# Traversable wormholes

Author: Oscar Michel González
*Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.*

Advisor: Bartomeu Fiol Núñez

**Abstract:** In this work we study firstly the general properties of wormholes in classical general relativity, and the constraints the metric must satisfy to support such structures. We will then describe some examples of classical solutions and find the issues they have before finally delving into more recent studies that show wormhole solutions supported by quantum fields and consistent with all known physical principles.

## I. INTRODUCTION

A wormhole is defined as a structure that connects two arbitrarily distant points in spacetime, acting as a bridge that allows fast travel and comunication between seemingly unreachable distances. The existence of such objects has been widely discussed by physicists almost since the inception of general relativity, the first popularised model being introduced by Einstein and Rosen in 1935 [1], and they have been invoked in the resolution of the Cosmological Constant problem [2] and the information loss paradox [3], among others. The concept has also been displayed in popular culture, one notable example being the novel *Contact* by Carl Sagan, which was the incentive for the development of a model for traversable wormholes by S.Morris and Kip S. Thorne, in 1987 [4].

Traversable wormholes are those that are suitable for objects, or humans to travel through. The essential requirement for a wormhole to be traversable is that the time to go through it must be reasonably small measured by all observers. This implies that there can be no horizons in the geometry. Other conditions can be required for the wormhole to be traversable in practice, such as making the tidal forces of the object traversing relatively small, or the wormhole to be big enough for large objects to traverse, but in this work we will focus in the most basic requirements.

In summary, if we take a wormhole to be spherically symmetric, the conditions the geometry will satisfy are:

1. It has a minimum radius $r_o$, that prevents the structure from collapsing into a point.

2. The metric has no horizons, and no naked singularities.

3. The space far away form the wormhole is asymptotically flat.

The method to find solutions that satisfy these conditions will consist on proposing a metric that alredy satisfies them and calculate the stress-energy tensor that generates it, and then we will see if the solution is physically reasonable. The result will be that for a traversable wormhole to exist, we need certain components of the stress-energy tensor to be negative, violating the so called "Null Energy Condition" (NEC), which states that

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \tag{1}$$

for any null vector $k^\mu$. Although classical models that violate the NEC can be constructed, all of them present problems, as there are no known fields in the standard model that violate it at the classical limit. However, as we will show, it is possible to find solutions supported by matter that violates NEC due to quantum effects [8].

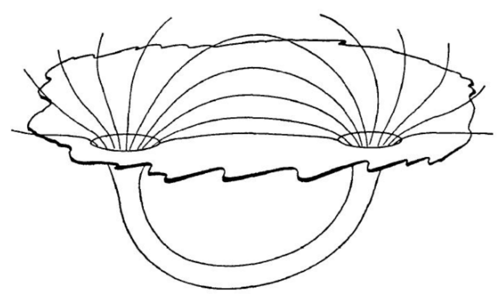

FIG. 1. Sketch of a traversable wormhole made by J. Wheeler in 1955 [9]. The lines between the two openings represent a magnetic field (See section 3)

## II. CLASSICAL WORMHOLES

### A. Metric

We will begin with an ansatz for a spherically symmetric and static metric

$$ds^2 = -e^{2\Phi}dt^2 + \left(1 - \frac{b}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

Where $\Phi = \Phi(r)$ and $b = b(r)$ with $(- + ++)$ signature. To deduce the stress-energy tensor we must first calculate the Riemann tensor and the curvature scalar, giving 24 non-zero components of the curvature tensor and 4 components of the Einstein tensor. Additionally the reference frame is switched to one that is at rest, that is $g_{\hat{\mu}\hat{\nu}} = diag(-1, 1, 1, 1)$, which will be more convenient to analyze the solutions. It is implemented in the calculation with the following coordinate transformation

$$e_{\hat{t}} = e^{-\Phi}e_t, \quad e_{\hat{r}} = (1-b/r)^{1/2}e_r, \\ e_{\hat{\theta}} = r^{-1}e_\theta, \quad e_{\hat{\phi}} = (r\sin\theta)^{-1}e_\phi \tag{3}$$

Then the 4 components of the Einstein tensor are

$$G_{\hat{t}\hat{t}} = b'/r^2 \tag{4}$$

$$G_{\hat{r}\hat{r}} = -b/r^3 + 2(1-b/r)\Phi'/r \tag{5}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \left(1-\frac{b}{r}\right)\left(\Phi'' - \frac{b'r-b}{2r(r-b)}\Phi' + \right. \\ \left. +(\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)}\right) \tag{6}$$

### B. Stress-energy tensor

Einstein's equations[1] $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$ show that there are 4 non-vanishing components of the stress-energy tensor, which take the form

$$T_{\hat{t}\hat{t}} = \rho(r), \quad T_{\hat{r}\hat{r}} = -\tau(r), \quad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p(r) \tag{7}$$

Where $\rho(r)$ is the energy density and $\tau(r)$ and $p(r)$ the radial and lateral pressures, respectively. Solving Einstein's equations the following relations are obtained.

$$\rho(r) = \frac{b'}{8\pi r^2} \tag{8}$$

$$\tau(r) = \frac{b/r - 2(r-b)\Phi'}{8\pi r^2} \tag{9}$$

$$p(r) = (2/r)[(\rho-\tau)\Phi' - \tau'] - \tau \tag{10}$$

Finally, the boundary conditions are the following. Considering a finite distribution of matter and energy $\rho, \tau$, and $p$ must be zero for $r > R$ where R is the radius of a sphere. This allows $\rho$ and $p$ to be discontinuous at $r = R$ but $\tau$, being a radial pressure, must go to zero at the frontier. This implies from equations (8) and (9) that the metric takes the Schwarzschild's form for $r > R$.

### C. General constraints of the wormhole

As described in the introduction a traversable wormhole requires two basic conditions. First that there is a minimum value of the radial coordinate, $b_0 > 0$, and second no horizons can exist. For a metric of the form (2), horizons are surfaces where the component $g_{00}$ of the metric tends to zero. In this metric it implies that $\Phi$ must be finite everywhere. Combining the the above conditions we can find a constraint on the stress-energy tensor. First we define the proper differential radial distance $dl(r)$ as

$$dl(r) = \frac{dr}{\sqrt{1-b/r}} \implies \frac{dr}{dl} = \pm\sqrt{1-\frac{b}{r}} \tag{11}$$

And differentiating with respect to $l(r)$ it is found that

$$\frac{d^2r}{dl^2} = \frac{1}{2r}\left[\frac{b}{r} - b'\right] \tag{12}$$

Now, since $r$ takes a minimum value at the throat and increases away from it, then $\frac{d^2r}{dl^2} > 0$ near the throat, which implies that

$$b' < \frac{b}{r} \implies b' < 1 \quad \text{near the throat} \quad (r \simeq b) \tag{13}$$

Taking now the following quantity: $T_{\hat{r}\hat{r}} - T_{\hat{t}\hat{t}}$ and using equations (8) and (9) near the throat and considering the result of (13) it is found that

$$T_{\hat{r}\hat{r}} - T_{\hat{t}\hat{t}} = \frac{1}{C}(b/r - 2(r-b)\Phi' - b') \xrightarrow[r,b\to b_0]{} \frac{1}{C}(1-b') > 0 \tag{14}$$

Where $C = 8\pi r^2 > 0$ and implies that $T_{\hat{t}\hat{t}} < T_{\hat{r}\hat{r}}$.

### D. NEC violation

The condition $T_{\hat{t}\hat{t}} < T_{\hat{r}\hat{r}}$ leads to an interesting result when we consider the measurement of a radially moving observer near the throat. Using a radial boost on $T_{\hat{t}\hat{t}}$ we find

$$T_{\hat{0}\hat{0}} = \gamma^2(T_{\hat{t}\hat{t}} - \beta^2 T_{\hat{r}\hat{r}}) = \gamma^2(T_{\hat{t}\hat{t}} - T_{\hat{r}\hat{r}}) + T_{\hat{r}\hat{r}} \tag{15}$$

And consequently, for large $\gamma$ the above result will be negative, meaning that an observer traversing the wormhole at sufficient speed will measure negative mass-energy densities in clear contradiction with the Null Energy Condition. The kind of substance with this property is refered to as "exotic matter" and it is indispensible for any wormhole solution to exist. The lack of such matter in classical theories will be the motivation for finding solutions in quantum field theory.

### E. Visual example of a wormhole

A particularly simple solution is the EMBT (Ellis-Bronnikov-Morris-Thorne) wormhole [4]. The shape of the metric is $\Phi = 0$ and $b(r) = \frac{b_o^2}{r}$. The main characteristic it has is that all components of stress-energy tensor are equal except for one sign.

$$-T_{\hat{t}\hat{t}} = T_{\hat{r}\hat{r}} = T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = \frac{b_o^2}{r^4} \tag{16}$$

[1] We will use natural units, $c = G = \hbar = 1$. throughout the whole paper

This can only be achieved with a field with negative kinetic energy and therefore it is not a realistic solution. The embedding diagram can be obtained taking a surface of constant $t$ and $\theta$ and doing a representation in cilyndrical coordinates leaving the curve

$$z(r) = \pm b_o ln\left(r/b_o + \sqrt{(r/b_o)^2 + 1}\right) \tag{17}$$

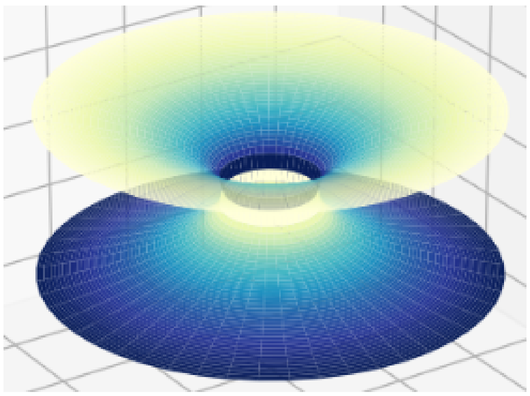

FIG. 2. Embedding diagam of a EBMT wormhole. It has a throat at $b_0$ and far from it the space-time tends to an asymptotically flat geometry

Other solutions of this kind can be obtained modifying the $\phi$ and $b$ functions in order to get wormholes with desirable properties. However all of them need the presence of exotic matter in greater or lesser quantities. One could also ask what happens if we take a more general metric, non-spherical and non-static, but we would find that the NEC is violated anyways by virtue of the Raychaudhuri's equation. (See Ref. [6], appendix F).

Another option is to consider classical fields that violate NEC such as the one from [7] where a coupling term is added to the Lagrangian of a scalar field. The issue here is that none of these fields fit in the standard model. However, it is relatively easy to find examples in quantum field theory that can violate NEC at a quantum level and still be permited by the standard model. This is the subject of the following section.

## III. TRAVERSABLE WORMHOLE FROM TWO MAGNETIC BLACK HOLES

In quantum mechanics, matter is not bound by the Null Energy Condition, but by the Achronal Averaged Null Energy Condition (AANEC), which states that the average energy along an achronal null geodesic must be positive, that is

$$\int_\gamma T_{\mu\nu} k^\mu k^\nu \geq 0 \tag{18}$$

This condition is less restrictive than the NEC and means that the stress-energy tensor can be negative locally, but not when averaged over the achronal null geodesic. The example from [8] described here consists in taking two near extremal magnetically charged black holes and putting a fermion field inside which will create a series of Landau levels, the lowest of which will have zero energy in the angular directions. Then it will be shown that the field has a negative Casimir energy able to support a wormhole throat.

### A. The geometry

The metric of a charged black hole is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r_e^2}{r}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{r_e^2}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{19}$$

Where $r_e^2 = \frac{\pi q^2}{g^2}$, with $M$ being its mass, $q$ its charge, and $g$ the coupling constant to the gauge field, which is

$$A = \frac{q}{2} \cos\theta d\phi \tag{20}$$

This geometry exhibits a horizon at $r = r_+$ where

$$r_\pm = M \pm \sqrt{M^2 - r_e^2} \tag{21}$$

At near extremality however, we approximate that $r_+ = r_- = r_e$ and in this limit the metric takes the form

$$ds^2 = r_e^2 \left[-(\rho^2 + 1)d\tau^2 + (\rho^2 + 1)^{-1} d\rho^2 + d\Omega^2\right] \tag{22}$$

Where

$$t = l\tau, \quad \rho = l(r - r_e)r_e^2/r_e^2 \tag{23}$$

This metric corresponds to an $AdS_2 \times S^2$ metric, which will be the throat of the wormhole, and $l$ is a parameter that will be related to its length. For a single black hole this means we would have an infinitely long throat at $r_+$. In this situation, however, we will consider two black holes separated by a distance $d$. The gauge field in this case will be

$$A = \frac{q}{2}(\cos\theta_1 - \cos\theta_2)d\phi \tag{24}$$

Where $\theta_1$ and $\theta_2$ are the angles measured from the two mouths with respect to the straight line connecting them. If we want our fermions to move along the magnetic field lines, then the magnetic field will be constant and this gives us a constraint,

$$(\cos\theta_1 - \cos\theta_2) = \nu \quad \text{with} \quad 0 \leq \nu \leq 2 \tag{25}$$

In summary, the geometry of the wormhole will be the following: The wormhole mouths will have the metric of (19) near its horizon, the wormhole between the two mouths will be $AdS_2 \times S^2$ , like (22), and the space in between, and far from the mouths will be flat.

### B. Fermions inside the wormhole

Consider a massless fermion field $\chi$ of charge one in the space described above. To find its energy we must calculate the eigenvectors of the Dirac's equation in curved spacetime.

$$i\gamma^a e^\mu_a D_\mu \Psi = 0 \tag{26}$$

Where $D_\mu = \partial_\mu - \frac{i}{4}\omega^{ab}_\mu \sigma_{ab} + iqA_\mu$. $e^\mu_a$ is a tetrad that defines a local rest frame, $\sigma_{ab}$ is the commutator of the gamma matrices, $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$, and $\omega^{ab}_\mu$ are the components of the spin connection.

If we consider a general metric of the form

$$ds^2 = -e^{2\sigma(t,r)}dt^2 + e^{-2\sigma(t,r)}dr^2 + R^2(t,r)d\Omega \tag{27}$$

Then the tetrad is

$$e^1 = e^\sigma dt, \quad e^2 = e^{-\sigma}dr, \quad e^3 = Rd\theta, \quad e^4 = R\sin\theta d\phi \tag{28}$$

And the only non zero components of the spin connection are

$$\begin{aligned} \omega^{12} &= \sigma' dt + \dot{\sigma}dx, \quad \omega^{32} = R' e^{-\sigma} d\theta, \\ \omega^{42} &= R'\sin\theta e^{-\sigma}, \quad \omega^{43} = \cos\theta d\phi \end{aligned} \tag{29}$$

Now we can express $\chi$ as a tensor product of two two-dimensional spinors, one in the radial and time dimensions, and a second one in the angular directions $\chi = \psi(r,t) \otimes \eta(\theta,\phi)$. Now substituting everything in (26), expressing the gamma matrices as a function of the Pauli matrices and using the following ansatz

$$\chi_{\alpha\beta} = \frac{e^{-\sigma/2}}{R}\psi_\alpha(r,t)\eta_\beta(\theta,\phi) \tag{30}$$

We obtain two equations

$$\left[\sigma_y \frac{\partial_\phi - iA_\phi}{\sin\theta} + \sigma_x\left(\partial_\theta + \frac{1}{2}cot\theta\right)\right]\eta = 0 \tag{31}$$

$$(i\sigma_x\partial_t + \sigma_y\partial_x)\psi = 0 \tag{32}$$

Solving the first equation we obtain the solutions

$$\eta_- = \left(\sin\frac{\theta}{2}\right)^{j-m}\left(\cos\frac{\theta}{2}\right)^{j+m} e^{i\phi}\begin{pmatrix}0\\1\end{pmatrix}, \quad \eta_+ = 0 \tag{33}$$

where $j = \frac{q-1}{2}$ is the angular momentum and $-j \leq m \leq j$ where $m$ is the $J^3$ quantum number. The key of this solution is to notice that at the lowest Landau level the energy along the $\theta$ and $\phi$ directions is zero so a 4-dimensional fermion field gives rise to $q$ 2-dimensional massless fermions in the radial and time directions.

### C. Energy along magnetic lines

To calculate the Casimir energy we first need to know the length of the path the fermions take, which will be the sum of the length of the wormhole and the length along the flat space between the mouths. First, we note that because the fermions move in a 2 dimensional plane, we can define a rescaled length $x$ such that the metric is

$$ds^2 = g_{tt}(-dt^2 + dx^2) \quad and \quad dx = \sqrt{\frac{g_{rr}}{-g_{tt}}}dr \tag{34}$$

Then the first term is obtained integrating $x$ along the wormhole region, which goes between the two points where the throat starts to open up $r - r_e \sim r_e$. At this point $\rho \sim \frac{l}{r_e}$, and we will suppose that $l \gg r_e$. The length of the wormhole throat is

$$L_{throat} = \int_{-l/r_e}^{+l/r_e} \frac{l}{(\rho^2+1)}d\rho = 2l\arctan(l/r_e) \sim \pi l \tag{35}$$

The second section of the path can be obtained parametrizing the constant magnetic field lines as a function of $\theta_1$ and $\theta_2$ and using (25) to express one angle as a function of the other one and $\nu$, which also gives us the integration limits. The length of the flat region is then $L_{flat} = df(\nu)$ with $d$ being the separation of the mouths and $f$ a function of $\nu$. But assuming that $l \gg d$ this term will be neglected so in the end $L = \pi l$. Finally, the energy of $q$ fermions moving on a circle of length L creates the following casimir energy [10]

$$E = -\frac{q}{12}\frac{2\pi}{L} = -\frac{1}{6}\frac{q}{l} \tag{36}$$

The total energy though, needs an extra term, due to the fact that the space in the wormhole is not flat, but conformally flat, and there is an extra contribution from the conformal anomaly. This energy is [11]

$$E = +\frac{q}{24}\frac{\pi}{L} \tag{37}$$

So the final energy of the wormhole is $E = -\frac{q}{8l}$ and the stress-energy tensor has to be one that gives this energy

$$\langle T_{tt}\rangle = \langle T_{xx}\rangle = -\frac{q}{8l^2}\frac{1}{4\pi^2 r_e^2} \tag{38}$$

### D. Solution of the Einstein's equations

We now proceed to incorporate the stress energy tensor into the Einstein's equations to prove that we have a genuine solution and find the length of the wormhole $l$, and its minimum energy. We will do that by expanding the metric (22) as a perturbation and comparing it with the metric from (19).

$$ds^2 = r_e^2 \left[ -(1+\rho^2+\gamma)d\tau^2 + (1+\rho^2+\gamma)^{-1}d\rho^2 + (1+\phi)d\Omega^2 \right] \tag{39}$$

Where $\gamma$ and $\phi$ are two parameters with small values. We can now find $\phi$ by aplying Einstein's equations to the radial component of the stress-energy tensor, which will have two contributions. One from the regular $AdS_2 \times S^2$ metric, and one from the Casimir effect computed in (38)

$$8\pi(T^{mag}_{\rho\rho} + \langle T_{\rho\rho}\rangle) = R_{\rho\rho} - \frac{1}{2}g_{\rho\rho}R \implies \implies 8\pi\langle T_{\rho\rho}\rangle = \frac{\rho\phi' - \phi}{1+\rho^2} \tag{40}$$

Now using (38) and the relations between $t$ and $\tau$ and between $x$,$\rho$ and $\tau$ from (34), we get

$$8\pi\langle T_{\rho\rho}\rangle = -\frac{\alpha}{1+\rho^2} \quad \text{where} \quad \alpha = \frac{q}{4\pi r_e} \tag{41}$$

Solving now the differential equation for $\phi$ and taking the limit for large $\rho$

$$\phi = \alpha(1+\rho\arctan\rho) \sim \frac{\pi\alpha}{2}\rho \tag{42}$$

And comparing the $d\Omega$ components of (19) and (39) we obtain

$$r^2 = r_e^2(1+\phi) \implies \frac{r-r_e}{r_e} = \phi = \frac{\pi\alpha}{2}\rho \tag{43}$$

For large $\rho$. Finally we can set the distance $l$ comparing the higher order terms of the time components and using that $dt = l d\tau$

$$\frac{r-r_e}{r_e}dt = r_e\rho d\tau \implies l = \frac{r_e^2\rho}{r-r_e} = 16\frac{r_e^2}{q} \tag{44}$$

With the above results we can also see that $(r-r_e)/r_e = \rho)$, so when comparing the time components of the metrics for $\gamma = 0$ the $\rho^2$ terms are identical and we only need to identify the the constant terms and find that

$$E_{min} = 2(M - r_e) = r_e^3 l^2 = -\frac{1}{256}\frac{q^2}{r_e^3} \tag{45}$$

Finding that the minimum energy of the system is negative. This condition, along with a minimum radius and no horizons guarantees that the connection between the mouths is in fact a traversable wormhole. It should be noted though, that the fact that the fermion field is considered massless restricts the size of the wormhole below the electroweak scale. Otherwise it would not fit in the standard model.

## IV. CONCLUSIONS

We have first shown the shape of the metric and the constraints on the stress-energy tensor for a spherical wormhole and found that we need the presence of exotic matter in order to meet the necessary conditions. We have proceeded to show a simple example of a wormhole supported by classical matter and justified the need for quantum effects to be taken into account in our solutions. Finally we have reproduced the example from [8] of a wormhole supported by the Casimir energy of fermion fields.

As a summary we can say that wormholes, although simple in appearance, are very intricated objects clashing whith multiple physical principles that force us to find creative ways to circumvent them. Although it is not quite clear if they can exist or be created artificially, they continue to provide insight about the workings of General Relativity and its connection with quantum field theory.

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