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Treball final de grau

On the stability of the Lagrangian points in the restricted circular 3-body problem

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Abstract

The 3-body problem is one of the most celebrated problems in mathematics. In this work we aim to find the equilibrium points of one of the three masses, which is considered infinitesimal, and study their stability in two different phase spaces. The question of stability is addressed using both analytical and numerical methods. Whereas the Lyapunov and KAM theories provide us with analytical proofs of the stable or unstable behaviour in the first phase space, analytical methods motivated by the Nekhoroshev theory allow us to compute practical bounds of the time until which the infinitesimal mass remains near the equilibria in the second. Finally, these bounds are applied to the case of a well known system: the Sun-Jupiter-Trojan system.

Resum

El problema dels 3 cossos és un dels problemes matemàtics més debatuts al llarg de la història. Aquest treball s'enfoca a trobar els punts d'equilibri d'una de les tres masses, la qual considerarem infinitessimal, i a estudiar-ne la seva estabilitat en dos espais de fase diferents. L'estudi de l'estabilitat d'aquests punts es tracta analíticament i numèrica, en funció de l'espai. D'una banda, les teories de Lyapunov i de KAM ens permeten donar demostracions analítiques del caràcter estable o inestable d'aquests punts en el primer cas. D'altra banda, mètodes numèrics motivats per la teoria de Nekhoroshev ens permeten estimar el temps en què la massa infinitessimal romandrà a prop dels punts d'equilibri en el segon. Finalment, aquestes darreres estimacions s'apliquen al cas particular d'un sistema ben conegut: el sistema Sol-Júpiter-Troià.

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Contents

Chapter 1

Introduction

The so-called N-body problem has been subject of study of many mathematicians and physicists in the recent centuries. Indeed, though the first contributions attributed to Isaac Newton date back to the early 17th century and countless outstanding scientists have faced the problem, little is known still nowadays.

After Newton's formulation of the laws of movement and the universal theory of gravity, the need to predict the motion of two massive objects (the Sun and the Earth, for instance) within their gravitational field arose, in the context of celestial mechanics. Naturally, if we call x_1 and x_2 the positions of the two bodies and consider only their gravitational interaction, the equations of movement read

$$F_{12}(x_1, x_2) = m_1 \ddot{x}_1, \qquad F_{21}(x_1, x_2) = m_2 \ddot{x}_2 \tag{1.1}$$

where m_i is the mass of the body i and F_{ij} is the force on the mass i due to the presence of the mass j. The addition of the equations shown above results in an equation describing the center of mass, whereas the subtraction of them results in an equation that describes the movement of $r = x_1 - x_2$ (or $r = x_2 - x_1$). In both cases, the new equations describe the motion of a body of mass $M = m_1 + m_2$ and one of mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ respectively, giving us two independent problems of a mass within a central field, which of course, can be solved analytically. From here it yields the solution of the 2-body problem (see H. Pollard [18] for more details).

The next intuitive step was to study the movement of the Moon under the influence of the Earth and the Sun. However, neither Newton nor subsequent mathematicians until Poincaré[1] succeeded in finding an analytical solution, which turned the 3-body problem into the most celebrated problem in mathematics.

In this work, we focus on a particular case of the *3-body problem*, called the *restricted circular* approach. Specifically, we aim to study the *stability* of those points where the third mass (which is taken infinitesimal) is in equilibrium. These points are often called *Lagrangian points* or *Libration points*.

The rest of this chapter is devoted to introducing some topics that are crucial for the understanding of the subsequent parts. Finally, the contents of each chapter are outlined.

[1] Henri Poincaré proved in fact that analytical solutions did not exist (see J. Barrow [3]).

The restricted circular approach

Consider a Newtonian reference system in $\mathbb{R}^3$ to describe the motion of N massive bodies (point particles) influenced only by their gravitational field. Analogously as we did in the case of two bodies, the relation between the force acting on each mass and its acceleration can be written. If we consider explicitly the expression of the force due to the gravity and let $x_i \in \mathbb{R}^3$ be the position of the mass m_i, we have that

$$m_i \ddot{x}_i = \sum_{j=1, j\neq i}^{N} \frac{\mathcal{G} m_i m_j (x_j - x_i)}{\|x_i - x_j\|^3}, \quad 1 \leq i \leq N \tag{1.2}$$

being $\mathcal{G}$ the universal gravitational constant. Notice that (1.2) defines a system of N ordinary differential not independent equations of second order, being the solution not trivial in general.

As stated above, the problem given by (1.2) and a complete set of boundary conditions is faced using a classical[2] vectorial mechanics approach, since forces, accelerations, and positions are of a vectorial nature. However, it is convenient for the ongoing work, to be developed on the frame of analytical mechanics. Even though a basic knowledge of analytical mechanics is assumed in this dissertation, the reader might find a clear but still exhaustive introduction in H. Goldstein et al. [10].

Now, let us fix $N = 3$. Furthermore, as a consequence of the analytical complexity of the 3-body problem, a special case can be considered. Here, we assume that one mass is infinitesimal, meaning that the presence of this mass (let us refer to it as the infinitesimal particle) in the system does not contribute to the gravitational field affecting the other two (which we call the primaries). This can be done since in the equation (1.2) it is noticeable that the acceleration of the third mass does not depend on its mass. Hence, we can vanish this mass and still have a non trivial acceleration. This approximation leads us to the restricted 3-body problem.

From this, it is clear that the trajectories of the primaries are given by the 2-body problem, which may have any conic section as a solution, depending on their initial conditions. In particular, we consider the case in which they describe circular orbits around the center of mass, which leads us to the restricted circular approach.

In this thesis we focus on the 3-body problem under the hypothesis described above. For a thorough description see V. Szebehely [20].

Hamiltonian systems

A Hamiltonian system is a dynamical system (a system that evolves along the variation of one variable, usually the time) that can be described by Hamilton's equations. More precisely, if we let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ be a phase space of a system having n degrees of freedom and $q, p \in \mathbb{R}^n$ the generalised coordinates and conjugated moments respectively, a Hamiltonian system is defined as follows:

[2] Classical in the sense that the effects of general relativity are not considered.

Definition 1.1. *A system is Hamiltonian if it exists a smooth function* $H\colon \mathcal{I} \times \Omega \to \mathbb{R}$ *where* $\mathcal{I} \subset \mathbb{R}$ *called the Hamiltonian function such that*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(t,q,p), \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}(t,q,p), \qquad 1 \leq i \leq n. \tag{1.3}$$

Let us assume for simplicity that $\mathcal{I} = \mathbb{R}$ henceforth in this dissertation. This means that the solutions are well defined for all time, which excludes the cases of collisions between masses.

Let now H be the Hamiltonian of a given system. Then, if we define

$$z = \begin{pmatrix} q \\ p \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad \nabla H = \begin{pmatrix} \frac{\partial H}{\partial z_1} \\ \vdots \\ \frac{\partial H}{\partial z_{2n}} \end{pmatrix} \tag{1.4}$$

the differential equations defining the dynamics of the system can be written in a compact manner as

$$\dot{z} = J\nabla H(t,z). \tag{1.5}$$

Given a complete set of initial conditions (t_0, z_0) with $z_0 \in \mathbb{R}^n$, a solution of this system is understood as a parameterised curve $\phi(t,t_0,z_0)\colon \mathbb{R} \to \Omega$ of parameter t. Indeed, since it yields from the existence and uniqueness theorem of differential equations that $\phi(t,t_0,z_0) = \phi(t-t_0,0,z_0)$, we can denote as $\phi(t,z_0)$ the solution satisfying $\phi(0,z_0) = z_0$.

First integrals of movement

A relevant characteristic of every dynamical system is whether or not it admits first integrals. Given a system in a phase space Ω, a first integral is a smooth function $I\colon \mathbb{R} \times \Omega \to \mathbb{R}$ which is constant along the solutions of the system. Besides, we say that a first integral is non-degenerated if it is not constant in $\mathbb{R} \times \Omega$.

From the Hamilton's equations it is not hard to see that

$$\frac{dH}{dt}(q,p,t) = \frac{\partial H}{\partial t}(q,p,t).$$

Then, if the Hamiltonian does not depend on time, (we say that the system is autonomous) the function H is itself a first integral. In the following chapters we manage to remove the dependence on time from our Hamiltonian by using an ingenious change of variables. In turn, the Hamiltonian will be a non-degenerated first integral thereafter.

A useful way to characterise the first integrals is by using the Poisson bracket bilinear operator, which we denote by $\{\cdot,\cdot\}$ and define:

Definition 1.2. *Let* $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ *be a phase space having n degrees of freedom and let* $G, F\colon \mathbb{R} \times \Omega \to \mathbb{R}$ *be two smooth functions. Then, we define*

$$\{G,F\} = \sum_{i=1}^{n} \left(\frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} \right). \tag{1.6}$$

Notice that $\{G, F\}$ is clearly a smooth function defined in $\mathbb{R} \times \Omega$. Now, if we let $I(q, p)$ be a first integral independent of time, by applying the chain rule it holds

$$\frac{dI}{dt}(q, p) = \sum_{i=1}^{n} \left(\frac{\partial I}{\partial q_i} \dot{q}_i + \frac{\partial I}{\partial p_i} \dot{p}_i \right) = 0$$

and recalling the Hamilton's equations to write the time derivatives of q_i and p_i in terms of the Hamiltonian H we have that

$$\{I, H\} = 0. \tag{1.7}$$

Therefore, the constants of movement independent of time together with the Hamiltonian vanish the Poisson bracket. Reciprocally, a time-independent function F defined in $\mathbb{R} \times \Omega$ that vanishes the linear operator $\{\cdot, H\}$ for every point in $\mathbb{R} \times \Omega$ is a first integral of the movement.

Contents of this work

As mentioned in the beginning of this chapter, the main goal is to study the stability of the equilibrium points of the restricted circular 3-body problem. For a better understanding, we first describe the problem introducing these equilibrium points. Afterwards, the problem of stability is considered in two different cases. The summarised contents of each chapter are:

Chapter 2. Here we introduce the Hamiltonian of the planar case and present its equilibrium points in a rotating frame of reference, following the steps made by K. Meyer et al. [15]. It includes a brief discussion concerning the possible regions of motion of the infinitesimal particle depending on its energy level, and it ends with a couple of comments about the linear stability, very useful afterwards. However, the results about the linear stability presented in this work are not proven for conciseness.

Chapter 3. This is a description of the stability of the equilibrium points in the planar case[3]. Some of them are proven to be unstable by using the Lyapunov stability theory, while the others are stable in some cases as yields from the KAM theory. Again, the results are presented in a similar scheme than in K. Meyer et al. [15], but here we include a developed idea of the invariant curve theorem's proof given by J. Moser [16]. Furthermore, we introduce concepts such as the Birkhoff normal form and the Arnold-Liouville theorem to emphasize on how the invariant curve theorem leads to the result of stability.

Chapter 4. This chapter is devoted to the study of the stability of the Lagrangian points as well, but now in the spatial case. By contrast to the previous chapter, the methods used in here are rather numerical, as analytical and concluding results regarding the stability in this case have not been found yet. We follow the scheme of A. Celletti et al. [5], motivated by the Nekhoroshev theory, in order to give a lower bound of the time until which some equilibrium points are stable. An explanation of the cited paper is completed with detailed proofs and an original

[3]The definition of this case is given in the next chapter.

implementation of the computation of the escape time lower bound for the Sun-Jupiter system using C++. This implementation is not only based on the results obtained in [5], but also on the work done by À. Jorba [11], which allowed a notable enhancement.

Chapter 5. The conclusions of the work and its main results are displayed here. Furthermore, we comment in this chapter how this work could be complemented.

Chapter 2

Description

This chapter is aimed to introduce the Hamiltonian system of our interest, working under the conditions mentioned in the introduction. Moreover, we also consider that the infinitesimal particle is initially placed in the plane containing the orbits of the two primaries. Then, since there is no force pushing the infinitesimal particle away from this plane, all the movement is restricted to this, which leads us to the planar restricted circular approach. Since we have 2 degrees of freedom, the phase space will be 4-dimensional.

In the sections that follow, we deduce the Hamiltonian and find its equilibrium points, the so-called Lagrangian points. Afterwards, we analyse the possible regions of motion of the infinitesimal particle for a given energy level, complementing the reasonings with a numerical computation of the zero-velocity curves, which are introduced later on. Finally, we make some comments about the linear stability of the equilibrium points, without giving details of their proofs.

2.1 Deduction of the Hamiltonian

First, we establish a units system in order to nondimensionalise the problem. Let us take the unit of mass the sum of the masses of the primaries, the unit of length the distance between the primaries and the unit of time such that the period of the orbits described by the primaries is 2π. It turns that with these units the universal gravitational constant is $\mathcal{G} = 1$, which comes from the Kepler's third law for a general elliptic orbit of the two masses.

Let us put the origin in the center of mass of the system. From this point the primary having mass $0 < \mu \leq \frac{1}{2}$ is always at a distance of $1 - \mu$ and initially at $(1 - \mu, 0)$, whereas the primary having mass $1 - \mu > 0$ is always at a distance of μ and initially at $(-\mu, 0)$. Since they both describe a 2π-period circular trajectory around the origin, these trajectories can be written as follows

$$(X_{11}, X_{12}) = (1 - \mu)(\cos t, \sin t), \qquad (X_{21}, X_{22}) = -\mu(\cos t, \sin t) \tag{2.1}$$

where the first sub indexes refer to each one of the primaries and the seconds to each one of the planar coordinates. Let now $X^T = (X_1, X_2)$ be the coordinates of the in-

finitesimal particle. Then, the square of the distances from this to the primaries are

$$d_1^2 = (X_1 - (1-\mu)\cos t)^2 + (X_2 - (1-\mu)\sin t)^2, \quad d_2^2 = (X_1 + \mu\cos t)^2 + (X_2 + \mu\sin t)^2. \tag{2.2}$$

Now, the kinetic and potential energies can be easily obtained and therefore, the Lagrangian of the system[1]:

$$L = T - U = \frac{1}{2}(\dot{X_1}^2 + \dot{X_2}^2) + \frac{\mu}{d_1} + \frac{1-\mu}{d_2}. \tag{2.3}$$

Note that this Lagrangian is time dependent as the distances d_1 and d_2 depend explicitly on time. However, a time-independent system can be obtained whether we consider a rotating frame of reference in which the primaries are still. Let $x^T = (x_1, x_2)$ be the coordinates of the infinitesimal particle in this new frame of reference. The relation between the coordinates in both frames is a canonical transformation given by a rotation:

$$X = A(t)\cdot x, \quad A(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{2.4}$$

Using (2.4) it is easy to obtain that $d_1^2 = (x_1 - 1 + \mu)^2 + x_2^2$ and $d_2 = (x_1 + \mu)^2 + x_2^2$. Along with the computation of $\dot{X}$ in terms of x, the Lagrangian can be expressed in the new frame of reference as follows

$$L = \frac{1}{2}\left((\dot{x_1} - x_2)^2 + (\dot{x_2} + x_1)^2\right) + \frac{\mu}{d_1} + \frac{1-\mu}{d_2} \tag{2.5}$$

where now the distances d_1 and d_2 do not depend explicitly on time, so the system is not time dependent in the new frame of reference. After that, the Hamiltonian can be found as defined in H. Goldstein [10]:

$$H = \sum_{i=1}^{2} \dot{x}_i y_i - L$$

where $y^T = (y_1, y_2)$ is the conjugated canonical momentum defined as $y_i = \frac{\partial L}{\partial \dot{x}_i}$. Hence, a simple calculation work leads us to

$$H = \frac{1}{2}\|\dot{x}\|^2 - \frac{1}{2}\|x\|^2 + U, \quad U = -\frac{\mu}{d_1} - \frac{1-\mu}{d_2} \tag{2.6}$$

which can also be written in terms of x and y as presented in K. Meyer et al. [15]

$$H = \frac{1}{2}\|y\|^2 - x^T K y + U, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.7}$$

The equations (2.6, 2.7) are different expressions of the Hamiltonian governing the dynamics of the infinitesimal particle. Since we are following the analytical mechanics' approach, the forces acting on the infinitesimal particle did not have to be considered, but only its kinetic and potential energies. However, one may notice that the term $x^T K y$ rises from the rotating coordinate system and is due to the Coriolis fictional forces.

[1] Note that the potential U is taken to be negative because of the attractive behaviour of the gravitational force.

2.2 The equilibrium points

Here we attempt to find the equilibrium points (the Lagrangian or Libration points) of the infinitesimal particle in the Cartesian plane using the expression of the Hamiltonian found in the previous section. It is well known that the equations of motion can be deducted from the Hamilton's equations, which in our case are

$$\dot{x} = \frac{\partial H}{\partial y} = y + Kx, \quad \dot{y} = -\frac{\partial H}{\partial x} = Ky - \frac{\partial U}{\partial x}. \tag{2.8}$$

The equilibrium condition will be achieved whenever the position and the momentum of the infinitesimal particle are constant. Imposing this on the equations (2.8) we can obtain that $y = -Kx$ from the first one. Substituting it in the second one and considering that $K^2 = -I$ we reach the following condition

$$x - \frac{\partial U}{\partial x} = 0. \tag{2.9}$$

In terms of the so-called amended potential $V = \|x\|^2 - 2U + \mu(1-\mu)$, the equation (2.9) can be rewritten in a compact manner as

$$\frac{\partial V}{\partial x} = 0. \tag{2.10}$$

On the other hand, it is easy to see that from the expressions of d_i^2 for $i = 1, 2$ it follows

$$\|x\|^2 = \mu d_1^2 + (1-\mu)d_2^2 - \mu(1-\mu). \tag{2.11}$$

Substituting now the expression of $\|x\|^2$ found above and recalling the definition of U, the amended potential reads

$$V(d_1, d_2) = \mu d_1^2 + (1-\mu)d_2^2 + 2\frac{\mu}{d_1} + 2\frac{1-\mu}{d_2}. \tag{2.12}$$

Now, using the chain rule, the condition $\frac{\partial}{\partial x}V(d_1, d_2) = 0$ can be expressed as

$$\begin{pmatrix} \frac{\partial V}{\partial x_1} \\ \\ \frac{\partial V}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial d_1}{\partial x_1} & \frac{\partial d_2}{\partial x_1} \\ \\ \frac{\partial d_1}{\partial x_2} & \frac{\partial d_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial V}{\partial d_1} \\ \\ \frac{\partial V}{\partial d_2} \end{pmatrix} = 0. \tag{2.13}$$

The equation (2.13) defines a homogeneous system. Hence, whether the rank of the matrix is proven to be maximum the condition $\frac{\partial}{\partial x}V = 0$ reads $\frac{\partial}{\partial d}V = 0$.

After a simple calculation, the determinant of the matrix can be computed to obtain

$$\begin{vmatrix} \frac{\partial d_1}{\partial x_1} & \frac{\partial d_2}{\partial x_1} \\ \\ \frac{\partial d_1}{\partial x_2} & \frac{\partial d_2}{\partial x_2} \end{vmatrix} = \frac{-x_2}{[(x_1 - 1 + \mu)^2 + x_2^2]^{\frac{1}{2}}[(x_1 + \mu)^2 + x_2^2]^{\frac{1}{2}}}$$

which clearly vanishes if and only if $x_2 = 0$. Therefore, for $x_2 \neq 0$ the rank is maximum and the equilibrium condition is $\frac{\partial}{\partial d}V = 0$, which leads us to the following conditions:

$$2\mu\left(d_1 - \frac{1}{d_1^2}\right) = 0, \quad 2(1-\mu)\left(d_2 - \frac{1}{d_2^2}\right) = 0. \tag{2.14}$$

Recalling that $0 < \mu \leq \frac{1}{2}$, the only possible solution is that $d_1 = d_2 = 1$. This means that an equilibrium state is located in each point of the Cartesian plane that is one unit away from each of the primaries. The geometrical place described here is actually the set of elements of $\mathbb{R}^2$ that together with the two primaries form an equilateral triangle. For historical reasons, let us call these two points $\mathcal{L}_4$ and $\mathcal{L}_5$. These points are often called equilateral points.

Let us now consider the case where $x_2 = 0$. Combining the equations (2.11, 2.12) the amended potential takes the form

$$V = x_1^2 \pm 2\frac{\mu}{x_1 - (1-\mu)} \pm 2\frac{1-\mu}{x_1 + \mu} + \mu(1-\mu) \tag{2.15}$$

where the plus-minus signs comes from the square root of d_i^2 for $i = 1, 2$. However, as d_i are distances, all the terms of the amended potential must be positive. Hence, the signs are chosen depending on the value of $x_1 \in \mathbb{R}$ in order to fulfil that restriction.

Therefore, one notices that $V \to \infty$ when $x \to \pm\infty$, $x \to -\mu$ and $x \to 1-\mu$. This, together with the fact that V is a continuous real map in the intervals $(-\infty, -\mu)$, $(-\mu, 1-\mu)$, and $(1-\mu, \infty)$ show us that V contains at least one critical point in each one of these intervals.

Furthermore, a the second derivative can be performed to study the convexity of V:

$$\frac{d^2V}{dx_1^2} = 2 \pm \frac{2\mu}{(x_1 - 1 + \mu)^3} \pm \frac{2(1-\mu)}{(x_1 + \mu)^3}. \tag{2.16}$$

It is clear that V is a convex function for all $x_1 \in \mathbb{R}$, since the signs are taken so that each term is positive. Hence, the convexity does not change, which implies that V has precisely one critical point in each interval. Recalling that the critical points of V are equilibrium points, we have proved that there are exactly 3 equilibrium points that are collinear to the primaries (since we are still in the case where $x_2 = 0$), which will be denoted $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ and called collinear points for obvious reasons. Although the position of these points can not be obtained analytically, since the condition $\frac{\partial V}{\partial x} = 0$ with $x_2 = 0$ results in the well known Euler's quintic equation, a suitable implementation of the Newton's method enables us to obtain a numerical approximation. However, for conciseness, this is not discussed in here.

2.3 Hill's regions

As it has been revealed previously, the Hamiltonian is independent of time in a suitable reference frame. Besides, as seen in the introduction, it is known that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$, so in our case the Hamiltonian is a conserved magnitude. Then, it is clear that the Jacobi constant $C = -2H + \mu(1-\mu)$ is also conserved in every trajectory. In terms of the amended potential, this constant can be written as follows

$$C = V - \|\dot{x}\|. \tag{2.17}$$

Since $\|\dot{x}\| \geq 0$, we get that $V \geq C$, which automatically gives us a restriction on the generalised coordinates of the infinitesimal particle in the configuration space. This naturally leads us to the following definition:

Definition 2.1. *Given a mass parameter μ and a Jacobi constant C, the Hill's region is the set defined by $\mathcal{H}(C,\mu) = \{x : V(x) \geq C\}$. Its boundary where the equality holds is a finite set of images of closed curves called zero-velocity curves.*

The study of the zero-velocity curves for a given μ and C, allows us to determine the regions of movement of the infinitesimal particle for a given energy level (equivalently, for a given Jacobi constant). Let us consider that $C \gg 1$. Recalling the equation (2.12), this implies that $d_1 \to 0$, $d_2 \to 0$, or $d_1, d_2 \to \infty$. In all three cases there is a dominant term in the amended potential, so the trajectory can be deducted easily. For example, consider the case where $d_1 \to 0$ (which automatically implies that $d_2 \to 1$). Then, we can approximate the amended potential as follows

$$V \approx 2\frac{\mu}{d_1}. \tag{2.18}$$

Imposing now $V = C$ to obtain the zero-velocity curve, we get that $d_1 \approx 2\frac{\mu}{C}$, which defines a circular trajectory around the first primary. Similarly, considering the remaining two cases one can approach the zero-velocity curves for big values of C.

A thorough description for all the possible values of C involves considering all the terms of the potential nevertheless, which is algebraically demanding for smaller values of C, where many terms are dominant at the same time. Hence, the zero-velocity curves have been obtained numerically instead. With that aim, an implementation of the numerical continuation method has been developed to find the path described by a zero-velocity curve (see Appendix A). Figure 2.1 shows some curves of a system with $\mu = 0.3$ for several values of the Jacobi constant.

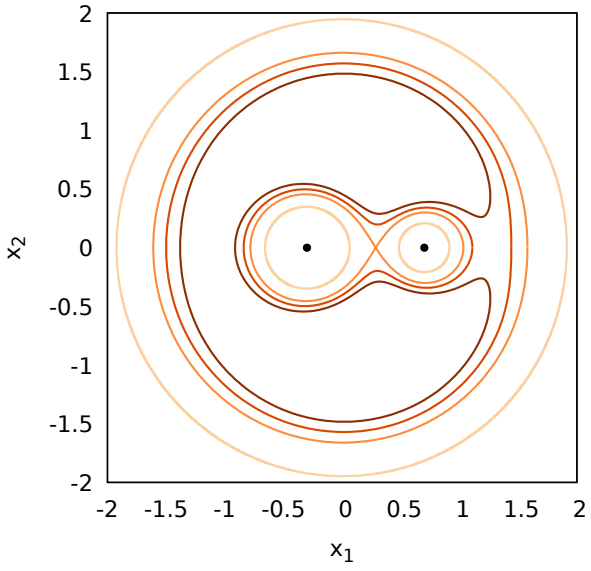

Figure 2.1: *The zero-velocity curves for a system with $\mu = 0.3$. From bright to dark, the Jacobi constants represented are $C = 5$, C_2, 3.9 and 3.7 where $C_2 = V(\mathcal{L}_2, \mu = 0.3) = 4.130149$. It also appears the position of the two primaries black dotted.*

The discussion made before about the curves for big values of C, together with the results shown in Figure 2.1, enables us to determine the allowed regions for the infinitesimal particle. For $C = 5$, the Hill's region is defined by the inner part of the two small circle-like curve and the outer part of the big one. According to the discussion for big values of C, for higher values the area of the inner circles decreases, whereas the outer circles increases its radius. Therefore, for big values of C we have that

$$\mathcal{H}(C_1, \mu) \subset \mathcal{H}(C_2, \mu), \qquad C_1 > C_2 > 5. \tag{2.19}$$

On the other hand, when the Jacobi constant decreases, the inner circles come closer until all the positions in the x_1 axis between the primaries are confined within the Hill's region, for $C = C_2$. By further decreasing C the transition of the curves is shown in the figure. For smaller values of C than $C = 3.7$, although not represented, the curves tend to encircle the equilateral Libration points $\mathcal{L}_4$ and $\mathcal{L}_5$ in a rather ellipse-like shape.

2.4 On the linear stability

Since the discussion of the linear stability involves only elemental calculation routines, it is not treated in here, but only the main results are outlined. For a review of this analysis see À. Jorba [12].

Let ξ_0 be a Libration point. In order to study the motion near this point, we translate to new symplectic coordinates (u_i, v_i) so ξ_0 is placed in the origin. Expanding the Hamiltonian in terms of the new variables we obtain

$$H = H_2 + H_3 + H_4 + \dots \tag{2.20}$$

where $H_j \in \mathbb{R}[u_i, v_i]$ is a j-degree polynomial. Note that the constant H_0 is irrelevant so it has been omitted and that there are no linear terms because an equilibrium point has been taken as the origin. Hence, the linear behaviour of the system is given by the H_2 term. Actually, the linearization of the Hamiltonian system $\dot{z} = J\nabla H$ is $\dot{z} = JHess(H_2)z$, so computing the eigenvalues of $JHess(H_2)$ leads us to the following result:

Lemma 2.2. *For $\xi_0 = \mathcal{L}_j$, $j = 1, 2, 3$ the matrix $JHess(H_2)$ has two real eigenvalues having opposite sign and two purely imaginary eigenvalues. For $\xi_0 = \mathcal{L}_j$, $j = 4, 5$ all the eigenvalues are purely imaginary for $0 < \mu < \mu_R$ where $\mu_R = \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right)$.*

From this lemma it follows that the collinear points are of the form center $\times$ saddle so they are linearly unstable. Hence, they are unstable, since higher order terms cannot make a linearly unstable point stable. On the other hand, the equilateral points are of the form center $\times$ center, so they are linearly stable. Then, to discuss the stability in these cases it is required to take into account also the non linear terms.

Chapter 3

Non-linear stability of the Lagrangian points in the plane

In the last section we have already seen that an analysis on the linear stability is not enough to draw general conclusions about the stability. For that reason, in this chapter we present alternative methods to discuss the stability of both the collinear and the equilateral points. Hereafter in this chapter, consider a phase space with two degrees of freedom (the infinitesimal particle is still in the orbit's plane).

On the one hand, in what follows the collinear points are proven to be unstable using the Lyapunov stability theory, which we introduce in the next section.

On the other hand, the equilateral points are proven to be stable under a certain assumption on the mass parameter by finding invariant objects that confine the trajectories near the equilibrium in its interior. To this end, we will introduce the well known invariant curve theorem.

3.1 Lyapunov stability theory

Here we present some parts of the classical Lyapunov stability theory that will be used later on to discuss the stability of the collinear points. Let us start introducing a couple of definitions that we use after on this chapter. To do so, consider the differential equation $\dot{z} = f(z)$ and let $\phi(t, \xi)$ be a solution. Recall that we say that ξ_0 is an equilibrium point if $\phi(t, \xi_0) = \xi_0$, for all $t \in \mathbb{R}$. For such a point, its exponents are defined as follows

Definition 3.1. *The exponents of an equilibrium point ξ_0 of $\dot{z} = f(z)$ are defined as the eigenvalues of $\frac{\partial f}{\partial z}(\xi_0)$. Moreover, we say that an equilibrium point is elementary if all the exponents are nonzero (equivalently, if $\frac{\partial f}{\partial z}(\xi_0)$ is nonsingular).*

Considering this definition it is quite straightforward to prove that the elementary equilibrium points are isolated. This result yields from the fact that $\frac{\partial f}{\partial z}(\xi_0)$ is nonsingular, so applying the implicit function theorem we obtain a neighbourhood of ξ_0 with no zeros.

Now, let us define the concept of stability in the sense of Lyapunov.

Definition 3.2. *ξ_0 is said to be positively (negatively) stable in the sense of Lyapunov if for all*

$\epsilon > 0$, it exists $\delta > 0$ such that $\|\phi(t,\xi) - \xi_0\| < \epsilon$, $\forall t \geq 0$ ($\forall t \leq 0$) when $\|\xi - \xi_0\| < \delta$. Moreover, ξ_0 is said to be stable in the sense of Lyapunov if it is both positively and negatively stable.

Definition 3.3. *ξ_0 is said to be asymptotically stable if it is positively stable in the sense of Lyapunov and it exists $\eta > 0$ such that for all $\xi \in \mathbb{R}$ satisfying $\|\xi - \xi_0\| < \eta$, then $\lim_{t\to\infty} \phi(t,\xi) = \xi_0$.*

Remember that the dynamical system of our interest is Hamiltonian. Thus, the set of differential equations defining the system is given by the equation (1.5). In such a system let us prove the necessary condition for stability that follows:

Theorem 3.4. ***(Dirichlet's stability theorem).*** *Whether ξ_0 is a strict local minimum or maximum of H, then ξ_0 is stable.*

Proof. Let us prove the stated result for the case of a minimum (the other case is analogous). Without loss of generality, let us assume that $\xi_0 = 0$ and $H(0) = 0$. As the origin is a local minimum, it exists $\eta > 0$ such that $H(\xi) > 0$ for all $0 < \|\xi\| \leq \eta$. Now, given $\epsilon > 0$, we define $\kappa = min(\epsilon, \eta)$ and $M = min\{H(\xi) \colon \|\xi\| = \kappa\}$. As H is a continuous map and $H(0) = 0$, it exists $\delta > 0$ such that for all $\|z\| < \delta$, then $H(z) < M$.

Recalling the existence and uniqueness theorem, it exists a solution ϕ with initial condition $z \in \mathbb{R}^n$. Moreover, as H is an integral, it holds that $H(z) = H(\phi(t,z))$, $\forall t \in \mathbb{R}$. Let us assume that $\|\phi(t,z)\| \geq \kappa$ for some t. Then, as $\phi(t,z)$ is a continuous map in t it exists t' such that $\|\phi(t',z)\| = \kappa$, which implies that $H(z) = H(\phi(t',z)) \geq M$, and it is a contradiction with the fact that $H(z) < M$ seen above, so $\|\phi(t,z)\| < \kappa < \epsilon$.

Hence, we have seen that for a given $\epsilon > 0$, it exists a $\delta > 0$ such that for all $\|z\| < \delta$, then $\|\phi(t,z)\| < \kappa$, $\forall t \in \mathbb{R}$. Finally, recalling that $\kappa < \epsilon$ by definition, the origin is proven to be stable. □

Above, it has been used that the Hamiltonian is a conserved magnitude along all the trajectories. The next theorem asserts that a system that admits a non degenerated first integral cannot have an asymptotically stable point, since it would imply that there is an open set where the first integral is constant, which is contradictory.

Theorem 3.5. *Let ξ_0 be an equilibrium point of the system $\dot{z} = f(z)$. Let $\mathcal{C}$ be a non degenerated integral of the system. Then, ξ_0 cannot be asymptotically stable.*

Proof. Let ξ_0 be an equilibrium point asymptotically stable. Then, it exists a neighbourhood N of ξ_0 such that if $\xi \in N$, then $\lim_{t\to\infty} \phi(t,\xi) = \xi_0$.

However, since $\mathcal{C}$ is an integral we have that $\mathcal{C}(\xi) = \mathcal{C}(\phi(t,\xi)) = C(\xi_0)$ for all $\xi \in N$. Therefore, $\mathcal{C}$ turns to be constant in an open set, which is contradictory with the fact that it is non degenerated. □

Now, we aim to prove the Lyapunov stability theorem. To this end, let $\xi_0 \in \mathbb{R}^n$ be an equilibrium point of $\dot{z} = f(z)$ and $\mathcal{O} \subset \mathbb{R}^n$ be an open neighbourhood of ξ_0 henceforth in this section. Then, a previous definition and the theorem follows:

Definition 3.6. *Let $V\colon \mathcal{O} \to \mathbb{R}$ be a smooth map. Then, V is said to be positive defined (with respect to ξ_0) if it exists an open neighbourhood $\mathcal{Q} \subset \mathcal{O}$ of ξ_0 such that $V(\xi_0) < V(\xi)$ for all $\xi \in \mathcal{Q} \setminus \{\xi_0\}$. That is, ξ_0 is a local minimum of V. Moreover, we define $\dot{V}\colon \mathcal{O} \to \mathbb{R}$ as $\dot{V}(\xi) = \nabla V(\xi) \cdot f(\xi)$.*

Theorem 3.7. ***(Lyapunov's stability theorem).*** *Suppose that it exists a smooth function $V\colon \mathcal{O} \to \mathbb{R}$ positively defined such that $\dot{V} \leq 0$ in a neighbourhood of ξ_0. Then, ξ_0 is positively stable in the sense of Lyapunov.*

Proof. Without loss of generality, let us assume that $\xi_0 = 0$ and $V(0) = 0$. By definition, $V(0) = 0$ is a local minimum, so it exists $\eta > 0$ such that $V(\xi) > 0$ for all $0 < \|\xi\| < \eta$. Given $\epsilon > 0$, taking η even smaller if necessary, can be ensured that $\dot{V}(\xi) \leq 0$ for $\|\xi\| < \eta$ (by hypothesis) and $\eta < \epsilon$.

Let $M = min\{V(\xi)\colon \|\xi\| = \eta\}$. Recalling that V is smooth and $V(0) = 0$, it exists $\delta > 0$ such that $\delta < \eta$ and $V(\xi) < M$ for all $\|\xi\| < \delta$. To prove stability for positive time we need to see that whether $\|\xi\| < \delta$, then $\|\phi(t, \xi)\| < \epsilon$, $\forall t > 0$.

Take now ξ such that $\|\xi\| < \delta < \eta$. Since $\|\phi(0, \xi)\| = \|\xi\| < \eta$ and ϕ is continuous in time, it exists $t' > 0$ such that $\|\phi(t, \xi)\| < \eta$ for $0 \leq t < t'$. Let t^* be the biggest t' fulfilling that condition. Assume now that t^* is finite, which implies that $\|\phi(t^*, \xi)\| = \eta$. Let us define $v(t) = V(\phi(t, \xi))$. It is clear that $v(0) = V(\phi(0, \xi)) = V(\xi) < M$ because $\|\xi\| < \delta < \eta$ and $\dot{v}(t) \leq 0$ for $0 \leq t \leq t^*$ by hypothesis. Thus, as v decreases, $v(t^*) < M$.

On the other hand, by definition: $v(t^*) = V(\phi(t^*, \xi)) \geq M$ because $\|\phi(t^*, \xi)\| = \eta$, which is a contradiction. Therefore, t^* is not finite and the theorem holds. □

As shown in Chapter 13 of K. Meyer et al. [15], a corollary that yields from this theorem is the Dirichlet's stability theorem 3.4 that we proved above.

The last two theorems we prove in this section are essential for the understanding of the analysis of the collinear points in the next section. Let us start with the Chetaev's theorem, which gives us a criteria of instability.

Theorem 3.8. ***(Chetaev's theorem).*** *Suppose that exists a smooth function $V\colon \mathcal{O} \to \mathbb{R}$ and an open set $\Omega \subset \mathcal{O}$ such that:*

1. *$\xi_0 \in \partial\Omega$.*
2. *$V(\xi) > 0$ for all $\xi \in \Omega$.*
3. *$V(\xi) = 0$ for all $\xi \in \partial\Omega$.*
4. *$\dot{V}(\xi) = \nabla V(\xi) \cdot f(\xi) > 0$ for all $\xi \in \Omega$.*

Then, ξ_0 is unstable. In particular, it exists a neighbourhood $\mathcal{Q}$ of ξ_0 such that all the solutions having initial condition in $\mathcal{Q} \cap \Omega$ leave $\mathcal{Q}$ in positive time.

Proof. Once again, let us assume that $\xi_0 = 0$. Let $\epsilon > 0$ be small enough so the adherence of the open ball centered in the origin of radius ϵ fits in $\mathcal{O}$, that is: $\overline{B(0, \epsilon)} \subset \mathcal{O}$. Let us define $\mathcal{Q} = \Omega \cap \{\|\xi\| < \epsilon\}$. We claim that there are trajectories starting at a point

arbitrarily close to the origin that reach a distance ϵ from the equilibrium in a finite time.

As $\mathcal{Q}$ clearly contains points arbitrarily close to the origin, for all $\delta > 0$ it exists $p \in \mathcal{Q}$ such that $\|p\| < \delta$. Let us now define $v(t) = V(\phi(t,p))$. There are only two possible options: $\phi(t,p) \in \mathcal{Q}$ for $t \geq 0$, or else $\phi(t,p)$ crosses the boundary of $\mathcal{Q}$ for the first time t^* instead.

If $\phi(t,p)$ remains in $\mathcal{Q}$ for all $t \geq 0$, then the trajectory restricts to the compact set $\overline{\mathcal{Q}}$ for all positive time. However, by hypothesis (4) $v(t)$ increases, since $\mathcal{Q} \subset \Omega$. Then, it exists some $\kappa > 0$ such that $\dot{v}(t) \geq \kappa$. Integrating, we get that $v(t) \geq \kappa t + v(0)$. Thus, it turns that $\lim_{t\to\infty} v(t) = \infty$, which is a contradiction because v is a continuous map defined in a compact set.

Hence, let us say that $\phi(t,p)$ crosses the boundary of $\mathcal{Q}$ for the first time at a time $t^* > 0$. Again, by hypothesis we have that $v(t^*) \geq v(0) > 0$. As $\phi(t^*,p) \in \partial\mathcal{Q}$, we have that $\phi(t^*,p) \in \partial\Omega$ or alternatively $\|\phi(t^*,p)\| = \epsilon$ (by definition of $\mathcal{Q}$). However, the first case implies that $v(t^*) = 0$ by hypothesis (3), which is a contradiction. Therefore, $\|\phi(t^*,p)\| = \epsilon$ and this proves the claim.

□

From now on, let us refer to the function defined in this theorem as a Chetaev function.

Lastly, the following theorem gives us a well-known result concerning the existence of periodic orbits near the collinear points. Consider still the differential equation $\dot{z} = f(z)$ and let $\phi(t,\xi_0)$ be a T-periodic solution. Then, the multipliers of this solution are defined as follows:

Definition 3.9. *The multipliers of $\phi(t,\xi_0)$ are the eigenvalues of $\frac{\partial\phi}{\partial z}(T,\xi_0)$.*

We now claim that at least one of the multipliers is $+1$. To prove this claim, we take the relation $\phi(\tau,\phi(t,\xi_0)) = \phi(t+\tau,\xi_0)$ which yields from the uniqueness theorem of solutions. Differentiating it with respect to the time and setting $t = 0$ and $\tau = T$ we obtain

$$\frac{\partial z}{\partial z}(T,\xi_0)\dot{\phi}(0,\xi_0) = \dot{\phi}(T,\xi_0) \tag{3.1}$$

Recalling that the solution ϕ is T-periodic we have that

$$\frac{\partial\phi}{\partial z}(T,\xi_0)f(\xi_0) = f(\xi_0) \tag{3.2}$$

and hence $+1$ is an eigenvalue with eigenvector $f(\xi_0)$, which proves the claim.

Finally, let us announce the Lyapunov center theorem.

Theorem 3.10. *Consider a system $\dot{z} = f(z)$ that admits a non-degenerated integral and has an equilibrium point with exponents $\pm i\omega$, $\lambda_3,\dots,\lambda_n$ where $\omega \in \mathbb{R}\setminus 0$. Then, if $\frac{\lambda_j}{i\omega}$ is not an integer for $j = 3,\dots,n$, it exists a family of periodic orbits emanating from the equilibrium point. Furthermore, when approaching the equilibrium point its period tends to $\frac{2\pi}{\omega}$ and its multipliers tend to $\exp(\frac{2\pi\lambda_j}{\omega})$ for $j = 3,\dots,n$.*

The existence of the family of periodic orbits emanating from the equilibrium point can be seen as follows: without loosing generality we can assume that the equilibrium point is $\xi_0 = 0$. Besides, we can write $f(z) = Az + g(z)$ where $g(0) = \frac{\partial g}{\partial z}(0) = 0$. If we now scale $x \to \epsilon x$, we obtain a system with the parameter ϵ given by

$$\dot{z} = Az + O(\epsilon). \tag{3.3}$$

It is clear that for $\epsilon = 0$ we get a linear system having exponents $\pm i\omega$. Moreover, the solution is $z_0 e^{At}$ where z_0 in a non-zero vector.

Recalling now the continuity of the solutions of an ordinary differential equation (ODE) with respect to the parameters, for $\epsilon \neq 0$ we have that $z_0 e^{At} + O(\epsilon)$ is a solution. For a complete proof of this theorem, see K. Meyer et al. [15]

This theorem can be applied to the collinear points, as seen in lemma 2.2. Hence, there is a one-parameter family of periodic solutions emanating from these equilibrium points.

3.2 Stability of the collinear points

It has already been stated (without being proven) in section 2.4 that the collinear equilibrium points are unstable, as they are linearly unstable. In this section we will focus on the same statement applying the main results of the Lyapunov's stability theory developed in the previous section.

First, let us obtain a normal form of the 2-degree term of the Hamiltonian centered in a collinear equilibrium point. Recalling the equation (2.20), an expansion of the Hamiltonian can be considered. Therefore, we can write $H = H_2((x,y)) + H^\dagger(x,y)$, being real analytic in a neighbourhood of the origin in $\mathbb{R}^4$ and $H^\dagger$ having at least degree 3.

To obtain a normal form of H_2, we remind the reader that in section 2.4 it has been pointed out that the collinear points are of the type center $\times$ saddle. Besides, as it is shown in the review by À. Jorba [12], the eigenvalues of H_2 are $\pm i\omega$ and $\pm\lambda$ with some $\omega, \lambda \in \mathbb{R}$.

It is well known that the elliptic matrix coming from the center behaviour of the Hamiltonian H_c can be reduced to $A = \omega J_2$. Considering now that the linearized equations are given by $\dot{z} = Az$ where $z = (x_1, y_1)^T$ and recalling the Hamilton's equations, we obtain that

$$\dot{x}_1 = \omega y_1 = \frac{\partial H_c}{\partial y_1}, \quad \dot{y}_1 = -\omega x_1 = -\frac{\partial H_c}{\partial x_1}. \tag{3.4}$$

An integration of the equations (3.4) leads us to the normal form of the center part, which can be written as $H_c = \frac{1}{2}\omega(x_1^2 + y_1^2)$.

Analogously, the part coming from the saddle is $H_s = \lambda I_2$, so the linearized equations in this case are

$$\dot{x}_2 = \frac{\partial H_s}{\partial y_2} = \lambda x_2, \quad \dot{y}_2 = -\frac{\partial H_s}{\partial x_2} = -\lambda y_2 \tag{3.5}$$

which implies that $H_s = \lambda x_2 y_2$. On the whole, we have found a normal form for H_2, so the Hamiltonian can be written as follows

$$H = H_c + H_s + H^\dagger = \frac{\omega}{2}(x_1^2 + y_1^2) + \lambda x_2 y_2 + H^\dagger(x, y). \tag{3.6}$$

Once having the normal form computed, we are ready to prove the following theorem.

Theorem 3.11. *The equilibrium located in the origin for a system with the Hamiltonian (3.6) is unstable. In fact, there is a neighbourhood $\mathcal{Q}$ of the origin such that any solution which begins off the Lyapunov center leaves the neighbourhood in both positive and negative time. That means, that the small periodic solutions given on the Lyapunov center are unstable.*

Proof. Without loosing generality, λ can be assumed positive. The equations of the movement can be easily obtained:

$$\dot{x}_1 = \frac{\partial H}{\partial y_1} = \omega y_1 + \frac{\partial H^\dagger}{\partial y_1}, \quad \dot{y}_1 = -\frac{\partial H}{\partial x_1} = -\omega x_1 - \frac{\partial H^\dagger}{\partial x_1},$$
$$\dot{x}_2 = \frac{\partial H}{\partial y_2} = \lambda x_2 + \frac{\partial H^\dagger}{\partial y_2}, \quad \dot{y}_2 = -\frac{\partial H}{\partial x_2} = -\lambda y_2 - \frac{\partial H^\dagger}{\partial x_2}.$$

Applying the Lyapunov's center theorem to the elliptic part of the Hamiltonian, we may assume that the Lyapunov center is in the coordinate plane $x_2 = y_2 = 0$. Hence, there is a parametric family of periodic orbits within the plane, so these solutions also hold that $\dot{x}_2 = \dot{y}_2 = 0$ (since both x_2 and y_2 are constants).

Regarding the equations of movement, this implies that $\frac{\partial}{\partial x_2} H^\dagger(x_2 = 0, y_2 = 0) = \frac{\partial}{\partial y_2} H^\dagger(x_2 = 0, y_2 = 0) = 0$, which means that $H^\dagger$ does not contain any term of the form $x_2(x_1^n y_1^m)$ or $y_2(x_1^n y_1^m)$, for $n, m \in \mathbb{Z}$.

Let us now define the Chetaev function that follows: $V(x_1, x_2, y_1, y_2) = \frac{1}{2}(x_2^2 - y_2^2)$. Our aim is to apply the Chetaev's theorem in order to prove the instability of the origin.

First, we claim that $\dot{V}$ is positive in a neighbourhood $\mathcal{Q}$ of the origin. By definition, $\dot{V} = \nabla V \cdot f$ where $f = J \cdot \nabla H$. An elemental procedure shows us that

$$\begin{gathered} J \cdot \nabla H = (\omega y_1 + \frac{\partial}{\partial y_1} H^\dagger, \lambda x_2 + \frac{\partial}{\partial y_2} H^\dagger, -\omega x_1 - \frac{\partial}{\partial x_1} H^\dagger, -\lambda y_2 - \frac{\partial}{\partial x_2} H^\dagger)^T, \\ \nabla V = (0, x_2, 0, -y_2)^T. \end{gathered} \tag{3.7}$$

Thus, we obtain that $\dot{V} = \lambda(x_2^2 + y_2^2) + x_2 \frac{\partial}{\partial y_2} H^\dagger + y_2 \frac{\partial}{\partial x_2} H^\dagger = \lambda(x_2^2 + y_2^2) + W(x, y)$, where we defined $W(x, y) = x_2 \frac{\partial}{\partial y_2} H^\dagger + y_2 \frac{\partial}{\partial x_2} H^\dagger$. Let us see that in fact $\|W(x, y)\| \leq \frac{\lambda}{2}(x_2^2 + y_2^2)$ in a neighbourhood $\mathcal{Q}$.

Consider the decomposition $H^\dagger = H_0^\dagger + H_2^\dagger + H_3^\dagger$ where

1. $H_0^\dagger$ does not depend on x_2, y_2.
2. $H_2^\dagger$ depends on x_2, y_2 quadratically.
3. $H_3^\dagger$ depends on x_2, y_2 at least cubically.

Note that there is no linear term, following the hypothesis made above. Let us see how does each term $H_j^\dagger$ contributes to W for a sufficiently small neighbourhood. It is clear that $H_0^\dagger$ is irrelevant, since both partials derivatives involved in W vanish for this term. Hence, we focus on the two remaining terms of $H^\dagger$.

On the one hand, it is clear that $x_2,\ y_2 \sim O((x_2^2+y_2^2)^{\frac{1}{2}})$. On the other hand, both $\frac{\partial}{\partial x_2}H_2^\dagger$ and $\frac{\partial}{\partial y_2}H_2^\dagger$ depends on $x_2,\ y_2$ at most linearly, and as $H_j^\dagger$ has at least degree 3, the previous derivatives also depend at least linearly on $x_1,\ y_1$ when they do not vanish. Similarly, for $\frac{\partial}{\partial x_2}H_3^\dagger$ and $\frac{\partial}{\partial y_2}H_3^\dagger$ we can ensure at least quadratic dependence in $x_2,\ y_2$ when they do not vanish. This little discussion brings us to

$$\begin{aligned} x_2\frac{\partial H_2^\dagger}{\partial y_2} + y_2\frac{\partial H_2^\dagger}{\partial x_2} &\sim O((x_2^2+y_2^2))\cdot O((x_1^2+y_1^2)^{\frac{1}{2}}),\\ x_2\frac{\partial H_3^\dagger}{\partial y_2} + y_2\frac{\partial H_3^\dagger}{\partial x_2} &\sim O((x_2^2+y_2^2)^{\frac{3}{2}}). \end{aligned} \tag{3.8}$$

In both cases, the growing rate near the origin is given by $O((x_1^2+y_1^2+x_2^2+y_2^2)^{\frac{3}{2}})$, while $\frac{\lambda}{2}(x_2^2+y_2^2) \sim O(x_2^2+y_2^2)$. Then, $\|W(x,y)\| \leq \frac{\lambda}{2}(x_2^2+y_2^2)$ in a neighbourhood $\mathcal{Q}$, and this proves the claim, since λ is assumed positive.

Now, let us define $\Omega = \{x_2^2 > y_2^2\} \cap \mathcal{Q} \subset \mathbb{R}^4$. It is clear that all the conditions of the Chetaev's theorem are hold (the only one being non-trivial is the condition (4), which is hold due to the claim we already proved). Likewise, reversing the time and taking the domain $\Omega' = \{x_2^2 < y_2^2\} \cap \mathcal{Q} \subset \mathbb{R}^4$ we can see that all the solutions starting in Ω' leave from $\mathcal{Q}$ in negative time. Therefore, all solutions which starts in Ω (Ω') leaves $\mathcal{Q}$ in positive (negative) time and the origin is proven to be unstable. □

All in all, we defined a Chetaev's function in a suitable domain to see that the Chetaev's theorem applies there, resulting it in the instability of the collinear points.

It can be noticed that in the previous proof the linear behaviour have been considered only to write the Hamiltonian in the form of (3.6).

3.3 Stability of the equilateral points

The stability of the equilateral points is far more complex than for the collinear points. We focus on the case where $0 < \mu < \mu_R$, since for $\mu_R < \mu < \frac{1}{2}$ the linear instability concludes the general case already as they are again unstable (see K. Meyer et al. [15]). However, in the case of our interest the points $\mathcal{L}_4$ and $\mathcal{L}_5$ are linearly stable, so it is not enough to discuss stability keeping only linear terms. Now, non linear terms may not change the stability or break it instead. For that reason, more complex theorems within the frame of the so-called KAM theory, called after Kolmogorov, Arnold, and Moser, are introduced.

First, let us momentarily move away from our Hamiltonian system of differential equations to focus on the fixed points of a diffeomorphism. Indeed, a close relation between flows and maps can be settled via the Poincare's map, which maps an $(m-1)$-dimensional manifold into itself, being m the dimension of the phase space. Furthermore, since we will fix an energy level in our 4-dimensional phase space, the dimensions will decrease in 1. Thus, a transversal region in this space is 2-dimensional,

so we will be considering maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ which are area-preserving, that is, maps $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $det(DF(\xi)) = 1$ for all $\xi \in \mathbb{R}^2$.

Next, we introduce the invariant curve theorem in order to see that many of the invariant curves of a particular map having particular frequencies persist under small perturbations. Afterwards, using this theorem invariant curves will be constructed in a cross section of our energy-fixed phase space, which will translate into invariant tori in the phase space with the fixed energy. Finally these tori will be used to prove the stability.

Although the pipeline presented in here distinguish two main theorems, they are often considered as a whole after the name of KAM theorem.

3.3.1 Moser's invariant curve theorem

There are countless ways to announce and prove the theorem that is the center of this section. We follow the scheme of J. Moser in [16] but omitting some details in the proof, as a detailed explanation of this theorem is itself an entire dissertation. A similar version of the theorem is presented in K. Meyer et al. [15] but working with action-angle variables instead of polar coordinates and skipping the proof. Finally, see J. Moser [17] for a different approach.

Let (θ, r) be the polar coordinates in $\mathbb{R}^2$ and let us define a particular map:

$$\begin{aligned} T_\phi\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ (\theta, r) &\mapsto (\theta + \phi, r) \end{aligned} \tag{3.9}$$

which we call ϕ-twist map (since it twists every point an argument of ϕ). We will work with a slightly perturbed twist map:

$$\begin{aligned} M\colon A &\to A \\ (\theta, r) &\mapsto (\theta + \alpha(r) + f(\theta, r), r + g(\theta, r)) \end{aligned} \tag{3.10}$$

where $A = \{(r, \theta)\colon 0 \leq a_0 \leq r \leq b_0,\ 0 \leq \theta \leq 2\pi\} \subset \mathbb{R}^2$ is an annulus. Moreover, we ask the functions involved in the definition of M to fulfil several conditions:

1. α is a real analytic monotone increasing function.
2. f and g are real analytic functions 2π-periodic in θ.
3. M has the following property: All curve $\Gamma = \{r = \Phi(\theta) = \Phi(\theta + 2\pi)\}$ intersects its image $M\Gamma$.

Note that taking $f \equiv g \equiv 0$ in M we get the map $T_{\alpha(r)}$. Hence, M is indeed a deformation of the twist map. Our aim is to construct M-invariant curves for a sufficiently "small" perturbation.

Remark: The condition (3) is considered in order to avoid the case in which M is an homotopy. For instance, for $f = 0$ and $g = \epsilon > 0$ sufficiently small, M is already a small perturbation of a twist map, yet it is clear that there are no M-invariant curves. To fulfil the condition (3), it is sufficient that M is an area-preserving map that preserves the inner boundary of A.

For simplicity, we will restrict ourselves to the case where $\alpha(r) = r$. After a change of variables the map M now reads:

$$\begin{aligned} &M\colon A \to A \\ &(x,y) \mapsto (x + y + f(x,y), y + g(x,y)). \end{aligned} \tag{3.11}$$

Furthermore, as f and g are assumed to be analytical, can be extended in a complex domain $D \subset \mathbb{C}$ where $|Imx| < r_0$ and $y \in D'$, being $0 < r_0 < 1$ and D' a complex neighbourhood of the interval $[a_0, b_0]$. Considering all this notation, the invariant curve theorem can be stated:

Theorem 3.12. ***(Moser's invariant curve theorem).*** *For all $\epsilon > 0$ it exists $\delta > 0$ depending on ϵ and D such that if $|f| + |g| < \delta$ in D, then M admits an invariant curve of the form*

$$x = \xi + u(\xi), \qquad y = v(\xi) \tag{3.12}$$

where u and v are real analytic functions in $|Im\xi| < \frac{1}{2}r_0$ having period 2π.

Furthermore, the parametrization is chosen so that the induced mapping on the curve (3.12) is given by $\xi_1 = \xi + \omega$ with ω a constant incommeasurable with 2π, and the functions u and v satisfying:

$$|u| + |v - \omega| < \epsilon. \tag{3.13}$$

As it has been mentioned in the beginning of this section, we only give a rough idea of the proof, because it is quite technical. Actually, it starts introducing some conditions to set the framework for the incoming lemma, which is used later on.

For $0 < s_0 < \frac{1}{4}$, let us choose ω such that $a_0 + s_0 < \omega < b_0 - s_0$, which also satisfies the diophantic condition

$$\left| \frac{\omega}{2\pi} q - p \right| \geq \frac{c_0}{q^\mu}, \qquad p, q = 1, 2 \ldots \tag{3.14}$$

where c_0 is a constant. The existence of such ω for $\mu \geq 2$ is proven in J. Moser [16] and can be justified with the following reasoning:

First, dividing by q, the diophantic condition reads

$$\left| \frac{\omega}{2\pi} - \frac{p}{q} \right| \geq \frac{c_0}{q^{\mu+1}}. \tag{3.15}$$

We claim that the set of values $\tilde{\omega} = \frac{\omega}{2\pi}$ not satisfying (3.15) has measure zero for a suitable c_0. It is clear that this set contains those points that are closer than $\delta(q) := \frac{c_0}{q^{\mu+1}}$ to $\frac{p}{q}$. If we arbitrarily take $\frac{p}{q} \in [0,1]$, then for each q from 1 to ∞, p takes the values $p = 0, 1, \ldots, q$. Hence, for each q there are $q+1$ intervals of measure $2\delta(q)$, and an upper bound of the measure of the set given by the intersection of the negation of the condition (3.15) with the interval $[0,1]$ is

$$\sum_{q=1}^{\infty} 2\delta(q)(q+1) = \sum_{q=1}^{\infty} 2\frac{c_0}{q^{\mu+1}}(q+1) = 2c_0 \left(\sum_{q=1}^{\infty} \frac{1}{q^\mu} + \sum_{q=1}^{\infty} \frac{1}{q^{\mu+1}} \right). \tag{3.16}$$

The first series converge if $\mu \geq 2$, while the second does if $\mu \geq 1$. Hence, both converge

for $\mu \geq 2$, proving that the measure of the set of $\tilde{\omega}$ not satisfying (3.15) is 0 for a suitable c_0 and for $\mu \geq 2$. Therefore, let us fix c_0 and ω hereafter.

In the complex plane, consider now the domain $\mathcal{U}$ defined by $|Imx| < r$ and $|y - \omega| < s$ and let us assume that $|f| + |g| < d$, where r, s and d are positive constants. Consider also $B \subset \mathcal{U}$ given by $|Imx| < \rho$ and $|y - \omega| < \sigma$, where $0 < \rho < r$ and $0 < \sigma < s$, and intermediate domains $\mathcal{U}^\nu$ for $\nu = 1, 2, 3$ defined by $|Imx| < r - \frac{r-\rho}{4}\nu$ and $|y - \omega| < s - \frac{s-\sigma}{4}\nu$ such that $B \subset \mathcal{U}^3 \subset \mathcal{U}^2 \subset \mathcal{U}^1 \subset \mathcal{U}$.

Let us assume that the previously defined constants satisfy:

$$0 < r \leq 1, \quad 0 < 3\sigma < s < \frac{r-\rho}{4}, \quad d < \frac{s}{6}. \tag{3.17}$$

Moreover, we also impose that

$$\zeta := c_3 (r-\rho)^{-2(\mu+1)} \frac{d}{s} < \frac{1}{7}, \tag{3.18}$$

being c_3 a positive constant to determine later on. Under these conditions, we can state the following lemma:

Lemma 3.13. *Under the assumptions made above, it exists a transformation U of the form*

$$x = \xi + u(\xi, \eta), \quad y = \eta + v(\xi, \eta) \tag{3.19}$$

where u and v are real analytic in $\mathcal{U}^1$ of period 2π in ξ such that the mapping $U^{-1}MU$ takes the form

$$\xi_1 = \xi + \eta + \phi(\xi, \eta), \quad \eta_1 = \eta + \varphi(\xi, \eta) \tag{3.20}$$

with ϕ and φ real analytic defined in B and satisfying:

$$|\phi| + |\varphi| < c_6 \left\{ (r-\rho)^{-2\mu-3} \left(\frac{d^2}{s} + sd \right) + \left(\frac{\sigma}{s} \right)^2 d \right\}. \tag{3.21}$$

for a certain c_6. Besides, in $\mathcal{U}^1$ the inequality $|u| + |v| < \zeta \cdot s$ is hold.

The proof of this lemma will be omitted (see J. Moser [16], where a whole section is devoted to this). Note that this lemma gives us a mapping $U^{-1}MU$ that has the same form than M but giving also an upper bound of the error[1]. Then, the proof proceeds as follows: we will apply iteratively this lemma in order to get a map defined each time in a smaller complex domain and closer to the twist map so the initial domain shrinks down to the pursued invariant curve.

Let us denote $M_0 = M$ and $\mathcal{U}_0$ be defined by $|Imx| < r_0$ and $|y - \omega| < s_0$. We assume that it exists $\delta > 0$ such that $|f| + |g| < \delta$ (in terms of the notation introduced before the lemma, $\delta = d_0$). Applying the lemma, we get that it exists a transformation U_0 such that $M_1 = U_0^{-1} M_0 U_0$ is defined in $\mathcal{U}_1 \subset \mathcal{U}_0$ given by $|Imx| < r_1$ and $|y - \omega| < s_1$ (being r_1 and s_1 the parameters ρ and σ of the lemma respectively). Applying this lemma iteratively with $r = r_n$, $s = s_n$, $d = d_n$, $\rho = r_{n+1}$ and $\sigma = s_{n+1}$, we obtain the map $M_{n+1} = U_n^{-1} M_n U_n$ defined in $\mathcal{U}_n \subset \ldots \subset \mathcal{U}_0$ given by $|Imx| < r_n$ and $|y - \omega| < s_n$. In

[1]The functions ϕ and φ are called the error functions as whether they vanish, then $U^{-1}MU$ is the twist map.

particular, let us set the parameters r_n, s_n and d_n of each iteration as follows

$$r_n = \frac{r_0}{2}\left(1+\frac{1}{2^n}\right), \quad s_n = d_n^{\frac{2}{3}}, \quad d_{n+1} = r_0^{-2\mu-3} c_7^{n+1} d_n^{\frac{4}{3}} \tag{3.22}$$

where c_7 is again a suitable constant.

Now, we would need to prove that M_n is well defined (which can be done by induction) and that M_n tends to the twist mapping when $n \to \infty$. Let us focus on the second.

It is clear that U_k maps $\mathcal{U}_{k+1}$ into $\mathcal{U}_k$. Hence, the transformation $V_n = U_0 U_1 \dots U_n$ is well defined and it enables us to write $M_{n+1} = V_n^{-1} M_0 V_n$. Moreover, we can write V_n as

$$x = \xi + p_n(\xi,\eta), \quad y = \eta + q_n(\xi,\eta) \tag{3.23}$$

where p_n and q_n are analytic in $\mathcal{U}_{n+1}$, since they are the composition of analytic functions. We claim now that p_n and q_n converge to analytic functions. By its definition, it is easy to see that $V_n = V_{n-1}U_n$, where the right sided transformation follows: U_n is given by $x = \xi + u_n$ and $y = \eta + v_n$, so the composition $V_{n-1}U_n$ reads

$$x = \xi + u_n + p_{n-1}(\xi + u_n, \eta + v_n), \quad y = \eta + v_n + q_{n-1}(\xi + u_n, \eta + v_n) \tag{3.24}$$

where u_n and v_n correspond to $\mathcal{U}_n$. Then, from $V_n = V_{n-1}U_n$ it turns that

$$p_n(\xi,\eta) = u_n + p_{n-1}(\xi + u_n, \eta + v_n), \quad q_n(\xi,\eta) = v_n + q_{n-1}(\xi + u_n, \eta + v_n). \tag{3.25}$$

Applying this process recursively and dismissing some arguments that are not needed afterwards, we finally obtain

$$p_n = u_n + u_{n-1} + \ldots + u_0, \quad q_n = v_n + v_{n-1} + \ldots + v_0 \tag{3.26}$$

and from the lemma we know $|u| + |v| < \zeta_n s_n < \frac{1}{7} s_n$, since $\zeta_n < \frac{1}{7}$.

It is not difficult to see that d_n converges to zero, so s_n converges uniformly. Then, from (3.26) together with the convergence of s_n, we get that p_n and q_n converge uniformly in the set given by $|Im\xi| < \frac{r_0}{2}$ and $\eta = \omega$ (since by (3.22) $r_n \to \frac{r_0}{2}$ and $s_n \to 0$) so we proved the claim. Hence, let $p_n \to u(\xi)$ and $q_n \to v(\xi) - \omega$ when $n \to \infty$, where $u(\xi)$ and $v(\xi)$ are real analytic 2π-periodic functions.

Now, it can be proved that $|p_n| + |q_n| < s_0$, so choosing d_0 (and hence s_0) small enough such that $s_0 < \epsilon$, we have that for all $\epsilon > 0$ exists $\delta > 0$ such that if $|p_n| + |q_n| < s_0$, then $|u| + |v - \omega| < \epsilon$, as u and $v - \omega$ are the limit of p_n and q_n respectively. Thus, whether we manage to see that the curve given by (3.12) is M_0-invariant the proof will be concluded.

Remark: Notice that since $d_n \to 0$ when $n \to \infty$ and $|f| + |g| < d_n$, the mapping M_n approaches to the twist mapping for increasing n, as asserted.

From $M_{n+1} = V_n^{-1} M_0 V_n$ we have that $V_n M_{n+1} = M_0 V_n$. Now, let us take this equality to the limit when $n \to \infty$. On the one hand, we already discussed that $M_{n+1} \to T_\omega$ when $n \to \infty$, so considering that $\eta = \omega$, the composition $V_n M_{n+1}$ when $n \to \infty$ is given by

$$\xi_1 = \xi + \omega + u(\xi + \omega), \quad \eta_1 = v(\xi + \omega). \tag{3.27}$$

On the other hand, using once again that V_n is known when $n \to \infty$ because the limits of p_n and q_n are $u(\xi)$ and $v(\xi) - \omega$ respectively, we have that in this limit $M_0 V_n$ is given by

$$\xi_1 = \xi + u(\xi) + v(\xi) + f(\xi + u, v), \qquad \eta_1 = v(\xi) + g(\xi + u, v). \tag{3.28}$$

Now, from (3.27) and (3.28) we finally obtain

$$\begin{aligned} \xi + \omega + u(\xi + \omega) &= \xi + u(\xi) + v(\xi) + f(\xi + u, v), \\ v(\xi + \omega) &= v(\xi) + g(\xi + u, v). \end{aligned} \tag{3.29}$$

Lastly, we claim that the conditions (3.29) imply that the curve (3.12), which will be denoted $\gamma(\xi)$, is M-invariant. Clearly, considering the relations (3.29) the curve $M\gamma(\xi)$ can be written as

$$\xi_1 = \xi + \omega + u(\xi + \omega), \qquad \eta_1 = v(\xi + \omega) \tag{3.30}$$

which is the curve $\gamma(\xi + \omega)$, that has the same image that $\gamma(\xi)$ as u and v are 2π-periodic. This proves the claim and therefore the theorem.

So far, this theorem leads us to the existence of an M-invariant curve. However, there is no impediment in applying the theorem for each sub interval of $[a_0, b_0]$, which eventually gives us an infinite number of invariant curves, all of them having particular frequencies.

It might be noticed that the condition (3) made on the map M has not been used so far, as it is required in the proof of the lemma, which we skipped. Nonetheless, it has been remarked that it is a necessary condition to avoid having an homotopy as a deformation.

The diophantic condition (3.14) asked on ω is used in the lemma as well. We are basically imposing $\frac{\omega}{2\pi}$ to be irrationally enough so the image of the map M is dense. Indeed, whether one tries to prove the theorem by expressing f and g as power series to seek the expansion of the curve (3.12) in the Fourier space, it turns that they converge if the term $e^{ik\omega} - 1$ for $k = \pm 1, \pm 2, \ldots$ do not approach zero too rapidly (see J. Moser [16]). This fact illustrates the need of $\frac{\omega}{2\pi}$ being irrational but fails to prove the result using these series since their convergence cannot be guaranteed.

3.3.2 Birkhoff normal form

Let us move momentarily apart from the Invariant curve theorem to focus on finding a normal form of the Hamiltonian centered in one of the equilateral equilibrium points. Afterwards, the statement of stability is proven using this normal form. Consider firstly, the following proposition:

Proposition 3.14. *Let $G(x, y)$ be a Hamiltonian function and $(x_0(t), y_0(t))$ a solution of the system. Then, for each smooth function $f = f(x, y)$ we have that*

$$\frac{d}{dt} f(x_0(t), y_0(t)) = \{f, G\}(x_0(t), y_0(t)). \tag{3.31}$$

Proof. We only have to apply the chain rule and then recall the relations given by the

Hamilton's equations:

$$\frac{d}{dt}f(x,y) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial G}{\partial x} = \{f, G\}(x,y).$$

□

After this preliminary result, consider our Hamiltonian H in the 4-dimensional phase space and a transformation $\tilde{H}$ given by a Taylor expansion around the origin (an equilateral point) evaluated in $t = 1$

$$\tilde{H} = H + H' + \frac{1}{2}H'' + \frac{1}{3!}H''' + \dots \tag{3.32}$$

Applying now the property given by proposition 3.14 the equation (3.32) reads

$$\tilde{H} = H + \{H, G\} + \frac{1}{2}\{\{H, G\}, G\} + \frac{1}{3!}\{\{\{H, G\}, G\}, G\} + \dots \tag{3.33}$$

where G is a generating function not determined yet and H can be written as in the equation (2.20) where H_j is a monomial of degree j. Besides, if we impose $G_1 = G_2 = 0$, where G_j is a j-degree monomial and recall that whether P and Q have degree r and s respectively, then $\{P, Q\}$ has degree $r + s - 2$, it follows that the 3-degree monomial of $\tilde{H}$ is

$$\tilde{H}_3 = H_3 + \{H_2, G_3\}. \tag{3.34}$$

Now we aim to simplify the form of $\tilde{H}$ by choosing G_3 in a manner that makes $\tilde{H}_3$ vanishes. In that direction, we use the known expression $H_2 = i\omega_1 y_1 x_1 + i\omega_2 y_2 x_2$ where $\omega = (\omega_1, \omega_2)$ is assumed to have its components independent over the rationals. This expression is deducted in the same way we obtained the term H_2 for the collinear points, and now a further transformation is applied to consider complex variables.

Let us denote $x^{k_x} = x_1^{k_{x_1}} x_2^{k_{x_2}}$ and similarly for y^{k_y}, where k_x, $k_y \in \mathbb{Z}^2$. Then, it is clear that considering some coefficients h_{k_x,k_y} and g_{k_x,k_y} we can write

$$H_3(x,y) = \sum_{|k_x|+|k_y|=3} h_{k_x,k_y} x^{k_x} y^{k_y}, \qquad G_3(x,y) = \sum_{|k_x|+|k_y|=3} g_{k_x,k_y} x^{k_x} y^{k_y} \tag{3.35}$$

where $|k_x| = k_{x_1} + k_{x_2}$ and analogously for $|k_y|$. Recalling now the definition of the Poisson brackets, after a simple calculation we obtain that

$$\{H_2, x^{k_x} y^{k_y}\} = i\langle k_y - k_x, \omega\rangle x^{k_x} y^{k_y}$$

where $k_y - k_x = (k_{y_1} - k_{x_1}, k_{y_2} - k_{x_2})$ and $\langle\cdot,\cdot\rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denotes the scalar product. This, along with the equation (3.34) an the expressions (3.35) implies that

$$\sum_{|k_x|+|k_y|=3} h_{k_x,k_y} x^{k_x} y^{k_y} + \sum_{|k_x|+|k_y|=3} g_{k_x,k_y} i\langle k_y - k_x, \omega\rangle x^{k_x} y^{k_y} = 0$$

which can be expressed using a single summation as

$$\sum_{|k_x|+|k_y|=3} \left(h_{k_x,k_y} x^{k_x} y^{k_y} + g_{k_x,k_y} i\langle k_y - k_x, \omega\rangle x^{k_x} y^{k_y} \right) = 0.$$

Now, vanishing each term of the summation, we obtain

$$g_{k_x,k_y} = \frac{-h_{k_x,k_y}}{i\langle k_y - k_x, \omega\rangle}. \tag{3.36}$$

Note that $k_y - k_x \neq 0$, since $|k_x| + |k_y| = 3$. Moreover, this together with the fact that ω_1 and ω_2 are linearly independent over the rationals, ensure us that $\langle k_y - k_x, \omega\rangle \neq 0$. Therefore, the coefficients g_{k_x,k_y} are well defined. So far, we have seen that G can be chosen such that $\tilde{H}_3 = 0$. Likewise, the same scheme can be applied for $d \in \mathbb{N}$, by choosing suitable G_j to vanish $\tilde{H}_{2d+1}$.

On the other hand, notice that for $|k_x| + |k_y| = 2d$, where $d \in \mathbb{N}$, the vector $k_y - k_x$ may vanish, so $\tilde{H}_{2d}$ can not generally be removed for $d \in \mathbb{N}$.

Thus, after this discussion we can reach the following normal form:

$$H = H_2 + H_4 + \ldots + H_{2N} + H^{\dagger}. \tag{3.37}$$

3.3.3 Invariant tori

Here we introduce a topological concept that will be a key point for incoming results. To this end, let us consider a system having n degrees of freedom to introduce the concept of integrable system, in which given a complete set of boundary conditions, the solution can be found analytically through the conserved quantities.

As we have seen, a first integral enables us to decrease the degrees of freedom of a system. However, the presence of two constants of movement not always reduces the degrees of freedom by a factor of two. Instead, it happens only if the two integrals are in involution.

Definition 3.15. *Being Ω a phase space, we say that n functions $F_1, \ldots, F_n \colon \mathbb{R} \times \Omega \to \mathbb{R}$ are in involution if it is hold*

$$\{F_i, F_j\} = 0, \quad 1 \leq i, j \leq n. \tag{3.38}$$

From the usually called Arnold-Liouville theorem (see V. Arnold [2]), we know that a system with n degrees of freedom is integrable if it admits n first integrals in involution. Also, this theorem proves that in such a case it exists a set of symplectic action-angle coordinates.

Now, supposing that our Hamiltonian system is integrable with an added perturbation and considering action-angle coordinates (I_i, ϕ_i), the Hamiltonian having n degrees of freedom can be written as $H = H(I_i) + \epsilon H_1(I_i, \phi_i)$, with $i = 1, \ldots, n$ where the second term is the perturbation. For now, consider only the part depending exclusively on the actions. Then, the equations of movement that yield from the Hamilton's equations read

$$\dot{I}_i = 0, \quad \dot{\phi}_i = \frac{\partial H}{\partial I_i} = \omega_j(I_i). \tag{3.39}$$

The integration of these equations leads us to $I_i(t) = I_i(0)$ and $\phi(t) = \omega_j(I_i(0))t + \phi_i(0)$.

Notice that the image of each couple of solutions $I_i(t)$ and $\phi_i(t)$ is $\mathbb{S}^1 \subset \mathbb{R}^2$. Hence, the movement in the phase space is defined by an n-dimensional torus

$$\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1. \tag{3.40}$$

having frequencies ω_j and depending on the action variables. On the whole, we have seen that the integrable part of the Hamiltonian make the phase space to be filled by solutions defining different tori. As we will see later in this chapter for a 4-dimensional phase space, the Moser's theorem ensures that the majority of these tori are deformed but not destroyed (those having sufficiently irrational frequencies) when considering the perturbative part.

Taking into account what we have just discussed in this section, the Hamiltonian (3.37) fulfils the following conditions:

1. H is real analytic in a domain containing the origin in $\mathbb{R}^4$.
2. H_{2k} is a homogeneous polynomial of degree k in I_1, I_2 for $1 \leq k \leq N$.
3. $H^\dagger$ admits a series expansion of terms having at least degree $N+1$.
4. $H_2 = \omega_1 I_1 - \omega_2 I_2$, with ω_i nonzero constants.

The last condition comes from the linearization of the Hamiltonian and from considering action-angle coordinates.

3.3.4 Arnold's stability theorem

The theorem that is stated and proved in this section, along with the normal form of our 2-degrees of freedom Hamiltonian obtained in section 3.3.2, enable us to prove that the Libration points $\mathcal{L}_4$ and $\mathcal{L}_5$ are stable when $\mu < \mu_R$.

To this end, we fix an energy level in our phase space $\Omega \subset \mathbb{R}^4$ (restricting ourselves to a 3-dimensional subspace) and consider there the Poincare's map, mapping a cross section $\Sigma \subset \mathbb{R}^2$ into itself. For this map get an invariant curve in Σ via the Moser's theorem, which results in an invariant torus on the phase space with the fixed level.

The solutions with initial conditions within these invariant tori are confined there for all positive and negative time, proving stability. In the proof of the following theorem this idea is formalized further.

Theorem 3.16. ***(Arnold's stability theorem).*** *Consider a Hamiltonian of the form (3.37) such that for some $1 \leq k \leq N$ it holds $D_{2k} = H_{2k}(\omega_1, \omega_2) \neq 0$ or, equivalently, H_2 does not divide H_{2k}. Then, the equilibrium point at the origin is stable. Moreover, arbitrarily close to the origin in $\mathbb{R}^4$ there are invariant tori and the flow on these invariant tori is the linear flow with irrational slope.*

Proof. Let us assume that $D_2 = \ldots = D_{2N-2} = 0$ and $D_{2N} \neq 0$. Then, since H_2 divides H_{2k} for $k = 2, \ldots, N-1$, there are homogeneous polynomials F_{2k} for these values of k having degree $2k$ such that $H_{2k} = H_2 F_{2k-2}$. Therefore, the Hamiltonian (3.37) can be rewritten as

$$H = H_2(1 + F_2 + \ldots + F_{2N-4}) + H_{2N} + H^\dagger. \tag{3.41}$$

Let us introduce action-angle coordinates in the phase space (which corresponds to a symplectic transformation). The actions and angles read respectively $I_i = \frac{1}{2}(x_i^2 + y_i^2)$ and $\phi_i = arctan(\frac{y_i}{x_i})$ for $i = 1, 2$. Moreover, consider a scaling in the action coordinates such that $I_i = \epsilon^2 J_i$, being ϵ the scaling parameter.

Notice that as H_{2k} is a homogeneous polynomial of degree k in I_1, I_2, then it is also a homogeneous polynomial of degree k in J_1, J_2 accompanied by the term ϵ^{2k} and by the multiplier of the transformation ϵ^{-2}. In addition, H reads:

$$\begin{aligned} H = H_2 + \epsilon^2 H_4 + \ldots + \epsilon^{2N-2} H_{2N} + O(\epsilon^{2N-1}) = \\ = H_2(1 + \epsilon^2 F_2 + \ldots + \epsilon^{2N-4} F_{2N-4}) + \epsilon^{2N-2} H_{2N} + O(\epsilon^{2N-1}). \end{aligned} \tag{3.42}$$

Let us focus on a bounded neighbourhood of the origin $\mathcal{O}$ given by $|J_i| \leq 4$ for $i = 1, 2$. In $\mathcal{O}$, we claim that the following equality holds:

$$H - \epsilon^{2N-1} \cdot h = KF \tag{3.43}$$

where $K = H_2 + \epsilon^{2N-2} H_{2N} + O(\epsilon^{2N-1})$, $F = 1 + \epsilon^2 F_2 + \ldots + \epsilon^{2N-4} F_{2N-4}$, and $h \in [-1, 1]$. For the left-sided part, it is clear that $-\epsilon^{2N-1} \cdot h$ can be included in $O(\epsilon^{2N-1})$, so this part is equal to H. On the other hand, the right-sided part reads

$$KF = H_2 F + \epsilon^{2N-2} H_{2N} F + O(\epsilon^{2N-1}) F = H_2 F + \epsilon^{2N-2} H_{2N} + O(\epsilon^{2N-1})$$

which again equals to H. Thus, we proved the claim.

Consider now that ϵ is sufficiently small to let F be positive. Then, imposing $K = 0$ in (3.43) we obtain $H = \epsilon^{2N-1} \cdot h$, so H is in a fixed level characterized by $K = 0$. Let us denote $z = (J_i, \phi_i)$ where $i = 1, 2$ and $\nabla = \nabla_z$ the gradient operator in these variables. Recalling that $H = KF$ and elemental properties of the nabla operator, the equations of movement are given by

$$\dot{z} = J\nabla H = J\nabla(KF) = J[(\nabla K)F + K(\nabla F)] = (J\nabla K)F + K(J\nabla F). \tag{3.44}$$

Thus, for $K = 0$ the equations read $\dot{z} = (J\nabla K)F$. Considering the reparametrization $d\tau = F dt$ we get:

$$\frac{dz}{d\tau} = z' = J\nabla K. \tag{3.45}$$

Notice that we proved that for a sufficiently small ϵ the system given by H in the level $H = \epsilon^{2N-1} h$ can be reparametrized by K in $\mathcal{O}$. Under these conditions, recalling the explicit form $H_2 = \omega_1 J_1 - \omega_2 J_2$, Hamilton's equations read

$$\begin{aligned} J_1' &= \frac{\partial K}{\partial \phi_1} = O(\epsilon^{2N-1}) = J_2', \\ \phi_1' &= -\frac{\partial K}{\partial J_1} = -\omega_1 - \epsilon^{2N-2} \frac{\partial H_{2N}}{\partial J_1} + O(\epsilon^{2N-1}), \\ \phi_2' &= -\frac{\partial K}{\partial J_2} = \omega_2 - \epsilon^{2N-2} \frac{\partial H_{2N}}{\partial J_2} + O(\epsilon^{2N-1}). \end{aligned} \tag{3.46}$$

We now want to find the Poincare's map in the cross section $\Sigma \subset \mathbb{R}^2$ given by $\phi_2 = 0 \pmod{2\pi}$ in the energy level $K = 0$. Integrating the last equation in (3.46) taking $\phi_2(0) = 0$ we obtain that

$$\phi_2 = \left[\omega_2 \left(1 - \frac{\epsilon^{2N-2}}{\omega_2} \frac{\partial H_{2N}}{\partial J_2} \right) + O(\epsilon^{2N-1}) \right] \tau.$$

Hence, the time that it takes for ϕ_2 to reach 2π for the first time is

$$T = \frac{2\pi}{\omega_2}\left(1 - \frac{\epsilon^{2N-2}}{\omega_2}\frac{\partial H_{2N}}{\partial J_2}\right)^{-1} + O(\epsilon^{2N-1}) = \frac{2\pi}{\omega_2}\left(1 + \frac{\epsilon^{2N-2}}{\omega_2}\frac{\partial H_{2N}}{\partial J_2}\right) + O(\epsilon^{2N-1}) \tag{3.47}$$

where in the last equality it has been used that $\frac{1}{1-x} = 1 + x + O(x^2)$ for x small.

Now, integrating the second equation in (3.46) from $\tau = 0$ to $\tau = T$:

$$\phi_1 = \phi_0 - \left(\omega_1 + \epsilon^{2N-2}\frac{\partial H_{2N}}{\partial J_1}\right) T + O(\epsilon^{2N-1}).$$

Substituting the expression of T found in (3.47) we finally obtain

$$\phi_1 = \phi_0 - 2\pi\left(\frac{\omega_1}{\omega_2}\right) - \epsilon^{2N-2}\frac{2\pi}{\omega_2^2}\left(\omega_2\frac{\partial H_{2N}}{\partial J_1} + \omega_1\frac{\partial H_{2N}}{\partial J_2}\right) + O(\epsilon^{2N-1}). \tag{3.48}$$

On the other hand, as we confined ourselves in the level $K = 0$, it follows that $H_2 + \epsilon^{2N-2}H_{2N} + O(\epsilon^{2N-1}) = 0$. Recalling once again the explicit form of H_2 and considering only the linear terms in ϵ, we get the relation

$$J_2 = \frac{\omega_1}{\omega_2}J_1 + O(\epsilon^2). \tag{3.49}$$

Finally, mixing the equations (3.48, 3.49) and applying Euler's theorem on homogeneous polynomials we reach that

$$\phi_1 = \phi_0 + \alpha + \epsilon^{2N-2}\beta J_1^{N-1} + O(\epsilon^{2N-1}) \tag{3.50}$$

where $\alpha = -2\pi\left(\frac{\omega_1}{\omega_2}\right)$ and $\beta = -2\pi\left(\frac{N}{\omega_2^{N+1}}\right)H_{2N}(\omega_1, \omega_2)$. Note that $\beta \neq 0$, since $D_{2N} = H_{2N}(\omega_1, \omega_2) \neq 0$ by hypothesis.

Observe, that the equation (3.50) enables us to define the Poincare's map M given by

$$\phi_1' = \phi_0 + \alpha + \epsilon^{2N-2}\beta J_1^{N-1} + O(\epsilon^{2N-1}), \qquad J_1' = J_1 + O(\epsilon^{2N-1}) \tag{3.51}$$

which is area-preserving and can be defined in the annulus given by $\frac{1}{2} \leq J_1 \leq 3$ for a sufficiently small ϵ. Hence, since this map M fulfils the conditions mentioned in the Moser's invariant curve theorem, it exists $\epsilon_0 > 0$ such that for all $0 \leq \epsilon \leq \epsilon_0$ it exists an M-invariant curve taking the form $J_1 = \rho(\phi_1, \epsilon)$ where ρ is continuous, 2π-periodic, and fulfilling that $\frac{1}{2} \leq \rho(\phi_1, \epsilon) \leq 3$, $\forall \phi_1$.

Until now, we have found an invariant curve of the Poincare's map in the cross section $\Sigma \subset \mathbb{R}^2$. Since this map is given by the intersection of the solutions of (3.46) with the cross section Σ, recalling that an invariant manifold cannot be crossed by any solution (which yields from the existence and uniqueness theorem of ordinary differential equations) it is clear that for $K = 0$, a solution with initial condition $J_1(0) < \frac{1}{2}$ will satisfy $J_1(\tau) < 3$, $\forall \tau$, still in Σ. Moreover, since for $K = 0$ we have already seen the relation (3.49), a bound of J_1 gives us a bound of J_2.

Therefore, there are constants C_1 and C_2 such that if $J_1(\tau)$ and $J_2(\tau)$ are solutions of (3.46) for $K = 0$ fulfilling $|J_i(0)| \leq C_1$ for $i = 1, 2$ then $|J_i(\tau)| \leq C_2$ $\forall \tau$, $h \in [-1, 1]$, $0 \leq \epsilon \leq \epsilon_0$.

Now, going back to the original action-angle coordinates (I_i, ϕ_i) for $i = 1, 2$ and to the original Hamiltonian, what we have seen so far is that given the fixed level $H = \epsilon^{2N-1} \cdot h$ and initial conditions such that $|I_i(0)| \leq \epsilon^2 C_1$ for $i = 1, 2$ the solutions $I_i(t)$ satisfy that $|I_i(t)| \leq \epsilon^2 C_2$ for $i = 1, 2$, $\forall t$, $h \in [-1, 1]$, $0 \leq \epsilon \leq \epsilon_0$. This concludes that the origin is stable. Furthermore, it can be noticed that any of the invariant curves on Σ is a transversal section of an invariant 2-dimensional torus in the phase space restricted to the level given by $K = 0$. □

Remark: It is important to note that the torus is not necessarily $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^3$, but something diffeomorphic, as the image of the invariant curve given by the Moser's theorem is in general a deformation of $\mathbb{S}^1$.

Remark: The fact that $D_{2N} \neq 0$ is crucial to prove the statement. If D_{2N} is zero at some point then β may vanishes. Paying attention at the map (3.51), this means that the invariant curves given by the Moser's theorem will have all the same frequency, resulting in invariant tori of the same frequency. Hence, in resonant condition all the invariant tori might be destroyed and stability can not be proved.

Now, this theorem can be used to prove the stability of the Lagrangian points $\mathcal{L}_4$ and $\mathcal{L}_5$. In our case, it can be seen, computing the normal form up to the fourth term that

$$D_4 = H_4 = \frac{1}{2}(A\omega_2^2 + 2\omega_1\omega_2 + C\omega_1^2) \tag{3.52}$$

where A, B and C are coefficients first calculated by Deprit and Deprit-Bartholome [6]. It turns that, D_4 vanishes for 3 critical values of $\mu < \mu_R$ (let us put μ_1, μ_2 and μ_c). However, for μ_c it was proven in Meyer and Schmidt [14] that $D_6(\mu_c) \neq 0$. Thus, we can ensure stability for all values of $\mu < \mu_R$ but for μ_1 and μ_2. In fact, for these values the Hamiltonian is unstable (see K. Meyer et al.[15] and A. Markeev [13]).

Chapter 4

On the non-linear stability of the equilateral Lagrangian points in the space

From now on we focus only on the stability of the Lagrangian points $\mathcal{L}_4$ and $\mathcal{L}_5$. Since we consider now the problem in the space (meaning that the infinitesimal particle is not anymore restricted to the plane containing the orbits of the primaries), we have one more degree of freedom.

First, let us remark that the KAM theory developed in the previous chapter can be applied in this case to prove the existence of invariant tori near the equilibrium as well. However, we can easily see that these tori do not enclose a region in the phase space, meaning that a trajectory starting near one of these tori may diverge in both positive and negative time.

Since our new Hamiltonian has 3 degrees of freedom, by fixing an energy level we obtain a 5-dimensional subspace of the 6-dimensional phase space Ω. Besides, the tori given by the Moser's and Arnold's theorems are $\mathbb{T}^3$, as seen in section 3.3.3. Hence, we have 3-dimensional invariant tori in $\mathbb{R}^5$, but since these tori are not 1-codimensional with $\mathbb{R}^5$, they do not enclose a subregion of $\mathbb{R}^5$.

Moreover, although these points were thought to be stable until the mid-1960s, V. Arnold conjectured in [1] that they are unstable instead. Actually, the phenomenon of diffusion pushing the infinitesimal particle away from the equilibrium is named after Arnold. However, it might take a long time (of the scale of the solar system's age) for a solution to move away from an invariant torus, so the term of effective stability is introduced: an equilibrium point is said to be effectively stable if a particle remains there for a long period of time.

In addition, this chapter is entirely devoted to obtaining a bound from bellow of the time that it would take for a trajectory within a ball of radius ρ_0 to leave from a ball of radius ρ, being $\rho > \rho_0$. The scheme followed in here is based on the Nekhoroshev theory. An introduction to this new theory can be found in D. Benest et al. [4] and in the references therein.

Bellow, the new Hamiltonian and the first integrals are introduced. Then, we announce and prove some technical lemmas that are useful to estimate some bounds of the Hamiltonian and the first integrals afterwards, following the steps made by A. Cel-

letti et al. [5]. Finally, we manage to obtain an estimation of the stability time and we implement a code in C++ to compute this bound for the Sun-Jupiter system, using here a slightly different approach, enriched with the open software developed by À. Jorba [11].

4.1 Theoretical framework

To start with, we introduce the Hamiltonian governing the dynamics of the problem. In section 2.1 we already introduced in a deductive manner the Hamiltonian for the same problem considering only two degrees of freedom.

Following our previous steps, consider the same units system and positions of the primaries as in section 2.1. Let now $X^T = (X_1, X_2, X_3)$ be the position of the infinitesimal particle. The expression of the Lagrangian must include now the component $\dot{X}_3$ of the velocity, so the equation (2.3) is rewritten as

$$L = T - U = \frac{1}{2}(\dot{X_1}^2 + \dot{X_2}^2 + \dot{X_3}^2) + \frac{\mu}{d_1'} + \frac{1-\mu}{d_2'} \tag{4.1}$$

being d_1' and d_2' the new distances from the infinitesimal particle to each of the primaries.

We take now the rotational frame of reference introduced in section 2.1 as well, which is now given by the generalised transformation

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad X_3 = x_3 \tag{4.2}$$

where $x^T = (x_1, x_2, x_3)$ are the coordinates of the infinitesimal particle in this new frame of reference. Now, analogously to the planar case, the Lagrangian in this system is written as follows

$$L = \frac{1}{2}\left((\dot{x_1} - x_2)^2 + (\dot{x_2} + x_1)^2 + \dot{x_3}^2\right) + \frac{\mu}{d_1'} + \frac{1-\mu}{d_2'} \tag{4.3}$$

with $d_1'^2 = (x_1 - 1 + \mu)^2 + x_2^2 + x_3^2$ and $d_2'^2 = (x_1 + \mu)^2 + x_2^2 + x_3^2$. Finally, from the definition of the Hamiltonian and writing the velocities in terms of the generalised moments, we reach that

$$H = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) + x_2 y_1 - x_1 y_2 - \frac{\mu}{d_1'} - \frac{1-\mu}{d_2'}. \tag{4.4}$$

However, for the treatment coming we are interested in the expression of the Hamiltonian centered in one of the equilateral Lagrangian points. It is quite straightforward that even considering the spatial problem, the equilibrium points remain in the orbit's plane, so both $\mathcal{L}_4$ and $\mathcal{L}_5$ are in the plane $x_3 = 0$ and fulfil the conditions $d_1' = d_2' = 1$. From here, if we aim to set the new origin in $\mathcal{L}_4$, for instance, it is easy to see that its

coordinates are $\mathcal{L}_4 = (\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}, 0)$. Then, if we apply the following change of variables

$$\begin{aligned} x &= x_1 - \left(\frac{1}{2} - \mu\right), & p_x &= y_1 + \frac{\sqrt{3}}{2}, \\ y &= x_2 - \frac{\sqrt{3}}{2}, & p_y &= y_2 - \left(\frac{1}{2} - \mu\right), \\ z &= x_3, & p_z &= y_3 \end{aligned} \tag{4.5}$$

the positions and moments vanishes in $\mathcal{L}_4$. After a short but still tedious computation and vanishing the constant terms by imposing a shift, the new Hamiltonian (which again we call H to keep the notation clear) reads

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y - \left(\frac{1}{2} - \mu\right)x - \frac{\sqrt{3}}{2}y - \frac{\mu}{d_1'} - \frac{1-\mu}{d_2'} \tag{4.6}$$

where d_1' and d_2' can be written in terms of the new variables as

$$d_1'^2 = 1 - x + \sqrt{3}y + x^2 + y^2 + z^2, \quad d_2'^2 = 1 + x + \sqrt{3}y + x^2 + y^2 + z^2. \tag{4.7}$$

Once we have the Hamiltonian centered in the desired point, we aim to find its series expansion. It can be noticed that only the last two terms in the expression (4.6) have to be expanded, indeed. To this end, we use the definition of the Legendre polynomials

$$\frac{1}{\sqrt{1 - 2\xi t + t^2}} = \sum_{n=0}^{\infty} P_n(\xi) t^n \tag{4.8}$$

and note that $d_1'^2 = 1 - x + \sqrt{3}y + r^2$ with $r^2 = x^2 + y^2 + z^2$, so by letting $t = r$ and $\xi = \frac{x - \sqrt{3}y}{2r}$ we obtain

$$\frac{1}{d_1'} = \sum_{n=0}^{\infty} P_n\left(\frac{x - \sqrt{3}y}{2r}\right) r^n \tag{4.9}$$

and proceeding similarly for $\frac{1}{d_2'}$ we get that the terms H_k having order higher than 2 are

$$H_k = r^k P_k\left(\frac{x - \sqrt{3}y}{2r}\right)\mu + r^k P_k\left(\frac{-x - \sqrt{3}y}{2r}\right)(1 - \mu), \quad k > 2. \tag{4.10}$$

One last change of variables is required: the one that turns H_2 into its normal form. It is proven in A. Celletti et al. [5] and in A. Giorgilli et al. [9], that in terms of a new set of variables $(x_1, x_2, x_3, y_1, y_2, y_3)$ given by a symplectic change $(x, y, p_x, p_y)^T = M(x_1, x_2, y_1, y_2)^T$, $z = x_3$, $p_z = y_3$ the term H_2 reads

$$H_2 = \frac{1}{2}\sum_{l=1}^{3} \omega_l (x_l^2 + y_l^2)$$

being ω_l the eigenvalues of H_2. Moreover, the papers cited above show explicitly that

the symplectic change is $M = (e_1', e_2', f_1', f_2')$ where

$$e_j' = \frac{e_j}{\sqrt{\omega_j\left(2\omega_j^4 + \frac{\omega_j^2}{2} - \frac{3}{4}\right)}}, \qquad f_j' = \frac{f_j}{\sqrt{\omega_j\left(2\omega_j^4 + \frac{\omega_j^2}{2} - \frac{3}{4}\right)}}$$

being $e_j = \left(a, -\frac{3}{4} - \omega_j^2, \frac{3}{4} - \omega_j^2, a\right)^T$, $f_j = \left(2\omega_j, 0, a\omega_j, (\frac{5}{4} - \omega_j^2)\omega_j\right)^T$ and $a = -(1 - 2\mu)\frac{3\sqrt{3}}{4}$. On its side, provided that $27\mu(1-\mu) < 1$ (which is hold in the system Sun-Jupiter, since $\mu = 9.5387536 \times 10^{-4}$), the eigenvalues of H_2 are given by

$$\omega_1^2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{27}{4} + 4a^2}, \qquad \omega_2^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{27}{4} + 4a^2}, \qquad \omega_3^2 = 1. \tag{4.11}$$

From now on, despite the fact that the spatial case implies 3 degrees of freedom, let us consider the general case of n degrees. Then, the term H_2 reads

$$H_2 = \frac{1}{2}\sum_{l=1}^{n} \omega_l(x_l^2 + y_l^2).$$

Furthermore, it admits the first integrals $\Phi_2^{(l)} = \frac{1}{2}(x_l^2 + y_l^2)$ for $1 \leq l \leq n$. This can be quickly proven by checking that $\{\Phi_2^{(l)}, H_2\} = 0$ for $1 \leq l \leq n$. Also, if we consider all the Hamiltonian H, we can define its first integrals $\Phi^{(l)}$ through the condition $\{\Phi^{(l)}, H\} = 0$, where $\Phi^{(l)}$ can be expressed as a power series

$$\Phi^{(l)} = \Phi_2^{(l)} + \Phi_3^{(l)} + \dots \tag{4.12}$$

being $\Phi_s^{(l)}$ a homogeneous polynomial having all terms of degree s and depending in general on x_l, y_l for $1 \leq l \leq n$ when $s > 2$. The fact that $\Phi^{(l)}$ are first integrals leads us to the following recursive system.

Lemma 4.1. *Under the notation introduced above, it holds that*

$$\{H_2, \Phi_s^{(l)}\} = \varphi_s^{(l)} \tag{4.13}$$

with

$$\varphi_3^{(l)} = -\{H_3, \Phi_2^{(l)}\}, \qquad \varphi_s^{(l)} = -\sum_{j=1}^{s-3}\{H_{s-j}, \Phi_{j+2}^{(l)}\} - \{H_s, \Phi_2^{(l)}\}, \quad s > 3. \tag{4.14}$$

Proof. To prove this result we only need to consider the linearity of the Poisson bracket operator and recall the following property used in previous chapters already: if f and g are two monomials of degree r and s respectively, then $\{f, g\}$ is a monomial of degree $r + s - 2$.

Then, isolating $\{H_2, \Phi_s^{(l)}\}$ from the condition $\{\Phi^{(l)}, H\} = 0$ and taking into account that all the terms of $\{H_2, \Phi_s^{(l)}\}$ have degree s, many of the terms vanish remaining only the ones having degree s. The lemma Follows from here. □

Besides, considering the relation (4.13) and a general form for $\varphi_s^{(l)}$, a concrete expression for $\Phi_s^{(l)}$ can be stated. Consider first the complex change of variables given by

$$x_l = \frac{1}{\sqrt{2}}(\xi_l + i\eta_l), \quad y_l = \frac{i}{\sqrt{2}}(\xi_l - i\eta_l) \tag{4.15}$$

for $1 \leq l \leq n$. Hence, in complex variables H_2 reads

$$H_2 = i\sum_{l=1}^{n} \omega_l \xi_l \eta_l.$$

Then, considering a polynomial expansion of $\varphi_s^{(l)}$ in $\mathbb{C}[\xi, \eta]$ as follows

$$\varphi_s^{(l)} = \sum_{|j+k|=s} c_{jk} \xi^j \eta^k,$$

with $\xi^j \eta^k = \xi_1^{j_1} \cdot \ldots \cdot \xi_n^{j_n} \cdot \eta_1^{k_1} \cdot \ldots \cdot \eta_n^{k_n}$ and $|j+k| = j_1 + \ldots + j_n + k_1 + \ldots + k_n$, $\Phi_s^{(l)}$ can be given by the following proposition.

Proposition 4.2. *Taking $\varphi_s^{(l)}$ as defined above, $\Phi_s^{(l)}$ takes the form*

$$\Phi_s^{(l)} = -i \sum_{|j+k|=s} \frac{c_{jk}}{(k-j)\omega} \xi^j \eta^k. \tag{4.16}$$

Proof. We only need to see if given the expression for $\Phi_s^{(l)}$, the relation (4.13) holds. Computing the Poisson bracket, we obtain:

$$\{H_2, \Phi_s^{(l)}\} = \sum_{l=1}^{n} \left(\frac{\partial H_2}{\partial \xi_l} \frac{\partial \Phi_s^{(l)}}{\partial \eta_l} - \frac{\partial H_2}{\partial \eta_l} \frac{\partial \Phi_s^{(l)}}{\partial \xi_l} \right) =$$

$$= \sum_{l=1}^{n} \left(i\omega_l \eta_l (-i) \sum_{|j+k|=s} \frac{c_{jk}}{(k-j)\omega} k_l \xi^j \frac{\eta^k}{\eta_l} - i\omega_l \xi_l (-i) \sum_{|j+k|=s} \frac{c_{jk}}{(k-j)\omega} j_l \frac{\xi^j}{\xi_l} \eta^k \right) =$$

$$= \sum_{l=1}^{n} \left(\omega_l \sum_{|j+k|=s} \frac{c_{jk}}{(k-j)\omega} \xi^j \eta^k (k_l - j_l) \right).$$

Finally, if we sum for all the values of l we get:

$$\{H_2, \Phi_s^{(l)}\} = \sum_{|j+k|=s} c_{jk} \xi^j \eta^k.$$

□

It is important to remark that our goal is to obtain a computational bound. Thus, working with infinite series is meaningless and therefore we will consider the first integrals truncated up to a finite order. In addition, $\Phi^{(l,r)}$ will stand for the first integral l up to order r. In the literature these truncated integrals are often called quasi-first integrals, since they are not absolutely conserved in a trajectory in general, as

$\{\Phi^{(l,r)}, H\}$ is not zero. Of course, when r gets bigger $\{\Phi^{(l,r)}, H\}$ becomes smaller since $\lim_{r\to\infty} \Phi^{(l,r)} = \Phi^{(l)}$.

Now, we introduce a norm that will enable us to establish bounds of the Hamiltonian and the first integrals. To this end, let us fix a vector $R = (R_1, \ldots, R_n) \in \mathbb{R}^n$.

Definition 4.3. *Let $f(x,y) = \sum_{j,k} f_{jk} x^j y^k$ be a $2n$-variables polynomial of arbitrary finite degree, where $x^j = x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}$ and similarly for y^k. Then, we define the norm of f with respect to $R \in \mathbb{R}^n$ as*

$$\|f\|_R = \sum_{j,k} |f_{jk}| R^{j+k} \tag{4.17}$$

where $R^{j+k} = R_1^{j_1+k_1} \cdot \ldots \cdot R_n^{j_n+k_n}$.

To illustrate the choice of this norm, we start by defining a reference domain in the phase space. Let $\rho \in \mathbb{R}$ and let R be the fixed vector defined above hereafter. Then, let us consider

$$\Delta_{\rho R} = \{(x,y) \in \mathbb{R}^{2n} \colon x_l^2 + y_l^2 \leq \rho^2 R_l^2, \;\; 1 \leq l \leq n\} \subset \Omega. \tag{4.18}$$

Essentially, the set $\Delta_{\rho R}$ is a $2n$-dimensional ellipsoid in Ω, since the components of $R \in \mathbb{R}^n$ might be different in general. The following proposition provides us with a simple bound of a polynomial defined in $\Delta_{\rho R}$:

Proposition 4.4. *Let $f(x,y)$ be a homogeneous polynomial of degree s defined in $(x,y) \in \Delta_{\rho R} \subset \Omega$. Then, one has*

$$|f(x,y)| \leq \rho^s \|f\|_R. \tag{4.19}$$

Proof. Let us denote $f(x,y) = \sum_{j,k} f_{jk} x^j y^k$. Then, it is clear that

$$|f(x,y)| \leq \sum_{j,k} |f_{jk}| |x|^j |y|^k.$$

Further, since $(x,y) \in \Delta_{\rho R}$, we have that $|x|,|y| \leq \rho R$, so the last term can be bounded as follows

$$\sum_{j,k} |f_{jk}| |x|^j |y|^k \leq \sum_{j,k} |f_{jk}| \rho^{j+k} R^{j+k} \leq \rho^s \|f\|_R$$

where in the last inequality we recalled both the definition of $\|\cdot\|_R$ and the fact that the degree of f is s, so $j + k \leq s$. □

Notwithstanding it can be noticed that the bounds $|x|,|y| \leq \rho R$ are hardly accurate, the bound of $|f(x,y)|$ given by proposition can still be saturated, (for instance, consider the polynomial $f(x,y) = x_1^s$ in $(x,y) = (\rho R_1, 0, \ldots, 0, 0, \ldots, 0) \in \partial\Delta_{\rho R}$). This fact makes it difficult to find a better bound, which would barely increase the accuracy and would be far less intuitive.

In the next section, it will be useful to obtain lower bounds of the norm of a Poisson bracket. With that in mind, we announce and prove the following lemma, stated in A. Giorgilli [8].

Lemma 4.5. *Let f and g be two homogeneous polynomials of degree s and r respectively. Then, the following condition is fulfilled*

$$\|\{f,g\}\|_R \leq sr\Lambda^2\|f\|_R\|g\|_R \tag{4.20}$$

where $\Lambda = (min_l R_l)^{-1}$, for $1 \leq l \leq n$.

Proof. Let us denote $f = \sum_{j,k} f_{jk}x^j y^k$ and $g = \sum_{j',k'} g_{j'k'}x^{j'}y^{k'}$. By definition, we have that

$$\{f,g\} = \sum_{l=1}^{n}\left(\frac{\partial f}{\partial x_l}\frac{\partial g}{\partial y_l} - \frac{\partial f}{\partial y_l}\frac{\partial g}{\partial x_l}\right) =$$

$$= \sum_{l=1}^{n}\left(\sum_{j,k,j',k'} f_{jk}g_{j'k'}x^{j+j'}y^{k+k'}\frac{j_l k'_l}{x_l y_l} - \sum_{j,k,j',k'} f_{jk}g_{j'k'}x^{j+j'}y^{k+k'}\frac{j'_l k_l}{x_l y_l}\right) =$$

$$= \sum_{j,k,j',k'} f_{jk}g_{j'k'}x^{j+j'}y^{k+k'}\sum_{l=1}^{n}\frac{j_l k'_l - j'_l k_l}{x_l y_l}.$$

Then, applying its definition, the norm of $\{f,g\}$ can be lower bounded by:

$$\|\{f,g\}\|_R \leq \sum_{j,k,j',k'}|f_{jk}||g_{j'k'}|R^{j+j'+k+k'}\sum_{l=1}^{n}\frac{j_l k'_l + j'_l k_l}{R_l^2} = \|f\|_R\|g\|_R\sum_{l=1}^{n}\frac{j_l k'_l + j'_l k_l}{R_l^2}.$$

Now, since the degree of f and g are s and r respectively, we have that $j_l < s$, $k_l < s$ and $j'_l + k'_l < r$. Thus, we obtain:

$$\sum_{l=1}^{n}\frac{j_l k'_l + j'_l k_l}{R_l^2} \leq s\sum_{l=1}^{n}\frac{k'_l + j'_l}{R_l^2} \leq sr\sum_{l=1}^{n}\frac{1}{R_l^2} \leq sr\Lambda^2.$$

Combining these chains of inequalities the lemma follows. □

A particular case of this lemma can be considered if $g = \Phi_2^{(l)}$. From here, it yields the following result:

Lemma 4.6. *Let f be a homogeneous polynomial of degree s. Then, it is fulfilled*

$$\|\{f,\Phi_2^{(l)}\}\|_R \leq s\|f\|_R. \tag{4.21}$$

Proof. Taking f as in the previous proof, it is simple to compute the Poisson bracket involved in the claim

$$\{\Phi_2^{(l)},f\} = \sum_{j,k} x_l f_{jk}x^j y^k\frac{k_l}{y_l} - \sum_{j,k} y_l f_{jk}x^j y^k\frac{j_l}{x_l} = \left(k_l\frac{x_l}{y_l} - j_l\frac{y_l}{x_l}\right)\sum_{j,k} j_{jk}x^j y^k.$$

Then, the norm can be bounded by

$$\|\{f,\Phi_2^{(l)}\}\|_R = \|\{\Phi_2^{(l)},f\}\|_R \leq (k_l + j_l)\sum_{j,k}|f_{jk}|R^{j+k} \leq s\sum_{j,k}|f_{jk}|R^{j+k} = s\|f\|_R$$

where in the last inequality we used that the degree of f is s, by hypothesis. □

Now, we introduce a result that is useful when bounding the first integrals. Let the absolute value of the terms $(k-j)\cdot\omega$ of the equation (4.16) be bounded from bellow by some real constants α_s. Then, using the result seen in proposition 4.2, it is straightforward to prove that

$$\|\Phi_s^{(l)}\|_R \leq \frac{1}{\alpha_s}\|\varphi_s^{(l)}\|_R. \tag{4.22}$$

It is clear that the use of this bound requires to determine computationally the constants α_s. Since $k-j=(k_1-j_1,\ldots,k_n-j_n)\in\mathbb{Z}^n$, we can redefine $t=k-j$ and ask $|t\cdot\omega_l|\geq\alpha_s$. Indeed, the constants

$$\alpha_s=\min_{t\in T_s}|t\cdot\omega_l| \tag{4.23}$$

fulfil that condition, so if T_s is a finite set, α_s can be found with a computer by testing $t\in T_s$ exhaustively. In addition, from $|k+j|=k_1+\ldots+k_n+j_1+\ldots+j_n\leq s$ and $k,j\in\mathbb{N}^n$, it follows $|t|<s$. In turn, $T_s=\{t\in\mathbb{Z}^n,\ 0<|t|\leq s\}$, which is a finite set.

Finally, we need to introduce in a simple way the time derivative of the first integrals. To do so, we recall that if f is an arbitrary function in a phase space Ω and H is a Hamiltonian, then $\dot{f}=\frac{\partial f}{\partial t}+\{f,H\}$. Thus, it holds that

$$\dot{\Phi}^{(l,r)}=\{\Phi^{(l,r)},H\}=\sum_{j=0}^{r-2}\sum_{s>r}\{\Phi_{j+2}^{(l)},H_{s-j}\}. \tag{4.24}$$

One may notice that in the previous summations there are only terms having degree bigger than r. In particular, the Poisson bracket can be decomposed as

$$\{\Phi^{(l,r)},H\}=\{\Phi^{(l,r)},H_2+\ldots+H_r\}+\{\Phi^{(l,r)},H_{r+1}\ldots\}$$

using its linearity. Then, since the first term clearly vanishes because of the definition of first integral, all the terms in (4.24) come from the second Poisson bracket, which have degree greater than r.

4.2 Estimations of the Hamiltonian and first integrals

As will be noticed in section 4.3, it is required to obtain lower bounds of the Hamiltonian and the first integrals to bound the escape time thereafter.

To this end, in this section several technical lemmas are introduced, giving us some practical bounds. Whereas lemmas 4.7, 4.8 focus on bounding the Hamiltonian, lemma 4.9 and proposition 4.10 enable us to bound the first integrals and their time derivatives respectively. To begin with, let us announce a lemma that bounds H_k for every k.

Since the expression (4.10) of the Hamiltonian is recalled bellow, let us redefine the variables

$$r=\sqrt{x^2+y^2+z^2},\qquad \gamma=\frac{x-\sqrt{3}y}{2r},\qquad \delta=\frac{-x-\sqrt{3}y}{2r}, \tag{4.25}$$

and the variables r_t, γ_t and δ_t for the the same quantities in terms of the coordinates $x_1,\ldots,x_n,y_1,\ldots,y_n$.

Lemma 4.7. *Assume that for some $m \geq 2$ and some constants S_{m-1} and S_m it holds that*

$$\begin{aligned}(1-\mu)\|r_t^{m-1}P_{m-1}(\gamma_t)\|_R + \mu\|r_t^{m-1}P_{m-1}(\delta_t)\|_R \leq S_{m-1},\\ (1-\mu)\|r_t^{m}P_{m}(\gamma_t)\|_R + \mu\|r_t^{m}P_{m}(\delta_t)\|_R \leq S_{m}.\end{aligned} \tag{4.26}$$

Then, we have that $\|H_k\|_R \leq S_k$, being $\{S_k\}_{k>m}$ recursively defined as follows:

$$S_{k+1} = \frac{2k+1}{k+1}c_1 S_k + \frac{k}{k+1}c_2 S_{k-1} \tag{4.27}$$

where $c_1 = max(\|\gamma_t r_t\|_R, \|\delta_t r_t\|_R)$ and $c_2 = \|r_t^2\|_R$.

Proof. Considering the norm of H_k as written in (4.10) we get the first bound

$$\|H_k\|_R \leq (1-\mu)\|r_t^k P_k(\gamma_t)\|_R + \mu\|r_t^k P_k(\delta_t)\|_R.$$

Recalling now the recursive property of the Legendre polynomials

$$P_{k+1}(x) = \frac{(2k+1)P_k(x)x - kP_{k-1}(x)}{k+1} \tag{4.28}$$

for $x = \gamma_t$ and multiplying each side by r_t^{k+1} we get

$$r_t^{k+1}P_{k+1}(\gamma_t) = \frac{2k+1}{k+1}r_t^{k+1}P_k(\gamma_t)\gamma_t - \frac{k}{k+1}r_t^{k+1}P_{k-1}(\gamma_t).$$

Now, using the definition of the constants c_1 and c_2 we obtain the bound

$$\|r_t^{k+1}P_{k+1}(\gamma_t)\|_R \leq \frac{2k+1}{k+1}c_1\|r_t^k P_k(\gamma_t)\|_R + \frac{k}{k+1}c_2\|r_t^{k-1}P_{k-1}(\gamma_t)\|_R$$

and a similar one for the variable δ_t. Putting these bounds into the one of the Hamiltonian we have

$$\begin{aligned}\|H_{k+1}\|_R \leq \frac{2k+1}{k+1}c_1\left[(1-\mu)\|r_t^k P_k(\gamma_t)\|_R + \mu\|r_t^k P_k(\delta_t)\|_R\right] +\\ +\frac{k}{k+1}c_2\left[(1-\mu)\|r_t^{k-1} P_{k-1}(\gamma_t)\|_R + \mu\|r_t^{k-1} P_{k-1}(\delta_t)\|_R\right].\end{aligned} \tag{4.29}$$

If we now take $k+1 = m+1$ we can bound the two terms in $[\cdot]$ by S_m and S_{m-1} respectively, by hypothesis. Doing so, we obtain that

$$\|H_{m+1}\|_R \leq \frac{2k+1}{k+1}c_1 S_m + \frac{k}{k+1}c_2 S_{m-1} = S_{m+1},$$

so the bound is proven for H_{m+1} already. Furthermore, the inequality (4.29) is true in general, so the bound can be proved for H_k for $k > m$ inductively. □

Notice that the bound given by this lemma depends on k, in a way that in order to compute the bound for $\|H_k\|_R$, the same bounds for H_{k-2} and H_{k-1} are needed. For that reason, we state the following lemma to find a more flexible bound, but less accurate.

Lemma 4.8. *Assume that for some $m \geq 2$ and some constants S_{m-1} and S_m it holds:*

$$\begin{aligned}(1-\mu)\|r_t^{m-1}P_{m-1}(\gamma_t)\|_R + \mu\|r_t^{m-1}P_{m-1}(\delta_t)\|_R \leq S_{m-1},\\ (1-\mu)\|r_t^{m}P_{m}(\gamma_t)\|_R + \mu\|r_t^{m}P_{m}(\delta_t)\|_R \leq S_{m}.\end{aligned} \tag{4.30}$$

Then it is hold that $\|H_k\|_R \leq h^{k-m+1}E$ for $k > m$ where $E = S_{m-1}$, c_1 and c_2 are defined in lemma (4.7) and $h = max\left(\frac{S_m}{S_{m-1}}, c_1 + \sqrt{c_1^2 + c_2}\right)$.

Proof. Taking the result of the previous lemma, we only need to prove that $S_k \leq h^{k-m+1}E$. Let us denote $a_k = \frac{2k+1}{k+1}c_1$ and $b_k = \frac{k}{k+1}c_2$. We will consider an inductive method, starting for $k = m$.

Note that $h \geq \frac{S_m}{S_{m-1}} = \frac{S_m}{E}$ by the definition of h. Thus, $S_m \leq hE$ and the result holds for $k = m$.

For $k > m$ we assume that the result is true for $k \leq l$ and we aim to prove it for $k = l + 1$. From the previous lemma we know

$$S_{l+1} = \frac{2l+1}{l+1}c_1 S_l + \frac{l}{l+1}c_2 S_{l-1}.$$

Applying now the definitions of a_l, b_l and the lemma for $k = l, l-1$ we obtain

$$S_{l+1} \leq \left(a_l h^{l-m+1} + b_l h^{l-m}\right) E,$$

so whether we prove that $a_l h^{l-m+1} + b_l h^{l-m} \leq h^{l-m+2}$ we have the result for $k = l+1$ already. Then, we claim that the following inequality holds

$$a_l h + b_l \leq h^2 \tag{4.31}$$

which gives us the desired inequality if multiplied by h^{l-m} in each side. To prove this claim it is sufficient to see that by definition, we have that

$$h \geq c_1 + \sqrt{c_1^2 + c_2} \;\Leftrightarrow\; (h - c_1)^2 \geq c_1^2 + c_2 \;\Leftrightarrow\; h^2 \geq 2hc_1 + c_2 \geq a_l h + b_l$$

where the last inequality holds for all values of $l \in \mathbb{N}$. □

Let us now compute a bound for the first integrals. To do so, we assume that a bound for $\|\Phi_j^{(l)}\|_R$ is already known for $3 \leq j \leq \tilde{r}$ given some $\tilde{r} > 3$. This is actually true since in the worst of the cases ($\tilde{r} = 3$) we have that

$$\|\Phi_3^{(l)}\|_R \leq \frac{1}{\alpha_3}\|\varphi_3^{(l)}\| = \frac{1}{\alpha_3}\|-\{H_3, \Phi_2^{(l)}\}\|_R \leq \frac{3}{\alpha_3}\|H_3\|_R \leq \frac{3}{\alpha_3}S_3$$

where we used the inequalities given by the equations (4.22, 4.21) and lemma 4.7 respectively. Thus, we can bound the first integrals up to degree $r > \tilde{r}$ as follows:

Lemma 4.9. *Let $\|H_k\|_R \leq S_k$ for $3 \leq k \leq r$ and $\|\Phi_j^{(l)}\|_R \leq F_j^{(l)}$ for $3 \leq j \leq \tilde{r}$, being $3 \leq \tilde{r} \leq r$, and S_k and $F_j^{(l)}$ some positive constants. Then, it holds that $\|\Phi_j^{(l)}\|_R \leq F_j^{(l)}$ for*

$\tilde{r} < j \leq r$ where

$$F_j^{(l)} = \frac{1}{\alpha_j}\left[\Lambda^2 \sum_{k=1}^{j-3}(k+2)(j-k)F_{k+2}^{(l)}S_{j-k} + jS_j\right]. \tag{4.32}$$

Proof. Using (4.22), we have that

$$\|\Phi_j^{(l)}\|_R \leq \frac{1}{\alpha_j}\|\varphi_j^{(l)}\|_R = \frac{1}{\alpha_j}\| - \sum_{k=1}^{j-3}\{H_{j-k}, \Phi_{k+2}^{(l)}\} - \{H_j, \Phi_2^{(l)}\}\|_R.$$

Recalling now the triangular inequality and lemmas 4.5, 4.6 for the Poisson bracket in the summation and the outside one respectively, it follows:

$$\frac{1}{\alpha_j}\| - \sum_{k=1}^{j-3}\{H_{j-k}, \Phi_{k+2}^{(l)}\} - \{H_j, \Phi_2^{(l)}\}\|_R \leq$$

$$\leq \frac{1}{\alpha_j}\left[\sum_{k=1}^{j-3}(j-k)(k+2)\Lambda^2\|H_{j-k}\|_R\|\Phi_{k+2}^{(l)}\|_R + j\|H_j\|_R\right].$$

Considering now the case $j = \tilde{r}+1$, the highest order of $\Phi^{(l)}$ involved in the last expression is $\Phi_{\tilde{r}}^{(l)}$, so it follows that $\|\Phi_{k+2}^{(l)}\|_R \leq F_{k+2}^{(l)}$ by hypothesis. This, together with the bounds of the terms of the Hamiltonian, conclude the proof for $j = \tilde{r}+1$. However, repeating the same procedure in an inductive way the result can be proven for $\tilde{r} < j \leq r$. Notice that if $j \geq r$ then we lack of a bound of $\|H_k\|_R \leq S_k$ so the scheme cannot be repeated. □

Finally, we will use the three lemmas introduced in this section to prove the following result, which again gives us a quite technical but still useful lower bound.

Proposition 4.10. *Given r and $\tilde{r}$ such that $3 \leq r \leq \tilde{r}$, let $\|H_k\|_R \leq S_k$ for $3 \leq k \leq \tilde{r}$, let $\|H_k\|_R \leq h^{k-\tilde{r}+1}E$ for $k > r$ and let $\|\Phi_j^{(l)}\|_R \leq F_j^{(l)}$ for $3 \leq j \leq r$. Furthermore, let us assume that $h\rho < 1$. Then, within the domain $\Delta_{\rho R}$ it holds that $|\dot{\Phi}^{(l,r)}| \leq \mathcal{R}^{(l,r)}(\rho)$, where:*

$$\begin{aligned}\mathcal{R}^{(l,r)}(\rho) &= \Lambda^2 \sum_{j=1}^{r-2}(j+2)\rho^j F_{j+2}^{(l)} \sum_{r-j<p\leq\tilde{r}} p\rho^p S_p + \\ &+\Lambda^2 \sum_{j=1}^{r-2}(j+2)\rho^j F_{j+2}^{(l)}\frac{E}{h^{\tilde{r}}}\cdot h\frac{(\tilde{r}+1)(h\rho)^{\tilde{r}+1} - \tilde{r}(h\rho)^{\tilde{r}+2}}{(1-h\rho)^2} + \\ &+ \sum_{r<s\leq\tilde{r}} sS_s\rho^s + \frac{E}{h^{\tilde{r}}}\cdot h\frac{(\tilde{r}+1)(h\rho)^{\tilde{r}+1} - \tilde{r}(h\rho)^{\tilde{r}+2}}{(1-h\rho)^2}.\end{aligned} \tag{4.33}$$

Proof. It will be useful to rewrite the equality (4.24) as follows

$$\dot{\Phi}^{(l,r)} = \sum_{j=1}^{r-2}\sum_{s>r}\{\Phi_{j+2}^{(l)}, H_{s-j}\} + \sum_{s>r}\{\Phi_2^{(l)}, H_s\}.$$

The absolute value of this last expression can be bounded using the norm, as seen in

proposition 4.4 and noticing that each Poisson bracket has degree s, we have that

$$|\dot{\Phi}^{(l,r)}| \leq \sum_{j=1}^{r-2} \sum_{s>r} |\{\Phi_{j+2}^{(l)}, H_{s-j}\}| + \sum_{s>r} \{\Phi_2^{(l)}, H_s\} \leq$$

$$\leq \sum_{j=1}^{r-2} \sum_{s>r} \rho^s \|\{\Phi_{j+2}^{(l)}, H_{s-j}\}\|_R + \sum_{s>r} \rho^s \|\{\Phi_2^{(l)}, H_s\}\|_R.$$

Whether we now take $p = s - j$, the last inequality reads:

$$|\dot{\Phi}^{(l,r)}| \leq \sum_{j=1}^{r-2} \sum_{p>r-j} \rho^{p+j} \|\{\Phi_{j+2}^{(l)}, H_p\}\|_R + \sum_{s>r} \rho^s \|\{\Phi_2^{(l)}, H_s\}\|_R. \tag{4.34}$$

Note that the first summation can be decomposed as it is shown:

$$\sum_{j=1}^{r-2} \left[\sum_{r-j<p\leq\tilde{r}} \rho^{p+j} \|\{\Phi_{j+2}^{(l)}, H_p\}\|_R + \sum_{p>\tilde{r}} \rho^{p+j} \|\{\Phi_{j+2}^{(l)}, H_p\}\|_R \right]. \tag{4.35}$$

We now proceed as follows: in order to bound the right term in (4.34) we first aim to bound the infinite summation in (4.35). Using lemmas 4.5, 4.9 we have that

$$\sum_{j=1}^{r-2} \sum_{p>\tilde{r}} \rho^{p+j} \|\{\Phi_{j+2}^{(l)}, H_p\}\|_R \leq \sum_{j=1}^{r-2} \sum_{p>\tilde{r}} \rho^{p+j} \Lambda^2 (j+2) p \|\Phi_{j+2}^{(l)}\|_R \|H_p\|_R \leq$$

$$\leq \sum_{j=1}^{r-2} \Lambda^2 (j+2) \rho^j F_{j+2}^{(l)} \sum_{p>\tilde{r}} \rho^p \cdot p \|H_p\|_R.$$

Now, recalling that $\|H_p\|_R \leq h^{p-\tilde{r}+1} E$ (given by lemma 4.8), the last term is bounded by:

$$\sum_{j=1}^{r-2} \Lambda^2 (j+2) \rho^j F_{j+2}^{(l)} \sum_{p>\tilde{r}} \rho^p \cdot p \|H_p\|_R \leq \sum_{j=1}^{r-2} \Lambda^2 (j+2) \rho^j F_{j+2}^{(l)} \frac{E}{h^{\tilde{r}-1}} \sum_{p>\tilde{r}} p \cdot \rho^p h^p.$$

Finally, if we consider the identity

$$\sum_{p>\tilde{r}} p x^p = \frac{(\tilde{r}+1) x^{\tilde{r}+1} - \tilde{r} x^{\tilde{r}+2}}{(1-x)^2} \tag{4.36}$$

with $x = h\rho < 1$, and complete the chain of inequalities, we find that:

$$\sum_{j=1}^{r-2} \sum_{p>\tilde{r}} \rho^{p+j} \|\{\Phi_{j+2}^{(l)}, H_p\}\|_R \leq \sum_{j=1}^{r-2} \Lambda^2 (j+2) \rho^j F_{j+2}^{(l)} \frac{E}{h^{\tilde{r}-1}} \frac{(\tilde{r}+1)(h\rho)^{\tilde{r}+1} - \tilde{r}(h\rho)^{\tilde{r}+2}}{(1-h\rho)^2}. \tag{4.37}$$

Let us focus now on the first summation in (4.35). Once more, using lemma 4.5 we

get to following bound:

$$\sum_{j=1}^{r-2}\sum_{r-j<p\leq\tilde{r}}\rho^{p+j}\|\{\Phi_{j+2}^{(l)},H_p\}\|_R\leq\sum_{j=1}^{r-2}\sum_{r-j<p\leq\tilde{r}}\rho^{p+j}\Lambda^2(j+2)p\|\Phi_{j+2}^{(l)}\|_R\|H_p\|_R\leq \\ \leq\Lambda^2\sum_{j=1}^{r-2}(j+2)\rho^j F_{j+2}^{(l)}\sum_{r-j<p\leq\tilde{r}}p\rho^p S_p \tag{4.38}$$

where in the last inequality we recalled the bounds of the first integrals and the Hamiltonian found previously in this section. Notice that this bound along with the inequality (4.37) give us a lower bound of the term (4.35). Then, to conclude the proof we only need to find a bound of the remaining term in (4.34). This can be done bounding the Poisson bracket involved:

$$\sum_{s>r}\rho^s\|\{\Phi_2^{(l)},H_s\}\|_R\leq\sum_{s>r}s\rho^s\|H_s\|_R\leq\sum_{s>r}sS_s\rho^s= \\ =\sum_{r<s\leq\tilde{r}}sS_s\rho^s+\sum_{s>\tilde{r}}sS_s\rho^s.$$

Applying now the result yield from lemma 4.8 to the second summation and the identity (4.36) we reach the final bound:

$$\sum_{s>r}\rho^s\|\{\Phi_2^{(l)},H_s\}\|_R\leq\sum_{r<s\leq\tilde{r}}sS_s\rho^s+\sum_{s>\tilde{r}}\frac{E}{h^{\tilde{r}-1}}s(h\rho)^s= \\ =\sum_{r<s\leq\tilde{r}}sS_s\rho^s+\frac{E}{h^{\tilde{r}-1}}\frac{(\tilde{r}+1)(h\rho)^{\tilde{r}+1}-\tilde{r}(h\rho)^{\tilde{r}+2}}{(1-h\rho)^2}. \tag{4.39}$$

The stated result follows from the combination of the inequalities found above. □

4.3 Estimation of the escape time

Once the theoretical framework has been settled and the expansions of the Hamiltonian and first integrals have been bounded, we are now in position to find a lower bound of the escape time of a trajectory initially within a concrete domain. To do so, we use proposition 4.10.

More precisely, let us consider a particle initially in the domain $\Delta_{\rho_0 R}$. For a given $\rho>\rho_0$, we aim to find a lower bound of the time that it takes for the particle to leave from $\Delta_{\rho R}$. Considering action variables, we ask for the distances in the phase space between the action coordinates at a time t and at a time $t_0=0$ to be smaller than the distances from $\Delta_{\rho R}$ to $\Delta_{\rho_0 R}$ in each axis. This condition reads:

$$|I_l(t)-I_l(0)|\leq\frac{1}{2}R_l^2(\rho^2-\rho_0^2),\qquad 1\leq l\leq n. \tag{4.40}$$

In fact, if this condition holds, it is easy to see that the trajectory remains within the domain $\Delta_{\rho R}$ at time t, since

$$|I_l(t)|\leq|I_l(t)-I_l(0)|+|I_l(0)|\leq\frac{1}{2}R_l^2(\rho^2-\rho_0^2)+\frac{1}{2}R_l^2\rho_0^2=\frac{1}{2}R_l^2\rho^2$$

and in terms of the Cartesian coordinates, this implies that $x_l^2(t) + y_l^2(t) \leq R_l^2\rho^2$. Actually, we will work with a bound less accurate than (4.40). For now, let us prove the following lemma:

Lemma 4.11. *Under the notation introduced in this section, we have that*

$$|I_l(t) - \Phi^{(l,r)}(t)| \leq \delta_r^{(l)}(\rho)$$

where $\delta_r^{(l)}(\rho) = \sum_{j=3}^{r} \|\Phi_j^{(l)}\|_R \rho^j$. *Moreover, for* $t = 0$ *we have the same result with* ρ_0 *instead of* ρ.

Proof. Recalling that $\Phi^{(l,r)} = \Phi_2^{(l)} + \ldots + \Phi_r^{(l)}$, noticing the fact that $I_l(t) = \frac{1}{2}(x_l^2 + y_l^2) = \Phi_2^{(l)}(t)$, and applying the triangular inequality we get that

$$|I_l(t) - \Phi^{(l,r)}(t)| = |-\Phi_3^{(l)}(t) - \ldots - \Phi_r^{(l)}(t)| \leq |\Phi_3^{(l)}(t)| + \ldots + |\Phi_r^{(l)}(t)|.$$

Applying now proposition 4.4, we obtain

$$|\Phi_3^{(l)}(t)| + \ldots + |\Phi_r^{(l)}(t)| \leq \rho^3 \|\Phi_3^{(l)}(t)\|_R + \ldots + \rho^r \|\Phi_r^{(l)}(t)\|_R = \delta_r^{(l)}(\rho)$$

and the lemma follows. □

Now, using this lemma we can write

$$|I_l(t) - I_l(0)| \leq |I_l(t) - \Phi^{(l,r)}(t)| + |\Phi^{(l,r)}(t) - \Phi^{(l,r)}(0)| + |\Phi^{(l,r)}(0) - I_l(0)| \leq$$

$$\leq \delta_r^{(l)}(\rho) + |\Phi^{(l,r)}(t) - \Phi^{(l,r)}(0)| + \delta_r^{(l)}(\rho_0).$$

It is clear that if we ask to the right side of the previous inequality to be smaller that $\frac{1}{2}R_l^2(\rho^2 - \rho_0^2)$, then the condition (4.40) is fulfilled, and so the trajectory remains in the domain $\Delta_{\rho R}$, as seen above. Hence, imposing this, we can bound the maximum variation of the truncated first integral from time 0 to t as

$$|\Phi^{(l,r)}(t) - \Phi^{(l,r)}(0)| \leq \frac{1}{2}R_l^2(\rho^2 - \rho_0^2) - \delta_r^{(l)}(\rho) - \delta_r^{(l)}(\rho_0). \tag{4.41}$$

We will be calling $\Delta_r^{(l)}(\rho_0, \rho) = \frac{1}{2}R_l^2(\rho^2 - \rho_0^2) - \delta_r^{(l)}(\rho) - \delta_r^{(l)}(\rho_0)$ from now on, and it represents the maximum variation of $\Phi^{(l,r)}(t)$ along the trajectory.

Now, recalling the mean value theorem we have that $|\Phi^{(l,r)}(t) - \Phi^{(l,r)}(0)| = |\dot{\Phi}^{(l,r)}(t^*)|t$, where $t^* \in (0, t)$. Hence, the time until which the particle is within $\Delta_{\rho R}$ fulfils

$$t \leq \frac{\Delta_r^{(l)}(\rho_0, \rho)}{|\dot{\Phi}^{(l,r)}|}.$$

Taking into account that there are l first integrals and recalling proposition 4.10, a lower bound of the escape time can be defined

$$\tau(\rho_0, \rho) = \min_l \frac{\Delta_r^{(l)}(\rho_0, \rho)}{\mathcal{R}^{(l,r)}(\rho)}. \tag{4.42}$$

4.4 Computational implementation

Here we use some of the results introduced so far to computationally obtain a lower bound of the effective stability time for an infinitesimal particle in the Sun-Jupiter system. To this end, we are provided with the work of À. Jorba [11] which is of open access. The software described in this reference allows us to compute the series expansion of the Hamiltonian centered in $\mathcal{L}_4$ in normal form already, as well as of the first integrals truncated up to a finite order.

The fact that we are provided with this software, enables us to introduce little nuances to the work of A. Celletti et al. [5] in order to obtain better lower bounds of the series expansions. However, in this paper the authors go one step further maximizing the escape time by considering many of the possible values of ρ for a given ρ_0.

Let us first introduce how the software of À. Jorba is used for our own purpose and how the time bound is computed afterwards.

4.4.1 Power series of the Hamiltonian and first integrals

These power series are obtained using the software mentioned. Therein, the power expansion of the Hamiltonian is implemented as detailed in section 4.1, whereas the computation of the normal form is conducted by implementing the Birkhoff normal form, as stated in section 3.3.2.

On the other hand, the calculation of the first integrals has not been described yet. Let $\Phi^{(l)}$ be the first integrals for $1 \leq l \leq 3$. Then, as seen in the introduction it holds $\{H, \Phi^{(l)}\} = 0$. Now, recalling once more that $\{H_2, \Phi_n^{(l)}\}$ consists of terms having degree n, we get

$$\{H_2, \Phi_n^{(l)}\} = -\sum_{j=3}^{n} \{H_j, \Phi_{n-j+2}^{(l)}\}. \tag{4.43}$$

Note that the left side can be recursively computed because the biggest term of $\Phi^{(l)}$ involved in the right side is $\Phi_{n-1}^{(l)}$, and $\Phi_2^{(l)}$ is known beforehand. Then, it is simple to isolate $\Phi_n^{(l)}$ from the equation (4.43) due to the simplicity of the term H_2 in normal form.

Finally, let us remark the fact that nothing prevents $\sum_{j\geq 2} \Phi_n^{(l)}$ from diverging. In fact, in the general case H is non-integrable and a first integral cannot be found. However, if we truncate $\Phi^{(l)}$ up to a finite order the previous sum always converges.

4.4.2 Computation of the time bound

Since we aim to model the effective stability for the system Sun-Jupiter, the parameter $\mu = 9.5387536 \times 10^{-4}$ is fixed hereafter. Moreover, let us fix the vector $R = (1, 1, 1)$ in order to work with spherical domains. As it has been advanced, the escape time is computed as the minimum of $\frac{\Delta_r^{(l)}(\rho_0,\rho)}{\mathcal{R}^{(l,r)}(\rho)}$ for $1 \leq l \leq 3$. Hence, the computation of the numerator and denominator are performed for each l.

First, let us introduce the input parameters of the program:

- r, $\tilde{r}$: Positive integers that satisfy the condition $\tilde{r} > r$. r is the highest order terms

in the series expansions of the Hamiltonian and the first integrals, while $\tilde{r}$ is the highest term until which we consider the bound of $\|H_k\|_R$ given by lemma 4.7. In this implementation we used $r = 16$ and $\tilde{r} = 32$.

- ρ, ρ_0: Reals that satisfy the condition $\rho > \rho_0$.
- S_j, F_j for $1 \leq j \leq r$: Reals. Terms of the series expansions of the Hamiltonian and the first integral respectively.

With these inputs, the numerator is computed using its definition. The routine in which this simple calculation is implemented is called `obtainDelta`, and it requires the norm of the terms $\Phi_s^{(l)}$ to be obtained beforehand by calling the routine `obtainF`.

Analogously, the denominator is computed by the routine `Rbound` using the expression of $\mathcal{R}^{(l,r)}$ obtained in proposition 4.10, but with a few nuances in the bounds mentioned in the hypothesis of the proposition. In concrete, the bounds we used follow:

- To bound $\|H_k\|_R$ for $3 \leq k \leq r$ we used the absolute values of the series expansion of the Hamiltonian.
- To bound $\|H_k\|_R$ for $r \leq k \leq \tilde{r}$ we used the lower bounds given by lemma 4.7. The implementation of this part is done by the routine `simplebound`.
- To bound $\|H_k\|_R$ for $k > \tilde{r}$ we used the lower bounds given by lemma 4.8. the implementation of this part is done by the routine `hbound`.
- To bound $\|\Phi_j^{(l)}\|_R$ for $3 \leq k \leq r$ we used the absolute values of the series expansion of the first integrals.

It is important to note that the routines `simplebound` and `hbound` require c_1 and c_2 as input parameters. However, once μ and R are fixed, these parameters can be numerically obtained. The resulting values are $c_1 = 8.0572153$ and $c_2 = 92.04853$.

4.4.3 Results

We show the values of τ in Figure 4.1, for a concrete relationship between ρ and ρ_0. It can be noticed that the dependence of the time with the parameter ρ_0 is the same as shown in A. Celletti et al. [5]. As could have been expected, the lower bound of the stability time decreases when we allow the infinitesimal particle to be further away from the equilibrium point (meaning that ρ_0 increases). Moreover, the linear behaviour in a log-log scale seen in Figure 4.1 gives us the relationship:

$$\tau \propto \rho_0^{-\lambda}$$

where λ is a real positive value. Then, recalling that $t \propto \Phi/\dot{\Phi}$ we obtain:

$$\dot{\Phi} \propto \rho_0^{\lambda} \cdot \Phi \tag{4.44}$$

From here, we may notice that for a given ρ_0 the drift in the first integral, which is responsible for the escapement of the particle, increases with the first integral. Also, the last equation shows that the first integral increases exponentially in time.

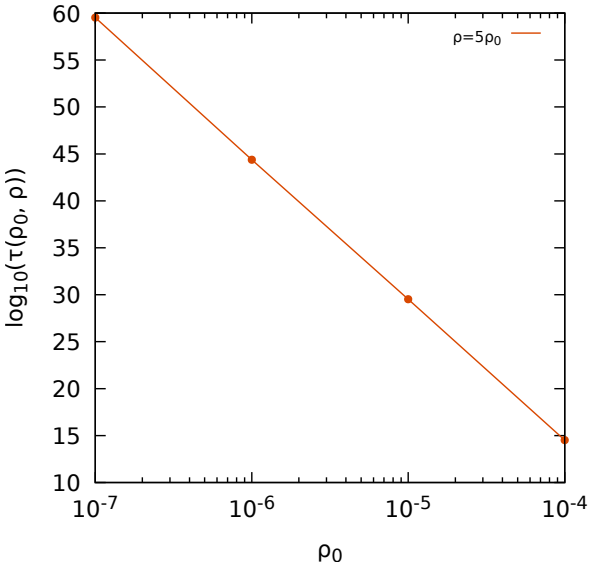

Figure 4.1: *Logarithm of the lower bound of the time escape as a function of the parameter ρ_0 for $\rho = 5\rho_0$. While the dots represent the values obtained with the C++ implementation described above, the lines are an interpolation to illustrate the linear behaviour.*

Let us remember that the units of the distance and the time are given by the units system that we have chosen. In particular, in the example of the system Sun-Jupiter, the unit of time is $\frac{1}{2\pi}$ times the revolution period of Jupiter around the sun and the unit of distance is the distance between the Sun and Jupiter.

A relevant application of the theory and the estimations made in this chapter is found in the Trojans. These, are asteroids located nearby the points $\mathcal{L}_4$ and $\mathcal{L}_5$ of the Sun-Jupiter system, and are thought to be effectively stable.

In fact, the proof of this statement was attempted by A. Celletti et al. in [5], among others, by computing a bound of the escape time, as we did in here. However, no concluding results have been found yet, since these methods proved effective stability for a time of the order of the age of the Universe only for a small region ρ_0, from which the Trojans are not only not inside it, but also so far away.

A final but still important remark is that the implementation described in this section does not provide us with a mathematical proof in the more strict sense. This fact, is mainly due to the uncertainty about the errors made by the computer when rounding terms in each operation.

In order to turn this implementation into a CAP (computer assisted proof) we would need to include an interval arithmetic, as done by À. Jorba in [11]. Briefly, this consists of a substitution of the real and integer magnitudes by intervals. Then, every time we carry out an operation, we do it for the two extremes, rounding the above extreme to a higher value and the bellow extreme to a smaller one. That way, we can guarantee that the value resulting from the operation lays within the resulting interval, giving us a control of the bounding errors of the computer. However, it is obvious that the size of the intervals increase with the number of operations, resulting it in a more imprecise interval each time.

Chapter 5

Summary and conclusions

In this work, we analysed the stability of the so familiar Lagrange points in the restricted circular 3-body problem (both in the planar and spatial cases), from a non-linear point of view. With that aim, the second chapter was devoted to introducing the Hamiltonian ruling the dynamics of the system and the equilibrium points were briefly deducted through analytical mechanics. Whereas the equilateral points were given by a compact condition, the collinears, on the other hand, could not be found analytically.

Concerning the allowed regions for the infinitesimal particle under the gravitational effects of the two primaries, it has been stated that these regions depend on the energy level of such a particle, or likewise, on the well known Jacobi constant. In order to illustrate this idea, an implementation of a numerical continuation method was done in C++ to draw some of the zero velocity curves enclosing these regions, which again depended on the energy level. The computation of these curves together with the discussion for big values of C enabled us to visualise the shapes of the Hill's regions, for a wide range of energy levels.

The problem of stability in the planar case, however, was not faced until the third chapter. There, we firstly introduced the basis of the Lyapunov stability theory, which already gave us important results such as the Dirichlet's and the Chetaev's theorems. Furthermore, a lot of results yielded from the linear behaviour already, such as the presence of a one-parameter family of periodic solutions emanating the collinear points and their instability, which also results from their linear behaviour. Nonetheless, we went one step further to prove this instability without seeking any eigenvalues, but using the Lyapunov theory instead.

After, the analysis of the stability of the equilateral points, still in the planar case, resulted more laborious. In fact, no conclusive information could be inferred from the linear regime in that case, leading us to the introduction of the KAM theory. Mainly developed by Kolmogorov, Moser, and Arnold, after several failed attempts from Poincare and many others who struggled with that problem, this theory provided a mathematical proof of the claim that the equilateral points in the planar case are stable, under certain conditions of the mass parameter.

In this dissertation, we briefly explained the celebrated Moser's theorem, enlarging on the link between the equilibrium points of a differential equation and the fixed points of a properly defined diffeomorphism. Afterwards, the Arnold's theorem enabled us to construct invariant tori emanating the equilibrium points. Connecting the

dots, altogether resulted in the stability behaviour of these points.

Accurately, we missed two cases of the mass parameter, in which the equilibrium points are actually unstable. Although it has not been explained in this work for conciseness, these cases correspond to certain integer resonance between the linear exponents (meaning that their ratio is a certain integer number) and further details can be found in K. Meyer et al. [15].

For what it concerns to the stability of the equilateral points in the spatial case, we saw that the added dimension prevented us from proving a stability result based on the KAM theory. Despite the fact that the theory was perfectly valid in that case too, it failed to prove the stability, since the invariant tori do not enclose a subregion in the new phase space. In addition, the infinitesimal particle is thought to leave from a small region near these equilibria, through a phenomenon currently known as Arnold's diffusion. Nevertheless, the time that it may take for this particle to leave from an equilibrium point might be of the order of the age of the solar system. Indeed, the concept of effective stability was introduced to address the cases where stability is not proven, but such a long time is needed for the trajectories nearby to leave. From that point, our reasonings where motivated by the Nekhoroshev theory, even though our intention was not to focus on this subject.

Analogously to the planar case, we faced the spatial problem by giving an overview to the Hamiltonian, which involved working in a 6-dimensional phase space. Then, this Hamiltonian was normalised and a change of variables was implemented to transfer the origin to one of the equilateral points. Furthermore, we introduced the n first integrals as a power series, which in turn enabled us to consider quasi first integrals by considering the first integrals truncated up to a finite order. This was vital for the computational implementation of the lower bound of the escape time (an approximation of the time of effective stability).

Naturally, this computational implementation was subsequent to some technical lemmas that aimed to bound the terms of the Hamiltonian and first integrals via the norm that we defined. We also used an open software made by À. Jorba to compute the series of the first integrals.

Although we developed the theory in order to obtain general bounds, we then applied it to a well known system: the Sun-Jupiter-Trojan system. Taking the corresponding mass parameter, we managed to obtain some bounds of the time until the effective stability holds, for a concrete relationship between the initial and final domains of the infinitesimal particle.

However, we explained how this failed to provide a result of stability for the Trojan asteroids for a time of the order of the age of the Universe. In fact, these asteroids are not sufficiently close to the equilibrium point to explain how can they be stable by using these bounds and approximations. Indeed, although it was widely studied in the 1990s, it just does not seem feasible to prove their stability using this type of bounds nowadays, since only one real asteroid has been proven to be effectively stable using these methods (see C. Skokos and A. Dokoumetzidis [19]).

Newer methods attempt to work with a normal form computed not in the equilibrium point, but in a point closer to the asteroids. This allows to compute invariant tori near the asteroids, and therefore, the escape time found is bigger, since the asteroids

are closer to the invariant tori, which are sticky[1]. One example of this method is found in the work of F. Gabern et al. [7], where this methodology is applied to find invariant tori near some of the asteroids assuming they are in the orbit's plane.

Still, there are a lot of related topics that we did not address. From the discussion of the linear behaviour to the proof of the lemma used in the proof of the Moser's theorem, there are many subjects that could have been developed further in a longer dissertation. Besides, we restricted ourselves to just a particular case of the general problem of 3 bodies, which was not our original intention but turned out as the most convenient option due to the deep analysis that it allowed.

But, what happens if the circular or the restricted approaches are not considered? How is the problem addressed for $N > 3$? Can it be faced with success if we consider the theory of General Relativity to describe the gravitational interaction between the particles? These are, among many others, some of the questions that have not been answered in this thesis and can serve as ideas for ongoing work.

[1]This term suggests that the orbits near the invariant tori remain there for long periods of time.

Appendix A

The continuation method to find zero-velocity curves

This method consists of an iteratively implementation of the well known Newton's method that finds ordered points for each zero-velocity curve.

For a given μ and C, such curves are given by the equation:

$$x_1^2 + x_2^2 + \frac{2\mu}{\sqrt{(x_1 - (1-\mu))^2 + x_2^2}} + \frac{2(1-\mu)}{\sqrt{(x_1 + \mu)^2 + x_2^2}} + \mu(1-\mu) - C = 0. \tag{A.1}$$

Let us assume we already know two points of the planar curve P_i and P_{i+1}. Let $\delta > 0$ be the inverse resolution parameter. Then, we are seeking a point $P_{i+2} \in \mathbb{R}^2$ that is a solution of (A.1) and is at a distance δ of P_{i+1}.

To do so, let us define the vector $v = P_{i+1} - P_i \in \mathbb{R}^2$. Then, as the curve given by (A.1) is smooth, the point $P_{i+1} + \delta \frac{v}{\|v\|}$ can be taken as a proper seed for a Newton's implementation to find P_{i+2}. Whenever P_{i+2} turns to be at a minor distance from P_0 than δ, the algorithm has reached the end, since it means that we have found our enclosed curve. Of course, this assumes that the corresponding parameterised curve visits the point P_0 only once, which is the case.

Hence, the remaining discussion focuses on how to find two points P_0 and P_1 that fulfil (A.1) and which are at a distance δ from each other. In order to find P_0, for the curves that crosses the x_1 axis, we set the trial $P_0 = (x_1, 0)$. Imposing $x_2 = 0$ in (A.1) we get:

$$x_1^2 \pm \frac{2\mu}{x_1 - (1-\mu)} \pm \frac{2(1-\mu)}{x_1 + \mu} + \mu(1-\mu) - C = 0. \tag{A.2}$$

Now, considering the change $\xi = x_1 - (1-\mu)$ we obtain that

$$(\xi + 1 - \mu)^2 \pm \frac{2\mu}{\xi} \pm \frac{2(1-\mu)}{\xi + 1} + \mu(1-\mu) - C = 0 \tag{A.3}$$

where the plus-minus signs are chosen so each term is positive (as they come from a distance). It is clear that the equation (A.3) is a quartic polynomial equalled to zero, so the solution can be obtained analytically, and therefore, P_0.

Afterwards, P_1 can be obtained using $P_0 + \delta(0, 1)$ as the seed point in a Newton's

implementation, due to the geometry of these curves.

A final remark concerning the resolution parameter can be stated; because of the geometrical properties of the curves, the resolution does not ever has to be the same. In those regions where the curvature is smaller, a high resolution is not needed to draw the curve, so the resolution parameter can be smaller (meaning that δ is bigger) in order to make the algorithm faster. This can be easily adjusted with the steps that each step of the Newton's method takes to find the solution.

Code implementation

Here we show the most relevant blocks of the code implementation done in C++ to obtain the zero velocity curves. First, since we aim to draw curves in two dimensions, it is useful to define a point in 2D and the basic operations of this objects.

```
struct point {
     double x1;
     double x2;
};

//Compute the distance
double Pdist (point a, point b) {
     return sqrt(pow(a.x1-b.x1,2)+pow(a.x2-b.x2,2));
}

//Add points
point Psum (point a, point b) {
     point sum;
     sum.x1=a.x1+b.x1;
     sum.x2=a.x2+b.x2;
     return sum;
}

//Subtract points
point Prest (point a, point b) {
     point rest;
     rest.x1=a.x1-b.x1;
     rest.x2=a.x2-b.x2;
     return rest;
}
```

In the main program, given the required parameters[1] `C, mu, delta, P0`, a second point of the curve is computed. Afterwards, an implementation of the numerical continuation algorithm is run in order to find the shape of the whole curve.

```
int main () {

     //Read the parameters

     //Find the second point using newton method
     point seedP1;
     seedP1.x1=P0.x1;
     seedP1.x2=P0.x2+delta;
     pair<point, int> P1 =newton(seedP1, P0, tol, maxIter, C, mu, delta);
```

[1]Note that the first point is a parameter. As explained, the procedure to find such a point is straightforward and it will not be explained in here.

```
    //Find all the others using the continuation method
    vector<point> curve = continuation(P0, P1.first, C, mu);

    //Write the results down on a file
    return 0;
}
```

The continuation algorithm is implemented in the 4-arguments routine `continuation`.

```
vector<point> continuation (point P0, point P1, double C, double mu) {

    //define delta, tol and maxIter for Newton's method
    double delta=Pdist(P0, P1);
    double tol=0.00001;
    int maxIter=2000;

    //Define a vector that will contain all the points
    vector<point> curve;

    curve.push_back (P0);
    curve.push_back (P1);

    //Define an index of curve
    int i=1;
    double d;

    //Find all the points
    do {

        //Find the seed
        point v=Prest(curve[i], curve[i-1]);
        double vmod=sqrt(pow(v.x1,2)+pow(v.x2,2));
        if (vmod < 0.00001) {
            cerr << "divide by 0 in seed " << endl;
            exit(1);
        }
        point vnew;
        vnew.x1=delta*(v.x1/vmod);
        vnew.x2=delta*(v.x2/vmod);
        point seed=Psum(curve[i], vnew);

        //Find the next point using Newton's method
        pair<point, int> newpoint=newton(seed, curve[i], tol, maxIter, C
   , mu, delta);
        curve.push_back (newpoint.first);
        i++;

        //Adjust delta depending on the steps required in the Newton's
   method

        //Compute the distance between the new point and P0
        d=Pdist(P0, curve[i]);

    } while (d>delta);

    return curve;
}
```

The reader may notice that both in the `main` and in `continuation` routine the routine

newton is called. This function returns both the aimed point of the curve and the number of iterations it takes.

```
pair<point, int> newton (point x0, point ant, double tol, int maxIter,
    double C, double mu, double delta) {
     point xn=x0;
     point sol;
     int i, res=0, meanwhile=0;
     for (i=0; i<maxIter && meanwhile==0; i++) {
          x0 = xn;

          //Compute new xn

          //Find the distance
          double dcompare=Pdist(xn, x0);
          if (dcompare < tol) {
               res = 1;
               sol = x0;
               meanwhile = 1;
          }
     }

     if (meanwhile == 0) {
          res = 1;
     }

     if (res == 0) {
          cerr << "No convergence reached in Newton " << endl;
          exit(1);
     }

     pair<point,int> newpoint;
     newpoint.first=sol;
     newpoint.second=i;

     return newpoint;
}
```

Appendix B

Code implementation of the computation of the escape time

In this last appendix we briefly explain the pseudo-code of the implementation of the escape time lower bound. To begin with, we defined a structure to work with polynomials. In fact, a polynomial is understood as a vector of monomials (`vector<Monomial>`), which contains the exponents of each variable and the corresponding coefficient.

```
 typedef struct Monomial{
     vector<int> x, y;
     double coef;
} Monomial;
```

Then, the norm of the series of the Hamiltonian and the first integrals were obtained by applying the norm to the series given by the software of À. Jorba [11]. This is done by the routines `obtainBoundH` and `obtainBoundF`, which receive the vector R, the reals r and ρ_0 and the name of the file containing the series expansion as parameters.

```
 //We compute the norm of each H_k
double* obtainBoundH (int r, vector<double> R, double rho0, string fname)
   {

     double* H;
    //Save memory for H

     ifstream fin;
     //Open file

     //Exponents sum
     int e1, e2, e3;

     double value;
     int current, old=2;

     int i=2;

     //Exponents
     vector<int> x(4,0), y(4,0);

     while (fin>>x1) {

        //Read other exponents and coefficient
```

```
          //Degree
          current=x1+x2+x3+y1+y2+y3;

          if(current>old) {
               i++;
               old++;
          }

          //Save monomial
          Monomial M;
          M.x=x;
          M.y=y;
          M.coef=value;

          e1=M.x[1]+M.y[1];
          e2=M.x[2]+M.y[2];
          e3=M.x[3]+M.y[3];

          H[i]+=fabs(M.coef)*pow(R[1],e1)*pow(R[2],e2)*pow(R[3],e3);

     }

     //Close file

     return H;

}
```

The routine `obtainBoundF` is basically the same but adapting the rows that have to be read from the input file.

In order to compute the bounds of the Hamiltonian, we implemented the corresponding lemmas in the routines `simplebound` and `hbound`. Specifically, this second routine was implemented to perform the computation of the parameter h.

```
 //Here we compute the bounds in lemma 1 in celletti1991
double* simplebound (double* H, int r, int rtilde, double c1, double c2)
    {

     double* S;
     S=(double*) malloc((rtilde+1)*sizeof(double));

     //From 2 to r the bound is the term in absolute value
     for(int i=2; i<=r; i++)
          S[i]=fabs(H[i]);

     //From r+1 to rtilde the bound is as in lemma 1 in celletti1991
     for(int i=r; i<rtilde; i++) {
          S[i+1]=((((2*double(i))+1)/(double(i)+1))*c1*S[i]) + (((double(i
    ))/(double(i)+1))*c2*S[i-1]);
     }

     return S;
}


//Here we find the parameter h
double hbound (double* S, int r, double c1, double c2) {

     double h=S[r]/S[r-1];
```

```
     double aux= c1 + sqrt((c1*c1) + c2);

     if (aux>h)
          h=aux;

     return h;

}
```

Now, we are in condition to compute the bound given by proposition 4.10. This is implemented in the routine called `Rbound`, and consists mainly of a simple translation of summations into loops.

```
 //Here we compute the bound of the derivative of the first integral
double Rbound (double* S, double* F, double h, double lambda, double E,
    double rho, int r, int rtilde) {

     double res=0;
     double term1=0,term2=0, term3=0, term4=0;
     double sumint=0, num=0, frac=0;

     //First line lemma 4 celletti1991
     for(int j=1; j<=r-2; j++) {
          sumint=0;
          for(int p=r-j+1; p<=rtilde; p++) {
               sumint+=p*pow(rho,p)*S[p];
          }
          term1+= (j+2)*pow(rho,j)*F[j+2]*sumint;
     }
     term1*=pow(lambda,2);

     //Second line lemma 4 celletti1991
     num=((rtilde+1)*(pow((h*rho),rtilde+1))) - (rtilde*(pow((h*rho),
    rtilde+2)));
     frac=(num)/(pow((1-(h*rho)),2));
     frac*=(E/pow(h,rtilde))*h;
     for(int j=1; j<=r-2; j++) {
          term2+= (j+2)*pow(rho,j)*F[j+2];
     }
     term2*=frac;
     term2*=pow(lambda,2);

     //Third line lemma 4 celletti1991 first term
     for(int s=r+1; s<=rtilde; s++) {
          term3+= s*S[s]*pow(rho,s);
     }

     //Third line lemma 4 celletti1991 second term
     term4=frac;

     res=term1+term2+term3+term4;

     return res;
}
```

Finally, one last computation remains: the computation of $\Delta(\rho_0, \rho)$. This is done in the routine `obtainDelta` and it is quite straightforward considering its definition.

```
 //We compute the Delta constant
```

```
double obtainDelta (double rho, double rho0, int r, double* F, double Rl)
    {

      double drho=0, drho0=0;
      double D=0;

      //compute the delta functions involved
      for(int j=3; j<=r; j++) {
           drho+=F[j]*pow(rho,j);
           drho0+=F[j]*pow(rho0,j);
      }

      //Compute Delta (D)
      D=pow(rho,2)-pow(rho0,2);
      D*=0.5*Rl*Rl;
      D-=drho;
      D-=drho0;

      return D;
}
```

On the whole, the `main` program of this implementation applies the routines described above to obtain the value of τ as shown in the expression 4.42. To do so, this program receive the parameters c_1 and c_2 as well as the parameters ρ and ρ_0. Also, the user must introduce the parameters r and $\tilde{r}$ beforehand.

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