Two weakly interacting Bose-Einstein condensates: a single component and a binary mixture

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Abstract: We analyze the tunneling dynamics between two weakly interacting Bose-Einstein condensates in a symmetric double-well potential within the two-mode approach. We consider a single component and a binary mixture as well. Using a two-mode ansatz, we obtain the corresponding equations of motion from the Gross-Pitaevskii equation. We investigate the Josephson dynamics and the macroscopic quantum self-trapping regimes for the single component case. On the other hand, for the binary mixture, we explore the small oscillations for the strongly polarized and the non-polarized case.

I. INTRODUCTION

A Bose-Einstein condensate (BEC) is a new state of matter that exhibits an ensemble of bosons at very low temperatures. A BEC is characterized by the macroscopic population of the lowest energy single-particle state. From the first experimental realization in 1995 [1], they have attracted an increasing interest since in these systems, quantum phenomena become macroscopic [2]. Some interesting properties that have been investigated in BECs are, for instance, superfluidity and phase coherence. The latter one has been experimentally observed in interference experiments [3] and in Josephson oscillations [4].

The aim of this project is to study the tunneling dynamics and the Josephson-like oscillations between two weakly interacting BECs confined in a double-well potential. We will consider a large number of bosons, at zero temperature, and we will assume a two-mode approach to obtain the coupled equations that describe the tunneling dynamics of the system. We will consider first a single component BEC and afterwards, we will study a binary mixture.

The paper is organized as follows. In Sect. II we introduce the Gross-Pitaevskii equation that provides the mean-field theoretical framework. Then, in Sect. III we investigate the single component case and the binary mixture is presented in Sect. IV. In both cases, we obtain the analytical equations of motion and present the results focusing on two different regimes for each case. Finally, in Sect. V we conclude.

II. GROSS-PITAEVSKII EQUATION

We consider a system of N weakly-interacting bosons at zero temperature, trapped in a double-well potential, $V_{ext}(\mathbf{r})$. The many-body Hamiltonian of the system is [5]:

$$\mathcal{H} = \sum_{i=1}^{N}\left[-\frac{\hbar^2\nabla_i^2}{2m} + V_{ext}(\mathbf{r}_i)\right] + \frac{1}{2}\sum_{i\neq j} V(r_{ij})\,. \qquad (1)$$

We assume that the potential barrier in $V_{ext}(\mathbf{r})$ is high enough to ensure the weakly interaction between the two components and low enough to allow tunneling. From Eq. (1) we can obtain the Gross-Pitaevskii (GP) equation in the mean-field approximation. It provides an accurate description when N is large, at very low temperatures and for weakly interacting systems.

The GP equation for a single component condensate reads:

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + gN|\Psi(\mathbf{r},t)|^2\right]\Psi(\mathbf{r},t)\,, \qquad (2)$$

where the coupling constant is $g = 4\pi\hbar^2 a_s/m$, and a_s is the scattering length. We have normalized the wave function as $\int d\mathbf{r}|\Psi(\mathbf{r},t)|^2 = 1$.

In a binary mixture there are two different wave functions, one for each component; $\Psi_i(\mathbf{r},t)$ with $i = a, b$. Such two-component BECs can be experimentally produced from two different hyperfine states of the same atomic species, for instance ^{87}Rb, or from two different atomic species. Here we consider the first situation. The binary mixture system is described by two coupled GP equations:

$$i\hbar\frac{\partial\Psi_i(\mathbf{r},t)}{\partial t} = \Big[-\frac{\hbar^2\nabla^2}{2m_i} + V_{ext}(\mathbf{r}) + g_{ii}N_i|\Psi_i(\mathbf{r},t)|^2 + g_{ij}N_j|\Psi_j(\mathbf{r},t)|^2\Big]\Psi_i(\mathbf{r},t)\,, \qquad (3)$$

where g_{ii} and g_{ij} are the intra- and inter-species interactions with $i \neq j$ and $i, j = a, b$. The number of particles of each component is N_i and satisfies $N = N_a + N_b$.

We will also use the following expression of the energy to obtain the Hamiltonian of the system:

$$E = \int d\mathbf{r}\left[\frac{\hbar^2}{2m}|\vec{\nabla}\Psi(\mathbf{r},t)|^2 + V_{ext}(\mathbf{r})|\Psi(\mathbf{r},t)|^2 + \frac{gN}{2}|\Psi(\mathbf{r},t)|^4\right]. \qquad (4)$$

III. SINGLE COMPONENT CASE

We study the system dynamics with a two-mode approximation that will allow us to find the equations of motion in an analytical form. In the single component case, we consider a single wave function, describing the condensate, composed by the superposition of two time-independent spatial wave functions $\Phi_L(\mathbf{r})$ and $\Phi_R(\mathbf{r})$ [6]:

$$\Psi(\mathbf{r},t) = \Psi_L(t)\Phi_L(\mathbf{r}) + \Psi_R(t)\Phi_R(\mathbf{r})\,, \tag{5}$$

where each wave function, $\Phi_L(\mathbf{r})$ and $\Phi_R(\mathbf{r})$, will be located on the left and right side of the potential barrier, respectively. The ansatz (5) is a good approximation provided the barrier is high enough for weakly interaction and low enough for tunneling effect between the two coupled condensates. Due to the phase coherence properties of the BECs one can write:

$$\Psi_{L(R)}(t) = \sqrt{\frac{N_{L(R)}(t)}{N}} e^{i\phi_{L(R)}(t)}\,, \tag{6}$$

where the total number of atoms is $N = N_L(t) + N_R(t)$. Each spatial wave function can be expressed as a superposition of the ground state and the first excited state of the double-well potential, $\Phi_+(\mathbf{r})$ and $\Phi_-(\mathbf{r})$, respectively:

$$\Phi_L(\mathbf{r}) = \frac{\Phi_+(\mathbf{r}) + \Phi_-(\mathbf{r})}{\sqrt{2}} \quad , \quad \Phi_R(\mathbf{r}) = \frac{\Phi_+(\mathbf{r}) - \Phi_-(\mathbf{r})}{\sqrt{2}}\,.$$

In a symmetric double-well, $\Phi_\pm$ are orthogonal for its parity, therefore $\Phi_{L(R)}$ are also orthogonal. It can be checked that $\Psi(\mathbf{r},t)$ is normalized to 1.

A. Equations of motion

To study the tunneling dynamics we consider the standard two-mode approximation (S2M) [7]. We replace the wave function ansatz (5) in the GP equation (2). The S2M approach consists of neglecting the overlap between left and right components due to the weak link. Therefore the integrals of the type $\int d\mathbf{r}\, \Phi_i\Phi_j \simeq 0$ with $i \neq j$ and $i,j = a,b$. Then, projecting the resulting equation to the left (right) mode we arrive at two new coupled equations [6]:

$$i\hbar\frac{\partial\Psi_L(t)}{\partial t} = \Psi_L(t)\left[E_L^0 + N_L(t)\,U_L\right] - \Psi_R(t)K\,,$$
$$i\hbar\frac{\partial\Psi_R(t)}{\partial t} = \Psi_R(t)\left[E_R^0 + N_R(t)\,U_R\right] - \Psi_L(t)K\,,$$

where

$$E^0_{L(R)} = \frac{\hbar^2}{2m}\int d\mathbf{r}\,|\vec{\nabla}\Phi_{L(R)}|^2 + \int d\mathbf{r}\,V_{ext}\Phi^2_{L(R)}\,,$$
$$K = -\frac{\hbar^2}{2m}\int d\mathbf{r}\,\vec{\nabla}\Phi_L\cdot\vec{\nabla}\Phi_R - \int d\mathbf{r}\,\Phi_L V_{ext}\Phi_R\,,$$
$$U_{L(R)} = g\int d\mathbf{r}\,\Phi^4_{L(R)}\,,\ \Delta E = \frac{E_L^0 - E_R^0}{2K} + \frac{U_L - U_R}{4K}N\,.$$

Notice that in a symmetric double-well $E_L^0 = E_R^0$, $U_L = U_R$ and $\Delta E = 0$. By defining the two new variables, the population imbalance, $z(t)$, and the phase difference, $\delta\phi(t)$,

$$z(t) = \frac{N_L(t) - N_R(t)}{N} \quad , \quad \delta\phi(t) = \phi_R(t) - \phi_L(t)\,,$$

one can obtain the coupled equations of motion in the S2M:

$$\dot{z}(t) = -\sqrt{1-z^2(t)}\sin\delta\phi(t)\,, \tag{7}$$
$$\delta\dot{\phi}(t) = \Lambda z(t) + \frac{z(t)}{\sqrt{1-z^2(t)}}\cos\delta\phi(t)\,, \tag{8}$$

where the time is in units of the Rabi frequency, $\omega_R = 2K/\hbar$. We have defined $\Lambda \equiv NU/(\hbar\omega_R)$, it is the ratio between the interaction of particles of the same mode and the coupling term K [8].

To find the Hamiltonian expression as a function of the new variables, we proceed in an analogous way as before, but starting with the equation of the energy (4). It yields:

$$H = \frac{\Lambda}{2}z^2(t) - \sqrt{1-z^2(t)}\cos\delta\phi(t)\,. \tag{9}$$

B. Results

With the coupled equations (7) and (8), we can represent the evolution of the population imbalance and the phase difference over time. But the initial conditions z^0, $\delta\phi^0$ and the Λ value have to be defined.

In order to find the stability points [6], we compute from the Hamiltonian (9):

$$\left.\frac{\partial H}{\partial z}\right|_{z_0,\delta\phi_0} = 0 \quad , \quad \left.\frac{\partial H}{\partial \delta\phi}\right|_{z_0,\delta\phi_0} = 0\,.$$

It is found for the phase difference; $\sin\delta\phi^0 = 0$, so we will consider $\delta\phi^0 = 0, \pi$. For the population imbalance we obtain: $z^0 = 0, \pm\sqrt{1-1/\Lambda^2}$.

We can distinguish two interesting regimes to study. The first one is the Josephson dynamics. It is characterised by the fast tunnelling of atoms, through the potential barrier which is caused by the phase difference. The population imbalance oscillates around 0, so its temporal mean will be $\langle z\rangle = 0$. Fig. 1 shows the characteristic behavior of Josephson dynamics. The top panel, represents the time evolution of the population imbalance and the phase difference. The trajectories in the $z - \delta\phi$ plane are depicted at the bottom panel. The initial conditions are $z^0 = 0.2$, $\delta\phi^0 = 0$ and $\Lambda = 0.2$. It is shown that the Josephson orbits are closed trajectories in the $z - \delta\phi$ plane around a minimum or a maximum. In this case they are around the minimum ($z^0 = 0, \delta\phi^0 = 0$).

There is another interesting regime: the macroscopic quantum self-trapping (MQST). In this case, the population imbalance does not oscillate around 0, but around a

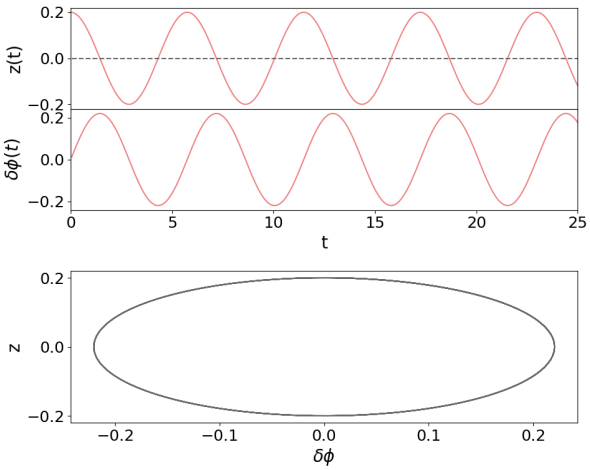

FIG. 1: Top panel: imbalance and phase difference as a function of time. Bottom panel: trajectory in the $z - \delta\phi$ plane. Initial conditions: $z^0 = 0.2$, $\delta\phi^0 = 0$ and $\Lambda = 0.2$, that correspond to Josephson dynamics.

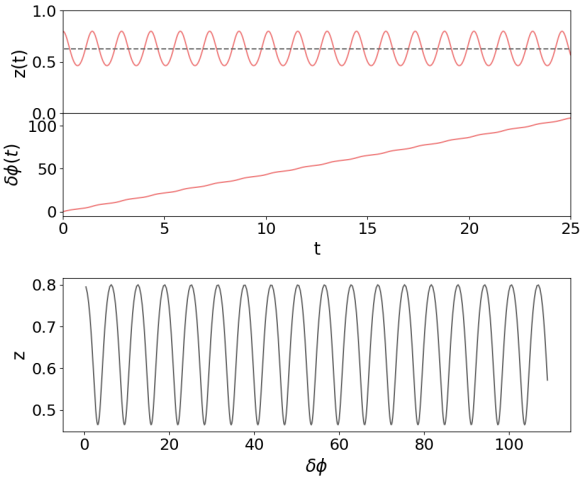

FIG. 2: Same as in Fig. 1, for $z^0 = 0.8$, $\delta\phi^0 = 0$ and $\Lambda = 7$. It corresponds to macroscopic quantum self-trapping.

positive or negative value, depending on which side of the well the self-trapping is located. This is caused by high initial z^0 values. In this regime, most particles do not reach the other side of the potential barrier through tunneling, hence they are trapped in one side of the well, as the name suggests. This effect comes from the non-linear interaction term of the GP equation [6]. This behaviour leads to a non-zero temporal mean value of the population imbalance, $\langle z \rangle \neq 0$. Fig. 2 shows this regime for the initial conditions $z^0 = 0.8$, $\delta\phi^0 = 0$ and $\Lambda = 7$. We can see that the phase difference does not oscillate, but keeps rising. This results can be compared with a pendulum with a high initial velocity that spins without oscillating. For this reason, the trajectories in the $z - \delta\phi$ plane are no longer closed orbits.

We are interested in the small oscillations around the minima ($z^0 = 0$, $\delta\phi^0 = 0, \pi$), so we can approximate $\sin\delta\phi^0 \sim \delta\phi^0$ and $\cos\delta\phi^0 \sim 0$. It allows to find the new equation of motion of the population imbalance $\ddot{z}(t) = -z(t)\,(1 + \cos\delta\phi_0\Lambda)$. We can see that we have recovered the expression of a harmonic oscillator with frequency $\omega_J = \omega_R\sqrt{1 + \cos\delta\phi(t)\Lambda}$.

We fix the initial conditions ($z^0 = \pm 0.9$, $\delta\phi^0 = 0$) and we consider different values of Λ. Then, we solve the time evolution of the population imbalance, and we compute its temporal mean value. These results are presented in Fig. 3. It shows that for small Λ values the system begins at the Josephson regime, oscillating around $z = 0$ either with $z^0 = \pm 0.9$. On the other hand, with larger values of Λ the system exhibits the MQST dynamics since the temporal average $\langle z \rangle \neq 0$. The positive branch corresponds to the initial value of $z^0 = 0.9$, whereas the negative one to $z^0 = -0.9$. It is interesting to point out that this representation has been used for experimental demonstration

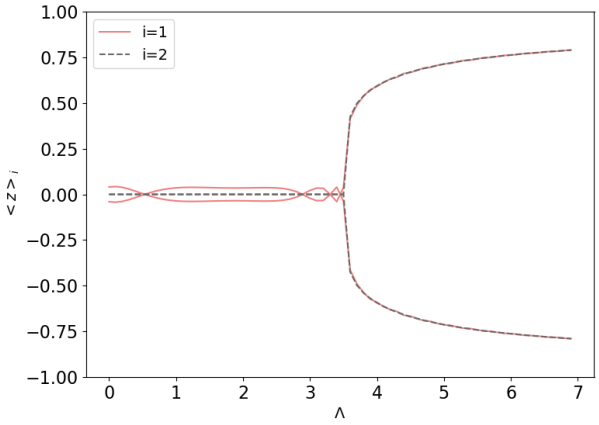

FIG. 3: Temporal mean value of the population imbalance with the initial conditions ($z^0 = \pm 0.9$, $\delta\phi^0 = 0$) for different values of Λ. $\langle z \rangle_2$ (grey dashed line) is calculated for a time average 500 larger than the $\langle z \rangle_1$ time (red solid line).

of the different regimes [8]. However, these results may be slightly different depending on how long the averaging is done. See for example the comparison between the two representations of Fig. 3. The grey dashed line is the temporal mean $\langle z \rangle_2$ for a time 500 larger than the red one, $\langle z \rangle_1$.

IV. BINARY MIXTURE CASE

In this section, we generalize the two-mode ansatz for a binary mixture. For each component (a) and (b) we assume an analogous wave function as in the single com-

ponent case:

$$\Psi_j(\mathbf{r},t) = \Psi_{jL}(t)\Phi_{jL}(\mathbf{r}) + \Psi_{jR}(t)\Phi_{jR}(\mathbf{r})\,,$$

where $i,j = a,b$ and it is fulfilled $\int d\mathbf{r}\,\Phi_{i\alpha}\Phi_{j\beta} = \delta_{ij}\delta_{\alpha\beta}$, where $\alpha,\beta = L,R$. As above, we can consider that each wave function has a well-defined quantum phase, therefore we can write the following ansatz:

$$\Psi_{j,\alpha}(t) = \sqrt{\frac{N_{j,\alpha}(t)}{N_j}}e^{i\phi_{j,\alpha}(t)}\,,$$

where $N_{j,\alpha}$ is the population of the j-component on each side of the potential barrier. It fulfils $N_j = N_{j,L}(t) + N_{j,R}(t)$. So the total number of particles is $N = N_a + N_b = N_{a,L}(t) + N_{a,R}(t) + N_{b,L}(t) + N_{b,R}(t)$.

A. Equations of motion

From the GP equation of the binary mixture (3), we can arrive at the corresponding equations of motion with a similar procedure as in the single component case:

$$i\hbar\frac{\partial\Psi_{i\alpha}(t)}{\partial t} = \Psi_{i\alpha}(t)\big[E^0_{i\alpha} + N_{i\alpha}(t)\,U_{ii\alpha} + \\ + N_{j\alpha}(t)\,U_{ij\alpha}\big] - \Psi_{i\beta}(t)K_i\,,$$

where

$$E^0_{j\alpha} = \frac{\hbar^2}{2m_j}\int d\mathbf{r}|\vec{\nabla}\Phi_{j\alpha}|^2 + \int d\mathbf{r}\Phi^2_{j\alpha}V_{ext}\,,$$

$$K_j = -\frac{\hbar^2}{2m_j}\int d\mathbf{r}\vec{\nabla}\Phi_{jL}\cdot\vec{\nabla}\Phi_{jR} - \int d\mathbf{r}\Phi_{jL}V_{ext}\Phi_{jR}\,,$$

$$U_{ij\alpha} = g_{ij}\int d\mathbf{r}\,\Phi^2_{i\alpha}\Phi^2_{j\alpha}\,,$$

$$\Delta E_{i,j} = \frac{E^0_{iL} - E^0_{iR}}{\hbar} + \frac{U_{iiL} - U_{iiR}}{2\hbar}N_i + \frac{U_{ijL} - U_{ijR}}{2\hbar}N_j\,,$$

with $i,j = a,b$ and $\alpha,\beta = L,R$ [6]. Since we considered a system where the two components have the same atomic mass, $m_a = m_b$, then $g_{aa} = g_{bb}$ and $g_{ab} = g_{ba}$. As the single component case, we use a symmetric double-well potential for each component. Hence, we can write:

$$\Phi_{L(R)} \equiv \Phi_{a,L(R)} = \Phi_{b,L(R)}\,,$$
$$E \equiv E^0_{aL} = E^0_{bL} = E^0_{aR} = E^0_{bR}\,,$$
$$U \equiv U_{aaL} = U_{bbL} = U_{aaR} = U_{bbR}\,,$$
$$\tilde{U} \equiv U_{abL} = U_{baL} = U_{abR} = U_{baR}\,,$$
$$K \equiv K_a = K_b\,,$$
$$\Delta E_{a,b} = \Delta E_{b,a} = 0\,.$$

The new variables that will describe the dynamics in the two-mode ansatz are again the population imbalance and the phase difference, one for each component:

$$z_i(t) = \frac{N_{iL}(t) - N_{iR}(t)}{N_i} \quad , \quad \delta\phi_i = \delta\phi_{iR}(t) - \delta\phi_{iL}(t)\,.$$

In this case, an interaction term between the atoms of different components appears in the GP equation, which leads to new parameters: $\Lambda = NU/(\hbar\omega_R)$ and $\tilde{\Lambda} = N\tilde{U}/(\hbar\omega_R)$. The first one is the ratio between the intra-species interaction (U_{ii}) with the coupling, and the second one, between the inter-species interaction (U_{ij}) and the coupling, where $i,j = a,b$ and $i \neq j$. The fraction of the population for each component is defined as $f_i = N_i/N$ [6].

Eventually, we arrive at the equations of motion in units of the Rabi frequency, $\omega_R = 2K/\hbar$:

$$\dot{z}_a(t) = -\sqrt{1 - z_a^2(t)}\sin\delta\phi_a(t)\,, \tag{10}$$

$$\delta\dot{\phi}_a(t) = f_a\Lambda z_a(t) + f_b\tilde{\Lambda}z_b(t) + \frac{z_a(t)}{\sqrt{1 - z_a^2(t)}}\cos\delta\phi_a(t)\,, \tag{11}$$

$$\dot{z}_b(t) = -\sqrt{1 - z_b^2(t)}\sin\delta\phi_b(t)\,, \tag{12}$$

$$\delta\dot{\phi}_b(t) = f_b\Lambda z_b(t) + f_a\tilde{\Lambda}z_a(t) + \frac{z_b(t)}{\sqrt{1 - z_b^2(t)}}\cos\delta\phi_b(t)\,. \tag{13}$$

These equations represent two non-rigid coupled pendulums [9].

B. Results

A stationary point is described by $\dot{z}_i = 0$ and $\delta\dot{\phi}_i = 0$ [6]. Therefore, we arrive at $\sin\delta\phi_i = 0$. The initial conditions about the phase difference have to be $\delta\phi_i = 0,\pi$. An obvious stationary point is $z^0_a = z^0_b = 0$. In Ref. [10], they study other stationary points, and they found a condition for the z^0 values. We are going to focus on small oscillations around $z^0_i = \delta\phi^0_i = 0$. There are two different interesting behaviours to study: the strongly polarized and the non-polarized case.

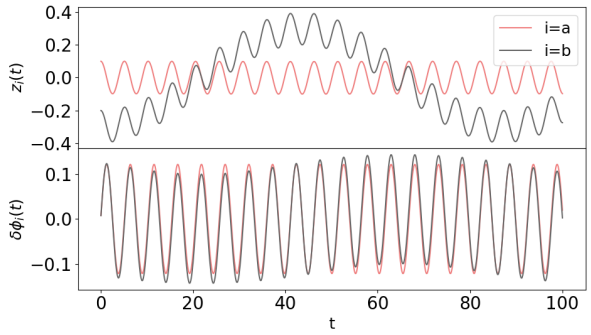

FIG. 4: Time evolution of population imbalance and phase difference for the two components of a strongly polarized mixture: $f_a = 0.99$, $f_b = 0.01$, $z^0_a = 0.1$, $z^0_b = -0.2$ and $\Lambda = 0.5$.

The strongly polarized case occurs when a component is much more populated than the other, for example

$f_a \sim 1$ and $f_b \sim 0$. The dynamics of the most populated component (a) is similar to the single component case and it drives the time evolution of the under-populated component (b). Fig. 4 shows the time evolution of the population imbalance and the phase difference of the two components in this situation. It corresponds to $f_a = 0.99$, $f_b = 0.01$, $z_a^0 = 0.1$, $z_b^0 = -0.2$ and $\Lambda = 0.5$ values. The initial condition z_b^0 is negative, so if there wasn't an interaction with the other component (a), z_b would tend to rise to reach equilibrium ($z_b = 0$). On the contrary, initially, its population imbalance goes down, making its absolute value grow. This is produced by the interaction between the two components [9]. It appears a new frequency for the under-populated component much lower than the frequency shared by the two components. This last one is similar to the single component case. The

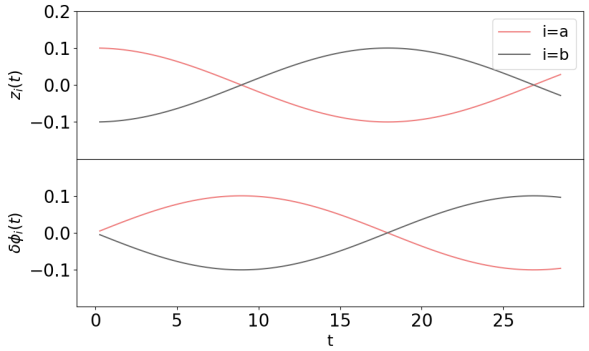

FIG. 5: Same as in Fig. 4 but for a non-polarized mixture: $f_a = f_b = 0.5$, $z_a^0 = -z_b^0 = 0.1$ and $\Lambda = 2$.

non-polarized case happens when both populations are nearly the same $f_a \simeq f_b \simeq 0.5$. If we consider the initial conditions of the population imbalance $z_a^0 = -z_b^0$, we can represent the evolution of the two variables with the equations of motion (10-13). We can see its behaviour in Fig. 5. In this case, both components oscillate with the same frequency caused by Josephson tunneling [9].

V. CONCLUSIONS

In this work, we have studied the tunneling dynamics of a system of N bosons in a double-well potential. Our theoretical framewok is the Gross-Pitaevskii equation for a single component and for a binary mixture.

We have started investigating the single component case. By using a two-mode approximation we have obtained the analytical equations of motion described by two conjugate variables: the population imbalance and the phase difference. We have also derived the expression of the Hamiltonian as a function of these variables. To analyze the evolution of the system, we have represented z and $\delta\phi$ as a function of time for two interesting regimes; the Josephson dynamics and the macroscopic quantum self-trapping. For small oscillations, we have recovered the expression of a harmonic oscillator for the z variable. In addition, we have represented the temporal mean value of the population imbalance for different values of Λ, which has been used for the experimental demonstration of the different dynamical regimes [8].

Furthermore, we have studied the binary mixture case. As the single component situation, from the Gross-Pitaevskii equation we have derived, with a two-mode approach, the equations of motion with the same variables, one for each component: z_a, z_b, $\delta\phi_a$ and $\delta\phi_b$. We have described two relevant behaviours for small oscillations around $z_i^0 = \delta\phi_i^0 = 0$: the strongly polarized and the non-polarized mixture. Finally, we have characterized each case presenting the evolution of the conjugate variables.

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