Quantum Mechanics and Path Integrals

Author: Lluís de Miguel Blasco.
*Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.**

Advisor: Josep Taron Roca

Abstract: Firstly, we introduce the concept of a Path Integral and see how to use it to formulate Quantum Mechanics. We see some classical examples and learn how to calculate Path Integrals for simple systems. We also show how Classical Mechanics arises from the Path Integral formulation of Quantum Mechanics in the classical limit. Then, we see the equivalence between Schrödinger's formulation of Quantum Mechanics and the formulation in terms of Path Integrals. Lastly, we study the Aharonov-Bohm effect.

I. Introduction

The concept of trajectory is of paramount importance in classical mechanics. When studying physical systems, one often attempts to find the position of one or more particles in space at every point in time, be it using Newton's vector-based formulation of mechanics or using the tools provided by analytical mechanics. At the beginning of the 20th century, however, Quantum Mechanics was being developed and it forced physicists to abandon many of the core ideas of classical mechanics, including the notion of trajectory. Heisenberg's uncertainty principle states that it is not possible to perfectly determine the position of a particle and thus it is not possible to determine its trajectory. In spite of this, some physicists still tried to formulate Quantum Mechanics in a way that the idea of trajectory did not need to be relinquished. A notable step towards this direction is Dirac's 1945 paper [2] in which he introduces a new function $S(q, q')$ defined by

$$\langle q|q'\rangle = \exp\left(\frac{iS(q,q')}{\hbar}\right)\cdot G\left(q,q'\right) \tag{1}$$

where q and q' represent the eigenvalues of the observables associated with the (generalised) coordinates of the system Q and Q', respectively, and $G\left(q,q'\right)$ is a product of delta functions depending each on a function of q and q'. He then shows that, under certain conditions, the corresponding momenta observables satisfy

$$P_r = \frac{\partial S(Q,Q')}{\partial Q_r} + \Delta_1 \qquad P'_r = -\frac{\partial S(Q,Q')}{\partial Q'_r} + \Delta_2$$

where Δ_i are functions of a dynamic variable and a function whose definition is not relevant for the purposes of this section. Afterwards, he shows that both Δ_i depend on the state of the system between Q and Q'. Feynman in [4] expands on this ideas and postulates that the amplitude probability for a particle to travel from one position to another in a certain amount of time is a sum of complex numbers corresponding each to one possible path the particle might take and the contribution from each path is just the exponential term in (1) where S corresponds to the classical action associated to that path. The goal of this work is to present Feynman's idea in more detail and to show it is equivalent to Schrödinger's formulation of Quantum Mechanics as Feynman does in that same article. Then we will see as an example the free particle. Finally we will study the Aharonov-Bohm effect using Feynman's Path Integral formulation.

II. The Path Integral

Let $a = (t_a, x_a)$ and $b = (t_b, x_b)$ be two points in spacetime and consider a particle that travels from a to b. This particle can take many paths. Let's define the probability amplitude for this particle to go from a to b, the kernel $K(b,a)$, as a combination of complex numbers $\Phi[x(t)]$ each associated to a path $x(t)$ joining these points:

$$K(b,a) = \sum_{x(t)} \Phi[x(t)] \tag{2}$$

where $\Phi[x(t)] \propto \exp\left(\frac{iS[x(t)]}{\hbar}\right)$ and $S[x(t)]$ is the classical action for the path $x(t)$. In order to formalise this summation over all paths we do the following: consider a partition of the interval $[t_a, t_b]$ formed by $N+1$ evenly spaced points $\{t_i\}$ and associate to each t_i a position in space x_i in such a way that $(t_0, x_0) = a$, $(t_N, x_N) = b$. By joining points (t_i, x_i) and (t_{i+1}, x_{i+1}) with line segments we create a polygonal path that goes from a to b. Now by varying each x_i except for x_0 and x_N we can go over all polygonal paths joining a and b so the summation at (2) becomes

$$\int ... \int \Phi[x(t)]\, dx_1 \, ... \, dx_{N-1}$$

Calling ε the distance between any t_i and t_{i+1}, if we take ε to be small, the polygonal path becomes less jagged and resembles more a smooth path between a and b. In order to better approximate a truly smooth path we would like to take the limit of this expression as ε approaches 0, but it would not converge. We need to add a constant to ensure convergence defined by

$$A^{-N} = \left(\left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{\frac{1}{2}}\right)^{-N} \tag{3}$$

*Electronic address: lldemigb7@alumnes.ub.edu

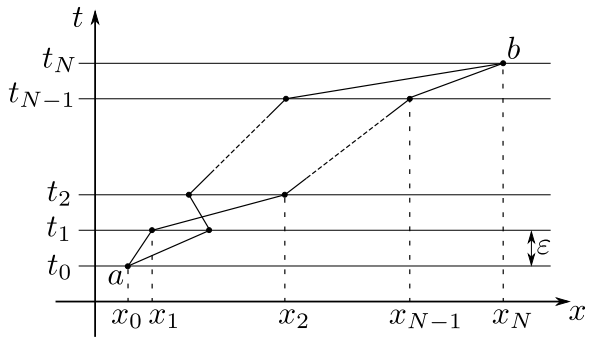

FIG. 1: Each path is defined by its position x_i at time t_i. To get all paths, we vary the position at every time t_i

Finally, we can define the path integral as

$$K(b,a) = \lim_{\varepsilon \to 0} \frac{1}{A} \int ... \int \exp\left(\frac{iS[b,a]}{\hbar}\right) \frac{dx_1}{A} ... \frac{dx_{N-1}}{A} \tag{4}$$

where $S[b,a]$ is the classical action over the path with endpoints a and b formed by joining (t_i, x_i) and (t_{i+1}, x_{i+1}) with line segments. Instead of with line segments, we can join them with sections of the classical path which goes from a to b passing through all (t_i, x_i). The calculation of $K(b,a)$ is to be performed in this order: first calculate the integrals and, only then, take the limit. Introducing the following notation we can write the Path Integral in a more compact form:

$$K(b,a) = \int_a^b \exp\left(\frac{iS[b,a]}{\hbar}\right) \mathcal{D}x(t) \tag{5}$$

III. Events in succesion

If the Lagrangian of the system does not depend on derivatives of the position of order higher than one, so it depends on at most the velocity, we can write $S[b,a] = S[b,c]+S[c,a]$, where $c = (t_c, x_c)$ and t_c is between t_a and t_b (this condition on the Lagrangian is required to avoid needing to specify values of velocity, acceleration... at the point c). Therefore, $\exp\left(\frac{iS[b,a]}{\hbar}\right) = \exp\left(\frac{iS[b,c]}{\hbar} + \frac{iS[c,a]}{\hbar}\right)$ and the path joining a and b can be divided into two paths, one joining a and c and the other joining c and b. We can now perform the integral (5) in two steps: first we integrate over all paths between a and c

$$K(b,a) = \int_{-\infty}^{\infty} \int_c^b \exp\left(\frac{iS[b,c]}{\hbar}\right) K(c,a) \mathcal{D}x(t)\, dx_c$$

and then we integrate over all paths between c and b leaving x_c fixed:

$$K(b,a) = \int_{-\infty}^{\infty} K(b,c)K(c,a)\, dx_c \tag{6}$$

From this, we can deduce that the kernel to go from a to b is the sum of the product of amplitudes to go from a to c and from c to b, over all values of x_c. We can also deduce that the kernel to go from a to c, measure the state of the system at this point in time, and then go from c to b is the product of the kernels to go from a to c and from c to b.

This can readily be extended to consider more than one point between a and b. In fact, it can even be used to define the Path Integral in an alternative way (see [5] p. 36, "Events occurring in succession").

IV. The classical limit

If the concepts we have introduced are a good model of nature, they must also hold in the classical limit, that is, when the action is very large. Let's briefly justify this is the case. Consider a classical system, a system which has dimensions, mass... so large that the action S is huge in comparison to $\hbar \propto 10^{-34}\,\mathrm{J\,s}$. For a given path this system might take $x(t)$ with associated action $S[x(t)]$, the action of each of its neighbouring paths will be almost equal to $S[x(t)]$. This difference, as small as it might be from a classical point of view, is huge when compared to $\hbar$. Therefore, $\exp\left(\frac{iS[x(t)]}{\hbar}\right)$ oscillates rapidly when small variations to $x(t)$ are performed. As a consequence, there exists a path $x'(t)$ close to $x(t)$ such that the quantity $\exp\left(\frac{iS[x'(t)]}{\hbar}\right)$ is in opposition to the same quantity for $x(t)$ and their contributions to the Path Integral cancel out. However, the classical path $\overline{x}(t)$ is an extrema of the action, so small variations to it do not cause changes in the action $S[\overline{x}(t)]$ to first order. Thus, the only paths that contribute to the Path Integral are the classical path and its close neighbours, so the only important path is the classical one.

In the classical setting, we would also expect that small changes in the endpoint b do not alter noticeably the probability that the system ends up in b. We can convince ourselves this is what happens with the following argument: small changes in b will cause big changes in S and strong oscillations in $\exp\left(\frac{iS[x(t)]}{\hbar}\right)$, and as a result $K(b,a)$ will be greatly altered. Nevertheless, from the previous reasoning we can see that

$$K(b,a) = f(\alpha) \exp\left(\frac{iS[\overline{x}(t)]}{\hbar}\right)$$

where α represents classical quantities of the system, and f is a function that only changes when variations in a classical scale are performed on α. So the probability of finding the system at b only depends on $f(\alpha)$ and therefore small changes to b do not affect it.

V. The Path Integral and the Schrödinger equation

Often we are just concerned with the probability amplitude for a particle to get to a point (t_b, x_b) and where the particle came from is of no interest to us. In this case, the kernel $K(x_b, t_b; x_a, t_a)$, which is the amplitude we are interested in, is written as $\psi(x_b, t_b)$ and is called a wave function. It is also often the case that calculating the path integral is very hard, so we would like to find a differential equation that the wave function must fulfill and calculate it by solving this equation. Let's begin

with noticing that since the wave function is a probability amplitude, for any two events in succession it must satisfy (6) which in terms of wave functions is

$$\psi\left(x_b,t_b\right)=\int_{-\infty}^{\infty}K(x_b,t_b;x_a,t_a)\psi\left(x_a,t_a\right)\,dx_a$$

Now, let ε be an infinitesimal amount of time and put $t_a=t$, $t_b=t+\varepsilon$, $x_b=x$ and $x_a=y$. As ε is very small, we can take the following approximation for the action for a path $x(t)$ joining a and b:

$$\begin{aligned}S[b,a]&=\int_{t_a}^{t_b}L\left(x(t),\dot{x}(t),t\right)\,dt\\&\simeq\varepsilon L\left(\frac{x+y}{2},\frac{x-y}{\varepsilon}\right)\end{aligned}$$

where L is the Lagrangian of the system. The kernel $K(x_b,t_b;x_a,t_a)$ is proportional to the exponential of $i/\hbar$ times the action $S[b,a]$, so substituting in the previous equation we get

$$\begin{aligned}&\psi\left(x,t+\varepsilon\right)=\\&\frac{1}{A}\int_{-\infty}^{\infty}\exp\left(\frac{i}{\hbar}\varepsilon L\left(\frac{x+y}{2},\frac{x-y}{\varepsilon}\right)\right)\psi\left(y,t\right)\,dy\end{aligned}$$

The Lagrangian for a particle of mass m in a potential $V(x,t)$ is $L(x,\dot{x},t)=\frac{m}{2}\dot{x}^2-V(x,t)$ and thus

$$\begin{aligned}\psi\left(x,t+\varepsilon\right)=\frac{1}{A}\int_{-\infty}^{\infty}&\exp\left(\frac{i}{\hbar}\frac{m\left(x-y\right)^2}{2\varepsilon}\right)\\&\times\exp\left(-\frac{i}{\hbar}\varepsilon V\left(\frac{x+y}{2},t\right)\right)\psi\left(y,t\right)\,dy\end{aligned}$$

If y is very different from x, the first exponential will oscillate rapidly due to the term $\left(x-y\right)^2$ so the only important contributions come from y near x. For this reason, we make the substitution $y=x+\eta$ and expand $\psi(y,t)=\psi(x+\eta,t)$ in a power series around x on the right-hand side of the equality. On the left-hand side, we expand $\psi(x,t+\varepsilon)$ around t. Keeping only terms of order ε and approximating $\varepsilon V\left(x+\frac{\eta}{2},t\right)$ by $\varepsilon V(x,t)$ because the error is of higher order than ε we arrive at

$$\begin{aligned}\psi(x,t)+\varepsilon\frac{\partial\psi}{\partial t}=\frac{1}{A}\int_{-\infty}^{\infty}&\exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\\&\times\left(1-\frac{i}{\hbar}\epsilon V(x,t)\right)\left(\psi(x,t)+\eta\frac{\partial\psi}{\partial x}+\frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}\right)\,d\eta\end{aligned}\tag{7}$$

By performing the products under the integral, we get the term

$$\psi(x,t)\frac{1}{A}\int_{-\infty}^{\infty}\exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\,d\eta$$

which has to equal $\psi(x,t)$ at the left-hand side in the limit ε approaching 0. Therefore, $A=\left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{1/2}$ as we have stated in (3). This method for finding the value of A can be adapted to be used in other similar problems. Returning to (7), we can carry out the products and take all the terms which do not depend on η outside of the integral. Performing the gaussian integrals that are left (see [5] p. 78 for details), equation (7) becomes

$$\psi+\varepsilon\frac{\partial\psi}{\partial t}=\psi-\frac{i}{\hbar}\varepsilon V\psi+\frac{i\hbar\varepsilon}{2m}\frac{\partial^2\psi}{\partial x^2}$$

Finally, this equation is satisfied to order ε if ψ satisfies

$$i\hbar\frac{\partial\psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x,t)\psi\right)$$

which is the Schrödinger equation. This argument and its conclusions can be readily generalised to the case of a particle in a 3-dimensional space.

With this, we have proved the equivalence between Schrödinger's formulation of Quantum Mechanics and the Path Integral formulation.

VI. The Free Particle

Consider a free particle. Its velocity is $v=\frac{x_i-x_{i-1}}{t_i-t_{i-1}}$ and it is constant, and its potential energy is 0 everywhere. Therefore, the Lagrangian is $L=\frac{1}{2}m\left(\frac{x_i-x_{i-1}}{t_i-t_{i-1}}\right)^2$ and the action is

$$S[b,a]=\int_{t_a}^{t_b}L\,dt=\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}L\,dt$$

We have, remembering that $\varepsilon=t_i-t_{i-1}$

$$\begin{aligned}\int_{t_{i-1}}^{t_i}L\,dt&=\int_{t_{i-1}}^{t_i}\frac{1}{2}m\left(\frac{x_i-x_{i-1}}{t_i-t_{i-1}}\right)^2\,dt\\&=\frac{1}{2}m\frac{\left(x_i-x_{i-1}\right)^2}{\varepsilon}\end{aligned}$$

Thus, $S[b,a]=\frac{m}{2\varepsilon}\sum_{i=1}^{N}\left(x_i-x_{i-1}\right)^2$ and (4) is

$$\begin{aligned}K(b,a)=\lim_{\varepsilon\to0}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}}\int...\\...\int\exp\left(\frac{im}{2\varepsilon\hbar}\sum_{i=1}^{N}\left(x_i-x_{i-1}\right)^2\right)dx_1...\,dx_{N-1}\end{aligned}\tag{8}$$

In order to calculate this integral, we first integrate with respect to x_1, then x_2 and so on. To perform the integral with respect to x_1 we split the exponential of the summation into the product of two exponentials, one containing all the terms of the summation that depend on x_1 (those corresponding to $i=1$ and $i=2$) and another exponential that does not depend on x_1 and can be taken out from the integral. The integral with respect to x_1 is

$$\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{2}{2}}\int_{-\infty}^{\infty}\exp\left(\frac{im}{2\hbar\varepsilon}\left(\left(x_2-x_1\right)^2+\left(x_1-x_0\right)^2\right)\right)dx_1$$

Expanding the squares of the sums and grouping all terms depending on x_1 we arrive at

$$\begin{aligned}&\frac{m}{2\pi i\hbar\varepsilon}\exp\left(\frac{im}{2\varepsilon\hbar}\left(x_2^2+x_0^2\right)\right)\\&\qquad\times\int_{-\infty}^{\infty}\exp\left(\frac{im}{2\hbar\varepsilon}\left(2x_1^2-2\left(x_2+x_0\right)x_1\right)\right)dx_1\end{aligned}$$

This is a gaussian integral. Calculating it we obtain

$$\frac{m}{2\pi i\hbar 2\varepsilon}\exp\left(\frac{im}{2\varepsilon 2\hbar}\left(x_2-x_0\right)^2\right) \tag{9}$$

Now we have to integrate with respect to x_2. This corresponds to integrating the terms $i=2$ and $i=3$ in (8). The term corresponding to $i=2$ is already included in (9) so we just have to multiply it by the term corresponding to $i=3$ and integrate using the same technique as before. We get:

$$\left(\frac{m}{2\pi i\hbar 3\varepsilon}\right)^{\frac{1}{2}}\exp\left(\frac{im}{3\varepsilon 2\hbar}\left(x_3-x_0\right)^2\right)$$

It can be proved that a recursive process is established and after $n-1$ steps we have

$$\left(\frac{m}{2\pi i\hbar n\varepsilon}\right)^{\frac{1}{2}}\exp\left(\frac{im}{n\varepsilon 2\hbar}\left(x_n-x_0\right)^2\right)$$

Since $n\varepsilon = t_n - t_0$, after the necessary $N-1$ steps to calculate the integrals with respect to $x_1, x_2,\ ...,\ x_{N-1}$ we obtain an expression which does not depend on ε and the limit does not have any effect on it. Finally, the kernel for the free particle is

$$\left(\frac{m}{2\pi i\hbar\left(t_b-t_a\right)}\right)^{\frac{1}{2}}\exp\left(\frac{im}{2\hbar}\frac{\left(x_b-x_a\right)^2}{t_b-t_a}\right)$$

which is equal to the wave function of a free particle calculated using the Schrödinger equation as it should be.

VII. The Aharonov-Bohm Effect

In [3] W. Ehrenberg and R. E. Siday considered an experiment consisting in shooting electrons at a screen through a double slit and studying the pattern imprinted on the screen. They noticed that if a magnetic field exists in a region of space this pattern is modified even when this region is not accessible to the electrons. That is, the existence of a magnetic field has an effect on the outcome of the experiment even when the beams of electrons are in field-free regions only. This effect was presented as a curiosity and not much importance was given to it. Some years later, Y. Aharonov and D. Bohm in [1] rediscovered this phenomenon and explained it more thoroughly. This effect is nowadays called the Aharonov-Bohm effect and in this section we are going to present it briefly.

Consider the setup presented in figure 2. When the solenoid is activated, a current flows through it and as a consequence a magnetic field $\vec{B}$ is established in the solenoid. The vector potential associated is $\vec{A}$ and it cannot be null everywhere outside the solenoid. The proof of this is as follows: assume $\vec{A}$ was null outside the solenoid and take a closed curve α circling the solenoid. Letting Σ be the region of space contained by α, the flux of $\vec{B}$ through Σ is

$$\int_\Sigma \vec{B}\cdot d\vec{S} = \varphi_B$$

for some non-zero constant φ_B since Σ contains the solenoid. However, using Stokes' Theorem and since $\vec{A}$ is null outside the solenoid, we have

$$\int_\Sigma \vec{B}\cdot d\vec{S} = \int_\Sigma \left(\nabla\times\vec{A}\right)\cdot d\vec{S} = \int_\alpha \vec{A}\cdot d\vec{l} = 0$$

which is a contradiction. Therefore, as the hamiltonian of the electron H depends on $\vec{A}$ it will not be the same if the solenoid is active or not and the wave function will differ between this cases.

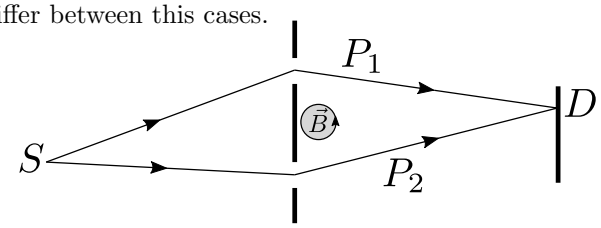

FIG. 2: Electrons from a source S are shot at a screen through a double slit and are detected on the screen at a point D

Let P_1 and P_2 be paths going through the first and second slits respectively. In the case $\vec{B}=0$, the wave function of the electron is

$$\psi(D) = \psi_{P_1}(D) + \psi_{P_2}(D)$$

where ψ_{P_k} is the wave function corresponding to the path P_k and its neighbours for $k=1,2$. Let's use the Path Integral to show how the wave function of the electron is altered if there exists a magnetic field. The lagrangian in this case is the sum of the langrangian for $\vec{B}=0$ L_0 plus and additional term $\Delta L = \frac{e}{c}\vec{v}\cdot\vec{A}$ where e is the elementary charge, c is the speed of light and $\vec{v}$ is the (classical) velocity of the electron. Therefore, for an electron traveling from a point in the source at some time a to a point in the screen b at a later time following the path P_i, the exponential in the Path Integral (5) is

$$\begin{aligned}&\exp\left(\frac{iS}{\hbar}\right) = \exp\left(\frac{i}{\hbar}\int L\,dt\right) = \exp\left(\frac{i}{\hbar}\int L_0\,dt\right)\\ &\times\exp\left(\frac{i}{\hbar}\int \Delta L\,dt\right) = \exp\left(\frac{iS_0}{\hbar}\right)\exp\left(\frac{i}{\hbar}\int\Delta L\,dt\right)\\ &= E_0\exp\left(\frac{ie}{\hbar c}\int_{t_a}^{t_b}\dot{\vec{x}}\cdot\vec{A}\,dt\right) = E_0\exp\left(\frac{ie}{\hbar c}\int_{P_i}\vec{A}\cdot d\vec{l}\right)\end{aligned} \tag{10}$$

where E_0 is the exponential of the action (times $i/\hbar$) in the case of no magnetic field S_0. Taking two paths P_i and $P_{i'}$, the difference $\Delta_{i,i'}$ in the argument of the exponential that weights their contribution in the Path Integral is

$$\int_{P_i}\vec{A}\cdot d\vec{l} - \int_{P_{i'}}\vec{A}\cdot d\vec{l} = \int_{P_{i,i'}}\vec{A}\cdot d\vec{l} = \int_{S_{i,i'}}\left(\nabla\times\vec{A}\right)\cdot d\vec{S}$$

where $P_{i,i'}$ is the closed path formed by joining P_i and $P_{i'}$ reversed and $S_{i,i'}$ is the surface whose boundary is $P_{i,i'}$. In the neighbourhood of P_1, since $\vec{B}=\nabla\times\vec{A}=0$, this

difference is 0. Thus, the neighbours of P_1 contribute to the Path Integral equally. The same holds true for P_2. It is interesting, then, to calculate this difference for paths surrounding the solenoid such as P_1 and P_2:

$$\int_{P_1} \vec{A}\cdot\vec{dl} - \int_{P_2} \vec{A}\cdot\vec{dl} = \int_{S_{1,2}} \left(\nabla\times\vec{A}\right)\cdot\vec{dS} = \int_{S_{1,2}} \vec{B}\cdot\vec{dS} = \varphi_B \tag{11}$$

Finally, we can calculate the wave function of the electron. The electron going through one of the slits at a point in its path (t_s, x_s) means $x_s \in I_1 \cup I_2$ where I_1, I_2 are two disjoint intervals of $\mathbb{R}$ representing each one of the slits. The path integral can then be split into two integrals corresponding each to the paths going through one of the slits. Letting $E[b,a] = 1/A^N \exp\left(\frac{iS[b,a]}{\hbar}\right)$ and using (10) we get:

$$\begin{aligned}
\psi(D) &= \lim_{\varepsilon\to 0} \int ... \int ... \int E[b,a] dx_1 \, ... \, dx_s \, ... \, dx_{N-1} \\
&= \lim_{\varepsilon\to 0} \int ... \int \int_{I_1} E[b,a] dx_s \, dx_1 \, ... \, dx_{N-1} \\
&\quad + \lim_{\varepsilon\to 0} \int ... \int \int_{I_2} E[b,a] dx_s \, dx_1 \, ... \, dx_{N-1} \\
&= \int_{x_s\in I_1} \exp\left(\frac{iS}{\hbar}\right) \mathcal{D}x(t) + \int_{x_s\in I_2} \exp\left(\frac{iS}{\hbar}\right) \mathcal{D}x(t) \\
&= \int_{x_s\in I_1} E_0 \exp\left(\frac{ie}{\hbar c}\int_{x(t)} \vec{A}\cdot\vec{dl}\right) \mathcal{D}x(t) \\
&\quad + \int_{x_s\in I_2} E_0 \exp\left(\frac{ie}{\hbar c}\int_{x(t)} \vec{A}\cdot\vec{dl}\right) \mathcal{D}x(t)
\end{aligned}$$

Since all paths in the integral in the first term go through the first slit, they are neighbours of P_1. Therefore, for all of them the integral in the exponential is $\int_{P_1}\vec{A}\cdot\vec{dl}$, independently of the path $x(t)$ in the Path Integral. Analogously, the integral in the second term becomes $\int_{P_2}\vec{A}\cdot\vec{dl}$ which, according to (11), is equal to $\int_{P_1}\vec{A}\cdot\vec{dl} - \varphi_B$. Substituting we obtain:

$$\begin{aligned}
\psi(D) &= \exp\left(\frac{ie}{\hbar c}\int_{P_1}\vec{A}\cdot\vec{dl}\right) \int_{x_s\in I_1} E_0 \, \mathcal{D}x(t) \\
&\quad + \exp\left(-\frac{ie}{\hbar c}\varphi_B\right) \exp\left(\frac{ie}{\hbar c}\int_{P_1}\vec{A}\cdot\vec{dl}\right) \int_{x_s\in I_2} E_0 \, \mathcal{D}x(t) \\
&= C\left(\psi_{P_1}(D) + \psi_{P_2}(D)\exp\left(-\frac{ie}{\hbar c}\varphi_B\right)\right)
\end{aligned}$$

The common phase $C = \exp\left(\frac{ie}{\hbar c}\int_{P_1}\vec{A}\cdot\vec{dl}\right)$ does not have any effect on the probability of finding the electron at the screen at D, but the factor $\exp\left(-\frac{ie}{\hbar c}\varphi_B\right)$ is a relative phase between the wave functions ψ_{P_1} and ψ_{P_2} that alters the way they interfere and, as a consequence, it alters the probability with respect to that of the case $\vec{B} = 0$.

Hence, the existence of a magnetic field modifies the outcomes of the experiment even when there's no apparent interaction between the electron and the field. For experimental confirmation of this fact see [6].

VIII. Conclusions

Formulating Quantum Mechanics in terms of Path Integrals is very interesting from a theoretical point of view as it allows us to use, to some extent, classical ideas and to deduce quantum mechanical rules of nature such as those for the composition of movements with ease. Its equivalence to other formulations means it is a great alternative to them when tackling problems which might be hard to study using other formulations. We have also succinctly showed its practical utility with the study of a free particle and the Aharonov-Bohm effect, although it is widely used in many areas outside the scope of this project such as Quantum Chromodynamics.

The Aharonov-Bohm effect makes apparent the quantum mechanical reality of nature by means of a simple thought experiment. This experiment has later been performed with different techniques and all results are in line with predictions, confirming Quantum Mechanics is a great model of natural phenomena and the Path Integral formulation is a correct way of formulating it.

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