Muon Decay and the Higgs Vaccum Expectation Value

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Abstract: The Higgs vacuum expectation value (vev) can be determined experimentally by measuring the muon lifetime. However, this determination can be modified by the introduction of a new virtual particle in the muon decay, for example a Z'. This new massive vector boson can change the mechanism by which the muon decays despite having an expression for the amplitude of the same form as the one predicted by the SM, where the process is mediated by a W boson. These two theories have been matched to the 4-fermion operator from a low-energy Effective Field Theory that describes the physics at low energies, resulting in a correlation between the Higgs vacuum expectation value and the mass of the new particle.

I. INTRODUCTION

The fundamental constants of the theories that aim at describing nature have their numerical values computed by fitting these theories to empirical observables. For instance, the Higgs field vacuum expectation value (vev) can be deduced from the measurement of the muon lifetime. Hence, in order to compute its value, a theoretical relation between the fundamental constant (vev) and the observable (muon lifetime) must be deduced. This is the topic that will be discussed in this work.

An effective field theory (EFT)[1] is a tool conceived to analyse phenomena at a certain low energy scale disregarding the dynamics of physics at high energies. With this in mind, one can relax the restrictions imposed by the symmetries fulfilled at high energies and only consider the ones that the low energy processes must also obey. Hence, an effective Lagrangian is obtained that accurately describes low energy processes without inquiring into how nature behaves at the ultra-violet scale. However, the two energy ranges are not disconnected, as a result, the Wilson coefficients or coupling constants that couple the fields considered in the effective Lagrangian of the EFT –namely, the fundamental couplings of the EFT– can be matched to fundamental constants of the high energy theory.

In this paper we will address the muon decay, thus we must build an EFT at the muon mass energy scale. As mentioned, this theory is determined by the Wilson coefficients, among which, one can be fixed by the measurement of the muon lifetime. In order to relate the EFT with the valid theory at high energies, the "matching" calculation must be made. If the high energy theory at hand is the Standard Model (SM), the matching of the Wilson coefficient of the operator relevant to the muon lifetime in the EFT into the SM gives an expression for the Higgs vacuum expectation value as a function of the Wilson coefficient. However, this determination of the Higgs vev is only valid if there is no physics beyond the SM (BSM physics) affecting the muon lifetime. If such is the case, the new contribution has to be taken into account.

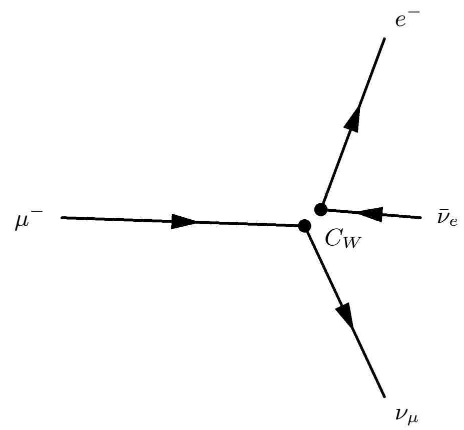

FIG. 1: Diagram depicting the muon decay, as described by the explicit operator shown in the Effective Lagrangian (1).

In this article, we discuss how to determine the Higgs vev from an EFT at the muon mass energy scale, and how this determination gets modified if there exists a particle beyond the SM: a neutral massive vector boson Z'.

II. LOW-ENERGY EFT AND MUON DECAY

A. Effective Lagrangian

The relevant theory at energy scales around the muon mass (much lower than the EW scale) is given by an EFT with a Lagrangian of the form:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QED+QCD}} + C_W (\bar{\nu}_\mu \gamma^\sigma P_L \mu)\ (\bar{e}\gamma_\sigma P_L \nu_e) + \cdots \quad (1)$$

where $P_L = (1-\gamma_5)/2$ is the left-handed chiral projector. We have shown only the dimension-six four fermion operator contributing to the muon lifetime in the SM. This 4-fermion operator is analogous to the one first introduced by Fermi to describe beta-decay [2].

B. Muon decay amplitude in the EFT

The amplitude of the process $\mu \to e\nu_\mu\bar{\nu}_e$ at the leading order in QED in the EFT given by Eq. (1), is given by the Feynman diagram in Fig. 1, and reads:

$$i\mathcal{M} = iC_W \left\{ \bar{u}_1^{s'} \gamma^\sigma P_L u_0^s \; \bar{u}_3^r \gamma_\sigma P_L v_2^{r'} \right\} , \qquad (2)$$

where the fields take the form of Dirac Spinors and their indexes $(0,1,2,3)$ refer to the muon, the muon neutrino, the electron anti-neutrino and the electron, respectively. In this expression the spin orientation of the fermions is encoded in the superscripts (s, s', r, r').

C. Total decay rate and lifetime

In order to compute the lifetime of the muon, we first need an expression for the total decay rate, which is obtained by integrating the differential decay rate over the whole phase space. The differential decay rate for a 3-body decay is given by [3]

$$d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{32M^3} \overline{|\mathcal{M}|^2} dm_{13}^2 dm_{23}^2 , \qquad (3)$$

where $m_{13}^2 = (p_1 + p_3)^2$, $m_{23}^2 = (p_2 + p_3)^2$, p_i is the 4-momentum of the i daughter particle, M is the mass of the decaying particle (the muon in our case), and $\overline{|\mathcal{M}|^2}$ is the squared decay amplitude averaged over the spin orientations of the decaying particle (unpolarized) and summed over the spin orientations of the final-state particles. We thus need an algebraic expression for this quantity, which in the case of muon decay reads:

$$\overline{|\mathcal{M}|^2} = 8\, C_W^2\, (p \cdot k')(p' \cdot k) , \qquad (4)$$

where p, p', k, k' are the 4-momenta of the muon, the muon neutrino, the electron and the electron anti-neutrino, respectively. With this, an expression for the muon decay rate and lifetime can be found.

By neglecting the electron mass, and keeping only the highest order in $M = m_\mu$, the expression for the total decay rate is:

$$\Gamma = \frac{C_W^2 m_\mu^5}{1536\pi^3} , \qquad (5)$$

and the muon lifetime is

$$\tau_\mu = \frac{\hbar}{\Gamma} = \frac{1536\hbar\pi^3}{C_W^2 m_\mu^5} , \qquad (6)$$

where the $\hbar$ is added to recover the SI units of time.

For the explicit derivation of the equations shown in this section, see Appendix A.

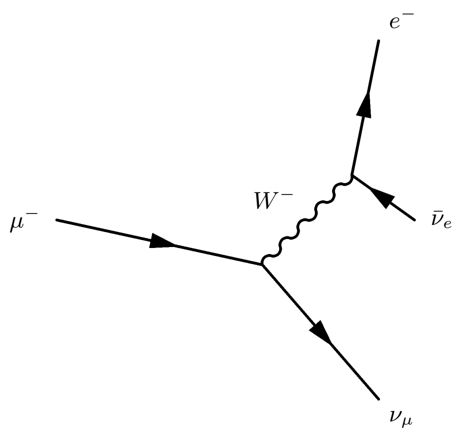

FIG. 2: Tree-level Feynman diagram describing muon decay in the SM, mediated by a W boson. The corresponding amplitude is given in Eq. (9).

D. Determination of the Wilson coefficient C_W

The determination of the Wilson coefficient C_W is done by fitting the experimental measurement of the muon lifetime [3]:

$$\tau_\mu = 2.1969811(22)\ \mu s \qquad (7)$$

to the theoretical prediction given in Eq. (6). Given the muon mass $m_\mu = 105.6583745(24)\,\mathrm{MeV}$ [3], we find the following result for the Wilson coefficient,

$$|C_W| = \sqrt{\frac{1536\hbar\pi^3}{\tau_\mu m_\mu^5}} = 3.2917764(17) \cdot 10^{-5}\ \mathrm{GeV}^{-2} . \quad (8)$$

III. DETERMINATION OF THE HIGGS VEV

A. Matching the EFT to the SM

At tree level in the SM, as illustrated in Fig. 2, the amplitude of the muon decay is given by [4]

$$i\mathcal{M} = \frac{ig}{\sqrt{2}} \bar{u}_1^{s'} \gamma^\sigma P_L u_0^s \left(-i \frac{g_{\sigma\rho}}{k_W^2 - M_W^2} \right) \frac{ig}{\sqrt{2}} \bar{u}_3^r \gamma_\rho P_L v_2^{r'} ,$$

where k_W and M_W are the W boson momentum and mass, and the subscripts $(0,1,2,3)$ stand for the same particles as in the previous section. By expanding the $1/(k_W^2 - M_W^2)$ factor at leading order in k_W^2/M_W^2, the amplitude reads:

$$i\mathcal{M} = \frac{-ig^2}{2M_W^2} \left(\bar{u}_1^{s'} \gamma^\sigma P_L u_0^s \right) \left(\bar{u}_3^r \gamma_\sigma P_L v_2^{r'} \right) . \qquad (9)$$

This amplitude has the same form as the EFT amplitude in Eq. (2) derived from the Effective Lagrangian. Therefore, the matching of the Wilson coefficient C_W and the fundamental constants that appear in the SM amplitude is evident. The following expression is obtained for the Wilson coefficient:

$$C_W = -\frac{g^2}{2M_W^2} \ . \tag{10}$$

This result is the tree-level matching condition for C_W in the SM, assuming no contribution from BSM physics.

B. Determination of the Higgs vev in the SM

As aforementioned, Fermi introduced the 4-fermion operator relevant for beta decay. The coupling constant that he used was G_F, the Fermi constant. This constant plays a big role in the SM and is defined as [5]

$$G_F \equiv \frac{\sqrt{2}g^2}{8M_W^2} = \frac{1}{\sqrt{2}v^2} \ , \tag{11}$$

where v is the vacuum expectation value of the Higgs field, or Higgs vev. As a result, assuming the SM, we can compute the Higgs vev, a fundamental constant within the SM, as a function of C_W, which is fixed by the muon lifetime as given in Eq. (6):

$$v = \sqrt{\frac{2}{|C_W|}} = 246.490300(94) \text{ MeV} \ . \tag{12}$$

Assuming the SM, the Fermi constant can also be computed from C_W:

$$G_F = \frac{|C_W|}{2\sqrt{2}} = 1.16381870(60) \cdot 10^{-5} \text{ GeV}^{-2} \ . \tag{13}$$

C. BSM contribution from a massive Z'

In this section we introduce a hypothetical new particle, a new massive vector boson Z' which can mediate muon decay. This particle is electrically neutral and its interaction with leptons is described by the following Lagrangian:

$$\mathcal{L}_{Z'} \supset \frac{1}{2}M_{Z'}^2 Z_\mu Z^\mu + \lambda_\ell \, \bar{e} \not{Z} P_L \mu + \lambda_\nu \, \bar{\nu}_\mu \not{Z} \nu_e \tag{14}$$

The fact that Z' is uncharged makes the muon decay into an electron at first, changing the lepton flavour (see Fig. 3). From this diagram, and analogously to the SM case (neglecting the particle momentum), the following amplitude is obtained:

$$i\mathcal{M} = \frac{-i\lambda_\ell \, \lambda_\nu}{M_{Z'}^2} \left(\bar{u}_3^r \gamma^\sigma P_L u_0^s\right) \left(\bar{u}_1^{s'} \gamma_\sigma P_L v_2^{r'}\right) \ . \tag{15}$$

Note that the P_L has been added to the left of the electron neutrino spinor (v_2) because this particle is left-handed, thus the effect of the projector is the same as

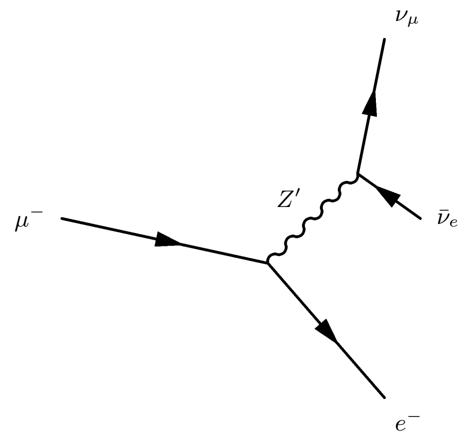

FIG. 3: New physics diagram of the muon decay, in this case the interaction is mediated by a virtual particle without charge: Z'. This diagram illustrates the Lagrangian (14).

the identity. Finally, a Fierz transformation [6] is used in order to recall the form of the EFT and SM amplitudes, resulting in

$$i\mathcal{M} = \frac{i\lambda_\ell \, \lambda_\nu}{M_{Z'}^2} \left(\bar{u}_3^r \gamma^\sigma P_L v_2^{r'}\right) \left(\bar{u}_1^{s'} \gamma_\sigma P_L u_0^s\right) \ . \tag{16}$$

Now, we can include the contribution of this new particle into C_W, because the amplitude takes the same form. As a result, the value of C_W, fixed by the muon lifetime in the EFT, is related to fundamental constants in the UV theory: the Higgs vev from the SM and the Z' particle mass and couplings. We can therefore write a relation between the Higgs vev and $M_{Z'}$, given the experimental determination of C_W, assuming some canonical values for $\lambda_\ell, \lambda_\nu$; this way, if the value of the Higgs vev should be proven to be different from the one predicted by the SM (Eq. (12)), we could immediately know the mass of the Z' particle. The relation between the Higgs vev and $M_{Z'}$ reads:

$$C_W = -\frac{2}{v^2} + \frac{\lambda_\ell \, \lambda_\nu}{M_{Z'}^2} \ . \tag{17}$$

From the experimental determination of C_W in Eq. (8) we can extract v as a function of the Z' mass and couplings. This is shown in Fig. 4, where we have fixed $\lambda_\ell = \lambda_\nu = 0.1$ and shown the corresponding value of the Higgs vev v as a function of the mass of the new Z' boson. We see that for large masses $M_{Z'} \gg M_W$ the value of v approaches the SM determination, since the BSM contribution becomes negligible. However, for lower values of $M_{Z'}$ the true value of v extracted from experiment would be significantly different from that obtained assuming the Standard Model.

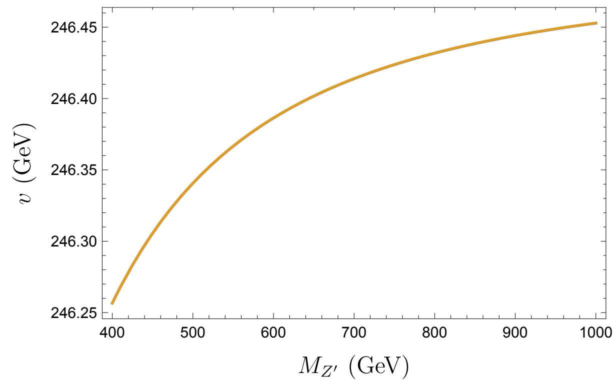

FIG. 4: Determination of the Higgs vev v as a function of the mass of the hypothetical Z' boson, for fixed couplings $\lambda_\ell = \lambda_\nu = 0.1$.

IV. CONCLUSIONS

The EFT we have built for the muon mass energy scale has let us obtain an expression for the amplitude of the muon decay which resembles a lot the amplitude the SM predicts at Tree-level. Therefore, the matching of the EFT to the SM is almost immediate, which lets us find a relation between the Wilson coefficient C_W and the Higgs vev. However, this expression is modified if the SM is not assumed. An example of this has been put forward with the introduction of a new mechanism of decay mediated by the virtual particle Z'. The amplitude of the process in this case has also the same form as the EFT and SM amplitudes, this can modify the Higgs vev. The explanation of such modification lays on the fact that the C_W of the EFT has a value that cannot be changed by the underlying theories except when these theories add new terms to the Effective Lagrangian, which is not the case at hand. In the theories mentioned in the article, the amplitude of the muon decay has the same form, this means that both the SM and the new physics model may contribute to the value of C_W. Therefore, in order to maintain this value, the Higgs vev has to change accordingly. As a result, we have concluded that the Higgs vev value depends on the mass of the new particle, $M_{Z'}$, following the relation

$$C_W = -\frac{2}{v^2} + \frac{\lambda_\ell \, \lambda_\nu}{M_{Z'}^2}$$

We have seen that this effect can modify the determination of v at the one-per-mille level for couplings of order 0.1 and Z' masses of several hundred GeV. This is a relevant effect given the precision with which v is determined in the SM, given in Eq. (12).

Acknowledgments

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Appendix A: Computation of the integrated decay rate

We first want to obtain the averaged squared amplitude [7]. To do so we need:

$$\begin{aligned}(\bar{u}_1 \gamma^\sigma P_L u_0)^\dagger = \left(u_1^\dagger \gamma^0 \gamma^\sigma P_L u_0\right)^\dagger = \\ = \left(u_0^\dagger P_L^\dagger \gamma^{\sigma\dagger} \gamma^{0\dagger} u_1\right) = (\bar{u}_0 \gamma^\sigma P_L u_1) \ ,\end{aligned}$$

where the last expression is obtained by using some gamma matrices properties as well as anti-commutation rules [7]. A similar deduction is conducted for the second term between brackets in (2), thus obtaining for the squared amplitude:

$$|\mathcal{M}|^2 = \frac{C_W^2}{16} \, T_1 \, T_2 \ , \tag{A1}$$

where

$$\begin{aligned} T_1 &\equiv 4\left\{\bar{u}_0 \gamma^\sigma P_L u_1 \bar{u}_1 \gamma^\rho P_L u_0\right\} , \\ T_2 &\equiv 4\left\{\bar{v}_2 \gamma_\sigma P_L u_3 \bar{u}_3 \gamma_\rho P_L v_2\right\} . \end{aligned}$$

However, as mentioned before, we should sum over the possible spin orientations of the involved particles in order to fit the result into experiments that cannot detect spin orientations,

The total decay rate will only depend on the spin-averaged squared amplitude (recalling that all external particles are spin 1/2 fermions). Precisely, we should average over the muon spins s, and sum over all final-state particle spins s', r, r',

$$\overline{|\mathcal{M}|^2} = \frac{1}{2} \sum_s \sum_{s'} \sum_r \sum_{r'} |\mathcal{M}(s \longrightarrow s' r' r)|^2 \tag{A2}$$

The steps from here involve trace technology, and the following relations for Dirac spinors [7]

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m ,$$
$$\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m .$$

The steps are shown explicitly for the term T_1, where in the first expression the matrix indexes are written in the form of Latin letter sub-indexes, so the spinors can be freely moved:

$$\sum_{s,s'} \left\{ \bar{u}_{0,a} \left[\gamma^\sigma(1-\gamma_5)\right]_{ab} u_{1,b} \bar{u}_{1,c} \left[\gamma^\rho(1-\gamma_5)\right]_{cd} u_{0,d} \right\} =$$
$$= \left\{ (\not{p}+M)_{da} \left[\gamma_\sigma(1-\gamma_5)\right]_{ab} (\not{p}') \left[\gamma_\rho(1-\gamma_5)\right]_{cd} \right\} =$$
$$= \mathrm{Tr}\left\{ (\not{p}+M)\gamma^\sigma(1-\gamma_5)(\not{p}')\gamma^\rho(1-\gamma_5) \right\} \equiv \mathrm{Tr}_1 .$$

Similarly,

$$\mathrm{Tr}_2 \equiv \mathrm{Tr}\left\{ (\not{k}')\gamma_\sigma(1-\gamma_5)(\not{k}+m)\gamma_\rho(1-\gamma_5) \right\}$$

Where m is the electron mass, M, the muon mass, and the neutrino masses have been neglected. By using Trace technology, one arrives at:

$$\mathrm{Tr}_1 = 8\left[p^\sigma p'^\rho + p^\rho p'^\sigma - g^{\sigma\rho}(p\cdot p') + i\epsilon^{\alpha\sigma\beta\rho} p_\alpha p'_\beta\right]$$
$$\mathrm{Tr}_2 = 8\left[k_\sigma k'_\rho + k_\rho k'_\sigma - g_{\sigma\rho}(k\cdot k') + i\epsilon^{\alpha\beta}_{\sigma\rho} k_\alpha k'_\beta\right] .$$

Hence,

$$\overline{|\mathcal{M}|^2} = \frac{C_W^2}{32}\mathrm{Tr}_1\mathrm{Tr}_2 . \tag{A3}$$

The following steps involve contracting the indices, and recalling the anti-symmetry of the Levi-Civita symbol. The final expression for the squared amplitude is:

$$\overline{|\mathcal{M}|^2} = 8\, C_W^2\, (p\cdot k')(p'\cdot k) , \tag{A4}$$

where p, p', k, k' are the 4-momenta of the muon, the muon neutrino, the electron and the electron anti-neutrino, respectively. With this, an expression for the muon decay rate and lifetime can be found.

From here, the decay rate can be integrated over the phase space:

$$d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{32M^3}\overline{|\mathcal{M}|^2} dm_{13}^2 dm_{23}^2 . \tag{A5}$$

Recalling that $m_{13}^2 = (p_1+p_3)^2$, $m_{23}^2 = (p_2+p_3)^2$, p_i is the 4-momentum of the i daughter particle, M is the mass of the decaying particle (the muon). We can express the 4-momenta products in Eq. (A4) as a function of m_{12} and m_{13}, where $(0,1,2,3)$ refers to $(\mu, \nu_\mu, e, \bar{\nu_e})$ the following way:

$$p\cdot k' = \frac{M^2 - m_{13}^2}{2} , \tag{A6}$$
$$p'\cdot k = \frac{m_{13}^2}{2} , \tag{A7}$$

where the electron mass has been neglected. Knowing that m_{13}^2 goes from 0 to $M^2 - m_{23}^2$, and that m_{23}^2 goes from 0 to M^2, we can integrate over m_{13}^2 first and afterwards over m_{23}^2. Arriving at the following equation for the total decay rate:

$$\Gamma = \frac{C_W^2 m_\mu^5}{1536\pi^3} , \tag{A8}$$

where we have only kept the dominant term. This is the expression we were looking for.

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