CORRIGENDUM TO *AN ENHANCED UNCERTAINTY PRINCIPLE FOR THE VASERSTEIN DISTANCE*

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One of the main results in the paper mentioned in the title is the following.

Theorem 1. *Let $Q_0 = [0,1]^d$ be the unit cube in $\mathbb{R}^d$ and let $f : Q_0 \to \mathbb{R}$ be a continuous function with zero mean. Let $Z(f)$ be the nodal set $Z(f) = \{x \in Q : f(x) = 0\}$. Let $H^{d-1}(Z(f))$ denote the $(d-1)$-dimensional Hausdorff measure of $Z(f)$. Then*

$$W_1(f^+, f^-)\, H^{d-1}(Z(f)) \left(\frac{\|f\|_\infty}{\|f\|_1}\right)^{2-1/d} \gtrsim \|f\|_1. \tag{1}$$

Here $W_1(f^+, f^-)$ indicates the Vaserstein distance between the positive and negative parts of f.

The proof, which we now recall briefly, was based on a recursive selection, through a stopping time argument, of dyadic squares Q where either the mass of $|f|$ is irrelevant, called *empty*, or $\int_Q f^+$ is much larger than $\int_Q f^-$ (or the other way around), called *unbalanced*.

Assume, without loss of generality, that $H^{d-1}(Z(f))$ is finite and that f is normalized so that $\|f\|_1 = 1$. Consider now the standard dyadic partition of the unit cube: first split Q_0 into 2^d subcubes of length $1/2$ and then, recursively, split each of the new subcubes into 2^d "descendants" each with side length half that of the "parent". A cube $Q \subset [0,1]^d$ is *balanced* if

$$\frac{1}{100\|f\|_\infty} \le \frac{V_f^+(Q)}{V_f^-(Q)} \le 100\|f\|_\infty,$$

where V indicates the volume and

$$V_f^+(Q) = V(Q \cap \{f > 0\}) \text{ and } V_f^-(Q) = V(Q \cap \{f < 0\}).$$

On the other hand, Q is *empty* if $\int_Q |f| < V(Q)/10$.

Recursively, at each generation we set aside the cubes $\widetilde{Q_i}$ such that either:

(i) The cube $\widetilde{Q_i}$ itself is empty;

(ii) One of the 2^d direct descendants Q_i of $\widetilde{Q_i}$ is full (non empty) and unbalanced. In this case, denote by Q_i the full, unbalanced descendant of $\widetilde{Q_i}$ for which $\int_{Q_i} |f|$ is maximal.

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This gives a decomposition

$$Q_0 = [0,1]^d = \left(\cup_{i\in\mathcal{E}}\,\widetilde{Q_i}\right) \cup \left(\cup_{i\in\mathcal{F}}\,\widetilde{Q_i}\right) \cup R,$$

where $\mathcal{E}$ and $\mathcal{F}$ indicate respectively the indices of the full and empty selected cubes, and where R is the remaining set (whatever has not been selected in the process above).

We proved (Proposition 2) that $V(R) = 0$ and that the full cubes $\widetilde{Q_i}$ carry most of the mass, i.e. $\sum_{i\in\mathcal{F}}\int_{\widetilde{Q_i}}|f| \geq 9/10$, and next we observed (Remark 1) that the corresponding descendants Q_i, $i\in\mathcal{F}$, still carry a significant part of that mass:

$$\sum_{i\in\mathcal{F}}\int_{Q_i}|f| \geq \frac{9}{10}\times 2^{-d}. \tag{2}$$

Benjamin Jaye has kindly pointed out that this last estimate may in principle not be true in general, in that the mass in the full, unbalanced descendants Q_i may not be comparable to the total mass of f in $\widetilde{Q_i}$ – this mass may lie predominantly in the balanced descendants of $\widetilde{Q_i}$. He suggested circumventing this difficulty by constructing the cubes Q_i by means of a continuous stopping time argument together with the Besicovitch covering theorem, replacing our original dyadic-based decomposition. With these new cubes, Propositions 3 and 4 remain valid, hence also the statement of Theorem 1. The dyadic decomposition used to prove Theorem 2 requires analogous modification. The result itself, as well as all other results, remains correct as stated.

Details of the modifications to the proof of Theorem 1 are as follows. The function f is extended by 0 from Q_0 to all of $\mathbb{R}^d$. The definitions of balanced and unbalanced cubes are modified so that a cube Q is *balanced* if

$$\frac{1}{100\times 5^d\,\|f\|_\infty} \leq \frac{V_f^+(Q)}{V_f^-(Q)} \leq 100\times 5^d\,\|f\|_\infty. \tag{3}$$

The choice of the factor 5^d is in accordance to the constant appearing in Besicovich covering theorem (see below).

A similar adjustment is made to the definition of full and empty cubes, in that a cube Q is *empty* whenever

$$\int_Q |f| \leq \frac{1}{10\times 5^d}\,V(Q\cap Q_0).$$

For every $x\in Q_0$ such that $f(x)\neq 0$, there exists $l(x)>0$ such that the open cube Q_x centred at x and of side length $l(x) = l(Q_x)$ is simultaneously balanced and unbalanced in that either $V_f^+(Q)/V_f^-(Q)$ or $V_f^-(Q)/V_f^+(Q)$ *equals* $100\times 5^d\,\|f\|_\infty$. This can be achieved by continuity, since for l very small the cube centred at x and of side length l is infinitely unbalanced, while for side length $l=2$ it is balanced. Then there must be an intermediate side length $l(x)$ for which one of the inequalities in (3) is actually an identity.

These cubes Q_x cover Q_0 (up to at most a zero-measure set). According to the Besicovich covering theorem [1, Theorem 18.1] (see also [2]), one can find 5^d sequences $(x_{i,j})_{i\geq 1}$,

$j = 1, \dots, 5^d$, such that for each j the cubes $(Q_{x_{i,j}})_{i\geq 1}$ are pairwise disjoint and together still cover Q_0:

$$Q_0 \subset \bigcup_{j=1}^{5^d} \bigcup_{i\geq 1} Q_{x_{i,j}}.$$

Since $\int_{Q_0} |f| = 1$, there is at least one family of cubes $(Q_{x_{i,j}})_{i\geq 1}$ such that

$$\sum_{i\geq 1} \int_{Q_{x_{i,j}}} |f| \geq 5^{-d}.$$

From this particular sequence of cubes we select those that are full, and further relabel the cubes themselves as $(Q_i)_{i\geq 1}$. These cubes are thus disjoint and carry most of the mass, since

$$\sum_{i:\ Q_{x_{i,j}} \text{ empty}} \int_{Q_{x_{i,j}}} |f| \leq \frac{5^{-d}}{10} \sum_{i:\ Q_{x_{i,j}} \text{ empty}} V(Q_{x_{i,j}} \cap Q_0) \leq \frac{5^{-d}}{10}.$$

In conclusion, estimate (2) holds for the cubes $(Q_i)_{i\geq 1}$, up to a change of constants. Lemma 1 follows as before with only minor modifications due to extending f by 0 to all of $\mathbb{R}^d$.

By the precise choice of the side length of the cubes Q_i, both the estimates for the size of the zero set (Proposition 3) and for the Vaserstein distance (Proposition 4) hold for the new Q_i, with the obvious modifications. The proof of Theorem 1 concludes then as before.

References

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