An examination of the tail contribution to distortion risk measures

Miguel Santolino[a*] Jaume Belles-Sampera[a] José María Sarabia[b]
Montserrat Guillen[a]

August 2021

[a]Dept. Econometrics, Riskcenter-IREA, Universitat de Barcelona
Av. Diagonal, 690, 08034 Barcelona, Spain
[b]Dept. of Quantitative Methods, CUNEF Universidad
C/ de los Pirineos, 55, 28040 Madrid, Spain

Abstract

Extreme losses are specially relevant in the finance and insurance sectors. Here, we analyze the tail behavior of risks and its influence on risk measures. Specifically, we examine the part of the risk value of a distortion risk measure that is attributable to extreme losses. We analyze the additive properties of tail contributions to risk values when several risks are aggregated. We show that the partial contributions are subadditive if the distortion function is concave in the tail. We examine the tail behavior for quantile-based distortion risk measures, including value-at-risk (VaR) and tail value-at-risk (TVaR). We conclude that decision makers obtain relevant information about the contribution of extreme losses to risk values and about the fraction of the diversification benefit attributable to the tails. An example is used to illustrate our results.

JEL classification: C18, G22, D81.
Keywords: Risk management, tail risk, extreme losses, diversification, Lebesgue-Stieltjes integral.

*[Corresponding author: msantolino@ub.edu] The authors thank the comments of anonymous referees and editor. Authors acknowledge support received from the Spanish Ministry of Science and Innovation under Grant PID2019-105986GB-C21, the Riskcenter research group members at the UB. MS acknowledge support from Catalan Government under Grant 2020-PANDE-00074. MG thanks ICREA Academia and the Consortium for Data Analytics in Risk (CDAR) at the University of California, Berkeley, for the generous hospitality.

1 Introduction

Risk is typically defined as the possibility of suffering future losses, where these losses are represented by a non-negative random variable, X. This risk can then be quantified using risk measures, which encapsulate the risk associated with the loss distribution in one single value. In calculating the solvency requirements of financial institutions, such risk measures are applied to determine their economic capital, i.e. the amount they need to hold to ensure that future liabilities are covered with an acceptable degree of confidence [McNeil et al., 2005], and, in so doing, they are influenced by the occurrence of extreme losses located in the right tail of the loss distribution.

Here, our goal is to analyze the influence of extreme losses in the risk value computed using a particular risk measure, specifically, the family of distortion risk measures [Denneberg, 1990; Wang, 1995, 1996]. Distortion risk measures (DRMs) are based on a function that distorts the probability measure applied to all subsets of a sample space. The class of DRMs includes the most frequently used quantile-based risk measures in finance and insurance, most notably value-at-risk (VaR) and tail value-at-risk (TVaR). We evaluate the importance of the tail in the risk values of distortion risk measures. It is our belief that this should help decision makers anticipate the impact of an extreme loss on the risk estimate and, therefore, on other amounts derived from it, such as economic capital and premium loadings. Finally, we investigate the aggregation properties of the tail contributions when several risks are considered.

Tail behavior of risks has gained much attention in the literature in recent years [Liu and Wang, 2019; Belles-Sampera et al., 2014; Landsman et al., 2016; Cai and Li, 2005; Yin and Zhu, 2016]. Belles-Sampera et al. [2013, 2014] defined a new class of distortion risk measures, named GlueVaR, and examined their tail aggregation properties. Authors pointed out that subadditivity might be a tough requirement on the determinations of premiums and regulatory capitals. Instead of subadditivity, they introduced a weaker concept of tail subadditivity and showed that this property is satisfied as long as the distortion function is concave in the common tail region. Cai et al. [2017] generalized the concept of common tail region considered by Belles-Sampera et al. [2014] and gave sufficient and necessary conditions for a distortion risk measure to be tail subadditive. These two theoretical articles analize mathematically the tail subadditivity condition in the field of non-additive measure theory [Denneberg, 1994].

The approach we adopt here, however, differs from previous works. This article investigates how the theoretical concept of tail subadditivity can be evaluated in practice. This is made in two steps. First, we show how to assess the contribution of the tail of the distribution of the random variable on the value reported by the DRM. Closed-form expressions are given to compute the contribution of the tail to the risk value for a set of distortion risk measures (VaR, TVaR and GlueVaR). Second, we show that if two risks are aggregated, the contributions of the tails to the DRM values satisfy the tail subadditivity property. To our knowledge, this practical analysis of the contribution of the tail on the value of these DRM's and the evaluation of their tail aggregation properties providing closed-form expressions has not been previously performed. We argue that our approach helps to provide a deeper understanding of the theoretical concept of tail subadditivity. This is a convenient property because it implies that tail risk can be diversified.

An interesting approach to investigate the behavior of the tail distribution is developed by Liu and Wang [2019] who define tail risk measures as those in which the risk measure value is solely determined by the distribution of the risk beyond its quantile. They define the generator of a tail risk measure as a law-invariant risk measure applied to the conditional distribution of the tail and analyze if some properties often required to risk measures are transferred from the generator to the tail risk measure. They showed that subadditivity and convexity can be passed when accompanied by other properties. Quantile based distortion risk measures, as VaR, TVaR and GlueVaR, would be a class of tail risk measures [Liu and Wang, 2019, Ex. E.2, pag.26]. Non-quantile based distortion risk measures such as the Dual Power or the Proportional Hazard Transform would be not covered by the class of tail risk measures.

Although both approaches focus on the tail of the distribution, our approach is different. The analysis carried out by Liu and Wang [2019] is based on the conditional distribution of the tail. Our analysis is based on the impact of the tail on the unconditional distorted distribution of the risk. We investigate how to measure the contribution of the tail of the distribution on the risk measure value and evaluate from a practical viewpoint the tail subadditivity property. As we deal with the unconditional distribution, the contribution of the tail to the value of the risk measure cannot be interpreted as a risk measure itself. We focus on DRM's and their properties from an axiomatic approach have been widely investigated (for instance, Balbás et al. [2009]; Wirch and Hardy [2002]; Kou and Peng [2016]). Our analysis covers all distortion risk measures, including the Dual Power and the Proportional Hazard Transform, as we show in an example.

Our practical approach has the advantage that the results are easily interpretable. Decision makers obtain information about the contribution of extreme losses to the risk value reported by a specific risk measure so that, when risks are aggregated, the part of the diversification benefit attributable to the tails of the loss distributions can be identified. DRMs are often used to compute the economic capital. For instance, Basel Committee on Banking Supervision sets the TVaR at 97.5% confidence level for computing the minimum capital requirements [BIS, 2019]. Decision makers could be interested in knowing the impact of the q% of highest losses on the value reported by the $\text{TVaR}_{97.5\%}$, with $0 < q < 0.025$, and the part of the diversification benefit attributable to these losses when risks are aggregated. In this article we show how to compute these.

The rest of the article is structured as follows. In Section 2 risk measures are introduced. In Section 3, we analyze the influence of tails on distortion risk measures and study tail aggregation properties, with particular attention to quantile-based risk measures. Finally, in Section 4, we present an example based on motor insurance claims to illustrate our results.

2 Risk quantification

2.1 Risk measures

Non-negative random variables are often deemed suitable for defining losses in the context of enterprise risk quantification. Let X be a non-negative random variable (r.v.) with finite expectation, called a 'loss'. The cumulative distribution function of X, denoted by F_X, is defined by $F_X(x) = P(X \le x)$ and it is often referred to as the 'loss distribution'. The survival function is denoted by $S_X(x) = P(X > x)$. The mathematical expectation can be written as

$$\mathbb{E}(X) = \int_0^{+\infty} x dF_X(x)$$

We follow standard notation, so that the derivative function of F_X, when it can be defined, is the density function f_X, so $dF_X(x) = f_X(x)dx$. In general, the mathematical expectation can also be obtained from the survival distribution as,

$$\mathbb{E}(X) = \int_0^{+\infty} S_X(x)dx.$$

The inverse function of F_X, F_X^{-1}, is known as the *quantile function*, i.e. $F_X^{-1}(\alpha) = \inf\{x \mid F_X(x) \ge \alpha\}$ where $\alpha \in (0,1)$. The mathematical expectation can be obtained from the quantile function as,

$$\begin{aligned}\mathbb{E}(X) &= \int_0^1 S_X^{-1}(u)du \\ &= \int_0^1 F_X^{-1}(1-u)du.\end{aligned} \tag{1}$$

A risk measure ρ assigns a value to X. Let Γ be the set of all random variables defined for a given probability space $(\Omega, \mathcal{A}, P)$. A risk measure is a mapping ρ from Γ to $\mathbb{R}$, so $\rho(X)$ is a real value for each $X \in \Gamma$. The goal of any risk measure is to summarize the risk associated with the loss distribution [McNeil et al., 2005].

The most frequently used risk measures in finance and insurance are the VaR and TVaR. The VaR at confidence level $\alpha \in (0,1)$ is defined as $\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha)$. The mathematical expectation can then be represented as $\mathbb{E}(X) = \int_0^1 \mathrm{VaR}_{1-u}(X)du$. The TVaR at confidence level α is defined as

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_0^{1-\alpha} \mathrm{VaR}_{1-u}(X)du$$

The TVaR can be understood as the mathematical expectation beyond VaR and expressed as $\mathrm{TVaR}_\alpha(X) = \mathbb{E}\left[X \mid X > \mathrm{VaR}_\alpha(X)\right]$. Interpreted in this way, TVaR is sometimes known as the 'expected shortfall' [McNeil et al., 2005]. Many other risk measures have been defined in literature, some of them can be found, for instance, in Denuit et al. [2005] and Furman et al. [2017].

2.2 Distortion risk measures

Distortion risk measures were first introduced by Denneberg [1990] and further developed by Wang [1995, 1996]. This class of risk measure is closely related to the distortion expectation theory [Yaari, 1987]. A key element in defining a DRM is its associated distortion function g, which can be defined as a left-continuous non-decreasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$.

Consider a non-negative random variable X and its survival function S_X, the function ρ_g defined by

$$\rho_g(X) = \int_0^{+\infty} g(S_X(x))\, dx \tag{2}$$

is called a DRM.

A DRM can be understood as the distorted expectation of X. The mathematical expectation is a distortion risk measure whose distortion function is the identity function, that is, $\rho_{id}(X) = \mathbb{E}(X)$ [Denuit et al., 2005]. DRMs can be expressed in terms of the quantile function with the Lebesgue-Stieltjes integral representation.

The distortion risk measure ρ_g is represented with the Lebesgue-Stieltjes integral as follows,

$$\rho_g(X) = \int_0^1 F_X^{-1}(1-u) dg(u). \tag{3}$$

Note that $g(S_X(x)) = \int_0^{S_X(x)} dg(u)$. Later, Fubini's theorem is applied to change the order of integrals [Dhaene et al., 2012, Theorem 6]. In case that the distortion function g is right-continuous, the DRM can be represented with the Lebesgue-Stieltjes integral in terms of $F_X^{-1+}(\alpha)$, where $F_X^{-1+}(\alpha) = \sup\{x \mid F_X(x) \le \alpha\}$ [Theorem 4 Dhaene et al., 2012; Wang et al., 2018, Lemma 2.6].

The VaR and TVaR risk measures can be represented as DRMs. A flexible family of four-parameter DRMs, known as GlueVaR risk measures, was introduced by Belles-Sampera et al. [2014]. GlueVaR risk measures includes VaR and TVaR as specific particular cases. The associated distortion function of the VaR, TVaR and GlueVaR risk measures are shown in Table 1.

The equivalence of (2) and (3) is illustrated graphically for the case of the TVaR measure in Figures 1 to 3. Figure 1 shows the functions involved in the computation of the $\mathrm{TVaR}_\alpha(X)$ when the risk measure is expressed as a DRM in (2). The value of the $\mathrm{TVaR}_\alpha(X)$ corresponds to the area under the solid-line function shown in Figure 1(c).

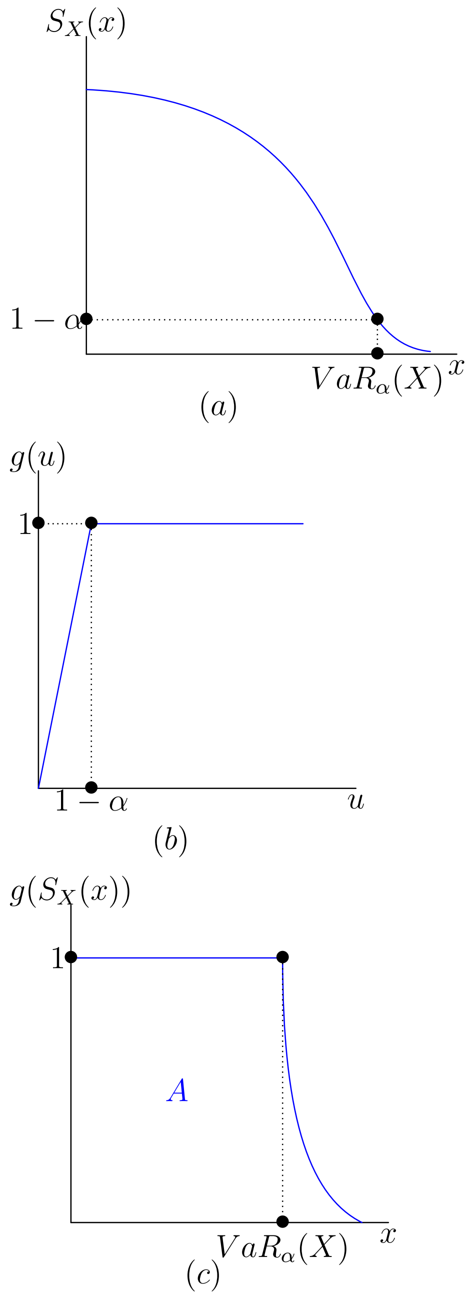


Figure 1: Functions associated with TVaR
(a) Survival distribution
(b) Distortion function
(c) Distorted survival distribution [$TVaR_\alpha(X) = Area \quad A$]

Table 1: Examples of distortion functions

Risk measure	Distortion function
VaR	$\psi_\alpha(u) = \begin{cases} 0 & \text{if} \quad 0 \le u \le 1-\alpha \\ 1 & \text{if} \quad 1-\alpha < u \le 1 \end{cases}$
TVaR	$\gamma_\alpha(u) = \begin{cases} \dfrac{u}{1-\alpha} & \text{if} \quad 0 \le u \le 1-\alpha \\ 1 & \text{if} \quad 1-\alpha < u \le 1 \end{cases}$
GlueVaR	$\kappa_{\beta,\alpha}^{h_1,h_2}(u) = \begin{cases} \dfrac{h_1}{1-\beta}u, & \text{if} \quad 0 \le u \le 1-\beta \\ h_1 + \dfrac{h_2-h_1}{\beta-\alpha}\left[u-(1-\beta)\right], & \text{if} \quad 1-\beta < u \le 1-\alpha \\ 1, & \text{if} \quad 1-\alpha < u \le 1 \end{cases}$

where $\alpha, \beta \in [0,1]$ such that $\alpha \le \beta$, $h_1 \in [0,1]$ and $h_2 \in [h_1, 1]$.

Figure 2 shows the functions involved in the computation of the $\mathrm{TVaR}_\alpha(X)$ when the risk measure is expressed as the DRM defined in (3). The value of the $\mathrm{TVaR}_\alpha(X)$ corresponds to the area under the solid-line function shown in Figure 2(c). The equivalence between the two forms to compute the TVaR is shown in Figure 3.

Many articles have examined DRMs [Zhu and Li, 2012; Belles-Sampera et al., 2014; Tsanakas and Millossovich, 2016; Goovaerts et al., 2012; Balbás et al., 2009], while the relationship between these measures and distortion expectation theory is investigated in Tsanakas and Desli [2005].

3 Tail contributions and their aggregation properties in distortion risk measures

In this section we analyze the role of the tail in distortion risk values. In other words, we study the contribution of the tail to the magnitude of a DRM.

3.1 Tail contribution to risk value

We refer to the Lebesque $q-$tail contribution in DRMs as a part of the risk measure value that can be associated to values located in the right tail of the loss distribution. Let us consider expression (3) and focus only on just one part of the integral. We define the Lebesgue $q-$tail contribution in distortion risk measures as follows.

Definition 3.1 ($q-$tail contribution)**.** *Given a $q \in [0,1]$, the $q-$tail contribution of a distortion*

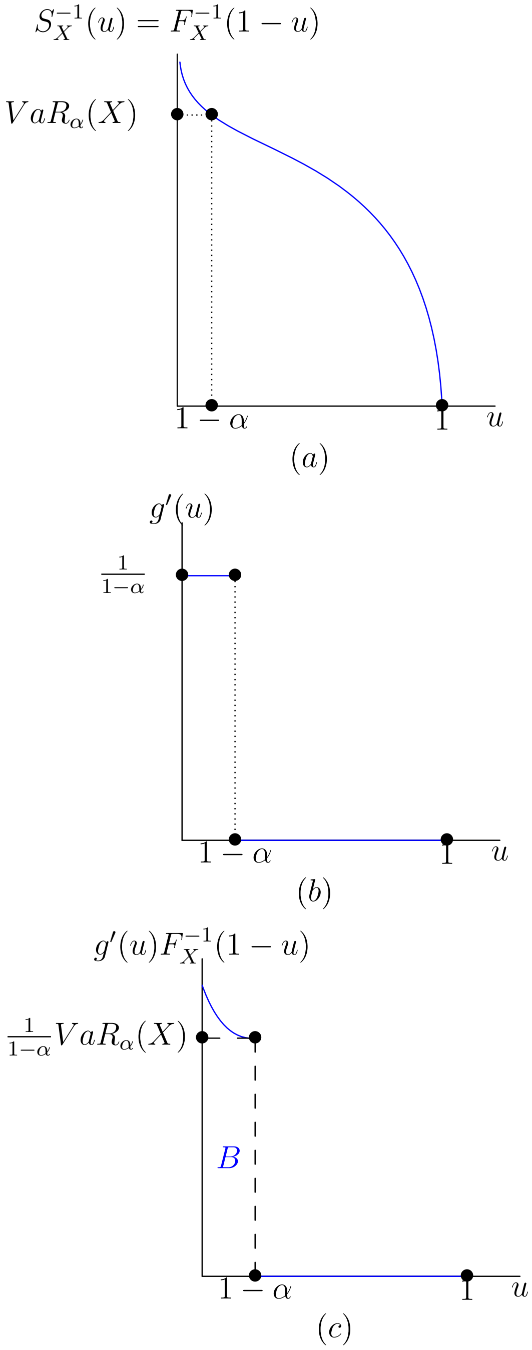


Figure 2: Functions associated with TVaR represented with the Lebesgue-Stieltjes integral
(a) Inverse survival distribution
(b) Derivative of the distortion function
(c) Product of the (a) and (b) $[TVaR_\alpha(X) = Area \quad B]$

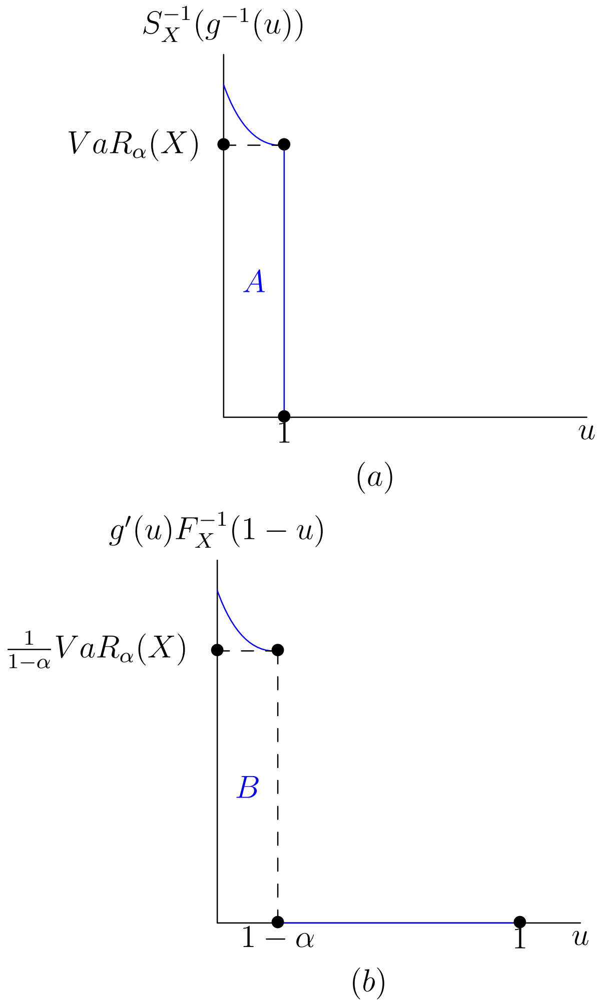


Figure 3: Equivalence of Area A and Area B
(a) Rotation (and re-escalation) of Figure 1(c)
(b) Figure 2(c)

risk measure $c^q_{\rho_g}$ *is represented by the Lebesgue-Stieltjes integral as,*

$$c^q_{\rho_g}(X) = \int_0^q F_X^{-1}(1-u)dg(u). \tag{4}$$

The information provided by expression (4) is of relevance to decision makers, because it serves as an indicator of the importance of the tail on the risk value. Computing the risk value according to the DRM defined in (3), the expression (4) provides information about the part of the risk value attributable to the q right tail of the loss distribution. The tail contribution can be then interpreted as an indicator of the sensitivity of the risk measure value to the q-right tail of the loss distribution. In calculating the solvency requirements of financial institutions, the selection of the DRM to apply in the computation of the economic capital is often set by regulators. In this context, risks managers will be interested in evaluating the part of the the economic capital estimate attributable to the $q\%$ of the highest losses of the loss distribution.

Proposition 3.1. *Given a random variable* X*, if the left-continuous function* g *is not differentiable at points in a countably finite set* $(u_1, u_2, ..., u_n)$*, the risk value of the distortion risk measure* ρ_g *can be expressed as*

$$\rho_g(X) = \int_0^q F_X^{-1}(1-u)dg(u) + \int_q^1 F_X^{-1}(1-u)dg(u),$$

regardless of the allocation of $(u_1, u_2, ..., u_n)$*, for* $q \in [0, 1]$.

Proof. Given a r.v. X, the risk value of the DRM ρ_g is computed as (3). When g is a left-continuous function, the following equivalence holds,

$$\int_0^1 F_X^{-1}(1-u)dg(u) = \int_0^1 F_X^{-1}(u)d\overline{g}(u),$$

where $\overline{g}(u) = 1 - g(1-u)$ is a right-continuous function [Dhaene et al., 2012]. For a right-continuous increasing function $\overline{g}$ that is differentiable on $\mathbb{R}$ except at points in a countably finite set $(u_1, u_2, ..., u_n)$, the Lebesgue-Stieltjes integral can be computed as follows,

$$\int_0^1 F_X^{-1}(u)d\overline{g}(u) = \int_0^{u_1} F_X^{-1}(u)\overline{g}'(u)du + \int_{u_1}^{u_2} F_X^{-1}(u)\overline{g}'(u)du + ...$$
$$+ \int_{u_n}^1 F_X^{-1}(u)\overline{g}'(u)du + \sum_{i=1}^n F_X^{-1}(u_i)\Delta\overline{g}(u_i),$$

where $\overline{g}'$ is the derivative of $\overline{g}$ and $\Delta\overline{g}(u_i) = \overline{g}(u_i) - \overline{g}({u_i}^-)$. For any given $q \in [0, 1]$, it is easy to check that the risk measure ρ_g can be represented with the Lebesgue-Stieltjes integral as

$$\int_0^{1-q} F_X^{-1}(u)d\overline{g}(u) + \int_{1-q}^1 F_X^{-1}(u)d\overline{g}(u),$$

or, equivalently, as

$$\int_q^1 F_X^{-1}(1-u)dg(u) + \int_0^q F_X^{-1}(1-u)dg(u).$$

□

Remark 3.1. *The first term on the right-hand side in proposition (3.1) can be interpreted as the part attributable to the q right-tail in the risk value returned by the risk measure ρ_g. In case of a jump of $\overline{g}$ in $(1-q)$, then $(1-q)$ contributes $F_X^{-1}(1-q)\left(\overline{g}(1-q) - \overline{g}((1-q)^-)\right)$ to $\int_0^{1-q} F_X^{-1}(u)d\overline{g}(u)$ (or, equivalently, to $\int_q^1 F_X^{-1}(1-u)dg(u)$) and $F_X^{-1}(1-q)\left(\overline{g}((1-q)^+) - \overline{g}(1-q)\right)$ to $\int_{1-q}^1 F_X^{-1}(u)d\overline{g}(u)$ (or, equivalently, to $\int_0^q F_X^{-1}(1-u)dg(u)$). The function $\overline{g}(u)$ is right-continuous, so the difference $\left(\overline{g}((1-q)^+) - \overline{g}(1-q)\right)$ is zero.*

Example 3.1. *Consider X is a uniform r.v. between 0 and 1. Assume that the proportional hazard transform distortion function is applied $g_{ph}(u) = u^r$ where $0 \leq r \leq 1$. The value of the risk measure is equal to $\rho_{g_{ph}} = \int_0^1 (1-u)^r du = \frac{1}{r+1}$. The q-tail contribution to the risk measure $c^q_{\rho_{g_{ph}}}(X)$ is equal to $c^q_{\rho_{g_{ph}}}(X) = \int_0^q F_X^{-1}(1-u)dg(u) = \int_0^q (1-u)ru^{r-1}du = q^r - \frac{r}{r+1}q^{r+1}$.*

3.2 Tail importance in quantile-based risk measures

We analyze the Lebesgue $q-$tail contribution for the three DRMs shown in Table 1.

VaR

We fix q in $[0, 1-\alpha]$. In the case of the VaR, the distortion function ψ_α is zero in $[0, q]$. So, it holds that

$$c^q_{VaR_\alpha}(X) = \int_0^q F_X^{-1}(1-u)d\psi_\alpha(u) = 0.$$

Here, it can be seen that expression (4) reflects the importance of the tail on the risk value. The VaR_α value does not take into account the q-tail of the random variable if $q \leq 1-\alpha$, so the impact of the q-right-tail on the risk value is null.

It is interesting to analyze the case $q > 1-\alpha$. Now, the q-tail contribution defined in (4) is exactly the VaR_α. Note that ψ_α is a step function. If the q-tail includes the point value in which ψ_α has the step, then the solution of the Lebesgue-Stieltjes integral is $F_X^{-1}(\alpha)$. That is, the $q-$tail contribution is the total value of VaR_α. So, only the value located in the α-quantile (step point) matters to determine the value of the risk measure.

An alternative interpretation when considering VaR_α risk measure is that $q-$tail contributions provide information about the extremes beyond a certain threshold in the right-tail of interest (the $1-\alpha$ right tail). In larger tails, i.e. when $q > 1-\alpha$, then the $q-$tail contribution is equal to the value of the risk measure, VaR_α.

TVaR

In the case of the TVaR, the distortion function γ_α takes the value $\frac{u}{1-\alpha}$, $\forall u$, when $q \in [0, 1-\alpha]$. Therefore, the q-tail contribution to the risk value of the TVaR_α is computed as,

$$c^q_{TVaR_\alpha}(X) = \int_0^q F_X^{-1}(1-u)d\gamma_\alpha(u) = \frac{1}{1-\alpha}\int_0^q F_X^{-1}(1-u)du.$$

Now, we replace $\dfrac{1}{1-\alpha} = \dfrac{q}{1-\alpha}\dfrac{1}{q}$, then the tail contribution can be expressed as,

$$c^q_{TVaR_\alpha}(X) = \frac{q}{1-\alpha}\text{TVaR}_{1-q}(X)$$

As in the previous case, when $q > 1-\alpha$, the Lebesgue q-tail contribution in the risk value of the TVaR_α is then $\dfrac{1}{1-\alpha}\displaystyle\int_0^{1-\alpha} F_X^{-1}(1-u)du$. Note that this is the definition of the TVaR. This means, when $q \geq 1-\alpha$, the Lebesgue q-tail contribution to the risk value of the TVaR_α represents the complete risk value. Let us recall that TVaR_α can be understood as the mathematical expectation beyond VaR_α. So, the q-tail contribution explains the whole risk value when the q-tail includes the $(1-\alpha)$-tail. Similarly to VaR_α, it can be interpreted as if our definition of $q-$tail contribution has an embedded threshold right tail of interest when considering the TVaR_α measure.

Figure 4 shows the Lebesgue q-tail contribution to the risk value of the TVaR_α when $q < 1-\alpha$. The Lebesgue q-tail contribution corresponds to the area labeled C. Note that the size of area C in Figure 4 is always smaller or equal to the size of area B in Figure 2(c).

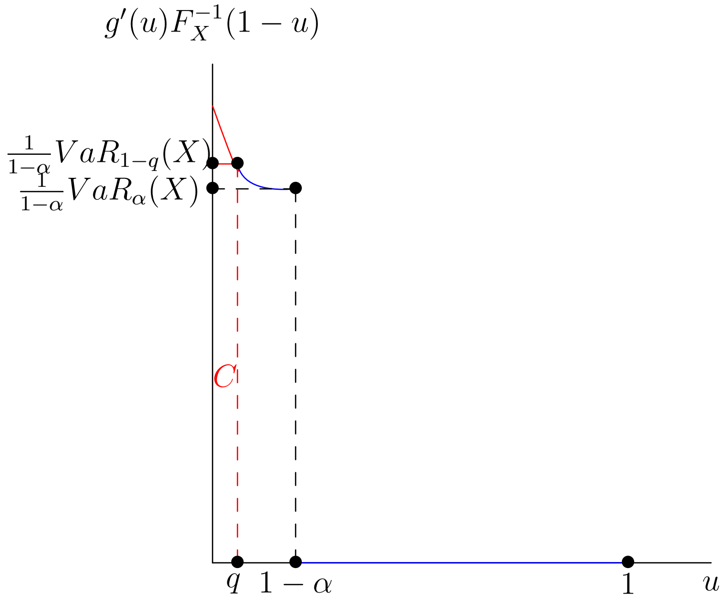


Figure 4: $q-$tail contribution to $TVaR_\alpha(X)$ $[Area \quad C]$.

GlueVaR

Finally, the Lebesgue $q-$tail contribution to GlueVaR measures is analyzed. As shown in Table(1), the distortion function for GlueVaR has an additional confidence level β. The shape of the GlueVaR distortion function is determined by the distorted survival probabilities h_1 and h_2 at levels $1-\beta$ and $1-\alpha$, respectively. Parameters h_1 and h_2 are the heights of the distortion function.

Let us consider the following notation,

$$\begin{cases} \omega_1 = & h_1 - \dfrac{(h_2-h_1)(1-\beta)}{\beta-\alpha} \\ \omega_2 = & \dfrac{h_2-h_1}{\beta-\alpha}(1-\alpha) \end{cases} \tag{5}$$

Belles-Sampera et al. [2014] showed that

$$\begin{gathered} \text{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X) = \omega_1 \text{TVaR}_\beta(X) + \\ \omega_2 \text{TVaR}_\alpha(X) + \omega_3 \text{VaR}_\alpha(X), \end{gathered} \tag{6}$$

where $\omega_3 = 1 - \omega_1 - \omega_2$.

We analyze the $q-$tail contribution to GlueVaR measures when $1-\beta < q \leq 1-\alpha$. The following proposition is shown.

Proposition 3.2. *The q-tail contribution of the* $\text{GlueVaR}_{\beta,\alpha}^{h_1,h_2}$ *when* $1-\beta < q \leq 1-\alpha$ *is equal to*

$$c^q_{\text{GlueVaR}_{\beta,\alpha}^{h_1,h_2}}(X) = \omega_1 \text{TVaR}_\beta(X) + \omega_2 \frac{q}{1-\alpha} \text{TVaR}_{1-q}(X).$$

Note that when $q = 1-\alpha$, then $c^q_{\text{GlueVaR}_{\beta,\alpha}^{h_1,h_2}}(X) = \omega_1 \text{TVaR}_\beta(X) + \omega_2 \text{TVaR}_\alpha(X).$

Proof. The $q-$tail contribution of the $\text{GlueVaR}_{\beta,\alpha}^{h_1,h_2}$ when $1-\beta < q \leq 1-\alpha$ can be expressed as,

$$\begin{gathered} c^q_{\text{GlueVaR}_{\beta,\alpha}^{h_1,h_2}}(X) = \int_0^q F_X^{-1}(1-u) d\kappa_{\beta,\alpha}^{h_1,h_2}(u) = \\ \frac{h_1}{1-\beta} \int_0^{1-\beta} F_X^{-1}(1-u) du + \frac{h_2-h_1}{\beta-\alpha} \int_{1-\beta}^q F_X^{-1}(1-u) du, \end{gathered}$$

where,

$$\begin{gathered} \frac{h_2-h_1}{\beta-\alpha} \int_{1-\beta}^q F_X^{-1}(1-u) du = \\ \frac{h_2-h_1}{\beta-\alpha} \left(\int_0^q F_X^{-1}(1-u) du - \int_0^{1-\beta} F_X^{-1}(1-u) du \right). \end{gathered}$$

Therefore,

$$c^q_{\text{GlueVaR}^{h_1,h_2}_{\beta,\alpha}}(X) =$$
$$\left(h_1 - \frac{(h_2-h_1)(1-\beta)}{\beta-\alpha}\right)\frac{1}{1-\beta}\int_0^{1-\beta} F_X^{-1}(1-u)du +$$
$$\frac{(h_2-h_1)(1-\alpha)}{\beta-\alpha}\frac{q}{1-\alpha}\frac{1}{q}\int_0^{q} F_X^{-1}(1-u)du.$$

This can be expressed as follows,

$$c^q_{\text{GlueVaR}^{h_1,h_2}_{\beta,\alpha}}(X) =$$
$$\left(h_1 - \frac{(h_2-h_1)(1-\beta)}{\beta-\alpha}\right)\text{TVaR}_\beta(X)+$$
$$\left(\frac{(h_2-h_1)(1-\alpha)}{\beta-\alpha}\right)\tfrac{q}{1-\alpha}\text{TVaR}_{1-q}(X).$$

And following notation (5), this is equivalent to

$$c^q_{\text{GlueVaR}^{h_1,h_2}_{\beta,\alpha}}(X) = \omega_1 \text{TVaR}_\beta(X) + \omega_2 \frac{q}{1-\alpha}\text{TVaR}_{1-q}(X).$$

□

To conclude, the q-tail contribution of the $\text{GlueVaR}^{h_1,h_2}_{\beta,\alpha}$ can be easily derived when $q \leq 1-\beta$ and $q > 1-\alpha$. If $q \leq 1-\beta$, then $c^q_{\text{GlueVaR}^{h_1,h_2}_{\beta,\alpha}}(X) = h_1 \frac{q}{1-\beta}\text{TVaR}_{1-q}(X)$. As in previous cases, if $q > 1-\alpha$, then the q-tail contribution of the risk measure is equal to the risk measure value.

3.3 Tail subadditivity

DRMs satisfy a set of properties, including positive homogeneity, translation invariance, comonotonic additivity and monotonicity [Balbás et al., 2009]. These properties of the DRMs are *desirable* in many contexts of risk quantification.

When aggregating risks, risk measures are often required to satisfy the subadditivity property, because the risk of the sum can be bounded by the sum of risks.

Definition 3.2. *Let X, Y be non-negative random variables. A distortion risk measure ρ_g is subadditive for X, Y if*

$$\int_0^{+\infty} g\left(S_Z(z)\right)dz \leq \int_0^{+\infty} g\left(S_X(x)\right)dx + \int_0^{+\infty} g\left(S_Y(y)\right)dy$$

where $Z = X + Y$.

In other words, a risk measure is subadditive if $\rho_g(X+Y) \leq \rho_g(X) + \rho_g(Y)$. Subadditivity distinguishes between VaR and TVaR measures, since this property is only satisfied by the latter. Embrechts et al. [2015] show several ways to demonstrate that the TVaR risk measure is subadditive. Based on the expression (6), the GlueVaR measure satisfies subadditivity property if -and only if- a null weight is given to the VaR and the TVaR's are not-negatively weighted.

The subadditivity property of DRMs is guaranteed when the distortion function g is concave in $[0, 1]$ [Denneberg, 1994; Wang and Dhaene, 1998; Wirch and Hardy, 2002]. Based on (3), the subadditivity property of a risk measure ρ_g can be expressed as,

$$\int_0^1 F_Z^{-1}(1-u)dg(u) \leq \int_0^1 F_X^{-1}(1-u)dg(u) + \int_0^1 F_Y^{-1}(1-u)dg(u).$$

where $Z = X + Y$.

Subadditivity is an appealing property for decision makers. Suppose a risk manager holds a pair of risks X and Y. A subadditive risk measure considers that the risk of holding X and Y together is lower than the risk of holding X and Y individually. Hence, subadditivity means that diversification benefits are reflected in the risk measure. Subaddtitivity in the whole domain is a strong condition. When dealing with fat tail losses (i.e. low-frequency and large-loss events), risk managers are especially interested in the tail region. Fat right-tails have been extensively studied in insurance and finance [Wang, 1998; Embrechts et al., 2009a,b; Degen et al., 2010; Nam et al., 2011; Chen et al., 2012]. Belles-Sampera et al. [2014] introduced a weaker concept of tail subadditivity and proved mathematillcy that this property is satisfied if the distortion function is concave in the common tail region. Later, Cai et al. [2017] gave sufficient and necessary conditions for a distortion risk measure to be tail subadditive. These two articles do not analyze how tail subadditivity can be evaluated for frequently used DRM's. We here introduce the definition of $q-$tail subadditivity for a pair of risks in terms of the tail contribution to the risk measure value, as follows:

Definition 3.3 ($q-$tail subadditivity)**.** *Given $q \in [0, 1]$ and non-negative random variables X, Y, a distortion risk measure ρ_g is $q-$tail subadditive if*

$$\int_0^q F_Z^{-1}(1-u)dg(u) \leq \int_0^q F_X^{-1}(1-u)dg(u) + \int_0^q F_Y^{-1}(1-u)dg(u),$$

where $Z = X + Y$ or, equivalently,

$$c^q_{\rho_g}(X + Y) \leq c^q_{\rho_g}(X) + c^q_{\rho_g}(Y).$$

The $q-$ tail subadditivity property is useful for decision makers to know the aggregate behavior of the loss distributions of a pair of risks in tails. If a subadditive risk measure is fixed as the regulatory risk measure in the computation of the economic capital, when risks X and Y are aggregated, decision makers may identify the portion of the diversification benefits attributable to the $q-$ tails of the loss distributions of both risks. If the regulatory risk measure is not subadditive but it is $q-$tail subadditive, diversification benefits of the $q\%$ of the extreme losses of both risks are captured by the risk measure even if the total diversification benefit can not be guaranteed to be positive.

Proposition 3.3. *Under the corresponding $q-$tail subadditivity property conditions, the $q-$tail subadditivity property is satisfied when the distortion function g is concave in $[0, q]$.*

The proof of proposition 3.3 is provided in the Appendix.

Example 3.2 (Continuation)**.** *Consider X and Y two independent uniform r.v.'s between 0 and 1. The density function of $Z = X + Y$ is*

$$f(z) = \begin{cases} z & if \quad 0 \leq z \leq 1 \\ 2 - z & if \quad 1 < u \leq 2. \end{cases}$$

The proportional hazard transform distortion function is applied. Assume that $r = 0.5$, the value of the risk measure for Z is equal to $\rho_{g_{ph}}(Z) = \int_0^1 (1 - \frac{u^2}{2})^{0.5} du + \int_1^2 (\frac{u^2}{2} - 2u + 2)^{0.5} du = \sqrt{\frac{1}{2}} \left(1 + \arcsin\left(\sqrt{\frac{1}{2}}\right)\right)$. From Ex.(3.1) we know that for X and Y the risk value is equal to $\rho_{g_{ph}}(X) = \rho_{g_{ph}}(Y) = \frac{1}{1.5}$ and the q-tail contribution to the risk measure value is equal to $c^q_{\rho_{g_{ph}}}(X) = c^q_{\rho_{g_{ph}}}(Y) = q^{0.5} - \frac{0.5}{1.5} q^{1.5}$. Subadditivity is satisfied since $\rho_{g_{ph}}(X) + \rho_{g_{ph}}(Y) - \rho_{g_{ph}}(Z) = 0.07$. If $q \leq 0.5$ the tail contribution to the risk measure $c^q_{\rho_{g_{ph}}}(Z)$ is equal to $c^q_{\rho_{g_{ph}}}(Z) = \int_0^q (2 - 2^{0.5}u) 0.5 u^{0.5-1} du = 2(q^{0.5}) - 0.5\sqrt{2}q$. Then, tail subadditivity is satisfied since $c^q_{\rho_{g_{ph}}}(X) + c^q_{\rho_{g_{ph}}}(Y) - c^q_{\rho_{g_{ph}}}(Z) = 0.5\sqrt{2}q - \frac{q^{1.5}}{1.5}$ is positive for any $0 \leq q \leq 0.5$.

3.4 Tail subadditivity for quantile-based distortion risk measures

We analyze the $q-$tail subadditivity property for the three risk measures shown in Table 1.

VaR

In the case of VaR, the distortion function ψ_α is concave in $[0, 1 - \alpha]$. Given a q in $[0, 1 - \alpha]$, it holds that for non-negative random variables X, Y, we can write,

$$\int_0^q F_Z^{-1}(1 - u) d\psi_\alpha(u) \leq \int_0^q F_X^{-1}(1 - u) d\psi_\alpha(u) + \int_0^q F_Y^{-1}(1 - u) d\psi_\alpha(u).$$

where $Z = X + Y$.

The VaR_α measure satisfies q-tail subadditivity when q is lower than or equal to $1 - \alpha$. It is straightforward to note that the integrals are equal to zero on both sides of the inequality.

TVaR

The distortion function of TVaR, γ, is concave in the whole interval $[0, 1]$. Therefore, from Proposition 3.3, for non-negative random variables X, Y, it holds that

$$\int_0^q F_Z^{-1}(1 - u) d\gamma_\alpha(u) \leq \int_0^q F_X^{-1}(1 - u) d\gamma_\alpha(u) + \int_0^q F_Y^{-1}(1 - u) d\gamma_\alpha(u),$$

where $Z = X + Y$, for any q in $[0,1]$. In other words, TVaR is $q-$tail subadditive for any q in $[0,1]$.

Given $q \in [0, 1-\alpha]$,

$$\frac{1}{1-\alpha}\int_0^q F_Z^{-1}(1-u)du \leq \frac{1}{1-\alpha}\int_0^q F_X^{-1}(1-u)du + \frac{1}{1-\alpha}\int_0^q F_Y^{-1}(1-u)du,$$

is equivalent to,

$$\frac{q}{1-\alpha}\mathrm{TVaR}_{1-q}(Z) \leq \frac{q}{1-\alpha}\mathrm{TVaR}_{1-q}(X) + \frac{q}{1-\alpha}\mathrm{TVaR}_{1-q}(Y),$$

where $\frac{q}{1-\alpha}$ is non-negative. Therefore, the inequality holds since $\mathrm{TVaR}_{1-q}(Z) \leq \mathrm{TVaR}_{1-q}(X) + \mathrm{TVaR}_{1-q}(Y)$.

When $q \in [1-\alpha, 1]$, then $\mathrm{TVaR}_{1-q}(X) = \mathrm{TVaR}_{\alpha}(X)$.

GlueVaR

The $q-$tail subadditivity of GlueVaR measures is now analyzed. In Figure 5, two examples of distortion functions of GlueVaR measures are shown.

Let us consider $\kappa_{\beta,\alpha}^{h_1,h_2}$ in Figure 5(a). The distortion function has a discontinuity in $1-\alpha$. Note that $\kappa_{\beta,\alpha}^{h_1,h_2}(1-\alpha) = h_2$ where $h_2 < 1$ and $\lim_{u \to (1-\alpha)^+} \kappa_{\beta,\alpha}^{h_1,h_2}(u) = 1$. This distortion function is concave in the interval $[0, 1-\alpha]$. However, the great flexibility of GlueVaR measures allows us to define a distortion function $\kappa_{\beta,\alpha}^{h_1,h_2}$ that is convex in the interval $[0, 1-\alpha]$. This is the case of the distortion function in Figure 5(b).

We fix $q = 1-\alpha$, so that for non-negative random variables X, Y, it holds that

$$\int_0^{1-\alpha} F_Z^{-1}(1-u)d\kappa_{\beta,\alpha}^{h_1,h_2}(u) \leq \int_0^{1-\alpha} F_X^{-1}(1-u)d\kappa_{\beta,\alpha}^{h_1,h_2}(u) + \int_0^{1-\alpha} F_Y^{-1}(1-u)d\kappa_{\beta,\alpha}^{h_1,h_2}(u).$$

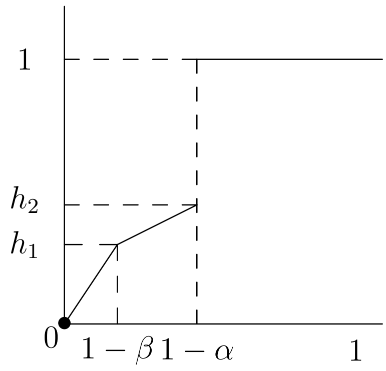
where $Z = X + Y$, if the distortion function $\kappa_{\beta,\alpha}^{h_1,h_2}$ is concave in $[0, 1-\alpha]$.

Our starting point is that

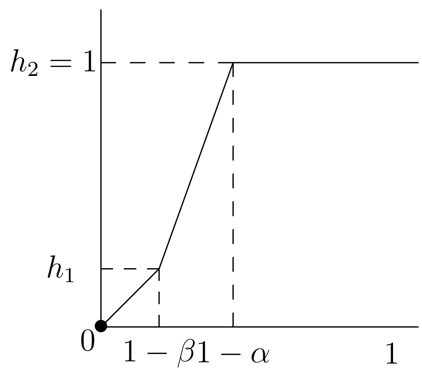
$$\int_0^{1-\alpha} F_X^{-1}(1-u)d\kappa_{\beta,\alpha}^{h_1,h_2}(u) = \omega_1 \mathrm{TVaR}_{\beta}(X) + \omega_2 \mathrm{TVaR}_{\alpha}(X).$$

Therefore, $q-$tail subadditivity is ensured if ω_1 and ω_2 are positive. The weights ω_1 and ω_2 are positive if and only if the distortion function κ is concave in $[0, 1-\alpha]$. Indeed, ω_1 is positive if and only if $\frac{h_1}{1-\beta} \geq \frac{(h_2 - h_1)}{\beta - \alpha}$. The distortion function κ is concave in $[0,1]$ if ω_1 is positive and $h_2 = 1$.

When q is in $(1-\beta, 1-\alpha)$, it can be shown that $q-$tail subadditivity is satisfied if g is concave in $[0, q]$ (see Proposition 3.2). Finally, $q-$tail subadditivity when $q \leq 1-\beta$ is directly derived.



(a)



(b)

Figure 5: Examples of GlueVaR distortion functions
(a) Distortion function is concave in $[0, 1-\alpha]$
(b) Distortion function is convex in $[0, 1-\alpha]$

4 Illustration

To illustrate the quantification of the tail contribution, we use the insurance data described in Belles-Sampera et al. [2017]. The data comprise three types of claim: property damage (X_1), bodily injuries (X_2) and medical expenses (X_3). The sample contains $n = 350$ observations of the cost of individual claims expressed in thousands of euros. The cost of bodily injuries contains compensation for bodily injuries. It is relatively low compared to that of property damage, because it does not include that part already covered by public health insurance.

In Table 2 a set of quantile-based risk measures, including three different GlueVaR, are shown. These GlueVaR measures reflect different risk attitudes. The $\text{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ corresponds to a balanced attitude between $\text{TVaR}_{99.5\%}$, $\text{TVaR}_{95\%}$ and $\text{VaR}_{95\%}$. Indeed, $h_1 = 11/30$ and $h_2 = 2/3$ corresponds to $\omega_1 = \omega_2 = \omega_3 = 1/3$. The $\text{GlueVaR}^{0,1}_{99.5\%,95\%}$ has associated weights $\omega_1 = -1/9$, $\omega_2 = 10/9$ and $\omega_3 = 0$. It corresponds to a extreme scenario in which the lowest feasible ω_1 is allocated to $\text{TVaR}_{99.5\%}$ and the highest ω_2 to $\text{TVaR}_{95\%}$. A zero weight is allocated to $\text{VaR}_{95\%}$, $\omega_3 = 0$. Finally, $\text{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ assigns a high weight to $\text{VaR}_{95\%}$

Table 2: Quantile-based risk measures and subadditivity in a motor insurance claims data set

	$\mathbf{X_1}$ (a)	$\mathbf{X_2}$ (b)	$\mathbf{X_3}$ (c)	$\mathbf{X_1+X_2+X_3}$ (d)	**Difference**$^{(*)}$ (a+b+c-d)
$\mathrm{VaR}_{95\%}$	2.5	0.6	1.1	5.9	-1.7
$\mathrm{TVaR}_{95\%}$	12.5	8.0	1.3	19.7	2.1
$\mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$	18.6	16.9	1.4	35.6	1.3
$\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$	4.9	2.9	1.1	10.2	-1.3
$\mathrm{GlueVaR}^{0,1}_{99.5\%,95\%}$	9.4	4.2	1.2	12.9	1.9

$^{(*)}$ Benefit of diversification.

$\mathbf{X_1}$ = property damage claims costs, $\mathbf{X_2}$ = bodily injury claims costs and $\mathbf{X_3}$ = additional medical expenses claims. All three are measured in thousands of euros.

($\omega_3 = 21/24$) and low weights to $\mathrm{TVaR}_{99.5\%}$ and $\mathrm{TVaR}_{95\%}$ ($\omega_1 = 1/24$ and $\omega_2 = 1/12$). Risk measures are estimated based on the empirical survival function.

Table 2 shows that $\mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ returns higher risk values than the other two selected GlueVaR measures. This result can be generalized to any random variable because the associated distortion function of $\mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ is greater than the other two distortion functions in the whole domain. It is also observed in Table 2 that $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%} \leq \mathrm{GlueVaR}^{0,1}_{99.5\%,95\%}$. Note that this outcome is a feature of this data set and cannot be generalized. We now analyze the subadditivity property. The only risk measure with a concave distortion function is the TVaR. The other risk measures shown in Table 2 do not have concave distortion functions in $[0, 1]$. The last column of Table 2 records the difference between the sum of risk values and the risk value of the sum. A negative value indicates that the risk value of the sum is higher, so the subadditivity is not satisfied, i.e. there is no benefit of diversification. The $\mathrm{VaR}_{95\%}$ and the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ fail to be subadditive for X_1, X_2 and X_3 since $2.5 + 0.6 + 1.1 < 5.9$ and $4.9 + 2.9 + 1.1 < 10.2$, respectively. The fact that risk values are subadditive for the $\mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ and $\mathrm{GlueVaR}^{0,1}_{99.5\%,95\%}$ is a characteristic attributable to this data set but cannot be generalized to all contexts.

In Table 3, the q-tail contribution to the risk values and tail subadditivity is analyzed for $q = 5\%$ and $q = 0.5\%$. We focus on column (d) containing the 5%- and 0.5%-tail contributions to the aggregate risk value, $\rho_g(X_1 + X_2 + X_3)$. The tail contribution indicates the sensitivity of the risk measure value to the q-right tail of the loss distribution. Both contributions are equal to zero when the $\mathrm{VaR}_{95\%}$ is analyzed. The contribution of the 5% of the most extreme losses to the $\mathrm{VaR}_{95\%}$ is null. The same occurs when the contribution of the 0.5% of the most extreme losses is considered, see lower part of Table 3. In other words, the information provided by the tail contribution to the $\mathrm{VaR}_{95\%}$ is that the 5% and 0.5% right tails of the loss distribution have not impact on the risk value.

In the case of the $\text{TVaR}_{95\%}$, losses located in the $5\%-$right tail of the distribution contribute the whole amount to the risk value, 19.7. As argued above, this is expected because the $5\%-$tail is, precisely, the threshold right-tail embedded in our $q-$tail contribution definition for $\text{TVaR}_{95\%}$. However, if we only consider the $0.5\%-$right tail of the loss distribution, the tail contribution is of 8.1 thousands of euros over 19.7 thousands of euros. That is, 0.5% of the highest aggregate losses contribute 41% (8.1/19.7) of the total risk value. This information is worthy in risk management. Mandatory reserves are often computed according to a risk measure set by regulators. Let consider that regulatory reserves must be computed based on the $\text{TVaR}_{95\%}$. Risk managers will know that the 41% of the regulatory reserve value is due to the $0.5\%-$right tail of the loss distribution. The same interpretation can be made for individual risks. If risks X_1, X_2 and X_3 are individually analyzed, losses located at the $0.5\%-$right tails of their loss distributions represent 33% (4.1/12.5), 52% (4.2/8.0) and 15% (0.2/1.3) of the $\text{TVaR}_{95\%}$ risk values, respectively. The tail contribution for the rest of risk measures can be interpreted in the same fashion.

Tail-subadditivity is now investigated. Non-negative values of the last column of the Table 3 reflect the part of the diversification benefit captured by the risk measures when the 5% and the 0.5% of the most extreme losses of all risks are considered. When looking at subadditivity in the tail, the associated distortion functions of $\text{VaR}_{95\%}$, $\text{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ and $\text{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ are concave in $[0, 0.05]$. Concavity of the distortion functions of the $\text{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$ and $\text{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$ in $[0, 0.05]$ is held because $\dfrac{h_1}{0.005} \geq \dfrac{(h_2 - h_1)}{0.995 - 0.95}$. The distortion functions of these three risk measures have a discontinuity at the point 0.05, i.e., $\psi_{0.05}(0.05) = 0$ and $\psi_{0.05}(0.05^+) = 1$ for $\text{VaR}_{95\%}$, $\kappa^{11/30,2/3}_{99.5\%,95\%}(0.05) = 2/3$ and $\kappa^{11/30,2/3}_{99.5\%,95\%}(0.05^+) = 1$ for $\text{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$, and $\kappa^{1/20,1/8}_{99.5\%,95\%}(0.05) = 1/8$ and $\kappa^{1/20,1/8}_{99.5\%,95\%}(0.05^+) = 1$ for $\text{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$. Finally, the associated distortion function of $\text{GlueVaR}^{0,1}_{99.5\%,95\%}$ is convex in the interval $[0, 0.05]$. In this case, concavity of the distortion function is only satisfied in the interval $[0, 0.005]$ where the distortion function $\kappa^{0,1}_{99.5\%,95\%}$ is equal to 0. So, only the $0.5\%-$tail subadditivity of the $\text{GlueVaR}^{0,1}_{99.5\%,95\%}$ can be guaranteed.

If we look at the $\text{TVaR}_{95\%}$, the proportion of the benefit of diversification associated to the 0.5% of the most extreme losses of all risks is the 19% of 2.1 millions of euros. In the case of the $\text{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$, a total diversification benefit of 1.3 millions of euros is considered when risks are aggregated (Table 2). The diversification benefit raises to 1.9 millions of euros when only the 5%-right tail of the loss distribution is considered (Table 3). The net diversification benefit in the region $(0.05, 1]$ is then equal to -0.6 millions of euros. If the focus is on the 0.5% of the highest losses, the diversification benefit is again 1.3 millions of euros. The information provided to decision makers is that a diversification benefit of 1.9 millions of euros is considered

Table 3: q-tail contribution and tail-subadditivity

		$\mathbf{X_1}$ (a)	$\mathbf{X_2}$ (b)	$\mathbf{X_3}$ (c)	$\mathbf{X_1+X_2+X_3}$ (d)	**Difference**$^{(*)}$ (a+b+c-d)
$q=5\%$	$\mathrm{VaR}_{95\%}$	0	0	0	0	0
	$\mathrm{TVaR}_{95\%}$	12.5	8.0	1.3	19.7	2.1
	$\mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$	17.8	16.7	1.0	33.6	1.9
	$\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$	2.7	2.4	0.1	5.0	0.2
	$\mathrm{GlueVaR}^{0,1}_{99.5\%,95\%}$	9.4	4.2	1.2	12.9	1.9
$q=0.5\%$	$\mathrm{VaR}_{95\%}$	0	0	0	0	0
	$\mathrm{TVaR}_{95\%}$	4.1	4.2	0.2	8.1	0.4
	$\mathrm{GlueVaR}^{11/30,2/3}_{99.5\%,95\%}$	15.0	15.4	0.7	29.7	1.3
	$\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$	2.0	2.1	0.1	4.1	0.2
	$\mathrm{GlueVaR}^{0,1}_{99.5\%,95\%}$	0	0	0	0	0

$^{(*)}$ Benefit of diversification.

by the risk measure in the region that the distortion function is concave $[0, 0.05]$, and the main part of this diversification benefit is due to the 0.5% of the highest losses, 1.3 (out of 1.9). This means that a relatively low weight is given to the likelihood of occurrence of the $0.5\%-$ highest losses of all three individual risks at the same time.

An interesting result is observed in the case of the $\mathrm{GlueVaR}^{1/20,1/8}_{99.5\%,95\%}$. The total diversification benefit is negative, -1.3 millions of euros. However, the diversification benefit associated to the 5% and the 0.5% of the highest losses is 0.2 millions of euros. The risk measure considers that there are benefits of diversification in the 5%-right tail of the loss distribution and these benefits are concentrated in the most extreme adverse scenarios (the 0.5% of the highest losses). To conclude, diversification benefits considered by the VaR and the $\mathrm{GlueVaR}^{0,1}_{99.5\%,95\%}$ in the regions that tail subadditivity is guaranteed, 5% and 0.5% respectively, are equal to zero.

5 Concluding remarks

We report a method for analyzing the influence of the tail in calculations of distortion risk measures. By concentrating on the tail, we define the $q-$tail contribution as being the size of the risk measure estimate that is attributable to the tail of the distribution. As such, the $q-$tail contribution represents the weight of the tail in the risk measure. For the VaR_α, TVaR_α and $\mathrm{GlueVaR}^{h_1,h_2}_{\beta,\alpha}$ measures, these weights are below 100% if the $q-$tail is smaller than some embedded threshold right tail (the $(1-\alpha)-$tail for each of these three quantile-based risk measures).

The tail contribution is a valuable information provided to decision makers. It reports the impact of the $q\%$ extreme losses on the risk measure value. The selection of the risk measure - and risk aversion coefficient- to compute economic reserves is often set by regulators. The $q-$tail contribution will inform to decision makers about the part of the regulatory reserve estimate attributable to the $q\%$ of the highest losses.

Under straightforward conditions, we have proven that tail subadditivity holds and that $q-$tail contributions satisfy the subadditivity property. So, if a subadditive distortion risk measure is fixed in the computation of the aggregate regulatory reserve, decision makers will identify the part of the diversification benefits attributable to the $q\%$ of the highest losses. If the regulatory risk measure is not subadditive but it is $q-$tail subadditive, diversification benefits due to the $q-$right tail of the loss distributions will be captured by the risk measure. In other words, even in the case that the total diversification benefit could be negative, the risk measure makes contemplate diversification benefits for extreme losses (in the $q-$right tail region), and decision makers are provided with an instrument to quantify the size of this diversification benefit in the tail.

In our example based on motor insurance claims, an examination of the risk of the severity of claims related to three dimensions - namely, property damage, bodily injury and additional medical expenses - allows us to conclude that the weight of the tail of each dimension of the final risk estimate can be assessed. Moreover, since subadditivity holds for that part of the domain, we can identify the role of each type of cost in the final risk. For instance, the contribution of additional medical expenses to risk is almost negligible compared to the contribution of the claims costs of property damage and bodily injury. This holds for all the distortion risk measures analyzed herein. However, because not all these DRMs satisfy subadditivity in general, the overall diversification of risk could not be analyzed.

Appendix

Proof. Tail subadditivity is first showed by means of the Choquet integral as previously done by Belles-Sampera et al. [2014] and Cai et al. [2017]. Later, we represent the result of the set theory in terms of the tail contribution to the risk measure value. Let us introduce the notion of measure and the Choquet integral. A measure μ is an σ-additive set function on an σ-algebra $\mathcal{S} \subset 2^{\Omega}$ where Ω denotes the basic set and $\mu(\emptyset) = 0$.

Let μ be a monotone measure on 2^{Ω}. The Choquet integral of a μ-measurable function $X : \Omega \to \mathbb{R}^{+} \cup \{0\}$ is denoted as $\int X d\mu$ and is equal to

$$\int X d\mu = \int_{0}^{+\infty} S_{\mu,X}(x) dx,$$

if $\mu(\Omega) < \infty$, where $S_{\mu,X}(x) = \mu(\{X > x\})$ is a decreasing function of X with respect to μ. A μ-measurable function X is, widely speaking, a function defined on Ω such that expressions like

$\mu(\{X > x\})$ or $\mu(\{X \leq x\})$ make sense. In the case that $\mu(\Omega) = 1$ then the measure μ is a probability measure and $S_{\mu,X}(x)$ denotes the *survival distribution function* [Denneberg, 1994].

Let us consider the two μ measurable functions $X, Y : \Omega \to \mathbb{R}$ being μ-essentially$> -\infty$. Subadditivity is satisfied if

$$\int (X + Y)\, d\mu \leq \int X d\mu + \int Y d\mu.$$

Denneberg [1994] showed that submodularity of the monotone measure μ is sufficient for subadditivity of the integral.

Let us provide at this stage of the proof some definitions. X is μ-essentially$> -\infty$ if $\lim_{x\to-\infty} S_{\mu,X}(x) = \mu(\Omega)$. A set function μ is submodular if $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$. A set function μ is monotone if $\mu(A) \leq \mu(B)$ for any $A \subseteq B$ in 2^{Ω}.

From the previous definitions, it is straightforward to see that for any random variable X, $\rho_g(X)$ is the Choquet integral of X with respect to the measure $\mu = g \circ P$, where P is the probability function associated with the probability space in which X is defined [Furman et al., 2017; Wang et al., 2018]. Let us define function g_q such that $g_q(u) := \min(g(q), g(u))$ for u in $[0, 1]$, so the measure $\mu_q = g_q \circ P$ is $\mu_q = \min(g(q), \mu)$, with $\mu = g \circ P$.

If μ_q is monotone and submodular and X and Y are μ_q-essentially$> -\infty$, then the subadditivity is satisfied. Let us consider a probability measure P on a σ-algebra $\mathcal{S} \subset 2^{\Omega}$. Given $A, B \in 2^{\Omega}$ suppose, without loss of generality, that $A \subseteq B$. Let us rename $a := P(A)$, $b := P(B)$, $i := P(A \cap B)$ and $u := P(A \cup B)$. Because P is monotone then it holds that $i \leq a \leq b \leq u$ due to $A \cap B \subseteq A \subseteq B \subseteq A \cup B$. The modularity of P implies that $i + u = a + b$, i.e. $[i, u]$ and $[a, b]$ have common centers, $\frac{i+u}{2} = \frac{a+b}{2}$. Then, when g_q is concave in $[i, u]$ it is satisfied that $g_q(u) + g_q(i) \leq g_q(a) + g_q(b)$ or, equivalently, that $g_q \circ P$ is submodular. Note that $\mu_q(\Omega) = g(q) = \lim_{x\to-\infty} S_{\mu_q,X}(x)$, so X and Y are μ_q-essentially bounded below.

Therefore, it is satisfied that

$$\int (X + Y)\, d\mu_q \leq \int X d\mu_q + \int Y d\mu_q.$$

To conclude the proof, note that

$$\int X d\mu_q = \int_0^{+\infty} g_q(S_X(x))\, dx,$$

where

$$\int_0^{+\infty} g_q(S_X(x))\, dx = \int_0^q F_X^{-1}(1-u) dg_q(u),$$

and

$$\int_0^q F_X^{-1}(1-u) dg_q(u) = \int_0^q F_X^{-1}(1-u) dg(u).$$

□

References

Balbás, A., Garrido, J., and Mayoral, S. (2009). Properties of distortion risk measures. *Methodology and Computing in Applied Probability*, 11(3, SI):385–399. https://doi.org/10.1007/s11009-008-9089-z.

Belles-Sampera, J., Guillen, M., and Santolino, M. (2017). *Risk quantification and allocation methods for practitioners*. Amsterdam University Press.

Belles-Sampera, J., Guillen, M., and Santolino, M. (2013). Generalizing some usual risk measures in financial and insurance applications. In Fernández-Izquierdo, M., Muñoz-Torres, M., and León, R., editors, *Modeling and Simulation in Engineering, Economics and Management. Proceedings of the MS 2013 International Conference*, volume 145 of *Lecture Notes in Business Information Processing*, pages 75–82. Springer-Verlag.

Belles-Sampera, J., Guillen, M., and Santolino, M. (2014). Beyond Value-at-Risk: GlueVaR distortion risk measures. *Risk Analysis*, 34(1):121–134. https://doi.org/10.1111/risa.12080.

BIS (2019). Basel Committee on Banking Supervision: Minimum capital requirements for market risk . *Bank for International Settlements*, ISBN: 978-92-9259-237-0.

Cai, J. and Li, H. (2005). Conditional tail expectations for multivariate phase type distributions. *Journal of Applied Probability*, 42:810–825. https://doi.org/10.1239/jap/1127322029.

Cai, J., Wang, Y. and Mao, T. (2005). Tail subadditivity of distortion risk measures and multivariate tail distortion risk measures. *Insurance: Mathematics and Economics*, 75:105–116. https://doi.org/10.1016/j.insmatheco.2017.05.004.

Chen, D., Mao, T., Pan, X., and Hu, T. (2012). Extreme value behavior of aggregate dependent risks. *Insurance: Mathematics and Economics*, 50(1):99–108. https://doi.org/10.1016/j.insmatheco.2011.10.008.

Degen, M., Lambrigger, D. D., and Segers, J. (2010). Risk concentration and diversification: Second-order properties. *Insurance: Mathematics and Economics*, 46(3):541–546. https://doi.org/10.1016/j.insmatheco.2010.01.011.

Denneberg, D. (1990). Premium calculation: why standard deviation should be replaced by absolute deviation. *ASTIN Bulletin*, 20(2):181–190. https://doi.org/10.2143/AST.20.2.2005441.

Denneberg, D. (1994). *Non-Additive Measure and Integral*. Kluwer Academic Publishers, Dordrecht.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). *Actuarial Theory for Dependent Risks. Measures, Orders and Models*. John Wiley & Sons Ltd, Chichester.

Dhaene, J., Kukush, A., Linders, D., and Tang, Q. (2012). Remarks on quantiles and distortion risk measures. *European Actuarial Journal*, 2(2):319–328. https://doi.org/10.1007/s13385-012-0058-0.

Embrechts, P., Lambrigger, D. D., and Wuethrich, M. V. (2009a). Multivariate extremes and the aggregation of dependent risks: examples and counter-examples. *Extremes*, 12(2):107–127. https://doi.org/10.1007/s10687-008-0071-5.

Embrechts, P., Neslehova, J., and Wuethrich, M. V. (2009b). Additivity properties for Value-at-Risk under Archimedean dependence and heavy-tailedness. *Insurance: Mathematics and Economics*, 44(2, SI):164–169. https://doi.org/10.1016/j.insmatheco.2008.08.001.

Embrechts, P., Liu, H., and Wang, R. (2015). Seven proofs for the subadditivity of Expected Shortfall. *Dependence modeling*, 30(1):126–140. https://doi.org/10.1515/demo-2015-0009.

Furman, E., Wang, R., and Zitikis, R. (2017). Gini-type measures of risk and variability: Gini shortafall, capital allocations, and heavy-tailed risks. *Journal of Banking & Finance*, 83:70–84. https://doi.org/10.1016/j.jbankfin.2017.06.013.

Goovaerts, M., Linders, D., Van Weert, K., and Tank, F. (2012). On the interplay between distortion, mean value and Haezendonck-Goovaerts risk measures. *Insurance: Mathematics and Economics*, 51(1):10–18. https://doi.org/10.1016/j.insmatheco.2012.02.012.

Kou, S., and Peng, X. (2016). On the Measurement of Economic Tail Risk. *Operations Research*, 64(5):1056–1072. https://doi.org/10.1287/opre.2016.1539.

Landsman, Z., Makov, U., and Shushi, T. (2016). Multivariate tail conditional expectation for elliptical distributions. *Insurance: Mathematics and Economics*, (70):216–223. https://doi.org/10.1016/j.insmatheco.2016.05.017.

Liu, F. and Wang, R. (2019). A Theory for Measures of Tail Risk. *Mathematics of Operations Research, forthcoming.* available at SSRN: https://ssrn.com/abstract=2841909.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management.* Princeton Series in Finance. Princeton University Press, New York.

Nam, H. S., Tang, Q., and Yang, F. (2011). Characterization of upper comonotonicity via tail convex order. *Insurance: Mathematics and Economics*, 48(3):368–373. https://doi.org/10.1016/j.insmatheco.2011.01.003.

Tsanakas, A. and Desli, E. (2005). Measurement and pricing of risk in insurance markets. *Risk Analysis*, 25(6):1653–1668. https://doi.org/10.1111/j.1539-6924.2005.00684.x.

Tsanakas, A. and Millossovich, P. (2016). Sensitivity analysis using risk measures. *Risk Analysis*, 36(1):30–48. https://doi.org/10.1111/risa.12434.

Wang, R., Wei, Y. and Willmot, G. (2018). Characterization, Robustness and Aggregation of Signed Choquet Integrals. *Available at SSRN:* https://ssrn.com/abstract=2956962.

Wang, S. S. (1995). Insurance pricing and increased limits ratemaking by proportional hazard transforms. *Insurance: Mathematics and Economics*, 17(1):43–54. https://doi.org/10.1016/0167-6687(95)00010-P.

Wang, S. S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bullletin*, 26(1):71–92. https://doi.org/10.2143/AST.26.1.563234.

Wang, S. S. (1998). An actuarial index of the right-tail risk. *North American Actuarial Journal*, 2(2):88–101. https://doi.org/10.1080/10920277.1998.10595760.

Wang, S. S. and Dhaene, J. (1998). Comonotonicity, correlation order and premium principles. *Insurance: Mathematics and Economics*, 22(3):235–242. https://doi.org/10.1016/S0167-6687(97)00040-1.

Wirch, J. L. and Hardy, M. R. (2002). Distortion risk measures: Coherence and stochastic dominance. IME Conference, Lisbon.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, 55(1):95–115. https://doi.org/10.2307/1911158.

Yin, C. and Zhu, D. (2016). New class of distortion risk measures and their tail asymptotics with emphasis on VaR. *arXiv:1503.08586v2*.

Zhu, L. and Li, H. (2012). Tail distortion risk and its asymptotic analysis. *Insurance: Mathematics and Economics*, 51(1):115–121. https://doi.org/110.1016/j.insmatheco.2012.03.010.