Electromagnetic radiation reaction in a classical binary system

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Abstract: We study the radiation reaction force in electrodynamics, reviewing a derivation of the Lorentz-Dirac equation. Then we treat the case of a non-relativistic binary system, which becomes unstable due to the radiation reaction force, and study the evolution of its orbital parameters and the decay time. Finally, we solve the relativistic case numerically and compare it with the non relativistic behaviour.

I. INTRODUCTION

When a charged particle is accelerated it emits radiation, leading to a loss of energy. In classical Electrodynamics, such phenomenon is described by the Lorentz-Dirac equation, which includes the radiation reaction force. If taken seriously, the equation predicts a fast decay of electron orbits around atomic nuclei, in contradiction with the existence of chemically stable matter. Of course, the behaviour of electrons in atoms is outside the scope of the classical theory, and requires a quantum description.

Nonetheless, the classical study of radiation reaction in two body systems of charged particles is simple enough to make some analytic progress, which may in fact be of more than just academic interest. There are physical situations where this can be a reasonable approximation. For instance, it may have interesting applications when studying classical Rydberg-like atoms [1]. Also, it can be used to describe the decay of "monopolonium" a bound state of magnetic monopole and antimonopole that may have formed in the early universe [2]. Monopoles are rather massive topological defects that behave classically, and hence the Lorentz-Dirac equation seems very appropriate to describe the evolution of monopolonium bound states. The radiation emitted by such bound states could be experimentally observed, yielding information about physical processes at energies ranging from current accelerators to the grand unification scale [2, 3].

In this work, we review a derivation of the Lorentz Dirac equation in its covariant form, commenting on certain issues it presents, such as pre-acceleration and runaway solutions. This is done in Section II. In Section III, we use this equation to study a binary system of two particles with opposite charges $+q$ and $-q$. We will study the behaviour of their orbits in the non-relativistic regime, and the secular changes in the orbital parameters (such as eccentricity and semi-major axis) due to the radiation reaction. Finally, in Section IV, we will perform a numerical treatment in the relativistic case, where the orbits precess, and quantify to what extent the results depart from the non-relativistic analytic treatment. Our conclusions are presented in Section V.

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II. THE LORENTZ-DIRAC EQUATION

In order to obtain the equations of motion of a charged particle we can use the Lorentz force caused by an electromagnetic field $F^{\alpha\beta}$. Knowing that the moving particle creates its own electromagnetic field, which we can call $F^{\alpha\beta}_{rr}$, we can consider the total field as the sum of the external field and a radiation reaction (rr) field

$$ma^{\alpha} = qF^{\alpha\beta}_{ext}u_{\beta} + qF^{\alpha\beta}_{rr}u_{\beta}. \tag{1}$$

Following [4], we need to calculate both the advanced and retarded potentials $A^{\alpha}_{ret}(x)$ and $A^{\alpha}_{adv}(x)$, because half the addition of both gives rise to the Coulomb field, and half the subtraction gives rise to the rr term [5]. They have the support in the past and future light cones respectively of the point x in which we are calculating the field.

We let u and v be the proper time in which the world line of the particle intersects the past and future light cones respectively, called the retarded and advanced times. Now we shall call $z^{\alpha}(u)$ and $u^{\alpha}(u)$ the position and velocity of the particle in these points. To define a notion of distance between these points and x we consider the scalars r and r_{adv} as:

$$\begin{aligned} r &= -[x^{\alpha} - z^{\alpha}(u)]u_{\alpha}(u) \\ r_{adv} &= -[x^{\alpha} - z^{\alpha}(v)]u_{\alpha}(v). \end{aligned} \tag{2}$$

If we compute these scalars in the momentarily comoving Lorentz frame (MCLF), in which the particle is momentarily at rest ($u^{\alpha} = \delta^{\alpha}_0$) we can see that they correspond to the distance between the points z and x, and so we call them the retarded and advanced distances respectively.

The retarded potential is given by the usual *Liénard-Wiechert* expression, which reads:

$$A^{\alpha}_{ret}(x) = q\frac{u^{\alpha}(u)}{r(x)}. \tag{3}$$

The form of the advanced potential will not be as simple, as we want to express it in the retarded coordinates. Since we are interested in points close to the world line, we will have to expand both $u^\alpha(v)$ and $r_{adv}(x)$ around $r = 0$. To do so we define $\Delta\tau \equiv v - u$. Furthermore, we introduce the null vector

$$k^\alpha(x) = \frac{1}{r}[x^\alpha - z^\alpha(u)], \tag{4}$$

and the scalars $a_k \equiv k^\alpha(x)a_\alpha(u)$, $\dot{a}_k \equiv k^\alpha(x)\dot{a}_\alpha(u)$, and $a^2 = a^\alpha(x)\ a_\alpha(u)$. Here the dot in $\dot{a}_\alpha$ represents a derivative with respect to the proper time. Keeping in mind that $z(v)$ is in the light cone of x we obtain the following relation between $\Delta\tau$ and r:

$$\Delta\tau = 2r\left[1 - a_k r + \left(a_k^2 - \frac{1}{3}a^2 - \frac{2}{3}\dot{a}_k\right)r^2 + \mathcal{O}(r^3)\right]. \tag{5}$$

The expansion of $z^\alpha(v)$ and $u^\alpha(v)$ in terms of $\Delta\tau$ is straightforward, and it reads:

$$\begin{aligned} z^\alpha(v) &= z^\alpha(u) + u^\alpha\Delta\tau + \frac{1}{2}a^\alpha\Delta\tau^2 + \frac{1}{6}\dot{a}^\alpha\Delta\tau^3 + \mathcal{O}(\Delta\tau^4), \\ u^\alpha(v) &= u^\alpha(u) + a^\alpha\Delta\tau + \frac{1}{2}\dot{a}^\alpha\Delta\tau^2 + \mathcal{O}(\Delta\tau^3). \end{aligned} \tag{6}$$

If we now substitute (6) and (5) into (2), after some algebra we obtain the desired expressions for r_{adv} and $u^\alpha(v)$:

$$\begin{aligned} r_{adv} &= r + \frac{2}{3}(a^2 + \dot{a}_k)r^3 + \mathcal{O}(r^4), \\ u^\alpha(v) &= u^\alpha(u) + 2a^\alpha r + 2(\dot{a}^\alpha - a_k a^\alpha)r^2 + \mathcal{O}(r^3). \end{aligned} \tag{7}$$

Plugging these into the advanced version of (3) we get the following vector potential:

$$\begin{aligned} A^\alpha_{adv}(x) = q\frac{u^\alpha}{r} + 2qa^\alpha + 2q[\dot{a}^\alpha - a_k a^\alpha \\ -\frac{1}{3}(a^2 + \dot{a}_k)u^\alpha]r + O(r^2). \end{aligned} \tag{8}$$

We want to calculate the potential on the world line, so all terms of order r^2 or higher will vanish after differentiation.

Considering that the term of the potential corresponding to radiation reaction is $A^\alpha_{rr}(x) = (A^\alpha_{ret} - A^\alpha_{adv})/2$ we now find that the rr field is

$$F^{\alpha\beta}_{rr} = -\frac{2}{3}q(\dot{a}^\alpha u^\beta - u^\alpha\dot{a}^\beta). \tag{9}$$

If we now substitute this into (1) we obtain

$$ma^\alpha = F^\alpha_{ext} + \frac{2}{3}q^2(\delta^\alpha_\beta + u^\alpha, u_\beta)\dot{a}^\beta, \tag{10}$$

which can also be expressed as

$$ma^\alpha = F^\alpha_{ext} + \frac{2}{3}q^2(\dot{a}^\alpha - a^2u^\alpha). \tag{11}$$

This is the Lorentz-Dirac equation. This equation presents a number of problems, which are nicely reviewed in [5].

First of all, we can see that it involves the time derivative of acceleration, meaning it is a third order differential equation. Therefore we should need more initial conditions than just the position and velocity, something quite problematic. Also, consider the non relativistic case where a force is turned on at $t = 0$ and then stays constant. We obtain the following solution:

$$\mathbf{a}(t) = e^{t/t_0}\left[\mathbf{b} - \frac{\mathbf{f}}{m}\left(1 - e^{-t/t_0}\right)\Theta(t)\right], \tag{12}$$

where $t_0 = 2q^2/3m$. As we can see for arbitrary values of $\mathbf{b}$ the acceleration grows exponentially with time even though the force is constant. These type of solutions are called runaway solutions. We can get rid of this behaviour by setting $\mathbf{b} = \mathbf{f}/m$, being this the third parameter we needed. The problem is that in this case the acceleration of the particle is affected by the force acting on it at later times, violating the principle of causality. This phenomenon is called preacceleration.

These problems are due to the fact that we considered the particle to be point-like, ignoring all of its multipole terms, which would not negligible in a regularized version of the world line where the particle has finite size. To avoid this problem we can use a technique known as reduction of order, discussed in [5] and [6], with which we obtain the modified Lorentz-Dirac equation:

$$ma^\alpha = F^\alpha_{ext} + \frac{2}{3}q^2(\delta^\alpha_\beta + u^\alpha u_\beta)F^\beta_{ext\ ,\gamma}u^\gamma, \tag{13}$$

where we have basically assumed that the time derivative of the acceleration is mostly due to the time derivative of the external force. This is justified by the fact that the radiation reaction force is suppressed by a factor (t_0/t_c), relative to the external force, where $t_c \sim a/\dot{a}$ is the timescale in which the acceleration changes significantly, and we assume $t_c \gg t_0$. In the non-relativistic regime, Eq. (13) reads:

$$m\mathbf{a} = \mathbf{F}_{ext} + \frac{2q^2}{3m}\dot{\mathbf{F}}_{ext}. \tag{14}$$

This equation presents no runaway solutions or preacceleration, and has the same degree of accuracy as the original, which is up to corrections of order $(t_0/t_c)^2$. Therefore this is a better candidate to describe the motion of a point charge.

III. NON-RELATIVISTIC LIMIT

Let us now consider two charges of opposite sign, bound to each other in non-relativistic motion. In this case the orbits are nearly elliptical. Following Landau and Lifshitz [4] (see also Peters [7] for the related case of gravitational radiation), the orbit will be described by its eccentricity e and its major semi-axis a. If there was no radiation reaction force, these two would be constants of the motion, but the emission of radiation causes a loss of energy and angular momentum, leading to a change in time of said parameters. From now on we will use the electrostatic CGS system of units, so the speed of light c will appear in the equations.

In an elliptical orbit the parameters a and e define the energy and the angular momentum as follows:

$$E = -\frac{\alpha}{2a} \qquad L = \sqrt{\mu\alpha a(1-e^2)}, \tag{15}$$

where α is the constant of the interaction between the particles. In the Coulomb case it reads $\alpha = |q_1 q_2|$.

For the following calculations we will restrict ourselves to the non-relativistic limit, and we will also treat the radiation reaction as a perturbative term. Therefore we will consider the orbiting charge to describe an unaltered trajectory, and then calculate the losses of energy and angular momentum upon the completion of such orbit.

The equation of motion for the particle will read:

$$\mu\ddot{\mathbf{r}} = -\alpha\frac{\mathbf{r}}{r^3}. \tag{16}$$

The expression for the intensity of the radiation of two moving charges in the non relativistic limit is given by the Larmor formula:

$$I = \frac{2}{3c^3}\ddot{\mathbf{d}}^2. \tag{17}$$

Where $\mathbf{d}$ is the dipole moment, which can be expressed as:

$$\mathbf{d} = \mu\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)\mathbf{r} \equiv \mu\beta\mathbf{r}. \tag{18}$$

μ is the reduced mass and $\mathbf{r}$ is the relative position between the charges $\mathbf{r}_1 - \mathbf{r}_2$. With the help of (16) and substituting (18) into (17) we obtain:

$$I = \frac{2\alpha^2}{3c^3}\beta^2\frac{1}{r^4}. \tag{19}$$

Now we have to average this intensity through a full period, $\langle I\rangle = \frac{1}{T}\int_0^T I dt$. Using the ellipse equation and the definition of the angular momentum, $L = \mu r^2\frac{d\phi}{dt}$ we can switch from an integration over time to an integration over the angular coordinate ϕ. This results in:

$$\left\langle\frac{dE}{dt}\right\rangle = -\langle I\rangle = -\frac{2\alpha^2\beta^2}{3c^3a^4(1-e^2)^{5/2}}\left[1+\frac{e^2}{2}\right]. \tag{20}$$

Now, the loss of angular momentum can be calculated from an equivalent definition, $\mathbf{L} = \sum\mathbf{r}\times\mathbf{p}$. We want to obtain its variation over time, $\dot{\mathbf{L}} = \sum\mathbf{r}\times\mathbf{f}$. The force acting upon the particle can be decomposed into the Coulomb force and the radiation reaction force, where the former will not cause a change in the angular momentum of the system. Thus, the change we are looking for will come from the latter, which acting on one charge reads:

$$\mathbf{f}_i = \frac{2q_i}{3c^3}\dddot{\mathbf{d}}. \tag{21}$$

Hence, the angular momentum will read:

$$\frac{d\mathbf{L}}{dt} = \frac{2}{3c^3}\sum q_i\mathbf{r}_i\times\dddot{\mathbf{d}} = \frac{2}{3c^3}\mathbf{d}\times\dddot{\mathbf{d}}. \tag{22}$$

$$\frac{d\mathbf{L}}{dt} = \frac{2}{3c^3}\left(\frac{d}{dt}(\mathbf{d}\times\ddot{\mathbf{d}}) - \dot{\mathbf{d}}\times\ddot{\mathbf{d}}\right). \tag{23}$$

The first term of the last expression will vanish upon averaging as we consider the charges to be in a periodic orbit. Therefore, the desired expression will be:

$$\frac{d\mathbf{L}}{dt} = -\frac{2\beta^2\mu^2}{3c^3}\left(\dot{\mathbf{r}}\times\ddot{\mathbf{r}}\right), \tag{24}$$

$$\frac{d\mathbf{L}}{dt} = -\frac{2\alpha\beta^2}{3c^3}\frac{\mathbf{L}}{r^3}. \tag{25}$$

Using a process similar as the one used in the calculation of the intensity we find:

$$\left\langle\frac{dL}{dt}\right\rangle = -\frac{2\alpha^{3/2}\mu^{1/2}\beta^2}{3c^3a^{5/2}(1-e^2)}. \tag{26}$$

Using (15) and the chain rule we can calculate the expressions for the major semi-axis and the eccentricity, and we obtain:

$$\left\langle\frac{da}{dt}\right\rangle = -\frac{4\alpha\beta^2}{3c^3a^2(1-e^2)^{5/2}}\left[1+\frac{e^2}{2}\right]. \tag{27}$$

$$\left\langle\frac{de}{dt}\right\rangle = -\frac{\alpha\beta^2}{c^3a^3(1-e^2)^{3/2}}e. \tag{28}$$

Thus, during the decay of the orbit we can relate the eccentricity and the major sem-axis with the following differential equation:

$$\left\langle\frac{da}{de}\right\rangle = \frac{4a}{3(1-e^2)e}\left[1+\frac{e^2}{2}\right]. \tag{29}$$

This equation can be resolved upon integration, and leads to:

$$a = a_0\left(\frac{e}{e_0}\right)^{4/3}\left(\frac{1-e_0^2}{1-e^2}\right). \tag{30}$$

It would also be interesting to calculate the decay time of the system, and to do that we shall substitute (30) into (28), obtaining:

$$\frac{de}{dt} = -\frac{\alpha\beta^2e_0^4}{c^3a_0^3(1-e_0^2)^3}\frac{(1-e^2)^3/2}{e^3}. \tag{31}$$

If we now integrate the differential equation from e_0 to 0, we will obtain the decay time $T(a_0, e_0)$

$$T(a_0, e_0) = \frac{c^3 a_0^3 (1 - e_0^2)^3}{\alpha \beta^2 e_0^4} \left[\frac{2 - e_0^2}{(1 - e_0^2)^{1/2}} - 2 \right]. \qquad (32)$$

Note that for a given initial major axis a_0 (or corresponding binding energy E_0) the life-time is significantly reduced if the orbit is very eccentric.

IV. NUMERICAL STUDY

To check the results obtained in the previous section we will compute the collapse of the system using numerical methods. The system will be treated relativistically, but it will have some other simplifying assumptions. We will consider the limit where one of the charged particles is much more massive than the other, and hence stationary, effectively transforming this into a one body problem. In this case, the dipole moment will become $\mathbf{d} = q\mathbf{r}$. Using the equation we obtained in (11) we obtain the following relativistic expression for the radiation reaction force:

$$\mathbf{F}_{reac} = \frac{2}{3} q^2 \left[\frac{d}{dt} \left(\gamma^2 \ddot{\mathbf{r}} + \gamma^4 \left(\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \right) \dot{\mathbf{r}} \right) - \left(\gamma^4 \ddot{\mathbf{r}}^2 + \gamma^6 \left(\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \right)^2 \right) \dot{\mathbf{r}} \right]. \qquad (33)$$

Where now the dots represent derivatives with respect to the coordinate time. The equation of motion reads:

$$\frac{d}{dt}(m\gamma\dot{\mathbf{r}}) = -\frac{e\mathbf{r}}{|r|^3} + \mathbf{F}_{reac}. \qquad (34)$$

We will also treat radiation reaction as a perturbation term, and assume the derivatives in (33) to come from (16). This way we obtain a second order differential equation which can be easily solved using a Runge-Kutta of order 4.

In order to obtain the parameters of the orbit a and e, the maximum and minimum distances to the focus r_{max} and r_{min} are calculated upon the completion of every orbit, and we define a and e as:

$$a = \frac{r_{max} + r_{min}}{2} \quad e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}. \qquad (35)$$

Where they do not have to be understood as the semiaxis and eccentricity of an ellipse, because the relativistic orbits are no longer ellipses and manifest precession (see e.g. [8]).

To check the results obtained in the last section we will first calculate the orbits in the non relativistic regime. We will start with an initial value of the major semiaxis of $a = 10^{-8} cm$ (which is comparable to the Bohr radius), the electron charge $q = 4.803 \cdot 10^{-10} statC$ and an orbiting particle with the electron mass $m = 9.109 \cdot 10^{-28} g$. If we plot the logarithm of the major semiaxis as a function of eccentricity, we obtain Fig. 1. The initial speed

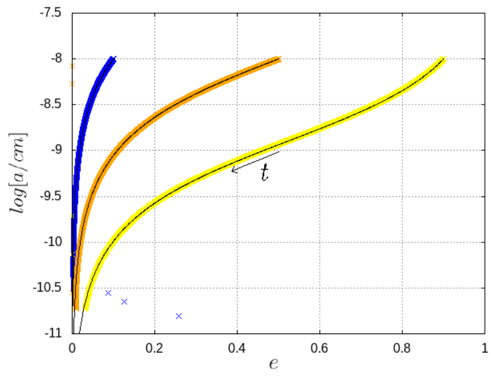


FIG. 1: Major semi-axis as a function of eccentricity for an electron orbiting a heavy nucleus of equal and opposite charge, for three different initial values of e. The initial size of the orbit is comparable to the Bohr radius, and the motion becomes mildly relativistic, with $(v/c) \sim 0.1$, towards the lower range of plotted sizes. Time evolution is indicated by an arrow. Thick coloured lines represent the numerical evolution, which agrees very well with the thin black lines corresponding to the analytic result Eq. (30).

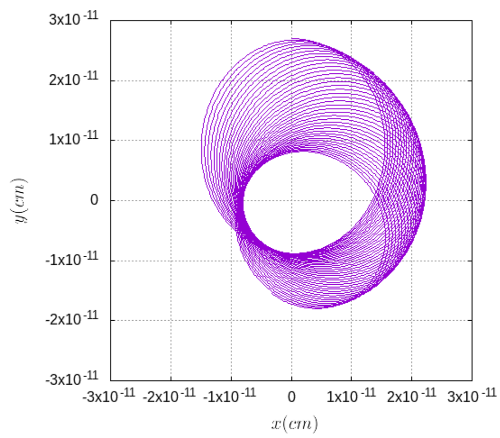
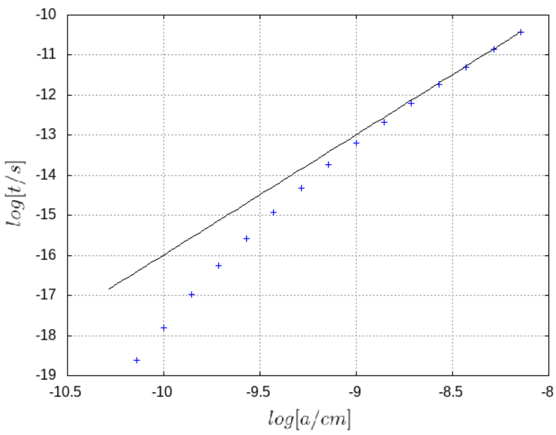


FIG. 2: Charged particle orbiting the nucleus, with an initial semiaxis of $a_0 = 1.8 \cdot 10^{-10} cm$. The orbits precess and their size decreases over many iterations.

in this case is of order $(v/c) \sim \alpha_{EM} \sim 10^{-2}$, where $\alpha_{EM} \propto q^2/\hbar c$ is the fine structure constant, and then scales as $(v/c) \propto a^{-1/2}$ as the radius decreases, with $(v/c) \lesssim 10^{-1}$ down to distances of order $a \gtrsim 3 \times 10^{-11} cm$. The numerical results, plotted in thick colored lines obtained are in very good agreement with the analytical results obtained in the previous Section, which, for comparison, are plotted as thin black lines for different values of the initial eccentricity.

The typical orbit is illustrated in Fig. 2, where we see that it precesses, and that its size shrinks over time. In Fig. 3 we plot the decay time as a function of the initial major semiaxis, for an orbit with negligible eccentricity [on a logarithmic scale, the effect of eccentricity



would only be noticeable for $(1 - e_0) \ll 1$]. The effect can be quite substantial even if the initial orbit is only mildly relativistic. The decay time departs from the non-relativistic expression (32), where it is proportional to a_0^3. In the relativistic regime the radiation reaction force in (33) is greater due to the gamma factors, and thus the particle spirals to the center at a faster pace.

V. CONCLUSIONS

We reviewed the derivation of the Lorentz-Dirac equation using covariant methods. This is an equation with third derivatives of position, and is riddled with problems such as runaway solutions and pre-acceleration. These can be addressed by using a reduction of order method, which leads to a modified version which does not have such pathologies and has the same level of accuracy.

We have then studied a classical binary system in the non relativistic limit, calculating the evolution over time of the orbit parameters and the decay time of the system, which spirals into the center radiating away its energy, and losing its eccentricity along the way too. Highly eccentic orbits decay much faster than circular ones, according to Eq. (32).

We then solved the relativistic equation using a Runge-Kutta method of order 4. The results agree with the analytic results in the case of an electron orbiting at an initial semi-axis of 1 Å. This is comparable to the Bohr radius and therefore a non-relativistic approach should be enough to describe Rydberg-like atoms, where the electrons are in highly excited states further away from the nucleus.

As we probe into smaller values of the initial size of the orbit, relativistic effects begin to take importance and the reaction force increases leading to smaller decay times. This observation is not particularly useful for the case of electrons in an atom, where orbitals of size comparable to the Bohr radius already require a quantum mechanical treatment. However, it could be of relevance in the case of monopolonium, where the magnetic charges behave classically even in the relativistic regime. The life-time of non-relativistic bound states scales like $T \propto M^2 q_m^{-4} a_0^3$, where $q_m \sim q^{-1}$ is the magnetic charge. Since the mass of a monopole in grand unified theories can be of order $M \gtrsim 10^{16} GeV$ (rather than $m \sim 0.5 MeV$ for the electron), the corresponding lifetime can easily be comparable to the age of the universe or larger, even for microscopic initial sizes. It would be interesing to consider the impact of relativistic effects on the lifetime of monopolonium in specific cosmological scenarios. This issue is left for further research.

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