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Time Consistent Pareto Solutions in Common Access Resource Games with Asymmetric Players

Albert de-Paz

Jesús Marín-Solano

Jorge Navas

Adreça correspondència:
Departament de Matemàtica econòmica, financera i actuarial
Facultat d'Economia i Empresa
Universitat de Barcelona
Avinguda Diagonal 690
08034 Barcelona (Spain)
Phone: 0034934029028
Email: jnavas@ub.edu

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Abstract

In the analysis of equilibrium policies in a differential game, if agents have different time preference rates, the cooperative (Pareto optimum) solution obtained by applying the Pontryagin's Maximum Principle becomes time inconsistent. In this work we derive a set of dynamic programming equations (in discrete and continuous time) whose solutions are time consistent equilibrium rules for N-player cooperative differential games in which agents differ in their instantaneous utility functions and also in their discount rates of time preference. The results are applied to the study of a cake-eating problem describing the management of a common property exhaustible natural resource. The extension of the results to a simple common property renewable natural resource model in infinite horizon is also discussed.

Abstract

En l'anàlisi de les polítiques d'equilibri d'un joc diferencial quan els agents decisors presenten taxes de preferència temporal diferents, la solució cooperativa (òptim de Pareto) obtinguda a partir del Principi del Màxim de Pontriaguin esdevé inconsistent temporalment. En aquest treball derivem un conjunt d'equacions de programació dinàmica (tant en temps discret com continu) per a les quals les seves solucions constitueixen regles d'equilibri consistents temporalment de jocs diferencials amb N jugadors i a on els agents decisors difereixen tant en les seves utilitats instantànies com en els tipus instantanis de preferència temporal. Els resultats obtinguts s'il·lustren estudiant un problema del tipus *cake-eating* on es descriu la gestió d'un recurs natural exhaurible de propietat comuna, estenent posteriorment el model al cas d'horitzó infinit.

Keywords: Cooperative solutions; differential games; heterogeneous discount rates; time-inconsistency; resource games; asymmetric players

JEL codes: C73, C71, C61, Q30, Q20

1 Introduction

Differential games study systems that evolve in continuous time and where the system dynamics can be described by differential equations, becoming an useful tool in order to analyze both non-cooperative and cooperative behavior between agents interacting over time. When players can communicate and coordinate their strategies in order to optimize their collective pay-off cooperative solutions are introduced. Although the natural framework for most of economic problems is to assume that the agents compete among them, in some models (for instance, related to environmental and resource economics) it is natural to look for mechanisms inducing the cooperation between economic agents. For example, the computation of cooperative solutions is useful in the study of coalition formation in the management of common property resources. Two recent references on the study of stability of coalitions are De Zeew (2008) (for the case of myopic stability) and Breton and Keoula (2010a) for the alternative assumption of farsightedness. In the cooperative case, when studying the Pareto solution in a differential game, some usual assumptions are introduced. For instance, if players are symmetric, it is customary to look for symmetric strategies with the simplest possible structure in order to obtain solutions in closed form. Another standard assumption is to consider that players have the same rate of time preference.

In the problem of extraction of exhaustible resources under common access (we refer to Van Long (2011) for a recent survey on dynamic games in the economics of natural resources), in a non-cooperative setting, feedback Nash equilibrium can be found for a class of problems when the utility of each agent depends only on her own extraction rates and all the players use the same discount rate of time preference (Clemhout and Wan (1985), Van Long and Shimomura (1998)). The non-cooperative problem with different power utility functions and different discount rates for each player was studied in Van Long et al (1999). It was proved that there exists a unique feedback equilibrium in linear strategies, although there is, in general, a continuum of feedback equilibria in non-linear strategies. With respect to the Pareto solution in the cooperative framework, if there is a unique discount rate for all agents, it is easily obtained by solving a standard optimal control problem. However, in the case of different discount rates, when looking for for time consistent cooperative solutions, standard dynamic optimization techniques fail.

In recent years papers departing from the standard discounting have received an increasing attention. Strotz (1956) called attention about the problem of time inconsistency when non-constant discount rates of time preference are introduced in a dynamic decision problem. More recently, another source of time inconsistency has been studied related to the case of a problem with heterogeneous discounting, referring to problems where the decision maker discounts differently instantaneous utilities and final gains. Equilibrium conditions for time consistent solutions have been derived for both kind of problems in a continuous time setting. With respect to models with non-constant discounting (or hyperbolic preferences), Karp (2007) introduced a dynamic programming approach in an infinite horizon setting. Later on, Marín-Solano and Navas (2009) extended the problem studied in Karp (2007) to the case of finite time horizon and free terminal time. For the problem with heterogeneous discounting, Marín-Solano and Patxot (2011) derived the corresponding dynamic programming equation.

Despite of the fact that non-standard discounting models have focused on individual agents, this framework has proved to be useful in the study of cooperative solutions for some standard discounting differential games. More precisely, if the players share the same instantaneous utility function (there is a representative agent) but have different rates of time preference, say $r_1 \neq \cdots \neq r_N$, the cooperative problem can be rewritten as a non-constant discounting problem. As a consequence, previous results in the literature can be applied in order to obtain a time consistent (subgame perfect) solution (see Remark 2 in Karp (2007)) as follows. Let us consider a N-player differential game where, as usual, the joint coalition maximizes the weighted sum of their respective pay-offs,

$$J\left(c(\cdot)\right) = \sum_{m=1}^{N} \alpha_m J^m \ ,$$

where

$$J^m = \int_0^T e^{-r_m s} U^m \left(x(s), c(s), s\right) \, ds$$

represents the individual pay-off of player m, $x(t)$ and $c(t)$ are the vector of state and control variables, respectively, and α_m accounts for the weight of player m in the coalition (strength bargaining), being $\alpha_m \geq 0$ and $\sum_{m=1}^{N} \alpha_m = 1$. Without loss of generality, we can assume that agents have equal weights and, normalizing, we can write $\alpha_m = 1$, for every $m = 1, \ldots, N$. Thus, the joint payoff is

$$J\left(c(\cdot)\right) = \sum_{m=1}^{N} \int_t^T e^{-r_m(s-t)} U^m \left(x(s), c(s), s\right) \, ds.$$

If all the agents share the same (joint) utility function $U(x, c, s)$, the joint payoff can be rewritten as

$$J\left(c(\cdot)\right) = \int_t^T \theta(s-t) U\left(x(s), c(s), s\right) ds \,,$$

where $\theta(s-t) = \sum_{m=1}^{N} e^{-r_m(s-t)}$ is the discount function, which can be also rewritten as

$$\theta(s-t) = e^{-\int_t^s \bar{r}(\tau - t) d\tau}$$

where the instantaneous time preference rate $\bar{r}(\cdot)$ is a non-constant function of its argument,

$$\bar{r}(\tau) = -\frac{\theta'(\tau)}{\theta(\tau)} = \frac{\sum_{m=1}^{N} r_m e^{-r_m \tau}}{\sum_{m=1}^{N} e^{-r_m \tau}}.$$

This case, as mentioned above, can be solved as a non-constant discounting problem, where the time consistent equilibrium is characterized as the solution to a functional equation.

Our main contribution in this paper is to provide a way in order to obtain time consistent cooperative solutions for N-person differential games with asymmetric players, i.e., we face the more general problem that consists in maximizing

$$J\left(c(\cdot)\right) = \sum_{m=1}^{N} \int_t^T e^{-r_m(s-t)} U^m\left(x(s), c(s), s\right) ds, \tag{1}$$

subject to

$$\dot{x}(s) = f(x(s), c(s), s), \qquad x(t) = x_t. \tag{2}$$

Hence, we focus on the case when agents exhibit different instantaneous pay-off functions and different (but constant) rates of time preference. This more general problem cannot be transformed into a problem with non-constant discounting. With this goal, we introduce two alternative approaches in order to find time consistent equilibria. Non-constant discounting models are typically very difficult to solve since the dynamic programming equation is not a standard (partial) differential equation. On the contrary, our approach to the problem with heterogeneous agents (and to problems with heterogeneous discounting) is more computationally tractable. In the first approach we transform a two-person cooperative differential game into a one-agent problem with heterogeneous discounting. Then, we can extend the idea in Karp (2007), Remark 2 (for cooperative solutions to two-person differential games with a representative agent and heterogeneous discounting) to our equivalent problem with one agent and heterogeneous discounting. As a result, we obtain two coupled dynamic programming equations which are more analytically tractable compared with the original problem. Our second approach allows us for studying problems with an arbitrary number of players. While our approach is mainly methodological, we will illustrate the effect of using different discount rates by solving an exhaustible resource extraction model with common access (see e.g. Clemhout and Wan (1985) and Dockner et al (2000)). We will prove that, for this problem, if all the agents use the same parameter σ in their utility function, $U^i(c_i) = \dfrac{c_i^{1-\sigma} - 1}{1-\sigma}$, the extraction rates of all agents in the time-consistent solution coincide. A similar result has been recently obtained in a discrete time setting in a fisheries model in the limit $\sigma = 1$ for a logarithmic utility function (see Breton and Keoula (2010b)). Next we extend our results to a basic common property renewable natural resource model (see e.g. Clark (1990)).

The paper is organized as follows. Section 2 introduces the problem of time inconsistency trough a nonrenewable resource extraction problem in finite planning horizon. In Section 3, we study the cooperative problem for the two-player case. We transform the problem into a heterogeneous discounting model and we derive the corresponding dynamic programming equation for this problem. The procedure is illustrated by solving the nonrenewable resource extraction model. In Section 4, we extend the two-player case to the N-player case, and obtain equilibrium conditions for the cooperative time consistent solutions. We illustrate the main result by solving a more general exhaustible resource extraction model. The extension to the infinite time horizon setting is studied in Section

5, and we extend previous results to a common access renewable resource model. Finally, Section 6 presents a resume of the main conclusions of the paper.

2 Cooperative solution for an exhaustible resource extraction model

In this section we will motivate the problem by analyzing the cooperative solution for a simple model of a common-property nonrenewable resource with two agents, $N = 2$, interacting over a finite time horizon T. Let $x(t)$ and $c_m(t)$, $m = 1, 2$, denote respectively the stock of the resource and player m's rate extraction at time t, while the evolution of the system follows

$$\dot{x}(t) = -c_1(t) - c_2(t), \quad x(0) = x_0, \quad x(T) = 0. \tag{3}$$

Each player m has an increasing and concave utility function $U^m(c_m)$. For our illustrative purposes, we assume that the utility function is of the logarithmic type, i.e., $U^m(c_m) = \ln(c_m)$, which is discounted at a constant time preference rate $r_m > 0$, with $r_1 \neq r_2$. In order to obtain the cooperative solution we assume that the two players have equal weight in the joint maximization problem, so that the objective for the coalitions is to maximize the sum of the integrals of discounted payoffs over the time interval $[0, T]$. The resulting optimal control problem consists in maximizing

$$\int_0^T \ln(c_1(s))\, e^{-r_1 s} ds + \int_0^T \ln(c_2(s))\, e^{-r_2 s} ds \tag{4}$$

subject to (3). If we solve problem (4) subject to (3) by means of Pontryagin's Maximum Principle we obtain that the optimal extraction rate at time $t = 0$ is given by

$$c_m^0(t) = \frac{e^{-r_m t}}{\sum_{i=1}^2 \frac{1-e^{-r_i T}}{r_i}} x_0 = \frac{e^{-r_m t}}{\sum_{i=1}^2 \frac{e^{-r_i t} - e^{-r_i T}}{r_i}} x_t \ , \tag{5}$$

where the superscript 0 in c_m^0 accounts for the moment at which the decision has been calculated. Now, consider that the coalition re-optimizes at some instant $t \in (0, T]$. The maximum of

$$\int_t^T \ln(c_1(s))\, e^{-r_1(s-t)} ds + \int_t^T \ln(c_2(s))\, e^{-r_2(s-t)} ds, \tag{6}$$

subject to

$$\dot{x}(s) = -c_1(s) - c_2(s), \quad x(t) = x_t \tag{7}$$

is given by

$$c_m^t(s) = \frac{e^{-r_i(s-t)}}{\sum_{i=1}^2 \frac{1-e^{-r_i(T-t)}}{r_i}} x_t \tag{8}$$

for $s \in [t,T]$. However, note that the solution $c_m^t(s)$ differs from that calculated from expression (5):

$$c_m^0(s) = \frac{e^{-r_m s}}{\sum_{i=1}^2 \frac{e^{-r_i s} - e^{-r_i T}}{r_i}} x_s \neq c_m^t(s) = \frac{e^{-r_m (s-t)}}{\sum_{i=1}^2 \frac{1-e^{-r_i (T-t)}}{r_i}} x_t \ ,$$

where x_t is calculated from (3). For instance, for $s = t$,

$$c_m^t(t) = \frac{1}{\sum_{i=1}^2 \frac{1-e^{-r_i (T-t)}}{r_i}} x_t \ ,$$

hence $c_1^t(t) = c_2^t(t)$, whereas $c_1^0(t) \neq c_2^0(t)$ for every $t > 0$. Thus, we have that the joint solution becomes time inconsistent as long as the coalition has the possibility of re-optimizing at any instant after $t = 0$. This can be explained because the two players have a different discount rate of time preference.[1]

This case is related in some way to the time inconsistent problem that arises in non-constant discount problems, first noted by Strotz (1956). Here, as long as we obtain a different solution at every moment t, we can also apply the standard terminology used in non-constant discounting models. Thus, we call a coalition taking decisions at time t the t-coalition, for every $t \in [0,T]$. Moreover, we can also define the usual three solutions concepts for non-constant discounting models - precommitment, naive and sophisticated solutions- according to the degree of sophistication or compromise of the coalition. For our particular problem, the precommitment solution is the optimal solution from the viewpoint of the 0-coalition, $c^P(t) = c^0(t)$, and can be associated with the existence of some binding agreement between the players at the beginning of the game, in the sense that both agents will follow the decision rule taken at time $t = 0$ despite they have incentives to deviate from the previously calculated cooperative decision rule. Somehow, we can see the precommitment solution as an open-loop solution in a sequencial game with a continuum of players (the t-coalitions, for $t \in [0,T]$). However, if such an agreement does not exist and players in the coalition can continuously re-calculate the cooperative solution, they will follow what we call the naive decision rule $c_m^N(t)$. Note that a coalition taking a decision at time t will choose the decision rule given by expression (8). However, at time $t' > t$ the coalition will re-compute the decision rule. Hence, $c_m^t(s)$ in (8) is followed only at the time $s = t$ at which the agents of the t-coalition have calculated the extraction rate, so that the actual extraction rate becomes

$$c_m^N(t) = c_m^t(t) = \frac{1}{\sum_{i=1}^2 \frac{1-e^{-r_i (T-t)}}{r_i}} x_t. \tag{9}$$

It is important to realize that the precommitment and naive solutions do not coincide unless $r_1 = r_2$. In fact, $c_1^P(t) \neq c_2^P(t)$, for every $t \in (0,T]$, whereas $c_1^N(t) = c_2^N(t)$ for every $t \in [0,T]$. The pre-commitment and naive solutions coincide just at the initial time

[1] For $r_1 = r_2 = r$ both solutions coincide for every instant t and the solution is time consistent.

$t = 0$. If the agents can split the resource at time $t = 0$ in an irreversible way so that $x_0 = x_0^1 + x_0^2$ where $x_0^i = \int_0^T c^i(t)\, dt$, $i = 1, 2$, then the precommitment solution becomes time-consistent. In general, naive agents are time-inconsistent for every $t \in [0, T)$.

The third solution concept is the sophisticated solution. In exactly the same spirit as in the case of hyperbolic preferences, the agents in the coalition recognize that they will be unable to precommit their future behavior beyond the next instant they decide and adopt a strategy of consistent planning by restricting their present behavior to their "optimal" (in fact, equilibrium) future joint behavior. As pointed out in Karp (2007): "... the problem is equivalent to a game amongst a succession of planners, each of whom chooses the control variable for a short (infinitesimal) amount of time".

In order to derive a time consistent solution for sophisticated agents, we need first to define what is understood by an equilibrium rule. In this paper we will follow the constructive and intuitive approach introduced by Karp (2007) for the problem with non-constant discounting (the discount function is of the form $\theta(s - t)$), consisting in looking first for the dynamic programming equation for a discretized version of the problem, and taking next the continuous time limit, provided that such a limit exists. In the derivation of the dynamic programming equation the decision rule is obtained by looking for the subgame perfect equilibria of the sequential game obtained when each t-coalition (i.e. the coalition taking a decision at time t) is considered as a different player. According to this approach, in the following we will address the problem of characterizing the time consistent solution to the cooperative dynamic game with asymmetric players in two different ways. The first one consists in transforming the problem, for the two-player case, in the problem with heterogeneous discounting introduced in Marín-Solano and Patxot (2011) in order to account for one decision maker who values in a different way two kind of goods. In this case, we will focus on the two-player case in problems with a finite horizon. The second approach faces the problem directly, by characterizing the time consistent cooperative equilibrium from the original problem, and can be applied for general N-player differential games. As should be expected, both approaches are equivalent for the case $N = 2$.

3 Heterogeneous discounting and time consistent solution for two-player cooperative differential games with asymmetric agents

Heterogeneous discounting problems were introduced in Marín-Solano and Patxot (2011) in order to study problems where the agent discounts in a different way the utilities enjoyed along the planning horizon (typically due to consumption in most of economic problems) and the final function (which has normally a different nature), i.e., the decision maker faces the problem of maximizing

$$\int_t^T e^{-r_1(s-t)} U(x(s), c(s), s)\, ds + e^{-r_2(T-t)} F(x(T), T) \tag{10}$$

subject to

$$\dot{x}(s) = f(x(s), u(s), s) \ , \ x(t) = x_t \ . \tag{11}$$

We refer to Marín-Solano and Patxot (2011) for an economic motivation of Problem 10-11, as well as a discussion on the time-inconsistency of these time preferences.

Next, for the two-player case, $N = 2$, we connect our cooperative problem with an heterogeneous discounting problem. In order to do that, we take one of the players' functional objective in (1) originally stated as a Lagrange problem, i.e., with only integral form, in an equivalent problem with only a terminal value term (Mayer form), in such a way that Problem 1-2 for the t-coalition becomes equivalent to maximizing

$$\int_t^T e^{-r_1(s-t)} U^1 \left(x(s), c_1(s), c_2(s), s\right) ds + e^{-r_2(T-t)} y(T)$$

subject to:

$$\begin{aligned} \dot{x}(s) &= f(x(s), c_1(s), c_2(s), s) \ , \\ \dot{y}(s) &= r_2 y(s) + U^2 \left(x(s), c_1(s), c_2(s), s\right) \ . \end{aligned}$$

With the addition of a new state variable y, we have rewritten the functional objective for one of the players in the Mayer form, and therefore, we have transformed the cooperative problem with asymmetric players into a Bolza problem for just one agent with integral and terminal value terms, but with different time preferences rates.

Although time-consistent equilibrium conditions for problems with heterogeneous discounting were already derived in Marín-Solano and Patxot (2011) following a variational approach, here we provide and alternative derivation to their main theorem, by obtaining the dynamic programming equation equation for a discretized version of Problem 10-11 and passing then to the continuous time limit, following a procedure similar to the one used in Karp (2007) or in Marín-Solano and Navas (2009) for non-constant/hyperbolic discounting models. In addition, we will propose a very simple method transforming the functional dynamic programing equation into a system of two partial differential equations, facilitating in this way the solution to the problem.

3.1 A set of dynamic programming equations

For Problem 10-11 let us assume the usual regularity conditions, i.e., functions L, F and f^i are continuously differentiable in all their arguments. Next, let us divide the interval $[0, T]$ into n periods of constant length $\epsilon = T/n$, in such a way that we identify $ds = \epsilon$, and $s = j\epsilon$, for $j = 0, 1, \ldots, n$. Then equation (11) becomes $x(s+\epsilon) - x(s) = f(x(s), c(s), s)\epsilon$. By denoting by $x(j\epsilon) = x_j$ and $c(k\epsilon) = c_k$ $(j, k = 0, \ldots, n-1)$, the objective of the agent in period $t = j\epsilon$ will be to maximize

$$V_j = \sum_{i=0}^{n-j-1} e^{-r_1(i\epsilon)} U(x_{(i+j)}, c_{(i+j)}, (i+j)\epsilon)\epsilon + e^{-r_2(n-j)\epsilon} F(x_n, n\epsilon) \tag{12}$$

subject to

$$x_{i+1} = x_i + f(x_i, c_i, i\epsilon)\epsilon \ , \ \ i = j, \ldots, n-1 \ , \ \ x_j \text{ given} \ , \qquad (13)$$

provided that future j' agents choose their best response actions. Let us state the dynamic programming algorithm for the discrete Problem 12)-(13. In the final period we define $V_n^* = F(x(T), T)$, as usual. For $j = n-1$, the optimal value for (12) will be given by the solution to the problem

$$V_{(n-1)}^*(x_{(n-1)}, (n-1)\epsilon) = \max_{\{c_{(n-1)}\}} \left\{ U(x_{(n-1)}, c_{(n-1)}, (n-1)\epsilon)\epsilon + e^{-r_2\epsilon} V_n^*(x_n, n\epsilon) \right\}$$

with $x_n = x_{(n-1)} + f(x_{(n-1)}, c_{(n-1)}, (n-1)\epsilon)\epsilon$. If $c_{(n-1)}^*(x_{(n-1)}, (n-1)\epsilon)$ is the maximizer of the right hand term of the above equation, let us denote

$$\bar{U}_{(n-1)}(x_{(n-1)}, (n-1)\epsilon) = U(x_{(n-1)}, c_{(n-1)}^*(x_{(n-1)}, (n-1)\epsilon), (n-1)\epsilon) \ .$$

In general, for $j = 1, \ldots, n-1$, the value $V_j^*(x_j, j\epsilon)$ in (12) can be written as

$$V_j^* = \max_{\{c_j\}} \left\{ U(x_j, c_j, j\epsilon)\epsilon + \sum_{k=1}^{n-j-1} e^{-r_1 k\epsilon} \bar{U}_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon + e^{-r_2(n-j)\epsilon} V_n^* \right\} \qquad (14)$$

with $x_{(j+1)} = x_j + f(x_j, c_j, j\epsilon)\epsilon$. Since

$$V_{j+1}^*(x_{(j+1)}, (j+1)\epsilon) = \sum_{i=0}^{n-j-2} e^{-r_1 i\epsilon} H_{(j+i+1)}(x_{(j+i+1)}, (j+i+1)\epsilon)\epsilon + e^{-r_2(n-j-1)\epsilon} V_n^* \ , \qquad (15)$$

then solving for $V_n^*(x_n)$ in (15) and substituting in (14) we obtain

$$\begin{aligned} V_j^*(x_j, j\epsilon) = \max_{\{c_j\}} \{ & U(x_j, c_j, j\epsilon)\epsilon \\ & + \sum_{k=1}^{n-j-1} e^{-r_1 k\epsilon} \left(1 - e^{-(r_2 - r_1)\epsilon}\right) \bar{U}_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon \\ & + e^{-r_2\epsilon} V_{j+1}^*(x_{(j+1)}, (j+1)\epsilon) \} \ , \end{aligned} \qquad (16)$$

with $x_{(j+1)} = x_j + f(x_j, u_j, j\epsilon)\epsilon, \quad , j = 0, \ldots, n-1$, and $V_n^* = F(x_n, n\epsilon)$.

For the continuous time case, we take the following definition.

Definition 1 *We define the value function for the Problem 10-11 as the solution to the dynamic programming equation obtained by taking the formal continuous time limit when* $\epsilon \to 0$ *of the dynamic programming equation (16) obtained for the discrete approximation to the problem, assuming that such a limit exists and that the solution is of class* C^1 *in all their arguments.*

In order to obtain the dynamic programming equation for the problem with heterogeneous discounting, let $W(x,t)$ represent the value function of the t-agent, with initial condition $x(t)=x$. We assume that $W(x,t)$ is continuously differentiable in all its arguments. Since $s=j\epsilon$ and $x(s+\epsilon)-x(s)=f(x(s),c(s),s)\epsilon$, then $W(x(t),t)=V_j(x_j,j\epsilon)$ and

$$W(x(t+\epsilon),t+\epsilon)=W(x(t),t)+\nabla_x W(x(t),t)\cdot f(x(t),c(t),t)\epsilon+\nabla_t W(x(t),t)\epsilon+o(\epsilon)\,,$$

where $\lim_{\epsilon\to 0} o(\epsilon)/\epsilon=0$. Substituting in (16), since $e^{-r_2\epsilon}=1-r_2\epsilon+o(\epsilon)$ we obtain

$$\begin{aligned} W(x(t),t)=\max_{\{c_t\}}\{&U(x(t),c(t),t)\epsilon+W(x(t),t)\\ &+\nabla_x W(x(t),t)\cdot f(x(t),c(t),t)\epsilon+\nabla_t W(x(t),t)\epsilon\\ &-r_2\epsilon W(x(t),t)-K(x(t),t)+o(\epsilon)\}\,, \end{aligned} \tag{17}$$

where

$$K(x(t),t)=-\sum_{k=1}^{n-j-1} e^{-r_1 k\epsilon}\left(1-e^{-(r_2-r_1)\epsilon}\right)\bar{U}_{j+k}(x_{(j+k)},(j+k)\epsilon)\epsilon. \tag{18}$$

Finally, since $\left(1-e^{-(r_2-r_1)\epsilon}\right)=(r_2-r_1)\epsilon+o(\epsilon)$, by dividing (17) and (18) by ϵ and taking the limit $\epsilon\to 0$, we obtain the following result:

Proposition 1 *Let $W(x,t)$ be a continuously differentiable function in (x,t) satisfying the dynamic programming equation*

$$r_2 W(x,t)+K(x,t)-\nabla_t W(x,t)=\max_{\{c\}}\{U(x,c,t)+\nabla_x W(x,t)\cdot f(x,c,t)\}\ , \tag{19}$$

with

$$W(x,T)=F(x,T)\ , \tag{20}$$

and

$$K(x,t)=(r_1-r_2)\int_t^T e^{-r_1(s-t)}\bar{U}(x,s)ds\ .$$

Then $W(x,t)$ is the value function for the Problem 10-11. If, for each pair (x,t), there exists $c^*=\phi(x,t)$, with corresponding state trajectory $x^*(t)$ (which is the solution to $\dot{x}(s)=f(x,\phi(x,s),s)$ with initial condition $x(t)=x$), such that c^* maximizes the right hand side term of (19), then $c^*=\phi(x,t)$ is called a Markov equilibrium rule for the problem with heterogeneous discounting.

Next, note that $\bar{U}(x,s)=U(x(s),\phi(x(s),s),s)$, where $x(s)$ is the solution to $\dot{x}(s)=f(x,\phi(x,s),s)$ with $x(t)=x$. Hence,

$$K(x,t)=(r_1-r_2)\int_t^T e^{-r_1(s-t)}U(x(s),\phi(x(s),s),s)\,ds \tag{21}$$

and, by differentiating K in (21) with respect to t we obtain the "auxiliary dynamic programming equation"

$$r_1 K(x,t) - \nabla_t K(x,t) = (r_1 - r_2) U(x, \phi(x,t), t) + \nabla_x K(x,t) \cdot f(x, \phi(x,t), t) \; . \qquad (22)$$

Hence we have

Corollary 1 *Let $W(x,t)$ be a continuously differentiable function in (x,t) satisfying the set of two dynamic programming equations (19) and (22) with boundary conditions $W(x,T) = F(x,T)$, $K(x,T) = 0$. Then $W(x,t)$ is the value function for Problem 10-11 and the corresponding solution $c^* = \phi(x,t)$ maximizing the right hand side term of (19) is a Markov equilibrium rule for the problem with heterogeneous discounting.*

3.2 The exhaustible resource model under common access: the case of two-asymmetric players

Let us consider again Problem 6-7. In order to look for a time-consistent equilibrium (i.e. the sophisticated solution), we reformulate first the problem as shown before by rewriting the second integral in (6) (the payoff of player 2) in the Mayer form. Working in this way, the objective functional becomes

$$\max_{\{c_1, c_2\}} \left\{ \int_t^T e^{-r_1(s-t)} \ln\left(c_1(s)\right) ds + e^{-r_2(T-t)} y(T) \right\}$$

subject to

$$\begin{aligned} \dot{x}(s) &= -c_1(s) - c_2(s) \; , \\ \dot{y}(s) &= r_2 y(s) + \ln\left(c_2(s)\right) \; . \end{aligned} \qquad (23)$$

Once we have transformed our original two-player cooperative problem for the grand coalition into a one-decision maker problem with heterogeneous discounting, we apply Proposition 1. We look for the solution of the DPE (19) subject to (20), i.e.,

$$\begin{aligned} & r_2 W(x,y,t) + K(x,y,t) - W_t(x,y,t) \\ & \quad = \max_{\{c_1,c_2\}} \left\{ \ln c_1 + W_x(x,y,t)(-c_1 - c_2) + W_y(x,y,t)\left(r_2 y + \ln(c_2)\right) \right\} , \end{aligned} \qquad (24)$$

where

$$K(x,y,t) = (r_1 - r_2) \int_t^T e^{-r_1(s-t)} \ln(c_1^*, s) ds.$$

We now guess for a value function of the form $W(x,y,t) = A(t)\ln(x) + B(t)y + C(t)$. If this choice proves to be consistent, then the decision rules for both agents in the coalition are given by $c_1(t) = 1/W_x = x/A(t)$ and $c_2(t) = W_y/W_x = B(t)x/A(t)$. Next, in order to solve (24) we need first to calculate the expression for $K(x,t)$. To do that, we solve (23) after substituting our "guessed" controls and obtain that $x(s) = x_t \exp\left(\Lambda_t(s)\right)$, with

$$\Lambda_t(s) = -\int_t^s \frac{1 + B(\tau)}{A(\tau)} d\tau$$

and, therefore,

$$K(x,y,t) = (r_1 - r_2)\int_t^T e^{-r_1(s-t)} \ln\left(\frac{x_t e^{\Lambda_t(s)}}{A(s)}\right) ds =$$

$$= \frac{r_1 - r_2}{r_1}\left(1 - e^{-r_1(T-t)}\right)\ln(x_t) + (r_1 - r_2)\int_t^T e^{-r_1(s-t)} \ln\left(\frac{e^{\Lambda_t(s)}}{A(s)}\right) ds \ .$$

By substituting in (24) and simplifying we obtain

$$\begin{aligned} r_2\left[A(t)\ln(x) + B(t)y + C(t)\right] - \left[A'(t)\ln(x) + B'(t)y + C'(t)\right] \\ + \frac{r_1 - r_2}{r_1}\left(1 - e^{-r_1(T-t)}\right)\ln(x) + (r_1 - r_2)\int_t^T e^{-r_1(s-t)} \ln\left(\frac{e^{\Lambda_t(s)}}{A(s)}\right) ds \\ = \ln(x) - \ln\left(A(t)\right) - 1 - B(t) + B(t)\left(r_2 y + \ln(x) + \ln\left(\frac{B(t)}{A(t)}\right)\right) . \end{aligned}$$

Since the equation above must be satisfied for every x and y, we must solve the differential equations system

$$r_2 A(t) - A'(t) + \frac{r_1 - r_2}{r_1}\left(1 - e^{-r_1(T-t)}\right) = 1 + B(t) \ , \quad B'(t) = 0 \ . \tag{25}$$

Since $B(T) = 1$, hence $B(t) = 1$ and $c_1(t) = c_2(t) = x/A(t)$, for every $t \in [0,T]$. With respect to $A(t)$ note that, if

$$A(t) = \sum_{i=1}^{2} \frac{1 - e^{-r_i(T-t)}}{r_i} \ ,$$

which describes the solution for a naive coalition (see expression (9)), then equation (25) is satisfied and, in addition, the solution to the state equation $\dot{x}(t) = -2x(t)/A(t)$, $x(0) = x_0$, satisfies the terminal condition $\lim_{t\to T} x(t) = 0$. Therefore, the solution obtained for the naive coalition is a time consistent policy. This feature, also arising in non-constant discounting models, is consequence of using logarithmic utility functions, and it does no longer hold when more general utility functions are considered. This fact often leads to the main limitation of Proposition 1 as a constructive procedure. Finding naive solutions is clearly easier than finding time consistent solutions. For these cases Corollary 1 provides an easier way to solve the problem. Next we solve the model for a general isoelastic utility function. Consider the problem of maximizing

$$\int_t^T \left(e^{-r_1 s} U^1(c_1) + e^{-r_2 s} U^2(c_2)\right) ds \tag{26}$$

subject to (3), where

$$U^i(c_i) = \frac{c_i^{1-\sigma} - 1}{1 - \sigma} \ ,$$

for $i = 1, 2$. If $\gamma_i = \frac{r_i}{\sigma}$, we can write the optimal solution from the viewpoint of the 0-agent (the precommitment solution) in feedback form as

$$c_i^P(t) = \frac{e^{-\gamma_i t}}{\sum_{j=1}^2 \frac{1}{\gamma_j}(e^{-\gamma_j t} - e^{-\gamma_j T})} x(t) \ .$$

If we compute the optimal solution from the viewpoint of an arbitrary t-agent, for $t \in [0, T)$ we obtain

$$c_i^t(s) = \frac{e^{-\gamma_i (s-t)}}{\sum_{j=1}^2 \frac{1}{\gamma_j}(1 - e^{-\gamma_j (T-t)})} x(t) \ .$$

The (time-inconsistent) naive solution is obtained taking $s = t$ in the above equation so that

$$c_i^N(t) = \frac{1}{\sum_{j=1}^2 \frac{1}{\gamma_i}(1 - e^{-\gamma_i (T-t)})} x(t) \ .$$

Note that the precommitment and naive solutions coincide if, and only if, $r_1 = r_2$.

For the calculation of the sophisticated solution we transform Problem 26 subject to (3) into the equivalent one-player problem

$$\max_{\{c_1, c_2\}} \left\{ \int_t^T e^{-r_1(s-t)} \frac{c_1^{1-\sigma} - 1}{1-\sigma} ds + e^{-r_2(T-t)} y(T) \right\}$$

subject to

$$\dot{x}(s) = -c_1(s) - c_2(s) \ ,$$

$$\dot{y}(s) = r_2 y(s) + \frac{c_2^{1-\sigma} - 1}{1-\sigma} \ .$$

¿From Corollary 1 we have to solve the set of two partial differential equations

$$\begin{aligned} & r_2 W(x,y,t) + K(x,y,t) - W_t(x,y,t) \\ & \quad = \max_{\{c_1, c_2\}} \left\{ \frac{c_1^{1-\sigma} - 1}{1-\sigma} + W_x(x,y,t)(-c_1 - c_2) + W_y(x,y,t) \left(r_2 y + \frac{c_2^{1-\sigma} - 1}{1-\sigma} \right) \right\} \ , \end{aligned} \quad (27)$$

$$\begin{aligned} & r_1 K(x,y,t) - K_t(x,y,t) \\ & = (r_1 - r_2) \frac{(c_1^*)^{1-\sigma} - 1}{1-\sigma} + K_x(x,y,t)(-c_1^* - c_2^*) + K_y(x,y,t) \left(r_2 y + \frac{(c_2^*)^{1-\sigma} - 1}{1-\sigma} \right) \ , \end{aligned} \quad (28)$$

where c_1^*, c_2^* in (28) are the maximizers to the right hand term in (27). Note that $(c_1^*)^{-\sigma} = W_x$ and $(c_2^*)^{-\sigma} = W_x / W_y$, hence $c_1^* = c_2^*$ if, and only if, $W_y = 1$. In fact, it is easy to prove that a solution exists with $W_y = 1$ (and $K_y = 0$) so that the extraction rules for the two agents coincide for every σ. It can be shown that, unless $\sigma = 1$ (the log-utility case), the naive and sophisticated solutions do not coincide in general. We will study this model in more detail in the following section, for a general N-player cooperative differential game.

4 Time consistent cooperative solutions with N asymmetric players

In this section we extend the two-player case analyzed above. Let us consider the case of N players who decide to form a coalition seeking for a time consistent solution maximizing

$$J(c(\cdot)) = \sum_{m=1}^{N} \int_t^T e^{-r_m(s-t)} U^m(x(s), c(s), s)\, ds \tag{29}$$

subject to

$$\dot{x}(s) = f(x, c, s) \text{ with } x(t) = x_t. \tag{30}$$

4.1 Dynamic programming equation

By proceeding in a similar way as in Section 3, we can discretize (29). The corresponding problem in discrete time is

$$\max_{\{c_1,\ldots,c_n\}} V_j = \sum_{m=1}^{N} V_j^m = \sum_{i=0}^{n-j-1} \sum_{m=1}^{N} e^{-r_m(i\epsilon)} U^m(x_{(i+j)}, c_{(i+j)}, (i+j)\epsilon)\epsilon \tag{31}$$

subject to

$$x_{i+1} = x_i + f(x_i, c_i, i\epsilon)\epsilon\ ,\ \ i = j, \ldots, n-1\ ,\ \ x_j \text{ given}\ . \tag{32}$$

Let us state the dynamic programming algorithm for the discrete time Problem 31-32. In the final period we define $V_n^* = 0$, as usual[2]. For $j = n-1$, the optimal value for (31) will be given by the solution to the problem

$$V^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon) = \max_{\{c_{(n-1)}\}} \left\{ \sum_{m=1}^{N} U^m(x_{(n-1)}, c_{(n-1)}, (n-1)\epsilon)\epsilon \right\}$$

with $x_n = x_{(n-1)} + f(x_{(n-1)}, u_{(n-1)}, (n-1)\epsilon)\epsilon$. If $c^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon)$ is the maximizer of the right hand term of the above equation, let us denote

$$\bar{U}^m_{(n-1)}(x_{(n-1)}, (n-1)\epsilon) = U^m(x_{(n-1)}, c^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon), (n-1)\epsilon)\ .$$

In general, for $j = 1, \ldots, n-1$, the value $V_j^*(x_j, j\epsilon)$ in (31) can be written as

$$V_j^* = \max_{\{u_j\}} \left\{ \sum_{m=1}^{N} U^m(x_j, c_j, j\epsilon)\epsilon + \sum_{k=1}^{n-j-1} \sum_{m=1}^{N} e^{-r_m k\epsilon} \bar{U}^m_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon \right\} \tag{33}$$

[2] Here, we assume that problem is stated in the Lagrange form, but the extension to the Bolza problem is straightforward by imposing as a boundary condition $V_n^* = \sum_{m=1}^{N} F^m$, being F^m the final function for player m.

with $x_{(j+1)} = x_j + f(x_j, c_j, j\epsilon)\epsilon$. Since

$$V^*_{j+1}(x_{(j+1)}, (j+1)\epsilon) = \sum_{i=0}^{n-j-2} \sum_{m=1}^{N} e^{-r_m i\epsilon} \bar{U}^m_{(j+i+1)}(x_{(j+i+1)}, (j+i+1)\epsilon)\epsilon,$$

then we can write

$$V^*_{j+1}(x_{(j+1)}, (j+1)\epsilon) - \sum_{i=0}^{n-j-2} \sum_{m=1}^{N} e^{-r_m i\epsilon} \bar{U}^m_{(j+i+1)}(x_{(j+i+1)}, (j+i+1)\epsilon)\epsilon = 0,$$

and adding the former expression to (33) we obtain

$$\begin{aligned} V^*_j(x_j, j\epsilon) = \max_{\{c_j\}} \Bigg\{ & \sum_{m=1}^{N} U^m(x_j, c_j, j\epsilon)\epsilon \\ & + \sum_{k=1}^{n-j-1} \sum_{m=1}^{N} (1 - e^{r_m\epsilon})\, e^{-r_m k\epsilon} \bar{U}^m_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon + V^*_{j+1}(x_{(j+1)}, (j+1)\epsilon) \Bigg\}, \end{aligned} \tag{34}$$

with $x_{(j+1)} = x_j + f(x_j, c_j, j\epsilon)\epsilon, \quad , j = 0, \ldots, n-1$, and $V^*_n = 0$.

Next, as in the previous section, we obtain a dynamic programming equation for the problem with heterogeneous discounting in continuous time by taking the limit $\epsilon \to 0$ in (34).

Definition 2 *We define the value function for the Problem 29-30 as the solution to the dynamic programming equation obtained by taking the formal continuous time limit when $\epsilon \to 0$ of the dynamic programming equation (34) obtained from the discrete approximation to the problem, assuming that it exists and that the solution is of class C^1 in all their arguments.*

Let $W^m(x,t)$ represent the value function of player m in the t-coalition and $W(x,t) = \sum_{m=1}^N W^m(x,t)$ the value function for the t-coalition, with initial condition $x(t) = x$. We assume that $W(x,t)$ is continuously differentiable in all its arguments. Since $s = j\epsilon$ and $x(s+\epsilon) - x(s) = f(x(s), c(s), s)\epsilon$, then $W(x(t), t) = V_j(x_j, j\epsilon)$ and

$$W(x(t+\epsilon), t+\epsilon) = W(x(t), t) + \nabla_x W(x(t), t) \cdot f(x(t), c(t), t)\epsilon + \nabla_t W(x(t), t)\epsilon + o(\epsilon)\,.$$

where $\lim_{\epsilon\to 0} o(\epsilon)/\epsilon = 0$. Substituting in (34) we obtain

$$\begin{aligned} W(x(t), t) = \max_{\{c_t\}} \Bigg\{ & \sum_{m=1}^{N} U^m(x(t), c(t), t)\epsilon + W(x(t), t) \\ & + \nabla_x W(x(t), t) \cdot f(x(t), c(t), t)\epsilon + \nabla_t W(x(t), t)\epsilon \\ & - \sum_{m=1}^{N} (1 - e^{r_m\epsilon})\, W^m(x(t), t) + o(\epsilon) \Bigg\}, \end{aligned} \tag{35}$$

where

$$W^m(x(t),t) = -\sum_{k=1}^{n-j-1} e^{-r_m k\epsilon} \bar{U}^m_{j+k}(x_{(j+k)},(j+k)\epsilon)\epsilon. \tag{36}$$

Finally, since $(1-e^{-r_m\epsilon}) = r_m\epsilon + o(\epsilon)$, by dividing (35) and (36) by ϵ and taking the limit $\epsilon \to 0$, we obtain the following result:

Proposition 2 *Let $W^m(x,t)$, $m=1,\dots,N$, be a set of continuously differentiable functions in (x,t), satisfying the dynamic programming equation*

$$\sum_{m=1}^{N} r_m W^m(x,t) - \sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial t} = \max_c \left\{ \sum_{m=1}^{N} U^m(x,c,t) + \left(\sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial x} \right) f(x,c,t) \right\} \tag{37}$$

with

$$W^m(x,T) = 0, \quad \forall m = 1,\dots,N\ ,$$

and

$$W^m(x,t) = \int_t^T e^{-r_m(s-t)} \bar{U}(x(s),s)\, ds\ , \tag{38}$$

where, if $c^*(t) = \phi(x(t),t)$ is the maximizer of the right hand term in Equation (37), $\bar{U}(x(s),s) = U(x(s),\phi(x(s),s))$. Then $W(x,t)$ is the value function of the whole coalition, the decision rule $c^* = \phi(x,t)$ is the (time-consistent) Markov Perfect Equilibrium, and $W^m(x,t)$, for $m=1,\dots,N$, is the value function of player m in the cooperative problem (29-30).

Remark 1 *Note that, along the equilibrium rule $c^* = \phi(x,t)$, for every player m, $W^m(x,t)$ in Equation (38) is a solution to the partial differential equation*

$$r_m W^m(x,t) - \frac{\partial W^m(x,t)}{\partial t} = U^m(x,\phi(x,t),t) + \frac{\partial W^m(x,t)}{\partial x} f(x,\phi(x,t),t), \qquad m=1,\dots,N\ , \tag{39}$$

with $W^m(x,T)=0$. Hence, we can compute the value function by determining, first, the decision rule solving the right hand term in Equation (37) and substituting, next, the decision rule into the system of N partial differential equations (39).

4.2 The exhaustible resource model under common access: the case of N-asymmetric players

Let us extend the results for the nonrenewable resource model in the previous section to the general case of N asymmetric players. If preferences of agent m, for $m=1,\dots,N$, are characterized by the (instantaneous) utility function $U^m(c_m) = \dfrac{c_m^{1-\sigma_m}-1}{1-\sigma_m}$ and the discount rate of time preference r_m then, at time t, in order to look for a Pareto solution we must solve

$$\max_{\{c_1,\ldots,c_n\}} \sum_{m=1}^{N} \int_t^T e^{-r_m(s-t)} \frac{(c_m(s))^{1-\sigma_m}-1}{1-\sigma_m}\, ds \tag{40}$$

subject to

$$\dot{x}(s) = -\sum_{m=1}^{N} c_m(s), \qquad x(t)=x_t, \quad x(T)=0\,. \tag{41}$$

We can easily obtain the precommitment and naive solutions for Problem 40-41, $c_m^P(s)$ and $c_m^N(t)$, respectively, for $m=1,\ldots,N$, by proceeding as in Section 2 in order to obtain

$$c_m^P(t) = \frac{e^{-\gamma_m t}}{\sum_{i=1}^N \frac{1}{\gamma_i}\left(e^{-\gamma_i t}-e^{-\gamma_i T}\right)} x_t\,,$$

if the agent can precommit her future behavior at time $t=0$, where $\gamma_m = \frac{r_m}{\sigma_m}$, and

$$c_m^N(t) = \frac{1}{\sum_{i=1}^N \frac{1}{\gamma_i}\left(1-e^{-\gamma_i (T-t)}\right)} x_t \tag{42}$$

if the coalition is naive.

However, if the agents cannot precommit their future behavior, the former solutions are time inconsistent. In order to look for a time consistent equilibrium for the whole coalition we can apply the results in Proposition 2 and Remark 1 to Problem 40-41. From equation (37) we have to solve

$$\sum_{m=1}^{N} r_m W^m(x,t) - \sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial t} = \max_{c_1,\ldots,c_N} \left\{ \sum_{m=1}^{N} \frac{c_m(s)^{1-\sigma_m}-1}{1-\sigma_m} + \left(\sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial x}\right)\left(-\sum_{m=1}^{n} c_m(s)\right)\right\}\,. \tag{43}$$

The maximizer of the left side term in (43) is

$$c_m^S(t) = \left(\sum_{j=1}^{N} \frac{\partial W^j(x,t)}{\partial x}\right)^{-\frac{1}{\sigma_m}}\,,$$

for $m=1,\ldots,N$. Therefore, as a first result we obtain that $c_m^S = c_{m'}^S$ (i.e. the extraction rates of agents m and m' coincide) if, and only if, $\sigma_m = \sigma_{m'}$. Since the extraction rate always coincide if all the members of the coalition are naive. Therefore, if there are two players m and m' such that $\sigma_m \neq \sigma'_m$, then the naive solution is time-inconsistent. Next we will prove that the naive solution is time consistent if $\sigma_m = 1$, for every $m=1,\ldots,N$.

In order to compute the actual decision rule if the agents of the coalition are sophisticated, we can solve the family of N coupled partial differential equations (39), which in our particular case becomes

$$r_m W^m(x,t) - \frac{\partial W^m(x,t)}{\partial t} = \frac{1}{1-\sigma_m}\left[\left(\sum_{j=1}^N \frac{\partial W^j(x,t)}{\partial x}\right)^{\frac{\sigma_m-1}{\sigma_m}} - 1\right] - \frac{\partial W^m(x,t)}{\partial x}\sum_{j=1}^N\left(\sum_{i=1}^N \frac{\partial W^i(x,t)}{\partial x}\right)^{-\frac{1}{\sigma_j}},$$

for $m = 1, \ldots, N$. If $\sigma_1 = \sigma_2 = \cdots = \sigma_N = \sigma$, the partial differential equation system above simplifies to

$$r_m W^m(x,t) - \frac{\partial W^m(x,t)}{\partial t} = \frac{1}{1-\sigma}\left[\left(\sum_{j=1}^N \frac{\partial W^j(x,t)}{\partial x}\right)^{1-\frac{1}{\sigma}} - 1\right] - N\frac{\partial W^m(x,t)}{\partial x}\left(\sum_{i=1}^N \frac{\partial W^i(x,t)}{\partial x}\right)^{-\frac{1}{\sigma}}, \quad (44)$$

for $m = 1, \ldots, N$. We guess $W^m(x,t) = A^m(t)\frac{x^{1-\sigma}-1}{1-\sigma} + B^m(t)$, $m = 1, \ldots, N$, with $A^m(t) > 0$ for every $t \in [0, T)$. By substituting in (44) we find that functions $A^m(t)$ are the solution to the system of ordinary differential equations

$$\dot{A}^m - r_m A^m = N(1-\sigma)A^m\left(\sum_{j=1}^N A^j\right)^{-\frac{1}{\sigma}} - \left(\sum_{j=1}^N A^j\right)^{1-\frac{1}{\sigma}}, \quad j = 1, \ldots, N. \quad (45)$$

For instance, in the limit case $\sigma = 1$ (which corresponds to a logarithmic utility function), the system above simplifies to

$$\dot{A}^m - r_m A^m + 1 = 0\,, \quad m = 1, \ldots, N\,.$$

Note that $A^m(t) = \frac{1}{r_m}[1 - e^{-r_m(T-t)}]$, which corresponds to the naive solution, satisfies the differential equation above. Hence, the naive solution becomes time-consistent also in the general case of N asymmetric players, extending in this way the result obtained in Section 3.2. Summarizing, we have proved:

Proposition 3 *In Problem 40-41, if the coalition is time-consistent, the extraction rates of two agents coincide if, and only if, they have the same marginal elasticity σ. In particular, if $\sigma_1 = \cdots = \sigma_N = 1$, then the naive solution (42) is time-consistent.*

If $\sigma \neq 1$ note that

$$c_m^S(t) = \frac{1}{\left(\sum_{j=1}^N A^j\right)^{\frac{1}{\sigma}}} \quad (46)$$

and the state equation becomes $\dot{x} = -N \left(\sum_{j=1}^{N} A^j \right)^{-\frac{1}{\sigma}} x$, $x(0) = x_0$, hence

$$x(t) = x_0 \exp \left\{ - \int_0^t \frac{N}{\left(\sum_{j=1}^{N} A^j \right)^{\frac{1}{\sigma}}} \, dt \right\} .$$

In order to achieve the terminal condition $x(T) = 0$, from the positivity of $A^m(t)$ for $t < T$ we obtain that $\lim_{t \to T} \sum_{j=1}^{N} A^j(t) = 0$ and then $A^m(T) = 0$, for every $m = 1, \ldots, N$.

If the naive and sophisticated solutions coincide for $\sigma \neq 1$ then, from (42) and (46),

$$\sum_{j=1}^{N} A^j = \left[\sum_{j=1}^{N} \frac{\sigma}{r_j} \left(1 - e^{-\frac{r_j}{\sigma}(T-t)} \right) \right]^{\sigma} \tag{47}$$

and by substituting in (45) we obtain the set of differential equations

$$\dot{A}^m - \left[r_m + N(1-\sigma)\alpha(t) \right] A^m = -\alpha(t)^{1-\sigma} , \quad A^m(T) = 0 ,$$

for $m = 1, \ldots, N$, where $\alpha(t) = \left[\sum_{j=1}^{N} \frac{\sigma}{r_j} \left(1 - e^{-\frac{r_j}{\sigma}(T-t)} \right) \right]^{-1}$. It can be shown that the solutions $A^m(t)$, $m = 1, \ldots, N$, to the above system of first order linear differential equations do not satisfy (47), in general, as we show in the following numerical illustrations.

Remark 2 *In fact, since all the agents extract the resource at the same rate, Problem 40-41 becomes equivalent to the problem of a representative agent using the discount function* $\sum_{m=1}^{N} e^{-r_m(s-t)}$. *The time-inconsistency of the naive solution if* $\sigma \neq 1$ *for the corresponding cake eating problem with nonconstant discounting was already shown in Marín-Solano and Navas (2009).*

Remark 3 *In the case of equal discount rates of time preference it is well-known that, if* $U^m(c_m) = \frac{c_m^{1-\sigma} - 1}{1-\sigma}$, *the symmetric open-loop Nash equilibrium is Pareto optimum. For the case of heterogeneous discount rates this result is no longer preserved. For instance, in the limit* $\sigma = 1$, *the open-loop Nash equilibria are given by* $c_m^{ON}(t) = \lambda_m e^{-r_i t}$, *where the coefficients* λ_m, $m = 1, \ldots, m$, *are the solutions to* $x_0 = \sum_{m=0}^{N} \frac{\lambda_m}{r_m}(1 - e^{-r_m T})$. *It is easy to check that these decision rules are not Pareto optima.*

Remark 4 *The analysis of feeedback Nash equilibra for asymmetric players has been already studied in the literature. It is easy to check that they are of the form* $c_m^{FN} = \nu_m x$, $m = 1, \ldots, N$, *where* ν_m *solve* $\dot{\nu}_m + \gamma_m \nu_m = \nu_m^2 - \frac{1-\sigma_m}{\sigma_m} \nu_m \sum_{j \neq m} \nu_j$.

4.3 Numerical illustrations

Next we illustrate numerically the above results. We consider as a baseline case the problem for three players, $N = 3$, exhibiting as time preference rates $r_1 = 0.03$, $r_2 = 0.06$ and $r_3 = 0.09$, respectively, i.e., agent 1 being the most patient and agent 3 the most impatient. Agents face the "optimal" exploitation of a common property exhaustible resource with an initial stock of $S_0 = 100$ during a time interval that extends from $t_0 = 0$ to $T = 50$ periods. Utilities from consumption are assumed to be of the iso-elastic type with equal intertemporal elasticity of substitution $(1/\sigma)$ for all three players in the coalition.

Figures 1 and 2 show the individual extraction rate for every agent in the coalition under the assumption of cooperation for the naive (dot dashed line) and the sophisticated solutions (dashed line), with $\sigma = 0.6$ (Figure 1) and $\sigma = 2$ (Figure 2). In both graphics, the solid line shows the extraction rate for logarithmic utilities.

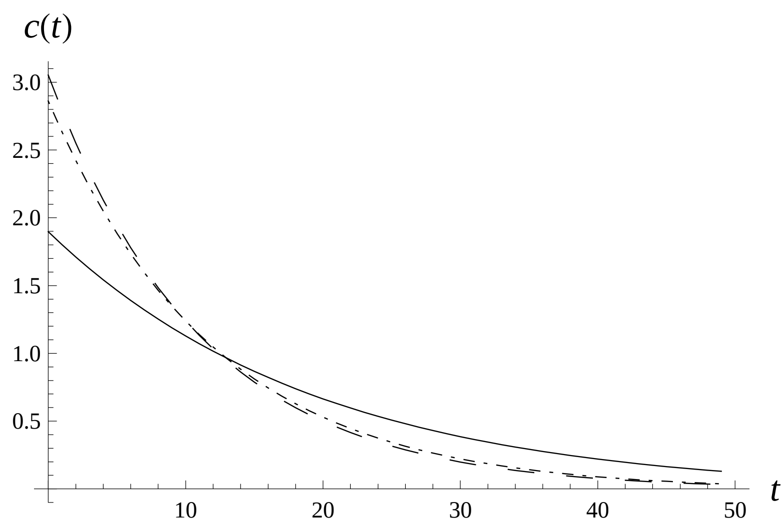

Figure 1: Extraction rates for naive and sophisticated agents ($\sigma = 0.6$) and logarithmic case.

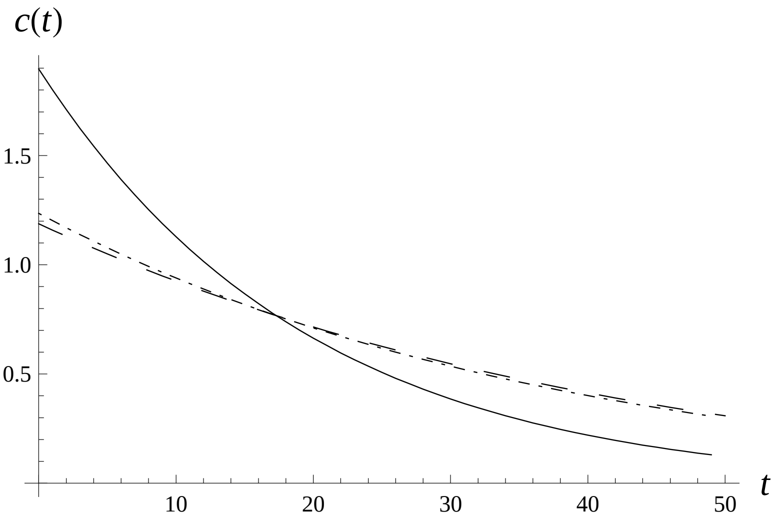

Figure 2: Extraction rates for naive and sophisticated agents ($\sigma = 2$) and logarithmic case.

Unless $\sigma = 1$ (logarithmic utilities), the solutions for naive and sophisticated agents do not coincide, as expected. For $\sigma = 0.6$ sophisticated agents' extraction rate is higher at initial periods compared with naive agents, being this behavior reversed for $\sigma = 2$.

Next, in Figure 3 we compare the extraction rates of sophisticated agents for different values of σ. We observe that higher values of σ lead agents to smooth their extraction rate path along the time horizon:

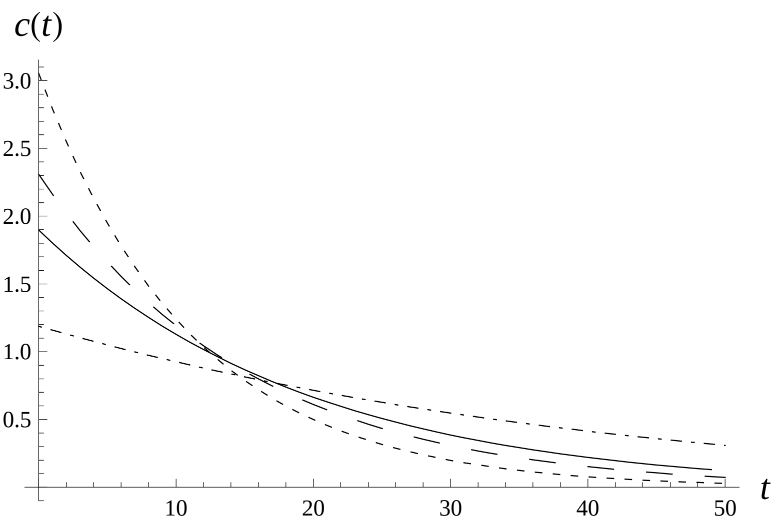

Figure 3: Extraction rates for a sophisticated agent in the coalition with $\sigma = 0.6$ (short dashed), $\sigma = 0.8$ (large dashed), logarithmic case (solid), and $\sigma = 2$ (dashed dot).

Finally, we compare at Figure 4 the cooperative precommitment solutions $(c_m^0(s), s \in [0, 50], m = 1, 2, 3)$ with the time consistent solution assuming now that utilities are of logarithmic type:

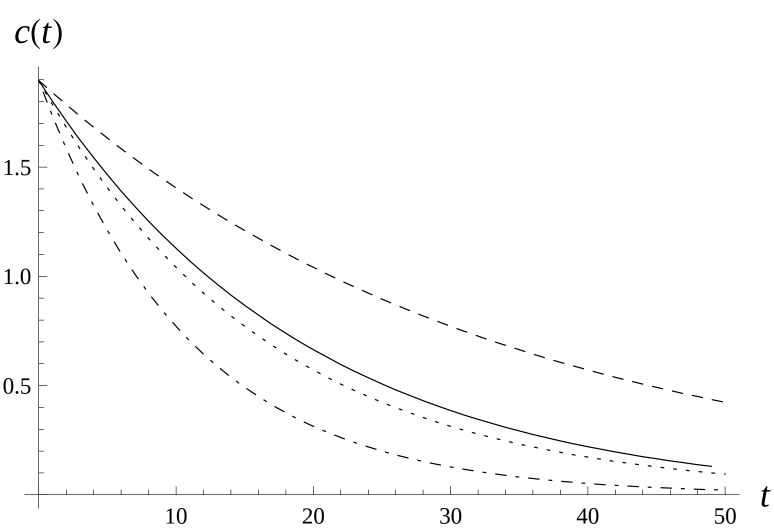

Figure 4: Extraction rates for sophisticated agents in the coalition (solid line) and individual extraction rates under precommitment at $t = 0$ (dashed, dotted and dot dashed lines correspond to players 1, 2 and 3, respectively). Logarithmic utility.

We observe that under the cooperative precommitment solution each player's extraction rate in the coalition is different, being (the patient) player 1 the agent in the coalition with higher aggregate extraction (and hence exploitation) of the resource. In the time-consistent solution, extraction rates are equal for all three players in the coalition, as shown with the solid line.

5 An extension: On the exploitation of a common-property renewable resource in infinite planning horizon

In the economic modeling of natural resources it is customary to work in an infinite horizon setting. For instance, an important issue in the management of natural resources (such as forests, aquifers or fish species) is the existence of positive steady state levels. In this section we briefly extend the results for the nonrenewable resource model in the previous section to a simple model of management of a common-property renewable resource. If preferences of agent m, for $m = 1, \ldots, N$, are characterized by the (instantaneous) utility function $U^m(c_m) = \dfrac{c_m^{1-\sigma_m} - 1}{1-\sigma_m}$ and the discount rate of time preference r_m then, at time t, in order to look for a Pareto solution we must solve

$$\max_{\{c_1,\ldots,c_n\}} \sum_{m=1}^{N} \int_t^{\infty} e^{-r_m(s-t)} \frac{(c_m(s))^{1-\sigma_m} - 1}{1-\sigma_m}\, ds \tag{48}$$

subject to

$$\dot{x}(s) = g(x) - \sum_{m=1}^{N} c_m(s)\ , \qquad x(t) = x_t\ , \tag{49}$$

where $c_m(t)$ is the harvest rate of agent m, for $m = 1, \ldots, N$, and $g(x)$ is the natural growth function of the resource stock x.

In the following, we use first an alternative approach for the derivation of a DPE in a cooperative setting with heterogeneous agents (see Marín-Solano and Shevkoplyas (2011)). This DPE is applied next to the study of the basic properties of Problem 48-49.

5.1 A Dynamic Programming Equation

Consider the problem of looking for the decision rule "maximizing"

$$J(c(\cdot)) = \sum_{m=1}^{N} \int_t^{\infty} e^{-r_m(s-t)} U^m(x(s), c(s), s)\, ds, \tag{50}$$

subject to (30).

¿From Proposition 2, a natural candidate for a DPE is given by (37-38) by taking $T = \infty$. However, in our derivation we assumed that T is finite. Next we provide a mathematical justification of this DPE by using a different procedure. Following the approach in Marín-Solano and Shevkoplyas (2010) (which is based on the one by Ekeland and Lazrak (2010)), if $c^*(s) = \phi(s, x(s))$ is the equilibrium rule, then the value function is given by

$$W(x,t) = \sum_{m=1}^{N} \int_t^{\infty} e^{-r_m(s-t)} U^m(x(s), \phi(x(s), s), s)\, ds \tag{51}$$

where $\dot{x}(s) = f(x(s), \phi(x(s), s), s)$, $x(t) = x_t$. Next, for $\epsilon > 0$ let us consider the variations

$$c_\epsilon(s) = \begin{cases} v(s) & \text{if} \quad s \in [t, t+\epsilon]\ , \\ \phi(x,s) & \text{if} \quad s > t+\epsilon\ . \end{cases}$$

If the t-agent can precommit her behavior during the period $[t, t+\epsilon]$, the value function for the perturbed control path c_ϵ is given by

$$W_\epsilon(x,t) = \max_{\{v(s),\, s\in[t,t+\epsilon]\}} \left\{ \sum_{m=1}^{N} \int_t^{t+\epsilon} e^{-r_m(s-t)} U^m\left(x(s), v(s), s\right) ds + \right.$$
$$\left. \sum_{m=1}^{N} \int_{t+\epsilon}^{\infty} e^{-r_m(s-t)} U^m(x(s), \phi(x(s), s), s)\, ds \right\}\ . \tag{52}$$

Definition 3 *Let W_ϵ be differentiable in ϵ in a neighborhood of $\epsilon = 0$. Then $c^*(s) = \phi(s, x(s))$ is called an equilibrium rule if*

$$\lim_{\epsilon \to 0^+} \frac{W(x,t) - W_\epsilon(x,t)}{\epsilon} \geq 0\ .$$

The definition above can be interpreted as follows. For ϵ sufficiently small, from the continuity of W_ϵ with respect to ϵ, the maximum of W_ϵ in the limit when $\epsilon = 0$ is precisely $W(x,t)$. In Definition 3 we are imposing an optimality condition. Although this notion of equilibrium is not as natural as in the approach described in the previous sections, it allows us to provide a mathematical justification to the DPE (37-38) with $T = \infty$.

Proposition 4 *If the value function (51) is of class C^1, then the solution $c = \phi(x,t)$ to the right hand term of the DPE*

$$\sum_{m=1}^{N} r_m W^m(x,t) - \sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial t} = \max_u \left\{ \sum_{m=1}^{N} U^m(x,c,t) + \left(\sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial x} \right) f(x,c,t) \right\} \tag{53}$$

with

$$W^m(x,t) = \int_t^{\infty} e^{-r_m(s-t)} \bar{U}(x(s), s)\, ds \tag{54}$$

is an equilibrium rule, in the sense that it satisfies Definition 3.

Proof: In order to prove that $c^*(t) = \phi(x,t)$ solving the right hand term in (53) is an equilibrium rule, we have to check Definition 3. We will do it in several steps.

If $\bar{x}(s)$ denotes the state trajectory corresponding to the decision rule $c_\epsilon(s)$, then

$$W(x,t) - W_\epsilon(x,t) = \sum_{m=1}^{N} \int_t^{t+\epsilon} e^{-r_m(s-t)} \left[U^m(x(s), \phi(x(s),s), s) - U^m(\bar{x}(s), v(s), s)\right] ds +$$

$$+ \sum_{m=1}^{N} \int_{t+\epsilon}^{\infty} e^{-r_m(s-t)} \left[U^m(x(s), \phi(x(s),s), s) - U^m(\bar{x}(s), \phi(\bar{x}(s),s), s)\right] ds \ .$$

Note that

$$\sum_{m=1}^{N} \int_{t+\epsilon}^{\infty} e^{-r_m(s-t)} U^m(x(s), \phi(x(s),s), s)\, ds \ = W(x(t+\epsilon), t+\epsilon) -$$

$$- \sum_{m=1}^{N} \int_{t+\epsilon}^{\infty} \left[e^{-r_m(s-t-\epsilon)} - e^{-r_m(s-t)}\right] U^m(x(s), \phi(x(s),s), s)\, ds \ .$$

In a similar way,

$$\sum_{m=1}^{N} \int_{t+\epsilon}^{\infty} e^{-r_m(s-t)} U^m(\bar{x}(s), \phi(\bar{x}(s),s), s)\, ds \ = W(\bar{x}(t+\epsilon), t+\epsilon) -$$

$$- \sum_{m=1}^{N} \int_{t+\epsilon}^{\infty} \left[e^{-r_m(s-t-\epsilon)} - e^{-r_m(s-t)}\right] U^m(\bar{x}(s), \phi(\bar{x}(s),s), s)\, ds \ .$$

Therefore,

$$\lim_{\epsilon \to 0^+} \frac{W(x,t) - W_\epsilon(x,t)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{\sum_{m=1}^{N} \int_t^{t+\epsilon} e^{-r_m(s-t)} \left[U^m(x(s), \phi(x(s),s), s) - U^m(\bar{x}(s), v(s), s)\right] ds}{\epsilon}$$

$$+ \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{m=1}^{N} \left[\int_{t+\epsilon}^{\infty} \left[e^{-r_m(s-t)} - e^{-r_m(s-t-\epsilon)}\right] \left[U^m(x(s), \phi(x(s),s), s) - U^m(\bar{x}(s), \phi(\bar{x}(s),s), s)\right] ds\right]$$

$$+ \lim_{\epsilon \to 0^+} \frac{W(x(t+\epsilon), t+\epsilon) - W(\bar{x}(t+\epsilon), t+\epsilon)}{\epsilon} = \sum_{m=1}^{N} \left[U^m(x(t), \phi(x(t),t), t) - U^m(x(t), v(t), t)\right]$$

$$+0 + \lim_{\epsilon \to 0^+} \frac{W(x(t+\epsilon), t+\epsilon) - W(x(t),t)}{\epsilon} - \lim_{\epsilon \to 0^+} \frac{W(\bar{x}(t+\epsilon), t+\epsilon) - W(x(t),t)}{\epsilon}$$

$$= \sum_{m=1}^{N} \left[U^m(x(t), \phi(x(t),t), t) - U^m(x(t), v(t), t)\right] + \left[\frac{\partial W(x,t)}{\partial t} + \nabla_x W(x,t) \cdot f(x, \phi(x,t), t)\right]$$

$$-\left[\frac{\partial W(x,t)}{\partial t}+\nabla_x W(x,t)\cdot f(x,v(t),t)\right]=\sum_{m=1}^{N}\left[U^m(x,\phi(x,t),t)+\nabla_x W^m(x,t)\cdot f(x,\phi(x,t),t)\right]$$

$$-\sum_{m=1}^{N}\left[U^m(x,v(t),t)+\nabla_x W^m(x,t)\cdot f(x,v(t),t)\right]\geq 0\ ,$$

since $c^*=\phi(x,t)$ is the maximizer of the right hand term in (53).

□

5.2 Time-consistent Pareto solution in the management of a renewable natural resource

Let us consider Problem 48-49. Since both the instantaneous utility functions and the state equation are autonomous, it seems natural to restrict our attention to time-independent value functions $W^m(x)$, for $m=1,\dots,N$. From Proposition 4 we have to solve

$$\sum_{m=1}^{N} r_m W^m = \max_{c_1,\dots,c_N}\left\{\sum_{m=1}^{N}\frac{c_m^{1-\sigma_m}-1}{1-\sigma_m}+\left(\sum_{j=1}^{N}W_x^j\right)\left(g(x)-\sum_{m=1}^{N}c_m\right)\right\}\ , \tag{55}$$

hence

$$c_m^*=\left(\sum_{j=1}^{N}W_x^j\right)^{-\frac{1}{\sigma_m}}\ .$$

Therefore, $c_m^*=c_{m'}^*$ if, and only if, $\sigma_m=\sigma_{m'}$. In general, along the equilibrium rule, $U'(c_m^*)=U'(c_{m'}^*)$, for all m,m'. In addition, we have the set of DPEs

$$r_m W^m=\frac{(c_m^*)^{1-\sigma_m}-1}{1-\sigma_m}+W_x^m\left(g(x)-\sum_{j=1}^{N}c_j^*\right)\ , \tag{56}$$

for all $m=1,\dots,N$.

Next, let us restrict our attention to the case of linear decision rules (an study on the limits to the use of linear Markov strategies in a non-cooperative setting can be found in Gaudet and Lohoues (2008)). Since $c_i^{-\sigma_i}=c_j^{-\sigma_j}$, for all $i,j=1,\dots,N$, if $c_m=\alpha_m x$ then $(\alpha_i x)^{-\sigma_i}=(\alpha_j x)^{-\sigma_j}$. Therefore, no linear decision rules exist unless $\sigma_i=\sigma_j$, for all i,j. For $\sigma_i=\sigma_j=\sigma$, then $\alpha_i=\alpha_j$ and the dynamic programming equation (55) becomes

$$\sum_{m=1}^{N} r_m W^m=\frac{N}{1-\sigma}\left(\alpha^{1-\sigma}x^{1-\sigma}-1\right)+\alpha^{-\sigma}x^{-\sigma}\left(g(x)-N\alpha x\right)\ .$$

The equation above has solution if $g(x)=ax$. In this case we obtain

$$\sum_{m=1}^{N} r_m W^m(x)=\left[\frac{N\sigma}{1-\sigma}\alpha^{1-\sigma}+a\alpha^{-\sigma}\right]x^{1-\sigma}-\frac{N}{1-\sigma}\ ,$$

together with $\sum_{m=1}^{N} W_x^m(x) = \alpha^{-\sigma} x^{-\sigma}$ and (56). If we try $W^m(x) = A^m \dfrac{x^{1-\sigma}-1}{1-\sigma} + B^m$, by simplifying we obtain that A^m, B^m and α are obtained by solving the equations system

$$[r_m - (1-\sigma)(a - N\alpha)]\, A^m = \alpha^{1-\sigma}\ , \tag{57}$$

$$r_m A^m - (1-\sigma) r_m B^m = 1\ , \tag{58}$$

$$\sum_{m=1}^{N} A^m = \alpha^{-\sigma}\ . \tag{59}$$

For instance, if $\sigma = 1$ (logarithmic utility) we have $A^m = \dfrac{1}{r_m}$ and $\alpha = \dfrac{1}{\sum_{m=1}^{N} \frac{1}{r_m}}$. If $r_1 = \cdots = r_N = r$ then $\alpha = \dfrac{r-(1-\sigma)a}{N\sigma}$.

The following proposition summarizes the main results of this section.

Proposition 5 *In Problem 48-49 along the equilibrium the marginal utility of all agents coincide. The extraction rates of two agents are equal if, and only if, they have the same marginal elasticity. If there are two players with different marginal elasticities, no linear decision rules exists. If the natural growth function is linear and all the agents have the same marginal elasticity* σ*, then the decision rules* $c_m = \alpha x$ *and the value functions* $W^m(x) = A^m \dfrac{x^{1-\sigma}-1}{1-\sigma} + B^m$*,* $m = 1, \ldots, N$ *solve Problem 48-49, where the coefficients* α*,* A^m *and* B^m *are the solutions to (57-59).*

6 Conclusions

In this paper we have studied the problem of looking for time-consistent solutions if heterogeneous agents (in the sense that they exhibit different instantaneous pay-off functions and different (but constant) discount rates of time preference) decide to cooperate. First we have restricted our attention to problems in a finite horizon setting. For this case we have introduced two alternative approaches in order to find time consistent equilibria. In the first approach we transform a two-player cooperative differential game into a one-agent problem with heterogeneous discounting. The search of time-consistent solutions for a problem with heterogeneous discounting has been recently studied in Marín-Solano and Patxot (2011). Whereas non-constant discounting models are typically very difficult to solve since the dynamic programming equation is not a standard (partial) differential equation, our problem with heterogeneous agents (and the problem with heterogeneous discounting) is more computationally tractable. The second approach allows us to study problems with an arbitrary number of players. After the derivation of a set of coupled dynamic programming equations, we have applied our results to the study of the effects of using different discount rates in the derivation of time-consistent extraction rates in a simple exhaustible resource extraction model with common access (see e.g. Clemhout and

Wan (1985) and Dockner et al (2000)). We have proved that, within the class of iso-elastic utility functions, if the agents decide to cooperate, if all the agents use the same parameter σ in their utility function, then the extraction rate of all players in the time-consistent solution coincide (although they are using different discount rates of time preferences). A similar result has been recently obtained in a discrete time setting in a fisheries model for a logarithmic utility function (see Breton and Keoula (2010b)). Next, we have extended our results to an infinite horizon setting, and a simple common access renewable natural resource model with heterogeneous agents has been also discussed.

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