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Muckenhoupt type weights and Berezin formulas for Bergman spaces ☆

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ABSTRACT

By means of Muckenhoupt type conditions, we characterize the weights ω on $\mathbb{C}$ such that the Bergman projection of $F^{2,\ell}_\alpha = H(\mathbb{C}) \cap L^2(\mathbb{C}, e^{-\frac{\alpha}{2}|z|^{2\ell}})$, $\alpha > 0$, $\ell > 1$, is bounded on $L^p(\mathbb{C}, e^{-\frac{\alpha p}{2}|z|^{2\ell}}\omega(z))$, for $1 < p < \infty$. We also obtain explicit representation integral formulas for functions in the weighted Bergman spaces $A^p(\omega) = H(\mathbb{C}) \cap L^p(\omega)$. Finally, we check the validity of the so called Sarason conjecture about the boundedness of products of certain Toeplitz operators on the spaces $F^{p,\ell}_\alpha = H(\mathbb{C}) \cap L^p(\mathbb{C}, e^{-\frac{\alpha p}{2}|z|^{2\ell}})$.

1. Introduction

One key tool in many problems in function theory is the existence of suitable integral representations (such as Bergman projections or Berezin integral operators) with precise estimates.

The main object of this paper is the study of such type of manageable representation formulas, and their accurate estimates for weighted Bergman spaces of entire functions related to Muckenhoupt type weights.

Our initial motivation comes from the following problem. Let $\alpha > 0$ and $\ell \geq 1$, and consider a real polynomial $\phi(x) = \frac{\alpha}{2}x^{2\ell} + \sum_{k=0}^{2\ell-1} c_k x^k = \phi_{\alpha,\ell}(x) + Q(x)$. Let $L^p_\phi = L^p(\mathbb{C}, e^{-p\phi(|z|)})$ (respectively, $L^{p,\ell}_\alpha = L^p_{\phi_{\alpha,\ell}}$) and let F^p_ϕ the corresponding Fock space of entire functions $f \in L^p_\phi$ (resp. $F^{p,\ell}_\alpha = F^p_{\phi_{\alpha,\ell}}$). The classical theory of Hilbert spaces gives the existence of the orthogonal projection P^ℓ_α from $L^{2,\ell}_\alpha$ onto $F^{2,\ell}_\alpha$, the so called Bergman projection, which is an integral operator with kernel K^α_ℓ, the Bergman kernel. This holds for any real number $\ell \geq 1$ (non necessarily integer). Indeed, $K^1_\alpha(z,w) = e^{\alpha z\overline{w}}$ and this neat expression easily gives the boundedness of P^1_α on $L^{p,\ell}_\alpha$, $1 \leq p \leq \infty$ (see [10]). For any real number $\ell > 1$, K^α_ℓ has an

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explicit expression in terms of Mittag-Leffler functions. Their asymptotic expansions give precise pointwise estimates of K_ℓ^α, from which the boundedness of P_α^ℓ on $L_\alpha^{p,\ell}$ is easily deduced (see, for instance, [4] and the references therein). We have to remark that these boundedness results are particular cases of a more general theory on doubling weighted Fock spaces [11]. These results suggest the following question: Is the Bergman projection P_α^ℓ also bounded on L_ϕ^p?

Since ϕ looks like a "perturbation" of $\phi_{\alpha,\ell}$, one might think that the answer is always affirmative. However, as we will show, this is not true. Surprisingly, P_α^ℓ is bounded on L_ϕ^p if and only if Q is a polynomial of degree less than or equal to ℓ.

The suitable setting for this problem is the study of the weights ω, that is, positive locally integrable functions on $\mathbb{C}$, such that P_α^ℓ is bounded from $L_\alpha^{p,\ell}(\omega) = L^p(\mathbb{C}, e^{-\frac{\alpha p}{2}|z|^{2\ell}}\omega)$ to $F_\alpha^{p,\ell}(\omega) = H(\mathbb{C}) \cap L_\alpha^{p,\ell}(\omega)$, for any real number $\ell \geq 1$. We obtain a characterization of such weights in terms of a Muckenhoupt type condition, that for the classical case $\ell = 1$ was obtained in [9].

We recall that, in the classical weighted Bergman setting on the unit ball of $\mathbb{C}^n$, the weights for which the Bergman projection is bounded are also characterized by a Muckenhoupt type condition. They are the so called Bekollé-Bonami weights (see [3] and [2]). There is a large literature on those weights, which have been a key tool in the study of weighted norm inequalities for the Bergman projection in different settings of complex analysis (see, for instance, [17], [19], [14], [15], and the references therein).

Before we state precisely our main results, we introduce the Muckenhoupt type weights we need. Let $\ell \geq 1$. For any $z \in \mathbb{C}$ and $\varrho > 0$, let $D_{z,\varrho}^\ell$ be the Euclidean disk of center z and radius $\varrho(1+|z|)^{1-\ell}$. We denote by $\mathcal{A}_{p,\varrho}^\ell$, $1 < p < \infty$, the set of all weights ω such that

$$[\omega]_{\mathcal{A}_{p,\varrho}^\ell} := \sup_{z\in\mathbb{C}} \frac{\omega(D_{z,\varrho}^\ell)\left(\omega'(D_{z,\varrho}^\ell)\right)^{p/p'}}{|D_{z,\varrho}^\ell|^p} < \infty.$$

Here p' is the conjugate exponent of p, $\omega' = \omega^{-p'/p}$, $|D_{z,\varrho}^\ell| := \int_{D_{z,\varrho}^\ell} dA = \varrho^2(1+|z|)^{2(1-\ell)}$ and $\omega(D_{z,\varrho}^\ell) := \int_{D_{z,\varrho}^\ell} d\omega$, where $d\omega := \omega\, dA$.

Recall that the Bergman projection on $F_\alpha^{2,\ell}$ is the orthogonal projection from $L_\alpha^{2,\ell}$ onto $F_\alpha^{2,\ell}$, which is the integral operator

$$P_\alpha^\ell\varphi(z) := \int_{\mathbb{C}} \varphi(w)\, K_\alpha^\ell(z,w)\, e^{-\alpha|w|^{2\ell}}\, dA(w).$$

Here K_α^ℓ is the Bergman kernel for $F_\alpha^{2,\ell}$. We also consider the operator $P_\alpha^{\ell,+}$, which is the "positive version" of P_α^ℓ and is defined by

$$P_\alpha^{\ell,+}f(z) := \int_{\mathbb{C}} f(w)|K_\alpha^\ell(z,w)|\, e^{-\alpha|w|^{2\ell}}\, dA(w).$$

Our first main result (which for $\ell = 1$ is proved in [9]) is the following.

Theorem 1.1. *Let $1 < p < \infty$. Then, for any weight ω, the following assertions are equivalent:*

(a) *$\omega \in \mathcal{A}_{p,\varrho}^\ell$, for some (any) $\varrho > 0$.*
(b) *$P_\alpha^{\ell,+}$ is bounded on $L_\alpha^{p,\ell}(\omega)$, for some (any) $\alpha > 0$.*
(c) *P_α^ℓ is bounded on $L_\alpha^{p,\ell}(\omega)$, for some (any) $\alpha > 0$.*

As a consequence of this theorem we obtain a complete answer of our initial question.

Corollary 1.2. *Let $\varphi(z) = \sum_{j=1}^{m} c_j |z|^{\ell_j}$, where $0 < \ell_1 < \ell_2 < \cdots < \ell_m$ and $c_1, \ldots, c_m \in \mathbb{R}$. Then, for any $\alpha > 0$, P^ℓ_α is a bounded operator from $L^p_\alpha(e^\varphi)$ onto $A^p_\alpha(e^\varphi)$ if and only if $\ell_m \leq \ell$.*

In order to prove Theorem 1.1, we need to develop the theory of $\mathcal{A}^\ell_{p,\varrho}$ weights, and for this reason the proof of that theorem will be postponed to the end of the paper.

Another important consequence of Theorem 1.1 is that the class $\mathcal{A}^\ell_{p,\varrho}$ does not depend on ϱ. Indeed, this fact will be deduced from a real version of the above theorem (see Theorem 1.4). This is why we maintain the "ϱ" in the notation $\mathcal{A}^\ell_{p,\varrho}$ until we state Theorem 1.4.

We remark that, for $\ell > 1$, the radius of the disk $D^\ell_{z,\varrho}$ which appears in the definition of the $\mathcal{A}^\ell_{p,\varrho}$ weights is a non-constant function of $|z|$, whereas for $\ell = 1$ this radius is constant. This difference makes the proof of Theorem 1.1 much more difficult when $\ell > 1$. For example, one key tool in this proof is the comparison of the mass of an $\mathcal{A}^\ell_{p,\varrho}$ weight on two disks $D^\ell_{z,\varrho}$:

Theorem 1.3. *Let $\ell \geq 1$, $\varrho > 0$ and $0 < \beta < \pi/(2\ell)$. Then for any $\omega \in \mathcal{A}^\ell_{p,\varrho}$ there exists a constant $C = C(\omega, \ell, \varrho, \beta) > 0$ such that*

$$\frac{\omega(D^\ell_{w,\varrho})}{\omega(D^\ell_{z,\varrho})} \leq C^{\phi^\ell_\beta(z,w)^{1/2}} \qquad (z, w \in \mathbb{C}).$$

Here, for $\ell = 1$, $\phi^\ell_\beta(z,w) = |z - w|^2$, whereas, for $\ell > 1$, $\phi^\ell_\beta(z,w) = |z^\ell - w^\ell|^2$, when z and w lie on a certain sector, and otherwise $\phi^\ell_\beta(z,w)$ is comparable to $|z|^{2\ell} + |w|^{2\ell}$ (see Section 2.2 for a precise definition).

Theorem 1.3 suggests to consider the following family of integral operators. For $\alpha > 0$, $0 < \beta < \pi/2\ell$ and $s \in \mathbb{R}$, let

$$T^\ell_{\alpha,\beta,s} f(z) := \int_{\mathbb{C}} T^\ell_{\alpha,\beta,s}(z,w) f(w) dA(w),$$

where

$$T^\ell_{\alpha,\beta,s}(z,w) := \frac{e^{-\frac{\alpha}{2}\phi^\ell_\beta(z,w)}}{|D^\ell_{z,\varrho}|} \left(\frac{|D^\ell_{w,\varrho}|}{|D^\ell_{z,\varrho}|} \right)^{s/(2\ell-2)}.$$

The importance of these operators is that their $L^p(\omega)$-boundedness characterize that $\omega \in \mathcal{A}^\ell_{p,\varrho}$.

Theorem 1.4. *Let ω be a weight and let $1 < p < \infty$. Then the following assertions are equivalent:*

(a) *$\omega \in \mathcal{A}^\ell_{p,\varrho}$, for some (any) $\varrho > 0$.*
(b) *$T^\ell_{\alpha,\beta,s}$ is bounded on $L^p(\omega)$, for some (any) $\alpha > 0$, $\beta \in (0, \pi/(2\ell))$ and $s \in \mathbb{R}$.*

In particular, the class $\mathcal{A}^\ell_{p,\varrho}$ does not depend on ϱ.

As we have said before, until the end of this introduction we will simply denote the class $\mathcal{A}^\ell_{p,\varrho}$ by $\mathcal{A}^\ell_p$.

The integral kernels $T^\ell_{\alpha,\beta,s}$ are related with the Bergman kernel by the pointwise estimate $|K^\ell_\alpha(z,w)| \lesssim T^\ell_{\alpha,\beta,s}(z,w)$. This estimate together with Theorem 1.4 and the fact that P^ℓ_α is bounded on L^p_α gives the following result on a Berezin type operator.

Theorem 1.5. *Let $\ell > 1$ and $\alpha > 0$. Let $\mathcal{B}^\ell_\alpha$ be the integral operator defined by*

$$\mathcal{B}^\ell_\alpha(\varphi)(z) = \int_{\mathbb{C}} \varphi(w)\, \mathcal{B}^\ell_\alpha(z,w)\, dA(w), \quad \text{where} \quad \mathcal{B}^\ell_\alpha(z,w) = \frac{|K^\ell_\alpha(z,w)|^2}{K^\ell_\alpha(z,z)} e^{-\alpha|w|^{2\ell}}.$$

Then, for any $1 < p < \infty$, *we have that:*

a) $\mathcal{B}^\ell_\alpha$ *is bounded on* $L^p(\omega)$ *if and only if* $\omega \in \mathcal{A}^\ell_p$.
b) $\mathcal{B}^\ell_\alpha f = f$, *whenever* $\omega \in \mathcal{A}^\ell_p$ *and* $f \in A^p(\omega) = H(\mathbb{C}) \cap L^p(\omega)$.

Finally, as an application, we obtain a characterization of the boundedness of products of certain Toeplitz operators on $F^{p,\ell}_\alpha$. This result was proved by similar methods for any p and $\ell = 1$ in [9] and by direct computation for $p = 2$ and any $\ell > 1$ in [5].

We consider the Toeplitz operator $T^\ell_\varphi h := P^\ell_\alpha(\varphi h)$, which is densely defined in $F^{p,\ell}_\alpha$ (see Section 7.2 for a precise definition). Then the boundedness of the composition operator $T^\ell_f T^\ell_{\overline{g}}$ on $F^{p,\ell}_\alpha$ can be completely characterized in terms of f and g as follows.

Theorem 1.6. *Let* $1 < p < \infty$ *and let* $f, g \in F^{1,\ell}_\gamma$, $0 < \gamma < 2\alpha$, *be two non-identically zero functions. Then the following assertions are equivalent:*

(a) $T^\ell_f T^\ell_{\overline{g}}$ *extends to a bounded operator on* $F^{p,\ell}_\alpha$.
(b) *The product* $f\,g$ *is a constant function and* $f = e^h$, *where* h *is a polynomial of degree* $m \le \ell$.
(c) *The product* $f\,g$ *is a constant function and* $|f|^p \in \mathcal{A}^\ell_p$.

This result clearly shows that Sarason's conjecture on the product of Toeplitz operators is trivially true for the Fock space $F^{p,\ell}_\alpha$, $\ell > 1$, as it happens in the $\ell = 1$ case (see [9]). The situation is completely different in the Hardy and Bergman settings, where it is known that this conjecture is false (see [12] and [1], respectively).

The paper is organized as follows. In Section 2, we prove some geometrical results related to the disks $D^\ell_{\zeta,\varrho}$, and we also state some basic properties of the $\mathcal{A}^\ell_p$ weights. Section 3 is devoted to the proof of Theorem 1.3. In the next section, we study the boundedness of the operators $T^\ell_{\alpha,\beta,s}$ and we prove Theorem 1.4. Section 5 describes several properties and examples of weights in $\mathcal{A}^\ell_p$. In this section it is also proved that, for $\omega \in \mathcal{A}^\ell_p$, the weighted Bergman space $A^p(\omega)$ coincides with $A^p(\Omega)$, where Ω is a more regular weight (slowly growing weight). The Berezin type results, including the proof of Theorem 1.5, are given in Section 6. Finally, in Section 7 we prove Theorems 1.1 and 1.6.

Notations:. We denote by $\mathbb{N}$ the set of all positive integers.

The notation $\Phi \lesssim \Psi$ ($\Psi \gtrsim \Phi$) means that there exists a constant $C > 0$, which does not depend on the involved variables, such that $\Phi \le C\,\Psi$. We write $\Phi \simeq \Psi$ if $\Phi \lesssim \Psi$ and $\Psi \lesssim \Phi$.

2. Preliminaries and $\mathcal{A}^\ell_{p,\varrho}$ weights

2.1. Properties of the disks $D^\ell_{z,\varrho}$

In this section we study some elementary properties of the disks

$$D^\ell_{z,\varrho} = \{w \in \mathbb{C} : |w - z| < \varrho(1 + |z|)^{1-\ell}\}.$$

Lemma 2.1.

$$\frac{1+|z|}{1+\varrho} \le 1 + |w| \le (1+\varrho)(1+|z|) \qquad (z \in \mathbb{C},\ w \in \overline{D}^\ell_{z,\varrho}). \tag{2.1}$$

Proof. If $w \in \overline{D}^\ell_{z,\varrho}$ then $|(1+|z|)-(1+|w|)| \le |w-z| \le \varrho \min(1+|z|, 1+|w|)$, and so $1+|z| \le (1+\varrho)(1+|w|)$ and $1 + |w| \le (1+\varrho)(1+|z|)$. □

Remark 2.2. The function $\eta^\ell_\varrho(z) := \varrho(1+|z|)^{1-\ell}$ is a radius function in the sense of [8, p. 1617-1618], that is, $\eta^\ell_\varrho(z) \simeq \eta^\ell_\varrho(w)$, for $z \in \mathbb{C}$ and $w \in D^\ell_{z,\varrho}$.

Lemma 2.3.

a) *Let $\varrho > 0$. For any $z \in \mathbb{C}$ and $w \in D^\ell_{z,\varrho}$, we have that $D^\ell_{w,\varrho} \subset D^\ell_{z,\tau_\ell(\varrho)}$ and $D^\ell_{z,\varrho} \subset D^\ell_{w,\tau_\ell(\varrho)}$, where $\tau_\ell(\varrho) = 2\varrho(1+\varrho)^{\ell-1}$.*

b) *Let $\varrho, \varrho', \varrho'' > 0$ such that $\tau_\ell(\varrho) \le \varrho'$ and $\tau_\ell(\varrho') \le \varrho''$. Then, for any $z \in \mathbb{C}$ and $w \in D^\ell_{z,\varrho}$ we have*

$$D^\ell_{z,\varrho} \subset D^\ell_{w,\varrho'} \subset D^\ell_{z,\varrho''} \qquad \textit{and} \qquad D^\ell_{w,\varrho} \subset D^\ell_{z,\varrho'} \subset D^\ell_{w,\varrho''}.$$

Proof. Part a) follows from (2.1). Indeed, for $\zeta \in D^\ell_{w,\varrho}$ we have that

$$\begin{aligned}|\zeta - z| &\le |\zeta - w| + |w - z| \le \varrho(1+|w|)^{1-\ell} + \varrho(1+|z|)^{1-\ell}\\ &\le \varrho\big((1+\varrho)^{\ell-1}+1\big)(1+|z|)^{1-\ell} \le \tau_\ell(\varrho)(1+|z|)^{1-\ell},\end{aligned}$$

and so $D^\ell_{w,\varrho} \subset D^\ell_{z,\tau_\ell(\varrho)}$. Similarly we show that $D^\ell_{z,\varrho} \subset D^\ell_{w,\tau_\ell(\varrho)}$.

Finally, since $\tau_\ell(\varrho) \ge \varrho$, for every $\varrho > 0$, part b) follows from part a). Indeed, $D^\ell_{z,\varrho} \subset D^\ell_{w,\tau_\ell(\varrho)} \subset D^\ell_{w,\varrho'} \subset D^\ell_{z,\tau_\ell(\varrho')} \subset D^\ell_{z,\varrho''}$ and the other chain of inclusions follow in a similar way. □

To finish the section we recall the following covering property (see, for instance, [8, Proposition 7] and [13, Lemma 6.6]).

Lemma 2.4. *For each $\varrho > 0$, there exists a sequence $\{z_k\}_k \in \mathbb{C}$ such that the sequence of disks $\{D^\ell_{z_k,\varrho}\}_k$ covers the whole plane $\mathbb{C}$, and, for any $\varrho' \ge \varrho$, the overlapping of the disks $\{D^\ell_{z_k,\varrho'}\}_k$ is finite, that is, there exists $N^\ell_{\varrho,\varrho'} \in \mathbb{N}$ so that $\sum_k \mathcal{X}_{D^\ell_{z_k,\varrho'}}(z) \le N^\ell_{\varrho,\varrho'}$, for every $z \in \mathbb{C}$.*

2.2. The functions ϕ^ℓ_β and $\psi^\ell_{\alpha,\beta}$

For $0 < \beta < \pi/(2\ell)$ and $z \ne 0$, let

$$S_{z,\beta} := \{w \in \mathbb{C} \setminus \{0\} : |\arg(z\overline{w})| \le \beta\} \cup \{0\} \qquad (z \in \mathbb{C}\setminus\{0\},\, 0 < \beta < \tfrac{\pi}{2}),$$

where $\arg\lambda$ denotes the principal branch of the argument of $\lambda \in \mathbb{C}\setminus\{0\}$, *i.e.* $-\pi < \arg\lambda \le \pi$. The following elementary property will be useful later on.

Lemma 2.5. *Let $\varrho > 0$ and let $0 < \beta < \pi/(2\ell)$. Then there is $R > 0$ such that $D^\ell_{z,\varrho} \subset S_{z,\beta}$ for $|z| > R$.*

Proof. It is clear that $R = \varrho/\sin\beta$ satisfies the statement. □

Let $\phi^\ell_\beta : \mathbb{C}\times\mathbb{C} \to [0,\infty)$ be the continuous function defined by

$$\phi^\ell_\beta(z,w) := \begin{cases} |z|^{2\ell} + |w|^{2\ell} - 2\,\mathrm{Re}((w\overline{z})^\ell), & \text{if } z \in \mathbb{C}\setminus\{0\} \text{ and } w \in S_{z,\beta},\\ |z|^{2\ell} + |w|^{2\ell} - 2|z|^\ell|w|^\ell\cos(\ell\beta), & \text{otherwise.}\end{cases}$$

Moreover, for $\alpha > 0$, let $\psi^\ell_{\alpha,\beta} := e^{-\frac{\alpha}{2}\phi^\ell_\beta}$. Observe that $\phi^\ell_\beta(z,w) = |z^\ell - w^\ell|^2$, for $z, w \in S_{z,\beta}$, where λ^ℓ denotes any branch of the ℓ-th power of λ on $S_{z,\beta}$. Hence $(\phi^\ell_\beta)^{1/2}$ defines a distance on any fixed sector $S_{z,\beta}$. The definition of the functions ϕ^ℓ_β directly shows the following symmetry properties.

Lemma 2.6. *For $0 < \beta < \pi/(2\ell)$ we have:*

a) $\phi^\ell_\beta(z,w) = \phi^\ell_\beta(ze^{i\theta}, we^{i\theta})$, *for any* $z, w \in \mathbb{C}$ *and* $\theta \in \mathbb{R}$.
b) $\phi^\ell_\beta(z,w) = \phi^\ell_\beta(w,z)$, *for any* $z, w \in \mathbb{C}$.

Next lemma states some useful simple estimates of the functions ϕ_β.

Lemma 2.7. *Let $z, w, u \in \mathbb{C}$, with $u \neq 0$, and let $0 < \beta < \pi/(2\ell)$.*

a) *If* $w \in \mathbb{C} \setminus S_{z,\beta}$, *with* $z \neq 0$, *then* $\phi^\ell_\beta(z,w) \simeq |z|^{2\ell} + |w|^{2\ell}$.
b) *If* $z, w \in S_{u,\beta}$ *then*

$$\phi^\ell_\beta(z,w)^{1/2} = |z^\ell - w^\ell| = \ell|z-w| \left| \int_0^1 \eta_t^{\ell-1} dt \right|,$$

where $\eta_t = (1-t)z + tw$, $0 \le t \le 1$. Here we consider the powers λ^ℓ and $\lambda^{\ell-1}$, for any $\lambda \in S_{u,\beta}$, defined as follows: $0^\ell = 0^{\ell-1} = 0$, $\lambda^\ell = e^{\ell \log \lambda}$ and $\lambda^{\ell-1} = e^{(\ell-1)\log\lambda}$, for any $\lambda \in S_{u,\beta} \setminus \{0\}$, where log is any branch of the logarithm on $S_{u,\beta} \setminus \{0\}$.
c) *If* $z, w \in S^\ell_{u,\beta}$ *and* $w \in D^\ell_{z,\varrho}$, *then* $\phi^\ell_\beta(z,w)^{1/2} = |z^\ell - w^\ell| \le \ell\varrho(1+\varrho)^{\ell-1}$.

Proof. Part a) is obvious, while part b) directly follows from Lemma 2.6 b). Finally, part c) is easily deduced, since, for $z, w \in S_{u,\beta}$ with $w \in D^\ell_{z,\varrho}$, we have

$$\begin{aligned}\phi^\ell_\beta(z,w)^{1/2} &= |w^\ell - z^\ell| \le \ell|w-z| \int_0^1 (|z| + t|w-z|)^{\ell-1} dt \\ &\le \ell\varrho(1+|z|)^{1-\ell}(|z| + \varrho(1+|z|)^{1-\ell})^{\ell-1} \le \ell\varrho(1+\varrho)^{\ell-1}. \quad \square\end{aligned}$$

Corollary 2.8. *If either $w \in D^\ell_{z,\varrho}$ and $z = 0$, or $w \in D^\ell_{z,\varrho} \cap S_{z,\beta}$ and $z \neq 0$, then $\phi^\ell_\beta(z,w) \lesssim 1$ and $\psi^\ell_{\alpha,\beta}(z,w) \simeq 1$.*

The following lemma states an important property of the functions $\psi_{\alpha,\beta}$, that roughly speaking says that, up to a change of the parameter α, the value $\psi_{\alpha,\beta}(z,w)$, for z and w in the disks $D_{u,\varrho}$ and $D_{v,\varrho}$, respectively, is comparable to the values at the centers.

Lemma 2.9. *Let $\varrho > 0$. Then for any $\alpha > 0$ there exist $\alpha', \alpha'' > 0$ such that*

$$\psi^\ell_{\alpha',\beta}(u,v) \lesssim \psi^\ell_{\alpha,\beta}(z,w) \lesssim \psi^\ell_{\alpha'',\beta}(u,v) \qquad (u, v \in \mathbb{C}, z \in D^\ell_{u,\varrho}, w \in D^\ell_{v,\varrho}). \tag{2.2}$$

Similarly, for any $\alpha > 0$ there exist $\alpha', \alpha'' > 0$ such that

$$\psi^\ell_{\alpha',\beta}(z,w) \lesssim \psi^\ell_{\alpha,\beta}(u,v) \lesssim \psi^\ell_{\alpha'',\beta}(z,w) \quad (u, v \in \mathbb{C}, z \in D^\ell_{u,\varrho}, w \in D^\ell_{v,\varrho}). \tag{2.3}$$

Proof. First, we prove (2.2). Since $\psi^\ell_{\alpha,\beta}(\zeta,\xi) = \psi^\ell_{\alpha,\beta}(\xi,\zeta)$, for any $\zeta, \xi \in \mathbb{C}$, it is enough to show the result for $z = u$, that is,

$$\psi^\ell_{\alpha',\beta}(u,v) \lesssim \psi_{\alpha,\beta^\ell}(u,w) \lesssim \psi^\ell_{\alpha'',\beta}(u,v) \tag{2.4}$$

Indeed, $\psi^\ell_{\alpha,\beta}(z,w) \lesssim \psi^\ell_{\alpha''_1,\beta}(z,v) = \psi^\ell_{\alpha''_1,\beta}(v,z) \lesssim \psi^\ell_{\alpha''_2,\beta}(v,u) = \psi^\ell_{\alpha''_2,\beta}(u,v)$, and similarly we obtain that $\psi^\ell_{\alpha',\beta}(u,v) \lesssim \psi^\ell_{\alpha,\beta}(z,w)$.

Since the function $\psi^\ell_{\alpha,\beta}$ is continuous, without loss of generality we may assume that $u, v, w \in \mathbb{C} \setminus \{0\}$. We consider four cases:

Case 1: $v, w \in S_{u,\beta}$.
Taking into account the notations in Lemma 2.7 b), we assume first that $|u^\ell - v^\ell| \ge 2\ell\varrho(1+\varrho)^{\ell-1}$. Then, by Lemma 2.7 c),

$$||u^\ell - w^\ell| - |u^\ell - v^\ell|| \le |w^\ell - v^\ell| < \ell\rho(1+\rho)^{\ell-1} < \tfrac{1}{2}|u^\ell - v^\ell|,$$

and so $\frac{1}{2}|u^\ell - v^\ell| \le |u^\ell - w^\ell| \le \frac{3}{2}|u^\ell - v^\ell|$, *i.e.* $\frac{1}{4}\phi^\ell_\beta(u,v) \le \phi^\ell_\beta(u,w) \le \frac{9}{4}\phi^\ell_\beta(u,v)$, which proves (2.4) in this situation.
Next assume that $|u^\ell - v^\ell| < 2\ell\varrho(1+\varrho)^{\ell-1}$. Again by Lemma 2.7 c) we get $|u^\ell - w^\ell| < 3\ell\varrho(1+\varrho)^{\ell-1}$. Therefore $\phi^\ell_\beta(u,v) \lesssim 1$ and $\phi^\ell_\beta(u,w) \lesssim 1$, and hence $\psi^\ell_{\alpha,\beta}(u,v) \simeq 1 \simeq \psi^\ell_{\alpha,\beta}(u,w)$.
The remaining cases will follow from Case 1.

Case 2: $v \in S_{u,\beta}$, $w \notin S_{u,\beta}$.
Assume $\arg(v\overline{u}) \ge 0$, and let $\widetilde{w} = |w|e^{i(\arg(u)+\beta)}$, which satisfies $\widetilde{w} \in S_{u,\beta}$ and $\psi^\ell_{\alpha,\beta}(u,\widetilde{w}) = \psi^\ell_{\alpha,\beta}(u,w)$. Since $|\widetilde{w} - v| < |w - v|$, we also have that $\widetilde{w} \in D^\ell_{v,\varrho}$. So (2.4) follows from Case 1, because $\psi^\ell_{\alpha',\beta}(u,v) \lesssim \psi^\ell_{\alpha,\beta}(u,\widetilde{w}) \lesssim \psi^\ell_{\alpha'',\beta}(u,v)$.
If $\arg(v\overline{u}) < 0$, a similar argument considering now $\widehat{w} = |w|e^{i(\arg(u)-\beta)}$ instead of $\widetilde{w}$, gives (2.4) in this case.

Case 3: $v \notin S_{u,\beta}$ and $w \in S_{u,\beta}$.
This case is similar to Case 2. Indeed, if $\arg(w\overline{u}) \ge 0$ then $w \in D^\ell_{\widetilde{v},\varrho}$, where $\widetilde{v} = |v|e^{i(\arg(u)+\beta)}$ satisfies $\widetilde{v} \in S_{u,\beta}$ and $\psi^\ell_{\alpha,\beta}(u,\widetilde{v}) = \psi^\ell_{\alpha,\beta}(u,v)$, so Case 1 gives (2.4). When $\arg(w\overline{u}) \le 0$ we consider $\widehat{v} = |v|e^{i(\arg(u)-\beta)}$ instead of $\widetilde{v}$.

Case 4: $v, w \notin S_{u,\beta}$.
In this case, using the above notations, $|\widetilde{w} - \widetilde{v}| = ||w| - |v|| \le |w - v|$, so $\widetilde{w} \in D^\ell_{\widetilde{v},\varrho}$. Hence we apply Case 1 to $\widetilde{v}$ and $\widetilde{w}$.

Finally, (2.3) is an immediate consequence of (2.2) because if $z \in D^\ell_{u,\varrho}$ and $w \in D^\ell_{v,\varrho}$ then $u \in D^\ell_{z,\tau(\varrho)}$ and $v \in D^\ell_{w,\tau(\varrho)}$ (see Lemma 2.3 a)). □

2.3. $\mathcal{A}^\ell_{p,\varrho}$ weights and doubling weights $\mathcal{D}^\ell_\varrho$

The following characterization of $\mathcal{A}^\ell_{p,\varrho}$ resembles a well known characterization of the classical Muckenhoupt weights.

Lemma 2.10. *For a weight ω on $\mathbb{C}$, the following assertions are equivalent:*

(a) $w \in \mathcal{A}^\ell_{p,\varrho}$.
(b) *There is a constant $C > 0$ such that*

$$\left(\frac{1}{|D^\ell_{z,\varrho}|}\int_{D^\ell_{z,\varrho}} f\,dA\right)^p \le \frac{C}{\omega(D^\ell_{z,\varrho})}\int_{D^\ell_{z,\varrho}} f^p\,d\omega, \tag{2.5}$$

for any non-negative measurable function f on $\mathbb{C}$ and for any $z \in \mathbb{C}$.

The best constant C in the above inequality coincides with $[\omega]_{\mathcal{A}^{\ell}_{p,\varrho}}$.

Proof. The proof is similar to the one for the classical Muckenhoupt weights (see, for instance, [18, Ch. 5]). The fact that (a) implies (b) with $C = [\omega]_{\mathcal{A}^{\ell}_{p,\varrho}}$ is a direct consequence of Hölder's inequality, while the opposite implication follows by applying (2.5) to $f = \omega' = \omega^{-p'/p}$. $\square$

As the classical Muckenhoupt weights, our $\mathcal{A}^{\ell}_{p,\varrho'}$ weights are doubling in a sense that we are going to state precisely.

Definition 2.11. The doubling class $\mathcal{D}^{\ell}_{\varrho}$ consists of all the weights ω satisfying

$$[\omega]_{\mathcal{D}^{\ell}_{\varrho}} := \sup_{z\in\mathbb{C}} \frac{\omega(D^{\ell}_{z,\tau(\varrho)})}{\omega(D^{\ell}_{z,\varrho})} < \infty, \quad \text{where} \quad \tau(\varrho) = 2\varrho(1+\varrho)^{\ell-1}.$$

Then Lemma 2.3 directly gives the following result.

Lemma 2.12. *For any $\omega \in \mathcal{D}^{\ell}_{\varrho}$ we have that*

$$\frac{1}{[\omega]_{\mathcal{D}^{\ell}_{\varrho}}} \le \frac{\omega(D^{\ell}_{w,\varrho})}{\omega(D^{\ell}_{z,\varrho})} \le [\omega]_{\mathcal{D}^{\ell}_{\varrho}} \qquad (z \in \mathbb{C},\ w \in D^{\ell}_{z,\varrho}). \tag{2.6}$$

Proposition 2.13.

a) *If $0 < \varrho < \varrho'$ and $\omega \in \mathcal{A}^{\ell}_{p,\varrho'}$, then $\omega \in \mathcal{A}^{\ell}_{p,\varrho}$, $[\omega]_{\mathcal{A}^{\ell}_{p,\varrho}} \le (\varrho'/\varrho)^{2p}\,[\omega]_{\mathcal{A}^{\ell}_{p,\varrho'}}$, and*

$$\omega(D^{\ell}_{z,\varrho'}) \le (\varrho'/\varrho)^{2p}\,[\omega]_{\mathcal{A}^{\ell}_{p,\varrho'}}\,\omega(D^{\ell}_{z,\varrho}) \qquad (z \in \mathbb{C}). \tag{2.7}$$

b) *If $\varrho, \varrho' > 0$ satisfy $\varrho' \ge \tau(\varrho)$, then $\mathcal{A}^{\ell}_{p,\varrho'} \subset \mathcal{A}^{\ell}_{p,\varrho} \cap \mathcal{D}^{\ell}_{\varrho}$.*

Proof. The first two assertions of a) easily follow from the definition of $\mathcal{A}^{\ell}_{p,\varrho}$. We obtain (2.7) by applying (2.5) (replacing ϱ by ϱ', and taking $C = [\omega]_{\mathcal{A}^{\ell}_{p,\varrho'}}$) to the characteristic function f of $D^{\ell}_{z,\varrho}$. Finally, b) is a direct consequence of a). $\square$

Remark 2.14. We point out that Theorem 1.4 will show that the class $\mathcal{A}^{\ell}_{p,\varrho}$ does not depend on ϱ. A direct and important consequence of that fact and assertion b) of the above proposition is that $\mathcal{A}^{\ell}_{p,\varrho} \subset \mathcal{D}^{\ell}_{\varrho'}$, for any $\varrho, \varrho' > 0$, which means that, as the classical Muckenhoupt weights, our $\mathcal{A}^{\ell}_{p,\varrho}$ weights are really doubling weights.

3. Proof of Theorem 1.3

In order to compare the ω-measure of two arbitrary disks $D^{\ell}_{z,\varrho}$ and $D^{\ell}_{w,\varrho}$, we consider suitable finite chains of overlapping disks linking $D^{\ell}_{z,\varrho}$ with $D^{\ell}_{w,\varrho}$.

Definition 3.1. Given $z, w \in \mathbb{C}$ and $\varrho > 0$, $U^{\ell}_{\varrho}(z,w)$ will denote the set of all finite sequences of points $\{\zeta_k\}_{k=0}^{N}$ such that $N \in \mathbb{N}$, $\zeta_0 = z$, $\zeta_N = w$, $\zeta_k \in \overline{D}^{\ell}_{\zeta_{k-1},\varrho}$ and $\zeta_{k-1} \in \overline{D}^{\ell}_{\zeta_k,\varrho}$, for $k = 1,\dots,N$. Moreover,

$$N^{\ell}_{\varrho}(z,w) := \min\{N \in \mathbb{N} : \{\zeta_k\}_{k=0}^{N} \in U^{\ell}_{\varrho}(z,w)\}.$$

Now we state some elementary properties of $U^{\ell}_{\varrho}(z,w)$ and $N_{\varrho}(z,w)$:

Lemma 3.2. *For $z, w, u \in \mathbb{C}$ and $\theta \in \mathbb{R}$, we have:*

a) *If $\{\zeta_k\}_{k=0}^N \in U_\varrho^\ell(z,w)$ then $\{\zeta_{N-k}\}_{k=0}^N \in U_\varrho^\ell(w,z)$, so $N_\varrho^\ell(z,w) = N_\varrho^\ell(w,z)$.*
b) *$N_\varrho^\ell(z,w) \le N_\varrho^\ell(z,u) + N_\varrho^\ell(u,w)$.*
c) *$N_\varrho^\ell(z,w) = N_\varrho^\ell(e^{i\theta}z, e^{i\theta}w)$.*

Observe that, if $\omega \in \mathcal{D}_\varrho^\ell$ and $\{\zeta_k\}_{k=0}^N \in U_\varrho^\ell(z,w)$, then (2.6) implies that

$$\omega(D_{\zeta_{k-1},\varrho}^\ell) \le [\omega]_{\mathcal{D}_\varrho^\ell}\, \omega(D_{\zeta_k,\varrho}^\ell) \le [\omega]_{\mathcal{D}_\varrho^\ell}^2\, \omega(D_{\zeta_{k-1},\varrho}^\ell).$$

Thus we have the following lemma, which compares the ω-measure of any two arbitrary disks $D_{z,\varrho}^\ell$.

Lemma 3.3. *If $\omega \in \mathcal{D}_\varrho^\ell$, then*

$$\omega(D_{w,\varrho}^\ell) \le [\omega]_{\mathcal{D}_\varrho^\ell}^{N_\varrho^\ell(z,w)}\, \omega(D_{z,\varrho}^\ell) \qquad (z, w \in \mathbb{C}).$$

It directly follows from Lemma 3.3 that, in order to prove Theorem 1.3, it is enough to show the following proposition.

Proposition 3.4. *Let $\ell \ge 1$, $\varrho > 0$ and $\beta < \pi/(2\ell)$. Then*

$$N_\varrho^\ell(z,w) \lesssim \phi_\beta^\ell(z,w)^{1/2} + 1 \qquad (z, w \in \mathbb{C}).$$

The rest of this section is devoted to prove Proposition 3.4.

Observe that $N_\varrho^\ell(z,w) \ge 1$ and $N_\varrho^\ell(z,w) = 1$ if and only if $w \in \overline{D}_{z,\varrho}^\ell$ and $z \in \overline{D}_{w,\varrho}^\ell$, that is,

$$|z - w| \le \varrho \min\{(1+|z|)^{1-\ell}, (1+|w|)^{1-\ell}\}. \tag{3.8}$$

The following lemma considers the case where the points z and w belong to a neighborhood of the origin only depending on ϱ:

Lemma 3.5. *For every $\varrho > 0$ let $r_\ell(\varrho)$ be the only positive solution to the equation $2r(1+r)^{\ell-1} = \varrho$. Then $N_\varrho^\ell(z,w) = 1$, for any $z, w \in \overline{D}_{0,r_\ell(\varrho)}^\ell$.*

In order to estimate $N_\varrho^\ell(z,w)$ when z and w do not satisfy (3.8), we will consider the cases where both z and w lay either on a circle centered at the origin or on a line passing through the origin.

Lemma 3.6. *Let $\gamma : [0,1] \to \mathbb{C}$ be a C^1 curve such that $\gamma(0) = z$ and $\gamma(1) = w$. Let $M = \sup\{|\gamma(t)| : t \in [0,1]\}$. If z and w do not satisfy (3.8), then*

$$N_\varrho^\ell(z,w) \le \frac{2(1+M)^{\ell-1}}{\varrho} \int_0^1 |\gamma'(t)|dt.$$

Proof. Since $(1+M)^{1-\ell} \le (1+|\gamma(t)|)^{1-\ell}$ and z, w do not satisfy (3.8), there exist $t_0 = 0 < t_1 < \cdots < t_N = 1$ such that

$$\frac{\varrho}{2}(1+M)^{1-\ell} \le |\gamma(t_{k-1}) - \gamma(t_k)| < \varrho(1+M)^{1-\ell}, \qquad k = 1, \cdots, N.$$

Then it is clear that $\{\zeta_k = \gamma(t_k)\}_{k=0}^N \in U_\varrho^\ell(z,w)$, and so $N_\varrho^\ell(z,w) \le N$. But

$$N \le \frac{2}{\varrho(1+M)^{1-\ell}} \sum_{k=1}^N |\zeta_{k-1} - \zeta_k| \le \frac{2(1+M)^{\ell-1}}{\varrho} \int_0^1 |\gamma'(t)|dt. \quad \square$$

Corollary 3.7. *For any $z, w \in \mathbb{C}$ such that $|z| = |w|$, we have*

$$N_\varrho^\ell(z,w) \le \frac{\pi}{\varrho}(1+|z|)^{\ell-1}|z-w| + 1.$$

Proof. If z and w satisfy (3.8), then $N_\varrho^\ell(z,w) = 1$. Otherwise, applying Lemma 3.6 to circle arcs $\gamma(t) = |z|e^{it}$ of length less than $\pi|z|$, we obtain

$$N_\varrho^\ell(z,w) \le \frac{\pi}{\varrho}(1+|z|)^{\ell-1}|z-w|. \quad \square$$

Now we consider the radial case $w = sz$, $s \in (0,1)$.

Lemma 3.8. *Assume that $z \in \mathbb{C}$ and $w = sz$, $s \in (0,1)$. Then*

$$N_\varrho^\ell(z,w) \le \frac{(1+2\varrho)^{\ell-1}}{\ell\, r_\ell(\varrho)^\ell}(|z|^\ell - |w|^\ell) + 2,$$

where $r(\varrho) > 0$ is as in Lemma 3.5.

Proof. Without loss of generality we may assume $N_\varrho^\ell(z,w) > 1$. Then we consider three cases.

Case 1: $r_\ell(\varrho) \le |w|$ and $|z| - |w| \le 2\varrho$.

By Lemma 3.6 applied to the line segment determined by z and w, we obtain

$$N_\varrho^\ell(z,w) \le \frac{2}{\varrho}(1+|z|)^{\ell-1}(|z|-|w|).$$

Since $1 + |z| \le (1+2\varrho)(1+|w|)$, by the mean value theorem we have that

$$N_\varrho^\ell(z,w) \le \frac{2(1+2\varrho)^{\ell-1}}{\ell\,\varrho}\left(\frac{1+|w|}{|w|}\right)^{\ell-1}(|z|^\ell - |w|^\ell) \le \frac{(1+2\varrho)^{\ell-1}}{\ell\, r_\ell(\varrho)^\ell}(|z|^\ell - |w|^\ell).$$

Case 2: $r(\varrho) \le |w|$ and $|z| - |w| > 2\varrho$.

Then there exist $t_0 = s < t_1 < \cdots < t_N = 1$ such that $\varrho < (t_k - t_{k-1})|z| \le 2\varrho$. Let $z_k = t_k z$, for $k = 0, \dots, N$. Thus we may apply case 1 to the points z_{k-1} and z_k, and obtain that

$$N_\varrho^\ell(z_{k-1}, z_k) \le \frac{(1+2\varrho)^{\ell-1}}{\ell\, r_\ell(\varrho)^\ell}(|z_k|^\ell - |z_{k-1}|^\ell) \qquad (k = 1, \dots, N).$$

Then, by Lemma 3.2,

$$N_\varrho^\ell(z,w) \le \sum_{k=1}^N N_\varrho(z_{k-1}, z_k) \le \frac{(1+2\varrho)^{\ell-1}}{\ell\, r_\ell(\varrho)^\ell}(|z|^\ell - |w|^\ell).$$

Case 3: $|w| < r_\ell(\varrho)$.

Then $\widetilde{w} = r_\ell(\varrho)\,(z/|z|) \in \overline{D}^\ell_{0,r_\ell(\varrho)}$, so $N^\ell_\varrho(z,w) \le N^\ell_\varrho(z,\widetilde{w}) + 1$, by Lemmas 3.2 and 3.5. In particular, $N^\ell_\varrho(z,w) \le 2$, if $N^\ell_\varrho(z,\widetilde{w}) = 1$. When $N^\ell_\varrho(z,\widetilde{w}) > 1$, the preceding cases give that

$$N^\ell_\varrho(z,w) \le \frac{(1+2\varrho)^{\ell-1}}{\ell\, r_\ell(\varrho)^\ell}(|z|^\ell - |w|^\ell) + 1. \quad \square$$

Proof of Proposition 3.4. By Lemmas 2.6 and 3.2, without loss of generality we may assume $z = |z| > 0$ and $|w| \le |z|$. Then Lemmas 3.2 and 3.8, and Corollary 3.7 show that

$$N^\ell_\varrho(z,w) \le N^\ell_\varrho(z,|w|) + N^\ell_\varrho(|w|,w) \lesssim |z|^\ell - |w|^\ell + (1+|w|)^{\ell-1}\big|w - |w|\big| + 1.$$

In order to conclude the proof we need to show that the right hand side term in the above inequality is bounded by

$$\phi^\ell_\beta(z,w)^{1/2} + 1 \simeq \begin{cases} \big||z|^\ell - w^\ell\big| + 1, & \text{if } w \in S_{|z|,\beta}, \\ |z|^\ell + |w|^\ell + 1, & \text{otherwise.} \end{cases}$$

Assume first that $w \in S_{|z|,\beta}$. It is clear that $|z|^\ell - |w|^\ell \le \big||z|^\ell - w^\ell\big|$. Moreover, then $(1+|w|)^{\ell-1}\big|w-|w|\big| \le 2^\ell$, if $|w| \le 1$. On the other hand, when $|w| > 1$ we have that

$$(1+|w|)^{\ell-1}\big|w - |w|\big| \le 2^{\ell-1}|w|^{\ell-1}\big|w - |w|\big| \le 2^{\ell-1}|w|^{\ell-1}\big|w - |z|\big|,$$

because $\operatorname{Re} w \le |w| \le \frac{1}{2}(|w| + |z|)$. But, since

$$(\cos(\ell\beta))^{1/2}\,|w|^\ell \le \min_{0\le t\le 1}\big|(1-t)w^\ell + t|z|^\ell\big|,$$

the mean value theorem shows in this case that

$$(1+|w|)^{\ell-1}\big|w - |w|\big| \lesssim |w|^{\ell-1}\big|w - |z|\big| \lesssim \big|w^\ell - |z|^\ell\big|.$$

Finally, when $w \notin S_{|z|,\beta}$ we have the estimate

$$|z|^\ell - |w|^\ell + (1+|w|)^{\ell-1}\big|w - |w|\big| \lesssim |w|^\ell + |z|^\ell + 1,$$

which ends the proof. $\square$

4. Integral operators on $L^p(\omega)$. Proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following three technical results. The first two give some useful estimates of integrals and sums related to the $\psi^\ell_{\alpha,\beta}$ functions and the $\mathcal{A}^\ell_{p,\varrho}$ weights.

Lemma 4.1. *Let $\alpha > 0$, $\beta \in (0, \pi/(2\ell))$ and $s \in \mathbb{R}$. Then*

$$\int_{\mathbb{C}} (1+|w|)^{s-2(1-\ell)}\,\psi^\ell_{\alpha,\beta}(z,w)\,dA(w) \simeq (1+|z|)^s \qquad (z \in \mathbb{C}). \tag{4.9}$$

Moreover, if $\{D^\ell_{z_k,\varrho}\}_k$ is a covering as in Lemma 2.4, then

$$\sum_j (1+|z_j|)^s\,\psi^\ell_{\alpha,\beta}(z_k,z_j) \simeq (1+|z_k|)^s. \tag{4.10}$$

Proof. First we prove (4.9). Taking into account Lemma 2.6 b), by making a rotation, we may assume that $z = |z| > 0$. Let I_z be the integral in (4.9). First note that, by Corollary 2.8, $\psi^\ell_{\alpha,\beta}(z,w) \simeq 1$, for $w \in D_{z,\varrho}$, and so

$$I_z \gtrsim \int\limits_{D^\ell_{z,\varrho}} (1+|w|)^{s-2(1-\ell)}\, dA(w) \simeq (1+|z|)^s.$$

Now we prove the opposite estimate. Since $I_z \lesssim 1$, for $|z| \le 1$, we may assume that $|z| > 1$. Now we split I_z into two integrals:

$$I_z = \Big\{ \int\limits_{\Omega} + \int\limits_{\mathbb{C}\setminus\Omega} \Big\} (1+|w|)^{s-2(1-\ell)}\, \psi^\ell_{\alpha,\beta}(z,w)\, dA(w) =: I'_z + I''_z,$$

where $\Omega := \{w \in S_{|z|,\beta} : |w^\ell - |z|^\ell| < \frac{1}{2}|z|^\ell\}$. We claim that $I'_z \lesssim (1+|z|)^s$ and there is some $\varepsilon > 0$ such that $I''_z \lesssim e^{-\varepsilon|z|^{2\ell}} \lesssim (1+|z|)^s$.

Note that $|w| \simeq |z|$, for $w \in \Omega$, so $I'_z \simeq (1+|z|)^s J_z$, where

$$J_z = \int\limits_{\Omega} (1+|w|)^{-2(1-\ell)} e^{-\frac{\alpha}{2}|w^\ell - |z|^\ell|^2} dA(w).$$

Now the change of variables $u = w^\ell - |z|^\ell$ shows that $J_z \lesssim 1$, which means that $I'_z \lesssim (1+|z|)^s$.

In order to prove the second claim, it is enough to check that there is some $\varepsilon > 0$ such that $\psi^\ell_{\alpha,\beta}(z,w) = e^{-\frac{\alpha}{2}\phi^\ell_\beta(z,w)} \lesssim e^{-\varepsilon(|w|^{2\ell}+|z|^{2\ell})}$, for $w \in \mathbb{C}\setminus\Omega$. And this can be deduced as follows:

If $w \in S_{z,\beta}$ and $\frac{1}{2}|z|^\ell \le |w^\ell - |z|^\ell| \le 2|z|^\ell$, then $|w|^\ell \le 3|z|^\ell \le 6|w^\ell - |z|^\ell|$, and so $\frac{1}{72}(|w|^{2\ell} + |z|^{2\ell}) \le |w^\ell - |z|^\ell|^2 = \phi^\ell_\beta(z,w)$. If $w \in S_{z,\beta}$ and $|w^\ell - |z|^\ell| > 2|z|^\ell$, then $|w|^\ell \le |w^\ell - z^\ell| + |z|^\ell \le \frac{3}{2}|w^\ell - z^\ell|$, and so $\frac{2}{9}(|w|^{2\ell} + |z|^{2\ell}) \le \phi^\ell_\beta(z,w)$. For $w \in \mathbb{C}\setminus S_{z,\beta}$, the estimate is a direct consequence of Lemma 2.7 a).

Finally, we prove (4.10). Since $\psi^\ell_{\alpha,\beta}(z_k,z_k) = 1$, the estimate $\gtrsim$ is clear. The opposite estimate follows from (4.9). Indeed, if $\{D^\ell_{z_k,\varrho}\}_k$ is a covering as in Lemma 2.4, then Lemma 2.9 implies that there is $\alpha' > 0$ such that

$$\begin{aligned}\sum_j (1+|z_j|)^s\, \psi^\ell_{\alpha,\beta}(z_k,z_j) &\lesssim \sum_j \int\limits_{D^\ell_{z_j,\varrho}} (1+|w|)^{s-2(1-\ell)}\, \psi^\ell_{\alpha',\beta}(z_k,w)\, dA(w)\\ &\lesssim \int\limits_{\mathbb{C}} (1+|w|)^{s-2(1-\ell)}\, \psi^\ell_{\alpha',\beta}(z_k,w)\, dA(w). \quad \square\end{aligned}$$

Proposition 4.2. *Let $s \in \mathbb{R}$. Then any weight ω satisfies*

$$\int\limits_{\mathbb{C}} (1+|w|)^s\, \psi^\ell_{\alpha,\beta}(z,w)\, d\omega(w) \gtrsim (1+|z|)^s\, \omega(D^\ell_{z,\varrho}). \tag{4.11}$$

Moreover, when $\omega \in \mathcal{A}^\ell_{p,\varrho}$, we have that

$$\int\limits_{\mathbb{C}} (1+|w|)^s\, \psi^\ell_{\alpha,\beta}(z,w)\, d\omega(w) \simeq (1+|z|)^s\, \omega(D^\ell_{z,\varrho}), \tag{4.12}$$

and, in particular, for every $1 \le q < \infty$, the following estimate holds

$$\|T^\ell_{\alpha,\beta,s}(z,\cdot)\|^q_{L^q(\omega)} \simeq (1+|z|)^{2(\ell-1)q}\,\omega(D^\ell_{z,\varrho}) \simeq \frac{\omega(D^\ell_{z,\varrho})}{|D^\ell_{z,\varrho}|^q}. \tag{4.13}$$

In order to prove Proposition 4.2 we need next lemma which is a direct consequence of Theorem 1.3.

Lemma 4.3. *Let $\omega \in \mathcal{A}^\ell_{p,\varrho}$, $\alpha > 0$, and $\delta \in \mathbb{R}$. Then there exists $\alpha' > 0$ so that*

$$\psi^\ell_{\alpha,\beta}(z,w)\left(\frac{\omega(D^\ell_{w,\varrho})}{\omega(D^\ell_{z,\varrho})}\right)^\delta \lesssim \psi^\ell_{\alpha',\beta}(z,w) \qquad (z,w\in\mathbb{C}).$$

Proof of Proposition 4.2. First note that $(1+|w|)^s\psi^\ell_{\alpha,\beta}z,w) \simeq (1+|z|)^s$, for $w \in D^\ell_{z,\varrho}$, and so (4.11) clearly holds. Next we prove (4.12). Assume that $\omega \in \mathcal{A}^\ell_{p,\varrho}$, and let $\{D^\ell_{z_k,\varrho}\}_k$ be a covering as in Lemma 2.4. Then Lemmas 2.9 and 4.3, and (4.10) give that there are $\alpha', \alpha'' > 0$ such that

$$\begin{aligned}\int_{\mathbb{C}} (1+|w|)^s\psi^\ell_{\alpha,\beta}(z,w)\omega(w)dA(w) &\lesssim \omega(D^\ell_{z,\varrho})\sum_k (1+|z_k|)^s\psi^\ell_{\alpha',\beta}(z,z_k)\frac{\omega(D^\ell_{z_k,\varrho})}{\omega(D^\ell_{z,\varrho})}\\ &\lesssim \omega(D^\ell_{z,\varrho})\sum_k (1+|z_k|)^s\psi^\ell_{\alpha'',\beta}(z,z_k)\\ &\lesssim (1+|z|)^s\omega(D^\ell_{z,\varrho}).\end{aligned}$$

Finally, (4.13) directly follows from (4.12) and the fact that $(\psi^\ell_{\alpha,\beta})^q = \psi^\ell_{q\alpha,\beta}$. □

Our third technical result states that if $T^\ell_{\alpha,\beta,s}$ is well defined on $L^p(\omega)$ (in the sense that $\psi^\ell_z(w) = f(w)\,T^\ell_{\alpha,\beta,s}(z,w) \in L^1(dA)$ a.e. $z \in \mathbb{C}$, for any $f \in L^p(\omega)$), then ω' is also a weight. Its proof follows the ideas in [2, Lemma 4].

Lemma 4.4. *If $T^\ell_{\alpha,\beta,s}$ is well defined on $L^p(\omega)$, then $\omega' = \omega^{-p'/p}$ is a weight.*

Proof. The proof is by contradiction. Assume that $T^\ell_{\alpha,\beta,s}$ is well defined on $L^p(\omega)$, but $\omega' \notin L^1(D^\ell_{0,R})$, for some $R > 0$. Then $\omega^{-1/p} \notin L^{p'}(D^\ell_{0,R})$, and so there is some $g \in L^p(D^\ell_{0,R})$ satisfying that $f = g\omega^{-1/p} \notin L^1(D^\ell_{0,R})$. On the other hand, since $f \in L^p(\omega)$ and $T^\ell_{\alpha,\beta,s}$ is well defined on $L^p(\omega)$, we have that $f(\cdot)\,T^\ell_{\alpha,\beta,s}(z,\cdot) \in L^1(D^\ell_{0,R})$ a.e. $z \in \mathbb{C}$. Finally, since $T^\ell_{\alpha,\beta,\rho}(z,w) \simeq 1$, for $z,w \in D^\ell_{0,R}$, we deduce that $f \in L^1(D^\ell_{0,R})$, which is a contradiction. □

Proof of (a) ⇒ (b) in Theorem 1.4. Assume that $\omega \in \mathcal{A}^p_{p,\varrho}$, for some $\varrho > 0$, and we want to show that $T^\ell_{\alpha,\beta,s}$ is bounded on $L^p(\omega)$, for any $\alpha > 0$, $\beta \in (0,\pi/(2\ell))$ and $s \in \mathbb{R}$. First observe that $\omega' \in \mathcal{A}^\ell_{p',\varrho}$, so, for any $\varphi \in L^p(\omega)$, Hölder's inequality, (4.13) and Tonelli's theorem give

$$\begin{aligned}&\|T^\ell_{\alpha,\beta,s}\,\varphi\|^p_{L^p(\omega)}\\ &\le \int_{\mathbb{C}}\left(\int_{\mathbb{C}}\left(|\varphi(w)|\omega(w)^{1/p}\right)\left(\omega'(w)^{1/p'}\right)T^\ell_{\alpha,\beta,s}(z,w)\,dA(w)\right)^p d\omega(z)\\ &\lesssim \int_{\mathbb{C}}\left(\int_{\mathbb{C}}|\varphi(w)|^p\,T^\ell_{\alpha,\beta,s}(z,w)\,d\omega(w)\right)\left(\frac{\omega'(D^\ell_{z,\varrho})}{|D^\ell_{z,\varrho}|}\right)^{p/p'} d\omega(z)\\ &= \int_{\mathbb{C}}|\varphi(w)|^p\,\omega'(D^\ell_{w,\varrho})^{p/p'}\left(\int_{\mathbb{C}}\frac{T^\ell_{\alpha,\beta,s}(z,w)}{|D^\ell_{z,\varrho}|^{p/p'}}\left(\frac{\omega'(D^\ell_{z,\varrho})}{\omega'(D^\ell_{w,\varrho})}\right)^{p/p'} d\omega(z)\right)d\omega(w).\end{aligned}$$

Now Lemma 4.3 shows that

$$\frac{T^\ell_{\alpha,\beta,s}(z,w)}{|D_{z,\varrho}|^{p/p'}}\left(\frac{\omega'(D^\ell_{z,\varrho})}{\omega'(D^\ell_{w,\varrho})}\right)^{p/p'} \lesssim (1+|w|)^s\,(1+|z|)^t\,\psi_{\alpha',\beta}(z,w),$$

where $t=2(\ell-1)p-s$, so, by (4.12),

$$\int\limits_{\mathbb{C}}\frac{T^\ell_{\alpha,\beta,s}(z,w)}{|D^\ell_{z,\varrho}|^{p/p'}}\left(\frac{\omega'(D^\ell_{z,\varrho})}{\omega'(D^\ell_{w,\varrho})}\right)^{p/p'} d\omega(z)\lesssim (1+|w|)^{2(\ell-1)p}\,\omega(D^\ell_{w,\varrho}).$$

Hence

$$\|T^\ell_{\alpha,\beta,s}\,\varphi\|^p_{L^p(\omega)}\lesssim\int\limits_{\mathbb{C}}|\varphi(w)|^p\,\frac{\omega(D^\ell_{w,\varrho})\,\omega'(D^\ell_{w,\varrho})^{p/p'}}{|D^\ell_{w,\varrho}|^p}\,d\omega(w)\lesssim\|\varphi\|^p_{L^p(\omega)}.\quad\square$$

Proof of (b) ⇒ (a) in Theorem 1.4. We assume that $T^\ell_{\alpha,\beta,s}$ is bounded on $L^p(\omega)$, for some $\alpha>0$, $\beta\in(0,\pi/(2\ell))$ and $s\in\mathbb{R}$, and we want to prove that $\omega\in\mathcal{A}^\ell_{p,\varrho}$, for any $\varrho>0$.

For $\zeta\in\mathbb{C}$ and $\varrho>0$, let χ_ζ be the characteristic function of $D^\ell_{\zeta,\varrho}$ and let $\varphi_\zeta(w):=\chi_\zeta(w)\,\omega'(w)$. Since $(\omega')^p\omega=\omega'$ is a weight, by Lemma 4.4, $\|\varphi_z\|_{L^p(\omega)}=^\ell\omega'(D_{z,\varrho})^{1/p}<\infty$, and so $\|T^\ell_{\alpha,\beta,s}\,\varphi_\zeta\|_{L^p(\omega)}\le\|T^\ell_{\alpha,\beta,s}\|\,\omega'(D^\ell_{\zeta,\varrho})^{1/p}$. Next, by Lemma 2.9,

$$(1+|z|)^{2(\ell-1)-s}\,(1+|\zeta|)^s\,\omega'(D^\ell_{\zeta,\varrho})\lesssim T^\ell_{\alpha,\beta,s}\,\varphi_\zeta(z)\qquad(\zeta\in\mathbb{C},\,z\in D^\ell_{\zeta,\varrho}).$$

Therefore

$$(1+|\zeta|)^{2(\ell-1)p}\omega'(D^\ell_{\zeta,\varrho})^p\omega(D^\ell_{\zeta,\varrho})\lesssim\int\limits_{D^\ell_{\zeta,\varrho}}|T^\ell_{\alpha,\beta,s}\varphi(z)|^p\,d\omega(z)\le\|T^\ell_{\alpha,\beta,s}\|^p\omega'(D^\ell_{\zeta,\varrho}),$$

and hence $[\omega]_{\mathcal{A}^\ell_{p,\varrho}}\lesssim\|T^\ell_{\alpha,\beta,s}\|^p$. □

As a consequence of Theorem 1.4 the class $\mathcal{A}^\ell_{p,\varrho}$ does not depend on ϱ, so from now on we will simply write $\mathcal{A}^\ell_p$ instead of $\mathcal{A}^\ell_{p,\varrho}$. Then, by Proposition 2.13 b), we deduce:

Corollary 4.5. *$\mathcal{A}^\ell_p\subset\mathcal{D}^\ell_\varrho$, for any $\varrho>0$.*

Finally, Theorem 1.4 and Lemma 4.4 directly imply the following result.

Corollary 4.6. *If $\omega\in\mathcal{A}^\ell_p$ then $\omega'\in\mathcal{A}^\ell_{p'}$.*

5. Properties and examples of weights

In what follows, the disk $D^\ell_{z,1}$ will be simply denoted by D^ℓ_z. Our first class of weights consists of weights that are "essentially constant" on those disks.

5.1. Slowly growing weights $\mathcal{S}^\ell$

Definition 5.1. The class of *slowly growing weights* $\mathcal{S}^\ell$ is composed of all the weights ω such that there is a constant $C\ge1$ satisfying $\dfrac{1}{C}\le\dfrac{\omega(w)}{\omega(z)}\le C$, for any $z\in\mathbb{C}$ and $w\in D^\ell_z$. Then we have:

Lemma 5.2. *If $\omega \in \mathcal{S}^\ell$, then $\omega \in \mathcal{A}_p^\ell$, for any $1 < p < \infty$, and $\omega' \in \mathcal{S}^\ell$.*

Proposition 5.3. *For every $\omega \in \mathcal{S}^\ell$ there exists a constant $\gamma = \gamma(\omega, \ell, p) > 0$ such that $\omega(z) \lesssim e^{\gamma|z|^\ell}$ and $\omega'(z) \lesssim e^{\gamma|z|^\ell}$.*

Proof. By Lemma 5.2 it is enough to prove the estimate for ω. Since $\omega \in \mathcal{S}^\ell$, we have that $\omega(z) \simeq \omega(D_z^\ell)/|D_z^\ell|$. Then, taking into account that $\omega \in \mathcal{A}_p^\ell$ (by Lemma 5.2), Theorem 1.3 shows that there exists a constant $C > 1$ such that $\omega(z) \lesssim \omega(D_0^\ell)(1 + |z|)^{2(\ell-1)} C^{|z|^\ell}$. Therefore the estimate $\omega(z) \lesssim e^{\gamma|z|^\ell}$ holds for any constant $\gamma > \log C$. □

5.2. Bergman spaces $A^p(\omega)$

The main result of this section shows that the weighted Bergman space $A^p(\omega)$, $\omega \in \mathcal{A}_p^\ell$, coincides with a weighted Bergman space $A^p(\Omega)$, where Ω is a more regular weight in the class $\mathcal{S}^\ell$. Namely:

Theorem 5.4. *Let $\omega \in \mathcal{A}_p^\ell$. Then:*

a) *The average $\Omega(z) := \dfrac{\omega(D_z^\ell)}{|D_z^\ell|}$ is a weight in $\mathcal{S}^\ell$.*
b) *For $1 < p < \infty$, $A^p(\Omega) = A^p(\omega)$ with equivalent norms.*

In order to prove this result we need the following lemma, which is a consequence of Lemma 2.10 and the mean value property for subharmonic functions.

Lemma 5.5. *Given $\omega \in \mathcal{A}_p^\ell$ and $\varrho > 0$, we have the following estimates:*

$$|f(w)|^p \lesssim \frac{1}{|D^\ell_{z,\tau(\varrho)}|} \int_{D^\ell_{z,\tau(\varrho)}} |f|^p \, dA \quad (f \in H(\mathbb{C}),\, z \in \mathbb{C},\, w \in D_{z,\varrho}) \tag{5.14}$$

$$|f(w)|^p \lesssim \frac{1}{\omega(D^\ell_{z,\varrho})} \int_{D^\ell_{z,\tau(\varrho)}} |f|^p \, d\omega \quad (f \in H(\mathbb{C}),\, z \in \mathbb{C},\, w \in D^\ell_{z,\varrho}) \tag{5.15}$$

Proof. The sub-mean-value property for the subharmonic function $|f|^p$ gives that

$$|f(w)|^p \le \frac{1}{|D^\ell_{w,\varrho}|} \int_{D^\ell_{w,\varrho}} |f|^p \, dA \qquad (f \in H(\mathbb{C}),\, w \in \mathbb{C}).$$

Moreover, the sub-mean-value property for $|f|$ and Lemma 2.10 show that

$$|f(w)|^p \le \left(\frac{1}{|D^\ell_{w,\varrho}|} \int_{D^\ell_{w,\varrho}} |f| \, dA \right)^p \lesssim \frac{1}{\omega(D^\ell_{w,\varrho})} \int_{D^\ell_{w,\varrho}} |f|^p \, d\omega \;\; (f \in H(\mathbb{C}),\, w \in \mathbb{C}).$$

Now note that $|D^\ell_{w,\varrho}| \simeq |D^\ell_{z,\tau_\ell(\varrho)}|$ and, by Lemma 2.3 a), $D^\ell_{w,\varrho} \subset D^\ell_{z,\tau_\ell(\varrho)}$, for every $w \in D^\ell_{z,\varrho}$. Furthermore, Corollary 4.5 and Lemma 2.12 imply that

$$\omega(D^\ell_{w,\varrho}) \simeq \omega(D^\ell_{z,\varrho}) \qquad (z \in \mathbb{C},\, w \in D^\ell_{z,\varrho}).$$

Therefore it is clear that both (5.14) and (5.15) hold. □

Proof of Theorem 5.4. Part a) is a direct consequence of Corollary 4.5 and Lemma 2.12. Next we prove part b).

Pick $\varrho > 0$ so that $\tau_\ell(\varrho) = 1$, and consider a covering $\{D^\ell_{z_k,\varrho}\}_k$ as in Lemma 2.4. Note that, since $\varrho < 1$, $\{D^\ell_{z_k}\}_k$ is also a covering of this type. By integrating (5.14) against $d\omega(w)$ on $D^\ell_{z,\varrho}$ and applying a), we obtain

$$\int_{D^\ell_{z,\varrho}} |f|^p \, d\omega \lesssim \int_{D^\ell_z} |f|^p \, d\Omega \qquad (f \in H(\mathbb{C}),\ z \in \mathbb{C}). \tag{5.16}$$

We also integrate (5.15) against $dA(w)/|D^\ell_{z,\varrho}|$ on $D^\ell_{z,\varrho}$, and use a) to get

$$\int_{D^\ell_{z,\varrho}} |f|^p \, d\Omega \lesssim \int_{D^\ell_z} |f|^p \, d\omega \qquad (f \in H(\mathbb{C}),\ z \in \mathbb{C}). \tag{5.17}$$

Finally, if we apply the estimates (5.16) and (5.17) to $z = z_k$ and sum on k, we deduce that

$$\int_{\mathbb{C}} |f|^p \, d\Omega \simeq \int_{\mathbb{C}} |f|^p \, d\omega \qquad (f \in H(\mathbb{C})),$$

which finishes the proof. $\square$

5.3. Doubling weights

In this section we will prove two technical properties of the doubling weights (see Section 2.3 for the precise definition), that we will use later on.

Proposition 5.6. *Let $\omega \in \mathcal{D}^\ell_\varrho$. Then there exists a constant $\gamma = \gamma(\omega, \ell, \varrho) > 0$ such that $\omega(z)e^{-\gamma|z|^\ell} \in L^1(dA)$. In particular, $\omega(z)e^{-\varepsilon|z|^m} \in L^1(dA)$, for any $\varepsilon > 0$ and $m > \ell$.*

Proof. Let $\{D^\ell_{z_k,\varrho}\}_k$ be a covering of $\mathbb{C}$ as in Lemma 2.4. By Theorem 1.3, there exists a constant $C = C(\omega, \ell, p) > 1$ such that $\omega(D^\ell_{z_k,\varrho}) \le \omega(D^\ell_{0,\varrho})C^{|z_k|^\ell}$. Moreover, if $w \in D^\ell_{z_k,\varrho}$ then $\big|\,|w|^\ell - |z_k|^\ell\,\big| \lesssim 1$. Therefore, for $\gamma > \log C$, we have that

$$\int_{\mathbb{C}} \omega(w)e^{-\gamma|w|^\ell} \, dA(w) \lesssim \sum_k e^{-\gamma|z_k|^\ell}\omega(D^\ell_{z_k,\varrho}) \lesssim \sum_k e^{-\gamma|z_k|^\ell} C^{|z_k|^\ell} < \infty. \quad \square$$

As a consequence of Proposition 5.6, we obtain the following embeddings.

Proposition 5.7. *Let $\omega \in \mathcal{D}^\ell_\varrho$ and $1 < p < \infty$ such that $\omega' = \omega^{-p'/p} \in \mathcal{D}^\ell_\varrho$. Then*

$$L^{\infty,\ell}_{\alpha_0} \hookrightarrow L^{p,\ell}_\alpha(\omega) \hookrightarrow L^{1,\ell}_{\alpha_1} \qquad (\alpha_0 < \alpha < \alpha_1).$$

In particular, the above embeddings hold when $\omega \in \mathcal{A}^\ell_p$.

Proof. Let $\alpha_0 < \alpha < \alpha_1$. Then, for any $f \in L^\infty_{\alpha_0}$, it is clear

$$\|f\|^p_{L^p_\alpha(\omega)} = \int_{\mathbb{C}} |f(w)|^p \omega(w) e^{-\frac{\alpha p}{2}|w|^{2\ell}} dA(w) \le \|f\|^p_{L^\infty_{\alpha_0}} \|\omega(w)e^{-\varepsilon|w|^{2\ell}}\|_{L^1(dA)},$$

where $\varepsilon = \frac{(\alpha-\alpha_0)p}{2}$. Therefore Proposition 5.6 gives that $\|f\|_{L^p_\alpha(\omega)} \lesssim \|f\|_{L^\infty_{\alpha_0}}$.

On the other hand, for any $f \in L^p_\alpha(\omega)$, Hölder's inequality shows that

$$\|f\|_{L^1_{\alpha_1}} = \int_{\mathbb{C}} |f(w)|e^{-\frac{\alpha_1}{2}|w|^{2\ell}} dA(w) \leq \|f\|_{L^p_\alpha(\omega)} \|\omega'(w)e^{-\frac{(\alpha_1-\alpha_0)p'}{2}|w|^{2\ell}}\|^{1/p'}_{L^1(dA)},$$

so, by Proposition 5.6, we conclude that $\|f\|_{L^1_{\alpha_1}} \lesssim \|f\|_{L^p_\alpha(\omega)}$. □

5.4. Radial $\mathcal{A}^\ell_p$ weights

We start this section by "rewriting" the definition of $\mathcal{A}^\ell_p$ weights for radial weights. For $R \geq 0$ let $I^\ell_R := [\max\{0, R-(1+R)^{1-\ell}\}, R+(1+R)^{1-\ell}]$.

Proposition 5.8. *Let $\omega : [0,\infty) \to \mathbb{R}$ be a positive locally integrable function, and define $\omega(z) := \omega(|z|)$, for $z \in \mathbb{C}$. Then the following assertions are equivalent:*

(a) $\omega \in \mathcal{A}^\ell_p$.
(b) $\sup_{R\geq 0} (1+R)^{\ell-1}\omega(I^\ell_R)^{1/p}\omega'(I^\ell_R)^{1/p'} < \infty$.

Proof. For any $z \in \mathbb{C}$, we consider the set

$$U^\ell_z := \begin{cases} \{w \in \mathbb{C} : |w| \in I^\ell_{|z|}\}, & \text{if } |z| \leq 2, \\ \{re^{i\theta} : r \in I^\ell_{|z|},\ |\theta - \arg z| \leq \theta_z\}, & \text{if } |z| > 2, \end{cases}$$

where $\theta_z = \arcsin((1+|z|)^{\ell-1}/|z|)$, which satisfies the estimate

$$\omega(U^\ell_z)^{1/p}\omega'(U^\ell_z)^{1/p'} \simeq (1+|z|)^{1-\ell}\omega(I^\ell_{|z|^\ell})^{1/p}\omega'(I^\ell_{|z|^\ell})^{1/p'} \qquad (z \in \mathbb{C}). \tag{5.18}$$

First we prove that (a) implies (b). Assume that $\omega \in \mathcal{A}^\ell_p$. Then we know that ω and ω' are locally integrable functions on $\mathbb{C}$ (see Corollary 4.6). Since $I^\ell_R \subset [0,3]$, for every $0 \leq R \leq 2$, it directly follows that

$$\sup_{0\leq R\leq 2} (1+R)^{\ell-1}\omega(I^\ell_R)^{1/p}\omega'(I^\ell_R)^{1/p'} \leq 3^{\ell-1}\omega([0,3])^{1/p}\omega([0,3])^{1/p'} < \infty.$$

On the other hand, if $R > 2$ then $U^\ell_R \subset D^\ell_{R,3}$, and so

$$\omega(U^\ell_R)^{1/p}\,\omega'(U^\ell_R)^{1/p'} \lesssim (1+R)^{2(1-\ell)} \quad (R > 2),$$

which, by (5.18), implies that

$$(1+R)^{\ell-1}\omega(I^\ell_R)^{1/p}\omega'(I^\ell_R)^{1/p'} \lesssim 1 \qquad (R > 2).$$

Therefore (b) holds.

Finally, the implication (b) ⇒ (a) follows from the inclusion $D^\ell_z \subset U^\ell_z$ and the estimate (5.18). Indeed, then we have that

$$\omega(D^\ell_z)^{1/p}\omega'(D^\ell_z)^{1/p'} \leq \omega(U^\ell_z)^{1/p}\omega'(U^\ell_z)^{1/p'} \simeq (1+|z|)^{1-\ell}\omega(I^\ell_{|z|})^{1/p}\omega'(I^\ell_{|z|})^{1/p'},$$

which clearly shows that (b) implies (a). □

Next we describe the $\mathcal{A}_p^\ell$ weights which are linear combinations of functions of the form $|z|^\gamma$, $\gamma > 0$.

Proposition 5.9. *Let $\varphi(r) = \sum_{j=1}^m c_j r^{\ell_j}$, where $0 < \ell_1 < \ell_2 < \cdots < \ell_m$ and $c_1, \dots, c_n \in \mathbb{R}$, and let $\omega(z) = (1+|z|)^s e^{\varphi(|z|)}$, where $s \in \mathbb{R}$. Then the following assertions are equivalent:*

(a) $\omega \in \mathcal{A}_p^\ell$.
(b) $\omega \in \mathcal{S}^\ell$.
(c) $\ell_m \le \ell$.

Proof. Note that the implication (b) $\Rightarrow$ (a) is a direct consequence of Lemma 5.2. Moreover, since, by Corollary 4.5, $\mathcal{A}_p^\ell \subset \mathcal{D}_\varrho^\ell$, Proposition 5.6 shows that (a) implies (c). Finally, the implication (c) $\Rightarrow$ (b) follows from two facts: $(1+|z|)^s$ is a radius function (by Lemma 2.1) and

$$\big|\, |w|^\beta - |z|^\beta \big| \lesssim 1 \qquad (z \in \mathbb{C}, w \in D_z^\ell),$$

whenever $0 < \beta \le \ell$. □

As a consequence of Proposition 5.9 we characterize the functions $|e^h|$, where $h \in H(\mathbb{C})$, which are $\mathcal{A}_p^\ell$ weights.

Proposition 5.10. *Let $h \in H(\mathbb{C})$ and let $\omega = |e^h|$. Then the following assertions are equivalent:*

(a) $\omega \in \mathcal{A}_p^\ell$.
(b) $\omega \in \mathcal{S}^\ell$.
(c) *h is a polynomial of degree $m \le \ell$.*

Proof. First, note that Lemma 5.2 shows that (b) implies (a). Now we are going to prove that (a) implies (c). Assume that $\omega \in \mathcal{A}_p^\ell$. Then, by Corollary 4.5 and Proposition 5.6, there exists $\gamma > 0$ such that

$$\int_{\mathbb{C}} e^{\operatorname{Re} h(w) - \gamma |w|^\ell} dA(w) < \infty.$$

By taking into account the subharmonicity of $e^{\operatorname{Re} h} = |e^h|$ and the fact that $e^{-\gamma|w|^\ell}$ is a slowly growing function (by Proposition 5.9), we deduce that $e^{\operatorname{Re} h(z) - \gamma|z|^\ell}(1+|z|)^{2(1-\ell)} \lesssim 1$. Therefore $e^{\operatorname{Re} h(z)} \lesssim e^{2\gamma|z|^\ell}$ and, by a classical theorem of Hadamard (see, for instance, [6, Corollary 6.33]), we obtain that h is a polynomial of degree $m \le \ell$.

Finally, we show that (c) implies (b). Since a finite product of slowly growing functions is a slowly growing function, without loss of generality we may assume that $h(z) = az^k$, where $a \in \mathbb{C}$ and $k = 0, 1, \dots, \ell$. Then

$$|\operatorname{Re}(h(w) - h(z))| \le |h(w) - h(z)| \le |a||w - z| \sum_{j=0}^{k-1} |z|^j |w|^{k-1-j} \quad (z, w \in \mathbb{C}),$$

and so

$$|\operatorname{Re}(h(w) - h(z))| \lesssim (1+|z|)^{k-\ell} \le 1 \qquad (z \in \mathbb{C},\, w \in D_z^\ell).$$

Since $\omega(w)/\omega(z) = e^{\operatorname{Re}(h(w)-h(z))}$, it follows that $\omega \in \mathcal{S}^\ell$, which finishes the proof. □

6. Proof of Theorem 1.5

In order to prove Theorem 1.5, we need some estimates of the Berezin kernel $\mathcal{B}^\ell_\alpha$, which will follow from the behavior of the Bergman kernel K^ℓ_α.

Recall that the Bergman kernel for $F^{2,\ell}_\alpha$ is given by

$$K^\ell_\alpha(z,w) = \ell\alpha^{1/\ell} E_{1/\ell,1/\ell}(\alpha^{1/\ell} z\overline{w}), \tag{6.19}$$

where $E_{1/\ell,1/\ell}(\lambda) = \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma((k+1)/\ell)}$ is a Mittag-Leffler function (see, for instance, [4] for the details). This function satisfies the following asymptotic expansion when $|\lambda| \to \infty$ (see, for instance, [16, Theorem 1.2.1]):

$$E_{1/\ell,1/\ell}(\lambda) = \begin{cases} \ell\lambda^{\ell-1} e^{\lambda^\ell} + O(\lambda^{-1}), & \text{if } \ |\arg\lambda| \le \frac{\pi}{2\ell}, \\ O(\lambda^{-1}), & \text{if } \ |\arg\lambda| > \frac{\pi}{2\ell}. \end{cases} \tag{6.20}$$

This asymptotic equality shows that, for a fixed $R > 0$ large enough, we have that

$$\begin{cases} |E_{1/\ell,1/\ell}(\lambda)| \simeq (1+|\lambda|)^{\ell-1} e^{\operatorname{Re}(\lambda^\ell)}, & \text{if } \ |\arg\lambda| \le \frac{\pi}{2\ell} \text{ and } |\lambda| > R, \\ |E_{1/\ell,1/\ell}(\lambda)| \lesssim 1, & \text{otherwise.} \end{cases}$$

Moreover, since $E_{1/\ell,1/\ell}(x) > 0$, for every $x \in [0,\infty)$, there exists $0 < \varepsilon < 1$ such that $|E_{1/\ell,1/\ell}(\lambda)| \simeq 1$, for $-\varepsilon \le \operatorname{Re}\lambda \le R+\varepsilon$, $-\varepsilon \le \operatorname{Im}\lambda \le \varepsilon$. Therefore there is $0 < \beta < \pi/(2\ell)$ small enough so that

$$\begin{cases} |E_{1/\ell,1/\ell}(\lambda)| \simeq (1+|\lambda|)^{\ell-1} e^{\operatorname{Re}(\lambda^\ell)}, & \text{if } \ |\arg\lambda| \le \beta \text{ or } |\lambda| < \varepsilon \\ |E_{1/\ell,1/\ell}(\lambda)| \lesssim 1, & \text{otherwise.} \end{cases} \tag{6.21}$$

Here $0^\ell = 0$ and $\lambda^\ell = |\lambda|^\ell e^{\cos(\ell\arg\lambda)}$, for any $\lambda \in \mathbb{C}\setminus\{0\}$. As a consequence of (6.21) we obtain global and local estimates of the Bergman kernel, which, for convenience, are written in terms of the so called *twisted Bergman kernel*

$$\mathbf{K}^\ell_\alpha(z,w) := e^{-\frac{\alpha}{2}|z|^{2\ell}} K^\ell_\alpha(z,w)\, e^{-\frac{\alpha}{2}|w|^{2\ell}}.$$

Lemma 6.1. *There is $0 < \beta < \pi/(2\ell)$ satisfying that*

$$|\mathbf{K}^\ell_\alpha(z,w)| \lesssim T^\ell_{\alpha,\beta,\ell-1}(z,w) \qquad (z,w \in \mathbb{C}). \tag{6.22}$$

Moreover, there is $\widetilde{\varrho} > 0$ such that

$$|\mathbf{K}^\ell_\alpha(z,w)| \simeq T^\ell_{\alpha,\beta,\ell-1}(z,w) \simeq (1+|z|)^{2(\ell-1)} \qquad (z \in \mathbb{C},\ w \in D^\ell_{z,\widetilde{\varrho}}). \tag{6.23}$$

In particular:

$$\mathcal{B}^\ell_\alpha(z,w) \lesssim T^\ell_{2\alpha,\beta,2(\ell-1)}(z,w) \qquad (z,w \in \mathbb{C}) \tag{6.24}$$

$$\mathcal{B}^\ell_\alpha(z,w) \simeq T^\ell_{2\alpha,\beta,2(\ell-1)}(z,w) \simeq (1+|z|)^{2(\ell-1)} \qquad (z \in \mathbb{C},\ w \in D^\ell_{z,\widetilde{\varrho}}). \tag{6.25}$$

Proof. Since $1+|z\overline{w}| \le (1+|z|)(1+|w|)$, (6.22) follows from (6.21). Moreover, (6.23) also follows from (6.21), because, for any $\varrho > 0$, we have that

$$1+|z\overline{w}| \simeq (1+|z|)(1+|w|) \qquad (z \in \mathbb{C},\ w \in D^\ell_{z,\varrho}),$$

and, given $0<\beta<\pi/2$ and $0<\varepsilon<1$, there is $\widetilde{\varrho}=\widetilde{\varrho}(\beta,\varepsilon)>0$ such that any $z\in\mathbb{C}$ and $w\in D_{z,\widetilde{\varrho}}$ satisfy either $|\arg(z\overline{w})|\le\beta$ or $|z\overline{w}|<\varepsilon$.

Indeed, if $\varrho>0$ and $|z|>\frac{\pi}{2\beta}\varrho$, then $|z|>\varrho\ge\varrho(1+|z|)^{1-\ell}$, and so

$$|\arg(z\overline{w})|\le\arcsin\frac{\varrho}{|z|}<\frac{\pi}{2}\,\frac{\varrho}{|z|}<\beta,\quad \text{for every } w\in D^\ell_{z,\varrho}.$$

On the other hand, if $|z|\le\frac{\pi}{2\beta}\varrho$ and $w\in D^\ell_{z,\varrho}$, then $|w|<|z|+\varrho\le\varrho(\frac{\pi}{2\beta}+1)$, so $|z\overline{w}|<\varrho^2(\frac{\pi}{2\beta}+1)^2$. Therefore $\widetilde{\varrho}=\varepsilon^{\frac12}(\frac{\pi}{2\beta}+1)^{-1}$ does the job.

Finally, by (6.23) we have that $K^\ell_\alpha(z,z)\simeq(1+|z|)^{2(1-\ell)}e^{\alpha|z|^{2\ell}}$, so

$$\mathcal{B}^\ell_\alpha(z,w)\simeq(1+|z|)^{2(1-\ell)}\,|\mathbf{K}^\ell_\alpha(z,w)|^2\qquad(z,w\in\mathbb{C}).$$

Hence it is clear that (6.24) and (6.25) are direct consequences of (6.22) and (6.23), respectively. Thus the proof is complete. □

The pointwise estimates (6.22) and (6.23) give, for $1\le p<\infty$, the $L^{p,\ell}_\alpha$-norm estimate for the Bergman kernel (see, for instance, [4] or [7]):

$$\|K^\ell_\alpha(\cdot,z)\|_{F^{p,\ell}_\alpha}\simeq(1+|z|)^{2(\ell-1)/p'}e^{\frac{\alpha}{2}|z|^{2\ell}}\qquad(z\in\mathbb{C}).\tag{6.26}$$

A direct consequence of (6.26) is the boundedness on $L^{p,\ell}_\alpha$ of the Bergman projection.

Proposition 6.2 *([4], [7, Lemma 2.15]).*

a) *For $1\le p<\infty$ the Bergman projection P^ℓ_α is bounded on $L^{p,\ell}_\alpha$.*
b) *If $\gamma<2\alpha$ then $P^\ell_\alpha f=f$, for every $f\in F^{1,\ell}_\gamma$.*

Corollary 6.3. *If $1\le p<\infty$ and $\gamma>0$, then*

$$|f(z)|\lesssim\|f\|_{F^{p,\ell}_\gamma}(1+|z|)^{2(\ell-1)}\,e^{\frac{\gamma}{2}|z|^{2\ell}}\qquad(f\in F^{p,\ell}_\gamma,\ z\in\mathbb{C}).$$

Now we give two technical lemmas needed in the proof of Theorem 1.5.

Lemma 6.4. *Assume $\widetilde{\varrho}>0$ satisfies (6.23). Then, for any $\omega\in\mathcal{A}^\ell_p$ and $0<\varrho\le\widetilde{\varrho}$, we have that*

$$\int_{\mathbb{C}}|\mathbf{K}^\ell_\alpha(z,w)|\,d\omega(w)\simeq\frac{\omega(D^\ell_{z,\varrho})}{|D^\ell_{z,\varrho}|}\qquad(z\in\mathbb{C}).$$

Proof. Let $I(z)$ be the integral at the statement. Then, by (6.23), we have

$$\frac{\omega(D^\ell_{z,\varrho})}{|D^\ell_{z,\varrho}|}\simeq(1+|z|)^{\ell-1}\int_{D^\ell_{z,\varrho}}(1+|w|)^{\ell-1}\,d\omega(w)\simeq\int_{D^\ell_{z,\varrho}}|\mathbf{K}_\alpha(z,w)|\,d\omega(w)\ \le I(z).$$

On the other hand, (6.22) and (4.13) show that

$$I(z)\lesssim\int_{\mathbb{C}}T^\ell_{\alpha,\beta,\ell-1}(z,w)\,d\omega(w)\simeq(1+|z|)^{2(\ell-1)}\omega(D^\ell_{z,\varrho})\simeq\frac{\omega(D^\ell_{z,\varrho})}{|D^\ell_{z,\varrho}|}.\quad\square$$

Lemma 6.5. *Let $1 < p < \infty$. If $\mathcal{B}^\ell_\alpha$ is well defined on $L^p(\omega)$, then ω' is a weight.*

Proof. We prove the assertion by contradiction. If ω' is not a weight, then there is $a \in \mathbb{C}$ such that $\omega' \notin L^1(D^\ell_{a,\widetilde{\varrho}})$, where $\widetilde{\varrho} > 0$ satisfies (6.25). Thus $\omega^{-1/p} \notin L^{p'}(D^\ell_{a,\widetilde{\varrho}})$ and so there exists $g \in L^p(D^\ell_{a,\widetilde{\varrho}})$ such that $f = g\omega^{-1/p} \notin L^1(D^\ell_{a,\widetilde{\varrho}})$. Now, since $f \in L^p(D^\ell_{a,\widetilde{\varrho}}; \omega\, dA)$, the hypothesis gives that $f\,\mathcal{B}^\ell_\alpha(z,\cdot) \in L^1(D^\ell_{a,\widetilde{\varrho}})$ for a.e. $z \in \mathbb{C}$. But (6.25) shows that $\mathcal{B}^\ell_\alpha(z,w) \simeq (1+|a|)^{2(\ell-1)}$, for $z,w \in D^\ell_{a,\widetilde{\varrho}}$, which implies that $f \in L^1(D^\ell_{a,\widetilde{\varrho}})$, and we get a contradiction. □

Proof of Theorem 1.5. Assume that $\omega \in \mathcal{A}^\ell_p$. Then (6.24) and Theorem 1.4 show that $\mathcal{B}^\ell_\alpha$ is bounded on $L^p(\omega)$. Moreover, the fact that $\mathcal{B}^\ell_\alpha f = f$, for every $f \in A^p(\omega) = H(\mathbb{C}) \cap L^p(\omega)$, follows from Propositions 5.7 and 6.2 b) by observing that if $f \in A^p(\omega)$ then $fK^\ell_\alpha(\cdot,z) \in F^1_\alpha$, for every $z \in \mathbb{C}$. Indeed, since $\omega \in \mathcal{A}^\ell_p$, we know that $\omega' = \omega^{-p'/p} \in \mathcal{A}^\ell_{p'}$ (see Corollary 4.6), and so Hölder's inequality, (6.22) and (4.13) show that, for any $f \in A^p(\omega)$, we have

$$\int_{\mathbb{C}} |f(w)||K^\ell_\alpha(w,z)|\, e^{-\frac{\alpha}{2}|w|^{2\ell}} dA(w) \le e^{\frac{\alpha}{2}|z|^{2\ell}} \|f\|_{L^p(\omega)} \|T^\ell_{\alpha,\beta,\ell-1}(z,\cdot)\|_{L^{p'}(\omega')} < \infty.$$

Conversely, assume that $\mathcal{B}^\ell_\alpha$ is bounded on $L^p(\omega)$. Then, by Lemma 6.5, ω' is a weight, and we may follow the argument in the proof of the implication (b) ⇒ (a) in Theorem 1.4, but replacing $T^\ell_{\alpha,\beta,s}$ and ϱ by $\mathcal{B}^\ell_\alpha$ and a radius $\widetilde{\varrho}$ satisfying (6.25), respectively, to conclude that $\omega \in \mathcal{A}^\ell_p$. □

7. Proof of Theorems 1.1 and 1.6

7.1. Proof of Theorem 1.1

By Theorem 1.4 and the pointwise estimate (6.22), it is clear that we only have to prove the implication (c) ⇒ (a).

Assume that P^ℓ_α is bounded on $L^{p,\ell}_\alpha(\omega)$, for some $\alpha > 0$, and we want to prove that $\omega \in \mathcal{A}^\ell_p$. In order to do that, we will follow the scheme of the proof of [9, Theorem 3.1].

First, we observe that the argument in the proof of Lemma 6.5 shows that, if P^ℓ_α is well defined on $L^{p,\ell}_\alpha(\omega)$, in the sense that, for every $f \in L^{p,\ell}_\alpha(\omega)$, we have

$$\psi_z(w) = f(w)\, K^\ell_\alpha(z,w)\, e^{-\alpha|w|^{2\ell}} \in L^1(dA) \quad \text{for a.e. } z \in \mathbb{C},$$

then ω' is a weight. Next, we prove that the boundedness of P^ℓ_α on $L^{p,\ell}_\alpha(\omega)$ implies that $\omega \in \mathcal{A}^\ell_p$, *i.e.* there is some $\varrho > 0$ satisfying

$$(\omega(D^\ell_{\zeta,\varrho}))^{1/p}(\omega'(D^\ell_{\zeta,\varrho}))^{1/p'} \lesssim |D^\ell_{\zeta,\varrho}| \qquad (\zeta \in \mathbb{C}). \tag{7.27}$$

Let $\varrho > 0$ and $R > 0$. Note that $D^\ell_{\zeta,\varrho} \subset D^\ell_{0,R+\varrho}$ and $|D^\ell_{\zeta,\varrho}| \ge \varrho^2(1+R)^{2(1-\ell)}$, for any $\zeta \in D_{0,R}$. Since ω and ω' are weights, we only have to show that there is some small $\varrho > 0$ and some large $R > 0$ such that (7.27) holds for $|\zeta| \ge R$.

Assume that $0 < \varrho < R/(4\ell)$ and $|\zeta| \ge R > 1$. Then $\frac{1}{2}|\zeta| \le |z| \le 2|\zeta|$ and $|\arg(z\overline{\zeta})| \le \frac{\pi}{8\ell}$, for any $z \in D^\ell_{\zeta,\varrho}$. In particular, $|\arg(z\overline{w})| \le \beta := \frac{\pi}{4\ell}$, for any $z, w \in D^\ell_{\zeta,\varrho}$. Now define the function

$$f_\zeta(w) := (\zeta\overline{w})^{1-\ell} e^{-i\alpha\,\mathrm{Im}\left((\zeta\overline{w})^\ell\right)} e^{\frac{\alpha}{2}|w|^{2\ell}} \chi_\zeta(w)\, \omega'(w) \qquad (w \in \mathbb{C}),$$

where χ_ζ denotes the characteristic function of the disk $D^\ell_{\zeta,\varrho}$. It is clear that

$$\|f_\zeta\|_{L^p_\alpha(\omega)} \lesssim |D^\ell_{\zeta,\varrho}| \left(\omega'(D^\ell_{\zeta,\varrho})\right)^{1/p} \qquad (|\zeta| \geq R),$$

and we make the following claim:

Claim. *There exist small enough $\varrho > 0$ and large enough $R > 0$ so that*

$$\omega'(D^\ell_{\zeta,\varrho}) \lesssim e^{-\frac{\alpha}{2}|z|^{2\ell}} |P^\ell_\alpha f_\zeta(z)| \qquad (|\zeta| \geq R,\ z \in D^\ell_{\zeta,\varrho}).$$

Taking for granted this claim, it is easy to prove (7.27) for $|\zeta| \geq R$:

$$\begin{aligned} \omega'(D^\ell_{\zeta,\varrho})(\omega(D^\ell_{\zeta,\varrho}))^{1/p} &\lesssim \left(\int_{D^\ell_{\zeta,\varrho}} e^{-\frac{p\alpha}{2}|z|^{2\ell}} |P^\ell_\alpha f_\zeta(z)|^p \, d\omega(z) \right)^{1/p} \\ &\leq \|P^\ell_\alpha\| \, \|f_\zeta\|_{L^p_\alpha(\omega)} \lesssim |D^\ell_{\zeta,\varrho}| \left(\omega'(D^\ell_{\zeta,\varrho})\right)^{1/p}. \end{aligned}$$

In order to prove the claim, note that $e^{-\frac{\alpha}{2}|z|^{2\ell}} P^\ell_\alpha f_\zeta(z) = \int_{D^\ell_{\zeta,\varrho}} \Phi_\zeta(z,w)\, dA(w)$, where

$$\Phi_\zeta(z,w) = f_\zeta(w)\, e^{-\frac{\alpha}{2}|z|^{2\ell}} K^\ell_\alpha(z,w) e^{-\alpha|w|^{2\ell}}.$$

Recall that $|z\overline{w}| > \frac{1}{4}|\zeta|^2 > \frac{1}{4}R^2$ and $|\arg(z\overline{w})| \leq \beta$, for every $z, w \in D^\ell_{\zeta,\varrho}$, so (6.19) and (6.20) show that, for large enough $R > 0$, we have

$$K^\ell_\alpha(z,w) = \alpha\ell(z\overline{w})^{\ell-1} e^{\alpha(z\overline{w})^\ell} + O((z\overline{w})^{-1}) \qquad (z, w \in D^\ell_{\zeta,\varrho},\ |\zeta| \geq R).$$

It follows that

$$\begin{aligned} \Phi_\zeta(z,w) &= \alpha\ell\,(z/\zeta)^{\ell-1}\, e^{i\alpha\,\mathrm{Im}((z\overline{w})^\ell - (\zeta\overline{w})^\ell)}\, e^{-\frac{\alpha}{2}(|z|^{2\ell} - 2\,\mathrm{Re}((z\overline{w})^\ell) + |w|^{2\ell})}\, \omega'(w) \\ &\quad + f_\zeta(w)\, e^{-\alpha|w|^{2\ell}}\, e^{-\frac{\alpha}{2}|z|^{2\ell}}\, O((z\overline{w})^{-1}) \\ &= \alpha\ell\,(z/\zeta)^{\ell-1}\, e^{i\alpha\,\mathrm{Im}((z\overline{\zeta})^\ell)}\, \psi_{\alpha,\beta}(z,w)\, \omega'(w) \\ &\quad - \alpha\ell\,(z/\zeta)^{\ell-1} \psi_{\alpha,\beta}(z,w) \left(e^{i\alpha\,\mathrm{Im}(z\overline{\zeta})^\ell)} - e^{i\alpha\,\mathrm{Im}((z\overline{w})^\ell - (\zeta\overline{w})^\ell)} \right) \omega'(w) \\ &\quad + (\zeta\overline{w})^{1-\ell} e^{-i\alpha\ \mathrm{Im}\left((\zeta\overline{w})^\ell\right)} e^{-\frac{\alpha}{2}(|z|^{2\ell} + |w|^{2\ell})}\, \omega'(w)\, O((z\overline{w})^{-1}) \\ &= \Phi^1_\zeta(z,w) + \Phi^2_\zeta(z,w) + \Phi^3_\zeta(z,w) \qquad (z, w \in D^\ell_{\zeta,\varrho},\ |\zeta| \geq R). \end{aligned}$$

Now we estimate the integrals $I^j_\zeta(z) = \int_{D^\ell_{\zeta,\varrho}} \Phi^j_\zeta(z,w)\, dA(w)$, for $|\zeta| \geq R$ and $z \in D^\ell_{\zeta,\varrho}$, as follows:

$$|I^1_\zeta(z)| \simeq \omega'(D^\ell_{\zeta,\varrho}). \tag{7.28}$$

$$|I^2_\zeta(z)| \leq \tfrac{1}{2}\,|I^1_\zeta(z)|, \quad \text{for small enough } \varrho > 0. \tag{7.29}$$

$$|I^3_\zeta(z)| \lesssim R^{-2\ell} e^{-\alpha(R/2)^{2\ell}}\ \omega'(D^\ell_{\zeta,\varrho}). \tag{7.30}$$

Observe that these estimates prove the claim, since, for large enough $R > 0$ and small enough $\varrho > 0$, we have that

$$e^{-\frac{\alpha}{2}|z|^{2\ell}} |P^\ell_\alpha(f_\zeta)(z)| \geq \tfrac{1}{2}|I^1_\zeta(z)| - |I^3_\zeta(z)| \gtrsim \omega'(D^\ell_{\zeta,\varrho}) \qquad (|\zeta| \geq R,\ z \in D^\ell_{\zeta,\varrho}).$$

The estimate (7.28) holds because $|I^1_\zeta(z)| = \int_{D_{\zeta,\varrho}} |\Phi^1_\zeta(z,w)|\,dA(w)$, and, by Corollary 2.8, $|\Phi^1_\zeta(z,w)| \simeq |z/\zeta|^{\ell-1}\psi_{\alpha,\beta}(z,w)\,\omega'(w) \simeq \omega'(w)$, for $z,w \in D^\ell_{\zeta,\varrho}$. The proof of (7.30) is also easy since, for $|\zeta| \geq R$ and $z,w \in D^\ell_{\zeta,\varrho}$, we have

$$|\Phi^3_\zeta(z,w)| \lesssim |\zeta|^{-2\ell} e^{-\alpha(|\zeta|/2)^{2\ell}}\,\omega'(w) \leq R^{-2\ell} e^{-\alpha(R/2)^{2\ell}}\,\omega'(w).$$

Finally, (7.29) is a consequence of the fact that $M_R(\varrho) \to 0$, as $\varrho \to 0^+$, where $M_R(\varrho) := \sup\{\Psi_\zeta(z,w) : |\zeta| \geq R,\ z,w \in D^\ell_{\zeta,\varrho}\}$, and

$$\Psi_\zeta(z,w) := |e^{i\alpha\,\mathrm{Im}((z\overline{\zeta})^\ell)} - e^{i\alpha\,\mathrm{Im}((z\overline{w})^\ell-(\zeta\overline{w})^\ell)}| = |1 - e^{i\alpha\,\mathrm{Im}((z\overline{w})^\ell-(\zeta\overline{w})^\ell-(z\overline{\zeta})^\ell)}|.$$

Indeed, since $z\overline{w} = u\overline{v}$, where $u = z\overline{\zeta}/|\zeta|$ and $v = w\overline{\zeta}/|\zeta|$ satisfy $|\arg u| + |\arg v| < \pi$, we have that $(z\overline{w})^\ell = u^\ell\overline{v}^\ell$, and so

$$(z\overline{w})^\ell - (\zeta\overline{w})^\ell - (z\overline{\zeta})^\ell = u^\ell\overline{v}^\ell - |\zeta|^\ell\overline{v}^\ell - |\zeta|^\ell u^\ell = (u^\ell - |\zeta|^\ell)(\overline{v}^\ell - |\zeta|^\ell) - |\zeta|^{2\ell}.$$

Moreover, since $z,w \in D^\ell_\zeta$, we have that $u,v \in D^\ell_{|\zeta|,\varrho}$, and so Lemma 2.7 shows that $|u^\ell - |\zeta|^\ell|$, $|\overline{v}^\ell - |\zeta|^\ell| \leq r(\varrho) := \ell\varrho(1+\varrho)^{\ell-1}$. Therefore

$$\Psi_\zeta(z,w) = |1 - e^{i\alpha\,\mathrm{Im}(u^\ell-|\zeta|^\ell)(\overline{v}^\ell-|\zeta|^\ell))}| \leq e^{\alpha r(\varrho)^2} - 1 \quad (|\zeta| > R,\ z,w \in D^\ell_{\zeta,\varrho}),$$

and hence $M_r(\varrho) \to 0$, as $\varrho \to 0^+$. □

As a direct consequence of Theorem 1.1 and Propositions 5.7 and 6.2 b), we obtain the following result.

Corollary 7.1. *If $\omega \in \mathcal{A}^\ell_p$, then P^ℓ_α maps $L^{p,\ell}_\alpha(\omega)$ onto $F^{p,\ell}_\alpha(\omega)$, for any $\alpha > 0$.*

Then Corollary 1.2, which answers the initial question in the introduction, follows from Corollary 7.1 and Proposition 5.9.

7.2. Sarason's conjecture for the spaces $F^{p,\ell}_\alpha$

Before proving Theorem 1.6 we precise the definition of the Toeplitz operators T^ℓ_φ. Given $\alpha > 0$, for $\varphi \in L^{1,\ell}_\gamma$, $0 < \gamma < 2\alpha$, we define the Toeplitz operator T^ℓ_φ on $X^\ell_\alpha := Span\{K^\ell_\alpha(\cdot,w) : w \in \mathbb{C}\}$ by $T^\ell_\varphi h := P^\ell_\alpha(\varphi h)$. Note that the condition $\gamma < 2\alpha$ assures that T^ℓ_φ is well defined, *i.e.* for any $h \in X^\ell_\alpha$ and $z \in \mathbb{C}$, we have that $f_z(w) = K^\ell_\alpha(z,w)\,\varphi(w)\,h(w) \in L^1(\mathbb{C}; e^{-\alpha|w|^{2\ell}}dA(w))$.

Moreover, if $g \in F^{1,\ell}_\gamma$ then $T^\ell_{\overline{g}}(K_\alpha(\cdot,z))(w) = \overline{g(z)}K^\ell_\alpha(w,z)$, and so, for $f,g \in F^{1,\ell}_\gamma$, the composition operator $T^\ell_f T^\ell_{\overline{g}}$ is well defined on X^ℓ_α and

$$T^\ell_f T^\ell_{\overline{g}}(K^\ell_\alpha(\cdot,z))(w) = f(w)\,\overline{g(z)}\,K_\alpha(w,z) \qquad (z,w \in \mathbb{C}). \tag{7.31}$$

Proof of Theorem 1.6. First we prove that (a) implies (b). By (7.31), we have that

$$T(z,w) := \langle T^\ell_f T^\ell_{\overline{g}}\,k_z, k'_w\rangle_\alpha = f(w)\,\overline{g(z)}S(z,w) \qquad (z,w \in \mathbb{C}),$$

where $k_z(u) = K^\ell_\alpha(u,z)/\|K^\ell_\alpha(\cdot,z)\|_{F^p_\alpha}$, $k'_w(u) = K^\ell_\alpha(u,w)/\|K^\ell_\alpha(\cdot,w)\|_{F^{p'}_\alpha}$ and

$$S(z,w) := \frac{K^\ell_\alpha(w,z)}{\|K^\ell_\alpha(\cdot,z)\|_{F^p_\alpha}\|K^\ell_\alpha(\cdot,w)\|_{F^{p'}_\alpha}}.$$

Since $T_f^\ell T_g^\ell$ is bounded on F_α^p, $T(z,w)$ is a bounded function on $\mathbb{C}^2$. By (6.26), $S(z,z)\simeq 1$, so $|T(z,z)|\simeq |f(z)g(z)|$. Thus the function fg is bounded, and, by Liouville theorem, it is a non-zero constant. In particular, $f=e^h$, for some entire function h. By Corollary 6.3, $|f(z)|\lesssim \|f\|_{F_\gamma^{1,\ell}}(1+|z|)^{2(\ell-1)}e^{\frac{\gamma}{2}|z|^{2\ell}}$. Hence, by a classical theorem of Hadamard (see, for instance, [6, Corollary 6.33]), we have that h is a polynomial of degree $m\le 2\ell$.

Now we prove that $m\le\ell$. Assume that $h(u)=\sum_{j=0}^m\lambda_j u^j$. Let $\widetilde{\varrho}>0$ be a radius satisfying (6.23), and let $\theta=\arg(\lambda_m)$. For $x>1$, we consider the points $w_x=e^{-i\theta/m}(x+\frac12\widetilde{\varrho}(1+x)^{1-\ell})$ and $z_x=e^{-i\theta/m}x$. Note that $w_x\in D_{z_x,\widetilde{\varrho}}$, and so (6.23) and (6.26) give that $S(z_x,w_x)\simeq 1$. It follows that

$$T(z_x,w_x)\simeq |f(w_x)||g(z_x)|=|f(w_x)||f(z_x)|^{-1}=e^{\varphi(x)},$$

where $\varphi(x)=\operatorname{Re}h(w_x)-\operatorname{Re}h(z_x)$. Next, for $j=1,\dots,m$, we have that

$$w_x^j-z_x^j=e^{-ij\theta/m}x^j\left((1+t(x))^j-1\right)=e^{-ij\theta/m}x^j\left(jt(x)+o(x^{-\ell})\right),$$

where $t(x)=\frac{\widetilde{\varrho}}{2}\frac{(1+x)^{1-\ell}}{x}$. Since $t(x)\simeq x^{-\ell}$, as $x\to\infty$, we deduce

$$\left|\sum_{j=0}^{m-1}\lambda_j(w_x^j-z_x^j)\right|=o(x^{m-\ell})\qquad(x\to\infty).$$

Hence $\varphi(x)=x^{m-\ell}(m\widetilde{\varrho}|\lambda_m|+o(1))$, and, since $|T(w_x,z_x)|=O(1)$, we conclude that $m\le\ell$.

By Proposition 5.10, (b) is equivalent to (c). Finally, the fact that (c) implies (a) follows from the boundedness of P_α^ℓ on $L_\alpha^{p,\ell}(\omega)$ whenever $\omega=|f|^p\in\mathcal{A}_p^\ell$ (by Theorem 1.1). Indeed, since (7.31) gives $T_f^\ell T_{1/\overline{f}}^\ell\varphi=P_\alpha^\ell(f\,T_{1/\overline{f}}^\ell\varphi)$, for any $\varphi\in X_\alpha^\ell$, we have that

$$\|T_f^\ell T_{1/\overline{f}}^\ell\varphi\|_{L_\alpha^{p,\ell}}=\|P_\alpha^\ell(\varphi/\overline{f})\|_{L_\alpha^{p,\ell}(\omega)}\le\|P_\alpha^\ell\|\,\|\varphi/\overline{f}\|_{L_\alpha^{p,\ell}(\omega)}=\|P_\alpha^\ell\|\,\|\varphi\|_{L_\alpha^{p,\ell}}.\quad\square$$

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