# An introduction to the Ryu-Takayanagi conjecture

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**Abstract:** The Ryu-Takayanagi conjecture proposes a holographic derivation of the entanglement entropy of a conformal field theory (CFT) in the context of the AdS/CFT correspondence. To understand the conjecture, we study the features of Anti-de Sitter (AdS) spacetime and explain the concept of the entanglement entropy for quantum mechanics. Finally, we use the conjecture to compute the entanglement entropy in $AdS_4$ and $AdS_3$, and compare the results with the obtained from CFT.

## I. INTRODUCTION

In November 1997, Juan Maldacena proposed a revolutionary conjecture in the context of string theory: the AdS/CFT correspondence, also known as the "Maldacena duality" or "gauge/gravity duality" [1]. The conjecture posits that there is a mapping between a conformal field theory and a theory of quantum gravity (QG), namely a string theory. The AdS/CFT has been groundbreaking principally because it is considered to be the most evident example of the holographic principle, (first suggested by Gerard 't Hooft and further interpreted by Leonard Susskind), which states that a certain $d$-volume of space can be described by the lower-dimensional boundary of the region [3].

From this point, and after all the attempts to show that such an equivalence could not stand, have failed, much of the effort has been focused on understanding how the correspondence works by establishing what is known as the "AdS/CFT dictionary" [7]. Its different entries posit an equivalence between a quantum property of the CFT sitting on the conformal boundary of an AdS spacetime (which is generally difficult to compute and can only be done in very few cases) and an (easier) calculation of the associated quantity on the gravitational side (i.e in the bulk).

It is in this context where the Ryu-Takayanagi conjecture is proposed, as a possible entry of the AdS/CFT dictionary. As Shinsei Ryu and Tadashi Takayanagi argue in [2], the conjecture relates the entanglement entropy between any two subsystems $A$ and $B$ of a CFT to an area ($\gamma_A$) in the AdS bulk:

$$S_A = \frac{\text{Area of } \gamma_\text{A}}{4G_N}, \tag{1}$$

Where $\gamma_A$ stands for the *Ryu-Takayanagi* (i.e. minimal) surface and $G_N$ for the Newton's constant.

## II. THE ANTI-DE SITTER SPACETIME

In order to comprehend the Ryu-Takayanagi conjecture it is compulsory to understand the properties of the Anti-de Sitter spacetime [5, 8] by studying its metric. For simplicity, we will first describe the 2-dimensional case ($AdS_2$) and further generalize to higher dimensions.

The defining properties of the AdS spacetime are that it is maximally symmetric and has a constant negative scalar curvature. Let us consider a $d = 2$ submanifold of equation

$$-X^2 - Y^2 + Z^2 = -L^2, \tag{2}$$

embedded in a $D = d + 1 = 3$ ambient flat space of signature $(-,-,+)$

$$dS^2_{Amb} = -dX^2 - dY^2 + dZ^2, \tag{3}$$

where $L$ is a positive real constant which stands for the radius of the minimal section of an hyperboloid. Since there is a hyperbolic symmetry, Eq.(2) can be parameterized in terms of two new variables $\rho$ and $t$ as

$$\begin{aligned} X &= L\cosh(\rho)\sin(t) \\ Y &= L\cosh(\rho)\cos(t) \\ Z &= L\sinh(\rho), \end{aligned} \tag{4}$$

where $\rho \in (-\infty, +\infty)$ and $t \in (-\infty, +\infty)$. Notice that even though $X$ and $Y$ are periodic in $t$, it is needed to extend the range of time to infinity so the spacetime can be globally hyperbolic, i.e. that there exists a Cauchy surface.

We can now compute the submanifold tangent vectors using $e_\mu = \frac{\partial X^\nu}{\partial x^\mu}\tilde{e}_\nu$ where $X^\nu$ are the ambient space coordinates, $x^\mu$ the submanifold coordinates and $\tilde{e}_\nu$ the Cartesian vectors of the ambient space. The resulting vectors are

$$\begin{aligned} e_\rho &= L\left[\sinh(\rho)\sin(t)\,\tilde{e}_1 + \sinh(\rho)\cos(t)\,\tilde{e}_2 + \cosh(\rho)\,\tilde{e}_3\right] \\ e_t &= L\left[\cosh(\rho)\cos(t)\,\tilde{e}_1 - \cosh(\rho)\sin(t)\,\tilde{e}_2\right]. \end{aligned} \tag{5}$$

Also, recalling that the elements of the metric tensor are defined as $g_{\mu\nu} = e_\mu \cdot e_\nu$ we obtain the non-vanishing coefficients of the metric that the ambient space induces to the embedded submanifold

$$\begin{aligned} g_{tt} &= e_t \cdot e_t = -L^2\cosh^2(\rho) \\ g_{\rho\rho} &= e_\rho \cdot e_\rho = L^2, \end{aligned} \tag{6}$$

where we have used $\eta_{11} = \tilde{e_1}\cdot\tilde{e_1} = -1$, $\eta_{22} = \tilde{e_2}\cdot\tilde{e_2} = -1$ and $\eta_{33} = \tilde{e_3}\cdot\tilde{e_3} = 1$. Thus, leading to the $AdS_2$ metric according to $dS^2 = g_{\mu\nu}dx^\mu dx^\nu$

$$dS^2_{AdS_2} = -L^2\,[\cosh^2(\rho)dt^2 + d\rho^2], \tag{7}$$

which describes a maximally symmetric spacetime of constant negative scalar curvature.

The decision of choosing the coordinate transformation from Eq.(4) to obtain $dS^2_{AdS_2}$ is not arbitrary because for this parameterization, the form taken by the metric tensor allows for a generalization to $AdS_d$. To do so, it can be proven that we only need to add a term which is proportional to the metric of a ($d$-2)-sphere ($d\Omega^2_{(d-2)}$)

$$dS^2_{AdS_d} = L^2\,[-\cosh^2\rho\; dt^2 + d\rho^2 + \sinh^2\rho\; d\Omega^2_{(d-2)}]. \tag{8}$$

Finally, Eq.(8) stands as the line element for $AdS_d$ spacetime.

### A. Killing vectors and curvature of AdS$_2$

We will show that $AdS_2$ is a maximally symmetric spacetime. One way to prove it is by calculating the Killing vectors of the metric (i.e the isometries of the spacetime), which are

$$\begin{aligned}
\zeta^{(1)} &= e_t \\
\zeta^{(2)} &= \tanh(\rho)\sin(t)e_t - \cos(t)e_\rho \\
\zeta^{(2)} &= \tanh(\rho)\cos(t)e_t + \sin(t)e_\rho.
\end{aligned} \tag{9}$$

Notice that $\zeta^{(1)} = e_t$ was to be expected since the $AdS_2$ metric coefficients $g_{\mu\nu}$ are time-independent. The maximum number of Killing vectors in a 2-dimensional maximally symmetric spacetime is $\#_{max} = \dfrac{d(d+1)}{2} = 3$. And, since it coincides with the number of Killing vectors of $AdS_2$ it is verified that it is a maximally symmetric spacetime.

Another way to prove it is by showing that the scalar curvature is constant. For maximally symmetric spacetimes, the curvature tensors can be written as

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= \frac{\pm1}{L^2}(g_{\rho\nu}g_{\sigma\mu} - g_{\rho\mu}g_{\sigma\nu}) \\
R_{\rho\sigma} &= \frac{\pm1}{L^2}(d-1)g_{\rho\sigma} \;\;;\;\; R = \frac{\pm1}{L^2}d(d-1).
\end{aligned} \tag{10}$$

For $d = 2$ the scalar curvature obtained from (10) is $R = \dfrac{\pm2}{L^2}$ but it still must be determined whether $R > 0$ or $R < 0$. Thus, we are forced to compute the Christoffel symbols for $AdS_2$. The non vanishing ones are

$$\Gamma^t_{\rho t} = \tanh(\rho) \;\;;\;\; \Gamma^\rho_{tt} = \cosh(\rho)\sinh(\rho), \tag{11}$$

thus leading to the Ricci tensor components

$$\begin{aligned}
&R_{tt} = \cosh^2(\rho) \;\;;\;\; R_{\rho\rho} = -1 \\
&R_{t\rho} = R_{\rho t} = 0.
\end{aligned} \tag{12}$$

Finally, since $R = g^{\mu\nu}R_{\mu\nu}$ we obtain the right sign

$$R = \frac{-2}{L^2}, \tag{13}$$

proving that $AdS_2$ is a spacetime with constant scalar curvature and thus, maximally symmetric.

### B. The Einstein field equations and the cosmological constant

Let us show that for AdS spacetime to be a solution of the Einstein field equations (EFE) it is needed a non-vanishing negative cosmological constant and no matter ($T_{\mu\nu} = 0$). Thus, the EFE become

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R + g_{\mu\nu}\,\Lambda = 0. \tag{14}$$

Substituting Eq.(10) in the latter, we obtain that the value of the cosmological constant for $d > 2$ is non-zero

$$\Lambda = -\frac{(d-1)(d-2)}{2L^2} < 0. \tag{15}$$

Notice that for $d = 2$ the cosmological constant is zero, but this is a special case in which the EFE in empty space are identically satisfied for any metric describing a maximally symmetric spacetime. However, for AdS in $d > 2$ to be a solution of the EFE in empty space, a negative cosmological constant is required.

### C. The Poincaré patch in AdS$_4$

The Poincaré patch is a new set of coordinates that will be very useful in the conformal boundary and will be used in section IV.

Similarly as we did in Eq.(2) we can define a 4-hyperboloid submanifold

$$-(X^0)^2 - (X^4)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = -L^2, \tag{16}$$

embedded in a $D = 5$ ambient flat space

$$dS^2_{Amb} = L^2[-(dX^0)^2 - (dX^4)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2]. \tag{17}$$

Also, consider the following coordinate transformation

$$\begin{aligned}
&X^\mu = rx^\mu \quad \text{for each} \quad \mu = 0,1,2 \\
&X^4 - X^3 = r \\
&X^4 + X^3 = \frac{1}{r} + r\,[-(x^0)^2 + (x^1)^2 + (x^2)^2],
\end{aligned} \tag{18}$$

where $\{X^0, X^1, X^2, X^3, X^4\}$ is the set of coordinates of the ambient space, and $\{x^0, x^1, x^2, r\}$ the coordinates of the 4-hyperboloid. Now, if we differentiate Eq.(18) and substitute it in Eq.(17) we obtain

$$dS^2_{AdS_4} = L^2[r^2\,[-(dx^0)^2+(dx^1)^2+(dx^2)^2]+\frac{dr^2}{r^2}]. \quad (19)$$

Finally, we define one last transformation: $r = \dfrac{1}{z}$ leading to

$$dS^2_{AdS_4} = \frac{L^2}{z^2}\,[-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dz)^2], \quad (20)$$

which is the $AdS_4$ metric restricted to the Poincaré patch. This result can be generalized for dimension $d$ as

$$dS^2_{AdS_d} = \frac{L^2}{(z)^2}\,[-(dx^0)^2+\sum_{i=1}^{d-2}(dx^i)^2+(dz)^2], \quad (21)$$

where $x^0$ can be interpreted as the time coordinate, $z$ the holographic coordinate (where $z > 0$ means we are inside the bulk) and $x^i$ the spatial (non-holographic) coordinates.

### D. The conformal boundary

In this section we will first define what a conformal boundary is and we will proceed to show that there actually is a conformal boundary in AdS spacetime. This is important because the CFT will be established in such a boundary.

We say that a spacetime has a conformal boundary at $\eta_0$ if the metric coefficients have the following structure

$$g_{\mu\nu} = \frac{1}{Z^2(\eta)}\,\tilde{g}_{\mu\nu}(\eta), \quad (22)$$

where the metric coefficients $\tilde{g}_{\mu\nu}$ are regular functions and $Z(\eta)$ is called the conformal factor, which depends on the holographic parameter $\eta$ and fulfils the following properties

$$\begin{aligned} &Z(\eta_0) = 0 \;\rightarrow\; \text{At the boundary} \\ &dZ(\eta_0) \neq 0 \;\rightarrow\; \text{At the boundary.} \end{aligned} \quad (23)$$

We choose $AdS_4$ to exemplify that the AdS spacetime has a conformal boundary. Since AdS has a spherical symmetry, the angular coordinates $\theta$ and $\phi$ can be set as constant because the conformal boundary will correspond to a certain radius.

We choose the coordinate transformation given for the expression $\sinh(\rho) = \tan(\eta)$ to obtain the $AdS_2$ metric in the form

$$dS^2_{AdS_2} = \frac{L^2}{\cos^2(\eta)}\left(-dt^2+d\eta^2\right), \quad (24)$$

which evinces the existence of a conformal boundary since $Z(\eta) = \cos(\eta)$ fulfils the properties given by Eq.(23) at $\eta_0 = \frac{\pi}{2}$.

Notice that the coordinate transformation $\sinh(\rho) = \tan(\eta)$ is exactly the equation of the null geodesics. Also, what has put us on track of what the transformations should be is the fact that it takes a finite time for light to reach infinity in AdS spacetime.

### E. Geodesics of an AdS spacetime restricted to the Poincaré patch - the upper half space (UHS)

The objective of this section is to obtain the spatial geodesics of $AdS_3$ and also, show that one type of spatial geodesics for $AdS_4$ are hemispheric regions with center in the conformal boundary. We will recover this result in section IV.A to easily compute the area of the Ryu-Takayanagi surface ($\gamma_A$) in a practical case.

Let us start from the metric of the $AdS_d$ spacetime, restricted to the Poincaré patch Eq.(21). Since we are interested in the spatial geodesics we will be considering only a time slice ($x^0$ = constant $\rightarrow$ $\mathrm{dx}^0 = 0$), thus leading to

$$dS^2_{AdS_{d(spatial)}} = \frac{L^2}{z^2}\,\left[\sum_{i=1}^{d-2}(dx^i)^2+(dz^2)\right]. \quad (25)$$

This line element coincides with the well known upper half space (UHS) in hyperbolic geometry. Then, the spatial part of $AdS_3$ is

$$dS^2_{AdS_{3(spatial)}} = \frac{L^2}{z^2}(dx^2+dz^2). \quad (26)$$

The spatial geodesics in hyperbolic geometry are also well known. There are two types: straight lines in the z direction (which will not be important for our purposes), and semicircles centered in the conformal boundary

$$x(\lambda) = R\cos(\lambda) \;\;;\;\; z(\lambda) = R\sin(\lambda), \quad (27)$$

where $\lambda \in [0, \pi]$ is a non-affine parameter. Next, by computing the non-vanishing Christoffel symbols $\{\Gamma^1_{2\,1} = \Gamma^2_{2\,2} = -\Gamma^2_{1\,1} = -\dfrac{1}{z}\}$, it can be proven that the parameterization Eq.(27) fulfils the geodesics equations $\dfrac{du^\mu}{d\lambda} + \Gamma^\mu_{\alpha\,\beta}u^\alpha u^\beta = C(\lambda)\,u^\mu$, where $C(\lambda)$ is a non-vanishing function of $\lambda$, and $u^\nu = \left(\dfrac{dx}{d\lambda}, \dfrac{dz}{d\lambda}\right)$ plays the *role* of the 4-velocity.

The generalization of Eq.(26) to $AdS_4$ is $dS^2_{AdS_{4(spatial)}} = \dfrac{dx^2+dy^2+dz^2}{z^2} = \dfrac{dr^2+r^2d\theta^2+dz^2}{z^2}$. Since we see that the coefficients of the metric are independent of the angular coordinate $\theta$, it means there is a Killing vector $\zeta^\theta = e_\theta$ associated to the rotational invariance. Thus, for the $AdS_4$ case, the geodesic semicircles become geodesic hemispheres centered in the conformal boundary.

## III. THE ENTANGLEMENT ENTROPY

In this section, we intend to review the concept of the entanglement entropy [4, 6]. Given a quantum system described by some pure state $|\Psi\rangle$, the von Neumann entropy is defined by

$$S = -\mathrm{tr}\,\rho \log \rho, \tag{28}$$

where $\rho = |\Psi\rangle\langle\Psi|$. If we divide the whole quantum system into two disjointed subsystems $A$ and $B$, and consider an observer who only has access to the subsystem $A$ and ignores any information from $B$, he will describe his system with the reduced density matrix $\rho_A = \mathrm{tr_B}\,|\Psi\rangle\langle\Psi|$. We can now define the entanglement entropy (of the subsystem A) as the von Neumann entropy of the reduced density matrix $\rho_A$

$$S_A = -\mathrm{tr}\,\rho_\mathrm{A} \log \rho_\mathrm{A}, \tag{29}$$

which is a measure of how entangled the subsystems $A$ and $B$ are.

In QFT though, things are not so simple. Not only because it is not true that the global Hilbert space can be described by the direct product of the Hilbert spaces of its parts, but also because the entanglement entropy diverges. As we will see, what is universal is *the way* it diverges.

As we will show in section IV, the Ryu-Takayanagi conjecture suggests a gravitational derivation of the entanglement entropy between two quantum subsystems of a CFT sitting at the conformal boundary of an AdS spacetime restricted to the Poincaré patch.

## IV. THE RYU-TAKAYANAGI CONJECTURE

The Ryu-Takayanagi conjecture states that the entanglement entropy of a CFT (a field theory defined on the conformal boundary of AdS) of a spatial sub-region A is proportional to the minimal area of the hypersurface in the bulk for a spatial slice of $AdS_d$ [2], i.e

$$S_A = \frac{\text{Area of } \gamma_\mathrm{A}}{4G_N}. \tag{30}$$

We refer to the minimal surface $\gamma_A$ as the Ryu-Takayanagi surface, which has the same boundary as A. In the next section we will use Eq.(30) in $AdS_3/CFT_2$ and $AdS_4/CFT_3$ and will compare the results with those obtained from CFT.

### A. Holographic derivation of the entanglement entropy for AdS$_4$/CFT$_3$ and AdS$_3$/CFT$_2$

As we know from the latter section, since one of the types of geodesics for $AdS_4$ are hemispheres which correspond to the minimal surfaces $\gamma_A$, we only need to compute their area.

In the Poincaré patch the spatial part of the $AdS_4$ metric is

$$dl^2 = \frac{L^2}{z^2}\left(dx^2 + dy^2 + dz^2\right), \tag{31}$$

or in cylindrical coordinates

$$dl^2 = \frac{L^2}{z^2}\left(dr^2 + r^2 d\theta^2 + dz^2\right). \tag{32}$$

We also know from section E that for $AdS_4$ the minimal surface $\gamma_A$ is a geodesic hemisphere of radius $R$ with its center in the boundary, namely

$$(x-x_0)^2 + (y-y_0)^2 + z^2 = R^2. \tag{33}$$

Also, since the coefficients of the metric Eq.(31) are independent of the coordinates $x$ and $y$, there is a translational invariance over both of these directions. Therefore, we can freely choose the position of the center of the hemispheres at $x_0 = y_0 = 0$ without loss of generality. Then, by differentiating Eq.(33) in cylindrical coordinates we get

$$z^2 = R^2 - r^2 \;\rightarrow\; dz^2 = \frac{r^2}{R^2 - r^2} dr^2. \tag{34}$$

By substituting Eq.(34) into Eq.(32) we obtain the induced metric of the hemisphere $d\sigma^2$ and its matrix form $h$

$$d\sigma^2 = \frac{L^2 R^2}{(R^2 - r^2)^2} dr^2 + \frac{L^2 r^2}{R^2 - r^2} d\theta^2$$
$$h = \begin{pmatrix} \frac{L^2R^2}{(R^2-r^2)^2} & 0 \\ 0 & \frac{L^2 r^2}{R^2 - r^2} \end{pmatrix}. \tag{35}$$

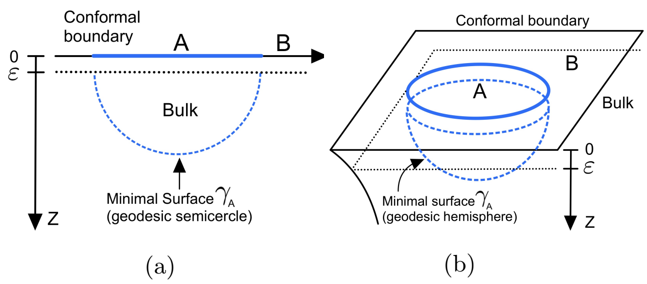


FIG. 1: Minimal surfaces $\gamma_A$ in the bulk for: (a) AdS$_3$ and (b) AdS$_4$.

The area can be computed as

$$Area\ of\ \gamma_A = \int_0^{2\pi} d\theta \int_0^R dr \sqrt{|h|}, \tag{36}$$

where $\sqrt{|h|} = \dfrac{L^2 R\, r}{(R^2 - r^2)^{3/2}}$ is the Jacobian. Now, substituting $\sqrt{|h|}$ and integrating we obtain

$$Area\ of\ \gamma_A = 2\pi L^2 R \left[\frac{1}{\sqrt{R^2 - r^2}}\right]_0^R. \tag{37}$$

Rewriting it in terms of $z = \sqrt{R^2 - r^2}$ leads to

$$Area\ of\ \gamma_A = 2\pi L^2 R\, \frac{1}{z}\bigg|_{z=0} - 2\pi L^2. \tag{38}$$

As we can see, the area diverges, but this is not surprising because in hyperbolic geometry the distances grow as we get closer to the boundary. What is actually important is *the way* it diverges. So, it is necessary to introduce a cutoff in z

$$Area\ of\ \gamma_A = \frac{2\pi L^2 R}{\epsilon} - 2\pi L^2. \tag{39}$$

Finally, from Eq.(30) we obtain the entanglement entropy as

$$S_A = \frac{\pi L^2 R}{2G_N}\frac{1}{\epsilon} - \frac{\pi L^2}{2G_N}. \tag{40}$$

Similarly, we can repeat the calculation for $AdS_3/CFT_2$ to finally obtain

$$S_A = \frac{L}{2\, G_N} \ln \frac{L}{\epsilon} = \frac{c}{3} \ln \frac{L}{\epsilon}, \tag{41}$$

where $c = \dfrac{3\, L}{2\, G_N}$ is the $CFT_2$ *central charge* which, in broad terms, counts the degrees of freedom of the theory.

Eq.(41) is one of the few cases (specifically, the easiest one) in which the entanglement entropy coincides exactly with that computed in CFT [6]. For Eq.(40) *the way* it diverges is exactly as it is expected by CFT. Still, even though the term $\dfrac{\pi L^2}{2G_N}$ is suspected to be related to the *F-Theorem* central charge, as far as we know, it has not yet been actually proven.

## V. CONCLUSIONS

In this paper we have proved that the Ryu-Takayanagi conjecture greatly simplifies the calculation of the entanglement entropy of a CFT. With this objective we have reviewed the most relevant tools from general relativity in order to characterize the AdS spacetime, as well as introducing the concept of entanglement entropy for quantum mechanics. It is important to highlight the fact that in order for AdS to be a solution of the EFE it needs a non-vanishing (and negative) cosmological constant. Finally, we have compared the results obtained to those computed directly from CFT. We also want to emphasise the fact that it is still a conjecture and not a theorem. The area law proposed by Shinsei Ryu and Tadashi Takayanagi is struggling to prove its validity for higher dimensions. E.g., for $AdS_4/CFT_3$ it has been possible to find the entanglement entropy by means of the area law conjectured, but the calculation from the CFT side, although it is suspected to be related with the so called *F-theorem*, as far as we know, it still remains to be exactly determined. Thus, making difficult to compare both of the results.

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