let $\Omega \subset \mathbb{C}^n$ be open, and $H\colon \big[0,1\big] \times \Omega \to \mathbb{C}^n$ be an isotopy. If $K \subset \Omega$ is a compact polynomially convex set satisfying that each $K_t = H(t, K)$ is polynomially convex, then there exists a basis of Stein neighborhoods U of K such that for every $t \in \big[0,1\big]$ $H(t,U)$ is Runge.

Proof. Because K is polynomially convex, there exists $\rho \geq 0$ a smooth plurisubharmonic exhaustion of $\mathbb{C}^n$ such that $\rho|_K = 0$. Now for each $\varepsilon > 0$ the set

$$U_\varepsilon = \{z \in \mathbb{C}^n \mid \rho(z) < \varepsilon\}$$

is pseudoconvex (because ρ is plurisubharmonic). In other words $(U_\varepsilon)_{\varepsilon>0}$ is a basis of Stein neighborhoods. We claim that for ε small enough $H(t, U_\varepsilon)$ is Runge.

Indeed, fix $t \in [0,1]$. Because $H(t,\cdot)$ is a biholomorphism, $\rho_t(z) = \rho\big(H^{-1}(t,z)\big) \geq 0$ is plurisubharmonic on $\Omega_t = H(t,\Omega)$, and it vanishes on K_t. Choose $V \Subset \Omega_t$ an open neighborhood of K_t. Since K_t is polynomially convex, we can find $\eta \geq 0$ a smooth plurisubharmonic exhaustion of $\mathbb{C}^n$ so that it is strongly plurisubharmonic on $\mathbb{C}^n \setminus V$ and vanishes on a smaller neighborhood $V_1 \Subset V$ os K_t. Consider now an auxiliary smooth function χ compactly supported on Ω_t so that $\chi|_V = 1$. We now can choose $\delta > 0$ small enough so that

$$\eta_t(z) = \eta(z) + \delta\chi(z)\rho_t(z), \quad z \in \mathbb{C}^n$$

is a strongly plurisubharmonic exhaustion of $\mathbb{C}^n$. This works because the Levi matrix of η_t will be the Levi matrix of η plus δ multiplied by something depending on χ and its derivatives (which are all uniformly bounded above and below by a constant M), thus taking $\delta < m/M$ where m is the minimum over the compact support of χ of the enrties of the Levi matrix of η, does the job.

Therefore we have that η_t is a smooth strongly plurisubharmonic exhaustion that vanishes on K_t, and equals $\delta\rho_t(z)$ whenever $z \in V_1$. We also have that the sets of the form

$$\{z \in \mathbb{C}^n \mid \eta_t(z) < \varepsilon\}$$

are Runge. In the end, taking $\varepsilon_0 > 0$ small enough so that for all $0 < \varepsilon < \varepsilon_0$ $H(t, U_\varepsilon) \subset V_1$ implies that

$$H(t, U_\varepsilon) = \{z \in V_1 \mid \rho_t(z) < \varepsilon\} = \{z \in \mathbb{C}^n \mid \eta_t(z) < \delta\varepsilon\},$$

hence $H(t, U_\varepsilon)$ is Runge, as we claimed.

Finally, the same ε_0 works for a neighborhood of the fixed t, thus using the compactness of $[0,1]$ we can choose ε_0 to be independent of t, thus proving the lemma. □

We are now ready to prove our auxiliary result.

Proposition 4.8. Let $\Omega \subset \mathbb{C}^n$ be open and H be an isotopy of biholomorphisms of Ω so that $H(0,\cdot) = \mathrm{Id}_\Omega$. Then for every compact polynomially convex set $K \subset \Omega$ the following are equivalent:

i) For every $t \in [0,1]$, $K_t = H(t,K)$ is polynomially convex.

ii) There exists a neighborhood U of K so that for all $t \in [0,1]$, the map $H(t,\cdot)$ can be uniformly approximated in U by automorphisms of $\mathbb{C}^n$.

Proof.
i)⇒ii)
By Lemma 4.7 we can find a neighborhood U of K so that each $H(t,U)$ is Runge. Then by the Local Andersén-Lempert Theorem, each map $H(t,\cdot)$ can be uniformly approximated by automorphisms of $\mathbb{C}^n$ on U.
ii)⇒i)
Applying Lemma 4.6 i) to each $H(t,\cdot)$ (we can do so because $H(t,\cdot)$ is approximable by automorphisms, by hypothesis) yields that each K_t is polynomially convex. □

We are now ready to prove Theorem 4.1.

4.1.3 Proof of Theorem 4.1

Because of Proposition 4.8 applied to $K = \bigcup_{j=1}^m K_j$, it is enough to find for each $1 \leq j \leq m$ an isotopy H_j defined on a neighborhood U_j of K_j satisfying:

a) $H_j(0, \cdot) = \mathrm{Id}_{U_j}$.

b) $H_j(1, \cdot) = \phi_j$

c) The sets $K_{j,t} = H(t, K_j)$ are pairwise disjoint, and their union $\bigcup_{j=1}^m K_{j,t}$ is polynomially convex for each $t \in [0, 1]$.

To find such an isotopy we first need some preparations.

Begin by considering for each $1 \leq j \leq m$, U_j a bounded neighborhood of K_j, star-shaped with respect to a point, call it a_j so that the $\overline{U}_j$'s are pairwise disjoint, and $\phi_j(U_j) = U_j'$ are also pairwise disjoint. We know that for each ϕ_j there is a smooth isotopy φ_j so that $\varphi_j(0, \cdot) = \mathrm{Id}_{\mathbb{C}^n}$, and $\varphi_j(1, \cdot) = \phi_j$ (the procedure to get this is really similar to the one done in Lemma 3.2 in Section 3). Modifying each φ_j by a family of translations (depending on t), we may assume that the points $b_{j,t} = \varphi_j(t, a_j)$ are all distinct for $1 \leq j \leq m$ and $t \in [0, 1]$.

Let now $\varepsilon > 0$ be small enough so that $\bigcup_{j=1}^m \overline{B(b_{j,t}, \varepsilon)}$ is polynomially convex for every $t \in [0, 1]$ (we can do this by Corollary 4.4). Let $\delta > 0$ be small enough so that $\varphi_j(t, \cdot)$ maps $B(a_j, \delta)$ to $B(b_{j,t}, \varepsilon)$ for every $t \in [0, 1]$ (we can do this because H is uniformly continuous on $[0, 1] \times K_j$). Finally, let $R >$ be large enough so that for every j, $U_j \subset B(a_j, R)$.

We are now ready to define our isotopy. Put c so that $1 - c/3 = \delta/R$, and define $H_j(t, z)$ by

$$H_j(t, z) = \begin{cases} a_j + (1 - ct)(z - a_j) & \text{if } 0 \leq t \leq 1/3 \\ \varphi\big(3t - 1, a_j + \delta/R(z - a_j)\big) & \text{if } 1/3 \leq t \leq 2/3 \\ \phi_j\Big(a_j + \big(1 + c(t - 1)\big)\big(z - a_j\big)\Big) & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

What this isotopy does is first contract U_j into $B(a_j, \delta)$, then apply φ_j to $B(a_j, \delta)$ and finally expand $B(b_{j,t}, \varepsilon)$ into U_j'. It remains to be seen that for each $t \in [0, 1]$, $\bigcup_{j=1}^m H_j(t, K_j)$ is polynomially convex.

For $1 \leq t \leq 1/3$ the union $\bigcup_{j=1}^m H_j(t, K_j)$ is polynomially convex because $K = \bigcup_{j=1}^m K_j$ is polynomially convex (by hypothesis), and $H_j(t, \cdot)$ is a contraction (thus a biholomorphism) therefore by Lemma 4.6 i) we have that $\bigcup_{j=1}^m H_j(t, K_j)$ is polynomially convex. A similar thing happens for $2/3 \leq t \leq 1$. And for $1/3 \leq t \leq 2/3$, $H_j(1/3, K_j)$ is star-shaped, because K_j is, and thus polynomially convex. Therefore again by Lemma 4.6 i) we have that $\bigcup_{j=1}^m H_j(t, K_j)$ is polynomially convex.

One could object that our isotopy H is not smooth on t, but that is no problem because we can reparametrise $[0, 1]$ so that $H_j(t, z)$ is smooth. This ends the proof. □

As an immediate corollary, using the Separation Lemma (Theorem 4.2) we have the following:

Corollary 4.9. Let B_1, B_2, B_3 and B_1', B_2', B_3' be two sets of pairwise disjoint closed balls in $\mathbb{C}^n$. Then there exists a sequence $(\psi_k)_k \subset \mathrm{Aut}(\mathbb{C}^n)$ so that ψ_k converges uniformly on a neighborhood of each B_j to an affine map sending B_j to B_j'.

4.2 Embedding holomorphic discs through discrete sets

In this section, we present yet another application of the Andersén-Lempert Theorem that can be found in [6], which schematically says that given a discrete sequence Z, we can always find a space biholomorphic to a disk, that crosses each one of the points in Z.

In more concrete terms we want to prove the following:

Theorem 4.10. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex Runge domain and $Z = (z_k)_{k\geq 0} \subset \Omega$ a discrete sequence (that is with no accumulation points). Then there exists a proper holomorphic embedding $f\colon \mathbb{D} \subset \mathbb{C} \to \Omega$ satisfying $Z \subset f(\mathbb{D})$.*

The idea of the proof of Theorem 4.10, is to construct a sequence $(f_k)_k$ of proper holomorphic embeddings $f_k\colon \mathbb{C} \to \mathbb{C}^n$ in a way that the points $z_0, \dots, z_k$ are contained in the same connected component of $f_k(\mathbb{C}) \cap \Omega$. The map f_{k+1} will be obtained as the composition $f_{k+1} = \phi_k \circ f_k$, where ϕ_k will be an automorphism of $\mathbb{C}^n$ such that ϕ_k is very close to the identity on a compact polynomially convex set $K_k \Subset \Omega$, it fixes $z_0, \dots, z_k$ and $z_{k+1} \in f_{k+1}(\mathbb{C})$.

In what follows we will need to refer both to points in $\mathbb{C}$ and points in $\mathbb{C}^n$. To make it easier for the reader to follow, w will denote a point in $\mathbb{C}$ while z will be a point in $\mathbb{C}^n$.

To prove Theorem 4.10 we need two major results:

Lemma 4.11. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex Runge domain and $f\colon \mathbb{C} \to \mathbb{C}^n$ be a proper holomorphic embedding. Then each connected component of $f(\mathbb{C}) \cap \Omega$ is simply connected and therefore is biholomorphic to the unit disk $\mathbb{D}$ or to $\mathbb{C}$.

Proof. Let A be a connected component of $f(\mathbb{C}) \cap \Omega$, and let $U = f^{-1}(A)$. Since f is an embedding, to prove A is simply connected, it is enough to prove that U is simply connected. Suppose it is not and let us arrive at a contradiction. Since U is not simply connected we can choose w_0 a point in a bounded component of $\mathbb{C} \setminus U$. Now consider the holomorphic function $F\colon A \to \mathbb{C}$ defined by

$$F(z) = \frac{1}{f^{-1}(z) - w_0},$$

which is well defined since $w_0 \notin U$. Now by Cartan's theorem A (Theorem 7.2.8 of [9]), F can be extended to a holomorphic function on Ω. Now because Ω is Runge, it follows that there exists $(P_k)_k$ a sequence of holomorphic polynomials converging uniformly on compact set to F. Then $P_k \circ f$ defines a sequence of entire functions converging uniformly on compact sets of U to the function $w \mapsto 1/(w - w_0)$ which is a contradiction. □

We also need the following proposition (which we think is interesting by itself).

Proposition 4.12 (Combing hair by Holomorphic Automorphisms)**.** Let $K \subset \mathbb{C}^n$ be a compact polynomially convex set and γ a parametrization of a $\mathcal{C}^r$-diffeomorphic image of $[0, 1]$ ($r \geq 3$) so that $\gamma^* \cap K = \{\gamma(0)\}$. Let $F\colon K \cup \gamma^* \to K \cup C' \subset \mathbb{C}^n$ be a homeomorphism so that $F|_{(K\cup\gamma^*)\cap U} = id$ for some open neighborhood U of K.

Then $\forall\, \varepsilon > 0\ \exists\, \psi \in \mathrm{Aut}(\mathbb{C}^n)$ so that $||\psi - F||_{K\cup\gamma^*} < \varepsilon$. Moreover, for each pair of finite subsets $A \subset \gamma^*$ $B \subset \mathbb{C}^n \setminus \gamma^*$, we can find ψ as above such that it also satisfies that $\psi|_A = F|_A$ and $\psi(b) = b$ for every $b \in B$.

Since the proof of the Combing hair by Holomorphic Automorphisms is rather long and to understand it we need to introduce different concepts, we choose to postpone it.

With Lemma 4.11 and the Combing hair by Holomorphic Automorphisms, we are ready to prove Theorem 4.10.

4.2.1 Proof of Theorem 4.10

We first consider the case where $\Omega \neq \mathbb{C}^n$.

Take $\rho\colon \Omega \to \mathbb{R}$ a smooth plurisubharmonic exhaustion function such that $\rho(z_k) \neq \rho(z_j)$ whenever $k \neq j$ (if needed modify ρ in a neighborhood of z_k). Assume that $\rho(z_k) < \rho(z_{k+1})$ for all $k \geq 0$ (if necessary we reorder the sequence Z). For each $k \geq 0$ choose r_k so that $\rho(z_k) < r_k < \rho(z_{k+1})$, and put $K_k = \{z \in \Omega \mid \rho(z) \leq r_k\}$. Because ρ is a plurisubharmonic exhaustion, it follows that K_k is a compact polynomially convex set (Theorem 5.2.10 of [9]). Then $(K_k)_k$ is an increasing sequence of compact polynomially convex sets such that $\bigcup_{k=0}^{\infty} K_k = \Omega$, $K_k \cap Z = \{z_0, \ldots, z_k\}$ and $(K_{k+1} \setminus K_k) \cap Z = \{z_{k+1}\}$.

Let us now define f_0. Fix a point $a \in \mathbb{C}^n \setminus \Omega$, and put $w_0 = 0 \in \mathbb{C}$. Let $f_0\colon \mathbb{C} \to \mathbb{C}^n$ be a proper holomorphic embedding so that $f_0(0) = z_0$, $f_0(1) = a$, and $z_1 \notin f_0(\mathbb{C})$. Set $L_{-1} = \Delta_{-1} = V_{-1} = \emptyset$.

Suppose now that for every $k \geq 0$ we already have a proper holomorphic embedding $f_k\colon \mathbb{C} \to \mathbb{C}^n$, a set of points in the complex plane $w_0, \ldots, w_k \neq 1$, a number M_{k-1} and a smooth bounded simply connected domain $\Delta_{k-1} \Subset \mathbb{C} \setminus \{1\}$ such that

i) $f_k(w_j) = z_j$ for all $0 \leq j \leq k$,

ii) $f_k(1) = a$,

iii) $z_{k+1} \notin f_k(\mathbb{C})$, and

iv) $\{w_0, \ldots, w_k\} \cup \overline{\Delta}_{k-1}$ is contained in U_k^0 which is a connected component of $f_k^{-1}(\Omega)$.

Observe that $\rho \circ f_k$ is an exhaustion function of $f_k^{-1}(\Omega)$. Let M_k be a so that

$$M_k \geq \max(r_k, M_{k-1}) + 1, \tag{6}$$

M_k is a regular value of $\rho \circ f_k|_{f_k^{-1}(\Omega)}$ and $\{w_0, \ldots, w_k\} \cup \overline{\Delta}_{k-1}$ is contained in one connected component of

$$V_k = \{w \in f_k^{-1}(\Omega) \mid \rho \circ f_k(w) < M_k\} \Subset f_k^{-1}(\Omega). \tag{7}$$

We can choose such an M_k because of Sard's Theorem, which tells us that the set of critical values of $\rho \circ f_k$ has Lebesgue measure 0. Thus we can always take M_k big enough to satisfy what we desire.

Let us denote the connected component of V_k containing $\{w_0, \ldots, w_k\} \cup \overline{\Delta}_{k-1}$ by Δ_k. By Lemma 4.11, V_k is made up of smooth bounded simply connected components $\Delta_k, \Delta_k^1, \ldots, \Delta_k^{j_k}$ which have disjoint closures (because M_k is a regular value). There are finitely many because f_k is an embedding (see Figure 1 to get an idea of $f_k(V_k)$ and $f_k(\Delta_k)$).

Put

$$L_k = K_k \cup (f_k(\mathbb{C}) \cap \Omega_{M_k}) = K_k \cup \overline{f_k(V_k)},$$

where $\Omega_{M_k} = \{z \in \Omega \mid \rho(z) \leq M_k\}$. Let us see that L_k is polynomially convex. To do that, suppose is not and let us arrive at a contradiction. Let $b \in \widehat{L}_k \setminus L_k$. Since $L_k \subset \Omega_{M_k}$ and Ω_{M_k} is polynomially convex, we have that $\widehat{L}_k \subset \Omega_{M_k}$, and therefore $b \notin f_k(\mathbb{C}) \cup K_k$. Using Cartan's A Theorem (Theorem 7.2.8 of [9]) there exists $g\colon \mathbb{C}^n \to \mathbb{C}$ holomorphic such that $g(b) = 1$ and $g = 0$ on $f_k(\mathbb{C})$. Because K_k is polynomially convex and $b \notin K_k$, we can find $h\colon \mathbb{C}^n \to \mathbb{C}$ holomorphic with $h(b) = 1$ and $||h||_{K_k} < 1$. Let now $N \in \mathbb{N}$ be so that

$$||h^N||_{K_k} < ||g||_{K_k}^{-1}.$$

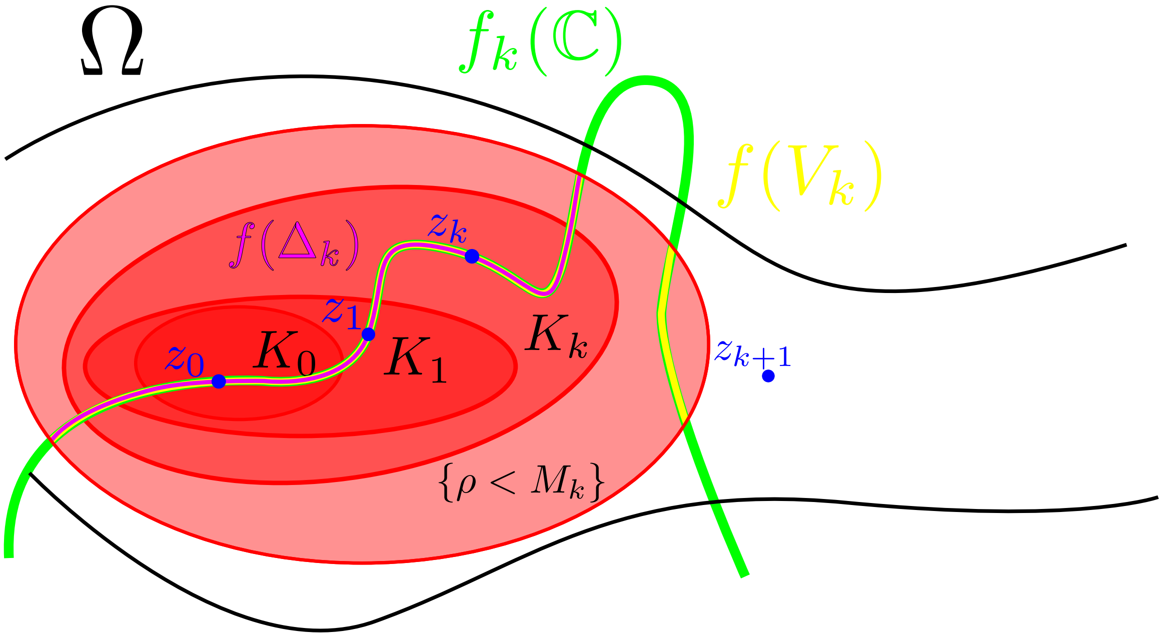

Figure 1: A schematic version of the disposition of the sets Ω, $f_k(\mathbb{C})$, $f(V_k)$, and $f(\Delta_k)$.

Then the function $G(z) = g(z)h(z)^N$ satisfies $G(b) = 1$ and

$$||G||_{L_k} = \max\{||G||_{K_k}, ||G||_{f_k(\mathbb{C})\cap\Omega_{M_k}}\} = ||G||_{K_k} = ||gh^N||_{K_k} < 1.$$

Thus we have found a holomorphic function G so that $||G||_{L_k} < G(b)$, this implies that $b \notin \widehat{L}_k$, which is a contradiction. Therefore L_k is polynomially convex.

We now want to apply Proposition 4.12 to the set L_k and some suitable arc C_k. For that, let α_k be a smooth arc in $U_k^0 \setminus \Delta_k$ with one endpoint attached to $\overline{\Delta}_k$ and not intersecting any other connected component of V_k. Call w_{k+1} the endpoint of α_k that is not attached to $\overline{\Delta}_k$. Then $C_k = f_k(\alpha_k)$ is an arc in $f_k(\mathbb{C}) \cap \Omega$ that connects $f_k(w_{k+1})$ to a point in $f_k(\overline{\Delta_k}) \subset L_k$.

Now by Proposition 4.12 applied to L_k and C_k, for any $\varepsilon_k > 0$ there exists $\phi_k \in \mathrm{Aut}(\mathbb{C}^n)$ satisfying:

a) $||\phi_k - \mathrm{Id}||_{L_k} < \varepsilon_k$

b) $\phi_k(f_k(w_{k+1})) = z_{k+1}$

c) $\phi_k(z_j) = z_j$ for all $0 \leq j \leq k$ and $\phi_k(a) = a$

d) $\phi_k(C_k) \subset \Omega$

e) $z_{k+2} \notin \phi_k(f_k(\mathbb{C}))$.

Now define $f_{k+1} := \phi_k \circ f_k \colon \mathbb{C} \to \mathbb{C}^n$. Since f_k is a proper holomorphic embedding and $\phi_k \in \mathrm{Aut}(\mathbb{C}^n)$, f_{k+1} is also a proper holomorphic embedding of $\mathbb{C}$ into $\mathbb{C}^n$. Moreover, f_{k+1} satisfies the same properties as f_k, that is properties i) to iv) above (replacing k with $k+1$).

With all of this, we have constructed a sequence $(f_k)_k$ of proper holomorphic embeddings from $\mathbb{C}$ to $\mathbb{C}^n$ with some desired properties. Observe that the ε_k's are arbitrary. We will impose some conditions on them now.

Set

$$V = \bigcup_{k=0}^{\infty} V_k \quad \text{and } \Delta = \bigcup_{k=0}^{\infty} \Delta_k. \tag{8}$$

Then we have $w_k \subset \Delta \subset V \subset \mathbb{C} \setminus \{1\}$ for all $k \geq 0$. Observe that by property a), for all $k \geq 0$ we have $||f_{k+1} - f_k||_{V_k} < \varepsilon_k$. Now in each step choose $\varepsilon_k > 0$ satisfying

A) $\varepsilon_k \leq \varepsilon_{k-1}/2$

B) $\varepsilon_k < \frac{1}{2}d(L_k, \mathbb{C}^n \setminus \Omega_{M_k+1})$

C) $\varepsilon_k < \frac{1}{2}d(K_{k-1}, \mathbb{C}^n \setminus K_k)$.

Because $M_{k+1} \geq M_k + 1$, by A) and B) we have that $V_k \Subset V_{k+1}$ for all k. We also have that $(f_k)_k$ is uniformly Cauchy on each V_k. Indeed, on V_k, for $j > k$

$$||f_j - f_k||_{V_k} \leq \sum_{l=k}^{\infty} ||f_{l+1} - f_l||_{V_k} < \sum_{l=k}^{\infty} \varepsilon_l \overset{\text{A)}}{\leq} \sum_{l=k}^{\infty} \frac{\varepsilon_k}{2^{l-k}} = 2\varepsilon_k.$$

Because the V_k's form an exhaustion of V, $(f_k)_k$ is uniformly Cauchy on compact sets of V.Then, there exists the limit $f = \lim_{k\to\infty} f_k$ (which is uniform on compact sets of V), and this limit defines a holomorphic function. By B), we also have that $f(V_k) \subset \Omega$ for all k, then $f(V) \subset \Omega$.

Now because each f_k is an embedding, we have that f is an injective immersion into Ω (if needed we take each ε_k smaller). If we see that f is proper, we will have that f is a proper injective immersion, and thus a proper embedding (Proposition 4.22 of [16]). Let us then see that f is proper.

Because $||\phi_k - \mathrm{Id}||_{K_k} < \varepsilon_k$, by A) and C), after the k-th step no point from $\mathbb{C}^n \setminus K_k$ will end up in K_{k-1}. Then

$$f(V \setminus V_k) \subset \Omega \setminus K_{k-1}$$

for all k. Thus f is proper.

Now because for each k Δ_k is a connected component of V_k and $V_k \Subset V_{k+1}$, we have that Δ is a connected component of V. More is true, $\Delta \subset \mathbb{C} \setminus \{1\}$ is a simply connected domain since $(\Delta_k)_k$ is an increasing sequence of simply connected domains.

In summary, we have that $f\colon \Delta \to \Omega$ is a proper holomorphic embedding from a simply connected domain $\Delta \subset \mathbb{C} \setminus \{1\}$ to Ω. If we see that $Z \subset f(\Delta)$ then we will be done. By properties b) and c) of ϕ_k, we have that for all k, $f(w_k) = \lim_{j\to\infty} f_j(w_k) = f_k(w_k) = z_k$. Then $Z \subset f(\Delta)$ as we wanted.

As we have claimed this is enough, because by the Riemann mapping theorem, since $\Delta \subset \mathbb{C} \setminus \{1\}$, Δ is biholomorphic to the disk $\mathbb{D}$, say by $F\colon \mathbb{D} \to \Delta$. Then the proper holomorphic embedding we want is the map $f \circ F$.

All of this takes care of the case $\Omega \neq \mathbb{C}^n$. To prove the case for $\mathbb{C}^n$, we take $U \subset \mathbb{C}^n$ a Fatou-Bieberbach Runge domain (that is, a Runge domain biholomorphic to $\mathbb{C}^n$), and $F\colon U \to \mathbb{C}^n$ a biholomorphism. We then apply the theorem to U with the discrete set $F^{-1}(Z)$ to find $f\colon \mathbb{D} \to U$ a proper holomorphic embedding such that $F^{-1}(Z) \subset f(\mathbb{D})$. Then the map $F \circ f$ is a proper holomorphic embedding of the disk such that $Z \subset F(f(\mathbb{D}))$. This finishes the proof (if we take the Combing hair by Holomorphic Automorphisms to be true). □

We take this opportunity to give two comments regarding the proof of Theorem 4.10. The first one is about the existence of Fatou-Bieberbach Runge domains, and the second one on the choices of the sequence $(w_k)_k$ in the previous proof.

Existence of Fatou-Bieberbach Runge domains: The existence of Fatou-Bieberbach Runge domains is not something to be taken for granted. Some work is required to prove that such objects exist. We just enunciate a result concerning this manner and give an example from a paper of Walter Rudin and Jean Pierre Rosay [22].

Theorem 4.13. *Let $F \in Aut(\mathbb{C}^n)$, and $p \in \mathbb{C}^n$ such that $F(p) = p$. Suppose the eigenvalues of $DF(p)$, satisfy $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ and*

$$|\lambda_1|^2 < |\lambda_n|.$$

Then

$$\Omega = \{z \in \mathbb{C}^n \ : \ \lim_{k\to\infty} F^k(z) = p\},$$

the basin of attraction of p, is a Fatou-Bieberbach region and there is a biholomorphic map $\psi \colon \Omega \to \mathbb{C}^n$, given by

$$\psi = \lim_{k\to\infty} (DF(p))^{-k} F^k,$$

where the limit is taken uniformly on compact subsets of Ω. Here, by F^k we mean $F \circ \overset{k}{\ldots} \circ F$.

Example 4.14. The map $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$F(z_1, z_2) = \left(\frac{z_1}{2} + \left(\frac{z_2}{2} + z_1^2\right)^2, \frac{z_2}{2} + z_1^2\right)$$

is a holomorphic automorphism of $\mathbb{C}^2$ whose eigenvalues at the origin are $1/2$ and $-1/2$ and that leaves the origin fixed (i.e. $F(0,0) = (0,0)$). In other words, F satisfies the hypothesis of Theorem 4.13. Thus the basin of attraction of $(0,0)$ is a Fatou-Bieberbach domain. But Ω is not all $\mathbb{C}$ because $F(1/2, 1/2) = (1/2, 1/2)$.

On the choice of $(w_k)_k$: With the same notations as in Theorem 4.10, one could ask that if the sequence $(w_k)_k$ in the proof of Theorem 4.10 is fixed, then can we also find $f \colon \mathbb{D} \to \mathbb{C}^n$ a proper holomorphic embedding satisfying $f(w_k) = z_k$? The answer depends on the choice of $(w_k)_k$ and f_k. The reason is that we needed the automorphisms ϕ_k to satify $\phi_k(f_k(w_j)) = z_j$ for all $j \leq k$ (with some additional properties). It is not clear at all that such an automorphism exists for an arbitrary sequence $(w_k)_k$. In fact, it was shown in [22] that there exist pairs of discrete sequences $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ on $\mathbb{C}^n$ such that there is no $\phi \in \mathrm{Aut}(\mathbb{C}^n)$ sending $(a_k)_{k\in\mathbb{N}}$ to $(b_k)_{k\in\mathbb{N}}$.

Having discussed the two previous topics, we now move to the proof of the proposition of Combing hair by Holomorphic Automorphisms.

4.2.2 Combing hair by Holomorphic Automorphisms

To fully understand the proof, it is important for the reader to be familiar with Section A.3, since we will use some techniques regarding the $\overline{\partial}$-equation.

Proof of Combing hair by Holomorphic Automorphisms
Firstly, since γ^* is a $\mathcal{C}^r$-arc and homeomorphisms are approximable by $\mathcal{C}^r$-diffeomorphisms, we then can think that F is a $\mathcal{C}^r$-diffeomorphism from γ^* to another arc and that F is the identity in a neighborhood U of K. Shrinking U if needed we can assume that $\gamma^* \cap \overline{U} = C' \cap \overline{U}$ (because $F|_U = id_U$). Now we extend F to the identity to $\overline{U}$. Let now $H \colon [0,1] \times (\overline{U} \cup C) \to \overline{U} \cup C_t$ be a $\mathcal{C}^r$ map connecting $id_{\overline{U} \cup \gamma^*}$ to F such that

i) $H(t,\cdot)|_{\overline{U}} = id_{\overline{U}}$ for all $0 \leq t \leq 1$.

ii) H and $\frac{dH}{dt}(t,z)$ are $\mathcal{C}^r\Big([0,1] \times (\overline{U} \cup \gamma^*)\Big)$.

Here $C_t = H(t,\gamma^*)$. Observe that $C_t \cap \overline{U} = \gamma^* \cap \overline{U}$ for all $0 \leq t \leq 1$. Let now X be the vector field defined by the equation

$$\frac{dH}{dt}(t,z) = X(H(t,z)).$$

By ii), X is of class $\mathcal{C}^r$ on (z,t). To make computations easier to follow we include t as a complex variable. For that let us define

$$S = \bigcup_{t\in[0,1]} \{t\} \times C_t \ , \quad L_0 = [0,1] \times K \ , \quad L = L_0 \cup S.$$

Now because the set $K \cup C_t \subset \mathbb{C}^n$ is polynomially convex for all $0 \leq t \leq 1$ (this follows from a result in [7]) then L_0 and L are also polynomially convex.

Choose $U' \Subset \mathbb{C}$ to be a neighborhood of the segment $[0,1] \subset \mathbb{C}$, and let $U_0 = U \times U' \Subset \mathbb{C}^{n+1}$ be the corresponding neighborhood of L_0. We then extend the map X to $U_0 \cup S$ as

$$X(\zeta) = X(t,z) \ , \text{ where } \zeta = (t,z) \in S \quad \text{ and } X|_{U_0} = 0.$$

Observe that $S \subset \mathbb{C}^{n+1}$ is a totally real $\mathcal{C}^r$-manifold. Because X is of class $\mathcal{C}^r$ on S and zero on U_0, we can extend X to a map $X\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ of class $\mathcal{C}^r$ with compact support. Then by Lemma A.32 we have that

$$\overline{\partial} X(\zeta) = o(d(\zeta,S)^{r-1}) \quad \text{as } \zeta \to S \tag{9}$$

uniformly on compact sets of S. In (9), by $\overline{\partial} X$ we mean $\overline{\partial} X_k$ for all $1 \leq k \leq n+1$. Now for each compact $K' \subset \mathbb{R}^m$ we denote $K'_\varepsilon = \{x \in \mathbb{R}^m |\ d(x,K) < \varepsilon\}$. The proposition follows now from the following two lemmas and the local Andersén-Lempert theorem.

Lemma 4.15. With the same notation as above and $r \geq 3$. There exists $\varepsilon_0 > 0$ and $\nu\colon [0,\infty) \to [0,\infty)$ such that $\nu(t) > 0 \ \forall \ t > 0$ and $\nu(0) = 0$, and for every $0 < \varepsilon < \varepsilon_0$ there exists $Y_{\varepsilon_0}\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ entire with

$$||X - Y_\varepsilon||_{L_\varepsilon} \leq \nu(\varepsilon)\varepsilon.$$

Lemma 4.16. Let X,Y be two time-dependent Lipschitz vector fields on $\mathbb{R}^m$ with local flows ϕ and ψ (respectively). Suppose that $\phi(t,x)$ is defined for all $x \in K \Subset \mathbb{R}^m$ and $0 \leq t \leq 1$. Denote $K_t = \phi(t,K)$, and

$$\begin{aligned} A(\varepsilon) &= \sup\{|X(t,x) - Y(t,x)| \ : \ x \in (K_t)_\varepsilon, 0 \leq t \leq 1\}, \\ B &= \sup\{|X(t,x) - Y(t,y)| \ : \ x,y \in (K_t)_\varepsilon, 0 \leq t \leq 1\}. \end{aligned}$$

If $A(\varepsilon)e^B \leq \varepsilon \leq 1$, then the flow $\psi(t,x)$ is defined for all $x \in K$ and $0 \leq t \leq 1$, and

$$|\phi(t,x) - \psi(t,x)| \leq A(\varepsilon)e^{Bt}, \quad x \in K, \ 0 \leq t \leq 1.$$

In particular, $\psi(t,x) \in (K_t)_\varepsilon$ for $x \in K$ and $0 \leq t \leq 1$.

With these two lemmas, the proof of the proposition is as follows.

Let $\varepsilon > 0$ be so that Lemma 4.15 holds. Let Y be the entire approximation of X. Then using Lemma 4.16 we can take ε small enough so that the flow of Y, $G(t, z)$, exists and remains in $(K_t)_\varepsilon$ for all $0 \leq t \leq 1$, where $K_t = K \cup C_t \subset \mathbb{C}^n$ and

$$|H(t, z) - G(t, z)| < \varepsilon, \quad z \in K \cup \gamma^*,\ t \in [0, 1].$$

Finally $|H(1, z) - G(1, z)| = |F(z) - G(1, z)| < \varepsilon$ for all $z \in K \cup \gamma^*$. Then by the Local Andersén-Lempert Theorem applied to $G(1, \cdot)$, we find $\psi \in \mathrm{Aut}(\mathbb{C}^n)$ that $||\psi - G(1, \cdot)||_{K\cup\gamma^*} < \varepsilon$, therefore $||\psi - F||_{K\cup\gamma^*} < 2\varepsilon$. □

It now then remains to prove Lemma 4.15 and Lemma 4.16. To prove Lemma 4.15 we first need the following:

Lemma 4.17. With the notations of Proposition 4.12, $\exists \rho \geq 0$ a continuous plurisubharmoni exhaustion of $\mathbb{C}^{n+1}$ so that

i) $\rho^{-1}(0) = L = L_0 \cup S$.

ii) $\rho(z) \leq d(z, L)^2$ for z in a neighborhood of L.

iii) $\rho(z) = d(z, S)^2$ in a neighborhood of $S \setminus U_0$.

Proof. Because L is polynomially convex, there exists $\rho_1 \geq 0$ a smooth plurisubharmonic exhaustion of $\mathbb{C}^{n+1}$ so that $\rho_1^{-1}(0) = L$ and ρ_1 is strongly plurisubharmonic outside L. Now because ρ_1, ρ_1' and ρ_1'' vanish on L, by replacing ρ_1 by $c\rho_1$ with $c > 0$ small enough (if necessary) we have that $\rho_1(z) \leq d(z, L)^2$ in a small neighborhood of L. Thus ρ_1 satisfies both i) and ii). We now modify ρ_1 so that iii) also holds. Observe that $d(z, S)^2$ is strongly plurisubharmonic on a sufficiently small neighborhood $V = (S \setminus U_0)_\varepsilon$ of $S \setminus U_0$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{C}^{n+1})$ be real with support contained in $U_0 \cap V$ so that $\chi|_{\partial V\cap S} > 0$. Choose $\delta > 0$ small enough so that $\rho_2 = d(z, L)^2 - \delta\chi(z)$ is strongly plurisubharmonic in V and $\rho_2 = d(z, L)^2$ in a neighborhood of $S \setminus U_0$ (this can be achieved in a similar manner as in Lemma 4.7). Observe now that in a neighborhood of $\partial V \cap S \Subset U_0$ we have $\rho_2 < 0 \leq \rho_1$. Then the function

$$\rho_3 = \max(\rho_1, \rho_2)$$

is well defined, continuous and plurisubharmonic in a neighborhood $W \subset U_0 \cup V$ of L even smaller. We then have $\rho_3 = \rho_1$ near L_0, $\rho_3 = \rho_2$ in $W \setminus U_0$, and $\rho_3^{-1}(0) = L$.

Finally, we only need to choose $C > 0$ large enough and $c > 0$ small enough so that $\rho = \max(\rho_3, C(\rho_1 - c))$ is a plurisubharmonic extension of ρ_3, to get what we want. □

We now move to the proof of Lemma 4.15.

Proof. By the previous Lemma we can find $\rho \geq 0$ a continuous plurisubharmonic exhaustion of $\mathbb{C}^{n+1}$ so that $\rho^{-1}(0) = L$, $\rho(z) \leq d(z, L)^2$ in a neighborhood of L and $\rho(z) = d(z, S)^2$ in a neighborhood of $S \setminus U_0$. For $\varepsilon > 0$ let $\omega_\varepsilon = \{z \in \mathbb{C}^{n+1} |\ \rho(z) < \varepsilon^2\}$. Let $\varepsilon_0 > 0$ be such that $\omega_{\varepsilon_0} \subset L_{\varepsilon_0} \cup U_0$ and $\rho(z) \leq d(z, L)^2$, $\rho(z) = d(z, S)^2$ holds for $z \in \omega_{\varepsilon_0}$. Then for $0 < \varepsilon < \varepsilon_0$ we have

$$L_\varepsilon \subset \omega_\varepsilon \subset L_\varepsilon \cup U_0 \quad \text{and} \quad \omega_\varepsilon \setminus U_0 = L_\varepsilon \setminus U_0 = S_\varepsilon \setminus U_0.$$

Remember that $f = \overline{\partial} X$ satisfies

$$|f(z)| = o(d(z, L)^{r-1}) \ \text{ as } z \to L \quad \text{and } f|_{U_0} = 0.$$

Now we get that as $\varepsilon \to 0$

$$\int_{\omega_{3\varepsilon}} |f|^2 dV = \int_{L_{3\varepsilon}\setminus U_0} |f|^2 dV + \int_{U_0} |f|^2 dV = \int_{L_{3\varepsilon}\setminus U_0} o(d(z,L)^{r-1})^2 dV =$$
$$= o(\varepsilon^{2(r-1)}) \int_{S_{3\varepsilon}\setminus U_0} dV = o(\varepsilon^{2(r-1)}) O(\varepsilon^{2n}) = o(\varepsilon^{2(n+r-1)}).$$

Now fix $0 < \varepsilon \leq \varepsilon_0/3$. Let $\phi_\varepsilon = h_\varepsilon \circ \rho$, where $h_\varepsilon \colon \mathbb{R} \to \mathbb{R}_+$ is a convex increasing function with $h_\varepsilon(t) = 0$ for $t \leq 2\varepsilon$, and h_ε is sufficiently rapidly increasing in $t > 2\varepsilon$ so that

$$\int_{\mathbb{C}^{n+1}\setminus\omega_{3\varepsilon}} |f|^2 e^{-\phi_\varepsilon} dV \leq \int_{\omega_{3\varepsilon}} |f|^2 dV.$$

Then

$$\int_{\mathbb{C}^{n+1}} |f|^2 e^{-\phi_\varepsilon} dV = o(\varepsilon^{2(n+r-1)}).$$

Now applying Theorem A.35 to f (we can do this because $f \in L^2(\mathbb{C}^{n+1}, \phi_\varepsilon)$ and $\overline{\partial} f = \overline{\partial}\overline{\partial} X = 0$) we can deduce that there exists $u_\varepsilon \in L^2_{\text{loc}}(\mathbb{C}^{n+1})$ satisfying $\overline{\partial} u_\varepsilon = f$ and

$$\int_{\mathbb{C}^{n+1}} |u_\varepsilon(z)|^2 e^{-\phi_\varepsilon(z)} \frac{dV}{(1 + ||z||^2)^2} = o(\varepsilon^{2(n+r-1)}).$$

Now, because $L_{2\varepsilon} \subset \omega_{2\varepsilon}$, using Lemma A.33 we can deduce that $||u_\varepsilon||_{L_\varepsilon} = o(\varepsilon^{r-1})$ (we first apply the Lemma to f to get $||f||_{L_{2\varepsilon}} = o(\varepsilon^{r-2})$ and then apply it again to u_ε).

Finally, $Y_\varepsilon = X - u_\varepsilon \colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ satisfies $\overline{\partial} Y_\varepsilon = 0$, thus Y_ε is entire and $||X - Y_\varepsilon||_{L_\varepsilon} = o(\varepsilon^{r-2}) = \varepsilon o(\varepsilon^{r-3})$ This proves the lemma because $r \geq 3$. □

We complete the proof of Proposition 4.12 by proving Lemma 4.16.

Proof. Fix $x \in K$ and set $f(t) = |\phi(t,x) - \psi(t,x)|$, which is defined for $0 \leq t \leq t_0$ for some $t_0 > 0$. Then for all $0 \leq t \leq t_0$ we have

$$f(t) = \Big| \int_0^t X(s, \phi(s,x)) - Y(s, \psi(s,x)) ds \Big| \leq$$
$$\leq \Big| \int_0^t X(s, \phi(s,x)) - X(s, \psi(s,x)) ds \Big| + \Big| \int_0^t X(s, \psi(s,x)) - Y(s, \psi(s,x)) ds \Big| \leq$$
$$\leq B \int_0^t f(s) ds + A(\varepsilon).$$

Then using Gronwall's inequality (Proposition 2.4) we get

$$f(t) \leq A(\varepsilon) e^{Bt}$$

for all $0 \leq t \leq 1$ where the flow $\psi(t,x)$ is defined. Since by hypothesis we have that $A(\varepsilon)e^B \leq \varepsilon$, the previous inequality tells us that $\psi(t,x) \in (K_t)_\varepsilon$ where it is defined. Because x is arbitrary and K is compact we get that $\psi(t,x)$ is defined for all $x \in K$ and all $0 \leq t \leq 1$ thus proving the Lemma. □

4.3 Approximate straightening

In this section, we want to tackle the problem of approximately straighten a curve. More precisely, given $\gamma\colon [0,1] \to \mathbb{C}^n$ a smooth arc we want to find a sequence $(\psi_j)_j$ of automorphisms of $\mathbb{C}^n$ such that $\psi_j \circ \gamma$ converges uniformly on compact sets of $[0,1]$ to the map $\nu\colon [0,1] \to \mathbb{C}^n$ defined by $\nu(t) = (t, 0, \ldots, 0)$. To do that we follow J-P.Rosay's paper [17].

Here we do not prove the case where γ is smooth but the case where it is real analytic (which is easier for us).

Theorem 4.18. *Let $\gamma\colon [0,1] \to \mathbb{C}^n$ be an injective real analytic arc with $\gamma' \neq 0$. Then there is a sequence $(\psi_k)_k$ of automorphisms of $\mathbb{C}^n$ such that $\psi_k \circ \gamma$ converges uniformly on $[0,1]$ to the map $\nu(t) = (t, 0, \ldots, 0)$ ($t \in [0,1]$).*

The idea of the proof is first to extend γ to a neighborhood of $[0,1]$. Then consider a plurisubharmonic exhaustion ρ of $\mathbb{C}^n$ so that it vanishes on $[0,1]$. Then for $\varepsilon > 0$ small, the image by γ of the sets $\{\rho(z)\varepsilon\}$ will be Runge. This will put us in a position where we can use the Local Andersén-Lempert Theorem to get the result.

Proof. Since γ is real analytic, we can extend γ to $U \subset \mathbb{C}^n$ a neighborhood of $[0,1]$ (viewed as a subset of $\mathbb{C}^n$). Moreover, the extension of γ, from now on $\widetilde{\gamma}$, can be chosen to be holomorphic and injective in U (if needed, we choose U smaller).

Let now $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ be an auxiliary convex function vanishing in $[0,1]$. Let $\rho\colon \mathbb{C}^n \to \mathbb{R}^+$ be defined by

$$\rho(z_1, \ldots, z_n) = \varphi(x_1) + y_1^2 + \sum_{k=2}^{n} |z_k|^2,$$

where $x_1 = \mathrm{Re}(z_1)$ and $y_1 = \mathrm{Im}(z_1)$. Observe that by our choice of φ, ρ is a plurisubharmonic exhaustion of $\mathbb{C}^n$ vanishing only on $[0,1] \times \{(0, \ldots, 0)\}$. Then for every $\varepsilon > 0$, the set $U_\varepsilon = \{z \in \mathbb{C}^n \mid \rho(z) < \varepsilon\}$ is pseudoconvex (in fact it is convex). From now on consider $\varepsilon > 0$ small enough so that $U_\varepsilon \subseteq U$. We claim that for ε small enough, the set $\widetilde{\gamma}(U_\varepsilon)$ is Runge. Indeed, since γ^* is an arc of finite length, it is polynomially convex (Chapter 3 of [16]). Then γ^* has a basis of Stein neighborhoods that are Runge. Let $V \subset \widetilde{\gamma}(U)$ be such a neighborhood. Take $\varepsilon > 0$ so that $\widetilde{\gamma}(U_\varepsilon) \subset V$. Now $\rho \circ \widetilde{\gamma}^{-1}$ is plurisubharmonic, thus $\widetilde{\gamma}(U_\varepsilon)$ is pseudoconvex (because $\widetilde{\gamma}(U_\varepsilon) = \{z \in \mathbb{C}^n \mid \rho \circ \widetilde{\gamma}^{-1}(z) < \varepsilon\}$). In the end, because $\widetilde{\gamma}(U_\varepsilon) \subset V$ and both are pseudoconvex, we deduce that $\widetilde{\gamma}(U_\varepsilon)$ is Runge in V (here we are using Theorem A.28, in Section A) But because V is Runge, it follows that $\widetilde{\gamma}(U_\varepsilon)$ is also Runge, just as we wanted.

Finally, by our choices of $\widetilde{\gamma}$ and U_ε, the map $\widetilde{\gamma}\colon U_\varepsilon \to \widetilde{\gamma}(U_\varepsilon)$ is a biholomorphism. And because $\widetilde{\gamma}(U_\varepsilon)$ is Runge, using the Local Andersén-Lempert Theorem we find a sequence $(\varphi_k)_k$ of automorphisms of $\mathbb{C}^n$ such that $\varphi_k \to \widetilde{\gamma}$ uniformly on compact sets of U_ε, as $k \to \infty$. Putting $\psi_k = \varphi_k^{-1}$ gives us the desired sequence. That is, a sequence $(\psi_k)_k$ of automorphisms of $\mathbb{C}^n$ such that $\psi_k \circ \gamma$ converges uniformly on $[0,1]$ to the map $t \mapsto (t, 0, \ldots, 0)$ (in fact $\psi_k \circ \gamma$ converges to $\mathrm{Id}_{U_\varepsilon}$ uniformly on compact sets of U_ε). □

Remark 4.19. As mentioned earlier, the previous theorem can be generalized to smooth arcs. But one has to be careful because by generalizing the arguments we have given, the result weakens a bit replacing uniform convergence for convergence in the $\mathcal{C}^\infty$ topology. Although it seems likely that the theorem could be true for uniform convergence in the smooth case (even in the case $\mathcal{C}^r$ with $r \geq 3$) using the Combing Hair Proposition (Proposition 4.12) of the previous section.

5 Conclusions

We consider that we have met our principal objectives, to prove the different versions of the Andersén-Lempert Theorem and give some relevant consequences of these theorems. We hope that we have been able to illustrate the difficulties one can encounter when working with automorphisms of $\mathbb{C}^n$.

Of course, there is a lot more that could be said about the space $\mathrm{Aut}(\mathbb{C}^n)$. In fact, automorphisms of $\mathbb{C}^n$ have been studied recently, as shown in [13], [8], and [21] (for example). If the reader wishes to know more about this topic, we highly recommend taking a look at [11], where a lot of results surrounding automorphism of $\mathbb{C}^n$ are presented.

Appendices

A Preliminaries

This section is devoted to defining the necessary concepts and key properties from the theory of several complex variables that we need in order to understand and follow this work. If the reader is familiar with the theory of several complex variables, he or she may skip this section.

Because we are only interested in results, we will not give most of the proofs and we will only present those that are relevant to our work (if the reader is interested in the proofs, we highly recommend taking a look at [9], [15] and [14]).

A.1 Holomorphic functions of several variables

We begin by defining the concept of a holomorphic function of several variables.

Definition A.1. Let $U \subset \mathbb{C}^n$ be open. A map $f\colon U \to \mathbb{C}$ is said to be *holomorphic in* U if f is holomorphic in each variable.

This definition is not enough for our work since we need to talk about holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ (or some subsets of $\mathbb{C}^n$). For that we have the following:

Definition A.2. Let $U \subset \mathbb{C}^n$ be open. Then a map $f\colon U \to \mathbb{C}^m$ is said to be *holomorphic in* U if each component of f is holomorphic. More concretely, if

$$f(z) = \big(f_1(z), \dots, f_m(z)\big), \quad z = (z_1, \dots, z_n) \in U$$

then f is holomorphic if each f_j is holomorphic (with $1 \leq j \leq m$).

As in the theory of one complex variable, we say that f is holomorphic on a closed set C if there exists an open set $U \supset C$ such that f can be extended to U and the extension is holomorphic. We will write $\mathcal{H}(A)$ to denote the space of holomorphic functions in $A \subset \mathbb{C}^n$.

We can now define the object of our study.

Definition A.3. Let $U \subset \mathbb{C}^n$ be open with $n \geq 1$. A map $f\colon U \to \mathbb{C}^m$ with $m \geq 1$, is called a *biholomorphism of* U if f is holomorphic, injective and it has a holomorphic inverse defined on the range of f. If in addition the domain and the range of f coincide, we then say that f is an *automorphism of* U. We will denote the space of automorphisms of U as $\mathrm{Aut}(U)$.

A.1.1 The Cauchy integral formula

As in the theory of one complex variable, in several complex variables there is also a Cauchy integral formula. With one complex variable, the Cauchy integral formula holds on disks, but with several complex variables, the Cauchy integral formula holds on *polydisks*.

Definition A.4. Let $n > 1$ be an integer, $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$. The *polydisk* of center a and radius r is the set

$$\begin{aligned}\Delta^n(a, r) &= \big\{(z_1, \dots, z_n) \in \mathbb{C}^n \;\big|\; |z_1 - a_1| < r_1, \dots, |z_n - a_n| < r_n\big\} = \\ &= D(a_1, r_1) \times \cdots \times D(a_n, r_n)\end{aligned}$$

where $D(a_j, r_j)$ is the disk of center a_j and radius r_j in $\mathbb{C}$.

The *distinguished boundary* (also called *skeleton*) is the set

$$\mathrm{b}\Delta^n(a, j) = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n \;\big|\; |z_1 - a_1| = r_1, \dots, |z_n - a_n| = r_n\big\}$$

which is the product of the n circles with center a_j and radius r_j in $\mathbb{C}$.

It is important to not confuse the concept of polydisk in $\mathbb{C}^n$ with that of a ball in $\mathbb{C}^n$. For example $(1/\sqrt{2}, 1/\sqrt{2}) \in \Delta^2(0,(1,1))$ since $1/\sqrt{2} < 1$, but $(1/\sqrt{2}, 1/\sqrt{2}) \notin B(0,1)$ since $||(1/\sqrt{2}, 1/\sqrt{2})|| = 1$ (here $B(a,r)$ denotes the ball of center $a \in \mathbb{C}^2$ and radius $r > 0$). In fact we always have the inclusion $B(a,r) \subset \Delta^n(a,(r,\ldots,r))$, but not the other way around.

In a similar manner, the skeleton of a polydisk is not the same as the boundary. As an example, consider the polydisk $\Delta^2(0,(1,1))$. Then $(1,0) \in \partial\Delta^2(0,(1,1))$ but $(1,0) \notin \mathrm{b}\Delta^2(0,(1,1))$. We always have that $\mathrm{b}\Delta^n(a,r) \subset \partial\Delta^n(a,r)$, but not the other way around.

Having discussed what a polydisk is, we are now able to state the Cauchy integral formula.

Theorem A.5 (Cauchy's integral formula for polydisks)**.** *Let $a \in \mathbb{C}^n$, $r \in \mathbb{R}^n_+$. If $f \in \mathcal{H}(\overline{\Delta^n(a,r)})$ then for every $z \in \Delta^n(a,r)$*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\mathrm{b}\Delta^n(a,r)} \frac{f(w)}{w-z} dw = \frac{1}{(2\pi i)^n} \int_{\mathrm{b}\Delta^n(a,r)} \frac{f(w_1,\ldots,w_n)}{(w_1-z_1)\cdots(w_n-z_n)} dw_1 \ldots dw_n.$$

From this theorem, there can be deduced two major corollaries.

Corollary A.6. Let $U \subset \mathbb{C}^n$ be open, $f \in \mathcal{H}(U)$. Then $f \in \mathcal{C}^\infty(D)$ and for every multi-index α, $\partial^\alpha f \in \mathcal{H}(U)$. Moreover, for every polydisk $\Delta^n(a,r) \Subset U$ (i.e. $\overline{\Delta^n(a,r)}$ is compact in U)

$$\partial^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{\mathrm{b}\Delta^n(a,r)} \frac{f(w)}{(w-z)^\alpha} dw, \quad z \in \Delta^n(a,r)$$

where $\alpha + 1 = (\alpha_1 + 1, \ldots, \alpha_n + 1)$.

Corollary A.7 (Cauchy inequalities)**.** Let $a \in \mathbb{C}^n$, $r \in \mathbb{R}^n_+$, and $f \in \mathcal{H}\Big(\overline{\Delta^n(a,r)}\Big)$. Then for every multi-index α

$$|\partial^\alpha f(a)| = \left|\frac{\partial^\alpha f}{\partial z^\alpha}(a)\right| \leq \frac{\alpha!}{r^\alpha} \sup_{z \in \overline{\Delta^n(a,r)}} |f(z)|.$$

A.1.2 Series expansion

We end this section of holomorphic functions by stating the following theorem regarding holomorphic functions and their Taylor series expansion.

Theorem A.8. *Let $U \subset \mathbb{C}^n$ be open and $f \in \mathcal{H}(U)$. Then f has a Taylor expansion locally in U, i.e. for every $w \in U$ there is a neighborhood V of w so that for every $z \in V$ we have*

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha}(w)(z-w)^\alpha,$$

where the series converges uniformly to f on compact sets of U.

The converse is also true, that is, every power series that converges uniformly on compact sets of an open set U, defines a holomorphic function whose Taylor expansion coincides with the series.

A.2 Domains

Another important concept (or rather concepts) in the theory of several complex variables are the different types of domains and how they relate to each other.

A.2.1 Runge domains

In the theory of one complex variable Runge's approximation theorem ([5]) asserts that given a holomorphic function $f\colon K \subset \mathbb{C} \to \mathbb{C}$, if K is a compact set such that $\mathbb{C} \setminus K$ is connected, then f can be uniformly approximated in K by polynomials. The importance of this theorem is that we only need a topological property on K to ensure that f can be approximated by polynomials uniformly. This is, in general, not the case in $\mathbb{C}^n$. That is why the following notion is useful:

Definition A.9. We say that a domain $\Omega \subset \mathbb{C}^n$ is *Runge* if for every $f \in \mathcal{H}(\Omega)$, f can be uniformly approximated on compact sets of Ω by polynomials.

More generally, given two domains $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$, we say that Ω_1 is *(relatively) Runge in* Ω_2 if for every $f \in \mathcal{H}(\Omega_1)$, f can be uniformly approximated on compact sets of Ω_1 by functions in $\mathcal{H}(\Omega_2)$.

Observe that since entire functions can be uniformly approximated on compact sets (by their Taylor expansion), saying that Ω is Runge is the same to say that Ω is Runge in $\mathbb{C}^n$.

Another useful observation is that if $\Omega_1 \subset \Omega_2 \subset \Omega_3$ are domains in $\mathbb{C}^n$ such that Ω_1 is Runge in Ω_2, and Ω_2 is Runge in Ω_3, then Ω_1 is Runge in Ω_3.

Example A.10. Every star-shaped domain Ω is a Runge domain ([1]).

A.2.2 Plurisubharmonicity

To give all the different types of domains, we need to talk about plurisubharmonic functions, which are a generalization of subharmonic functions to higher dimensions. Let us first recall what a subharmonic function is, for that we first need to define upper semicontinuous functions.

Definition A.11. Let X be a topological space. We say that $u\colon X \to [-\infty, +\infty)$ is *upper semi-continuous* if for every $a \in \mathbb{R}$, the set $u^{-1}\big([-\infty, a)\big)$ is open in X.

Definition A.12. Let $U \subset \mathbb{C}$ be open. A map $u\colon U \to [-\infty, +\infty)$ is *subharmonic* if it is upper semicontinuous and satisfies the following. For every $z \in U$ there exists $\rho > 0$ (depending on z) such that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt$$

for all $0 \leq r < \rho$.

The next proposition gives us a wide range of examples of subharmonic functions.

Proposition A.13. Let $U \subset \mathbb{C}$ be open, and $f \in \mathcal{H}(U)$. Then $\log |f(z)|$ is a subharmonic function on U.

We can now begin to talk about plurisubharmonic functions.

Definition A.14. Let $\Omega \subset \mathbb{C}^n$ be open. A map $u\colon \Omega \to [-\infty, +\infty)$ is *plurisubharmonic* if

i) u is upper semi-continuous.

ii) For every $z_0 \in \Omega$ and $a \in \mathbb{C}^n$, the restriction of u to the set $\{w \in \mathbb{C} |\ z_0 + wa \in \Omega\}$ is a subharmonic function.

We will denote $\mathcal{P}(\Omega)$ the set of all plurisubharmonic functions on Ω.

We have a similar proposition as in the case of one dimension.

Proposition A.15. Let $\Omega \subset \mathbb{C}^n$ be open and $f \in \mathcal{H}(\Omega)$. Then $\log|f|$ and $|f|^p$, for $p \geq 1$, are plurisubharmonic functions in Ω.

This proposition gives a wide family of examples of plurisubharmonic functions. Another big family of plurisubharmonic functions is the family of convex functions.

If we require $u \in \mathcal{C}^2(\Omega)$, checking if u is plurisubharmonic is far easier, as the following proposition shows.

Proposition A.16. Let $\Omega \subset \mathbb{C}^n$ be open, and $u \in \mathcal{C}^2(\Omega)$. Then u is plurisubharmonic if and only if for every $z \in \Omega$, the hermitian matrix $L_u = \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(z)\right)_{1 \leq i,j \leq n}$ (called the *Levi matrix of* u) is positive semidefinite.

Some useful properties of plurisubharmonic functions are the following:

Proposition A.17. Let $\Omega \subset \mathbb{C}^n$ be open, $u_1, u_2 \colon \Omega \to [-\infty, +\infty)$ be plurisubharmonic. Then

i) If $c > 0$ then cu_1 is plurisubharmonic.

ii) $u_1 + u_2$ is plurisubharmonic.

iii) If $\phi \colon \mathbb{R} \to \mathbb{R}$ is a monotonically increasing convex function, then $\phi \circ u_1$ is plurisubharmonic.

iv) $u(z) = \max\big(u_1(z), u_2(z)\big)$ is plurisubharmonic.

Having introduced plurisubharmonic functions, we introduce Levi pseudoconvex sets.

Definition A.18. Let $\Omega \subset \mathbb{C}^n$ be a domain with $\mathcal{C}^2$ boundary. Let $\rho \colon \mathbb{C}^n \to \mathbb{R}$ be a defining function for Ω, that is ρ is such that

$$\Omega = \{z \in \mathbb{C}^n \mid \rho(z) > 0\}$$

and $\nabla\rho(z) \neq 0$ for $z \in \partial G$. Then the point $p \in \partial\Omega$ is called *Levi pseudoconvex* if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \overline{w}_k \geq 0, \tag{10}$$

for all $w \in T_p^{(1,0)}(\partial\Omega)$. We say that p is *strongly Levi pseudoconvex* if the inequality in (10) is strict.

We will say that Ω is (*strictly*) *Levi pseudoconvex* if every point in $\partial\Omega$ is (strictly) Levi pseudoconvex.

Note that in particular, all convex domains are Levi pseudoconvex.

A.2.3 Domains of holomorphy, polynomial convexity, and pseudoconvexity

We now present the concept of a *domain of holomorphy*.

Definition A.19. Let $\Omega \subset \mathbb{C}^n$ be a domain. We say that Ω is a *domain of holomorphy* if for every connected domain U that intersects the boundary $\partial\Omega$ and for every component Ω_1 of $U \cap \Omega$, there is $f \in \mathcal{H}(\Omega)$ whose restriction $f|_{\Omega_1}$ has no holomorphic extension to U.

We also say that $\widetilde{\Omega}$ is an *envelope of holomorphy* for Ω if

i) $\Omega \subset \widetilde{\Omega}$ and every $f \in \mathcal{H}(\Omega)$ can be extended to $\widetilde{\Omega}$.

ii) For every $a \in \partial\widetilde{\Omega}$ there exists $f \in \mathcal{H}(\widetilde{\Omega})$ which has no holomorphic extension to a neighborhood of a.

Roughly speaking, this definition tells us that not every holomorphic function on Ω can be extended outside of Ω. The following proposition gives us some properties of domains of holomorphy.

Proposition A.20. i) If $\Omega_1, \ldots, \Omega_k \subset \mathbb{C}^n$ are domains of holomorphy, then $\Omega = \bigcap_{j=1}^k \Omega_j$ is also a domain of holomorphy.

ii) If $(\Omega_j)_{j\in\mathbb{N}}$ is an increasing sequence of domains of holomorphy in $\mathbb{C}^n$, then $\Omega = \bigcup_{j=1}^\infty \Omega_j$ is also a domain of holomorphy (this is refered to as the Behnke-Stein Theorem).

iii) If Ω_1 and Ω_2 are domains of holomorphy in $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$ (respectively), then $\Omega_1 \times \Omega_2$ is a domain of holomorphy in $\mathbb{C}^{n_1+n_2}$.

To get an idea of how domains of holomorphy look like, we give some examples.

Example A.21. i) In $\mathbb{C}$ every domain Ω is a domain of holomorphy. Take for $a \in \partial\Omega$ the function $f(z) = 1/(z-a)$.

ii) In $\mathbb{C}^n$ every "polydomain" $\Omega = \Omega_1 \times \cdots \times \Omega_n$, with $\Omega_j \subset \mathbb{C}$ a domain, is a domain of holomorphy. It is enough to consider the functions $f(z) = 1/(z_j - a_j)$ for $a_j \in \Omega_j$ (this fact can also be deduced from the first example and iii) of the previous proposition).

iii) Every convex domain in $\mathbb{C}^n$ is a domain of holomorphy.

The latter example raises the question of whether every domain of holomorphy is convex. The answer is no, but all domains of holomorphy have some type of convex property.

Definition A.22. Let Ω be a domain in $\mathbb{C}^n$, and $K \Subset \Omega$ a compact set. The set

$$\widehat{K}_\Omega = \{z \in \mathbb{C}^n \mid |f(z)| \le ||f||_K, \text{ for all } f \in \mathcal{H}(\Omega)\} \tag{11}$$

is called the *holomorphically convex hull* of K. We say that K is *holomorphically convex* if and only if $K = \widehat{K}_\Omega$. More generally, we say that Ω is *holomorphically convex* if for every $K \Subset \Omega$ we have $\widehat{K}_\Omega \Subset \Omega$.

It turns out that $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy if and only if Ω is holomorphically convex. From this, it is natural to ask if the same happens when in (11) we take f to belong to another family instead of $\mathcal{H}(\Omega)$. This leads to the following two definitions.

Definition A.23. Let Ω be a domain in $\mathbb{C}^n$, and $K \Subset \Omega$ a compact set. The set

$$\widehat{K} = \{z \in \mathbb{C}^n \mid |P(z)| \le ||P||_K, \text{ for all } P \text{ a holomorphic polynomial}\} \tag{12}$$

is called the *polynomially convex hull* of K. We say that K is *polynomially convex* if and only if $K = \widehat{K}$. More generally, we say that Ω is *polynomially convex* if for every $K \Subset \Omega$ we have $\widehat{K} \Subset \Omega$.

Definition A.24. Let Ω be a domain in $\mathbb{C}^n$, and $K \Subset \Omega$ a compact set. The set

$$\widehat{K}_{\mathcal{P}(\Omega)} = \{z \in \mathbb{C}^n \mid |h(z)| \le ||h||_K, \text{ for all } h \in \mathcal{P}(\Omega)\} \tag{13}$$

is called the *pseudoconvex hull* of K. We say that K is *pseudoconvex* if and only if $K = \widehat{K}_{\mathcal{P}(\Omega)}$. More generally, we say that Ω is *pseudoconvex* if for every $K \Subset \Omega$ we have $\widehat{K}_{\mathcal{P}(\Omega)} \Subset \Omega$.

It can be seen that if instead of $\mathcal{H}(\Omega)$ we put L (the set of linear functions in $\mathbb{C}^n$), then Ω is L-convex if and only if Ω is convex.

Observe that for $K \Subset \Omega$ compact we always have the inclusions $K \subset \widehat{K}_\Omega \subset \widehat{K} \subset \widehat{K}_L$, and $K \subset \widehat{K}_{\mathcal{P}(\Omega)}$. Therefore, every convex set is also polynomially convex. Another example of interesting polynomially convex set is the following:

Example A.25. Every finite arc C in $\mathbb{C}^n$ is polynomially convex ([19]).

The following theorem shows us how these different types of convexity relate to each other.

Theorem A.26. *Let $\Omega \subset \mathbb{C}^n$ be a domain. The following statements are equivalent:*

i) Ω is a domain of holomorphy.

ii) Ω is holomorphically convex.

iii) Ω is pseudoconvex.

iv) Ω is Levi pseudoconvex (assuming Ω has $\mathcal{C}^2$ boundary).

v) Ω has a continuous ($\mathcal{C}^\infty$) plurisubharmonic exhaustion ρ such that every $\Omega_c = \{z \in \Omega \mid \rho(z) < c\} \Subset \Omega$ (each Ω_c being pseudoconvex).

vi) Ω has a smooth ($\mathcal{C}^\infty$) plurisubharmonic exhaustion ρ such that every $\Omega_c = \{z \in \Omega \mid \rho(z) < c\} \Subset \Omega$ (each Ω_c being pseudoconvex).

In the two last statements, the sets Ω_c are also relatively Runge to each other. We will call such a family $(\Omega_c)_c$ a *basis of Stein neighborhoods of* Ω.

The last theorem does not tell us how the notions of pseudoconvexity relate to polynomial convexity. For that we have the following theorems:

Theorem A.27. *Let $K \subset \mathbb{C}^n$ be a polynomially convex compact set. Then there exists a non-negative plurisubharmonic exhaustion $\rho\colon \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ such that*

a) $K = \rho^{-1}(0)$.

b) ρ is strictly plurisubharmonic on $\mathbb{C}^n \setminus K$

c) $\lim\limits_{||z|| \to \infty} \rho(z) = \infty$.

Conversely, if ρ is a non-negative plurisubharmonic exhaustion of $\mathbb{C}^n$ such that $\lim\limits_{||z|| \to \infty} \rho(z) = \infty$, then $\rho^{-1}(0)$ is polynomially convex.

Theorem A.28. *Let Ω be a domain of holomorphy, and $K \subset \Omega$ a compact pseudoconvex set. Then K is Runge in Ω.*

Theorem A.29. *Let $\Omega \subset \mathbb{C}^n$ be a domain of holomorphy. Then the following are equivalent:*

i) Ω is Runge.

ii) Ω is a polynomially convex domain.

From these theorems, we can see that there is a close relation between polynomially convex domains and Runge domains.

On the one-dimensional setting, this last theorem and Runge's approximation theorem tells us that every polynomially convex set in $\mathbb{C}$ is simply connected. Thus the polynomially convex hull of $K \subset \mathbb{C}$ is obtained by "filling the holes" in K.

One final concept related to polynomial convexity is the following:

Definition A.30. Let $\Omega \subset \mathbb{C}^n$ be a domain. We say that Ω has a *basis of Stein neighborhoods* if there exists a sequence $(U_k)_k$ of pseudoconvex domains in $\mathbb{C}^n$ with $\overline{\Omega} \subset U_k$ and $\Omega = \bigcap_{k=1}^{\infty} U_k$.

The relationship between this new concept and pseudoconvexity is given by the following theorem.

Theorem A.31. *Every $\Omega \subset \mathbb{C}^n$ pseudoconvex Runge domain has a basis of Stein neighborhoods which are Runge.*

With this result, we end the section on domains and move to the next one.

A.3 The $\overline{\partial}$-equation

In the theory of one complex variable one can define the operator $\overline{\partial}$ as

$$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big),$$

where $z = x + iy$. This operator turns out to be very important since it characterizes holomorphic functions in the sense that f is holomorphic if and only if $\overline{\partial} f = 0$ (if f is supposed to be of class $\mathcal{C}^1$). The same notion can be generalized to several complex variables.

Given a function $f \in \mathcal{C}^1(\Omega)$ where Ω is a domain in $\mathbb{C}^n$, we define $\overline{\partial} f$ as the $(0,1)$-form

$$\overline{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z}_k}(z) d\overline{z}_k$$

where

$$\frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\Big) \tag{14}$$

and x_k, y_k are the real and imaginary parts of the coordinate z_k. Observe that in this setting f is holomorphic in Ω if and only if $\overline{\partial} f = 0$.

If we consider the differential operators in (14) to be defined in the sense of distributions, then we can extend the notion of $\overline{\partial} f$ to a more general setting where $f \in L^2(\Omega)$. We can also extend the $\overline{\partial}$ operator in another useful way, namely to (p,q)-forms. Let f be (p,q)-form with coefficients in $L^2(\Omega)$ (we will write $f \in L^2_{(p,q)}(\Omega)$). If f is given by

$$f = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} f_{\alpha\beta} dz^{\alpha} \wedge d\overline{z}^{\beta}$$

where α and β are multi indices in $\mathbb{N}^n$ in ascending order. Then we define $\overline{\partial} f$ as the $(p, q+1)$-form

$$\overline{\partial} f = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \Big(\sum_{k=1}^{n} \frac{\partial f_{\alpha\beta}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz^{\alpha} \wedge d\overline{z}^{\beta}\Big).$$

Observe that in the case where $f \in \mathcal{C}^1_{(p,q)}(\Omega)$, the fact that $\overline{\partial} f = 0$ does not imply that f is a holomorphic form. This is only true in the case $q = 0$. For example, the form given by

$$f(z) = ||z|| d\overline{z}_1 \wedge d\overline{z}_2 \wedge \ldots \wedge d\overline{z}_n \;\;, \quad z \in \mathbb{C}^n$$

clearly satisfies $\overline{\partial} f = 0$, but by no means $||z||$ is a holomorphic function. So we need to be careful when working with (p,q)-forms. Next, we enunciate some results that will be useful for our study.

The following lemmas are Lemma 4.3 and Lemma 4.4 of [10].

Lemma A.32. Let S be a closed subset of an open set $\Omega \subset \mathbb{C}^n$, and let K be a compact subset of S such that $S \setminus K$ is a totally real $\mathcal{C}^r$-manifold, with $r \geq 1$ (here totally real means that the tangent space at each point is real). Let $f \in \mathcal{C}^r(\Omega)$ and $\overline{\partial} f = 0$ in a neighborhood of K. Then there exists $u \in \mathcal{C}^r(\Omega)$ so that $u = f$ on S and

$$\overline{\partial} u(z) = o\big(d(z,S)^{r-1}\big) \quad \text{as } z \to S,$$

uniformly on compact sets of S.

Lemma A.33. Let $B(0,\varepsilon) = \{z \in \mathbb{C}^n |\ ||z|| < \varepsilon\}$. Let $u \in L^2(B(0,\varepsilon))$ satisfy $\overline{\partial} u = f$ in the sense of distributions. If f is continuous, then u is also continuous and

$$|u(0)| \leq C(\varepsilon^{-n}||u||_{L^2(B(0,\varepsilon))} + \varepsilon||f||_{B(0,\varepsilon)}).$$

This estimate can be generalized provided some additional assumptions are made (as the following remark explains).

Remark A.34. If $\Omega \subset \mathbb{C}^n$ is open and $u \in L^2(\Omega)$ satisfies $\overline{\partial} u = f$ on Ω, (with $f \in \mathcal{C}(\Omega)$) then for a fixed $z \in \Omega$ and a $\varepsilon > 0$ such that $B(z,\varepsilon) \subset \Omega$, we can apply Lemma A.33 to the function $u \circ \tau_{-z}$ where $\tau_{-z}(w) = w + z$ because $u \circ \tau_{-z} \in L^2(B(0,\varepsilon))$ and $\overline{\partial}(u \circ \tau_{-z}) = f \circ \tau_{-z}$. Then keeping in mind that $u(\tau_{-z}(0)) = u(z)$, we get

$$|u(z)| \leq C(\varepsilon^{-n}||u||_{L^2(B(z,\varepsilon))} + \varepsilon||f||_{B(z,\varepsilon)}) \leq C(\varepsilon^{-n}||u||_{L^2(\Omega)} + \varepsilon||f||_{\Omega}).$$

Of course this estimate is not independent of z since ε depends on z. But for every compact $K \subset \Omega$, by covering K with enough balls and using the compactness of K, we can make the estimate independent of z (say with $\varepsilon = \varepsilon_M$) to finally get

$$||u||_K \leq C(\varepsilon_M^{-n}||u||_{L^2(K)} + \varepsilon_M||f||_K).$$

Another useful result (the last one we mention) in estimating the solution to the equation $\overline{\partial} u = f$ is the following:

Theorem A.35. *Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and φ a plurisubharmonic function in Ω. For every $f \in L^2_{(p,q+1)}(\Omega,\varphi)$ with $\overline{\partial} f = 0$ there exists $u \in L^2_{(p,q),loc}(\Omega)$ such that $\overline{\partial} u = f$ and*

$$\int_\Omega |u(z)|^2 \frac{e^{-\varphi(z)}}{(1+||z||^2)^2} dV \leq \int_\Omega |f(z)|^2 e^{-\varphi(z)} dV.$$

Here $u \in L^2_{(p,q+1)}(\Omega,\varphi)$ if u is a $(p,q+1)$-form such that $||ue^{-\varphi}||_2 < \infty$, and $L^2_{(p,q),\text{loc}}(\Omega)$ is the space of locally square-integrable (p,q)-forms.

This ends Appendix A.

References

[1] El Kasimi Abdeslem. “Approximation polynômiale dans les domaines étoilés de $\mathbb{C}^n$). (Polynomial approximation on starshaped sets in $\mathbb{C}^n$”. French. In: *Complex Variables, Theory Appl.* 10.2-3 (1988), pp. 179–182. ISSN: 0278-1077; 1563-5066/e.

[2] Erik Andersén. “Volume-preserving automorphisms of $\mathbb{C}^n$”. English. In: *Complex Variables. Theory and Application* 14.1-4 (1990), pp. 223–235. ISSN: 0278-1077; 1563-5066/e.

[3] Erik Andersén and László Lempert. “On the group of holomorphic automorphisms of $\mathbb{C}^n$”. English. In: *Invent. Math.* 110.2 (1992), pp. 371–388. ISSN: 0020-9910; 1432-1297/e.

[4] Vladimir I. Arnold. *Ordinary Differential Equations.* Springer-Verlag Berlin Heidelberg, 1992. ISBN: 978-3-540-34563-3.

[5] John B.Conway. *Functions of One Complex Variable I.* 2nd ed. Vol. 11. Graduate Texts in Mathematics. Springer-Verlag New York, 1978. ISBN: 978-0-387-90328-6. DOI: 10.1007/978-1-4612-6313-5.

[6] F.Forstneric, J.Globevnik, and B. Stensønes. “Embedding holomorphic discs through discrete sets.” In: *Mathematische Annalen* 305.none (1996), pp. 559–569. DOI: 10.1007/BF01444237. URL: https://doi.org/10.1007/BF01444237.

[7] G.Stolzenberg. “Polynomially and rationally convex sets”. In: *Acta Mathematica* 109.none (1963), pp. 259 –289. DOI: 10.1007/BF02391815. URL: https://doi.org/10.1007/BF02391815.

[8] Hengnan Hu, Ser Peow Tan, and Ying Zhang. “Polynomial automorphisms of $\mathbb{C}^n$ preserving the Markoff-Hurwitz polynomial”. English. In: *Geom. Dedicata* 192 (2018), pp. 207–243. ISSN: 0046-5755; 1572-9168/e.

[9] L. Hörmander. *An Introduction to complex analysis in several variables, 3d edition.* North-Holland mathematical library ; 7. Amsterdam [etc.] : North-Holland, 1990, 1990. ISBN: 0444884467.

[10] L. Hörmander and J.Wermer. “Uniform approximation on compact sets in $\mathbb{C}^n$”. In: *Mathematica Scandinavica* 23.1 (1968), pp. 5–21. URL: http://www.jstor.org/stable/24489786.

[11] J-P.Roay. “Automorphisms of $\mathbb{C}^n$, A Survey of Andersén-Lempert Theory and Applications.” In: *Contemporary Mathematics* 222.none (1999), pp. 131–145.

[12] E. Kallin. “Polynomial convexity: The three spheres problem”. In: (1964), pp. 301–304. DOI: 10.1007/978-3-642-48016-4_26. URL: https://doi.org/10.1007/978-3-642-48016-4_26.

[13] Marek Karaś. “Multidegrees of tame automorphisms of $\mathbb{C}^n$)”. English. In: *Diss. Math.* 477 (2011), p. 55. ISSN: 0012-3862; 1730-6310/e.

[14] Jaap Korevaar and Jan Wiegerinck. *Several Complex Variables*. 2017. URL: https://staff.science.uva.nl/j.j.o.o.wiegerinck/edu/scv/scvboek.pdf.

[15] Steven G. Krantz. *Function Theory of Several Complex Variables*. AMS Chelsea Publishing, 2000. ISBN: 978-0-8218-2724-6. DOI: https://doi.org/10.1090/chel/340.

[16] John M.Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics ; 218. Springer-Verlag New York, 2003. ISBN: 978-0-387-21752-9. DOI: 10.1007/978-0-387-21752-9.

[17] J-P. Rosay. “Straightening of arcs”. In: *Colloque d'analyse complexe et géométrie. Marseille, France, 13-17 janvier 1992* (1993), pp. 217–225.

[18] Daniel Azagra Rueda. *Intorducción a la Teoría de Ecuaciones Diferenciales Ordinarias.* 2018. URL: http://www.mat.ucm.es/~dazagrar/docencia/ManualEDIFDanielAzagra.pdf.

[19] Edgar Lee Stout. *Polynomial Convexity.* Vol. 261. Progress in Mathematics. Birkhäuser Basel, 2007. ISBN: 978-0-8176-4537-3. DOI: 10.1007/978-0-8176-4538-0.

[20] Masakazu Suzuki. "Propriétés topologiques des polynômes de deux variables complexes et automorphismes algébriques de l'espace $\mathbb{C}^2$". French. In: *J. Math. Soc. Japan* 26 (1974), pp. 241–257. ISSN: 0025-5645; 1881-1167; 1881-2333/e.

[21] Riccardo Ugolini. "A parametric jet-interpolation theorem for holomorphic automorphisms of $\mathbb{C}^n$". English. In: *J. Geom. Anal.* 27.4 (2017), pp. 2684–2699. ISSN: 1050-6926; 1559-002X/e.

[22] W.Rudin and J-P.Rosay. "Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$". In: *Transactions of the American Mathematical Society* 310.1 (1988), pp. 47–86. ISSN: 0002-9947; 1088-6850/e.