Advertising and Quality Improving Strategies in a Supply Chain When Facing Potential Crises

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Abstract

In this paper, we consider a supply chain that faces a potential brand crisis, with one manufacturer deciding quality improvement and global advertising levels, and one retailer determining local advertising effort. The goodwill model proposed by Nerlove and Arrow (1962) is adopted here under the assumption that when the crisis happens, the companies suffer a sharp decrease in the goodwill. We characterize the feedback Nash equilibrium, and then we compare the corresponding quality and advertising strategies and outcomes with those of the case where the potential crises are absent, and where the companies do not invest in quality. The effects of the instantaneous crisis rate and the short-term and long-term damages are also evaluated. Our results reveal that the pre-crisis quality improvement accelerates the goodwill build-up before the crisis, and also helps the recovery in post-crisis regime. Its twofold function suggests that one of the pre- and post-crisis regimes/instants ought to be matched with more intense investment in both quality and global advertising, depending on the overall effect of instantaneous crisis rate, short-term damage and long-term damage. This carryover effect also brings a non-monotonicity of quality improvement effort and value functions with respect to the instantaneous crisis rate. These properties leave the chance to mitigate the loss by anticipating crisis for both members under certain circumstances.

Keywords: OR in marketing; Crisis Management; Quality Improvement; Supply Chain Management; Piecewise Deterministic Differential Game

1 Introduction

Companies face uncertainties. Brand crises can occur at random times. They may be related to defective products, for example, Samsung had to recall the Galaxy Note 7 because of critical failures in the batteries that could result in fires and Ford issued several recalls for millions of cars/trucks with loose steering wheels and an unseated gear shift cable locking clip. They could be associated with

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corporate social responsibility, like Nike's child labor scandal or Volkswagen's *Dieselgate*. Firms might even find themselves at the center of controversies that they are not responsible for, such as negative publicity generated by their celebrity endorser. Since the occurrence of such unexpected crises may damage a brand's goodwill, sales and profitability, farsighted managers should take them into account when making business plans.

Our primary research interest lies in the interaction between brand crisis and advertising because of their close connection. It is well-known that a strong commitment to advertising is the key to success. This marketing communication helps to increase brand and product awareness, build brand images, differentiate products from those of other companies, and so on. The importance of advertising is evidenced by the large and increasing amount of money spent by successful corporations. However, the intangible asset built up by advertising can be fragile and a brand crisis is among the biggest threats. Moreover, the effects of a brand crisis is not limited to the current period, but also in the future, as the effectiveness of advertising can be impacted (MacKenzie & Lutz, 1989; Van Heerde et al., 2007). Advertising is also considered an effective tool for firms to get through a crisis. For instance, British Petroleum Company spent neared $100 million in advertising during the Gulf of Mexico oil spill (Shelley DuBois, 2010), and Samsung apologized for the Note 7 defects through full-page ads (Alex Hern, 2016). Rubel et al. (2011) extended the model of Sethi (1983), in which a monopoly influences sales by advertising, to a stochastic setting that a product-harm crisis can occur at an unknown instant, and analyzed the effects of a crisis theoretically and empirically. Nonetheless, little academic attention has been paid to the context of a supply chain. The advertising of marketing channel members has a range of objectives and functions. In general, the manufacturer's global advertising aims to improve brand equity, whereas the retailer's local advertising contributes directly to consumer demand. Consequently, they may react in different ways when anticipating a crisis. In fact, contrary effects of brand and promotional advertising are found in Liu et al. (2017).

Another instrument that might help to navigate a brand through a crisis is quality improvement, whose relevance to a product-harm crisis is beyond doubt. It should not be taken lightly even when coping with other kinds of brand crisis, as quality plays a vital role in building and maintaining brand equity. It is agreed that quality can strengthen goodwill (for instance, see Nair & Narasimhan, 2006; De Giovanni, 2011; Liu et al., 2015) or slow down its depreciation (Reddy et al., 2016; De Giovanni, 2019). However, quality management is somehow reliant on advertising, in that information about quality needs to be delivered via this one-way communication from brands to customers. Products that are heavily advertised receive disproportionately more good ratings (Marquardt & McGann, 1975), since quality ratings/perceived quality is positively affected by advertising spending (Archibald et al., 1983; Moorthy & Zhao, 2000). Rubel et al. (2011) also discussed, briefly, the case where firms can choose between a low and a high type of quality investment, which corresponds to a high and a low instantaneous crisis rate, respectively. Their analysis suggested that firms can be better off with high quality investment only if sales are sufficiently high.

Hence, brand crisis, advertising, and quality improvement seem to be closely connected, and the interface among them deserves consideration. To this end, in this paper we provide an analytical framework to conduct a thorough analysis. Specifically, we consider a supply chain consisting of a single manufacturer and a single retailer, where the manufacturer controls global advertising and quality improvement, while the retailer focuses on local advertising, in the presence of potential crises. We try to answer the following questions:

1. What are the optimal global and local advertising plans when a supply chain faces potential crises?

2. How should a manufacturer coordinate the quality improvement effort and global advertising when envisioning a crisis?

3. What are the adjustments made by both marketing channel members when a crisis occurs?

4. What are the effects of the crisis intensity rate and the short-term and long-term damage?

5. Is it possible to mitigate the loss caused by a crisis?

We characterize rules to decide in which regime (pre-crisis or post-crisis) to allocate more quality improving resources, and to adjust global and local advertising when a crisis happens, which involve the consideration of short-term and long-term damage and the hazard rate of a crisis. We generalize the results of Rubel et al. (2011) and obtain a broader perspective of the effects of crises on strategies and payoffs. Moreover, with the combination of quality and advertising, we identify some circumstances under which enterprises can mitigate crisis damage by proactively anticipating the event, thus offering some theoretical support for the benefits of voluntary recalls.

Our contribution is fourfold.

First, we study the strategic use of different kinds of advertising in a marketing channel when potential crises exist. The manufacturer's brand advertising highlights the firm's image and could be useful in reputation restoration, whereas the retailer's repetitive promotion might be perceived as a signal of poor quality (Jørgensen et al., 2003). Such opposite consequences have inspired the differentiation between these two advertising types.

Second, we introduce quality management into the supply chain in an intertemporal setting. As the interplay among members in the supply chain differs from that in another market structure, the quality strategies may also differ. Special attention is required, yet studies placing this strategy in a supply chain environment are scarce (we refer the readers to Leng & Parlar, 2005, for a survey of game theoretic models in supply chain management). Most of the previous work has studied this issue using a static setting (see, for example, Reyniers & Tapiero, 1995; Wang et al., 2017), whereas El Ouardighi et al. (2008), El Ouardighi & Kim (2010), De Giovanni (2011), El Ouardighi & Kogan (2013), El Ouardighi (2014), Lambertini (2018) and Buratto et al. (2019) are among the rather few studies that apply a dynamic approach.

Third, we explore the interaction between marketing and operations management. As pointed out by Jørgensen (2018), although marketing plans affect and are affected by activities conducted in other functional areas, studies tackling these intersections that use optimal control or a differential game approach are scarce (some exceptions are Ringbeck, 1985; Colombo & Lambertini, 2003; El Ouardighi & Pasin, 2006; El Ouardighi et al., 2008; Liu et al., 2015 and Vörös, 2019). Undoubtedly, firms can benefit from a highly integrated organizational structure, and research exploring how to coordinate business functions could have some managerial implications.

Lastly, we study crisis management policies as a piecewise deterministic dynamic game, where strategic moves, dynamic evolution and uncertainty could be captured together. In fields like finance or environmental economics, the piecewise deterministic process has been extensively used to model problems with inherent uncertainty (for example, see Josa-Fombellida & Rincón-Zapatero, 2012; Polasky et al., 2011). However, this is not the case in the study of brand crisis management. To the best of our knowledge, Rubel et al. (2011) and Rubel (2018, 2020) are the only exceptions.

The rest of the paper proceeds as follows. First, we describe a piecewise deterministic game in which the supply chain members face a potential brand crisis in Section 2. The feedback equilibria are characterized in the following section. We then make a detailed analysis of the strategies and payoffs obtained for different regimes, followed by some numerical simulations to cast light on how the crisis influences the agents' behaviors and payoffs. Finally, a discussion on possible extensions and managerial insights is presented in Section 5.

2 Model Formulation

2.1 Advertising and Quality Management in a Supply Chain

There have been various game theoretic attempts to incorporate quality control into management activities. One common way to achieve this is to consider quality investment as a control variable contributing to goodwill build-up, customer retention, demand, potential market size and so on (see, e.g., Ringbeck, 1985; Nair & Narasimhan, 2006; De Giovanni, 2011; He et al., 2016; Caulkins et al., 2017; Buratto et al., 2019; Guan et al., 2020).

In another research area, it is considered that quality improvement activities can create intangible stock, which evolves over time and whose dynamics are subject to investment and depreciation. In line with this idea, some researchers mainly address conformance quality. For instance, Chand et al. (1996) adopted the "adaptation function" proposed by Levy (1965) to capture the conformance quality evolution with a decreasing rate of improvement,

$$\dot{Q}(t) = q(t)[1 - Q(t)] \ ,$$

and studied the allocation of capacity between production and process enhancement activities. El Ouardighi & Pasin (2006) reinterpreted this term in the way that firms work only on improving defective products to increase the perfection rate. They also assumed that only customers who have experienced defective products can be attracted by the rival company. Other extensions related to conformance quality include the incorporation of inventory management (El Ouardighi et al., 2008), interaction with design quality (El Ouardighi & Kogan, 2013), word-of-mouth effect (El Ouardighi et al., 2016), and the appraisal and prevention of non-defective products (De Giovanni, 2019).

Another common research stream, which we follow in this paper, takes a more general accumulation structure (for example, Vörös, 2006; Roselli & De Giovanni, 2012; El Ouardighi, 2014; Liu et al., 2015; Reddy et al., 2016; Xue et al., 2017),

$$\dot{Q}(t) = k_q q(t) - \epsilon Q(t), \quad Q(t) \geq 0 \; \forall \, t, \quad Q(0) = Q_0 \;, \tag{1}$$

where k_q and ϵ are positive constants representing the effectiveness of quality investment and the depreciation rate, respectively. Here, we look at quality stock from a more integrated, knowledge-alike aspect. The state variable $Q(t)$ can be considered total quality, and its evolution is attributed to all kinds of quality improvement effort $q(t)$ made on product quality, product process quality, service quality, service quality process, and business planning (Juran & Godfrey, 1951). Like any other intangible asset, it also suffers depreciation proportional to the current state.

The marketing implications of quality are well-supported in the business literature. A higher quality level, together with increased advertising, can improve a firm's reputation and thereby lead to a higher market share. Moreover, greater demand can be achieved by enhancing certain aspects of quality, such as fitness for use, aesthetics and conformance (Garvin, 1984).

We adopt the Nerlove & Arrow (1962) goodwill model to incorporate the market gains caused by quality. Unlike what Liu et al. (2015) do in their study, in which quality contributes to goodwill in the same linear way as advertising, we propose the following dynamics,

$$\dot{G}(t) = k_m A_M(t) \sqrt{Q(t)} - \delta G(t) \;, \tag{2}$$

where $k_m > 0$ denotes advertising effectiveness, and $\delta \in (0, 1)$ measures the consumers' forgetting effect. In (2), we suggest that the quality level determines how effective advertising can be. The main idea here is drawn from Nelson (1974), who argues that firms generally deliver quality information via advertisements, whereas consumers receive such information and validate it by searching or experiencing. For a firm that provides products of very poor quality ($Q(t) = 0$), a build-up of goodwill would be extremely unlikely. Although advertising providing wrong quality information can induce trial purchases, it does not contribute to, and may even damage, the seller's reputation. In contrast, repeat purchases usually happen to a firm that offers high-quality products, and therefore its advertising is more effective in the long run.

We extend the demand function in Lu et al. (2019) and define demand as follows:

$$D(t) = \theta + \mu G(t) + \gamma A_R(t) \sqrt{G(t)} + \eta Q(t) \;, \tag{3}$$

where the extra term $\eta Q(t)$ stands for the direct contribution of improved quality to demand, with effectiveness parameter η, θ represents the baseline sales, and μ and γ refer to the influential factors of goodwill and the synthetic product of goodwill and retailer's advertising.

The advertising and quality cost are assumed to be of quadratic form,

$$C(q) = \frac{c_q}{2}q^2 \ , \quad C(A_M) = \frac{c_m}{2}A_M(t)^2 \ , \quad C(A_R) = \frac{c_r}{2}A_R(t)^2 \ ,$$

where c_q, c_m, and c_r are positive constant cost parameters.

2.2 A Two-regime Game with Crisis

Now we proceed to incorporate crisis management.

There are many ways to introduce uncertainty into a differential game. The most frequent approaches consist in the employment of piecewise deterministic processes and Wiener processes (for more details, see Dockner et al., 2000). The Wiener process, also known as Brownian motion and white noise process, takes into account the continuous stochastic fluctuation of the state of the system, which is caused by unknown/external factors that are beyond the control of the decision makers. For instance, Prasad & Sethi (2004) analyze an advertising competition, in which the dynamics of market share are subject to disturbances caused by the inherent randomness of customers' purchasing behaviors, the lack of product differentiation, the forgetting effect, and so on.

In this paper, since our focus is to capture the possible occurrence of a crisis, we describe the uncertainty in the form of a piecewise deterministic process. In this framework, the unexpected events take place at discrete, random moments, and the system jumps from one mode to another. A sudden change in state, payoff and/or dynamics can happen due to these events, but in each mode, the state evolution is governed by a deterministic law.

For the sake of simplicity, we assume that a crisis happens just once, at a random time instant τ, which can take any value in $[0, \infty)$. This is a common setting in the literature related to regime shifts (for example, Polasky et al., 2011; Rubel et al., 2011). The common way to model this continuous random variable τ is through the hazard rate $\lambda(t)$, defined as

$$\lambda(t) = \lim_{\delta t \to 0} \frac{Pr\{t \leq \tau < t + \delta t \mid \tau \geq t\}}{\delta t} \ ,$$

which is the conditional probability that the crisis will take place in the interval $[t, t + \delta t)$, given that it has not occurred before. If we confine our interest to the case of constant hazard rate $\lambda(t) = \lambda$, then the corresponding probability distribution of the duration of the pre-crisis regime is exponential $F(t) = 1 - e^{-\lambda t}$. In addition, $1/\lambda$ (the mean of the exponential probability distribution) is the expected time when the crisis takes place, i.e., $E(\tau) = 1/\lambda$ (see Kiefer, 1988, for an explanation of distributions of duration and hazard functions).

The crisis results in an instantaneous goodwill downturn (a shock):

$$G(\tau+) = (1 - \Phi)G(\tau-) \ , \tag{4}$$

which in turn implies a loss of demand. In addition, this short-term damage also incorporates the lump-sum cost induced by the crisis. Take a product-harm crisis, for example. The firms may need to pay recall expenses, consumer compensation and a lawsuit, among other costs. Since the goodwill level somehow captures the market size, it is appropriate to assume that this cost is linear in goodwill. A similar idea appears in Rubel et al. (2011), where, instead of goodwill, they focus on the sales dynamics, and the sales drop-down caused by a crisis is also proportional to the state at the time τ.

The crisis might also make the subsequent goodwill accumulation less efficient. Consumers could be more skeptical when receiving information from advertisements, as the firm loses its credibility (MacKenzie & Lutz, 1989). Furthermore, dissatisfied customers may share their bad experience with potential customers. In other words, the negative word-of-mouth effect increases the difficulty in attracting new clients (El Ouardighi et al., 2016). The decrease in advertising effectiveness was confirmed in the empirical test run by Van Heerde et al. (2007), Zhao et al. (2011) and Liu & Shankar (2015).

Thus, the game is divided into pre-crisis and post-crisis regimes, with the following goodwill dynamics:

$$\dot{G}(t) = \begin{cases} k_{m1} A_M(t)\sqrt{Q(t)} - \delta_1 G(t) & \text{for} \quad 0 \le t \le \tau, \\ k_{m2} A_M(t)\sqrt{Q(t)} - \delta_2 G(t) & \text{for} \quad t \ge \tau, \end{cases} \qquad G(t) \ge 0 \; \forall \, t, \quad G(0) = G_0 \,, \tag{5}$$

where $k_{m1} \ge k_{m2}$ and $\delta_1 \le \delta_2$, denoting greater advertising effectiveness and a lower depreciation rate in the pre-crisis regime.

Assuming that the revenue is split between the manufacturer and retailer via a fixed transfer price (as in Chintagunta & Jain, 1992; Jørgensen et al., 2003; Jørgensen & Zaccour, 2003), and, without loss of generality, normalizing the retail price to one, the manufacturer and retailer aim to maximize their expected profits over time, given by

$$J_M = E\left[\int_0^\tau e^{-\rho t}\left(\pi D(t) - \frac{c_m}{2}A_M(t)^2 - \frac{c_q}{2}q(t)^2\right)dt + e^{-\rho\tau}V_M(2,G,Q)\right], \tag{6}$$

and

$$J_R = E\left[\int_0^\tau e^{-\rho t}\left((1-\pi)D(t) - \frac{c_r}{2}A_R(t)^2\right)dt + e^{-\rho\tau}V_R(2,G,Q)\right], \tag{7}$$

where $\pi \in (0,1)$ denotes the revenue sharing rate, $D(t)$ is defined in (3), and $V_M(2,G,Q)$ and $V_R(2,G,Q)$ stand for the manufacturer's and retailer's post-event value functions, respectively.

Equations (1), (4), (5), (6) and (7) define a two-player piecewise differential game with two state variables $Q(t) \ge 0$ and $G(t) \ge 0$, where the manufacturer controls $q(t) \ge 0$ and $A_M(t) \ge 0$, and the retailer controls $A_R(t) \ge 0$. Note that our problem is a stationary infinite time horizon model since τ, the time of occurrence of the crisis, is defined on the interval $[0,\infty)$ (see, for instance, Rubel et al., 2011).

In order to focus clearly on the impact of the crisis, we consider a noncooperative game where the manufacturer and retailer choose their policies simultaneously and independently. The Nash equilibrium is often studied in the literature when cooperation between the members of a supply chain (in the form of a cost-sharing program or vertical integration) is absent (e.g. Jørgensen et al., 2003; El Ouardighi & Kogan, 2013). However, in our model, the results would be the same if the game were played *à la Stackelberg* (a brief discussion will be presented in Remark 1).

3 Determination of Feedback Nash Equilibria

Following the approach in Seierstad (2013), the Hamilton-Jacobi-Bellman (HJB) equations for the two players are:

$$
\begin{aligned}
\rho V_M(1,G,Q) = \max_{\{A_M(1)\geq 0,\, q(1)\geq 0\}} & \Big\{ \pi \left[\theta + \mu G + \gamma A_R(1)\sqrt{G} + \eta Q\right] \\
& - \frac{c_m}{2} A_M(1)^2 - \frac{c_q}{2} q(1)^2 + \frac{\partial V_M(1,G,Q)}{\partial G}\left[k_{m1} A_M(1)\sqrt{Q} - \delta_1 G\right] \\
& + \frac{\partial V_M(1,G,Q)}{\partial Q}\left[k_q q(1) - \epsilon Q\right] \\
& + \lambda \left[V_M\left(2,(1-\Phi)G,Q\right) - V_M(1,G,Q)\right]\Big\},
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\rho V_R(1,G,Q) = \max_{\{A_R(1)\geq 0\}} & \Big\{ (1-\pi) \left[\theta + \mu G + \gamma A_R(1)\sqrt{G} + \eta Q\right] - \frac{c_r}{2} A_R(1)^2 \\
& + \frac{\partial V_R(1,G,Q)}{\partial G}\left[k_{m1} A_M(1)\sqrt{Q} - \delta_1 G\right] + \frac{\partial V_R(1,G,Q)}{\partial Q}\left[k_q q(1) - \epsilon Q\right] \\
& + \lambda \left[V_R\left(2,(1-\Phi)G,Q\right) - V_R(1,G,Q)\right]\Big\},
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\rho V_M(2,G,Q) = \max_{\{A_M(2)\geq 0,\, q(2)\geq 0\}} & \Big\{ \pi \left[\theta + \mu G + \gamma A_R(2)\sqrt{G} + \eta Q\right] - \frac{c_m}{2} A_M(2)^2 - \frac{c_q}{2} q(2)^2 \\
& + \frac{\partial V_M(2,G,Q)}{\partial G}\left[k_{m2} A_M(2)\sqrt{Q} - \delta_2 G\right] + \frac{\partial V_M(2,G,Q)}{\partial Q}\left[k_q q(2) - \epsilon Q\right]\Big\},
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\rho V_R(2,G,Q) = \max_{\{A_R(2)\geq 0\}} & \Big\{ (1-\pi) \left[\theta + \mu G + \gamma A_R(2)\sqrt{G} + \eta Q\right] - \frac{c_r}{2} A_R(2)^2 \\
& + \frac{\partial V_R(2,G,Q)}{\partial G}\left[k_{m2} A_M(2)\sqrt{Q} - \delta_2 G\right] + \frac{\partial V_R(2,G,Q)}{\partial Q}\left[k_q q(2) - \epsilon Q\right]\Big\},
\end{aligned} \tag{11}
$$

where 1 and 2 in parenthesis following strategies/value functions denote the pre-crisis and post-crisis regimes, respectively. Notice that the post-crisis regime game is equivalent to a deterministic game, so $V_M(2,G,Q)$ and $V_R(2,G,Q)$ can be computed using the corresponding method. Different from the usual HJB, equations (8) and (9) have an additional term $\lambda\left[V_i\left(2,(1-\Phi)G,Q\right) - V_i(1,G,Q)\right]$ $(i = M, R)$, indicating the expected change in profits by jumping from pre-crisis to post-crisis regime. Maximizing the right-hand-side of equations (8) - (11) yields

$$
q(j)^* = \frac{k_q}{c_q}\frac{\partial V_M}{\partial Q}(j,G,Q),\ A_M(j,Q)^* = \frac{k_{mj}}{c_m}\frac{\partial V_M}{\partial G}(j,G,Q)\sqrt{Q},\ A_R(j,G)^* = \frac{(1-\pi)\gamma}{c_r}\sqrt{G},\ (j=1,2).
$$

We guess that value functions in both regimes are linear in G and Q. After substituting them into (8) - (11), by identifying the parameters of G, Q and constant parts, the feedback Nash equilibrium strategies in both regimes are given in the following two propositions.

Proposition 1. *A Feedback Nash equilibrium in the post-crisis regime is given by the strategies*

$$
q(2) = \frac{k_q}{c_q}\beta_{M2}\,, \tag{12}
$$

$$
A_M(2,Q) = \frac{k_{m2}}{c_m}\alpha_{M2}\sqrt{Q}\,, \tag{13}
$$

$$A_R(2,G) = \frac{(1-\pi)\gamma}{c_r}\sqrt{G} \ , \tag{14}$$

and the corresponding value functions are given by

$$V_M(2,G,Q) = \alpha_{M2}G + \beta_{M2}Q + \tau_{M2} \ , \tag{15}$$

$$V_R(2,G,Q) = \alpha_{R2}G + \beta_{R2}Q + \tau_{R2} \ , \tag{16}$$

where $\alpha_{M2} = \dfrac{c_r\pi\mu + \pi(1-\pi)\gamma^2}{c_r(\rho+\delta_2)}$, $\beta_{M2} = \dfrac{(k_{m2})^2}{2c_m(\rho+\epsilon)}(\alpha_{M2})^2 + \dfrac{\pi\eta}{\rho+\epsilon}$, $\tau_{M2} = \dfrac{(k_q)^2}{2c_q\rho}(\beta_{M2})^2 + \dfrac{\pi\theta}{\rho}$,

$\alpha_{R2} = \dfrac{2c_r(1-\pi)\mu + (1-\pi)^2\gamma^2}{2c_r(\rho+\delta_2)}$, $\beta_{R2} = \dfrac{(k_{m2})^2}{c_m(\rho+\epsilon)}\alpha_{M2}\alpha_{R2} + \dfrac{(1-\pi)\eta}{\rho+\epsilon}$,

and $\tau_{R2} = \dfrac{(k_q)^2}{c_q\rho}\beta_{M2}\beta_{R2} + \dfrac{(1-\pi)\theta}{\rho}$.

All three strategies given by (12) - (14) are proportional to the ratio of their corresponding effectiveness parameter to cost parameter k_q/c_q, k_{m2}/c_m and γ/c_r. The investment in quality improvement $q(2)$ is increasing in μ, γ and η, i.e., when i) goodwill, ii) the synergistic effect of goodwill and local advertising, and/or iii) quality contribute to demand to a larger extent, the manufacturer will invest more. It also increases in k_{m2}, which measures how large the quality level's influence is on the dynamics of goodwill. On the contrary, higher cost parameters of marketing (of both agents), larger depreciation rates of quality and goodwill, and greater discount rates will result in a decrease of quality improvement expenditure. Regarding the manufacturer's advertising strategy $A_M(2,Q)$, it is increasing in the quality level with an elasticity of 0.5: 1% increase of Q will lead to an increase of 0.5% in A_M.

Concerning the retailer's advertising $A_R(2,G)$, it is goodwill-state dependent in a similar way as A_M with respect to Q. Besides, when she takes a higher part of revenue, she spends more in local advertising.

If the manufacturer only had a marketing tool A_M, as in Lu et al. (2019), she would invest in a constant way, whereas the retailer would decide local advertising depending on the goodwill level. These properties might be a result of the influential mechanism: the demand is highly dependent on the goodwill level, and global advertising A_M is the unique way to enhance it. The manufacturer's strategies change qualitatively when she has the option to improve quality. On the one hand, since the quality level determines the advertising's effectiveness, it is beneficial to make a positive effort on it. On the other hand, she gains sort of responsiveness by being able to adapt her advertising budget depending on the quality level achieved. The retailer reacts in exactly the same way (for a given goodwill level) in these two models because she has limited influential power in both settings, in the sense that A_R works solely together with goodwill.

Proposition 2. *A Feedback Nash equilibrium in the pre-crisis regime is given by the pair of strategies*

$$q(1) = \frac{k_q}{c_q}\beta_{M1} \ , \tag{17}$$

$$A_M(1,Q) = \frac{k_{m1}}{c_m}\alpha_{M1}\sqrt{Q} \ , \tag{18}$$

$$A_R(1,G) = \frac{(1-\pi)\gamma}{c_r}\sqrt{G}\ , \tag{19}$$

and the corresponding value functions are given by

$$V_M(1,G,Q) = \alpha_{M1}G + \beta_{M1}Q + \tau_{M1}\ , \tag{20}$$

$$V_R(1,G,Q) = \alpha_{R1}G + \beta_{R1}Q + \tau_{R1}\ , \tag{21}$$

where $\alpha_{M1} = \frac{\rho+\delta_2+\lambda(1-\Phi)}{\rho+\delta_1+\lambda}\alpha_{M2}, \quad \tau_{M1} = \frac{(k_q)^2}{2c_q(\rho+\lambda)}\left[(\beta_{M1})^2 + \frac{\lambda}{\rho}(\beta_{M2})^2\right] + \frac{\pi\theta}{\rho},$

$\beta_{M1} = \frac{1}{2c_m(\rho+\epsilon+\lambda)}\left[(k_{m1})^2(\alpha_{M1})^2 + \frac{\lambda}{(\rho+\epsilon)}(k_{m2})^2(\alpha_{M2})^2\right] + \frac{\pi\eta}{\rho+\epsilon},$

$\alpha_{R1} = \frac{\rho+\delta_2+\lambda(1-\Phi)}{\rho+\delta_1+\lambda}\alpha_{R2}, \quad \tau_{R1} = \frac{(k_q)^2}{c_q(\rho+\lambda)}\left(\beta_{M1}\beta_{R1} + \frac{\lambda}{\rho}\beta_{M2}\beta_{R2}\right) + \frac{(1-\pi)\theta}{\rho},$

$\beta_{R1} = \frac{1}{c_m(\rho+\epsilon+\lambda)}\left[(k_{m1})^2\alpha_{M1}\alpha_{R1} + \frac{\lambda}{(\rho+\epsilon)}(k_{m2})^2\alpha_{M2}\alpha_{R2}\right] + \frac{(1-\pi)\eta}{\rho+\epsilon},$

and α_{M2}*,* β_{M2}*,* α_{R2} *and* β_{R2} *are defined in Proposition 1.*

Strategies taken in the pre-event regime show similar structures as those in the post-event regime. Some properties such as being proportional to the efficiency ratio and being positive state-dependent with decreasing marginal effect also apply here. It is worth mentioning that parameters in the second regime are also involved in the decision making. For example, $A_M(1,Q)$ is decreasing in both δ_1 and δ_2 at different rates (but only increasing in k_{m1}). Besides, high advertising effectiveness in both regimes (k_{m1} and k_{m2}) would induce a higher investment in quality $q(1)$. Note that when the crisis happens, it is the goodwill $G(t)$ which suffers a sharp decrease, whereas the quality stock remains and will continue serving as a booster to goodwill accumulation. Thus the manufacturer would be also motivated to invest more in quality by high advertising effectiveness in the second regime. Besides, in the pre-crisis regime the manufacturer has to take into account the potential crisis, namely, the hazard rate λ, and the shock in goodwill Φ while deciding quality improvement and global advertising.

The response in global advertising $A_M(1,Q)$ to λ is straightforward. As

$$\frac{\partial A_M(1,Q)}{\partial\lambda} = -\frac{k_{m1}[\Phi\rho + (\delta_2-\delta_1) + \Phi\delta_1]}{c_m(\rho+\delta_1+\lambda)^2}\alpha_{M2}\sqrt{Q} < 0\ ,$$

the manufacturer invests less in marketing when anticipating a greater chance of the crisis happening. Since the shock is proportional to the goodwill state and $A_M(1,Q)$ directly acts on goodwill accumulation, it is reasonable to slower the build-up process before crisis in order to minimize the loss.

However, the case of quality investment $q(1)$ is more complicated. By rewriting

$$\begin{gathered} q(1) = \frac{k_q}{c_q}(\alpha_{M2})^2\left[f_1(\lambda)(k_{m1})^2 + f_2(\lambda)(k_{m2})^2\right] + \frac{k_q\pi\eta}{c_q(\rho+\epsilon)}\ , \\ \text{with } f_1(\lambda) = \frac{[\rho+\delta_2+\lambda(1-\Phi)]^2}{2c_m(\rho+\epsilon+\lambda)(\rho+\delta_1+\lambda)^2}\ ,\ f_2(\lambda) = \frac{\lambda}{2c_m(\rho+\epsilon+\lambda)(\rho+\epsilon)}\ , \end{gathered} \tag{22}$$

we can observe a conflicting influence of λ on $q(1)$ in (22), since $\frac{\partial f_1(\lambda)}{\partial\lambda} < 0$ whereas $\frac{\partial f_2(\lambda)}{\partial\lambda} > 0$. Let us consider the dual contribution of $q(1)$. On the one hand, it accelerates the goodwill build-up before the crisis, and a larger hazard rate is intuitively harmful and lowers the quality investment. On the other hand, it has a carryover effect on recovery after the crisis, and the player has incentives to increase the budget. The manufacturer needs to balance these two impacts while deciding on the quality improvement effort. A larger hazard rate corresponds to an earlier expected crisis occurrence time, therefore a shorter pre-crisis regime and a longer post-crisis period, which prioritizes the carryover effect, and *vice versa*. Consequently, the overall effect turns out negative when the hazard rate is small, and positive for larger hazard rates. The only exception is when the advertising effectiveness decreases so much after the crisis ($k_{m1} \gg k_{m2}$) that the carryover effect is trivial. In this case, the manufacturer would always invest less in quality improvement in the first regime when they face a larger hazard rate. We will offer some numerical illustration of $q(1)$ in Section 4.2.

As to the crisis magnitude Φ, it manifests a negative influence in both quality improvement $q(1)$ and global advertising $A_M(1, Q)$, due to the same idea of slowing down goodwill accumulation so that the crisis would be less catastrophic.

Notwithstanding, unlike the reactive manufacturer, the retailer's policies are similar before and after crisis (though the global expenditure varies as the goodwill level is evolving), because her highly limited influencing power makes her not able to respond to the crisis, though we can observe that at the moment of the crisis happening, she adjusts her local advertising through a reduction of $(1 - \pi)\gamma \left(1 - \sqrt{1 - \Phi}\right) \sqrt{G}/c_r$.

Remark 1. *In the literature of supply chain, another common setting is the Stackelberg mode of play. Nonetheless, in this paper, as no strategic interdependence occurs via the instantaneous payoff functions, the first-mover advantage disappears no matter which player is the leader (the coincidence of feedback Nash and Stackelberg equilibria also appears in De Giovanni, 2011, and Rubio, 2006 offers a technical explanation on this). Take, for example, the case of manufacturer being the leader. A stage-wise feedback Stackelberg equilibrium can be obtained following the approach in Başar & Haurie (1984). In both regimes, we first compute the retailer's optimal strategies by maximizing the right-hand-side of her HJB equations. Then we substitute them into the manufacturer's HJB equations, perform the maximization on the right-hand-side to get the leader's policies. Replacing all these strategies and the guessed linear value functions into the HJB equations yields a feedback Stackelberg equilibrium which is identical to the one of Nash characterized in Proposition 1 and 2.*

4 Analysis of the Results

We start our analysis by presenting two benchmark cases in order to have a better understanding about the changes of introducing quality management and crisis.

We first compare how the supply chain members adjust their strategies when the manufacturer gets an additional operational tool, the quality improvement, and if they benefit from it. To do so, we extend the model of Lu et al. (2019) to a crisis setting. The extended model and its equilibria are briefly given in the Appendix, and the comparison is summarized in Remark 2.

Remark 2. *Let $A_M^{NQ}(i)$ and $A_R^{NQ}(i)$ $(i = 1, 2)$ denote the manufacturer's advertising, the retailer's advertising in pre-crisis $(i = 1)$ and post-crisis $(i = 2)$ regimes with the absence of quality. The agents' behaviors and payoffs in feedback Nash equilibria can be related as follows:*

1. $A_M^{NQ}(i) \leq A_M(i, Q)$ *if* $Q \geq 1$ $(i = 1, 2)$.

2. $A_R^{NQ}(i, G) = A_R(i, G)$.

3. $V_j^{NQ}(1, G) \leq V_j(1, G, Q)$ *if* $Q \geq \frac{\rho}{\rho+\epsilon} - \frac{\tau_{j1}}{\beta_{j1}}$ $(j = M, R)$.

Proof. See Appendix. □

In general, if the manufacturer's product has a superior quality compared to the industry standard, she would invest more in global advertising. The retailer uses the same responsive strategy, thought the goodwill accumulation would be of different paths. In most of the cases, both members are better off even if the game starts with a zero quality level[1].

Next, we analyze how the existence of potential crisis could affect the agents' behaviors and the payoffs. We start by studying the deterministic model where the supply chain does not face a potential crisis. It can be represented by the special case of $\lambda = 0$. The equilibrium corresponding to such deterministic game is characterized in Proposition 2 with $\lambda = 0$.

Remark 3. *Let $\alpha_i = \alpha_{i1}|_{\lambda=0}$, $\beta_i = \beta_{i1}|_{\lambda=0}$, $\tau_i = \tau_{i1}|_{\lambda=0}$ $(i = M, R)$, and q, A_M and A_R denote the quality improvement effort, the manufacturer's advertising, the retailer's advertising when facing no potential crisis. The agents' behaviors and payoffs can be related as follows:*

1. $q > q(1)$ *for all* $\lambda > 0$.

2. $A_M(Q) \geq A_M(1, Q)$ *for all* $\lambda > 0$ *(the equality holds if and only if* $\Phi = 0$ *and* $\delta_1 = \delta_2$*)*.

3. $A_R(G) = A_R(1, G)$ *for all* $\lambda > 0$.

4. $V_M(G, Q) > V_M(1, G, Q)$ *and* $V_R(G, Q) > V_R(1, G, Q)$ *for all* $\lambda > 0$.

Proof. See Appendix. □

If there is no potential crisis, the manufacturer would invest more in quality improvement, and both types of advertising budgets are higher for a given quality/goodwill level. Furthermore, both the manufacturer and the retailer are better off, which is intuitive.

[1]It can be easily checked that $Q < \frac{\rho}{\rho+\epsilon} - \frac{\tau_{j1}}{\beta_{j1}}$ $(j = M, R)$ can hold only under extremely unreasonable parameters setting.

4.1 Pro-Efficiency vs. Pro-Recovery

We now compare the players' strategies before and after crisis. As the quality and goodwill levels are dynamic, so are both agents' advertising budgets (which are state dependent). Instead of comparing the advertising strategies in both regimes from a global aspect, as how we dealt with $q(1)$ and $q(2)$, we focus on the moment τ and we are able to show the players' immediate reaction in their marketing strategies when come up against a crisis.

Proposition 3. *The agents' strategies in the two regimes are related as follows:*

1. $q(1) = q(2) + \dfrac{k_q}{2c_m c_q(\rho+\epsilon+\lambda)}\left[(k_{m1}\alpha_{M1})^2 - (k_{m2}\alpha_{M2})^2\right]$, *and*
   $q(1) \geq q(2) \quad \textit{if} \quad \dfrac{k_{m2}}{k_{m1}} \leq \Omega = \dfrac{\rho+\delta_2+\lambda(1-\Phi)}{\rho+\delta_1+\lambda}$,
   $q(1) < q(2) \quad \textit{otherwise.}$

2. *When crisis occurs,* $A_M(1, Q(\tau)) = \left(\dfrac{k_{m1}\alpha_{M1}}{k_{m2}\alpha_{M2}}\right) A_M(2, Q(\tau))$, *and*
   $A_M(1, Q(\tau)) \geq A_M(2, Q(\tau)) \quad \textit{if} \quad \dfrac{k_{m2}}{k_{m1}} \leq \Omega = \dfrac{\rho+\delta_2+\lambda(1-\Phi)}{\rho+\delta_1+\lambda}$,
   $A_M(1, Q(\tau)) < A_M(2, Q(\tau)) \quad \textit{otherwise.}$

3. *When crisis occurs,* $A_R(1, G(\tau-)) \geq A_R(2, G(\tau+))$ *(with strict inequality for* $\Phi > 0$*).*

Proof. It follows from (12), (13), (14), (17), (18) and (19). □

As shown in Proposition 3, the difference between $q(1)$ and $q(2)$ is a constant, whereas at the moment when crisis happens, the *ex-post* global advertising $A_M(2, Q(\tau)$ is proportional to the *ex-ante* $A_M(1, Q(\tau)$. Although we compare the manufacturer's strategies in the pre- and post-crisis regimes in different ways, the manufacturer has quite a clear regime-based-priority, in the sense that in one of the pre- and post-event regimes/instants, she invests more in both quality and global advertising. This consistency stems from the two-sided effects of pre-crisis quality improvement. As explained previously, $q(1)$ contributes to goodwill accumulation in both pre- and post-event regimes, since the quality state is not affected by the crisis. Therefore, while deciding in which regime to allocate more quality investment, the manufacturer must consider the advertising effectiveness. Specifically, when k_{m2}/k_{m1}, the fraction of post-crisis global advertising effectiveness per unit of its pre-crisis value is smaller than the threshold Ω, which is decreasing in λ, Φ and $\delta_2 - \delta_1$, the priority will be to invest before the crisis occurs, and *vice versa.*

We call this *pro-efficiency* if the quality and global advertising budget are reduced when a crisis occurs, and *pro-recovery* in the opposite case.

The retailer always reduces local advertising when a crisis happens, as long as it harms the product's reputation. Greater advertising intensity could make consumers even more impressed by the crisis. Furthermore, the main activity of retail's advertising is promotion, which is likely to worsen the situation for certain types of crises. For instance, price deals are frequently associated with unstable or inferior quality (Raghubir & Corfman, 1999; Yoo et al., 2000; Villarejo-Ramos & Sánchez-Franco, 2005). Hence, it would not be a good idea to increase their intensity when a product-harm crisis is brewing. For a crisis related to corporate social responsibility (CSR), price promotion might raise consumers' suspicions of the firm's policies, since the prices of products labeled with CSR are generally higher (Creyer, 1997; Ferreira et al., 2010).

However, this does not apply to the manufacturer. Andina-Díaz et al. (2019) analyzed the advertising expenditure of Volkswagen Group in Spain from January 2014 to December 2016, and observed an average monthly decrease from €788,211 to €562,333 after the *Dieselgate* scandal. On the contrary, as shown in Rubel et al. (2011), the weekly brand advertising of Ford was increased after the recall of the Explorer sports utility vehicle (SUV). Although increased brand advertising may also attract unwanted attention from customers to the event, it is still one of the most powerful communication tools in a turbulent environment. The manufacturer can use global advertising to deliver messages involving an apology, a compensation plan, future commitment, and so on. Thus, brand advertising has positive impacts on restoring brand image (Cowden & Sellnow, 2002) and mitigating crisis damage (Sharpe & Hanson, 2020). These contradictory effects require manufacturers to take more factors into account when making advertising decisions during a crisis, as we will explain below.

Since

$$\frac{\partial \Omega}{\partial \lambda} = \frac{\delta_1(1-\Phi) - \delta_2 - \rho(1+\Phi)}{(\rho+\delta_1+\lambda)^2} \leq 0 \, ,$$

we have

$$1 - \Phi \leq \Omega \leq \frac{\rho+\delta_1}{\rho+\delta_2} \, ,$$

and we can classify some special cases as described in Remark 4.

Remark 4. *Depending on the short-term and long-term damages caused by the crisis, we have some special cases:*

(I) If $\dfrac{k_{m2}}{k_{m1}} = 1$ *and* $\delta_1 = \delta_2$*, then*
$q(1) \leq q(2)$ *and* $A_M(1, Q(\tau)) \leq A_M(2, Q(\tau))$ *for all* $\lambda > 0$ *(with strict inequality for* $\Phi > 0$*).*

(II) If $\Phi \neq 0$ *and* $1 - \Phi < \dfrac{k_{m2}}{k_{m1}} < 1$*, then*
$q(1) > q(2)$ *and* $A_M(1, Q(\tau)) > A_M(2, Q(\tau))$ *for* $0 < \lambda < \hat{\lambda}$*, and*
$q(1) < q(2)$ *and* $A_M(1, Q(\tau)) < A_M(2, Q(\tau))$ *for* $\lambda > \hat{\lambda}$*, where* $\hat{\lambda} = \dfrac{k_{m2}(\rho+\delta_1) - k_{m1}(\rho+\delta_2)}{k_{m1}(1-\Phi) - k_{m2}}$.

(III) If $\dfrac{k_{m2}}{k_{m1}} \leq 1 - \Phi$*, then*
$q(1) \geq q(2)$ *and* $A_M(1, Q(\tau)) \geq A_M(2, Q(\tau))$ *for all* $\lambda > 0$ *(with strict inequality for* $\Phi > 0$*).*

Case (I) is a special case where after the crisis the goodwill stock evolves exactly in the same way as how it is before ($k_{m1} = k_{m2}$ and $\delta_1 = \delta_2$), namely, the crisis does instantaneous harm to the companies (if $\Phi \neq 0$) without causing any other long-term effect. Under these circumstances, it is worthier to make relatively more effort after the crisis in order to recover from the shock as soon as possible. As a consequence, independently of the crisis hazard rate, the manufacturer always invests more in quality in the post-crisis regime, and increases global advertising when the crisis happens. In the empirical study by Cleeren et al. (2013), they recommend an advertising increase in a high-publicity product-harm case where the fault is not attributed to the firm (and thus there should not be long-term damages).

On the contrary, case (III) describes a situation where the long-term damage dominates the short-term loss. Accordingly, the emphasis will be always placed in the first regime regardless of the crisis intensity rate, in this way the manufacturer can profit from the high efficiency before crisis. It is interesting to note that the influence of the change in advertising effectiveness k_{m2}/k_{m1} is larger than that in goodwill depreciation $\delta_2 - \delta_1$, for the reason that k_{m1} and k_{m2} modify directly the effect of the strategies. This situation coincides with the empirical results of Cleeren et al. (2013), which suggest an advertising decrease when the product-harm crisis is of low publicity but the firm needs to acknowledge the fault.

As an intermediate case, in (II) the crisis causes an instantaneous loss, and also reshapes the goodwill accumulation path in the way that it becomes more difficult to strengthen the goodwill by advertisement and/or the goodwill suffers a faster depreciation. However, it is hard to conclude which impact is more destructive. The manufacturer exhibits higher interests in the first regime for a sufficiently small instantaneous crisis rate $\lambda < \hat{\lambda}$, which indicates a later expected occurrence time and thus a longer pre-crisis period. Whereas she would switch to *pro-recovery* strategies if the crisis is estimated to happen in the early stage ($\lambda > \hat{\lambda}$).

To sum up, the instantaneous injury generated by the crisis makes the manufacturer incline towards a set of *pro-recovery* strategies, while a strong long-term damage may lead to *pro-efficiency* policies. When making decisions, the manufacturer has to face a trade-off between higher efficiency in the pre-event regime and faster recovery in the post-event regime, apart from considering the crisis instantaneous rate.

4.2 The Three "Impact Factors" of Crisis: Numerical Illustration

In this section we present some numerical illustrations to throw light on how the crisis influences the quality and advertising strategies. There are three underlying "impact factors" capturing the nature of the crisis: the hazard rate λ, which is inversely proportional to the average time when the crisis occurs; the shock Φ, which exhibits the immediate loss; and the changes in effectiveness and in depreciation, which picture the permanent damage. Specifically, based on the previous analysis in Remark 4, the change in advertising effectiveness k_{m2}/k_{m1} is more representative of the long-term injury. The parameters used are summarized in Table 1.

Figures 1-3 show how the quality improvement expenditure changes with the hazard rate. Quality investment policies in a game without crisis (q) are also graphed to serve as a benchmark. To interpret better these figures, we firstly discuss two extreme points: $\lambda = 0$ and $\lambda \to \infty$. For zero hazard rate,

Table 1: Parameter Setting

ρ	θ	μ	γ	η	c_m	c_q	c_r
0.1	1	1	1	1	2	2	2
k_q	k_r	π	ϵ	δ_1	δ_2	Φ	k_{m1}
1	1	0.75	0.1	0.05	0.05	0.3	0.5
Figure		**1**	**2**	**3**	**4**	**5**	**6**
k_{m2}		0.5	0.43	0.3	0.45	0.45	0.45
G		–	–	–	5	20	70
Q		–	–	–	3	3	3

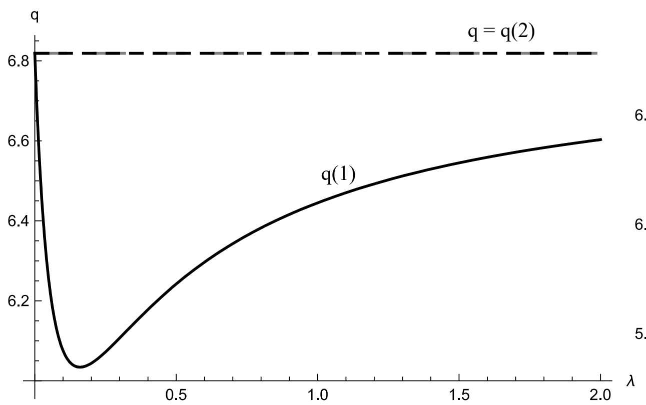

Figure 1: Quality Investment. Case (I)

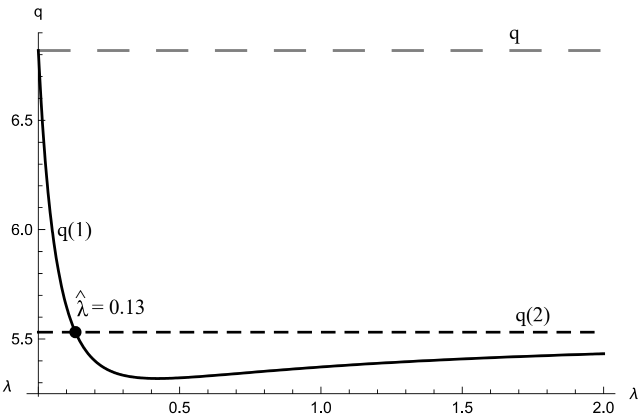

Figure 2: Quality Investment. Case (II)

$q(1)$ coincides with q. As to the other extreme point, from Proposition 3,

$$\lim_{\lambda\to\infty}[q(1)-q(2)] = \lim_{\lambda\to\infty}\left\{\frac{k_q}{2c_m c_q(\rho+\epsilon+\lambda)}\left[(k_{m1}\alpha_{M1})^2-(k_{m2}\alpha_{M2})^2\right]\right\} = 0\ , \tag{23}$$

so $q(1)$ converges to $q(2)$ as λ approaches ∞. Note that the case in which the crisis is expected to happen in the beginning of the time horizon can be considered as another deterministic game of the same set of parameters in the post-crisis regime, which explains why (23) holds. With the two extreme points fixed, we can observe how the manufacturer adapts $q(1)$ taking into account the hazard rate. These three figures correspond to the three cases described in Remark 4. Recall that $q(1)$ has double effects in both regimes, as discussed in Section 3, and in some cases, it exhibits non-monotonic tendency under the conflicting influences from hazard rate, as shown in Figure 1 and 2. However, when the long-term damage absolutely wins over the instantaneous loss (case (III) in Remark 4), the *pro-efficiency* strategies are applied independently of the hazard rate. It is also in this case when the carryover effect of $q(1)$ is irrelevant and the quality investment in pre-crisis regime is monotonically decreasing.

Next we discuss the effect of λ on value functions. Following the same idea of analyzing the quality

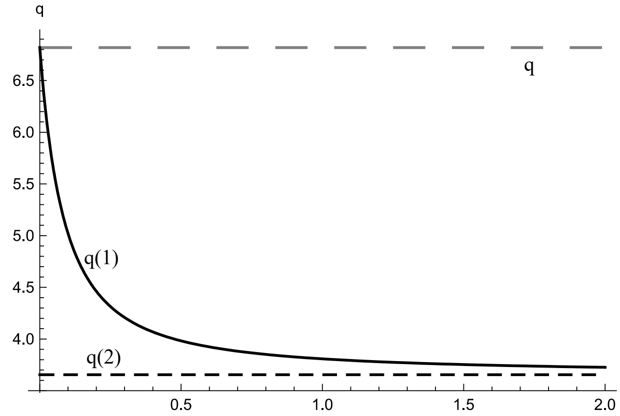

Figure 3: Quality Investment. Case (III)

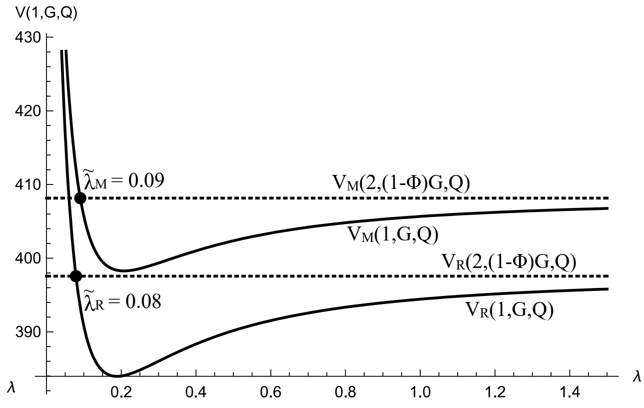

Figure 4: Value Functions. Case (IIb)

investment, we also focus on two extreme cases, $\lambda = 0$ and $\lambda \to \infty$, representing situations where there is no potential crisis, and the crisis is estimated to occur at the beginning of planning horizon, respectively. On the one hand, it is clear that, for both agents in the supply chain, the payoffs of a game with crisis are always inferior to that of the situation without crisis, as explained in Remark 3. Moreover,

$$\lim_{\lambda\to\infty} V_i(1, G, Q) = (1-\Phi)\alpha_{i2}G + \beta_{i2}Q + \tau_{i2} = V_i(2, (1-\Phi)G, Q)\ , \quad i = M, R.$$

If the crisis occurs immediately, the players will get as much as that in a deterministic game with initial goodwill state $(1-\Phi)G$ and under the parameters setting in the post-crisis regime.

Since $V_i(2, (1-\Phi)G, Q) < V_i(G, Q)$ holds for all $\lambda > 0$[2], the behaviors of $V_i(1, G, Q)$ will be determined by its value when λ tends to infinity. Take the manufacturer as an example (the retailer's value function has a similar behavior so it suffices to concentrate our analysis on the manufacturer), and compute

$$\begin{aligned} V_M(1, G, Q) - V_M(2, (1-\Phi)G, Q) &= [\alpha_{M1} - (1-\Phi)\alpha_{M2}]G + (\beta_{M1} - \beta_{M2})Q + (\tau_{M1} - \tau_{M2}) \\ &= \frac{\rho\Phi + \delta_2 - (1-\Phi)\delta_1}{\rho + \delta_1 + \lambda} G + (\beta_{M1} - \beta_{M2})Q + \frac{(k_q)^2}{2c_q(\rho+\lambda)}\left[(\beta_{M1})^2 - (\beta_{M2})^2\right]\ , \end{aligned} \tag{24}$$

and let $\tilde{\lambda}_M$ be the solution to $V_M(1, G, Q) = V_M(2, (1-\Phi)G, Q)$, when it exists. From (24), it is clear that the existence of $\tilde{\lambda}_M$ mainly depends on the initial goodwill level and the relationship between β_{M1} and β_{M2}, which determine $q(1)$ and $q(2)$ respectively. In particular, we can characterize three scenarios of zero, positive and very high initial goodwill levels, which are summarized, together with the three cases in Remark 4, in Table 2.

It is straightforward that for zero initial goodwill[3], which could be the case of a start-up manufacturer, the sign of (24) depends only on the relationship between β_{M1} and β_{M2}. Accordingly, there are

[2]Except for $\Phi = 0$, $k_{m1} = k_{m2}$ and $\delta_1 = \delta_2$. However, the crisis would have no effect in this case, so we do not include it into our discussion.

[3]Note that in this case, $V_M(2, (1-\Phi)G, Q)$ is "immune" from the instantaneous loss of the crisis.

Table 2: Existence of $\tilde{\lambda}_M$

	Scenario a: $G = 0$ or $\Phi = 0$, $\delta_2 = \delta_1$	Scenario b: $G > 0$, $\Phi \geq 0$, $\delta_2 \geq \delta_1$ (at least one $\geq$ is strict)	Scenario c: $G \gg Q$, $\Phi \geq 0$, $\delta_2 \geq \delta_1$ (at least one $\geq$ is strict)
Case (I): $\frac{k_{m2}}{k_{m1}} = 1$, $\delta_1 = \delta_2$	(24) < 0, no solution	(24) > 0 for $0 < \lambda < \tilde{\lambda}_M$ (24) < 0 for $\lambda > \tilde{\lambda}_M$	(24) > 0, no solution
Case (II): $1 - \Phi < \frac{k_{m2}}{k_{m1}} < 1$	(24) > 0 for $0 < \lambda < \tilde{\lambda}_M$ (24) < 0 for $\lambda > \tilde{\lambda}_M$ ($\tilde{\lambda}_M = \hat{\lambda}$)	(24) > 0 for $0 < \lambda < \tilde{\lambda}_M$ (24) < 0 for $\lambda > \tilde{\lambda}_M$ ($\tilde{\lambda}_M > \hat{\lambda}$)	(24) > 0, no solution
Case (III): $\frac{k_{m2}}{k_{m1}} \leq 1 - \Phi$	(24) > 0, no solution	(24) > 0, no solution	(24) > 0, no solution

three special cases, which are consistent with those described in Remark 4. If the initial goodwill level is positive, then even in the case (I) where the crisis has no long-term damage, there is also a single $\tilde{\lambda}_M$ solving $V_M(1, G, Q) = V_M(2, (1 - \Phi)G, Q)$. As to the case (II) where none of the short-term and long-term damages is strictly dominant for all possible hazard rates, we can find a unique solution $\tilde{\lambda}_M$, which is greater than $\hat{\lambda}$, the solution to $q(1) = q(2)$. Moreover, if the initial goodwill level is much higher than the initial quality level, it can happen that $V_M(1, G, Q) > V_M(2, (1 - \Phi)G, Q)$ holds for all $\lambda > 0$, no matter which is the dominance between short-term and long-term damages.

Figure 4 represents the case (IIb) and Figures 5 and 6 demonstrate the case (IIc)[4]. When (24)

[4]Note that the behaviors of $V_M(1, G, Q)$ in Figure 4 coincides with that of the case (Ib) and (IIa), whereas those in

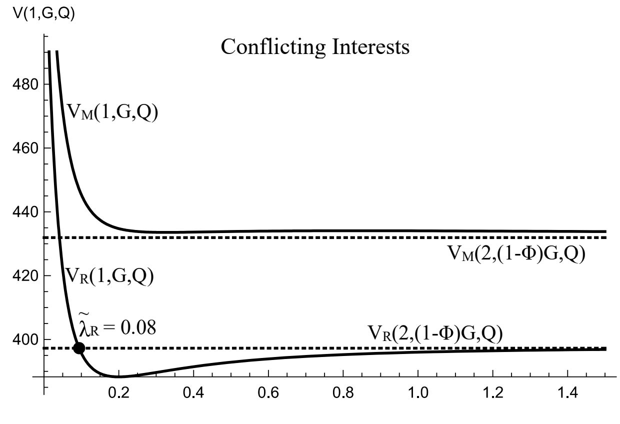

Figure 5: Value Functions. Case (IIc)

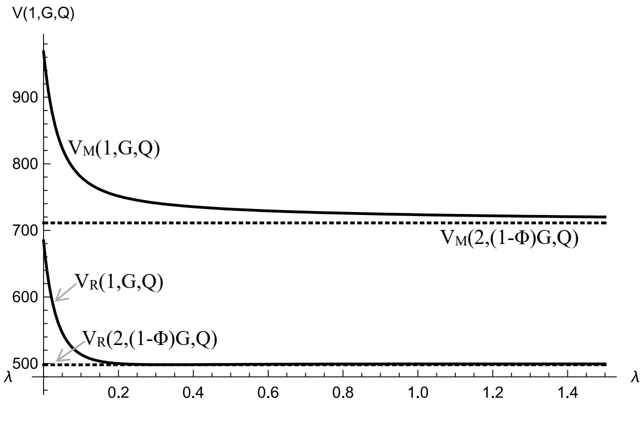

Figure 6: Value Functions. Case (IIc)

has a solution, as shown in Figure 4, there exists a λ minimizing the manufacturer's value function. This non-monotonicity derives from the binary effect of $q(1)$. It is worthwhile noting that, under these circumstances, a larger λ may be beneficial to the players. Compared with any instantaneous crisis rate such that $\lambda > \tilde{\lambda}_M$, the manufacturer would prefer $\lambda \to \infty$. A managerial implication derived from the discussion above is that, in this situation (case (IIb) or other similar cases), if the manufacturer is facing a hazard rate greater than $\tilde{\lambda}_M$, she would be interested in anticipating the crisis. The retailer also has incentives to anticipate the crisis, although she has another benchmark $\tilde{\lambda}_R$ different from $\tilde{\lambda}_M$.

To explain better this surprising result, we need to discuss some underlying properties. In this model we assume that the crisis happens only once, which means that if for some reason, the players anticipate the crisis, they can get rid of it forever. Besides, this kind of strategic anticipation does not work when the long-term damage plays a determinant role (Case III). It also seems more feasible to company that is not well known, as it does not work either when the firm has a strong initial goodwill (Scenario c). Lastly, V_M and V_R increasing in λ does not happen when λ is very small, which is, by intuition, more preferred by the players. This phenomenon only appears for λ being moderately small or large, that is to say, the crisis is expected to happen in the short run with a considerable chance. Under all these assumptions, it is true that the players would get better off when λ takes a greater value.

A higher risk level could also, in some situations, lead to higher profits in the context of pricing competition, according to Rubel (2018). He analyzed the pricing strategies in a duopolistic market under the threat of possible product-harm crisis, assuming that the involved firm has to recall all the defective products and withdraw from the competition (thus the market becomes monopolistic). His results show that a tiny risk level (λ is sufficiently small) is preferred compared with risk free ($\lambda = 0$), as it can soften price competition. Such preference is similar to but slightly different from ours. A mere anticipation in the future is recommended in pricing competition, whereas a radical anticipation to the present moment is suggested for a supply chain equipped with advertising and quality improving.

In the other case represented in Figures 5 and 6, the manufacturer would prefer the hazard rate to be as little as possible, which is coherent to our intuition. However, as we can see in Figure 5, the manufacturer and the retailer could have conflict of interests, as the retailer would prefer to anticipate the crisis.

Finally, we summarize the indirect effect in Figure 7 to illustrate how the crisis affects (indirectly) some strategies. From Section 3 we can see that the retailer's advertising in both regimes $A_R(1, G)$ and $A_R(2, G)$, as well as the post-crisis global advertising $A_M(2, Q)$ are not determined by crisis. However, as the crisis changes the goodwill and quality trajectories in the pre-crisis regime, these state-dependent strategies will also change accordingly. Summarizing, higher instantaneous damage rate will imply lower local advertising in pre- and post-crisis regime, and lower *ex-post* global advertising, which coincides what is found in Rubel et al. (2011). However, different from the study mentioned above, the

Figure 5 and 6 are consistent with that in the case (Ic), (IIIa), (IIIb) and (IIIc).

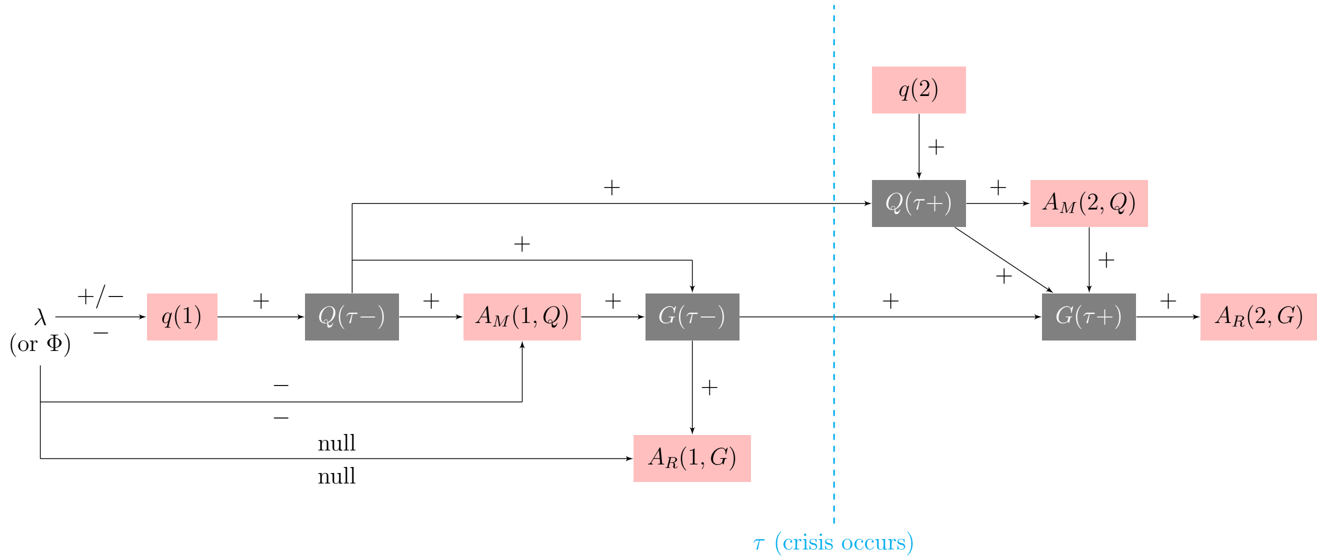

Figure 7: Indirect Effect

indirect effect of hazard rate λ on these policies can be positive when the carryover effect of pre-event quality investment $q(1)$ is strong enough and the crisis intensity rate is large.

5 Discussion and Conclusions

In this study, we have developed a piecewise deterministic differential game in a two-level supply chain, in which the manufacturer can decide quality improvement and global advertising levels, and the retailer determines local advertising effort. We have enriched the discussion about the quality management in supply chain operations, and have contributed to the research line of crisis management by using a differential game framework. Moreover, we have analyzed the interaction among quality control, advertising and crisis management, which, as far as we know, is new in the literature. The feedback Nash equilibria for both pre- and post-crisis regimes have been determined, the agents' behavior has been analyzed in detail, and the impacts of a crisis have been illustrated graphically.

Our results reveal that when the supply chain faces a potential crisis, strategies change accordingly under the overall interactive effect of crisis intensity rate, short-term damage and long-term damage. Besides, since pre-crisis quality investment also helps recovery in the post-crisis regime, the manufacturer will invest more in both quality and advertising in one of the regimes. Specifically, if the advertising effectiveness decreases sufficiently after the crisis, the manufacturer needs to apply *pro-efficiency* strategies, otherwise *pro-recovery* strategies are preferred.

The carryover effect of pre-event quality investments also gives rise to non-monotonicity of the quality improvement effort and value functions with respect to the instantaneous crisis rate. These properties allow both agents in the supply chain to strategically choose the crisis occurrence time under certain circumstances. Particularly, if the initial goodwill level is not much higher than the initial quality level, and the crisis long-term damage is not dominant over the short-term damage, the players could have the chance to reduce loss by anticipating the crisis. However, in some cases, the supply chain members may not agree on the policy of anticipation, as it is only beneficial to one of them.

We next discuss some possible extensions and managerial implications that might be of interest.

5.1 Possible Extensions

By focusing on the case of constant hazard rate, we have been able to solve the stochastic problem analytically and provide a detailed analysis. Naturally, one might wonder what happens if the hazard rate is non-constant. For instance, instead of a general crisis, we could focus on the product-harm crisis, whose intensity rate depends on the quality of products. Firms can reduce the instantaneous crisis rate through quality improvement, i.e., λ decreases in quality investment (q) or product quality (Q). Another possibility is to consider other probability distributions of the duration of the pre-crisis regime, such as the *Weibull distribution* and the *log-logistic distribution*, which give rise to time-varying crisis intensity rates. For example, a product-harm crisis is related to facility deterioration, and thus influenced by time. However, these modifications entail more work and effort to solve the problem. An inspection of the HJB equations (8) - (11) shows that the structures of value functions and optimal policies would be quite complicated. In particular, the nature of equilibria under endogenous hazard rate is considerably different from that obtained in this paper. Although analytical solutions may not be available, it is still possible to characterize the equilibria. Then, by comparing the optimal policies and steady states under different settings of hazard rate (zero/exogenous/endogenous), we can obtain some sketchy information about the agents' behavior and payoffs (as in Polasky et al., 2011; de Zeeuw & Zemel, 2012; Sakamoto, 2014).

Pricing strategies can be included in this model to enrich the discussion. In fact, as neither the wholesale price nor the retail price appear in dynamics (1) and (5), and the integrands in (6) and (7) are implicitly dependent on time, the extended model could remain analytically tractable with constant optimal prices. For example, the demand defined in (3) can be scaled up or down by retail price in a multiplicative structure (as in Jørgensen et al., 2001; Liu et al., 2015). This specification would lead to the same qualitative results as obtained in this paper.

It would be interesting to consider whether cooperation makes the supply chain more resistant to crisis. If supply chain members decide to carry out vertical integration, i.e., they form a coalition and act cooperatively aiming to maximize the collective profit, the equilibrium can be characterized by solving the optimal control problem with the objective function $J_M + J_R$. Under vertical integration, the coalition also clearly emphasizes in one of the regimes, exactly as in the non-cooperative case. Besides, in both pre- and post-crisis regimes, the supply chain increases the budget of all types of investment, thus the stationary levels of quality and goodwill reach a higher level and larger total profits are generated. The outcome of vertical integration is Pareto superior to that of non-cooperation. Another cooperation scheme is the adoption of a binding contract, for instance, a cost-sharing program in which the manufacturer subsidizes the retailer's local advertising. Such a cooperation mechanism can be modeled as a Stackelberg game, and the strategy interplay would yield a different outcome and possibly a first-mover advantage.

Lastly, we discuss the functional forms of this model and possible relaxations. Instead of the prevalent settings in the literature, we have adopted a square-root structure in different places to capture the decreasing marginal effect. However, some other well-known classes of analytically tractable functional forms would apply too, e.g., a linear-quadratic differential game. Nonetheless, the set of coefficients that need to be determined will be larger, and the interaction among parameters will be more complex. Therefore, it might be more difficult to obtain clear interpretation and insights.

5.2 Managerial Insights

We believe that the implications of our research could possibly support decision makers.

First, we offer manufacturers guidelines on the coordination between quality improvement and brand advertising. Chenavaz & Jasimuddin (2017) suggested that the linkage between advertising and product quality varies according to the nature of the goods and the cost. For an experience good (such as movie) and a low-cost industry, this relationship is positive. Our study has provided a new insight into the interaction of these two elements in a turbulent supply chain environment. As quality improves advertising effectiveness, it would be more efficient to couple intensive advertising with high quality commitment.

Second, we provide some suggestions for the supply chain on how to adjust resource allocation in a turbulent environment where crises can happen at any moment. When a crisis takes place, retailers should decrease local advertising effort in order to avoid undesirable attention and the possible linkage between price deals and poor product quality or lack of CSR commitment. However, due to the multiple contradictory impacts of brand advertising, farsighted manufacturers need to analyze the crisis characteristics and give preferential treatment to one of the pre- or post-crisis regimes. Particularly, evaluations of factors that determine the short-term and long-term damages of the crisis (such as crisis type, whether the company is blameworthy, media coverage, severity and relevance, consumers' prior expectations and beliefs, etc.), together with the instantaneous occurrence rate, will serve as ground rules for choosing between *pro-efficiency* and *pro-recovery* strategies.

Finally, firms can mitigate the loss caused by a brand crisis via advancing its occurrence, as long as the initial product goodwill does not far exceed the quality, and the crisis long-term effect is not much severer than the instantaneous one. The break-out of a crisis can be interpreted as the moment when negative information is disclosed to the public, has a broad impact and causes an unpleasant reaction, which is usually later than when the problem occurs. In practice, firms and customers often have asymmetric access to certain information. Therefore, the company can anticipate a crisis by sending out private (negative) information, for example, announcing a recall. Consider the case of a company that predicts a quality-related crisis in the near future due to some private information, such as quality test reports, consumer complaints, and so on, which is not evident to the public. In this situation, the best strategy would be to announce a recall policy. Although a recall is generally considered a crisis and harms the cooperation's reputation and sales, a voluntary recall can result in less loss and faster recovery of normal stock returns (Kong et al., 2019), and can have positive impacts on a firm's image, consumer loyalty and purchase intentions (Souiden & Pons, 2009).

To sum up, this study has painted a more complex, clearer picture of the interplay of different functional areas under uncertainty. Our results indicate that a supply chain could navigate through a brand crisis by coordinating advertising and quality improvement, prioritizing efficiency or recovery, and, under certain circumstances, anticipating its occurrence.

Acknowledgments

The authors would like to thank the editor and the two anonymous reviewers for their insightful comments and suggestions. The second author's research is supported by MINECO [ECO2017-82227-P (AEI/FEDER,UE)].

Appendix

The current model without quality.

Let $V_M(2,G)$ and $V_R(2,G)$ denote the manufacturer's and retailer's post-crisis value functions, the game is defined by

$$\max_{A_M(t)\geq 0} E\left[\int_0^\tau e^{-\rho t}\left[\pi\left(\theta+\mu G(t)+\gamma A_R(t)\sqrt{G(t)}\right)-\frac{c_m}{2}A_M(t)^2\right]dt+e^{-\rho\tau}V_M(2,G)\right],$$

$$\max_{A_R(t)\geq 0} E\left[\int_0^\tau e^{-\rho t}\left[(1-\pi)\left(\theta+\mu G(t)+\gamma A_R(t)\sqrt{G(t)}\right)-\frac{c_r}{2}A_R(t)^2\right]dt+e^{-\rho\tau}V_R(2,G)\right],$$

subject to

$$\dot{G}(t)=k_m A_M(t)-\delta G(t)\ .$$

The strategies in regime 1 (pre-crisis) and in regime 2 (post-crisis) are given by

$$A_M^{NQ}(1)=\frac{k_{m1}}{c_m}\alpha_{M1}\ ,\quad A_M^{NQ}(2)=\frac{k_{m2}}{c_m}\alpha_{M2}\ , \tag{25}$$

$$A_R^{NQ}(1,G)=A_R^{NQ}(2,G)=\frac{(1-\pi)\gamma}{c_r}\sqrt{G}\ , \tag{26}$$

and the corresponding value functions are determined by

$$V_M^{NQ}(1,G,Q)=\alpha_{M1}G+\frac{\rho}{\rho+\epsilon}\beta_{M1}\ , \tag{27}$$

$$V_R^{NQ}(1,G,Q)=\alpha_{R1}G+\frac{\rho}{\rho+\epsilon}\beta_{R1}\ , \tag{28}$$

where α_{M2} and α_{R2} are defined in Proposition 1, α_{M1}, α_{R1}, β_{M1} and β_{R1} are defined in Proposition 2.

Proof of Remark 2. 1. It follows from (13), (18) and (25).

2. It follows from (14), (19) and (26).

3. It follows from (20), (21), (27) and (28).

□

Proof of Remark 3.

$$\begin{aligned}
\alpha_M - \alpha_{M1} &= \frac{\lambda\left[\rho\Phi + \delta_2 - (1-\Phi)\delta_1\right]\left[c_r\pi\mu + \pi(1-\pi)\gamma^2\right]}{c_r(\rho+\delta_1)(\rho+\delta_2)(\rho+\delta_1+\lambda)} \geq 0 \,, \\
\alpha_M - \alpha_{M2} &= \frac{(\delta_2 - \delta_1)\left[c_r\pi\mu + \pi(1-\pi)\gamma^2\right]}{(\rho+\delta_1)(\rho+\delta_2)} \geq 0 \,, \\
\beta_M - \beta_{M1} &= \frac{(k_{m1})^2(\rho+\epsilon+\lambda)(\alpha_M)^2 - (k_{m1})^2(\rho+\epsilon)(\alpha_{M1})^2 - (k_{m2})^2\lambda(\alpha_{M2})^2}{2c_m(\rho+\epsilon)(\rho+\epsilon+\lambda)} \\
&= \frac{(k_{m1})^2(\rho+\epsilon)\left[(\alpha_M)^2 - (\alpha_{M1})^2\right] + \lambda\left[(k_{m1})^2(\alpha_M)^2 - (k_{m2})^2(\alpha_{M2})^2\right]}{2c_m(\rho+\epsilon)(\rho+\epsilon+\lambda)} \geq 0 \,, \\
\beta_M - \beta_{M2} &= \frac{(k_{m1})^2(\alpha_M)^2 - (k_{m2})^2(\alpha_{M2})^2}{2c_m(\rho+\epsilon)} \geq 0 \,, \\
\tau_M - \tau_{M1} &= \frac{(k_q)^2\left[(\rho+\lambda)(\beta_M)^2 - \rho(\beta_{M1})^2 - \lambda(\beta_{M2})^2)\right]}{2c_q\rho(\rho+\lambda)} \\
&= \frac{(k_q)^2\left\{\rho\left[(\beta_M)^2 - (\beta_{M1})^2\right] + \lambda\left[(\beta_M)^2 - (\beta_{M2})^2\right]\right\}}{2c_q\rho(\rho+\lambda)} \geq 0 \,.
\end{aligned}$$

Similarly, we have $\alpha_R - \alpha_{R1} \geq 0$, $\beta_R - \beta_{R1} \geq 0$ and $\tau_R - \tau_{R1} \geq 0$. Moreover, $\alpha_j = \alpha_{j1}$ holds if and only if $\Phi = 0$ and $\delta_1 = \delta_2$. $\beta_j = \beta_{j1}$ or $\tau_j = \tau_{j1}$ holds if and only if $\Phi = 0$, $\delta_1 = \delta_2$ and $k_{m1} = k_{m2}$ (and the crisis has no effects) $(j = M, R)$. Thus the results follow. □

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