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Treball final de grau

Teichmüller Spaces via Fenchel-Nielsen Coordinates

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Abstract

The main goal of this work is to study Teichmüller spaces of Riemann surfaces and hyperbolic surfaces via Fenchel-Nielsen coordinates. To do this, we first determine the universal holomorphic covering of every Riemann surface and briefly study Fuchsian groups. Then, we introduce moduli and Teichmüller spaces for Riemann surfaces and use the previous characterization to imbue the Teichmüller space with the so-called algebraic topology. We also compute said spaces for the torus and give a short remark on the mapping class group. Afterwards, we introduce hyperbolic geometry as the natural geometry compatible with the complex structure of the complex unit disc and use this along our previous effort to geometrize Riemann surfaces. We go on to study hyperbolic surfaces with and without boundary with a special emphasis on building the necessary machinery to prove the existence and uniqueness of closed geodesics in a given homotopy class via the axis of hyperbolic transformations. Finally, we undergo a thorough study of pairs of pants and X-pieces in order to demonstrate the main theorem about the Fenchel-Nielsen coordinates. Also, this study provides the necessary background from hyperbolic geometry for the short paper that has grown from this project. We conclude with some applications of these efforts, like the collar lemma.

Abstract en català

L'objectiu principal d'aquest treball és estudiar els espais de Teichmüller de superfícies de Riemann i superfícies hiperbòliques mitjançant les coordenades de Fenchel-Nielsen. En primer lloc, es determina el recobridor universal de cada superfície de Riemann i s'estudien grups Fuchsians breument. Després, introduïm espais de moduli i de Teichmüller de superfícies de Riemann i fem servir el que s'ha estudiat prèviament per dotar l'espai de Teichmüller de l'anomenada topologia algebraica. També calculem els espais en qüestió pel cas del torus i esmentem algunes connexions amb el *mapping class group*. Tot seguit, introduïm la geometria hiperbòlica com la geometria que és compatible de manera natural amb l'estructura complexa del disc unitat. Això, juntament amb el treball ja fet, ens permet geometritzar les superfícies de Riemann. A continuació, estudiem superfícies hiperbòliques amb i sense vora amb la intenció de construir la maquinària necessària per demostrar l'existència i unicitat de geodèsiques tancades en cada classe d'homotopia via eixos de transformacions hiperbòliques. Finalment, estudiem superfícies conegudes com a pantalons i peces X per tal de demostrar el teorema principal sobre coordenades de Fenchel-Nielsen. A més a més, aquest estudi dóna el bagatge de geometria hiperbòlica necessari per entendre el breu article de recerca que ha nascut d'aquest projecte. Concloem el treball amb aplicacions de les diverses tècniques que s'han introduït, com per exemple el *collar lemma*.

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I would also like to extend my gratitude to my friends, specially to Genís, who has holted me straight many times, made me *laugh* many more and made these last four years *shine* brighter. Also, I would like to thank my girlfriend, for *smiling* at me when sometimes all I could do is frown.

I feel it is only fair to also express my gratitude to the ineffable ways of life, to the things we cannot control nor forsee; to those things that make us stumble and to those that make us walk straight; to the dreams of broken chains and to the weight of walking free; to the beauty of a tainted shadow and to the awe found in a simple thing... Oh I am thankful, in short, for the ways of life, for they have been kind to me, taught me how to *learn* from them and how to find *beauty* in everything.

And finally... to my family, specially my mom, dad and sister, to whom I owe everything: how to dream, laugh and shine, smile, learn and find the beauty around me. Not only was I given life by them, I was shown how to live it.

Introduction

Classification problems crowd the mathematical landscape as a tool for understanding: one has a class of mathematical objects and a notion of sameness between said objects and thus wants a way to distinguish them. For example, take the class of connected compact orientable surfaces, declared the same if they are homeomorphic. Those objects are classified by the genus. That is, the number of "holes" the surface has completely determines it up to homeomorphism. Another example is the category of smooth compact surfaces with diffeomorphisms as the notion of sameness. It is known (see, for example, [Haf13]) that the classification is the same as the topological one, i.e. given two homeomorphic surfaces, they are also diffeomorphic. One may refer to this as "the topological type determines the smooth structure". Those are quite "simple" classifications since the set of equivalence classes is "parametrized" by the natural numbers. If we consider surfaces with complex structures and biholomorphisms[3] the classification is not so simple.

Definition 1 We declare two Riemann surfaces equivalent if they are biholomorphic. Now, fix a topological surface S, we refer to the set of equivalence classes of Riemann surfaces that topologically are S as the Riemann moduli space and write $\mathcal{M}(S)$. This is the same as saying that $\mathcal{M}(S)$ is the set of non-biholomorphic complex structures on S. If S is a connected compact orientable surface of genus g, we write $\mathcal{M}_g$ instead of $\mathcal{M}(S)$ (which makes sense by the topological classification).

To illustrate the point before the definition, consider the complex plane $\mathbb{C}$ and the open unit disc of the complex plane $\mathbb{D}$. Clearly, they are both Riemann surfaces with the identity and inclusion charts respectively. Even though they are homeomorphic they are not biholomorphic. Indeed, if they were, there would be a non-constant holomorphic map $f : \mathbb{C} \longrightarrow \mathbb{D}$ which by Liouville's theorem cannot be bounded, but it is. In fact, the uniformization theorem asserts that all simply connected Riemann surfaces are biholomorphic to $\mathbb{C}$, $\mathbb{D}$ or the Riemann sphere $\widehat{\mathbb{C}}$. This proves that the moduli space of $\mathbb{R}^2$, $\mathcal{M}(\mathbb{R}^2)$, has two elements and the moduli space of the sphere, $\mathcal{M}_0$, has only one. This theorem yields an example of a classification of Riemann surfaces with a fixed topological condition (simple connectivity).

What happens with surfaces with a more interesting topology? For example, the torus $S = \mathbb{T}$ can be regarded as a Riemann surface (that is, it has at least one complex structure) and, at first, it is a surprising fact that there are uncountably many non-equivalent complex structures on it. This is the first example of an uncountable moduli space $\mathcal{M}_1$, which puts forward another quirk of classifications. One does not aim to only tell objects of $\mathcal{M}_1$ apart, one wants that the classification has some kind of geometric meaning. In other words, we do not want to study $\mathcal{M}_1$ as a "point-set" but as a space; we are interested in a notion of closeness between non-biholomorphic complex structures and, ideally, we would like the moduli space to encode geometric information about $\mathbb{T}$. From the getgo, one would like the moduli space to inherit a complex structure. The good news are that in Section 1.2.1 we will see that $\mathcal{M}_1$ is a Riemann surface biholomorphic to the complex plane $\mathbb{C}$.

The not so good news are that the way we describe $\mathcal{M}_1$, as a quotient of the upper half-plane

[3] Those are smooth surfaces regarding the plane as the complex plane so that chart transition maps of the smooth atlass are biholomorphic. A holomorphic map is a map that is holomorphic in local coordinates and a biholomorphism is a bijective holomorphic map with holomorphic inverse. These surfaces are also called Riemann surfaces.

by the modular group, introduces some “singularities” that obstruct good geometric properties. For the case of the torus this problem can be somewhat remedied but when we consider higher genus Riemann surfaces (g-holed torus F_g, $g \geq 1$) the situation is worse. It turns out that the moduli space $\mathcal{M}_g$ can be described by $6g - g$ real parameters but it is not a manifold! The singularities that occur in $\mathcal{M}_1 = \mathbb{C}$ get much more complicated as the genus grows, so much so that it is not locally Euclidean. We discuss this further in Section 1.2.2. The solution is to parametrize a finer structure on Riemann surfaces. Namely:

Definition 2 We declare two Riemann surfaces equivalent if they are biholomorphic *and* that biholomorphism is isotopic to the identity. The set of equivalence classes is called the Teichmüller space and denoted $\mathcal{T}_g$. Though this definition is intuitive, it is not very usable. We usually define a marked Riemann surface as a Riemann surface S equipped with a homeomorphism $\varphi : F_g \to S$ and declare two Riemann surfaces (S, φ) and (R, ψ) equivalent if there is a biholomorphism $h : S \to R$ such that $h \circ \varphi$ is isotopic to ψ. The set of equivalence classes is also $\mathcal{T}_g$ (see Section 1.2.5). Note that with this definition the Teichmüller space is the “moduli space” of *marked* Riemann surfaces.

It turns out that the Teichmüller space does not have the aforementioned singularities and it provides the “best” geometric classification of structures. Algebraic geometers would refer to the Teichmüller space as a fine moduli space and the Riemann moduli space as a coarse moduli space. We discuss this shortly in Section 1.2.2. The study of this spaces is an active multidisciplinary field of research. For example, an important property is that the Teichmüller space of F_g is a $3g - 3$ dimemensional complex manifold. Our main goal, however, is to parametrize $\mathcal{T}_g$ in a real analytic way that is geometrically meaningful, the Fenchel-Nielsen coordinates. Before explaining the bare-bone of this constructions we have to go backwards a little.

As said before, every simply connected Riemann surface is biholomorphic to $\mathbb{C}$, $\mathbb{D}$ or $\widehat{\mathbb{C}}$. This implies that every Riemann surface is (holomorphically) universally covered by one the these surfaces. As a matter of fact, in Section 1.1 we will show that, in the complex category, $\mathbb{C}$ universally covers itself, the punctured plane and the torus; $\widehat{\mathbb{C}}$ can only cover itself; and $\mathbb{D}$ is the universal covering space of all other Riemann surfaces. In particular, every compact Riemann surface of genus $g \geq 2$ is the quotient of $\mathbb{D}$ by a discrete subgroup of automorphisms of $\mathbb{D}$. We will call this subgroup a group model for the corresponding Riemann surface. This builds a bridge from Riemann surfaces to group theory, in Section 1.2.4 we will use this connection to topologize $\mathcal{T}_g$.

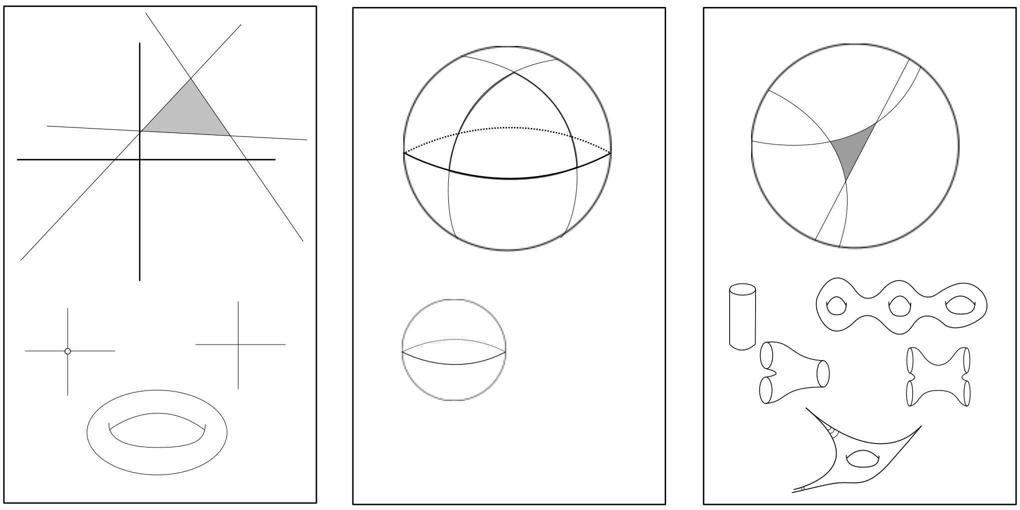

Figure 1: In the left the complex plane and the Riemann surfaces it covers: itself, the punctured plane and tori; In the center the sphere and the Riemann surfaces it covers: only itself; In the right the disc and some Riemann surfaces it covers: all the rest. Geodesic triangles are shown to give the idea of geometrization.

However, the fact above has another interpretation, one that has lead to the most important advances in low-dimensional topology in the last century. The sphere, plane and disc have natural geometries: spherical (elliptic), flat (parabolic) and hyperbolic.[4] This remarkable result enables the geometrization of surfaces, which endows them with a nice overall behaviour. In particular, most Riemann surfaces (those covered by $\mathbb{D}$) can be regarded as hyperbolic 2-manifolds.[5]

This constitutes yet another mathematical bridge for Riemann surfaces universally covered by $\mathbb{D}$: there is a one to one correspondence between biholomorphism classes of complex structures on F and isometry classes of hyperbolic metrics on F. Therefore, we can also regard[6] $\mathcal{M}_g$ as the set of classes of isometric hyperbolic surfaces and $\mathcal{T}_g$ as the set of marked hyperbolic surfaces; here the equivalence is defined in the same way as in Definition 2 switching biholomorphic by isometric. This geometrization procedure and equivalence of structures is explained in Section 2.2.1.

Therefore, our geometric study of $\mathcal{T}_g$ will come considering hyperbolic surfaces. We will prove that $\mathcal{T}_g$ is a real-analytic $(6g-6)$-dimensional ball via Fenchel-Nielsen coordinates, which we now outline. To fix ideas we consider the 3-holed torus. The main idea is to decompose it into 6 pairs of pants by cutting along geodesics like in the figure below.

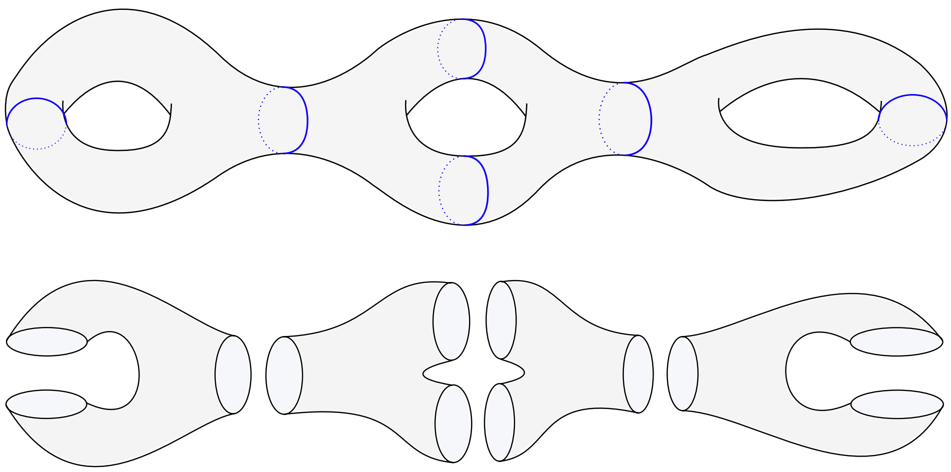

Figure 2: Decomposition into pairs of pants of a genus $g = 3$ compact Riemann surface along the geodesics colored blue.

We denote the length of the six decomposing geodesics by $\ell_1, \ldots, \ell_6$. The wonderful thing is that varying the length of those geodesics we get different hyperbolic surfaces. For example, letting ℓ_1 vary we get a path in $\mathcal{T}_g$. Notice that this provides a good notion of closeness of hyperbolic structures. In the figure below we can see two different points on $\mathcal{T}_3$.

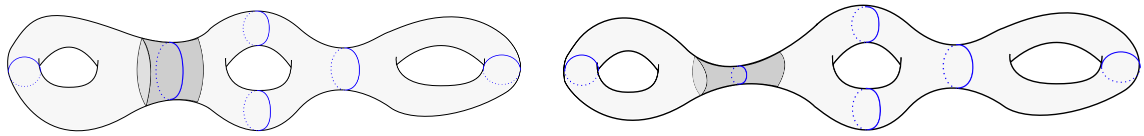

Figure 3: Recall that a point on $\mathcal{T}_3$ is an equivalence class of marked Riemann surfaces of genus g. This Figure shows two different points on $\mathcal{T}_g$: the geodesics in the gray collar have different widths.

One may be wondering what does it mean to vary the length of a geodesic, if tweaking the length changes the hyperbolic metric, shouldn't the geodesics of this new hyperbolic surface be different? This is a good question and its answer is what enables this construction to work: in Section 2.2.4 we

[4]For this work it is not necessary that the reader is previously acquainted with hyperbolic geometry. In fact, we will introduce it as "the geometry" that is "compatible" with the complex structure of $\mathbb{D}$.

[5]This is a Riemannian 2-manifold with constant sectional curvature -1. It is a standard result of differential geometry that this is equivalent to being locally isometric to $\mathbb{D}$ with the Poincaré metric.

[6]Although we will not use this fact in this work, the moduli space also parametrizes the isomorphism classes of smooth complex algebraic structures on F making it a very important object in Algebraic Geometry. Also, it parametrizes classes of conformal metrics on smooth surfaces as well, which makes it a hugely multidisciplinary object.

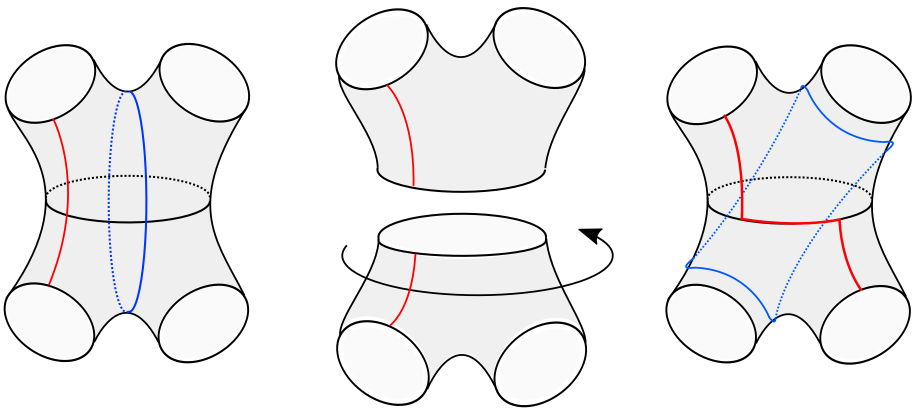

Figure 4: Pasting two different ways, no twist (left) and a twist of a quarter turn (right). The curves hint at how the hyperbolic metric may change.

show that in the homotopy class of every closed curve there is a unique geodesic representative for each hyperbolic metric. This allows us to draw the surface decomposition above for any hyperbolic metric on a fixed topological model.

Having said this, it makes sense that increasing those lengths the marked hyperbolic structure changes since we are increasing distances. In this heuristic spirit, one may wonder what happens if we twist one of the sleeves before pasting them together? Intuitively this should change the lengths of curves in the resulting surface, should it not? and if so, should it not change the hyperbolic marking? See Figure 4.

Indeed, before re-assembling the pairs of pants as in Figure 2, for each of the six decomposing geodesics we can twist before pasting, this introduces six twist parameters $\theta_1, \ldots, \theta_6$. The Fenchel-Nielsen coordinates are given by the length parameters and the twist parameters. Our main goal is to build the necessary techniques to define all this rigorously and prove the following theorem, which we do along Section 2.3 culminating in Subsection 2.3.5.

Theorem *The Fenchel-Nielsen coordinates FN as defined above parametrize $\mathcal{T}_g$ bijectively and homeomorphically:*

$$\begin{aligned} FN : \mathcal{T}_g \longrightarrow &\quad \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \\ S \longmapsto &\ (\ell_1(S), \ldots, \ell_{3g-3}(S), \theta_1(S), \ldots, \theta_{3g-g}(S)) \end{aligned}$$

Also, there are $9g-9$ homotopy classes of curves such that their lengths (or, more precisely, the lengths of their unique geodesic representatives) uniquely determine the point of $\mathcal{T}_g$. This determines an injection $\iota : \mathcal{T}_g \longrightarrow \mathbb{R}_+^{9g-9}$ given by said lengths which, in fact, is a topological embedding.

In Figure 5 we show 9 (homotopy classes of) curves that determine the Teichmüller space of the genus two compact hyperbolic surfaces. Parametrizing $\mathcal{T}_g$ only by length of curves (avoiding twist parameters) is in line with the good "moduli" philosophy, since we have obtained a geometric classification using objects that are intrinsically geometric, namely, lengths of geodesics. In fact, our proof consists of obtaining the injection ι first and then determining twist parameters from lengths of curves to obtain the map FN.

These coordinates are important for other reasons besides the geometric heuristics of the Teichmüller space. For example, the mapping class group of a surface (the group of classes of homeomorphisms modulo isotopy) is a fundamental topological object of great interest, specially in low-dimensional topology. It turns out that it acts naturally on the Teichmüller space and the quotient by this action is the moduli space. Therefore, the study of $\mathcal{T}_g$ and the mapping class group help us study the moduli space. In fact, it turns out that the action is properly discontinuous (this implies that we can push-forward a structure on $\mathcal{T}_g$ to $\mathcal{M}_g$) even though it is not free (it acts with fixed points, causing

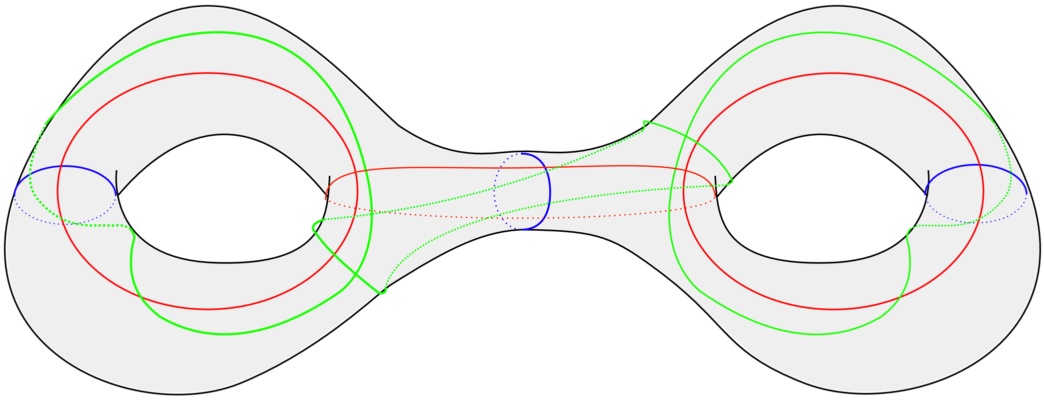

Figure 5: The 9 homotopy class of curves such that their lengths uniquely determine the Teichmüller space.

the aforementioned singularities). One way to prove this fact and actually give a sort of fundamental domain for $\mathcal{M}_g$ is by using the Fenchel-Nielsen coordinates.

The Fenchel-Nielsen parametrization also works for non-compact hyperbolic surfaces (we are allowed to remove discs and add punctures). This makes it ideal in a broad spectrum of topics which we only mention in passing. For example, the Fenchel-Nielsen coordinates define a natural symplectic form[7] $\frac{1}{2}\sum_i d\ell_i \wedge d\theta_i$ on $\mathcal{T}_g$ that is invariant under the action of the mapping class group, thus defining a symplectic form on $\mathcal{M}_g$, allowing us to compute volumes on the moduli space. Another fruitful application comes from the fact that one can compactify (not at all trivially) the space $\mathcal{T}_g$ so that it is homeomorphic to a $(6g-6)$-dimensional closed ball. This yields Thurston's celebrated classification of surface homeomorphism via the study of the fixed points of the action of the the mapping class group on the boundary of the ball.

Teichmüller theory is a vast and exciting field of study with connection to many branches of mathematics. In this survey we hope to introduce the reader to Riemann surfaces and the intuition behind moduli spaces, study the basic properties of the geometry of hyperbolic surfaces and give a full introduction to Fenchel-Nielsen coordinates seeking to hint at all of those rich connections.

Some outcomes of this project: Finally, we mention a couple of academic outcomes that have grown out of this project (other than this main expository text, of course). In the first place and most importantly, in [Pen88], R.C. Penner found conformal coordinates to the Teichmüller space of pairs of pants and used it to solved an analogue of the classical Schottky problem on this surface. This paper has been thoroughly studied (and hence references like [AS60], [Ahl78] and [Ahl73] have been examined) and two big natural questions have arisen: Can this new coordinates and approach to the Schottky problem be generalized to X-pieces and even to more general surfaces (which is a matter raised by the author himself)? Also, what is the relationship with the usual hyperbolic coordinates of the Teichmüller space of pairs of pants (which is the one shown in Section 2.3.2) and Penner's conformal coordinates? These questions are being explored, the latter is already answered and the former is still ongoing. As a proof of progress, a preliminary written account of this research has been handed in alongside this document. Secondly, a longer expository text covering some more of the subjects studied during this project has been written [Bar21]. Most of the proofs that have been omitted here due to space considerations can be found in said extended version.

[7] Surprisingly, this does not depend on the pairs of pants decomposition, Wolpert showed that this form is the so-called Weil-Peterson symplectic form, hence intrinsic, see [Wol10, Theorem 3.14]

Chapter 1

Riemann Surfaces

In this chapter we particularize the theory of topological covering spaces to holomorphic coverings, characterize the universal coverings of all Riemann surfaces and briefly explore Fuchsian groups. Then, we will be ready to present moduli and Teichmüller spaces and, amongst other things, compute those spaces for the torus, make dimensional counts and introduce a topology on the Teichmüller space.

References: For a brief introduction to Riemann surfaces we refer the reader to an extended version of this work [Bar21], for a full introduction we recommend [Mir95] or [For81]. Section 1.1 mainly draws from [IT92], [JS87], [Car92]; finally, Section 1.2 combines the approach of [IT92] with the approach [FM12], with some specific things taken from [Bus10], [Mar16], [Ji15], [Ji18] and [Ben].

1.1 Universal Covering and Group Models of Riemann Surfaces

The crux of this section is that the characterization of universal coverings in Corollary 1.1.2.7 provides us with a group-theoretic view of Riemann surfaces helpful to study moduli and Teichmüller spaces; and it unveils a way to find natural geometries on Riemann surfaces that will be needed in chapter 2.

1.1.1 Holomorphic Theory of Covering Spaces

We assume that the reader is familiar with the classical theory of topological covering spaces that can be found, for example, in [Lee11, Chapters 11 and 12] and [Mas91, Chapter 5]. This sections serves to establish some conventions about covering spaces and, most importantly, to make the necessary remarks to adapt this theory to the context of Riemann surfaces. Proofs in this section are omitted and we refer the interested reader to the extended version of this work [Bar21].

Notation 1.1.1.1. Let X and $\tilde{X}$ be a topological spaces and $p : \tilde{X} \to X$ a continuous map. The fiber of a set $S \subset X$ by p is the preimage of this set, denoted $\mathrm{Fib}_p(S) := p^{-1}(S)$, when no confusion may arise we simply write $\mathrm{Fib}(S)$. An open set $U \subset X$ is evenly covered by p if its fiber is a disjoint union of connected open sets, each of which is mapped homeomorphically onto U by p. We say that $(X, p : \tilde{X} \to X)$ is a covering space of X and p is a covering map if p is surjective, every point of X has an evenly covered neighbourhood by p and $\tilde{X}$ is connected and locally-path connected (which, in turn, implies X is locally and globally path-connected).

Notation 1.1.1.2. If Y is another topological space and $f : Y \to X$ any continuous map, a lift of f is a continuous map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$. Let $(\tilde{X}', p')$ be another covering space of X, a continuous map $f : \tilde{X} \to \tilde{X}'$ is said to be fiber-preserving or a covering homomorphism if $p' \circ f = p$. A covering isomorphism is a covering transformation with an inverse that is also a covering transformation (which is the same as saying a fiber-preserving homeomorphism). In the case the covering homomorphism $f : \tilde{X} \to \tilde{X}$ is a self map and a homeomorphism it is said to be a covering transformation or a deck transformation. The group (under composition) of covering transformations associated to the covering $(\tilde{X}, p)$ of X is denoted $\text{Deck}(\tilde{X}, p)$ and called deck group or covering transformations group. We say that a covering space $(\tilde{X}, p)$ is regular (also known as normal or Galois) if its deck group acts transitively on each fiber, that is, for every $q \in X$ and any $\tilde{q}, \tilde{q}' \in \text{Fib}(q) = p^{-1}(q)$ there is a covering transformation f such that $f(\tilde{q}) = \tilde{q}'$.

Notation 1.1.1.3. Finally, the action of a group G on X is said to be a covering space action if for every $q \in X$ there is a neighbourhood $q \in U \subset X$ such that $g \cdot U \cap U = \varnothing$ for every $g \in G \setminus \{Id\}$.[1]

Proposition 1.1.1.4. *Let M be a n-manifold and $(\tilde{M}, p : \tilde{M} \to M)$ a covering space of M. Then, $\tilde{M}$ is a n-manifold.*

Proposition 1.1.1.5. *Let $\tilde{S}, S$ and R be Riemann surfaces such that $p : \tilde{S} \to S$ is a holomorphic covering map and $f : R \to S$ is holomorphic. Then, any lift $\tilde{f} : R \to \tilde{S}$ of f is holomorphic. In particular, the deck group is a subgroup of the group of biholomorphic automorphisms of $\tilde{S}$.*

Theorem 1.1.1.6 (Pull-back Structure). *Let S be a Riemann surface covered by a (topological) surface $\tilde{S}$ and $p : \tilde{S} \to S$ be the covering map. Then, there is a unique complex structure on $\tilde{S}$ such that p is holomorphic.*[2]

Example 1.1.1.7. From topology it is known that the torus is homeomorphic to the orbit space $\mathbb{R}^2/\mathbb{Z}^2$. A group of the form $z_1\mathbb{Z} \times z_2\mathbb{Z}$ with $z_1, z_2 \in \mathbb{C}$ not zero and linearly independent over $\mathbb{R}$ is called a lattice. It is known that all quotients of $\mathbb{R}^2$ by lattices are tori. Figure 1.1 shows the identifications being made for $\mathbb{Z} \times w\mathbb{Z}$ with $w = e^{i\pi/3}$. From this, one can readily define a complex atlas on $\mathbb{C}/\mathbb{Z}^2$, the figure highlights how one would do it. In contrast to the fact that all lattices give rise to homeomorphic tori, not all yield biholomorphic tori. We will investigate this further in Section 1.2. Lastly, in Section 1.1.2 we will see that all complex structures on a torus come from this quotienting procedure.

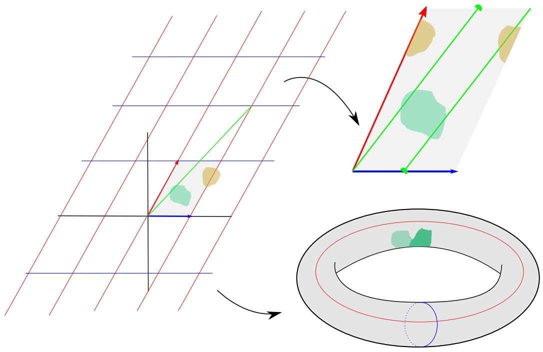

Figure 1.1: Two possible charts are highlighted in green and orange. The green line and the orange region show how identifications would be made in the quadrilateral, just like it is done in point-set topology. The red and blue loops on the torus represent the red and blue arrows in $\mathbb{C}$.

The following lemma is a generalization of the procedure indicated in the example above.

Lemma 1.1.1.8 (Push-Forward Structure). *If G acts on M by biholomorphisms and as a covering space action, the quotient M/G inherits a complex structure such that the projection $\pi : M \to M/G$ is a holomorphic covering map.*[3]

Theorem 1.1.1.9. *If $(\tilde{S}, p : \tilde{S} \to S)$ is a regular covering surface of the Riemann surface S and G is its deck group, then the map $z \mapsto [\tilde{z}]$ with $\tilde{z} \in \text{Fib}(z)$ from S to $\tilde{S}/G$ is a homeomorphism. Considering $\tilde{S}$ with the*

[1] This condition is sometimes referred as proper discontinuity but the author prefers to follow Massey's and Lee's much less ambiguous convention of calling it a covering space action.

[2] The classical theory of covering spaces, under weak conditions on the base topological space X, establishes an order-reversing bijective correspondence between $\mathcal{CS}(X)$ (the set of covering spaces of X modulo covering isomorphism) and the conjugacy classes of the fundamental group of X at a point. If X is a Riemann surface, the same statement is true in the complex category (switching the word covering space by covering surface and asking maps to be holomorphic). This follows directly from Propositions 1.1.1.4 and 1.1.1.5 and Theorem 1.1.1.6.

[3] It could happen that M/G was not Hausdorff, it would not be a manifold but it would have a complex structure. To avoid this, a "properness" condition on the action is usually considered.

pull-back structure of the previous theorem, G acts by biholomorphisms and $\tilde{S}/G$ inherits a complex structure from $\tilde{S}$ such that $\tilde{S}/G$ is biholomorphic to S.

Remark 1.1.1.10. Note that we have seen two methods to transfer a complex structure: the pull-back and push-forward structures. For example, Theorem 1.1.1.9 says that pulling back the complex structure of S to $\tilde{S}$ and then pushing forward to the quotient space $\tilde{S}/G$ produces a complex structure equivalent to the original one on S. An important remark about the push-forward structure lemma is that it is valid in the smooth category: if the structure at hand were smooth (as opposed to complex) and the group acted by diffeomorphisms, the quotient would inherit a smooth atlas

Since universal coverings are regular, the universal covering of every Riemann surface S is a simply connected *Riemann surface*, therefore, by the uniformization theorem (see [Hub06, Chapter 1]), it is either $\tilde{\mathbb{C}}, \mathbb{C}$ or $\mathbb{H}$ (or $\mathbb{D}$, since $\mathbb{H}$ and $\mathbb{D}$ are biholomorphic). From this, every Riemann surface will be biholomorphic to a quotient by a subgroup G of automorphisms of one of these spaces (where G is the covering transformations group). Now we proceed to study those groups closely.

Definition 1.1.1.11. If a Riemann surface S is biholomorphic to $\mathbb{H}/G$, $\mathbb{C}/G$ or $\widehat{\mathbb{C}}/G$ we will say that G is a group model for S.

Remark 1.1.1.12. Let S be a connected surface and $p : \tilde{S} \to S$ its universal covering. For the reader's convenience, we recall the identifications that make the fundamental group of S isomorphic to the covering transformations group G. Let $q \in S$ be a point on the base surface, on the one hand, given two points in the $\mathrm{Fib}(q) = p^{-1}(q)$ there is a unique element of G that takes one to the other. On the other hand, given a loop γ at q and a point $\tilde{q} \in \mathrm{Fib}(q) = p^{-1}(q)$, γ lifts to a unique path $\tilde{\gamma}$ at $\tilde{q}$ (that is, beginning at $\tilde{q}$) in $\tilde{S}$. The monodromy lemma asserts that the endpoint of this $\tilde{\gamma}$ does not depend on the path-homotopy class of γ. Then, for any $q \in S$ and $\tilde{q} \in p^{-1}(q)$, the following correspondence is a group isomorphism: to each element of $\pi(S, q)$ corresponds the element $g \in G$ that sends $\tilde{q}$ to the endpoint of the lift of γ at $\tilde{q}$. For details see the chapters about covering spaces in [Mas91], [Jän84] or [Lee11].

1.1.2 Characterization of Universal Coverings

Definition 1.1.2.1. A Riemann surface that is biholomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or a torus is said to be of exceptional type.

The following theorem asserts that the only way the Riemann sphere holomorphically covers a Riemann surface is if the surface itself is the Riemann sphere. After that, Theorem 1.1.2.4 characterizes the types of surfaces holomorphically covered by the complex plane, which are the other surfaces of exceptional type. Therefore, all other Riemann surfaces will have to be universally covered by $\mathbb{H}$. The general idea of the proofs in this section is the following: the automorphism groups of $\widehat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{H}$ are known[4] and the covering transformation group of a given covering is a subgroup of one of these groups. However, it may not be *any* subgroup, it must be isomorphic to a fundamental group of a Riemann surface (because of Remark 1.1.1.12). This algebraico-topological condition is enough to discern between the three possible universal covering spaces a Riemann surface admits.

Theorem 1.1.2.2. *A Riemann surface R has a universal covering surface $\tilde{R}$ biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$ if and only if R is itself biholomorphic to $\hat{\mathbb{C}}$.*

Proof. Given that $\tilde{R}$ is biholomorphic to $\hat{\mathbb{C}}$, we can assume that $(\hat{\mathbb{C}}, p)$ is *the* universal covering surface of R. Because the covering transformations group $\mathrm{Deck}(\hat{\mathbb{C}}, p) \subset \mathrm{Aut}(\hat{\mathbb{C}})$, every covering transformation is a Möbius transformation, in particular, it should have fixed points. However, non-identity covering transformations don't have fixed points, this implies $\mathrm{Deck}(\hat{\mathbb{C}}, p) = \{Id\}$. Finally, by Theorem 1.1.1.9 R is biholomorphic to $\tilde{R}/\,\mathrm{Deck}(\hat{\mathbb{C}}, p) = \tilde{R}/\{Id\} \cong \tilde{R}$. The converse implication is obvious; $(\hat{\mathbb{C}}, Id)$ is a universal covering surface for $\hat{\mathbb{C}}$. □

[4] We strongly recommend that the reader reads the appendix about Möbius transformation up to Theorem 4. A working knowledge on the content in the appendix will suffice for our purposes but, if the reader is interested, she can find the proofs in the extended version of this project [Bar21].

Lemma 1.1.2.3 (Characterization of Discrete $\mathbb{Z}$-modules in $\mathbb{C}$)**.** *A discrete $\mathbb{Z}$-module $M \subset \mathbb{C}$ is either trivial or the integral multiples of a non-zero complex number w_1 or all linear combinations $n_1 w_1 + n_2 w_2$ with integral coefficients $n_1, n_2 \in \mathbb{Z}$ of two non-zero $\mathbb{R}$-linearly independent complex numbers w_1, w_2.*

Proof. This is a standard fact in group theory about free abelian groups of rank $r \leq n$ in $\mathbb{R}^n$. For an elementary proof using only complex arithmetic see the extended version [Bar21]. □

Theorem 1.1.2.4 (Characterization of the Universal Covering of the Plane, the Punctured Plane and Tori)**.** *A Riemann surface R has a universal covering surface $\tilde{R}$ biholomorphic to the complex plane $\mathbb{C}$ if and only if R is biholomorphic to either one of the follwing: $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or a torus.*

Proof. We can assume $\tilde{R} = \mathbb{C}$ and let $\Gamma = \text{Deck}(\tilde{R}, p)$ be the deck group of the covering. Every element f of Γ can be written as $f(z) = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$ because $\Gamma \subset \text{Aut}(\mathbb{C})$. Being a covering transformations it has no fixed point on $\mathbb{C}$, thus $a = 1$. Hence, we can identify Γ with $\mathbb{Z}$-module of $\mathbb{C}$ (by $z + b \longleftrightarrow b$). Recall that Γ acts on $\mathbb{C}$ as a covering space action. In particular, it is direct that points cannot accumulate in Γ. This allows us to think about Γ as a discrete module over the integers and, therefore by the lemma above, one of the following possibilities occurs:

(i) $\Gamma = \{Id\}$.

(ii) $\Gamma = < f_0 >$ with $f_0(z) = z + b$, for some $b \in \mathbb{C}^*$.

(iii) $\Gamma = < f_0, f_1 >$ with $f_0(z) = z + b_0$, $f_1(z) = z + b_1$, b_0, b_1 non-zero complex numbers that are linearly independent over $\mathbb{R}$.

For the first case, $\tilde{R}/\Gamma \cong \mathbb{C}$ which is biholomorphic to R. For the second, the orbit space $\mathbb{C}/\Gamma$ is actually $\mathbb{C}/b\mathbb{Z}$ which is biholomorphic to $\mathbb{C}^*$. For this, consider continuous homomorphism of topological groups $e : \mathbb{C} \to \mathbb{C}^*$ defined by $e(z) = e^{2\pi i z/b}$. As groups, we have that $\mathbb{C}^*$ is isomorphic to $\mathbb{C}/ker(e) = \mathbb{C}/b\mathbb{Z}$, call such isomorphism E. We have that $e = E \circ p$. Recall that p is holomorphic and a local homeomorphism, we have that *locally* $E = e \circ p^{-1}$, which is locally holomorphic. E is locally holomorphic, thus holomorphic, and bijective, thus biholomorphic. Finally, for the third possibility, Γ is a lattice group and, hence, $\mathbb{C}/\Gamma$ is a torus.

Conversely, $(\mathbb{C}, Id)$ and $(\mathbb{C}, e^z)$ clearly are the universal covering spaces of $\mathbb{C}$ and $\mathbb{C}^*$. Let us now take a torus $\mathbb{S}^1 \times \mathbb{S}^1$ with a complex structure. The map $p : \mathbb{C} \to \mathbb{S}^1 \times \mathbb{S}^1$ defined by $p(a + ib) = (e^{2\pi i a}, e^{2\pi i b})$ is a covering map, we know there will be a unique complex structure on $\mathbb{C}$ such that p is holomorphic. Therefore, this torus is either universally covered by $\mathbb{C}$ or by $\mathbb{H}$. We want to show that it cannot be $\mathbb{H}$. If it was, the deck group of the covering would be a subgroup of $\text{Aut}(\mathbb{H})$ isomorphic to the fundamental group of the torus, which is $\mathbb{Z} \times \mathbb{Z}$. The following lemma tells us that this is not possible and hence we conclude that the torus is holomorphically covered by $\mathbb{C}$, as we wanted. □

Lemma 1.1.2.5. *Let Γ be a subgroup of $\text{Aut}(\mathbb{H})$ acting on $\mathbb{H}$ as a covering space action. If Γ is abelian, then it is cyclic.*

Proof. Assume that Γ is not the trivial group. Let $f \in \Gamma \setminus \{Id\}$, since by hypothesis it has no fixed points on $\mathbb{H}$, it should be parabolic or hyperbolic. If it is parabolic, via conjugation, we can assume that $f(z) = z + b_0$, $b_0 \in \mathbb{R} \setminus \{0\}$. An element of $\text{Aut}(\mathbb{H})$ commutes with f if and only if it is of the form $z + b$ with $b \in \mathbb{R}$. This comes from a simple computation imposing commutativity because Γ is abelian. We have thus found that Γ is an additive subgroup of the real numbers (by identifying $z + b \longleftrightarrow b$). Moreover, the hypothesis also implies discreteness. It is well know that all such subgroups are infinite cyclic.

On the other hand, if f is hyperbolic we can assume it takes the form $\lambda_0 z$ ($\lambda_0 > 0$). An element of $\text{Aut}(\mathbb{H})$ commutes with f if and only if it is of the form λz, $\lambda \in \mathbb{R}^{>0}$. Similarly as before, we have found that Γ is a multiplicative discrete subgroup of $\mathbb{R}^*$ by identifying $\lambda z \longleftrightarrow \lambda$. All such subgroups are infinite cyclic.[5] □

[5] This follows from the fact that a discrete subgroup of $(\mathbb{R}, +)$ is infinite cyclic and that the map e^x is a group monomorphism of $(\mathbb{R}, +)$ into $(\mathbb{R}^*, \cdot)$, which implies that discrete subgroup of $(\mathbb{R}^*, \cdot)$ it is also so of $(\mathbb{R}, +)$.

Corollary 1.1.2.6 (Tori as Quotients of Lattices)**.** *For every Riemann surface R of genus 1 there exists a lattice group Γ such that R is biholomorphic to $\mathbb{C}/\Gamma$.*

Corollary 1.1.2.7 (Characterization Theorem of the Universal Coverings of Riemann Surfaces)**.** *A Riemann surface has $\mathbb{H}$ as holomorphic universal covering if and only if it is not of exceptional type. The complex plane $\mathbb{C}$ universally covers and can only cover the surfaces of exceptional type, except $\hat{\mathbb{C}}$ which is covered and only covers itself.*

Remark 1.1.2.8. Since $\mathbb{C}$, $\widehat{\mathbb{C}}$ and $\mathbb{H}$ are natural models for flat, elliptic and hyperbolic geometry, we can push forward the Riemannian metric to the surfaces they cover. This provides extra structure that helps to unveil purely topological facts about surfaces, the next chapter is completely devoted to the study of the hyperbolic case.

1.1.2.1 Fuchsian Groups

In this section we will state some important facts about the theory of Fuchsian groups and Riemann surfaces. As seen in the previous sections, most Riemann surfaces are a quotient of $\mathbb{H}$ by a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. This opens the door to the study these subgroups and the exploration of Riemann surfaces via their group models (see).

Definition 1.1.2.9. Recall that $\mathrm{PSL}(2,\mathbb{R})$ is a Lie group with the topology induced by the identification of the 2×2 real matrices with $\mathbb{R}^4$. We say that a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group.

The following theorem serves as a converse to the theorems in the last section and describes a fruitfull connection between Riemann surfaces and group theory: biholomorphic Riemann surfaces (not of exceptional type) have conjugate group models (see Definition) and, surprisingly, the converse is also true:

Theorem 1.1.2.10. *Let Γ be a Fuchsian group without elliptic elements. Then, $\mathbb{H}/\Gamma$ is a connected Riemann surface and the projection $\mathbb{H}\to\mathbb{H}/\Gamma$ is a covering. Moreover, if Γ_1 and Γ_2 are Fuchsian groups without elliptic elements, $\mathbb{H}/\Gamma_1$ is biholomorphic to $\mathbb{H}/\Gamma_2$ if and only if Γ_1 and Γ_2 are conjugate by an element of* $\mathrm{PSL}(2,\mathbb{R})$*. This two facts together with the theorems in the last section give the following bijective correspondence:*

$$\left\{\begin{array}{l}\textit{Riemann surfaces not}\\ \textit{of exceptional type}\\ \textit{modulo biholomorphism}\end{array}\right\}\longleftrightarrow\left\{\begin{array}{l}\textit{Conjugacy classes of}\\ \textit{elliptic-free discrete}\\ \textit{subgroups of } \mathrm{PSL}(2,\mathbb{R})\end{array}\right\}$$

Proof. In the first place, connectedness comes from $\mathbb{H}$ being connected and the being Hausdorff follows from discreteness. The remark below, discreteness and Lemma 8 imply that the quotient has a complex structure such that the projection map is a covering. This proves the first statement. For the second, assume that Γ_1 and Γ_2 are conjugate by $g\in\mathrm{PSL}(2,\mathbb{R})$, that is, $g\Gamma_1 g^{-1}=\Gamma_2$, the map $[z]_{\Gamma_1}\mapsto [g(z)]_{\Gamma_2}$ is a biholomorphism from $\mathbb{H}/\Gamma_1$ to $\mathbb{H}/\Gamma_2$. Conversely, suppose that $H:\mathbb{H}/\Gamma_1\to\mathbb{H}/\Gamma_2$ is a biholomorphism, then it lifts to a biholomorphism $h:\mathbb{H}\to\mathbb{H}$ such that $H([z]_{\Gamma_1})=[h(z)]_{\Gamma_2}$. If $g\in\Gamma_1$, then $[z]_{\Gamma_1}=[g(z)]_{\Gamma_1}$, applying H we have that $[hg(z)]_{\Gamma_2}=[h(z)]_{\Gamma_2}$. This, in turn, implies that there is a $g'\in\Gamma_2$ such that $hg(z)=g'h(z)$ for all $z\in\mathbb{H}$ and so, $hgh^{-1}=g'\in\Gamma_2$. That is, $h\Gamma_1 h^{-1}\subseteq\Gamma_2$. The same argument applies to H^{-1} to get $h^{-1}\Gamma_2 h\subseteq\Gamma_1$. We have proved that $h\Gamma_1 h^{-1}=\Gamma_2$, like we wanted to show. ☐

When Fuchsian group Γ contains elliptic elements it cannot act on $\mathbb{H}$ as a covering space action since covering transformations act without fix points. Then, it is not obvious how one can give a complex structure to the topological quotient $\mathbb{H}/\Gamma$ since Lemma 8 no longer applies here. This makes the following theorem all the more surprising.

Theorem 1.1.2.11. *Let Γ be a Fuchsian group, then $\mathbb{H}/\Gamma$ is a connected Riemann surface and the projection $\mathbb{H}\to\mathbb{H}/\Gamma$ is holomorphic (but not necessarily a covering).*

Proof. That Γ is discrete is equivalent to the following properness property: for every $z\in\mathbb{H}$ there exists a vicinity V of z such that for all $g\in\Gamma$ if $g(V)\cap V\neq\varnothing$ then $g(z)=z$. This is not at all obvious and it does not happen in other more general contexts. Using this equivalence one can construct an atlas around the possible fixed point of elements of Γ. For an exploration of this topic see [JS87], in particular Theorem 5.9.1. □

We mention only in passing that the study of Fuchsian groups together with their fundamental domains can be used to prove a very important theorem of Hurwitz: $|\operatorname{Aut}(S)| \leq 84(g-1)$ for a compact Riemann surface S of genus $g \geq 2$. See [JS87] for this approach.

1.2 Moduli and Teichmüller Spaces

This section introduces moduli and Teichmüller spaces for compact Riemann surfaces. To fix ideas and intuition we compute the moduli space of the torus and ponder on weather this provides a good geometrical classification of complex structures. We use this to motivate the introduction of Teichmüller spaces. Then, our goal will be to introduce the so-called algebraic topology on Teichmüller spaces. In order to do this, we introduce fundamental group markings to develop intuition and to compute the Teichmüller space of the torus. The algebraic topology will topologize the Teichmüller space in a way that is coherent with our work with group models as well as allowing us to have a preliminary count of its dimension. After that, we will introduce different (and equivalent) way of marking Riemann surfaces, which is the one we will use in the rest of this text. Finally, we will define the mapping class group and explain how it acts on the Teichmüller space to produce the moduli space.

The Riemann moduli space of a topological surface is the set of non-equivalent complex structures that the surface admits. Since only homeomorphic surfaces can have biholomorphic complex structures and topological surfaces are classified by the genus, we will talk about the moduli space of a compact topological surface of fixed genus.

Definition 1.2.0.1. Throughout this text we will write F_g for a fixed an orientable closed surface of genus g. For surfaces S and R homeomorphic to F_g, let $(S, \mathcal{A}_1)$ and $(R, \mathcal{A}_2)$ be two complex structures (or maximal atlases), we say they are equivalent if there exists a biholomorphism $h : (S, \mathcal{A}_1) \to (R, \mathcal{A}_2)$. The set of complex structures modulo this equivalence is the moduli space of F_g, denoted $\mathcal{M}_g$.

An immediate consequence of our previous efforts in the last section yields the following characterization.

Proposition 1.2.0.2 (Group Model Description of the Moduli Space)**.** *For $g \geq 2$, there is the following bijective correspondence:*

$$\left\{\begin{matrix}\textit{Conjugacy classes of elliptic-free discrete sub-}\\ \textit{-groups of } \operatorname{Aut}(\mathbb{H}) \textit{ that admit the presentation}\\ \Gamma = \langle a_1, b_1, \dots, a_g, b_g \,|\, \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle\end{matrix}\right\} \longleftrightarrow \mathcal{M}_g$$

Proof. This is a restatement of Theorem 1.1.2.10 for details see the extendedv version [Bar21]. □

In the following section we study the $g = 1$ case. We find result similar result to the proposition above that allows us to describe $\mathcal{M}_1$ explicitly.

1.2.1 Moduli Space of the Torus $\mathcal{M}_1$

We will now see that for each element of the genus one moduli space we can take a representative determined by a $\tau \in \mathbb{H}$.

Lemma 1.2.1.1. *Every torus is biholomorphic to a torus of the form $\mathbb{C}/\Gamma_\tau$, with Γ_τ a lattice group with basis 1 and τ so that $Im(\tau) > 0$*

Proof. Recall that every (complex) torus can be written as a quotient Riemann surface $\mathbb{C}/\Gamma$ where Γ is a lattice group of $\mathbb{C}$. Moreover, if w_1 and w_2 generate Γ, the map h_i defined as $z \mapsto z/w_i$ transforms $\mathbb{C}$ biholomorphically onto itself and transforms Γ in a lattice group with basis 1 and τ. For either h_1 or h_2 we have that $Im(\tau) > 0$. The map h induces a bijective holomorphic map when descending to the quotients. □

We will write $\mathbb{T}_\tau$ for $\mathbb{C}/\Gamma_\tau$ and p_τ for the projection or covering map $p_\tau : \mathbb{C} \longrightarrow \mathbb{T}_\tau$. The following lemma characterizes holomorphic maps between tori.

Lemma 1.2.1.2. *Let $f : \mathbb{T}_{\tau'} \longrightarrow \mathbb{T}_\tau$ be holomorphic. Then, there exists a holomorphic map $\tilde{f} : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\tilde{f}(0) = 0$ and $f([z]) = [\tilde{f}(z)]$.*[6] *If f is a biholomorphism, then $\tilde{f}(z) = \alpha z$ for $\alpha \in \mathbb{C}^*$.*

Proof. Consider the map $f \circ p_{\tau'} : \mathbb{C} \longrightarrow \mathbb{T}_\tau$ and note that $p_\tau(0) = [0]$. By the lifting criterion of covering maps, there exists a unique lifting $\tilde{f}$ of $f \circ p_{\tau'}$ that sends 0 to 0. Moreover, liftings of holomorphic maps are holomorphic. Finally, if f is a biholomorphism then so is $\tilde{f}$, i.e. $\tilde{f} \in \mathrm{Aut}(\mathbb{C})$. This implies that $\tilde{f}(z) = \alpha z + \beta$, since 0 has to be a fixed point, we get that $\tilde{f}(z) = \alpha z$ (with $\alpha \in \mathbb{C}^*$). □

Theorem 1.2.1.3. *For any two τ and τ' in $\mathbb{H}$, two tori $\mathbb{T}_{\tau'}, \mathbb{T}_\tau$ are biholomorphically equivalent if and only if τ and τ' are related by a unimodular transformation, that is, there exists an $h \in \mathrm{PSL}(2,\mathbb{Z})$ such that $h(\tau) = \tau'$*

Proof. Let us assume that $f : \mathbb{T}_{\tau'} \longrightarrow \mathbb{T}_\tau$ is a biholomorphism and that $\tilde{f}$ is the lifting given by the lemma above, we have $f([z]) = [\alpha z]$. Moreover, note that the images of $1, \tau'$ by $\tilde{f}$ are congruent to 0 in Γ_τ. This is indeed true: $[\tilde{f}(1)] = f([1]) = f([0]) = [\tilde{f}(0)] = [0]$ and $[\tilde{f}(\tau')] = f([\tau']) = f([0]) = [\tilde{f}(0)] = [0]$ given that $[1] = [\tau'] = [0]$ (they are the vertices of the fundamental parallelogram of the torus and hence they identified as a single point). This produces the following system:

$$\tilde{f}(\tau') = \alpha\tau' = a\tau + b$$
$$\tilde{f}(1) = \alpha = c\tau + d$$

for a, b, c, d integers. Therefore, $\tau' = \frac{a\tau+b}{c\tau+d}$. Applying the same argument to f^{-1} yields $\tau = \frac{a'\tau'+b'}{c'\tau'+d'}$ for a', b', c', d' integers. By computing $f^{-1} \circ f(1) = 1$ and $f^{-1} \circ f(\tau') = \tau'$ we have that $ad - bc = \pm 1$. Finally, since $Im(\tau') = \frac{ad-bc}{|c\tau+d|^2} Im(\tau) > 0$, we can conclude that $ad - bc = 1$.

Conversely, if we have that $\tau' = \frac{a\tau+b}{c\tau+d}$ for integers a, b, c, d and $ad - bc = 1$, then the biholomorphism $\mathbb{T}_\tau \to \mathbb{T}_{\tau'}$ is given by $f([z]) = [(c\tau + d)z]$. □

Corollary 1.2.1.4. *The moduli space of the torus is the Riemann surface $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$, which is biholomorphic to $\mathbb{C}$.*

Proof. By the theorem above, the moduli space is the set of equivalence classes $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$. The relationship with $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ and $\mathbb{C}$ is more difficult, it requires the presentation of fundamental domains: for a brief introduction see the extended version [Bar21] and for a full presentation see [JS87]. □

1.2.2 Geometric Classification Problem

We will now inquire in the nature of moduli spaces. As was discussed in the introduction, the moduli space of a fixed topological surface aims to be the *geometric* classification of the Riemann surfaces with the same topology. The "geometric" part means that we would like the moduli space to reflect geometric properties of the surfaces. Therefore, it is a good sign that the moduli space of complex tori is a Riemann surface. However, it is not a "perfect" moduli space. To explain this we need the following remark.

Remark 1.2.2.1. It can be seen that $\mathrm{PSL}(2,\mathbb{Z})$ is the free product of $S(z) = -1/z$ (of order 2) and ST (of order 3), with $T(z) = z + 1$ (for a proof of this using only abstract algebra see [Alp93] or [Con]). Therefore, since S fixes i and ST fixes $e^{2\pi i/3}$, they are both elliptic Möbius transformations. In Subsection 1.1.2.1 we proved in a natural way that the quotient of $\mathbb{H}$ by an elliptic-free Fuchsian group is a Riemann surface. We also mentioned that the quotients by Fuchsian groups with elliptic elements are also Riemann surfaces but that fixed points behave like singularities making it difficult to build an atlas. In fact, that the quotient of a space M by any discrete group of $\mathrm{Aut}(M)$ is a manifold is quite a special property of $M = \mathbb{H}$. Therefore, it makes sense to say that $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ has "singularities" at the class of the points i and $e^{2\pi i/3}$.[7]

[6] Here there is a usual abuse of notation, the brackets represent different equivalence relations.

[7] For deeper and more geometric reasoning about why these points are singularities in the space $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ we refer the reader to the extended version of this work [Bar21]

A consequence of Theorem 1.2.13 is that the only tori $\mathbb{T}_\tau$ that admit non-trivial automorphisms are the ones for which τ is fixed by an element of the modular group, that is, there is an $f \in \mathrm{PSL}(2,\mathbb{Z})$ such that $f(\tau) = \tau$. This, together with the remark above, tells us that the "singularities" i and $e^{2\pi i/3}$ (that make it difficult for $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ to be locally euclidean) represent tori that have non-trivial automorphisms. For higher genus Riemann surfaces the situation is more dire, the moduli can also be obtained from a quotient of a complex manifold (see Section 1.2.6) but the singularities caused by points representing surfaces with non-trivial automorphisms make the moduli not be locally-euclidean. This phenomena are the tip of the iceberg of a general theory which we now very briefly, and rather informally, introduce.

What do we mean when we say that a moduli space should reflect the geometric properties of the structures it classifies (in our case, complex structures)? Let B be be a complex manifold, a holomorphic family of compact Riemann surfaces is a holomorphic surjective map $\pi : X \to B$ where X is a complex manifold, π is a submersion and each fiber $\pi^{-1}(b)$ is a compact Riemann surface. The idea is that $\{\pi^{-1}(b) | b \in B\}$ describes a family of compact Riemann surfaces that varies holomorphically amongst a complex structure. These are the objects that we want to control with a moduli space. The first property a good moduli space M should have is a complex structure such that for every holomorphic family $\pi : X \to B$ there is a holomorphic map $i : B \to M$ so that $\pi^{-1}(b)$ is a representative of the class $i(b)$. This roughly says that we can represent holomorphically in M all holomorphic families. This behaviour is what defines M to be a coarse moduli space. This notion is not "good" enough since, for example, it does not imply that the complex structure on M is unique or it does not pinpoint all holomorphic families well enough. The property that would make it a very good moduli space is the existence of a holomorphic family $U \to M$, called the universal family, such that any complex family $X \to B$ is the pull-back from U by a unique holomorphic map $B \to M$. A moduli space M with this property is called a fine moduli space and, amongst other things, the complex structure on M has to be unique modulo isomorphism. In particular, finding a complex structure would be finding the "right" complex structure. Note that this definitions apply *mutatis mutandi* to other contexts, such as topological, smooth or real-analytic structures and families.

One important notion that was understood before the distinction between fine and coarse moduli space was around was that non-trivial automorphisms disable the possibility of a moduli space being fine. The reason is that non-trivial automorphisms allow the construction of families with isomorphic fibers which makes them unable to be uniquely recovered from the universal family. However, one can add more structure to the objects we want to classify in order to vanquish the automorphisms and enable the existence of a fine moduli space.

In what follows we apply this idea to our problems. We introduce a finer structure than the complex structure to kill automorphisms and get a fine moduli space, which will be the Teichmüller space. In this chapter we give this space a topology and count its dimension to be $6g-6$. In the next chapter, we prove it is homeomorphic to an open $(6g-6)$-dimensional ball and has a well defined real-analytic structure. However, we will not get into why it is actually a $(3g-3)$-dimensional complex manifold. At the end of this section we show how to get the moduli space from the Teichmüller space and how it can be seen that a complex/real-analytic structure on the Teichmüller space gives a complex/real-analytic structure on the moduli space (which will still not be a manifold).

A note on history and references: From Riemann's introduction of the moduli space (which was not in the form explained above) up until Teichmüller, moduli space theory was not very developed and generally known informally. One of the great contributions of Teichmüller[8] in [ACa+] was the

[8]The reader may be aware that Oswald Teichmüller was a Nazi. It is not clear weather he was forced to be so or we chose to (and even so, in those times the word choice was quite void of meaning), when he was relieved from the Nazi military forces to work in academia he eventually decided to go back to war (after of the call to arms of the German forces after their defeat in the Battle of Stalingrad in 1942-1943). He died shortly after on September 11, 1943. However, on the brighter side, as L. Ahlfors and F. Gehring said in the preface to Teichmüllers Collected Works [TAG15]: "Oswald Teichmüller deserves our respect and admiration for his mathematics. His life is another matter. The charitable explanation is that he was a politically naive victim of

formalization of the ideas that had been cooking in the mathematical community and the understanding of the moduli philosophy explained above. For a survey on this history and more, see [Ji15]. To go deeper into why being a good geometric classification is translated into properties of the representation of holomorphic families see the article [Ben]. For a rigorous introduction of complex families, fine and coarse moduli spaces, why the Teichmüller space with its complex structure is the fine moduli space we refer the reader to [Ji18].

1.2.3 Fundamental Group Marking

1.2.3.1 Teichmüller Space of the Torus

We are aiming toward an algebraic characterization of Teichmüller spaces that will be easy to deal with topologically. Our study, however, begins with a very natural way to mark Riemann surfaces (to obtain a finer structure) that will add a geometrical flavour to the algebraic characterization. To fix ideas, we study this for the torus first.

Example 1.2.3.1. The torus $\mathbb{C}/\Gamma_\tau$ is equipped with a canonical choice of fundamental group generators: as shown in Figure 1.2, the lines segments given by 1 and τ in $\mathbb{C}$ become the "meridian and equator" loops in $\mathbb{C}/\Gamma_\tau$. With this choice, we can regard 1 as $(1,0)$ and τ as $(0,1)$ in $\mathbb{Z}\times\mathbb{Z} \cong \Gamma_\tau \cong \pi(\mathbb{C}/\Gamma_\tau, [0])$. Now, take $\tau' = \tau + 1$, we know that $\mathbb{C}/\Gamma_\tau$ and $\mathbb{C}/\Gamma_{\tau'}$ are biholomorphic. However, it is clear that τ' represents $(1,1)$ in $\mathbb{Z}\times\mathbb{Z}$ hence τ' and 1 form a different pair of generators of the fundamental group. In the Teichmüller space, $\mathbb{C}/\Gamma_\tau$ and $\mathbb{C}/\Gamma_{\tau'}$ will be different torus. That is, we mark Riemann surfaces by choosing certain generators of the fundamental group. All this motivates the following definition.

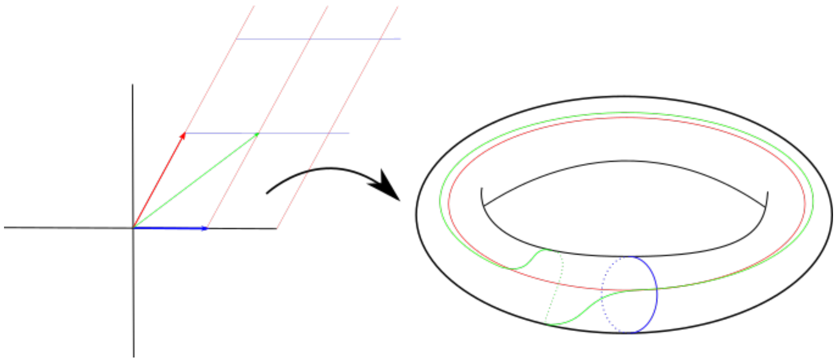

Figure 1.2: Here τ is red, 1 is blue and τ' is green. In the torus the green curve is a curve homotopic to the projection of the τ' segment (it is drawn that way to emphasize the twist.)

Definition 1.2.3.2. Let R be a Riemann surface. A marking Σ_p is a set of generators of the fundamental group of R based at $p \in R$. Two markings Σ_p and Σ'_q are said to be equivalent if there is a continuous path in R between p and q that induces an isomorphism $\pi(R,p) \to \pi(R,q)$ by conjugation that sends Σ_p to Σ'_q. Finally, (R,Σ_p) and (S,Σ'_q) are equivalent if there is a biholomorphism h such that $h_*(\Sigma_p)$ is equivalent to Σ'_q. The class of (R,Σ_p) is denoted by $[R,\Sigma_p]$ and it's called a marked Riemann surface. The Teichmüller space of F_g is defined as $\mathcal{T}_g = \{[R,\Sigma_p]\}$ with R homeomorphic to F_g. In particular, the Teichmüller space $\mathcal{T}_1$ of the torus is the set of all marked tori.

Also, for every $\tau \in \mathbb{H}$, let $\Sigma(\tau) = \{A(\tau), B(\tau)\}$ be the marking on $\mathbb{T}_\tau$ at $[0]$ determined by the generators induced by 1 and τ. For instance, in the discussion above $\Sigma(\tau)$ would be the blue and red loops and $\Sigma(\tau')$ would be the blue and green loops. The markings $\Sigma(\tau)$ and $\Sigma(\tau')$ in the example are not equivalent.

Theorem 1.2.3.3. *We have that* $[\mathbb{T}_\tau, \Sigma(\tau)] = [\mathbb{T}_{\tau'}, \Sigma(\tau')]$ *in* $\mathcal{T}_1$ *if and only if* $\tau = \tau'$. *Therefore, there is a bijection between* $\mathcal{T}_1$ *and* $\mathbb{H}$.

the disease that was rampant in his country. A redeeming feature is that he did not stoop to racial slurs in his scientific papers, which shows that his regard for mathematics was stronger than his prejudices."

Proof. If $\tau = \tau'$ the claim is trivial. If $[\mathbb{T}_\tau, \Sigma(\tau)] = [\mathbb{T}_{\tau'}, \Sigma(\tau')]$ we have a biholomorphism $h : \mathbb{T}_{\tau'} \to \mathbb{T}_\tau$ such that $h_*(\Sigma(\tau'))$ is equivalent to $\Sigma(\tau)$. Note that both $h_*(\Sigma(\tau'))$ and $\Sigma(\tau)$ are markings at $[0]$ given that $h([0]) = [\tilde{h}(0)]$. Then, the hypothesis tells us that $h_*(A(\tau')) = A(\tau)$ and $h_*(B(\tau')) = B(\tau)$. Therefore, $\tilde{h}$ sends the basis $1, \tau'$ to $1, \tau$. Recall from Theorem 1.2.1.3 that $\tilde{h}(z) = \alpha z$ hence $\tilde{h}(\tau') = \alpha\tau' = \tau$ and $\tilde{h}(1) = \alpha = 1$. We conclude that $\tau = \tau'$. The claim about the bijection between $\mathcal{T}_1$ and $\mathbb{H}$ is deduced from what we have just proved and from the fact that every torus is of the form $\mathbb{T}_\tau$ for some $\tau \in \mathbb{H}$ (the normalization biholomorphism of Lemma 1.2.1.1 preserves markings). □

We have that the following relationship between the moduli and the Teichmüller space of the torus:

$$\mathcal{M}_1 = \mathcal{T}_1 / \operatorname{PSL}(2, \mathbb{Z}).$$

1.2.4 The Algebraic Topology

Marking a surface of genus $g \geq 2$ by use of the fundamental group can be regarded as choosing discrete faithful representations of its fundamental group into $\operatorname{PSL}(2, \mathbb{R})$. This point of view has the advantage that $\mathcal{T}_g$ inherits a natural topology called the algebraic topology that is relatively easy to deal with.

Definition 1.2.4.1. A representation of the fundamental group in $\operatorname{PSL}(2, \mathbb{R})$ is a group homomorphism $\rho : \pi(F_g, p) \longrightarrow \operatorname{PSL}(2, \mathbb{R}) \in Hom(\pi(F_g, p), \operatorname{PSL}(2, \mathbb{R}))$. It is said to be discrete if $\rho(\pi(F_g, p))$ is discrete with the natural topology of $\operatorname{PSL}(2, \mathbb{R})$ and it is said to be faithful if it is injective. The discrete faithful representations are denoted by $DF(\pi(F_g, p), \operatorname{PSL}(2, \mathbb{R}))$.

There is a natural action of $\operatorname{PSL}(2, \mathbb{R})$ on $DF(\pi(F_g, p), \operatorname{PSL}(2, \mathbb{R}))$ given by conjugation. That is

$$\begin{aligned} \operatorname{PSL}(2, \mathbb{R}) \times DF(\pi(F_g, p), \operatorname{PSL}(2, \mathbb{R})) &\longrightarrow DF(\pi(F_g, p), \operatorname{PSL}(2, \mathbb{R})) \\ (h, \rho) &\longmapsto h \cdot \rho(\gamma) = h \circ \rho(\gamma) \circ h^{-1} \end{aligned}$$

Theorem 1.2.4.2. *For $g \geq 2$, there is a bijective correspondence between the orbit space of the action above $DF(\pi(F_g, p), \operatorname{PSL}(2, \mathbb{R})) / \operatorname{PSL}(2, \mathbb{R})$ with $\mathcal{T}_g$.*[9]

Proof. We fix a set of generators $\mathcal{C}$ in $\pi(F_g, p)$. Let Σ_ρ be the canonical generators of the fundamental group of $\mathbb{H}/\rho(\pi(F_g, p))$ (which is a Riemann surface because ρ is discrete) induced by $\rho(\mathcal{C})$ via the projection map $\pi_\rho : \mathbb{H} \longrightarrow \mathbb{H}/\rho(\pi(F_g, p))$ (as in Example 1.2.3.1). In other words, $\Sigma_\rho = \pi_\rho(\rho(\mathcal{C}))$. The naturally defined map

$$\begin{aligned} \Psi : DF(\pi(F_g, p), \operatorname{PSL}(2, \mathbb{R})) / \operatorname{PSL}(2, \mathbb{R}) &\longrightarrow \mathcal{T}_g \\ [\rho] &\longmapsto [\mathbb{H}/\rho(\pi(F_g, p)), \Sigma_\rho] \end{aligned}$$

is a well defined bijection. Indeed, if $\rho' = h \cdot \rho$ then $\Psi([\rho]) = \Psi([\rho'])$. This is a consequence of Theorem 1.1.2.10, the discrete subgroups $\rho(\pi(F_g, p))$ and $\rho'(\pi(F_g, p))$ are conjugate by h and hence $H([z]) = [h(z)]$ is a biholomorphism. Moreover, the markings are equivalent by this map because $H \circ \pi_\rho = \pi_{\rho'} \circ h$, hence $H(\Sigma_\rho) = \pi_{\rho'} \circ h(\rho'(\mathcal{C}))$ which is equivalent to $\Sigma_{\rho'} = \pi_{\rho'}(\rho'(\mathcal{C}))$. Analogous considerations show injectivity. For surjectivity we indicate how the inverse ought to be defined. Given (R, Σ), R is biholomorphic to a quotient of $\mathbb{H}$ by a Fuchsian group Γ (which is unique modulo conjugacy) and there is an isomorphism of Γ and $\pi(F_g, p)$. The image of Σ by this correspondence gives the desired discrete faithful representation of $\pi(F_g, p)$, which is well defined up to conjugacy. □

Definition 1.2.4.3 (The Algebraic Topology of $\mathcal{T}_g$)**.** Choose a set of generators for $\pi(F_g, p)$, there is a natural inclusion of $Hom(\pi(F_g, p), \operatorname{PSL}(2, \mathbb{R}))$ into the product of $2g$ copies of $\operatorname{PSL}(2, \mathbb{R})$ since each homomorphism from $\pi(F_g, p)$ to $\operatorname{PSL}(2, \mathbb{R})$ is determined by the image in $\operatorname{PSL}(2, \mathbb{R})$ of each of the

[9] We would like to note that the author has not seen the relationship between fundamental group marking and this representation space in the literature. He has decided to present the fundamental group marking (well developed in [IT92]) as an intuitive notion and then prove its equivalence with the practical representation space characterization of $\mathcal{T}_g$ (well developed in [FM12]).

$2g$ generators of the fundamental group. By endowing $\mathrm{PSL}(2,\mathbb{R})^{2g}$ with the natural Lie group topology both $DF(\pi(F_g,p),\mathrm{PSL}(2,\mathbb{R})) \subset Hom(\pi(F_g,p),\mathrm{PSL}(2,\mathbb{R}))$ have well defined natural topologies. It is readily checked that different choices of generators of the fundamental group give rise to equivalent topologies. The algebraic topology of $\mathcal{T}_g$ is the quotient topology of the topological quotient $DF(\pi(F_g,p),\mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$.

Example 1.2.4.4 (A dimension count of $\mathcal{T}_g$)**.** Now we give a heuristic computation of the number of parameters needed to describe $\mathcal{T}_g$. In the first place, similarly to the definition above, fixing $2g$ generators $a_1, b_1, \dots, a_g, b_g$ of the fundamental group and taking into account that $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1$, to specify a representation of the fundamental group we only need $2g-1$ parameters in $\mathrm{PSL}(2,\mathbb{R})$, i.e. we can think $DF(\pi(F_g,p),\mathrm{PSL}(2,\mathbb{R}))$ in $\mathrm{PSL}(2,\mathbb{R})^{2g-1}$. The action of $\mathrm{PSL}(2,\mathbb{R})$ on this space will subtract one free parameter, we summarize this in a informal diagram:

$$\mathcal{T}_g \longleftrightarrow DF(\pi(F_g,p),\mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{PSL}(2,\mathbb{R})^{2g-1}/\mathrm{PSL}(2,\mathbb{R}) \longleftrightarrow \mathrm{PSL}(2,\mathbb{R})^{2g-2}$$

Now, since $\mathrm{PSL}(2,\mathbb{R})$ is a 3-dimensional Lie group (alternatively, it is readily computed that 3 parameters suffice to describe it), we have found that we need $3 \times (2g-2) = 6g-6$ real parameters to describe $\mathcal{T}_g$.

Example 1.2.4.5. If $\gamma \in \pi(F_g,p)$ then we define the following map

$$\begin{aligned} \mathrm{tr}_\gamma : Hom(\pi(F_g,p),\mathrm{PSL}(2,\mathbb{R})) &\longrightarrow \mathbb{R} \\ \rho &\longmapsto \mathrm{tr}(\rho(\gamma)) \end{aligned}$$

We want to show the continuity of tr_γ with respect to the topology introduced in $Hom(\pi(F_g,p),\mathrm{PSL}(2,\mathbb{R}))$. In particular, they will define continuous functions in the algebraic topology of $\mathcal{T}_g$ because traces are a conjugacy invariant in $\mathrm{PSL}(2,\mathbb{R})$. It will turn our that elements of group models induce closed geodesics on surfaces in a way their length is related to these trace functions, this example will then yield continuity of length functions. Now, continuity of tr_γ (with $\gamma = \prod_{x\in\{a_i b_i\}_{i\in I}} x$) follows from the continuity of $\mathrm{tr}(\prod_{k\in I} p_i(X))$ (which is clearly continuous), where $p_i : \mathrm{PSL}(2,\mathbb{R})^{2g} \to \mathrm{PSL}(2,\mathbb{R})$ is the i-th coordinate projection, $X \in \mathrm{PSL}(2,\mathbb{R})^{2g}$ and I is any ordered finite index set.

Example 1.2.4.6. In the literature one often finds claims along the lines "continuous variation of the deck transformations yield a continuous variation in $\mathcal{T}_g$". This makes sense since two group models that are "close together" (with respect to the matrix topology of $\mathrm{PSL}(2,\mathbb{R})$) should give Riemann surface close together in $\mathcal{T}_g$ (with respect to the algebraic topology). We know frame this reasoning in our construction as follows. As in the definition of the algebraic topology, we think about $DF(\pi(F_g,p),\mathrm{PSL}(2,\mathbb{R}))$ in $\mathrm{PSL}(2,\mathbb{R})^{2g}$. Take a discrete faithful representation $\rho \in \mathrm{PSL}(2,\mathbb{R})^{2g}$ and consider $c : [0,1] \to \mathrm{PSL}(2,\mathbb{R})^{2g}$ a continuous path such that $c(0) = \rho$. We restrict ourselves to paths c such that $c(t)$ is a discrete faithful representation for each $t \in [0,1]$ (such a path exists, we can vary continuously one generator and adjust the rest by the relation of the fundamental group, which would give a path). In particularly, we "have traveled continuously" from ρ to $\rho' = c(1)$. This induces a continuous path from $[\rho]$ to $[\rho']$ in $\mathcal{T}_g$. In particular, a continuous variation on the group models ρ and ρ' yields a continuous variation on the marked surfaces they represent in the Teichmüller space.

1.2.5 Homeomorphism Marking

If we are to approach Teichmüller spaces as we have done so far, the marking we have introduced is useful but relies heavily on our control over the fundamental group and group models. However, in what is to come we will need to control decomposing curves and we will have to track geodesics under isometries. We now introduce a marking of surfaces via maps instead of via the fundamental group that will be useful for these purposes.

Definition 1.2.5.1. Let F_g be an orientable connected compact genus g surface (a priori, there is no need that F_g has a complex structure). A marking on a Riemann surface S of genus g is a homeomorphism $f : F_g \to S$ and we denote the marking by (S,f). We say that two markings (S,f) and (R,g) are equivalent if and only if $g \circ f^{-1}$ is isotopic to a biholomorphism $S \to R$. An equivalence class is denoted as $[S,f]$ and the set of all equivalence classes momentarily denoted by $\mathcal{T}^g$ (as opposed to $\mathcal{T}_g$).

Remark 1.2.5.2. In the literature one finds many variations of this definition: homeomorphism instead of diffeomorphism and/or isotopy instead of homotopy. Since in every isotopy class of homeomorphisms there is an orientation preserving diffeomorphism (see the paper [Hat13]) we could have marked surfaces with orientation-preserving diffeomorphisms. On the other hand, homotopic maps will be isotopic in the cases we care about, see Corollary 2.3.4.5.

Remark 1.2.5.3. In fact, there is a way to define markings that makes it very clear that equivalence of points in the Teichmüller space is a finer distinction than the equivalence of points in the moduli space. Fix a topological model F_g, we write $\mathcal{A}$ for a complex structure on this surface. On the set of non-biholomorphic complex structures on F_g (i.e. the moduli) we declare that $(F_g, \mathcal{A})$ and $(F_g, \mathcal{A}')$ are Teichmüller equivalent if and only if they are biholomorphic and the biholomorphism is isotopic to the identity. One can see that both definitions of markings are equivalent by checking that the following assignment is bijective: $[S, f]$ gets mapped to the equivalence class $(F_g, f^*(\mathcal{A}))$, where $\mathcal{A}$ denotes the complex structure of S and $f^*(\mathcal{A})$ the pull-back of this structure via f.

Theorem 1.2.5.4. *Marking by homeomorphism is equivalent to marking by fundamental group, that is,* $\mathcal{T}_g \longleftrightarrow \mathcal{T}^g$ *for* $g \geq 1$*.*

Proof. There are several ways we could approach this, for $g \geq 2$ one could prove that

$$DF(\pi(F_g, p), \mathrm{PSL}(2, \mathbb{R})) / \mathrm{PSL}(2, \mathbb{R})$$

is in bijection with $\mathcal{T}^g$, which would give a bijection of $\mathcal{T}_g$ and $\mathcal{T}^g$, for this approach see [FM12, Proposition 10.2]. The other way to approach this is generally more difficult for $g \geq 2$ but tractable for $g = 1$. Moreover, it gives an explicit relation between the markings. Let Σ be a fixed fundamental group marking on F_g, the marking on S given by the homeomorphism $f : F_g \to S$ induced a fundamental group marking $(S, f_*(\Sigma))$. The map

$$\begin{aligned} \mathcal{T}^g &\longrightarrow \mathcal{T}_g \\ [S, f] &\longmapsto [S, f_*(\Sigma)] \end{aligned}$$

is well defined (it is direct from the definition of each marking). What is more difficult to prove is that it is one-to-one and onto, for $g > 1$ we direct the reader to [IT92, Theorem 1.4]. The case $g = 1$ is explained in detail in the extended version [Bar21]. □

In the rest of the text we will regard the Teichmüller space as the set of homeomorphism marked Riemann surfaces with the algebraic topology and we will always refer to it as $\mathcal{T}_g$.

1.2.6 The Mapping Class Group

Definition 1.2.6.1. Let S be a genus g surface (possibly with b open discs removed). The group of orientation preserving homeomorphisms of S that restrict to the identity on ∂S will be denoted by $Homeo^+(S, \partial S)$. The mapping class group of S, written $Mod(S)$, is the group of isotopy classes of elements of $Homeo^+(S, \partial S)$.

It is a theorem, that we will not prove and use only here, that the mapping class group of the torus is the special linear group of integer 2×2 matrices, that is, $Mod(F_1) = SL(2, \mathbb{Z})$. Notice that the natural action of this group on $\mathbb{H}$ has kernel $\{\pm Id\}$, and hence the quotient $\mathbb{H}/SL(2, \mathbb{Z})$ is the same as $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$. Therefore, putting this together with previous sections, we have found that the mapping class group of the torus acts naturally on $\mathcal{T}_1 = \mathbb{H}$ to obtain $\mathcal{M}_1 = \mathbb{C}$. What is actually happening is the following.

The mapping class group of F_g acts naturally on $\mathcal{T}_g$ by translating the marking:

$$\begin{aligned} Mod(F_g) \times \mathcal{T}_g &\longrightarrow \mathcal{T}_g \\ ([h], [S, f]) &\longmapsto [S, f \circ h^{-1}] \end{aligned}$$

This action is well defined since isotopic homeomorphisms determine equivalent markings. We use h^{-1} instead of h so it is an actual group action. The most interesting thing about this is that the orbit space "forgets" the marking. Indeed, the orbit of a point $[S, \varphi]$ in $\mathcal{T}_g$ is the set of points $[S, \psi]$ where the marking ψ ranges over all isotopy classes of homeomorphisms $F_g \to S$. In particular, the marking becomes irrelevant since in every orbit there will be all Riemann surfaces biholomorphic to S and there will not be two non-biholomorphic ones, hence we can make the identification

$$\mathcal{M}_g = \mathcal{T}_g / Mod(F_g).$$

This is also true for Riemann surfaces of arbitrary signature (which we will define in the next chapter). In particular, we can study the moduli space from the Teichmüller space and we can study the mapping class group from its action over the Teichmüller space.

For example, it is an important result that the mapping class group acts on the Teichmüller space properly discontinuously.[10] If one manages to prove that the Teichmüller space has a $(3g-3)$-dimensional complex structure and that the mapping class group acts by biholomorphisms, as in our push-forward structure lemma, one gets a complex structure on the moduli space. Using the techniques of the next chapter it can be proven that the action is properly discontinuously and by real analytic diffeomorphisms, obtaining a real-analytic structure on the moduli space. It is important to note that, similar to the case of the torus, non-trivial automorphisms make the action not be free and hence the moduli space is not an actual manifold but what is known as an orbifold.

The main application in mapping class group theory of this action is the classification of surface homeomorphism, a great accomplishment of Thurston, see [FM12]. As mentioned in the introduction we will prove that $\mathcal{T}_g$ is homeomorphic to an open $(6g-6)$-dimensional real ball, $\mathcal{T}_g$ can also be compactified in a meaningful way into a closed ball so that the action of the mapping class group extends to the boundary of the ball. The fixed points of the surface homeomorphisms on the boundary of $\mathcal{T}_g$ serve as a geometrically relevant classification of elements in $Mod(S, \partial S)$. A similar idea, in a much simpler situation, has been employed in the appendix to classify elements of $\mathrm{PSL}(2, \mathbb{R})$ by the number of fixed points on $\mathbb{R} \cup \{\infty\}$. Considering the compactification of $\mathbb{H}$ given by $\bar{\mathbb{H}} = \mathbb{H} \cup \partial\bar{\mathbb{H}}$ with $\partial\bar{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$, we have classified the elements of $\mathrm{PSL}(2, \mathbb{R})$ by their fixed points on $\partial\bar{\mathbb{H}}$.

[10] A group G acts properly discontinuously on a topological space X if for any compact set $K \subset B$ the number of $g \in G$ such that $g \cdot B \cap B = \varnothing$ is finite.

Chapter 2

Hyperbolic surfaces

This chapter has three parts: hyperbolic geometry, hyperbolic surfaces and Teichmüller spaces. The first of these serves as a brief introduction to the basic features of "plane" hyperbolic geometry that we require. The second section deals with basic facts about hyperbolic surfaces: surfaces locally modelled after the hyperbolic plane. We study the deep connection between hyperbolic surfaces and Riemann surfaces, hyperbolic surfaces with boundary, how to glue such surfaces together and then further inquire into group models of hyperbolic and Riemann surfaces. Then we prove some very important features of closed geodesics in hyperbolic surfaces via (somewhat) algebraic means. We conclude with some basic facts about geodesic arcs and perpendiculars. The last section proves the main theorem of this text, namely, $\mathcal{T}_g$ is homeomorphic to a $(6g-6)$-dimensional ball and each point in this space is determined by the length of $9g-9$ (homotopy classes) of curves. To prove this we thoroughly study Y-pieces (or pairs of pants) and X-pieces.

References: Along this chapter, though not strictly required, some familiarity with Riemannian geometry is assumed, see [Car92], [Kli78] or [Lee97]. The first section mainly draws from [Ser], [Hub06], [Bea95], [Bus10] and [Ahl73] and we recommend the first three for a comprehensive introduction to hyperbolic geometry. The most basic facts in the second one are taken from [Bus10] and the necessary tools to study closed geodesics algebraically come from [FM12] and [Mar16]. Finally, the last section is mostly based on [Bus10]'s proof of the main theorem.

2.1 Hyperbolic Geometry

We would like to "geometrize" $\mathbb{D}$ and $\mathbb{H}$ in order to introduce more tools to understand Riemann surfaces or, in other words, find a compatible geometric structure. For now, we will turn our attention to $\mathbb{D}$ and find a Riemannian metric tensor on it in a way the group of isometries is the group of holomorphic automorphisms. This last condition is capital in order to geometrize all surfaces covered by $\mathbb{H}$, moreover, intuitively, two "mirror" structures should have the same corresponding symmetry group.

In order to find the metric tensor on the disc we find a length element that is invariant under conformal maps. Consider $\delta(z,w) = |\frac{z-w}{1-\bar{z}w}|$ and let $f(z) = \frac{az-b}{\bar{b}z-\bar{a}}$ be a holomorphic automorphism of $\mathbb{D}$ (see Theorem 7 in the Appendix). Using the fact that $|a|^2 - |b|^2 = 1$, we compute the following:

$$f(z) - f(w) = \frac{z-w}{(\bar{b}z-\bar{a})(\bar{b}w-\bar{a})}, \qquad 1 - \bar{f}(z)f(w) = \frac{1-\bar{z}w}{(b\bar{z}-a)(\bar{b}w-\bar{a})}.$$

Dividing both expressions and taking modulus we obtain that δ is invariant under automorphisms of the disc, that is, $\delta(z,w) = \delta(f(z), f(w))$. Letting w approach z we obtain the following length element equality

$$\frac{|dz|}{1-|z|^2} = \frac{df(z)}{1-|f(z)|^2}.$$

This suggests that the length element $ds = \frac{2|dz|}{1-|z|^2}$ is invariant under biholomorphisms of the disc, where $|dz|^2 = dx^2 + dy^2$. We define the Poincaré disc to be the Riemannian manifold

$$(\mathbb{D}, ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}).$$

The $2^2 = 4$ factor is introduced so that the curvature is -1. Now we shall determine the full isometry group $\mathrm{Is}(\mathbb{D})$ of the Poincaré disc.

Theorem 2.1.0.1. *Let $\mathrm{Is}^+(\mathbb{D})$ be the group of orientation-preserving (also known as direct) isometries of the Poincaré disc. We have that the elements of $\mathrm{Is}^+(\mathbb{D})$ are precisely the biholomorphisms of the disc, that is,*

$$\mathrm{Is}^+(\mathbb{D}) = \mathrm{Aut}(\mathbb{D}) = \{f(z) = \frac{az+b}{\bar{b}z+\bar{a}} \mid a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1\}.$$

The full group of isometries $\mathrm{Is}(\mathbb{D})$ consists of $\mathrm{Is}^+(\mathbb{D})$ and the elements of $\mathrm{Is}^+(\mathbb{D})$ composed with the orientation reversing isometry c defined by $x + yi \mapsto x - yi$ (complex conjugation).

Proof. The heuristic discussion above justifies that the elements of $\mathrm{Aut}(\mathbb{D})$ are (orientation-preserving) isometries (since they leave the metric invariant). Rigorous justification of that and the fact that $\mathrm{Aut}(\mathbb{D})$ composed with c preserve the Riemannian tensor is a standard exercise in geometry. For the converse take an isometry f, since it preserves angles in the hyperbolic metric it also preserves angles in the euclidean metric (they are multiple of each other by $1/(1-|z|^2)^2$, hence “conformal” metrics). Now, by composing with the map c above we can assume that f is an orientation preserving diffeomorphism that preserves angles, that is, a conformal diffeomorphism, hence a biholomorphism. We have proved that either f or $c \circ f$ is in $\mathrm{Aut}(\mathbb{D})$, which is what we wanted. □

Now we would like to do the same for the upper half plane. Recall that $f(z) = \frac{i-z}{i+z}$ maps $\mathbb{H}$ to $\mathbb{D}$ biholomorphically, to geometrize $\mathbb{H}$ we just declare this homeomorphism an isometry. In other words, we pull back the tensor ds^2 on $\mathbb{D}$ to $\mathbb{H}$ via the map f (promoting it to an isometry). It is a standard exercise in differential geometry that this yields the tensor $\frac{1}{Im(z)^2}|dz|^2$. Henceforth, we define the Poincaré upper half plane to be the Riemannian surface

$$(\mathbb{H}, ds^2 = \frac{1}{Im(z)^2}|dz|^2).$$

Theorem 2.1.0.2. *The orientation-preserving isometries of $\mathbb{H}$ are precisely the real Möbius transformations with positive determinant, that is, $\mathrm{Is}^+(\mathbb{H}) = \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$. The full group of isometries $\mathrm{Is}(\mathbb{H})$ consists of $\mathrm{Is}^+(\mathbb{H})$ and the elements of $\mathrm{Is}^+(\mathbb{H})$ composed with c' defined by $c'(z) = -c(z)$, i.e. $x + iy \mapsto x - iy$.*

Proof. This is a direct consequence of Theorem 2.1.0.1, Theorem 4 in the Appendix and the fact that the map f given by $\frac{i-z}{i+z} : \mathbb{H} \to \mathbb{D}$ is both an isometry and a biholomorphism. Note that $f^{-1}(z) = i\frac{1-z}{1+z}$ and so it is computed that $c = f \circ c' \circ f^{-1}$. □

It is an easy exercise to check that the curvature of this isometric spaces is -1 at every point. The geometric interpretation of this fact is that each point is a saddle point. Recall from Riemannian geometry that the length of a piece-wise differentiable curve $\gamma : [a, b] \to \mathbb{H}$ is given by

$$\ell(\gamma) = \int_\gamma ds = \int_a^b ||\gamma'(t)||dt = \int_a^b \frac{|\gamma'(t)|}{Im(\gamma(t))}dt$$

where $||\cdot||$ denotes the norm induced by the Poincaré metric and $|\cdot|$ the complex modulus. Also, recall that the function $d(z, w)$, defined as the infimum of the length of piece-wise differentiable curves joining z and w, is a distance function that induces a topology equivalent to the topology of the manifold, in our case, the distance function induces the usual euclidean topology. In particular, the Poincaré spaces introduced are complete as metric spaces. The following theorem lists different expressions for the distance function in $\mathbb{H}$ and the one that follows enumerates the most important properties of geodesics in $\mathbb{H}$ and $\mathbb{D}$, which play an important role in this text.

Theorem 2.1.0.3. *Let d be the distance function associated to the Poincaré metric on $\mathbb{H}$. We have that*

(1) $d(z,w) = \log(\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|})$;

(2) $\cosh d(z,w) = 1 + \frac{|z-w|^2}{2Im(z)Im(w)}$.

Proof. See [Bea95, Theorem 7.2.1.]. □

Theorem 2.1.0.4 (Geodesics in the Poincaré Models)**.** *Let $\mathbb{H}$ and $\mathbb{D}$ be the Poincaré plane and disc.*

(1) The geodesics on $\mathbb{H}$ are the vertical lines and the semicircles of arbitrary radius with center on the line $y = 0$ (which is the same as saying semicircles that meet $Im(z) = 0$ orthogonally.[1])

(2) The geodesics on $\mathbb{D}$ are the diameters of $\mathbb{D}$ and all circles in $\mathbb{D}$ meeting $\partial\mathbb{D} = \mathbb{S}^1$ orthogonally.

(3) There is a unique geodesic through any two distinct points.

(4) Given a point and a geodesic, there is a unique geodesic through that point and perpendicular to the original geodesic.

(5) Given two geodesics at a positive distance, there is a unique geodesic perpendicular to both.

Proof. The proof of the first two statements is a calculus exercise using the metric tensor, see for example [Ser, Proposition 2.3] for a elementary computation. The rest follows easily from either one. □

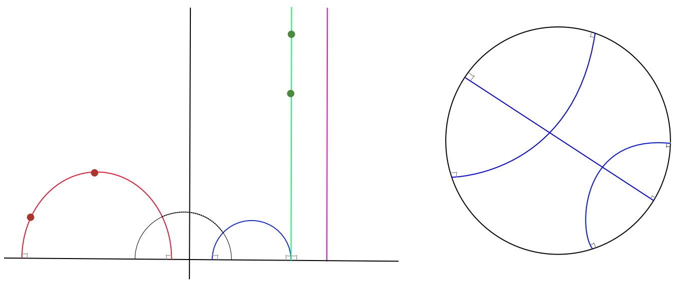

Figure 2.1: Left: geodesics in $\mathbb{H}$. The red one is the unique geodesic through the red points, the green one the unique geodesic through the green points, the black one joining the red and the blue ones is the unique geodesic joining both geodesics. The blue and green lines do not intersect but are distance 0. The same is true for the green and purple lines. Right: geodesics in $\mathbb{D}$.

2.1.1 Fermi Coordinates

So far we have described the hyperbolic plane using the usual "euclidian" coordinates, now we introduce a more intrinsic pair of coordinates that will be important to parametrize vicinities of geodesics on hyperbolic surfaces. Choose a geodesic η in $\mathbb{H}$ and a point $p \in \mathbb{H}$. If $p \notin \eta$ there is a unique geodesic γ perpendicular to η through p. Let us say that γ meets η at $\eta(t_p)$, that is, "at time t_p". Moreover, η divides $\mathbb{H}$ into two half spaces, a "right" (positive) hand side and a "left" (negative) hand side. It thus makes sense to speak of signed distance (also known as directed distance). The signed distance from $\eta(t_p)$ to p, call it ρ_p, is given by the length of the corresponding segment of γ (which is the unique geodesic joining both points) and its sign is given by which side of η the point p is in: *right hand side is positive and left hand side is negative*. If $p \in \eta$, we have $\rho_p = 0$ and $p = \eta(t_p)$. The pair (ρ_p, t_p) uniquely determines any point $p \in \mathbb{H}$. We call this coordinates the Fermi coordinates associated to η.

We fix η to be the euclidean y-axis (which is a geodesic in $\mathbb{H}$) and parametrize it at unit speed, $\eta(s) = ie^s$. Figure 2.2 shows how this parametrizes $\mathbb{H}$. In fact, these coordinates are a particular case of geodesic coordinates, for constant curvature spaces and geodesic coordinates the metric tensor is easy to compute, see [Kli78, Chapter 4, Section 3]. In our case we have that the metric tensor of the Fermi coordinates on $\mathbb{H}$ associated to η is

$$ds^2 = d\rho^2 + \cosh^2(\rho)dt^2.$$

[1]Note that since the Poincaré metrics are multiples of the euclidean metrics the angles they determine are the same.

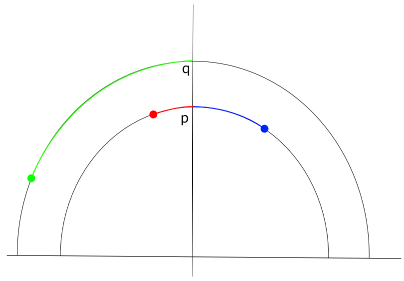

Figure 2.2: The blue and red point have the same time coordinate, $e^t = p$ and the green point has time coordinate $e^t = q$. The ρ component will be the length of the colored arcs. The green and red points will have $\rho < 0$ and the blue point $\rho > 0$.

2.1.2 Hyperbolic Trigonometry

Now we will give some trigonometric formulas for the hyperbolic plane that we will need later. While the proofs are not difficult we will not do them here due to space considerations: we would have to introduce the hyperboloid model (see Figure 2.3) for hyperbolic geometry, in which trigonometry is quite tractable[2]. We refer the reader to see [Bus10, Chapter. 2].

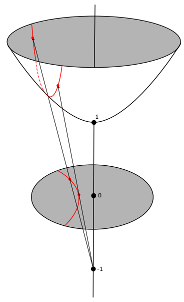

Figure 2.3: Isometry between the Poincaré disc and this upper leaf of a hyperboloid that makes it into a model of hyperbolic geometry.

Definition 2.1.2.1. A polygon is a piece-wise geodesic oriented closed curve (we allow self-intersections).

Theorem 2.1.2.2. *For any convex right-angled geodesic hexagon with consecutive sides $a, \gamma, b, \alpha, c, \beta$ the following holds (the same letters denote the corresponding side lengths):*

- $\cosh c = \sinh a \sinh b \cosh \gamma - \cosh a \cosh b,$
- $\frac{\sinh a}{\sinh \alpha} = \frac{\sinh b}{\sinh \beta} = \frac{\sinh c}{\sinh \gamma}.$

For any right-angled geodesic hexagons with consecutive sides $a, \gamma, b, \alpha, c, \beta$ such that c and γ intersect we have

- $\cosh c = \sinh a \sinh b \cosh \gamma + \cosh a \cosh b.$

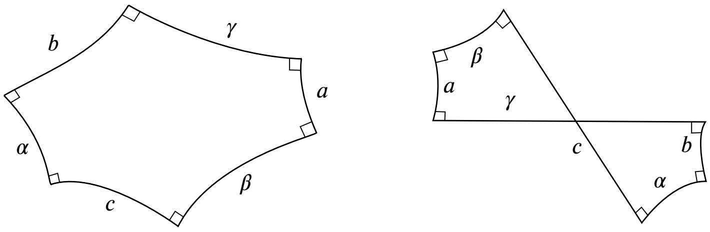

Figure 2.4: Left: Hexagon. Right: Crossed Hexagon.

The last and most important fact about hyperbolic geometry we will need is the existence and uniqueness of convex hexagons given only 3 non-consecutive side lengths (again, notice the similarities with triangles).

Theorem 2.1.2.3. *There exists a unique right-angled geodesic hexagon (modulo isometry) in the hyperbolic plane with non-adjacent sides of length $a, b, c \in \mathbb{R}_+$.*

[2]In the hyperbolic plane triangles and right-angled hexagons (some quadrilaterals and pentagons too) behave quite similarly, so much so that there are several configurations unifying those objects as the same type of objects. Buser introduces a new unifying configuration that uses the hyperboloid model to define a "generalized triangle" and obtains a general relation on the sides of a generalized triangle, see [Bus10, Theorem 2.2.6.]. This relation can be applied case by case to obtain many different formulas, like the ones we need.

Proof. For the existence we outline a construction in $\mathbb{D}$. For any $t > 0$, let a_t and b_t be a pair of geodesics in $\mathbb{D}$ at a distance t apart and let γ_t' be the unique perpendicular geodesic segment realizing this distance. Let α_t' and β_t' be geodesics on the same side of γ_t' such that α_t' has a perpendicular intersection with b_t at a distance b away from the intersection of γ_t' and β_t' has perpendicular intersection with a_t at a distance a away from γ_t'. Finally, if γ_t' is oriented from a_t to b_t, we require that α_t' and β_t' lie on the left of γ_t'.

There is a value of t_0 such that α_{t_0}' and β_{t_0}' are at zero distance but don't intersect.[3] For $t > t_0$ let c_t be the unique geodesic segment intersecting α_t' and β_t' perpendicularly (it exists because they are at a positive distance). As t varies from t_0 to infinity, the length of c_t varies continuously and monotonically from zero to infinity. Therefore, there is unique t such that the length of c_t is exactly c. This concludes the construction of the desired hexagon. For uniqueness we simply observe that the first formula in Theorem 2.1.2.2 determines the length of the other sides. □

2.2 Hyperbolic Surfaces

This section serves as preparation for the main section of this text about Teichmüller theory and the Fenchel-Nielsen coordinates. We will define hyperbolic surfaces, study their relationship with Riemann surfaces, explain how hyperbolic surfaces can be pasted/glued along boundaries and study the interplay of group models for hyperbolic surfaces and their geometry with the subsection about closed geodesics as our main goal.

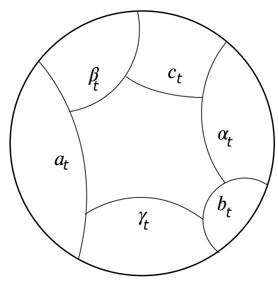

Figure 2.5

2.2.1 Definition and Relationship to Riemann Surfaces

A hyperbolic surface will be a surface that locally looks like the hyperbolic plane. Clearly, in order to be able to look like the hyperbolic plane the surface has to have a metric structure beforehand. Therefore:

Definition 2.2.1.1. A hyperbolic surface is a two dimensional orientable[4] Riemannian manifold that is locally isometric to the hyperbolic plane. We will also call this metric a hyperbolic structure.

Remark 2.2.1.2. Observe that the local isometry condition implies that the curvature of a hyperbolic surface is identically -1. In fact, we could have defined a hyperbolic surface as a Riemannian surface with sectional curvature constantly -1. Indeed, it is a known fact that a surface of constant curvature -1 is universally (locally isometrically) covered by $\mathbb{H}$ and hence locally isometric to the hyperbolic plane (see [Car92, Proposition 4.3]).

A hyperbolic structure can be thought of in terms of an atlas (see Subsection 2.2.1.1). The local isometries in definition 2.2.1.1 constitute charts that are isometrically compatible and can be taken to be orientation preserving. In particular, every hyperbolic atlas will have isometric transition maps, as such they will be conformal diffeomorphisms, hence biholomorphisms of open sets of the plane. *This implies that every hyperbolic surface can be given the structure of a Riemann surface. Moreover, every Riemann surface R except the plane, punctured plane, tori and Riemann sphere is a hyperbolic surface.* As was remarked in the beginning of this chapter, that biholomorphisms of the upper-half plane became isometries of the Poincaré metric was capital to introduce a geometric structures compatible with the complex ones. Now, this will be made apparent when we push-forward the Poincaré metric of the plane to surfaces and when the word isometry becomes "synonym" to biholomorphism. Let us prove the claim in italics: in the first place, by Theorem 1.1.2.2 said Riemann surfaces are holomorphically universally covered by $p : \mathbb{H} \to R$. In the second place, the covering transformation groups will be subgroups of the real

[3] Another way to say this is that they share an endpoint in $\partial\mathbb{D}$.
[4] This condition is not necessary but we have no need for non-orientable surfaces.

Möbius transformations with positive determinant, in particular, isometries (since $\text{Aut}(\mathbb{H}) = \text{Is}^+(\mathbb{H})$). We conclude that the Poincaré metric ds^2 is invariant under the covering transformation action and hence it descends to a well-defined metric g^R on R (see the lemma below). Moreover, the covering map p is a local isometry, i.e. the pull-back of g^R is the Poincaré metric $ds^2 = p^*(g^R)$. We can conclude that biholomorphisms of complex structures correspond to isometries of hyperbolic structures (which is straightforward by lifting the maps to $\mathbb{H}$).

Lemma 2.2.1.3 (Push-Forward Riemannian Structure)**.** *Let (M, g) be a Riemannian manifold and G a subgroup of the group of isometries of M that acts like a covering space action on M and let $\pi : M \to M/G$ the projection map. Then, M/G has a smooth structure and a Riemannian metric $\bar{g}$ such that π is a local isometry and*[5] $g = \pi^*(\bar{g})$.

Proof. The smooth structure comes from the push-forward structure Lemma 1.1.1.8. For the Riemannian metric consider $q \in M/G$ and $\tilde{q} \in \pi^{-1}(q)$, for every pair $u, v \in T_q(M/G)$, we set

$$\bar{g}_q(u, v) = g_{\tilde{q}}(d\pi^{-1}(u), d\pi^{-1}(v))$$

which is well defined because G acts by isometries. Indeed, since the action is a covering space action, from the topological theory of covering spaces, we know that π is regular covering, i.e. G acts transitively on the fiber of q. Therefore, for any $\tilde{q}' \in \pi^{-1}(q) \setminus \{\tilde{q}\}$ there is an isometry f in G such that $f(\tilde{q}) = \tilde{q}'$, hence the definition does not depend on the choice of the point on the fiber. The definition of $\bar{g}$ above is directly rephrased as $g = \pi^*(\bar{g})$, this implies that π is an isometry where it is a diffeomorphism. Since it is a local diffeomorphism it is a local isometry. □

Now, we synthesize into a theorem what we proved in the discussion above together with our work on Riemann surfaces.

Theorem 2.2.1.4 (Geometrization Theorem for Riemann Surfaces)**.** *Every hyperbolic manifold can be given a Riemann surface structure. Every Riemann surface different from the complex plane, punctured plane, tori and Riemann sphere can be given a hyperbolic structure.*[6] *Moreover, biholomorphisms correspond to isometries by this assignment. Also, every hyperbolic surface is universally covered by $\mathbb{H}$, the covering transformation group acts by isometries and the covering map is a local isometry. Furthermore, each hyperbolic surface is isometric to a quotient of $\mathbb{H}$ by a discrete subgroup of* $\text{Aut}(\mathbb{H})$ *without elliptic elements.*

Remark 2.2.1.5 (and Corollary)**.** The moduli and Teichmüller spaces for hyperbolic surfaces are defined analogously to moduli and Teichmüller spaces for Riemann surfaces (see Definitions 1.2.0.1 and 1.2.5.1) by changing the word "biholomorphism" by the word "isometry". The theorem above implies that both the moduli and Teichmüller space for Riemann surfaces and hyperbolic surfaces are identified bijectively, hence they parametrize two types of structures on surfaces: complex structures and hyperbolic structures. We will denote a connected compact orientable surface of genus g with n open discs removed by $F_{g,n}$. Below we will define hyperbolic surfaces with boundary so it makes sense to talk about the Teichmüller space and moduli space of $F_{g,n}$. The moduli and Teichmüller space will be written as $\mathcal{M}(F_{g,n}) = \mathcal{M}_{g,n}$ and $\mathcal{T}(F_{g,n}) = \mathcal{T}_{g,n}$, if $n = 0$, $\mathcal{M}_{g,0} = \mathcal{M}_g$ and $\mathcal{T}_{g,0} = \mathcal{T}_g$ as usual.

While one can define a Riemann surface with boundary, the (apparent) lack of a distinguished metric makes dealing with boundaries difficult. However, hyperbolic surfaces with boundary will have geodesic boundary, which make them a nicer object. To define said object, we need the following preliminary definitions.

- A side point of $\mathbb{H}$ is a point on a geodesic and a side sector in $\mathbb{H}$ is an open neighbourhood of a side point in a half-space of the geodesic.
- A vertex point is the intersection point of two geodesics and a vertex sector of angle θ in $\mathbb{H}$ is an open neighbourhood of a vertex point in the intersection of two half-spaces of the two geodesics that cut each other with angle θ. We only admit $0 < \theta \leq \pi$ with the understanding that $\theta = \pi$ is a side point.

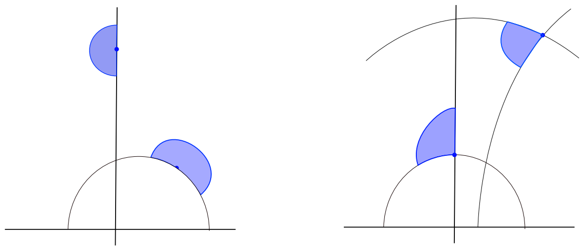

Figure 2.6: Left: side points in blue. Right: vertex points in blue.

Definition 2.2.1.6. A hyperbolic surface S with boundary is a two dimensional Riemannian manifold with piece-wise smooth boundary such that:

- If p is in the interior of S there is an open neighbourhood of p locally isometric to an open set in $\mathbb{H}$;
- if p is a boundary point where the boundary is smooth there is a local isometry to a side point and sector;
- if p is a boundary point where the boundary is not smooth there is a local isometry to a vertex point and sector of angle $\theta \in (0, \pi)$.

Here we have allowed "corners" with vertex sectors so polygons fit into the definition of being a hyperbolic surface. Note that the condition about local isometries implies that boundaries are at least piece-wise geodesic. The case that is most important to us is when the boundary is a closed geodesic, that is, a geodesic closed curve such that the tangent vectors at its terminal point are the same. We will study closed geodesics later in Section 2.2.4 but we need a working definition for the following sections.

Definition 2.2.1.7. A closed geodesic γ is a geodesic that is a smooth embedding of $\mathbb{S}^1$. In particular, the parameter t of $\gamma(t)$ varies in $\mathbb{S}^1 = \mathbb{R}/[t \mapsto t+1]$.

Definition 2.2.1.8. A compact hyperbolic surface S of genus g that has (possibly empty) boundary made of n (possibly $n = 0$) closed geodesics is said to be a hyperbolic surface of signature (g, n).

2.2.1.1 The Hyperbolic Atlas

As mentioned before, we are interested in gluing surfaces along their boundaries. In the case Riemannian structures can be described in terms of an atlas, this procedure is relatively simple. In the previous section, we showed how a hyperbolic structure yields a hyperbolic atlas. The converse is true, but we should define hyperbolic atlas properly first.

Definition 2.2.1.9. Let S be a surface with (possibly empty) boundary. A hyperbolic atlas of S is collection of pairs $\{(U_i, \varphi_i)\}_i$ called charts such that $\{U_i\}_i$ is an open cover of S and for every i the map $\varphi_i : U_i \to V_i \subset \mathbb{C}$ is homeomorphism where:

- (interior point chart) If p is in the interior of S there is a chart $\varphi_i : U_i \to V_i \subset \mathbb{C}$ so that $p \in U_i$ and V_i is an open set of $\mathbb{H}$;
- (side point chart) if p is a boundary point, we either have a chart $\varphi_i : U_i \to V_i \subset \mathbb{C}$ so that $p \in U_i$ is side point and V_i is a side sector of $\mathbb{H}$ or

[5] Here π^* indicates pull-back the of the tensor by π. We are pushing forward the structure of M to M/G so that the metric tensor pulls back correctly.

[6] In fact, our reasoning transfers perfectly to geometrize the plane, punctured plane and tori with plane/parabolic geometry and the sphere with spherical geometry, providing a geometrization of all Riemann surfaces.

- (vertex point chart) a chart $\varphi_i : U_i \to V_i \subset \mathbb{C}$ so that $p \in U_i$ is vertex point and V_i is a vertex sector of $\mathbb{H}$.

Also, we ask that the transition maps are restrictions of isometries $\mathrm{Is}^+(\mathbb{H})$, more concretely: if $U_i \cap U_j \neq \varnothing$ then the restriction of the transition map $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ to each connected component of $\varphi_j(U_i \cap U_j)$ is an orientation-preserving isometry of $\mathbb{H}$. As usual, a hyperbolic structure given by an atlas is an equivalence class of isometrically compatible atlases (isometrically compatible means that transition maps of charts from two different atlases are restriction of isometries of $\mathrm{Is}^+(\mathbb{H})$).

Since each chart is a homeomorphism we can pull back the metric tensor on $\mathbb{H}$ through the chart. Since they are all isometrically compatible, this defines a Riemannian metric on S. With this metric tensor each chart is an isometry with the hyperbolic plane, hence this definition yields a hyperbolic surface as defined in the previous section. We will not differentiate both definitions in what follows.

In the definition above we have asked the transition maps to be the restriction of an isometry of $\mathbb{H}$. We could have simply asked them to be an isometry of open sets of $\mathbb{H}$. This is because of the following fact.

Proposition 2.2.1.10 (Extension Lemma)**.** *Every orientation preserving isometry $m : U \to V$ between non-trivial connected open sets of $\mathbb{H}$ is the restriction a direct isometry of $\mathbb{H}$, $\mathrm{Is}^+(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$.*

Proof. Though the author has seen this fact stated, he did not find a proof of this. In the extended version [Bar21] he gives a proof using the Riemannian exponential map and Proposition 10 in the Appendix. □

2.2.2 Pasting, the Hyperbolic Cylinder and Collars

Let S and R be hyperbolic surfaces, the usual procedure to attach manifolds along boundary consists of taking an homeomorphism $f : \partial S \to \partial R$ and identifying points by the rule $p \sim q$ if and only if $q = f(p)$. Then, the space $S \cup_f R = S \sqcup R/\sim$ turns out to be a manifold as well, see [Lee11, Theorem 3.79]. In our case, hyperbolic surfaces have simple boundaries so we can adapt the definition in virtue of simplicity. Moreover, the upcoming simplification will allow us to give a detailed proof of the pasting theorem, which is looked over in the literature. Let us begin with a preliminary example that will be important for the proof of the pasting theorem.

Example 2.2.2.1. Let P and P' be two polygons in $\mathbb{H}$ (piece-wise geodesic closed curves) oriented clockwise. Let γ and γ' be sides of P and P' of the same length and parametrized at the same constant speed on the interval $[0, 1]$. By the 2-transitivity of $\mathrm{Is}^+(\mathbb{H})$ on equidistant points (see Proposition 10 in the Appendix) we can take P' to P via $m \in \mathrm{Is}^+(\mathbb{H})$ so that γ and γ' coincide. The points will overlap so $\gamma(t) = m(\gamma'(1-t))$ and, this way, the new polygon will have a boundary oriented clock-wise. Abstractly, we are identifying $\gamma(t) \sim \gamma'(1-t)$ in $P \sqcup P'$ to obtain a new polygon. This is the simplest case of pasting hyperbolic surfaces along boundaries.

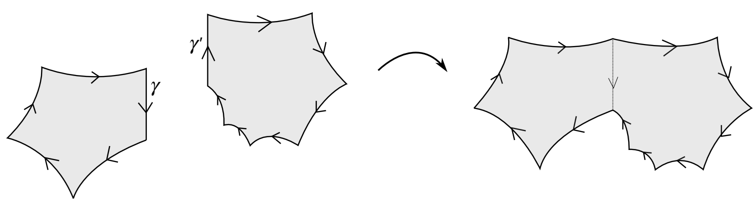

Figure 2.7

In this example we have made a choice in the orientations of P and P': we have oriented P' so that when we take it to P via $\mathrm{Is}^+(\mathbb{H})$ the resulting surface is oriented and have made the pertinent identifications in γ and γ', namely, $\gamma(t) \sim \gamma'(1-t)$. For general surfaces, we will implicitly make this choices of orientation in a way things work as with the case of P and P'. In the example above, it would be understood that the identification to make would be $\gamma(t) \sim \gamma'(t)$ orienting curves as necessary.

Definition 2.2.2.2. Let S and R be hyperbolic surfaces as before. Let $\gamma : [0,1] \to S$ and $\beta : [0,1] \to R$ be geodesics in ∂S and ∂R respectively that are parametrized at constant speed and have the same length. A pasting condition (or pasting scheme if we perform several identifications) is an identification of the form $\gamma(t) \sim \beta(t)$ for $t \in [0,1]$, it will be denoted by $\gamma(t) = \beta(t)$.[7] The resulting space $S \sqcup R / \sim$ will be written as $S + R$ or, if the situation demands it for clarity, as $S + R$ modulo $\gamma(t) = \beta(t)$.

Theorem 2.2.2.3 (Pasting Theorem)**.** *The space $S + R$ modulo $\gamma(t) = \beta(t)$ defined above is a hyperbolic surface such that, if $\pi : S \sqcup R \to S + R = S \sqcup R/ \sim$ denotes the projection map, π is a local isometry and S and R are isometric to $\pi(S)$ and $\pi(R)$ respectively.*

Proof. The proof consists of constructing hyperbolic charts around the pasting geodesic $\eta = \pi(\gamma) = \pi(\beta)$. A detailed account of how this would be done can be found in the extended version [Bar21]. □

Definition 2.2.2.4. The inverse process is called "cutting", with the notation of the theorem, S and R are obtained by cutting $S + R$ along η.

The following example is good for fixing ideas and because, around closed geodesics, hyperbolic surfaces all look like a piece of a hyperbolic cylinder.

Example 2.2.2.5 (The Hyperbolic Cylinder)**.** Consider the geodesic $\eta(u) = ie^u$ in $\mathbb{H}$ and let γ and γ' be two geodesics intersecting η perpendicularly at $a \in \mathbb{C}$ and $b \in \mathbb{C}$ ($|a| < |b|$). We can regard the closed strip $S = \{z \in \mathbb{H} \mid |a| \leq |z| \leq |b|\}$ as a hyperbolic surface. Simple computation shows that the segment of η in S has length $\ell = \log |b/a|$.

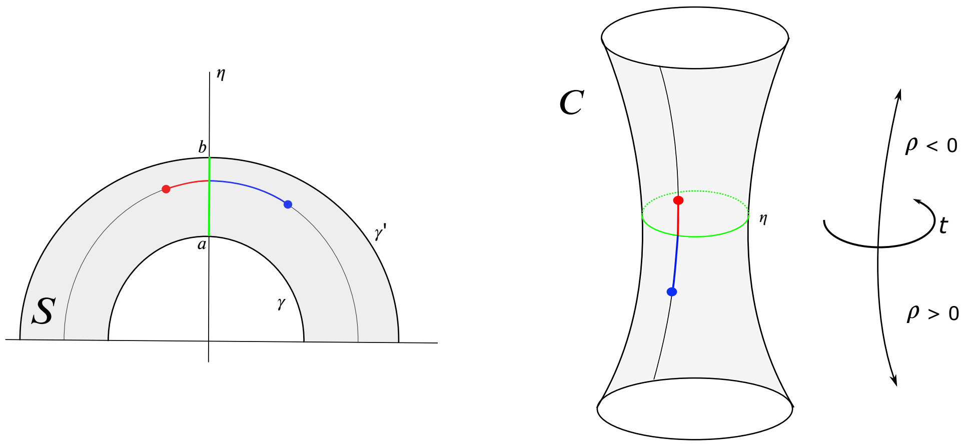

Figure 2.8: Hyperbolic cylinder; the red and blue lines show how Fermi coordinates are transferred from $\mathbb{H}$ to C.

We parametrize γ and γ' with unit speed and with clock-wise orientation (which is the same as saying with opposite boundary orientation, S will be at the right of γ' and the left of γ) such that $\gamma(0) = a$ and $\gamma'(0) = b$. The isometry $m(z) = \frac{b}{a} z$ is a hyperbolic transformation of axis η satisfying $m(\gamma(t)) = \gamma'(t)$ for $t \in \mathbb{R}$. In particular, m induces the pasting condition $\gamma(t) = \gamma'(t)$, which gives rise to a hyperbolic surface C. Moreover, we can identify it with the hyperbolic surface $\mathbb{H}/\Gamma$ where $\Gamma = \{m^k | k \in \mathbb{Z}\}$. Note that S is a fundamental domain (see Section **??**) of Γ. The surface C is called a hyperbolic cylinder.

Now we parametrize C and compute the metric in these coordinates. In the first place, note that the projection map $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ induces an isometry between the interior of S and C except a line (the projection of γ and γ'). Therefore, to find coordinates in C we find them in S. Consider the Fermi coordinates on $\mathbb{H}$ given by (ρ, t) (see Subsection 2.1.1), this induces the metric $d\rho^2 + \cosh^2 \rho dt^2$ on S. Since S is dense in C, we have the metric $d\rho^2 + \cosh^2 \rho dt^2$ in C. Then, let C be a hyperbolic cylinder as above and $\bar{\eta}$ the projection of η, every point p in C is uniquely determined by (ρ_p, t_p) where $\rho_p = \sigma(p) d_C(p, \bar{\eta}) = \sigma(p) d_C(p, \bar{\eta}(t_p))$ (here $\sigma(p)$ signs the distance). These are called the *Fermi coordinates* of C based at $\bar{\eta}$. Moreover note that $C = \mathbb{R} \times (\mathbb{R}/[t \mapsto t + \ell])$ and the metric tensor in

[7] In fact, this identification defines an isometry of those geodesics. This is why this is a particularization of the usual procedure to attach manifolds along boundaries $S \cup_f R$.

the coordinates (ρ, t) is $d\rho^2 + \cosh^2 \rho dt^2$. If we parametrize $\bar{\eta}$ with speed ℓ we get $d\rho^2 + \ell^2 \cosh^2 \rho dt^2$. From this tensor the geodesic γ is the only closed geodesic in C.

Definition 2.2.2.6. A collar of width (sometimes called length) $\epsilon > 0$ of a closed geodesic γ in a hyperbolic surface S is a neighbourhood of γ isometric to $\{(\rho, t) \in C| -\epsilon \leq \rho \leq \epsilon\}$, where C is a hyperbolic cylinder and the corresponding closed geodesic η of C has the same length as γ. We define a half collar of γ analogously with the neighbourhood $\{(\rho, t) \in C| \ 0 \leq \rho \leq \epsilon\}$. Finally, since this isometry gives a local parametrization around γ in S, the Fermi coordinates (ρ, t) in S will be called the Fermi coordinates in S or the collar coordinates.

Proposition 2.2.2.7. *Let S be a hyperbolic surface with non-empty boundary that consists of closed geodesics. Each boundary geodesic has a half collar of width ϵ for some $\epsilon > 0$. Moreover, every closed geodesic has a collar of width ϵ' for some $\epsilon' > 0$. As a consequence, all boundary closed geodesics of the same length have isometric neighbourhoods and all closed geodesics of the same length have isometric vicinities.*

Proof. In the extended version [Bar21] a proof, that the author has not been able to find in the literature, is given based on the so-called developing map. □

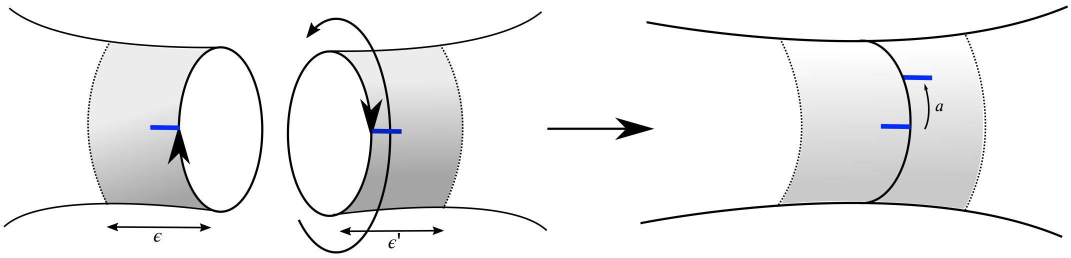

Figure 2.9: Pasting of closed geodesics $S +_a S'$, collars and half-collars.

Example 2.2.2.8. (The pasting/docking of closed geodesics) Let S and S' be two hyperbolic surfaces (not necessarily distinct) and let γ and γ' be boundary closed geodesic of S and S' respectively (distinct if $S = S'$) . If they have the same length ℓ we can paste them together. Choose a parametrization of both curves in $\mathbb{S}^1$ with speed ℓ. If the geodesics have the same orientation (in Figure 2.2.2.8 the surfaces are to the left of the corresponding boundary geodesics), the pasting condition $\gamma(t) = \gamma'(-t)$ gives rise to a hyperbolic surface $S + S'$ where γ and γ' project to a closed geodesic $\bar{\gamma}$ of length ℓ in $S + S'$. One can visualize this pasting two half-collars obtaining a collar. We remark that the geodesic $\bar{\gamma}$ will have a neighbourhood $\{p \in S + S' | d(p, \bar{\gamma}) \leq \epsilon\}$ isometric to a collar which allows us to parametrize this set with "collar/Fermi" coordinates. Finally, note the following interesting phenomenon. Since $\gamma'(t)$ is parametrized in $\mathbb{S}^1$, $\gamma'(t - a)$ is another parametrization of the same closed geodesic and hence we could paste S and S' imposing $\gamma(t) = \gamma'(a - t)$, let us call it $S +_a S'$. This does not change the local hyperbolic structure (it does not change the length of the geodesic, hence there will be isometric collars) but the hyperbolic structure of $S + S'$ and $S +_a S'$ need not be the same.[8] We will explore this further in Section 2.3.3.

Remark 2.2.2.9. We have found the existence of collars qualitatively. In Theorem 2.3.6.1 we give a quantitative bound on the width of collars. A consequence of that theorem to keep in mind is that the shorter the closed geodesic, the wider the collar and the other way around too.

2.2.3 Group Models of Hyperbolic Surfaces

In Theorem 2.2.1.4 we have showed that the group model theory for Riemann surfaces translates identically to hyperbolic surfaces changing the word biholomorphism by isometry. However, we have introduced a class of hyperbolic surfaces (those with boundary) where this group model theory is not known yet. Fortunately everything is quite the same considering hyperbolic-convex regions in $\mathbb{H}$ as universal coverings instead of the whole plane $\mathbb{H}$. We will mainly focus on hyperbolic surfaces with boundary consisting of closed geodesics, the following theorem is key.

[8]Note that $\gamma'(t) = \gamma'(t + 1)$ because $t \in \mathbb{S}^1 = \mathbb{R}/[t \mapsto t + 1]$. Therefore, we have that $S + S' = S +_1 S'$ for example.

Theorem 2.2.3.1. *Let S be a complete hyperbolic surfaces with boundary consisting of closed geodesics. There exists a complete hyperbolic surface S^* without boundary such that S is both isometrically embedded and a deformation retract of S^*.*

Proof. The idea is to sow sleeves to the closed boundary geodesics. Let γ be a length ℓ boundary closed geodesic in ∂S. We consider the truncated hyperbolic cylinder $[0,\infty) \times \mathbb{S}^1$ and η the closed geodesic of image $\{0\} \times \mathbb{S}^1$. For graphical intuition recall that γ has a width ϵ collar isometric to $[-\epsilon, 0] \times \mathbb{S}^1$, we will paste it with the cylinder to get $[-\epsilon, \infty) \times \mathbb{S}^1$. The pasting condition $\gamma(t) = \eta(-t)$ for $t \in \mathbb{S}^1$ sows the cylinder to S. Doing this to every geodesic boundary we get S^*. The other statements are readily seen. □

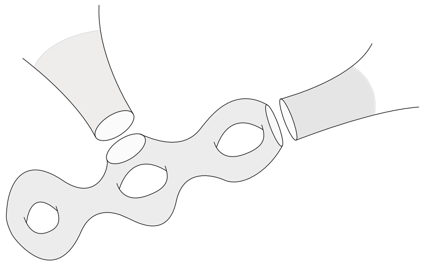

Figure 2.10: Attaching hyperbolic cylinders or "sleeves".

Theorem 2.2.3.2. *Let S be a complete hyperbolic surface with boundary of closed geodesics. The universal cover $\tilde{S}$ of S is isometric to a convex*[9] *subset of $\mathbb{H}$ whose boundary consists of geodesics and S isometric to a quotient of $\tilde{S}$ by a discrete subgroup of* $\mathrm{Is}^+(\mathbb{H})$.

Proof. We know that the hyperbolic surface S^* given by the theorem above is universally covered by $\mathbb{H}$. Let $\pi : \mathbb{H} \to S^*$ be the covering map, define $\tilde{S}$ to be a connected component in the fiber of S, that is, a connected component of $\pi^{-1}(S)$. The boundary of $\tilde{S}$ consists of the lifts of ∂S and hence of a countable collection of disjoint complete geodesics in $\mathbb{H}$. As such, its a countable intersection of half spaces (which are convex) and hence convex. The group model of S will be the group model of S^*. □

Remark 2.2.3.3. Notice that this reasoning proves that if S is a compact simply connected hyperbolic surface with piece-wise geodesic boundary (that is, an abstract hyperbolic geodesic polygon) then it is isometric to a polygon in $\mathbb{H}$. Indeed, as in Theorem 2.2.3.1 we paste pieces of non-compact 3 sided polygons to each side of S to obtain a non-compact simply connected hyperbolic surface S^* without boundary. Since it is universally covered by $\mathbb{H}$ and S^* is simply connected, $\mathbb{H}$ and S^* are isometric. In particular, this proves our claim about S being a polygon in $\mathbb{H}$.

Now that we have understood group models for surfaces with boundary, we will prove that the group model of a compact hyperbolic surface (possibly with closed geodesics as boundary) consists only of hyperbolic elements of $\mathrm{Is}^+(\mathbb{H}) = \mathrm{Aut}(\mathbb{H})$. In particular, all compact Riemann surfaces only have hyperbolic transformations in their deck group. This is a well known fact that will be important in the next section. Here we provide a very telling geometrical proof, found in [Mar16], relating minimum displacement of transformations and the injectivity radius.

For now, let S be any hyperbolic manifold without boundary, as a reminder, we define $\mathrm{inj}_p S$ as the supremum of all $r > 0$ such that $B(p,r) = \{q \in S : d_S(p,q) < r\}$ is isometric to a ball of radius r in $\mathbb{H}$. The injectivity radius[10] of S is $\mathrm{inj}\, S = \inf_{p \in S} \mathrm{inj}_p S$. Now, since $\mathrm{inj}_p S$ is positive for hyperbolic surfaces and $\mathrm{inj}_p S$ varies continuously with p, if S is compact, $\mathrm{inj}_p S$ has a minimum and a maximum. In particular, $\mathrm{inj}\, S > 0$ for a compact hyperbolic surface. If $X \subset \mathbb{H}$ is a discrete set we define $d(X)$ to be the infimum of $d(z,w)$ amongst all different points z, w in X. The following relationship between the injectivity radius and the fiber of each point in the universal covering is the key to our proof.

[9] A convex subset of $\mathbb{H}$ is a subset of $\mathbb{H}$ such that the length minimizing geodesic connecting any two points on this subset has its support inside this subset.

[10] This is a well know magnitude of Riemannian manifolds defined using the exponential map but for hyperbolic surfaces this description is simpler.

Proposition 2.2.3.4. *Let $S = \mathbb{H}/\Gamma$ be a hyperbolic surface and $\pi : \mathbb{H} \to S$ the projection map. For every $p \in S$ we have that*

$$\operatorname{inj}_p S = \frac{1}{2} d(\pi^{-1}(p)).$$

Proof. Firstly, $\operatorname{inj}_p S$ is the supremum over r of the $B(p,r)$ that are isometric to open balls of radius r in $\mathbb{H}$. Secondly, $B(p,r)$ is actually isometric to an open ball of radius r in $\mathbb{H}$ if and only if its counterimage via π consists of *disjoint* balls of radius r. This can only be if any two distinct points on $\pi^{-1}(x)$ stay at distance at least $2r$. □

The reader is encouraged to finish reading the appendix (in particular the last page titled "PSL(2, $\mathbb{R}$) and Hyperbolic Geometry") because its content will be needed in this section and the following.

Corollary 2.2.3.5. *Let $S = \mathbb{H}/\Gamma$ be a hyperbolic surface and let d also be the minimum displacement of an element of* $\operatorname{Aut}(\mathbb{H})$. *We have that*

$$\operatorname{inj} S = \frac{1}{2} \inf\{d(f) | f \in \Gamma \setminus \{Id\}\}.$$

If S is compact, every non-trivial element of Γ is hyperbolic.

Proof. Note that $d(\pi^{-1}(p)) = \inf\{d(q, f(q)) | f \in \Gamma,\ q \in \pi^{-1}(p)\}$ because the elements of the deck group Γ run through the entire fiber. The injectivity radius will be the half the infimum of this infimum, which gives the desired equality. For the last statement, recall that Γ may only contain parabolic and hyperbolic elements and that parabolic elements have zero minimal displacement and hyperbolic elements have positive minimal displacement. Since the injectivity radius of a compact surface is positive, the equality implies that the elements of Γ are hyperbolic. □

In the following theorem we remark our main accomplishment in this section and we adapt it for surfaces of signature (g, n), which it is not often done in the literature.

Theorem 2.2.3.6. *Let S be a compact hyperbolic surface (possibly with boundary of closed geodesics), the non-identity transformations of its deck group are hyperbolic. In particular, group models of Riemann surfaces only have hyperbolic elements (other than the identity).*

Proof. The case with empty boundary is done in the corollary above. If S has boundary of closed geodesics we sow hyperbolic cylinders and obtain S^*. The injectivity radius of S^* is well defined and positive on $S \subset S^*$ since S is compact. We show it is also positive in S^*. First, it is straightforward that hyperbolic cylinders have positive injectivity radius. Second, pasting a hyperbolic cylinder to S is as pasting a half-width $[-\epsilon, 0] \times \mathbb{S}^1$ to $[0, \infty) \times \mathbb{S}^1$, this means that the injectivity radius in the sleeves of S^* (i.e. $[0, \infty) \times \mathbb{S}^1$ of S^*) is positive. In conclusion, the injectivity radius of S^* is positive. The formula in Corollary 2.2.3.5 implies that the group model of S^* is made up of transformations with positive minimum displacement, hence of hyperbolic transformations. The group model of S^* is the same as the group model of S by Theorem 2.2.3.2. □

2.2.4 Closed Geodesics

In this section we study closed geodesics closer than before. We define them again for clarity.

Definition 2.2.4.1. Let S be a hyperbolic surface, a simple closed curve is a continuous embedding $\gamma : \mathbb{S}^1 \to S$ and a closed curve is a continuous immersion $\gamma : \mathbb{S}^1 \to S$. A closed geodesic in a hyperbolic surface is a smooth map $\gamma : \mathbb{S}^1 \to S$ such that $\gamma \circ \pi : \mathbb{R} \to S$ (with $\pi(t) = e^{2\pi i t} : \mathbb{R} \to \mathbb{S}^1$ the universal covering of $\mathbb{S}^1$) is a non-constant geodesic (in particular, γ is an embedding). We implicitly regard two closed curves γ, β to be equal when $\beta \circ \pi(t) = \gamma \circ \pi(t + a)$, $a \in \mathbb{R}$, (note that a whole turn is $t = 1$ and not $t = 2\pi$). Sometimes, we say that two closed curves are freely homotopic instead homotopic to accentuate that the homotopy is not a path-homotopy, there are no base-points involved here. We will denote the set of free homotopy classes of continuous maps from $\mathbb{S}^1$ to S as $[\mathbb{S}^1, S]$.

Theorem 2.2.4.2 (Unique Geodesic Representative)**.** *Let S be a compact hyperbolic surface possibly with boundary of closed geodesics, that is is a surface of signature (g, n). Then, there is a unique geodesic representative in each non-trivial free homotopy class of closed curves. In other words,*

$$\left\{ \text{Closed geodesics in } S \right\} \longleftrightarrow [\mathbb{S}^1, S].$$

Moreover, if a closed geodesic γ is freely homotopic to a simple curve, then γ is also simple.

This is a most remarkable fact with quite an elegant proof (which we carry out here) that puts together algebraic and geometric aspects of hyperbolic surfaces. As a matter of fact, this is the "technical" result that enables the Fenchel-Nielsen coordinates to work. For every homotopy class of curves in $[\mathbb{S}^1, S]$, a topological invariant of S, we find different geodesic representatives for each hyperbolic metric, a hyperbolic invariant. Therefore, each metric associates a number to each homotopy class, the length of the unique geodesic representative. This enables the definition of length functions.

Definition 2.2.4.3. Let S be a hyperbolic surface, the function that assigns to each element γ of $[\mathbb{S}^1, S]$ the length of its unique geodesic representative is denoted by $\ell_\gamma(S)$. The spectrum of a set X of homotopy classes is the set $\{\ell_\gamma(S) | \gamma \in X\} \subset \mathbb{R}_+$.

Hence every hyperbolic surface has a natural set of real numbers associated to it, the spectrum of $[\mathbb{S}^1, S]$. Could this set of invariants determine the hyperbolic structure? Surprisingly, it does. Even more surprisingly, only finitely many classes of curves are needed, this is the basis for the Fenchel-Nielsen coordinates.

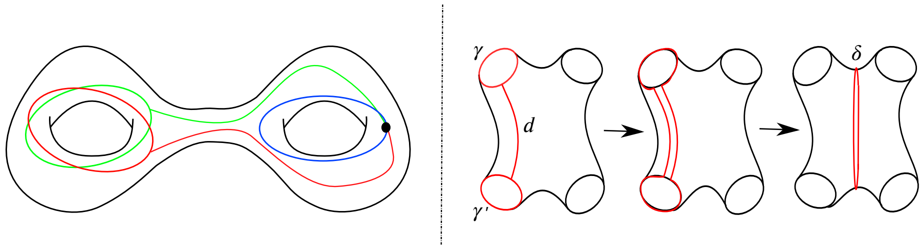

Figure 2.11: Left: The green and red curves are not path-homotopic but they are homotopic. If we allow the base point to vary during the homotopy it is clear that they are homotopic. Alternatively, they are homotopic because they are conjugate by the blue curve, hence in the same conjugacy class in the fundamental group, which corresponds to a unique homotopy class. Right: the red curve in the first figure is $d\gamma d^{-1}\gamma'$, the process shows how it is freely homotopic to δ.

From topology we know that the natural map $\pi_1(S, p) \to [\mathbb{S}^1, S]$ induces a bijection between the conjugacy classes of $\pi_1(S, p)$ and the set $[\mathbb{S}^1, S]$, see Figure 2.11. Moreover, if we have a hyperbolic surface $S = C/\Gamma$ (with $C = \mathbb{H}$ or, if S has non-empty boundary, C is a convex subset of $\mathbb{H}$) , we know that the fundamental group if isomorphic to Γ (see Remark 1.1.1.12). Therefore, the conjugacy classes of Γ are in bijection with $[\mathbb{S}^1, S]$. In particular, since being parabolic or hyperbolic and the minimum displacement are conjugacy class invariants, this allows us to speak about hyperbolic and parabolic elements of $[\mathbb{S}^1, S]$ and their minimum displacement.

Proposition 2.2.4.4. *Let S be a hyperbolic surface (possibly with boundary of closed geodesics). Every hyperbolic element of $[\mathbb{S}^1, S]$ is represented by a unique closed geodesic of length equal to its minimum displacement. Trivial and parabolic elements are not represented by closed geodesics.*

Proof. We prove it first in the case S has empty boundary. Let $S = \mathbb{H}/\Gamma$ and f a hyperbolic transformation of Γ. The axis of f descends to a closed geodesic on S. Indeed, note that f identifies points of A_f at distance $d(f)$ (minimum displacement of f), making it a closed curve in the quotient. Moreover, since A_f is a geodesic and the projection map is a local isometry, A_f descends to a geodesic of length d. Moreover, conjugate transformations determine the same closed geodesic since $A_{g \circ f \circ g^{-1}} = g(A_f)$ and they both have the same minimum displacement. On the other hand, a closed geodesic lifts to an arc connecting $z \in \mathbb{H}$ to $f(z)$ for an $f \in \Gamma$ that preserves the line through z and $f(z)$. This concludes the proof since only hyperbolic transformations fix lines.

For the case S has boundary of closed geodesics, as usual, we consider S^* given by Theorem 2.2.3.1. Here our previous discussion applies and it implies the thesis of the proposition for S. Indeed, the main thing to note is that S is a deformation retract of S^*, which implies that $[\mathbb{S}^1, S] = [\mathbb{S}^1, S^*]$. Moreover, hyperbolic cylinders only have one closed geodesic, which is the one sown to S. We conclude that each closed geodesic in S^* is actually in S. □

Corollary 2.2.4.5. *Let $S = C/\Gamma$ be a compact hyperbolic surface possibly with boundary of closed geodesics. We have the following bijections:*

$$\left\{ \text{Closed geodesics in } S \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Conjugacy classes of} \\ \text{hyperbolic elements of } \Gamma \end{array} \right\} = \left\{ \text{Conjugacy classes of } \Gamma \right\} \longleftrightarrow [\mathbb{S}^1, S]$$

Proof. The proposition above proves the first bijection. The equality comes from Theorem 2.2.3.6 and the last bijection has already been discussed. □

Corollary 2.2.4.6. *The length functions on $\mathcal{T}_g$ are continuous for each $\gamma \in [\mathbb{S}^1, F_g]$:*

$$\begin{aligned} \ell_\gamma : \mathcal{T}_g &\longrightarrow \mathbb{R}_+ \\ S &\longmapsto \ell_\gamma(S) \end{aligned}$$

In particular, fixing a free homotopy class of curves the length of the geodesics in this class varies continuously as we vary the marked hyperbolic metric.

Proof. The trace function is a continuous function on $\mathcal{T}_g = DF(\pi(F_g, p), \mathrm{PSL}(2, \mathbb{R})) / \mathrm{PSL}(2, \mathbb{R})$ and by Proposition 9 in the Appendix the trace functions and minimum displacement are related analytically. That, together with the fact that closed geodesics have length equal the minimum displacement of the hyperbolic transformation that give place to them, we conclude that the length functions are continuous on $\mathcal{T}_g$. □

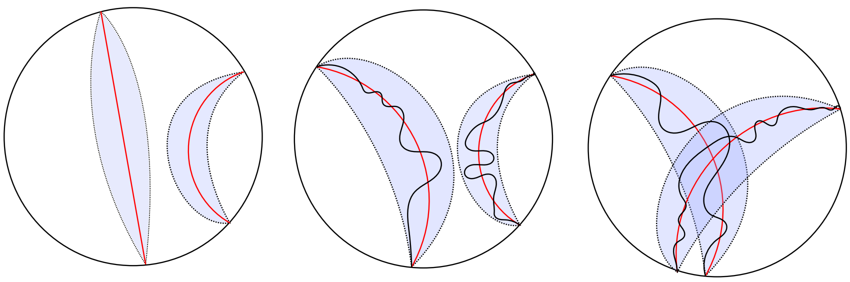

Figure 2.12: Left: R-neighbourhoods of the geodesics in red. Center: how lifts of c (in black) stay withing R-neighbourhoods of lifts of γ (in red). Right: how if lifts of γ intersect, lifts of c do too.

Proof of Theorem 2.2.4.2. The existence and uniqueness have already been proven. Let $S = \mathbb{H}/\Gamma$ be a compact hyperbolic surface. We assume it has empty boundary, in case it does not, the proof is the same. Let c be a simple closed curve homotopic to the closed geodesic γ. Let $\tilde{c}$ and $\tilde{\gamma}$ be lifts of c and γ, all other lifts are the orbits of those by Γ, that is the orbit is the fiber of the curve. Since geodesics have to lift to geodesics, and geodesics in $\mathbb{H}$ are simple geodesic arcs (with no self-intersections), to prove that γ is simple it suffices to see that all lifts are disjoint and that γ is primitive.[11] Let us prove that the fiber of γ is made up of disjoint arcs. Suppose it is not, we show that then two arcs on the fiber of c have to intersect. If this happened, then c would have self-intersections, against the simplicity hypothesis.

Let H be the homotopy from c to γ, it lifts to homotopies from arcs on the fiber of c to arcs on the fiber of γ. Since the image/support of H in S is compact, no point is dragged more than a certain number $R > 0$. In particular, the lifts of γ have R-neighbourhoods[12] and the corresponding homotopic

[11] A curve γ is said to be primitive if there is no curve ν such that $\gamma = \nu^n$ for $n \in \mathbb{Z}^{>1}$, here ν^n indicates the concatenation of the curve ν with itself n times.

[12] An R-neighbourhood of a curve β is $\{p \in \mathbb{H} | d_{\mathbb{H}}(\beta, p) \leq R\}$.

lifts of c have to be inside this neighbourhood, see the figure. Therefore, if two arcs in the fiber of γ intersect the corresponding arcs in the fiber of c also intersect. This concludes the disjointness proof.

That γ is primitive is a general fact that admits simpler proofs for hyperbolic manifolds. Ones reduces the primitivity of curves to the primitivity of the transformations in Γ that birth them. For details see [FM12, Proposition 1.4]. Another approach can be found in the first chapter of [Bus10]. □

2.2.5 Geodesic Arcs and Perpendiculars

In Theorem 2.1.0.4 we characterized geodesic and perpendiculars in $\mathbb{H}$ and $\mathbb{D}$, we would like to generalize this to hyperbolic surfaces. For example, given two different points in $\mathbb{H}$, there is always a geodesic joining them. Since the hyperbolic plane is simply connected, for general surfaces we should somewhat expect existence of geodesics in each homotopy class of curves instead of a unique geodesic given any two points, see Figure 2.13. Moreover, the notion of homotopy "with fixed endpoints" is too restrictive for our purposes, we will allow endpoints to glide. For example, observe that given a simple primitive curve connecting two geodesics at a positive distance in $\mathbb{H}$ we will only be able to find a perpendicular to both geodesics homotopic to the given simple curve if we allow endpoints to glide in the geodesics.

Definition 2.2.5.1. Let A and B be closed connected subsets of a hyperbolic surface S. Assume that $c, c' : [a.b] \to S$ are two curves with initial points $c(a), c'(a) \in A$ and endpoints $c(b), c'(b) \in B$. We say that c is homotopic to c' with endpoints gliding on A and B if there exists a homotopy $H : [0,1] \times [a,b] \to S$ between c and c' such that the terminal points along the deformations are only allowed to vary in A and B, i.e. $H(t,a) \in A$ and $H(t,b) \in B$ for $t \in [0,1]$

Theorem 2.2.5.2. *Let S be a compact hyperbolic surface with (possibly empty) boundary and let $c : [a,b] \to S$ be a curve with $c(a) \in A$ and $c(b) \in B$.*

(1) If A and B are points, then there exists a unique geodesic arc homotopic (with fixed endpoints) to c. If c is simple, the geodesic need not be simple (see Figure 2.13).

(2) Assume A and B are disjoint closed boundary geodesics of S. There exists a unique geodesic γ homotopic to c with endpoints gliding on A and B that meets A and B perpendicularly at its endpoints. All other points of γ lie in the interior of S. Moreover, if c is simple, then γ is simple.

Proof. We prove the existence claim in (1) because the existence for (2) is analogous and we prove the uniqueness claim in (2) because the uniqueness for (1) is a similar argument. We will also prove all other claims in (2) other than the simplicity assertion, this is done similarly as in Theorem 2.2.4.2. For details see [Bus10, Theorem 1.5.3].

To begin with, we cover S with finitely many charts that are isometries with regions of $\mathbb{H}$. In each of these we have a unique geodesic joining any two points. Now let M be the Lebesgue number of this cover: if two points are closer than M then they are joint by a unique geodesic. We go on to consider the path-homotopy class of c, that we denote H, and also let L be the infimum of lengths of piece-wise smooth curves in H. Choose a sequence of such curves $(c_n)_n$ in H such that $\ell(c_n) \to L$. Note that $d(c_n(x), c_n(y)) \leq \int_a^b ||c_n'||dt \leq K|x-y|$ because $||c_n'||$ is uniformly bounded since $\ell(c_n) \to L$. In virtue of this, we apply the Arzelà-Ascoli theorem[13] to obtain that the c_n converge to a curve $\gamma : [a,b] \to S$ (taking partial subsequences if necessary). We prove that this is the curve we were seeking.

We partition $[a,b]$ into $a = t_0 < \cdots < t_N = b$ finely enough so each arc $c_n([t_i, t_{i+1}])$ is of length M at most. Moreover, the endpoints $c_n(t_i)$ converge to $\gamma(t_i)$ and for n_0 large enough there is a c_{n_0} such that for each i, $c_{n_0}([t_i, t_{i+1}])$ is in the same open set of the cover than $\gamma([t_i, t_{i+1}])$. In particular, we can find a geodesic arc from $\gamma(t_i)$ to $\gamma(t_{i+1})$, that we call β_i, homotopic to c_{n_0}. Putting all of the β_i together we obtain a piece-wise geodesic curve β in H. The key part is that this curve β minimizes distance in H and hence it can be reparametrized into a geodesic. Now, since γ also minimizes distance we have that $\gamma = \beta$ and therefore $\gamma \in H$.

[13]A well known direct generalization of the classical Arzelà-Ascoli theorem is the following: if X is a compact metric space and we have K-Lipschitz maps $c_n : [a,b] \to X$, then there exists a subsequence converging uniformly on compact sets to a K-Lipschitz map $\gamma : [a,b] \to X$. The proof is almost the same as the classic result.

For (2), we prove the perpendicularity claim. We lift γ and A to $\mathbb{H}$ at a point in the fiber of $\gamma(a) \in S$. If we prove the lifts meet perpendicularly, since the projection is a local isometry, we will have proved that γ meets A perpendicularly. So, if the lifts did not meet at a right angle we could homotope the lift of γ into a curve that minimizes distance (because of Theorem 2.1.0.4) against the geodesic status of the lift of γ. In fact, the same argument proves that $\gamma(a,b) \subset \operatorname{Int} S$. Now we are ready to prove the uniqueness statement and conclude the proof.

Take $\tilde{A}, \tilde{B}$ and $\tilde{\gamma}$ lifts in $\mathbb{H}$ of A, B and γ such that $\tilde{\gamma}$ is the perpendicular from $\tilde{A}$ to $\tilde{B}$. Every homotopy with endpoints gliding on A and B in S lifts to a homotopy with endpoints gliding on $\tilde{A}$ and $\tilde{B}$. From the uniqueness of common perpendiculars in $\mathbb{H}$ follows our uniqueness claim in S. □

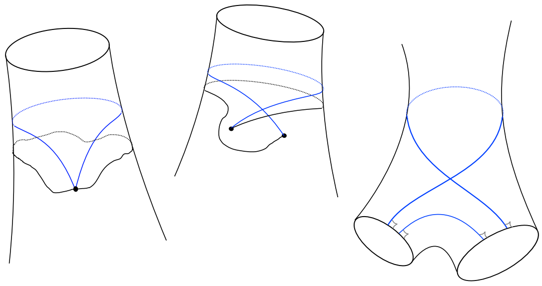

Figure 2.13: In the left we see an example for $A = B = \{p\}$, it is important to note that this proves nothing about closed geodesics, since here the homotopy is relative to p. In the center we see how γ need not be simple if c is simple if A and B are points. In the right we can see two instances of different homotopy classes yielding different perpendiculars.

Remark 2.2.5.3. Even though the theorem above does not say anything about closed geodesics, the same "local" arguments yield the same conclusion as Theorem 2.2.4.2, see [Bus10, theorem 1.6.6] for this proof. Moreover, note that this could be used, inversely to Theorem 2.2.4.2, to prove the hyperbolicity of elements in group models for compact hyperbolic surfaces.

2.3 The Teichmüller Space $\mathcal{T}_g$ and the Fenchel-Nielsen Coordinates

2.3.1 Pants Decompositions

This section is mostly topological in nature, we briefly explore the decomposition of surfaces (no extra structure) into pairs of pants. A pair of pants is a surface homeomorphic to a sphere with three open discs removed. Let S be a compact surface (possibly with boundary) with Euler characteristic $\chi(S) < 0$. A pair of pants decomposition of S is a collection of disjoint simple closed curves in S such that cutting S along those curves a disjoint union of pairs of pants is obtained. One can always find such a decomposition. The following, amongst other things, gives a reason why.

Remark 2.3.1.1. The pairs of pants decomposition is basically the only maximal decomposition of "its sort": indeed, one could also define a pants decomposition of S as a maximal collection of disjoint, essential[14] simple closed curves in S that are pair-wise non-isotopic. Let us prove that both definitions are equivalent. On the one hand, suppose we have a collection of simple closed curves that cut S into pairs of pants. Clearly, this means they are all essential. Furthermore, since any simple closed curve on a pair of pants is either homotopic to a point or to a boundary component the collection has to be maximal. On the other hand, assume we have a maximal collection of disjoint non-isotopic essential simple closed curves in S. Suppose that these curves do not cut S into a collection of pairs of pants. It follows from the classification of surfaces and the additivity of Euler characteristic that there is at least one component of the cut surface that either has positive genus or is a sphere with more than 3 boundary components. In both cases, there are essential simple closed curves that are not homotopic to a boundary component, contradicting the maximality condition.

[14]The closed curve c is essential if no component of $S \setminus c$ is a disc with possibly one puncture.

Now we count how many pairs of pants and curves a pants decomposition has. If we cut a surface along a collection of disjoint simple closed curves, the cut surface has the same Euler characteristic as the original surface. Since pairs of pants have Euler Characteristic -1, a pants decomposition must cut S into $-\chi(S)$ pairs of pants. This means that for a compact genus g surface S the pants decomposition has $2g-2$ pants and, since every trouser has 3 boundary components and they are pasted in pairs, there are $3\chi(S)/2 = 3g-3$ decomposing curves. By the same token, if S also had n boundary components the pants decomposition would have $3g+n-3$ curves and $2g-n-2$ pants.

Remark 2.3.1.2. The discussion so far has been centered around decomposition. Now we briefly mention the combinatorial scheme to glue pairs of pants together into a compact surface of genus g. Each 3-regular connected graph yields a way to put together pairs of pants, see Figure 2.14. A vertex represents a pair of pants and the edges represent glued curves. Note that a 3-regular graph (i.e. a graph such that each vertex has exactly 3 edges, counting loops twice) with $2g-2$ vertices always has $3g-3$ edges. One interesting application of this characterization is that we can use graph theory to give bounds on the number of essentially different ways to put pairs of pants together. If $m(g)$ denotes the number of pairwise non-isomorphic connected 3-regular graphs with $2g-2$ vertices, then

$$2^{g-3} \leq m(g) \leq g^{3g}$$

see [Bus10, Theorem 3.5.3] for a proof of this. Finally, note that for any $g > 1$ there is always a combinatorial scheme that does not glue together two curves on the same pair of pants. For this it's enough to give a graph with no loops, see Figure 2.14 again.

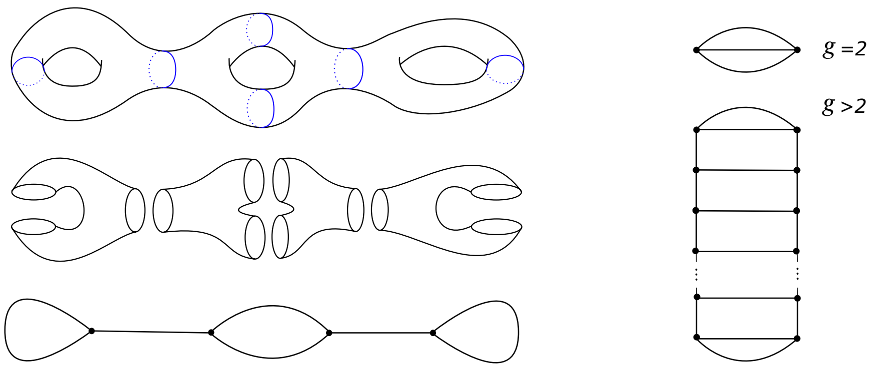

Figure 2.14: Left: A pants decomposition of a genus g surface and the corresponding graph. Right: An example for all $g > 1$ of a pants decomposition that does not paste geodesics on the same pant.

Now we fix some notation that we will use along this section when talking about this construction. From now on, we suppose that all Y-pieces are given a hyperbolic structure (in particular, the 3 boundary components are closed geodesics). Let us fix a genus g and consider $\{Y_i\}_{i=1,\ldots,2g-2}$ hyperbolic pairs of pants, each Y_i has boundary geodesics $\{\gamma_{ij}\}_{j=1,2,3}$. Now we decide on a way to assemble these pants, that is, we fix a 3-regular connected graph. Let us say that Y_i and Y_r are pasted together in this combinatorial model along the geodesics γ_{ij} and γ_{rs}. Since we would like the resulting surface to be hyperbolic we do the pastings according to Example 2.2.2.8. In particular, we ask that $\ell_k = \ell(\gamma_{ij}) = \ell(\gamma_{rs})$ and the pasting condition is $\gamma_{ij}(t) = \gamma_{rs}(a_k - t)$ for $t \in \mathbb{S}^1$ for some $a_k \in \mathbb{R}$. Thus, γ_{ij} and γ_{rs} yield a geodesic $\gamma_k(t)$ of length ℓ_k. Therefore, in the pasted surface we have $\{\gamma_k\}_{k=1,\ldots,3g-g}$ pasting geodesics and to assemble the surface we have to specify the length of each γ_k and the parameter a_k (and the combinatorial assembly model too but we already fixed it). That is, we specify the parameters

$$(L, A) = (\ell_1, \ldots, \ell_{3g-3}, a_1, \ldots, a_{3g-3}) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

We will write F^{LA} for the surface built according to this parameters and $\pi^{LA} : Y_1 \sqcup \cdots \sqcup Y_{3g-3} \to F^{LA}$ the corresponding pasting projection. The pasting geodesics $\{\gamma_k\}_{k=1,\ldots,3g-g}$ in F^{LA} will be written as $\{\gamma_k^{LA}\}_{k=1,\ldots,3g-g}$ unless $(L, A) = (1, \ldots, 1, 0, \ldots, 0)$, when we will omit the superindex LA and simply write $\{\gamma_k\}_{k=1,\ldots,3g-g}$.

The L and A parameters are respectively called the length and twist/angle parameters of F^{LA}. In the next section we study hyperbolic pairs of pants to understand what can be done with the L parameters and in Section 2.3.3 we study X-pieces to understand the A parameters.

2.3.2 Hyperbolic Pairs of Pants

Definition 2.3.2.1. From now on, a topological 3-holed sphere will be called a topological pair of pants as opposed to a pair of pants (or Y-piece or trouser), which will be a compact Riemann surface of signature $(0, 3)$ by default.

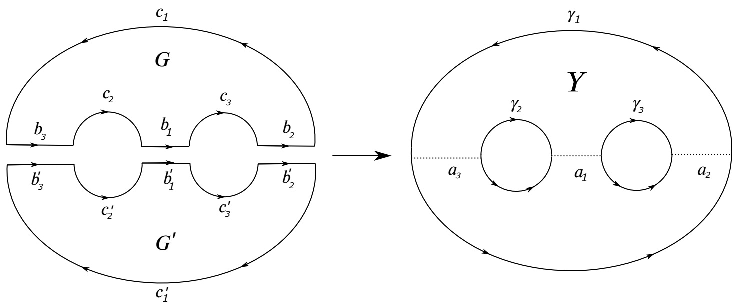

Figure 2.15

Example 2.3.2.2 (Construction of a Y-Piece via Hexagons)**.** Let G be a right-angled geodesic hexagon in the hyperbolic plane with consecutive sides $b_1, c_3, b_2, c_1, b_3, c_2$ and G' a copy with consecutive sides $b'_1, c'_3, b'_2, c'_1, b'_3, c'_2$. We parametrize all sides on the interval $[0, 1]$ with constant speed in a way that all sides form a closed boundary curve. The following pasting scheme:

$$(\wp) : a_i(t) := b_i(t) = b'_i(t), t \in [0, 1], i = 1, 2, 3$$

defines a hyperbolic surface Y that inherits the hyperbolic structure of G and G':

$$Y = G + G' \ mod \ \wp$$

To see that it is a pair of pants we have to prove that the boundary curves are closed geodesics. For $i = 1, 2, 3$ we define $\gamma_i(t)$ as $c_i(2t)$ in $0 \leq t \leq 1/2$ and $c'_i(2 - 2t)$ in $1/2 \leq t \leq 1$. These γ_i are the piece-wise geodesic boundary components, since all angles are right-angles these curves are closed geodesics. As such, in the rest of the text these boundary geodesics γ_i will be parametrized in $\mathbb{S}^1$.

Now we prove that all Y-pieces are obtained in this way:

Proposition 2.3.2.3. *Let S be an arbitrary Y-piece. For every pair of boundary geodesics of S there exists a unique simple common perpendicular. The three perpendiculars cut S into two isometric right-angles geodesic hexagons.*

Proof. First of all, we find a triplet a_1, a_2, a_3 of simple geodesics perpendiculars between boundary geodesics. Let us find a_1 the perpendicular connecting γ_2 and γ_3. We just have to note that there exists a simple arc in Y connecting γ_2 and γ_3 and a_1 will be the simple unique perpendicular in its homotopy class (with endpoints gliding on γ_2 and γ_3) given by Theorem 2.2.5.2. In fact, any such arc will be homotopic[15] (with gliding endpoints) to a_1.

We want to see that the perpendiculars are pairwise disjoint: by cutting along a_3, for example, we obtain a hyperbolic surface A. Using Theorem 2.2.5.2 again, there exists a simple perpendicular geodesic a'_1 in A between γ_2 and γ_3; a'_1 will also be a geodesic in Y. Since a_1 and a'_1 are geodesics in

[15] There is only one gliding homotopy class of simple arcs from γ_2 to γ_3, we sketch a proof. From fundamental group theory the path-homotopy classes of simple paths from γ_2 to γ_3 (with fixed endpoints) is characterized by the winding number around each one of the two holes in the Y-piece. Since we can glide endpoints around γ_2 and γ_3 we can make any path with endpoints in those curves not wind around any of the two holes. In conclusion, there is a unique homotopy class of paths joining γ_2 and γ_3 allowing the endpoints to glide.

the same gliding homotopy class, by uniqueness of geodesics, we have that $a_1' = a_1$. Therefore, a_1 does not intersect a_3 since it has not been cut. The same holds for the others.

Cutting along a_1, a_2, a_3 we obtain two simply connected right-angled geodesic hexagons G and G', in Remark 2.2.3.3 we proved that they have to be polygons in $\mathbb{H}$. Uniqueness follows from Theorem 2.1.2.3. □

Theorem 2.3.2.4. *For any triple of positive real numbers l_1, l_2, l_3 there exists a unique*[16] *pair of pants Y with boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ of lengths $(\ell(\gamma_1), \ell(\gamma_2), \ell(\gamma_3)) = (l_1, l_2, l_3)$.*

Proof. Using Theorem 2.1.2.3 there exist a geodesic hexagon in $\mathbb{H}$ with prescribed length of three non-adjacent sides. By the construction in example 2.3.2.2, existence of the desired Y-piece is proved. Let Y and $\tilde{Y}$ be two pairs of pants with boundary $\gamma_i, \tilde{\gamma}_i$, $i = 1, 2, 3$, as in the statement of the theorem. Using the proposition above we cut both pairs of pants into hexagons G, G' and $\tilde{G}, \tilde{G}'$. We can find isometries that take G to $\tilde{G}$ and G' to $\tilde{G}'$ that send a_i to $\tilde{a}_i'$ (following the notation of the proposition above). That defines an isometry from Y to $\tilde{Y}$ sending γ_i to $\tilde{\gamma}_i$. □

Remark 2.3.2.5. An interesting direct consequence of the theorem above is that $\mathbb{R}^3_+$ overdetermines the moduli space $\mathcal{M}_{0,3}$. In fact, it can be seen that the marking by homeomorphism, in the case of pairs of pants, is equivalent to labeling the boundary geodesics; boundary components cannot be permuted. Then, the theorem above shows that $\mathcal{T}_{0,3} = \mathbb{R}^3_+$. What's more, the action on $\mathcal{T}_{0,3}$ of permuting boundary components would give rise to the moduli space, that is $\mathcal{M}_{0,3} = \mathbb{R}^3_+ / S_3$ (here S_3 is the symmetric group on three elements).

We want to define a map that stretches Y-pieces into other Y-pieces homeomorphically (and quasi-isometrically, which is a concept we will introduce in the last section of this work). To do so, first we explain how to stretch hexagons into other hexagons. Let G and G' be a right-angled geodesic hexagons in the hyperbolic plane with consecutive sides $b_1, c_3, b_2, c_1, b_3, c_2$ and $b_1', c_3', b_2', c_1', b_3', c_2'$ respectively. We know define a map $\sigma(G, G')$ that stretches G into G'.

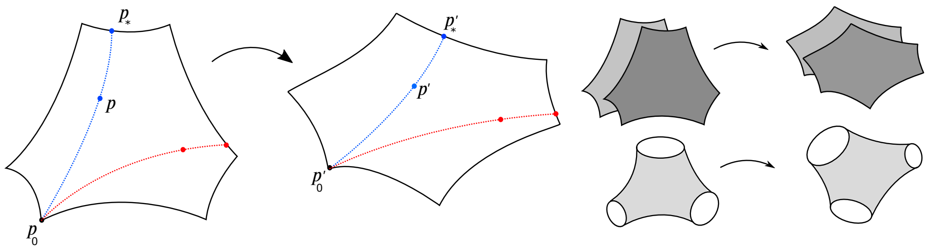

Figure 2.16: Left: Stretch maps for hexagons. Right: how those transfer (approximately) to trousers.

The idea is to take the geodesic sides as reference and send every point to points of "the same proportion" with respect to the reference. Let p_0 and p_0' common vertices of b_2 and c_1 and b_2' and c_1' respectively. For a point $p \neq p_0$ in G there exists a unique side b_i or c_i and a unique $t_p \in [0, 1]$ such that p lies on the geodesic ray from p_0 to $p_* = b_i(t_p)$ or $p_* = c_i(t_p)$. The point $\sigma(G, G')(p)$ will be the point p' on the geodesic ray from p_0 to $p_*' = b_i'(t_p)$ or $p_*' = c_i'(t_p)$ satisfying

$$\frac{d(p_0', p')}{d(p_0', p_*')} = \frac{d(p_0, p)}{d(p_0, p_*)}.$$

Finally, we set $\sigma(G, G')(p_0) = p_0'$. This is clearly continuous and the inverse is defined in the same way, so it is continuous too, hence those stretch maps are homeomorphisms.

Definition 2.3.2.6 (Stretching Trousers)**.** Let Y and T be two arbitrary Y-pieces and let G, G' and H, H' be decomposing hexagons as in example 2.3.2.2 or Proposition 2.3.2.3. We define $\sigma(Y, T) : Y \to T$ the stretching map of Y onto T by setting

$$\sigma(Y, T) = \begin{cases} \sigma(G, H) \text{ on } G \\ \sigma(G', H') \text{ on } G' \end{cases}$$

[16]Here uniqueness is understood modulo boundary preserving isometry. In other words, the boundary components of Y are ordered/labeled.

Since $\sigma(G, H)$ and $\sigma(G', H')$ preserve boundary parametrization they agree on $G \cap G'$. This implies that $\sigma(Y, T)$ is a well defined homeomorphism that preserves boundary parametrizations.

2.3.3 Twist Homeomorphisms and X-Pieces

From Theorem 2.3.2.4 it is beginning to be clear that decomposing a surface into pairs of pants and studying the length of the decomposing geodesics may be a nice way to study the variation of hyperbolic structures. However, note that when we paste two closed geodesics there is a twist parameter which needs to be taken into account. In this section we study hyperbolic surfaces made up of two trousers and twist parameters.

Definition 2.3.3.1. A hyperbolic surface of signature $(0, 4)$ is called an X-piece.

Example 2.3.3.2 (Construction of an X-Piece and Notation)**.** Let Y and Y' be two Y-pieces with boundary geodesics γ_i, γ_i' with $i = 1, 2, 3$ such that $\ell := \ell(\gamma_1) = \ell(\gamma_1')$ and let X^a be the surface that arises from pasting Y and Y' along γ_1 and γ_1' with a a as the twist parameter. That is, $X^a = Y + Y'$ according to the pasting condition $\gamma_1(t) = \gamma_1'(a - t)$ for $t \in \mathbb{S}^1$. In this section π^a will denote the projection map of this pasting and γ^a the closed geodesic in X^a that comes from γ_1 and γ_1', that is, $\gamma^a = \pi^a(\gamma_1(t)) = \pi^a(\gamma_1'(a - t))$. Since Y and Y' are compact hyperbolic surfaces, X^a is also a compact hyperbolic surface and has 4 boundary geodesics, which means X^a is an X-piece.When $a = 0$ we will write X, π and γ instead of X^0, π^0 and γ^0. It turns out that, analogously to Theorem 2.3.2.4, every X-piece is uniquely obtained by this construction.

Theorem 2.3.3.3. *Let X be an X-piece and fix $(\ell_1, \ldots, \ell_5, a) \in \mathbb{R}_+^5 \times [0, 1)$. Then, there are pairs of pants Y and Y' with boundary geodesics γ_i and γ_i' $(i = 1, 2, 3)$ respectively such that $\ell_1 = \ell(\gamma_1) = \ell(\gamma_1')$, $\ell_2 = \ell(\gamma_2)$, $\ell_3 = \ell(\gamma_3)$, $\ell_4 = \ell(\gamma_2')$ and $\ell_5 = \ell(\gamma_3')$ so that $X = Y + Y'$ modulo $\gamma_1(t) = \gamma_1'(a - t)$, $t \in \mathbb{S}^1$. Moreover, X is the only X-piece obtained this way.*[17]

Proof. Label the boundary geodesics of X by $\gamma_3, \gamma_2, \gamma_2', \gamma_3'$ as in Figure 2.18 (center-left) starting from the top left corner/hole clock-wise. We take a geodesic that separates two sets of boundary geodesics and cut along this curve, we obtain (topologically) two 3-holed spheres with a hyperbolic metric with boundary of closed geodesics, i.e. two compact Riemann surfaces of signature $(0, 3)$, two Y-pieces. This curve exists (there is more than one). For example, take a simple perpendicular g from γ_3 to γ_2 and define γ to be the unique closed geodesic in the free homotopy class of the curve $g\gamma_3 g^{-1}\gamma_2$. This way, we get two Y-pieces with boundary geodesics $\gamma_2, \gamma_3, \gamma$ and $\gamma_2', \gamma_3', \gamma$ respectively. In particular, this proves the existence claim.

To prove uniqueness we first fix $a \in [0, 1)$. Let X and $\tilde{X}$ be two X-pieces built according to $X = Y + Y'$and $\tilde{X} = \tilde{Y} + \tilde{Y}'$ where $Y, Y', \tilde{Y}$ and $\tilde{Y}'$ are pairs of pants with the prescribed lengths $\ell_1, \ldots, \ell_5$ as in the statement of the theorem (here both pastings are done with twist a). By Theorem 2.3.2.4, there are isometries $Y \to \tilde{Y}$ and $Y' \to \tilde{Y}'$. By construction, since a is the same in both pastings, they define an isometry $X \to X'$. Finally, we have to see what happens when we let a vary. Let Y and Y' be two pairs of pants as in the statement of the theorem, if $X^a = Y + Y'$ with twist parameter a, then it suffices to prove that $\{X^a\}_{a \in [0,1)}$ are pair-wise non-isometric.

In the first place, let $k \in \mathbb{Z}$, then observe that $X^a = X^{a+k}$. This is direct from the fact that $\gamma(t) = \gamma(t + k)$. Clearly, $X^a \neq X^b$ for $a \neq b \in [0, 1)$, but that they are not isometric is still unclear. So, let $m : X^0 \to X^a$ be an isometry that preserves the boundary (as always, uniqueness is conditional to this), then the pairs of pants Y and Y' in X^0 are sent to pairs of pants $m(Y)$ and $m(Y')$ that decompose X^a. Here we are using the fact that m is a homeomorphism sending "topological" three-holed spheres to three-holed spheres and that m is an isometry to see that the three-holed spheres are pairs of pants. Therefore, since m fixes the boundary components and $X^a = Y + Y'$ with twist a, $m(Y) = Y$ and $m(Y') = Y'$. In particular, we have that $m \circ \pi^0$ has to be of the form π^b for some $b \in [0, 1)$, which would imply that $m = \pi^b \circ (\pi^0)^{-1}$. This is a contradiction because the map $\pi^b \circ (\pi^0)^{-1}$ cannot be an isometry (it does not preserve distance between points, below we study this map more carefully, see Figure 2.17 for example). □

[17] As with Theorem 2.3.2.4, uniqueness is understood up to the order of the boundary components.

Remark 2.3.3.4. As with Remark 2.3.2.5, the theorem above proves that $\mathbb{R}^5_+ \times [0,1)$ overdetermines the moduli space $\mathcal{M}_{0,4}$. In other words, the moduli is determined by the length of 5 curves and 1 twist parameter.

Now we are interested in understanding the twist parameter. Without loss of generality, we consider X-pieces of the form $\{X^a\}_{a\in\mathbb{R}}$ and use the notations introduced in example 2.3.3.2. Recall that around the pasting geodesic γ^a there is a collar $\mathcal{C}[\gamma^a]$ isometric to $[-\epsilon,\epsilon]\times\mathbb{S}^1$ and because of that we can talk about the Fermi coordinates (ρ,t) in $\mathcal{C}[\gamma^a]$. We re-estate how this coordinates are taken: if $p\in\mathcal{C}[\gamma^a]$, there exists a unique length minimizing perpendicular from p to γ^a that meets γ^a at $\gamma^a(t_p)$ and has length $|\rho_p|$, if p is above γ^a the distance is signed negative and below is singed positive (corresponding to left $(-)$ and right $(+)$ with respect to the orientation of γ^a), hence $p=(\rho_p,t_p)\in[-\epsilon,\epsilon]\times\mathbb{S}^1$. Moreover, since γ^a has speed $\ell(\gamma^a)$ we have the following metric tensor in $\mathcal{C}[\gamma^a]$: $ds^2=d\rho^2+\ell(\gamma^a)^2\cosh^2(\rho)dt^2$, see Section 2.2.2 for details.

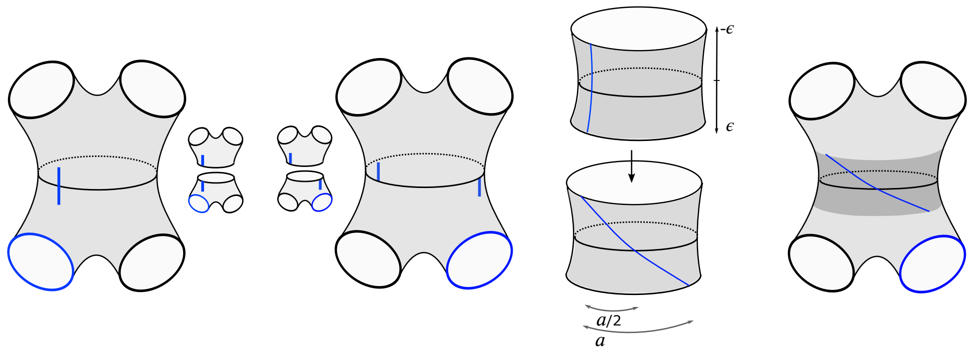

Figure 2.17: Four figures, first from the left: X^0 big and Y and Y' small. Second from the left: X^a big Y and Y' before pasting with twist a small. These two represent the map $\pi^a\circ(\pi^0)^{-1}$. Third from the left: the map T^a. Fourth from the left: the map τ^a.

Now we introduce two ways to twist X-pieces. The first one is taking $X=X^0$, then separating the two Y-pieces, twisting a parts of a turn (i.e. $2\pi a$ radians) the bottom one and the putting them together again, this is represented by $\pi^a\circ(\pi)^{-1}: X\to X^a$, see Figure 2.17. The other way to twist is similar but without separating the two Y-pieces, therefore points in the middle of the X-piece get dragged and twisted, those will be the twist homeomorphisms. Let us introduce them properly.

Take collars in $X=X^0$ and in X^a around γ and γ^a of width 2ϵ. We define a homeomorphism of collars $T^a:\mathcal{C}[\gamma]\to\mathcal{C}[\gamma^a]$ as follows:

$$T^a(\rho,t)=(\rho,t+a\frac{\epsilon+\rho}{2\epsilon}).$$

Observe that for $\rho=-\epsilon$ we have $T^a(\rho,t)=(-\epsilon,t)$, for $\rho=0$ we have $T^a(\rho,t)=(0,t+\frac{1}{2}a)$ and for $\rho=\epsilon$ we have $T^a(\rho,t)=(\epsilon,t+a)$. That is, we have linearly twisted the collar from 0 to a. Moreover, since at $\{-\epsilon\}\times\mathbb{S}^1$ the map T^a agrees with π and at $\{\epsilon\}\times\mathbb{S}^1$ the map T^a agrees with π^a, the following map is a well defined homeomorphism:

$$\tau^a=\begin{cases}T^a \text{ on } \mathcal{C}[\gamma]\\ \pi^a\circ(\pi)^{-1} \text{ on } X\setminus\mathcal{C}[\gamma]\end{cases}$$

These maps τ^a are called *twist homeomorphisms*. It is easy to realize that they are not isometries. Note that, because $X^0=X^1$ the map $\tau^1:X^0\to X^1$ can be regarded as a self-homeomorphism $\tau^1:X^0\to X^0$. In this case we denote τ^1 by $\mathcal{D}$ and it is called an *elementary Dehn twist of* X^0. For $m\in\mathbb{Z}$, any map isotopic to $\mathcal{D}^m$ that fixes the boundary ∂X point-wise is a *Dehn twist of order m*. We define a *Dehn twists of order m for* X^a by $\mathcal{D}^m_a=\tau^a\circ\mathcal{D}^m\circ(\tau^a)^{-1}$. Intuitively, a Dehn twist of order m is just twisting the bottom part of an X-piece m full turns.

Question 2.3.3.5. Given a family of X-pieces $\{X^a|a\in\mathbb{R}\}$, is there a way to know what the value of a is for any given X-piece? More importantly, can we find closed geodesics on X-pieces such that their lengths determine a? This is interesting for geometrical reasons and also because we know length functions are continuous with respect to the algebraic topology of Teichmüller spaces, hence if a is expressed continuously as length functions, it will also be a continuous function on the Teichmüller space.

The answer is yes and it is of capital importance for our construction of the Fenchel-Nielsen coordinates. Take $X = X^0 = \pi(Y \sqcup Y')$, let b_2 and b_2' in Y and Y' be the simple joining perpendiculars as in Figure 2.15 and also in Proposition 2.3.2.3 (where they are denoted with a_2 instead of b_2) and $d := b_2' b_2^{-1}$ a perpendicular between γ_3 and γ_3'. Let δ be the unique closed geodesic in the free homotopy class of $d\gamma_3 d^{-1}\gamma_3'$ (the homotopy can be visualized in Figure 2.18). As in the existence part of Theorem 2.3.3.3, δ is a separating curve of γ_2 and γ_2' from γ_3 and γ_3'. Now define η to be the unique closed geodesic in the homotopy class of $\mathcal{D}(\delta)$ (here $\mathcal{D}$ is the Dehn twist). An equivalent way to define η is as the unique closed geodesic in the homotopy class of $\bar{d}\gamma_3\bar{d}^{-1}\gamma_3'$ where $\bar{d} = b_2'\gamma b_2^{-1}$.

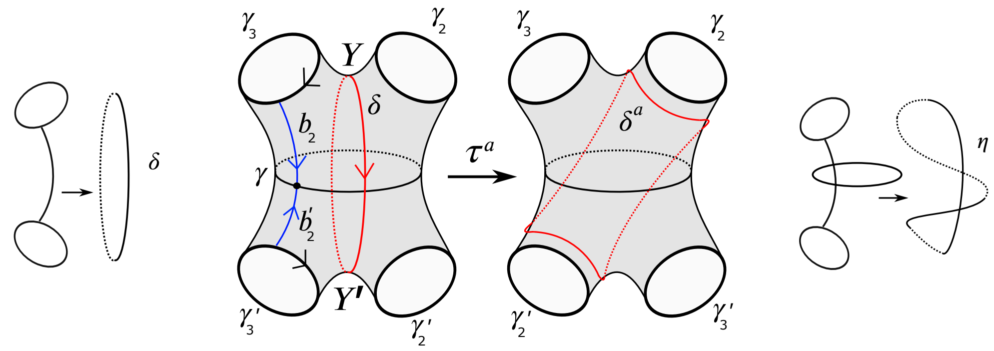

Figure 2.18: Left: short representation of the homotopy of $d\gamma_3 d^{-1}\gamma_3'$ to δ, a longer representation can be found in Figure 2.11. Center: some labels, δ and δ^a in red and the map τ^a. Right: similar rough representation for η.

Definition 2.3.3.6. On X^a, we define δ^a and η^a to be the unique simple closed geodesics in the homotopy classes of $\tau^a(\delta)$ and $\tau^a(\eta)$, where τ^a is the twist homeomorphism defined above.

Remark 2.3.3.7. Considering Remark 2.3.3.4, the question of how many curves are needed to determine the twist parameter $a \in [0,1)$ is closely connected to the question of how many curves are needed to determine $\mathcal{M}_{0,4}$. The following theorem will prove that it is in fact controlled by the length of six curves: the four boundary geodesics and the "γ" and "δ" curves, see Figure 2.18. The reason is that the theorem yields the formula $\cosh(a\ell(\gamma)) = \frac{1}{v}(\cosh(\frac{1}{2}\ell(\delta^a)) - u)$ and, since the hyperbolic cosine is one to one in $[0,1)$, we can invert it to obtain a from the left hand side. However, while these six curves are enough to overdetermine the moduli space, they are not enough for the Teichmüller space. Note that since cosh is two-to-one in $\mathbb{R}$, if we want to determine the value of a varying in $\mathbb{R}$ (as opposed to only in $[0,1)$) we would need two formulas like the one above to pin-point its value. We will see that $\mathcal{T}_{0,4} = \mathbb{R}_+^5 \times \mathbb{R}$ and therefore we need two algebraic relations with cosh to determine a. This is the reason why we have introduced the "η" curve. In the corollary below, we make explicit the fact that the length of the δ and η curve determine the parameter $a \in \mathbb{R}$ real-analitically.

The following lemma is very intuitive and explains an important way to measure a with curves that will be needed a couple of times in this text.

Lemma 2.3.3.8. *As shown in Figure 2.19, the curve $\tau^a(d)$ is homotopic in X^a with endpoints gliding on γ_3 and γ_3' to the curve $b_2' b b_2^{-1}$, where b is a parametrized arc of γ^a and the signed*[18] *length of b is given by*

$$\ell(b) = a\ell(\gamma^a)$$

Proof. In the same way as we did above, take a collar $\mathcal{C}[\gamma^a]$ around γ^a. Observe that a curve (written in Fermi coordinates) of the form $b(s) = (0, t_p + sa)$ for $0 \le s \le 1$ has length $a\ell(\gamma^a)$. This is computed using the metric tensor $d\rho^2 + \ell(\gamma^a)^2 dt^2$ which we know is the metric in collars for $\rho = 0$. Now, let $p = (0, t_p)$ be the intersection point $\gamma \cap d$ on X.

[18] Here the signed distance is assigned as follows: positive if b leads to the left hand side of b_2 and negative otherwise.

By definition of τ^a, $\tau^a(d)$ will be homotopic to a curve $b_2' c b_2^{-1}$ with c an arc of γ. We want to prove that $c = b$ (where b is defined in the paragraph above). To see this, we show that $b_2 \cap \gamma^a = (0, t_p)$ and $b_2' \cap \gamma^a = (0, t_p + a)$. This is true by definition of X^a. Indeed, the curves b_2 and b_2' are defined as the geodesic perpendicular in each trouser joining γ_3 to γ and γ_3' to γ. Roughly said, the "upper" pant in X^0 and X^a is the "same" while the "lower" pant is rotated a parts of a turn, see Figure 2.17. Formally, by definition of X^a: the curve b_2 in X^0 will be the same as in X^a but the curve b_2' will intersect γ^a at precisely the point $(0, t_p + a)$. □

Theorem 2.3.3.9. *For the above family $\{X^a | a \in \mathbb{R}\}$ we have that*

$$\cosh(a\ell(\gamma)) = \frac{1}{v}(\cosh(\frac{1}{2}\ell(\delta^a)) - u)$$
$$\cosh((a+1)\ell(\gamma)) = \frac{1}{v}(\cosh(\frac{1}{2}\ell(\eta^a)) - u)$$

where u and $v > 0$ are real analytic functions of the lengths of γ, γ_2, γ_3, γ_2' and γ_3' that do not depend on the twist parameter a.

Proof. As a matter of fact, we will prove something stronger. First, we simplify notation by writing the length of a curve by the name of the curve, that is $\ell(\gamma) = \gamma$. We prove that $F(a) = \cosh \frac{1}{2}\delta^a$ and $F(a+1) = \cosh \frac{1}{2}\eta^a$ where

$$F(a) = \sinh \frac{1}{2}\gamma_3 \sinh \frac{1}{2}\gamma_3' \Big(\sinh b_2 \sinh b_2' \cosh a\gamma + \cosh b_2 \cosh b_2' \Big) - \cosh \frac{1}{2}\gamma_3 \cosh \frac{1}{2}\gamma_3'. \qquad (2.1)$$

A simple computation then shows that this implies the formulas written above. It is important to note that the lengths b_2 and b_2' are determined by the lengths of γ, γ_2, γ_3, γ_2' and γ_3' because, decomposing X^a into pants and the pants into hexagons, b_2 and b_2' are sides of a geodesic hexagon, hence completely determined. Moreover, if we show that $F(a) = \cosh \frac{1}{2}\delta^a$ then, since the length of η^a in X^a is the same as the length of δ^{a+1} on X^{a+1} (by definition and uniqueness of closed geodesics in homotopy classes), we will have that $F(a+1) = \cosh \frac{1}{2}\delta^{a+1} = \cosh \frac{1}{2}\eta^a$. We now proceed to prove formula 2.1.

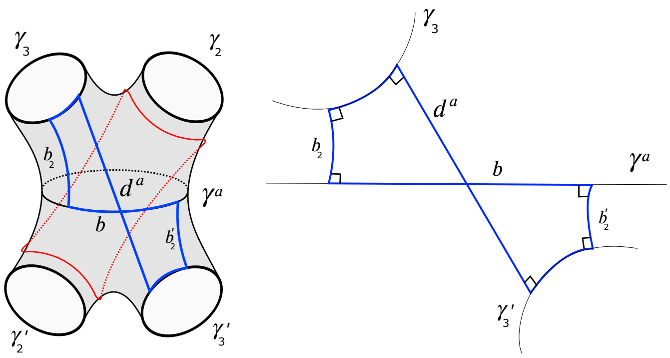

Figure 2.19: How the hexagon in X^a lifts to a crossed hexagon.

Take the curve $b_2' b b_2^{-1}$ from the lemma above with $\ell(b) = a\ell(\gamma^a) = a\ell(\gamma) = a\gamma$ (according to our notational agreement). By Theorem 2.2.5.2 there is a unique perpendicular d^a with endpoints gliding on γ_3 and γ_3'. In particular, we have a piece-wise geodesic right-angled closed curve, by lifting it to the universal cover of X^a, we get a closed piece-wise geodesic curve; a polygon, see Figure 2.19. In this case, it is a crossed right-angled geodesic hexagon. We abuse notation and denote the lifts with the same letters. By Theorem 2.1.2.2 we have that

$$\cosh d^a = \sinh b_2 \sinh b_2' \cosh a\gamma + \cosh b_2 \cosh b_2'.$$

This relates a to the length of d^a, hence we need to focus on how the length of d^a relates to δ^a. In the first place, we note that δ^a is freely homotopic to the curve $d^a \gamma_3 (d^a)^{-1} \gamma_3'$. In the second place, δ^a does not intersect d^a. Indeed, if we cut X^a open along d^a, in the interior of this new hyperbolic surface the closed curve $d^a \gamma_3 (d^a)^{-1} \gamma_3'$ will be homotopic to a unique closed geodesic. By uniqueness in X^a this geodesic is precisely δ^a and, hence, δ^a does not cut d^a.

Now we cut X^a along δ^a and obtain two Y-pieces, we call $\mathcal{Y}$ the one with boundary geodesics γ_3', γ_3 and δ^a. As shown above, d^a is the simple perpendicular from γ_3' to γ_3 in the pair of pants $\mathcal{Y}$. In particular, decomposing $\mathcal{Y}$ into two right-angled hexagons and applying the first formula in Theorem 2.1.2.2 we have that

$$\cosh\frac{1}{2}\delta^a = \sinh\frac{1}{2}\gamma_3 \sinh\frac{1}{2}\gamma_3' \cosh d^a - \cosh\frac{1}{2}\gamma_3 \cosh\frac{1}{2}\gamma_3'$$

which is exactly $\cosh\frac{1}{2}\delta^a = F(a)$. This concludes the proof. □

Corollary 2.3.3.10. *The twist parameter is a real analytic function of the lengths of the seven curves in the theorem above.*

Proof. The trick is to write $\cosh(a\ell(\gamma))$ as $\cosh(\sqrt{(a\ell(\gamma))^2})$ and invert it using that $\cosh\sqrt{z}$ is holomorphic in $\mathbb{C}$ and has positive first derivative on $(-\pi^2, \infty)$. With this and the theorem above we obtain $a^2 = h(\ell(\delta^a, u, v))$ and $(a+1)^2 = h(\ell(\eta^a, u, v))$ where h is a real analytic function. Subtracting the first equation to the second we obtain

$$a = \frac{1}{2}\big(h(\ell(\delta^a, u, v)) - h(\ell(\eta^a, u, v)) - 1\big)$$

which is what we wanted. □

2.3.4 X-Piece Decomposition and the Canonical System of Curves

Observe that not all closed curves on a surface made of pairs of pants come from closed curves of the pairs of pants; in pasting, new homotopy classes of closed curves are created. To characterize $\mathcal{T}_g$ with closed geodesics we need a decomposition that "accounts for all possible closed curves". In this section we define such a decomposition and introduce a very important collection of curves Ω that will be fundamental in our proof of the Fenchel-Nielsen coordinates theorem. From now on, the genus g will be fixed and we use the notation introduced in the end of Section 2.3.1.

Each point in $\mathcal{T}_g$ is represented by (S, φ) where $\varphi : F_g \to S$ is the marking homeomorphism. So far we have not assumed F_g to have more than a topological structure. However, a "canonic" hyperbolic structure on $F = F_g$ would be convenient. For this, we set $F = F_g := F^{L_0 A_0}$ for $L_0 = (1, \dots, 1)$ and $A_0 = (0, \dots, 0)$ (this surface exists by Theorem 2.3.2.4). Choose a pasting geodesic γ_k for a $k = 1, \dots, 3g-3$ and suppose that γ_k comes from the pasting of Y_i and Y_r along γ_{ij} and γ_{rs}. We define $X^k = Y_i + Y_r$ according to the pasting scheme $\gamma_{ij}(t) = \gamma_{rs}(-t)$. Note that there is a canonical isometric inclusion $\iota_k : X^k \to F$ that follows from the canonical isometric inclusion of Y_i and Y_j inside F. If $i = r$, X^k will be a signature $(1, 1)$ surface and if $i \neq r$ we get an X-piece. In order to avoid case by case studies we take a combinatorial model to assemble the surfaces F^{LA} that does not paste curves of the same trousers (this can be done as explained in Remark 2.3.1.2).

Definition 2.3.4.1. For $k = 1, \dots, 3g-3$ we let δ^k and η^k be the curves introduced in X^k in definition 2.3.3.6 in X^k. We denote δ_k and η_k their (isometric) images in F, that is, $\delta_k = \iota_k(\delta^k)$ and $\eta_k = \iota_k(\eta^k)$. The set

$$\Omega = \{\gamma_1, \dots, \gamma_{3g-3}, \delta_1, \dots, \delta_{3g-3}, \eta_1, \dots, \eta_{3g-3}\}$$

is called the canonical curve system. If $\phi : F \to S$ is a homeomorphism, we write $\Omega(S)$ for the collection of curves that are the unique geodesic representatives of the curves in $\varphi(\Omega)$. Lastly, $\ell\Omega(S)$ is the ordered tuple of lengths of the curves in $\Omega(S)$.

Remark 2.3.4.2. This curve system depends heavily in the assembly model of Y-pieces. As said before, we have taken this model so that we do not paste geodesics on the same trouser. However, Ω could still be defined in this case (even though not all curves would come from X-pieces). In Figure 6 one can see an example of Ω for such a combinatorial model.

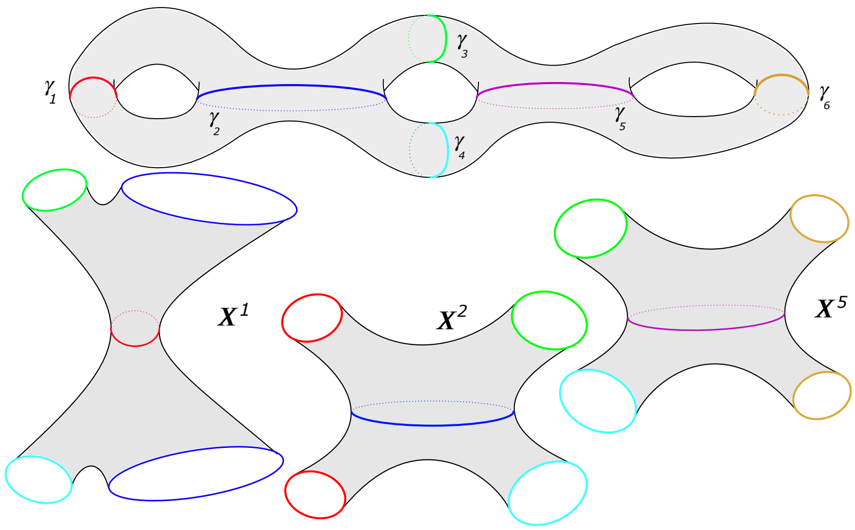

Figure 2.20: Three examples of how the decomposition of F into X^k happens. Contrary to a pants decomposition, while X^1 and X^5 are disjoint, X^1 and X^2 are not (which is necessary if we are to recover all homotopy classes of curves of F through a decomposition).

Remark 2.3.4.3. In the proof of the main theorem we will need the following fact. Let $c_1, \ldots, c_m$ be pair-wise disjoint simple closed curves on a topological surface S. If $\gamma_1, \ldots, \gamma_m$ are curves with the same properties, there exist a homeomorphism $\psi : S \to S$ that is *isotopic to the identity* such that $\psi \circ c_k = \gamma_k$ for $k = 1, \ldots, m$. The main application to this fact is that given a marking homeomorphism $\varphi : F \to S$, we can control the image of a set of disjoint simple closed curves via φ by post-composing with ψ and, since ψ is isotopic to the identity, this does not change the class of the marking. This is a non-trivial topological fact, for a proof we refer the reader to [Bus10, Theorem A.3], which the author names Baer-Zieschang's theorem.

The first $6g - 6$ curves in the canonical system of curves Ω can be used to characterize isotopy classes of homeomorphisms.

Theorem 2.3.4.4. *Let $\varphi, \varphi' : F_g \to S$ be marking homeomorphisms such that, for $k = 1, \ldots, 3g - 3$, we have that*

$$\varphi \circ \gamma_k \text{ is homotopic to } \varphi' \circ \gamma_k \text{ and } \varphi \circ \delta_k \text{ is homotopic to } \varphi' \circ \delta_k.$$

Then φ and φ' are isotopic.

Proof. The proof of this important theorem requires certain knowledge about the mapping class group of a 3-holed sphere that we have had no space to develop. We refer the reader to the extended version [Bar21] for a sketch of this proof and to [Bus10, Theorem 6.1.7 and appendix] for a full study. □

Corollary 2.3.4.5. *Two homeomorphisms $\varphi, \varphi' : F_g \to S$ are homotopic if and only if they are isotopic.*

Proof. If they are homotopic the conditions on the theorem above are obviously satisfied. □

2.3.5 The Fenchel-Nielsen Coordinates

Theorem 2.3.5.1. *The Fenchel-Nielsen coordinates FN (as defined below) parametrize $\mathcal{T}_g$ bijectively and homeomorphically:*

$$\begin{aligned} \mathrm{FN} : \mathcal{T}_g &\longrightarrow \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \\ S &\longmapsto (\ell_1(S), \ldots, \ell_{3g-3}(S), a_1(S), \ldots, a_{3g-g}(S)) \end{aligned}$$

Moreover, there are $9g - 9$ homotopy classes of curves such that their lengths uniquely determine the point of $\mathcal{T}_g$. In other words, there is an injection $\iota_{\mathcal{T}} : \mathcal{T}_g \longrightarrow \mathbb{R}_+^{9g-9}$ and this injection is a topological embedding.

Proof. Throughout this section we carry out the construction and proof of this theorem. In the first place, we should define the map. To do so we will find canonical homeomorphisms φ^{LA} from $F = F^{L_0A_0}$ to F^{LA} and then show that every point in $\mathcal{T}_g$ is represented by a *unique* (F^{LA}, φ^{LA}). This will prove that the map is well defined and bijective.

STEP 1: *Construction of* φ^{LA}. First we define a map σ^L that stretches $F = F^{F_0A_0}$ into F^{LA_0} and then a map τ^{LA} that twists F^{LA_0} into F^{LA}, φ^{LA} will be the composition of both.

For every $i = 1, \dots, 2g-2$ we let $\sigma_i^{LA_0} : Y_i \to Y_i^{LA_0}$ be the stretch map introduced in definition 2.3.2.6. Since each $\sigma_i^{LA_0}$ preserves the boundary parametrization of the boundary geodesics and all twists parameters are zero the following map is a well defined homeomorphism:

$$\sigma^L := \left\{ \pi^{LA_0} \circ \sigma_i^{LA_0} \circ \pi^{-1} \text{ for every } i = 1, \dots, 2g-2 \right.$$

Moreover, the homeomorphism $\sigma^L : F \to F^{LA_0}$ sends $\gamma_k(t)$ to $\gamma_k^{LA_0}(t)$ for each $t \in \mathbb{S}^1$ and each $k = 1, \dots, 3g-3$. Now, for a surface F^{LA}, consider collars $\mathcal{C}_k^{LA}$ around each γ_k^{LA} of width ϵ_k so that they are disjoint. To define the map τ^{LA} we first define maps from the collars $\mathcal{C}_k^{LA_0}$ in F^{LA_0} to $\mathcal{C}_k^{LA}$ in F^{LA} (note that since the length of $\gamma_k^{LA_0}$ is the same as the length of γ_k^{LA}, the collars $\mathcal{C}_k^{LA_0}$ and $\mathcal{C}_k^{LA}$ can be taken to be isometric). So, in each of this collars we define twists like in Section 2.3.3. That is, for every k we define $T_k^{LA} : \mathcal{C}_k^{LA_0} \to \mathcal{C}_k^{LA}$ as follows

$$T_k^{LA}(\rho, t) = (\rho, t + a_k \frac{\epsilon_k + \rho}{2\epsilon_k}).$$

Finally, if we let I^{LA} be the natural identification $I^{LA} : Y_1^{LA_0} \sqcup \cdots \sqcup Y_{2g-2}^{LA_0} \to Y_1^{LA} \sqcup \cdots \sqcup Y_{2g-2}^{LA}$, then the following map is a well defined homeomorphism from F^{LA_0} to F^{LA}:

$$\tau^{LA} = \begin{cases} T_k^{LA} & \text{on } \mathcal{C}_k^{LA}, k = 1, \dots, 3g-3 \\ \pi^{LA} \circ I^{LA} \circ (\pi^{LA_0})^{-1} & \text{elsewhere} \end{cases}$$

We have proven that the map $\varphi^{LA} = \tau^{LA} \circ \sigma^L : F \longrightarrow F^{LA}$ is a homeomorphism. We note that $\varphi^{LA} \circ \gamma_k(t) = \gamma_k^{LA}(t + a_k/2)$ for $t \in \mathbb{S}^1$ and $k = 1, \dots, 3g-3$.

STEP 2: *Surjectivity: given a marked hyperbolic surface* $(S, \varphi : F \to S)$ *there exists a marking equivalent surface* (F^{LA}, φ^{LA}) *for some* $(L, A) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$.

Step 2, part 1: *Finding a suitable candidate for* (L, A). For every $k = 1, \dots, 3g-3$, we will denote the unique geodesic in the homotopy class of $\varphi(\gamma_k)$ by $\gamma_k(S)$. By Remark 2.3.4.3 we can, in fact, assume that $\varphi(\gamma_k)$ is exactly $\gamma_k(S)$. Clearly, the candidate for L is $L(S) = (\ell(\gamma_1(S)), \dots, \ell(\gamma_{3g-3}(S)))$. Now we wish to determine what the twist parameters of S might be. Recall from Section 2.3.4 that for every k we have X-pieces X^k isometrically embedded in F by ι_k. Intuitively, this "X-piece hyperbolic decomposition" on F induces a hyperbolic decomposition on S via φ. Rigorously, we pull back the structure of the hyperbolic surface $\varphi(\iota_k(X^k))$ onto X^k, then we obtain an X-piece denoted $X^k(S)$ isometrically embedded in S by $\iota'_k : X^k(S) \to S$ and a homeomorphism $\varphi^k : X^k \to X^k(S)$ such that the following diagram commutes.

$$\begin{array}{ccc} X^k & \longrightarrow & X^k(S) \\ \downarrow & & \downarrow \\ F & \longrightarrow & S \end{array}$$

The whole idea (as illustrated in Figure 2.21) is that the maps φ^k twist the X-pieces so we would like to find an analogue of the b curve in Lemma 2.3.3.8 to measure the twist, let us do so now. Observe that the map φ^k takes Y-pieces to Y-pieces. Next, let $\gamma_k(S)$, $\delta_k(S)$ and $\eta_k(S)$ be the unique closed geodesics in S homotopic to $\varphi(\gamma^k)$, $\varphi(\delta^k)$ and $\varphi(\eta^k)$ respectively. Likewise, let $\gamma^k(S)$, $\delta^k(S)$ and $\eta^k(S)$ be the unique closed geodesics in $X^k(S)$ homotopic to $\varphi^k(\gamma^k)$, $\varphi^k(\delta^k)$ and $\varphi^k(\eta^k)$ respectively. In particular, since ι'_k is an isometry, by uniqueness of geodesics, we have that $\iota'_k(\gamma^k(S)) = \gamma_k(S)$, $\iota'_k(\delta^k(S)) = \delta_k(S)$ and $\iota'_k(\eta^k(S)) = \eta_k(S)$.

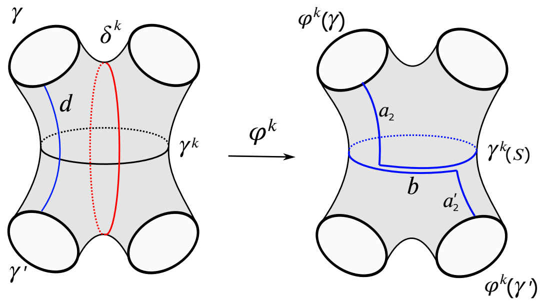

Figure 2.21

Let Y_i and Y_r the Y-pieces that decompose X^k in the pants decomposition of $F = F^{L_0 A_o}$, let γ and γ' be boundary geodesics of Y_i and Y_r in X^k respectively that are not separated by δ^k (as in Figure 2.21). As in Section 2.3.3, let $d = b_2' b_2^{-1}$ and recall that δ^k is freely homotopic to $d\gamma d^{-1}\gamma'$. The arc $\varphi^k(\delta)$ in $X^k(S)$ is homotopic with endpoints gliding on the boundary to a unique curve $a_2' b a_2^{-1}$ with the following properties (shown in Figure 2.21): φ^k induces an trousers decomposition of $X^k(S) = \varphi^k(Y_i) + \varphi^k(Y_r)$ along the geodesic $\gamma^k(S)$, we let a_2 be the shortest simple perpendicular connecting the boundary $\varphi^k(\gamma)$ with $\gamma^k(S)$, a_2' likewise connecting $\varphi^k(\gamma')$ with $\gamma^k(S)$, and b be a geodesic arc on $\gamma^k(S)$ (which is not simple in general). This measures how much φ^k has twisted X^k. Therefore, with Lemma 2.3.3.8 in mind, we define $a_k(S) = \ell(b)/\ell(\gamma_k(S))$, where, for emphasis, we denote $\bar{\ell}$ the signed length of b; positive if b leads to the left hand side of a_2' and negative otherwise. Therefore, we have found the following candidate for (L, A).

$$(L(S), A(S)) = (\ell(\gamma_1(S)), \ldots, \ell(\gamma_{3g-3}(S)), a_1(S), \ldots, a_{3g-3}(S)).$$

Step 2, part 2: *Checking that* $(F^{L(S)A(S)}, \varphi^{L(S)A(S)})$ *is marking equivalent to* (S, φ). We will write $\widehat{S} = F^{L(S)A(S)}$ and $\hat{\varphi} = \varphi^{L(S)A(S)}$ to simplify notation. If we repeat the process in *Step 2, part 1* for the surface $\widehat{S}$ we obtain X-pieces along isometric embedding $\hat{\iota}_k : X^k(\widehat{S}) \to F^{LA}$ such that the following diagram commutes:

$$\begin{array}{ccc} X^k & \xrightarrow{\hat{\varphi}^k} & X^k(\widehat{S}) \\ \downarrow{\scriptstyle \iota_k} & & \downarrow{\scriptstyle \hat{\iota}_k'} \\ F & \xrightarrow{\hat{\varphi} = \varphi^{L(S)A(S)}} & \widehat{S} = F^{L(S)A(S)} \end{array}$$

The analogue of $a_2' b a_2'$ will be $\hat{a}_2' \hat{b} \hat{a}_2'$. Now, by Lemma 2.3.3.8 and the construction above both $X^k(S)$ and $X^k(\widehat{S})$ have the same twist parameters. Moreover, since their boundary geodesics and "γ" curve have the same lengths by definition, Theorem 2.3.3.3 tells us that they are isometric, let $m^k : X^k(S) \to X^k(\widehat{S})$ denote the isometry. How do we make sure this isometries define an isometry $m : S \to \widehat{S}$? Recall that the X-pieces X^k overlap a lot (see Figure 2.20), so one should be careful here. The trick is to use the uniqueness in Theorem 2.3.3.3 to assert that m^k is the unique isometry that induces the same permutation of boundary geodesics as $\hat{\varphi}^k \circ (\varphi^k)^{-1}$ does. In other words, the action of m^k on the pants decomposition is the same as $\hat{\varphi}^k \circ (\varphi^k)^{-1}$. Therefore, the isometries

$$\hat{\iota}_k \circ m^k \circ (\iota_k')^{-1} : \iota_k'(int X^k(S)) \to \hat{\iota}_k(int X^k(\widehat{S}))$$

for every k together define a single isometry $m : S \to \widehat{S}$. Moreover, $m \circ \gamma_k(S) = \gamma_k(\widehat{S})$ and $m \circ \delta_k(S) = \delta_k(\widehat{S})$. The former equality we already know, for the latter note that δ^k is homotopic to $d\gamma d^{-1}\gamma'$, it follows that $m^k(\delta^k(S))$ is homotopic to $\delta^k(\widehat{S})$ and, by uniqueness, we have that $m^k \circ \delta^k(S) = \delta^k(\widehat{S})$. All in all, by Theorem 2.3.4.4, S and $\widehat{S} = F^{LA}$ are marking equivalent, which concludes the proof of this step.

STEP 3: *Injectivity: if F^{LA} and $F^{L'A'}$ are marking equivalent, then* $(L, A) = (L', A')$. From the definition of marking equivalence it follows that there is an isometry $m : F^{LA} \to F^{L'A'}$ such that $m \circ \varphi^{LA}$ is isotopic to $\varphi^{L'A'}$. Following the notation of definition 2.3.4.1, m takes $\Omega(F^{LA})$ to $\Omega(F^{L'A'})$ curve by curve. In particular, for every $k = 1, \dots, 3g-3$:

$$\ell(\gamma_k(F^{LA})) = \ell(\gamma_k(F^{L'A'})), \ \ \ell(\delta_k(F^{LA})) = \ell(\delta_k(F^{L'A'})), \ \ \ell(\eta_k(F^{LA})) = \ell(\eta_k(F^{L'A'})).$$

This implies directly that $L = L'$ and Theorem 2.3.3.9 implies that $A = A'$ since the lengths of the δ and η curves determine the twist parameters.

Observe that this proves that FN is a well defined bijection. Moreover, it also proves the "$9g-9$ curves" part of the theorem since we have found that two surfaces S and S' are marking equivalent if and only if the lengths of $\Omega(S)$ and $\Omega(S')$ agree curve to curve, that is $\ell(\Omega(S)) = \ell(\Omega(S'))$.

STEP 4: *FN is a homeomorphism and $\iota_{\mathcal{T}}$ and embedding.* Theorem 2.3.3.9 tells us that twist parameters are determined continuously by length functions and the length functions of the δ and η are continuously determined by the twist parameter. As a consequence, since length functions are continuous by Corollary 2.2.4.6, FN is continuous and $\iota_{\mathcal{T}} : \mathcal{T}_g \to \mathbb{R}^{9g-9}$ is also continuous. If we see that ι is a homeomorphism of $\mathcal{T}_g$ with its image it will imply that FN is also a homeomorphism. Indeed, since ι is injective, $FN \circ \iota_{\mathcal{T}}^{-1} : \iota_{\mathcal{T}}(\mathcal{T}_g) \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ is a homeomorphism (it is in fact a real-analytic diffeomorphism) because of what was said in the beginning of this paragraph. Therefore, if $\iota_{\mathcal{T}}$ is a hoemomorphism with its image, FN will also be a homeomorphism.

We have to show that $\iota_{\mathcal{T}}^{-1} : \iota_{\mathcal{T}}(\mathcal{T}_g) \to \mathcal{T}_g$ is continuous. This comes from the fact that continuous variation of the length of one of the curves in the Fundamental System of Curves Ω yields a continuous variation on the axi of the respective hyperbolic transformations (see Section 1.2.4 and 2.2.4). In particular, deck transformations vary continuously in the matrix topology of $\mathrm{PSL}(2, \mathbb{R})$. This gives us continuity with respect to the algebraic topology (see Example 1.2.4.6).

This concludes the proof of the theorem. □

Remark 2.3.5.2 (Well Defined Analytic Structure on $\mathcal{T}_g$)**.** The identifications made by the Fenchel-Nielsen coordinates FN allow us to make $\mathcal{T}_g$ into a real-analytic manifold. However, note that for each combinatorial model in the assembly of Y-pieces we get a different set of Fenchel-Nielsen coordinates FN', one has to check (it is non-trivial) that $FN' \circ FN^{-1} : \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ is real analytic. The idea is the following: Theorem 2.3.3.9 implies that $FN(S) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ is a real analytic function of $\ell(\Omega(S))$ since the length parameters clearly are real analytic and, by Theorem 2.3.3.9, the twist parameters are too. It can also be seen that all length functions (of other curves not necessarily in Ω) are real analytic in $FN(\mathcal{T}_g) = \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$, which ends up proving that the real-analytic structure on $\mathcal{T}_g$ is well defined since the transition maps will be real-analytic. For a detailed proof of this see [Bus10, Section 6.3].

2.3.6 A Brief Mention of Some Applications

In this section we sketch some interesting applications of our work. The first two are quite immediate.

Theorem 2.3.6.1 (Collar Theorem)**.** *Let S be a compact Riemann surface of genus $g \geq 2$ and let $\gamma_1, \dots, \gamma_m$ be a maximal collection of disjoint essential simple closed geodesics. Then $m = 3g-3$ and the collars*

$$\mathcal{C}(\gamma_i) = \{p \in S | d(p, \gamma_i) \leq w_i\}, \text{ where the widths are } w_i = \frac{1}{\sinh \frac{1}{2}\ell(\gamma_i)}$$

are pairwise disjoint for $i = 1, \dots, 3g-3$. In particular, the shorter the geodesic the wider the collar and the longer the geodesic the thinner the collar, see Figure 3.

Proof. From Section 2.3.1 we know that such a decomposition is a pants decomposition. So we only have to compute half-collars in each trouser. This is reduced to computing "half-collars" on the hexagon decomposition which is an exercise in hyperbolic geometry. Figure 2.22 is representative of this exercise. □

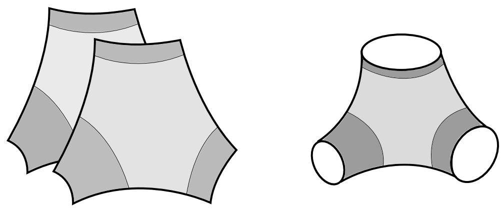

Figure 2.22: Half-collars on two isometric hexagons and how they make half-collars in the pasted Y-piece.

Let us now ponder on a distance function on $\mathcal{T}_g$. We say that (S, φ) and (S', φ') are equivalent if there is an isometry m such that $\varphi' \circ \varphi^{-1}$ is isotopic to m. This means that they are the same point in $\mathcal{T}_g$, they are at "distance" 0. What if there exists no such isometry and only a map that distorts distances a little, intuitively, both marking should be points close together in $\mathcal{T}_g$. This is indeed true and formalized as follows. For $q \geq 1$, a homeomorphism between metric spaces $\phi : A \to B$ is said to be a q-quasi isometry if for all $x, y \in A$ we have that

$$\frac{1}{q} d_A(x, y) \leq d_B(\phi x, \phi y) \leq q d_A(x, y).$$

For a quasi-isometry ϕ we denote by $q[\phi]$ the infimum of these $q \in \mathbb{R}$ for which ϕ is a q-quasi isometry. This $q[\phi]$ represents the maximal length distortion. Therefore, according to common sense, if instead of isometries m we have quasi-isometries, the number $q[\phi]$ should help us measure distance. Moreover, note that stretch maps and twist homeomorphisms are quasi-isometries.

Theorem 2.3.6.2. *For $S = (S, \varphi)$ and $S' = (S', \varphi')$ points of $\mathcal{T}_g$, the function δ defined as*

$$\delta(S, S') := \inf \log q[\phi]$$

where ϕ runs through quasi-isometries $\phi : S \to S'$ in the isotopy class of $\varphi' \circ \varphi^{-1}$, is a distance function compatible with the topology of $\mathcal{T}_g$.

Proof. If we look a little close, this is a direct consequence of our construction. For pants we have that the stretch maps $\sigma(Y, Y')$ are clearly quasi-isometries and it's easy to prove that if a sequence of pants $\{Y_n\}_n$ converges to Y (in the sense that the lengths of the boundary geodesics converge) then $q[\sigma(Y_n, Y)] \xrightarrow{n} 1$. Similarly, each twist homeomorphism is also a quasi-isometry and we have that if $a_n \xrightarrow{n} a$ then $q[\tau^{a_n} \circ (\tau^a)^{-1}] \xrightarrow{n} 1$. Now, using the notion of convergence given by the map FN it is easy to see that $\{(S_n, \varphi_n)\}_n$ converges to (S, φ) implies that $q[\varphi_n \circ \varphi^{-1}] \to 1$, hence the distance δ converges to 0. For the converse, one proves that $\delta(S_n, S) \to 0$ implies that $\ell\Omega(S_n) \to \ell\Omega(S)$ by bounding the length of curves under quasi-isometries. Convergence in the curve embedding implies convergence in $\mathcal{T}_g$. □

We note that this approach is reminiscent (and equivalent) to quasi-conformal mappings, a way to measure maximal distortion amongst complex structures of Riemann surfaces. Finally, the last thing we have gotten for free from our construction is a generalization to surfaces of signature (g, n). In Section 2.3.1 the pants decomposition for such surfaces was discussed, careful review of our proofs yields:

Theorem 2.3.6.3. *The Fenchel-Nielsen coordinates for hyperbolic surfaces $F_{g,n}$ are*

$$(\ell_1, \ldots, \ell_{3g-3+n}, \ell_1^\partial, \ldots, \ell_n^\partial, a_1, \ldots, a_{3g-3+n})$$

where ℓ and a are as before and ℓ^∂ are the lengths of the n boundary geodesics. Hence $\mathcal{T}_{g,n}$ is homeomorphic to a ball of dimension $3(g+n) - 3$.

Finally, we mention in passing one astonishing aspect of Wolpert's formula. The form given by

$$w = \sum_{i=1}^{3g-3} d\ell_i \wedge da_i$$

is a symplectic form well defined for *any* pants decomposition and independent of the specific Fenchel-Nielsen coordinates taken. For a proof of this remarkable fact see Theorem 3.14 in [Wol10].

Appendix A

Möbius transformations and $\mathrm{PSL}(2,\mathbb{R})$

This appendix is meant to summarize some features of Möbius transformations and the group $\mathrm{PSL}(2,\mathbb{R})$, specially those outside the usual undergraduate curricula. If the reader is interested in the proofs of this facts and some more, they can be found in the extended version of this work [Bar21]. A comprehensive and deep study of this can be found in [Bea95] or [Ser].

Basic properties

A map $f : \mathbb{C} \longrightarrow \mathbb{C}$ of the form $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ it is called a Möbius transformation. We can extend the map to the Riemann sphere $\hat{\mathbb{C}}$ by defining $f(\infty) = a/c$ (with the usual convention of $a/0 = \infty$) and $f(-d/c) = \infty$. Therefore, we regard f as a holomorphic function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.[1] It can be easily computed that $\frac{dz-b}{-cz+a}$ is the inverse Möbius transformation of f. We can conclude that f is a biholomorphism of the Riemann sphere onto itself. We have reasoned the well definedness of the following definition.

Definition 1. A Möbius transformation f is a biholomorphic automorphism of the Riemann sphere given by $f(z) = \frac{az+b}{cz+d}$ such that $ad - bc \neq 0$ for complex numbers a, b, c, d, which are usually called coefficients of the Möbius transformation f (the letters a, b, c and d canonically denote the coefficients). In fact, we can assume that $ad - bc = 1$ because multiplication of the coefficients of f by a non-zero constant $\lambda \in \mathbb{C}$ does not alter f. A real Möbius transformation will be a Möbius transformation with real coefficients, i.e. $a, b, c, d \in \mathbb{R}$.

It is important to note that one cannot always take the $ad - bc = 1$ for real Möbius transformations. For example, $f(z) = 1/z$ satisfies $ad - bc = -1$, if we wanted $ad - bc = 1$, then we would have to multiply the coefficients by i: $f(z) = i/iz$. However, this would no longer be a real Möbius transformation. Hence, real Möbius transformations can be normalized (so that they still have real coefficients) to either have $ad - bc = 1$ or $ad - bc = -1$.

Proposition 2. Möbius transformations satisfy the following properties:

(i) Each non-identity Möbius transformation is the composition of a translation, followed by an inversion, a homothety, a rotation and a translation.

(ii) Circles and lines of the Riemann sphere[2] are sent to circles and lines of the Riemann sphere.

(iii) As a self-mapping of the complex plane it is conformal wherever it is defined.

(iv) They have one or two fixed points (save for the identity).

(v) There is a unique Möbius transformation taking any three distinct points of $\hat{\mathbb{C}}$ to any three distinct points of $\hat{\mathbb{C}}$, that is, the action of Möbius transformations on $\hat{\mathbb{C}}$ is sharply 3-transitive.

It is worth noting that the so-called Caley transformation $\frac{i-z}{z+i}$ maps the upper half plane biholomorphically onto the open unit disc (with inverse $i\frac{1-z}{1+z}$).

[1] If $ad - bc = 0$ and a, b, c and d are not all 0, then f is constant, which is a case of no interest.

[2] Lines of $\hat{\mathbb{C}}$ are lines of $\mathbb{C}$ with the point at infinity.

Matrix representation

Let $\mathfrak{M}$ be the group of Möbius transformations with the composition as the operation. Recall that we can take $\mathfrak{M}$ to be the Möbius transformations such that $ad - bc = 1$. Therefore, we have a map from the special linear group into $\mathfrak{M}$:

$$\pi : SL(2,\mathbb{C}) \longrightarrow \mathfrak{M}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az+b}{cz+d}$$

It can be readily checked that it is a group homomorphism. Moreover, it is clear that its kernel is $\{\pm Id\}$. By the isomorphism theorem, $\mathfrak{M} \cong SL(2,\mathbb{C})/\{\pm Id\} =: \mathrm{PSL}(2,\mathbb{C})$. We will denote this isomorphism by M, that is, if $f(z) = \frac{az+b}{cz+d}$ then $M(f) = \{\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}\}$. Usually we take $M(f)$ to be a representative instead of the class. We have proved:

Proposition 3. The Möbius transformations $\mathfrak{M}$ are group-isomorphic to the projective linear group $\mathrm{PSL}(2,\mathbb{C})$ (recall that $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{PGL}(2,\mathbb{C})$ are isomorphic).

Similar regards can be used to see that the group of real Möbius transformations with positive determinant $\mathfrak{M}^+_{\mathbb{R}}$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. Lastly, note that $\mathrm{PSL}(2,\mathbb{R})$ is *not* isomorphic to $\mathrm{PGL}(2,\mathbb{R})$

Canonical form

Two transformations $f, g \in \mathfrak{M}$ are said to be conjugate in $\mathfrak{M}$ or $\mathfrak{M}$-conjugate if there is an $h \in \mathfrak{M}$ such that $f = h \circ g \circ h^{-1}$. Every Möbius transformation can be put into a simple form called the canonical form:

Proposition 4. Let $f(z) = \frac{az+b}{cz+d} \in \mathfrak{M}$, $f \neq Id$. Then,

(i) if f has a single fixed point it is conjugate to: $z + \alpha$, $\alpha \neq 0$. Its matrix representation is $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$,

(ii) if f has a two fixed points it is conjugate to: λz, $\lambda \neq 0, 1$. Its matrix representation is $\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$.

To conclude this section, we note that if a real Möbius transformation has fixed points in $\mathbb{R} \cup \{\infty\}$ then it is conjugate in the group of real Möbius transformation to one of the above canonical forms with $\alpha, \lambda \in \mathbb{R}$.

Classification of Möbius Transformations

We know classify Möbius transformations.

(1) A Möbius transformation is called *parabolic* if it has one fixed point or, equivalently if it is conjugate to a transformation of the form $z + \alpha$, $\alpha \neq 0$.

(2) A Möbius transformation is called *elliptic* if it is conjugate to $e^{i\theta}z$ for $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. This can be regarded as rotations of the Riemann sphere, in its canonical form is a rotation of angle θ with respect to the vertical axis.

(3) A Möbius transformation is called *hyperbolic* if it is conjugate to λz with $\lambda > 0$ and $\lambda \neq 1$. This is a dilation. Note that the "case" $\lambda = -1$ is in fact elliptic.

(4) Not all elements in $\mathfrak{M}$ fall into one of the previous categories. Transformation that have two distinct fixed points (non-parabolic) and are not rotations (not elliptic) are called loxodromic. Those include hyperbolic transformations.

Recall that the trace of a matrix is invariant under conjugation. This implies that every Möbius transformation has "two" traces associated to it. However, given that they only differ by sign we can define $\mathrm{tr}^2(f) = (\mathrm{tr}(M(f)))^2$. Using the canonical form to compute the trace we have proved:

Proposition 5. Let $f \neq Id$ be a a Möbius transformation.

(i) f is parabolic if and only if $\mathrm{tr}^2(f) = 4$.

(ii) f is elliptic if and only if $\mathrm{tr}^2(f) \in [0,4)$.

(iii) f is hyperbolic if and only if $\mathrm{tr}^2(f) > 4$.

(iv) f is loxodromic if and only if $\mathrm{tr}^2(f) \in \mathbb{C} \setminus [0,4]$.

It is easy to see that *real Möbius transformations that are not the identity are one of the first three types*. Studying the "fixed point" equation $f(z_0) = z_0$ the following is easily proved.

Proposition 6. If $f \neq Id$ is a real Möbius transformation:

(i) f is parabolic if and only if f has one fixed point in $\mathbb{R} \cup \{\infty\}$

(ii) f is elliptic if and only if f has two different conjugate fixed points.

(iii) f is hyperbolic if and only if f has two fixed point in $\mathbb{R} \cup \{\infty\}$

Biholomorphic Automorphism Groups of Canonical Domains

The following theorem characterizes Möbius transformations as groups of conformal automorphism of the simply connected Riemann surfaces.

Theorem 7 (Autobiholomorphism Groups of $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{H}$)**.** Let $\mathrm{Aut}(S)$ denote the group of autobiholomorphism of a Riemann surface S, $\mathfrak{M}$ the group of Möbius transformations and $\mathfrak{M}^+_{\mathbb{R}}$ the group of real Möbius transformations with positive determinant.

(1) $\mathrm{Aut}(\hat{\mathbb{C}}) = \mathfrak{M} \cong \mathrm{PSL}(2,\mathbb{C})$,

(2) $\mathrm{Aut}(\mathbb{C}) = \{f(z) = az + b|\ a, b \in \mathbb{C}, a \neq 0\}$,

(3) $\mathrm{Aut}(\mathbb{D}) = \{f \in \mathfrak{M} | f(\mathbb{D}) = \mathbb{D}\} = \{f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}|\ \theta \in \mathbb{R}, a \in \mathbb{D}\} = \{f(z) = \frac{az+b}{\bar{b}z+\bar{a}}|\ a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1\}$,

(4) $\mathrm{Aut}(\mathbb{H}) = \{f \in \mathfrak{M} | f(\mathbb{H}) = \mathbb{H}\} = \mathfrak{M}^+_{\mathbb{R}} \cong \mathrm{PSL}(2,\mathbb{R})$.

In Section 2.1 we show that $\mathrm{Is}^+(\mathbb{H}) = \mathrm{Aut}(\mathbb{H})$ and now we have seen that $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R})$. Then, it makes sense to study this group from the point of view of hyperbolic geometry. Conversely, as seen in the main text, the study of this group is very fruitful for the study of hyperbolic surfaces.

$\mathrm{PSL}(2,\mathbb{R})$ and Hyperbolic Geometry

Now we introduce a new invariant for real Möbius transformations, the minimum displacement, which will play an important role in this text. Let $f \in \mathrm{PSL}(2,\mathbb{R})$, we define the minimum displacement $d(f)$ to be the infimum over $z \in \mathbb{H}$ of how much f has displaced z, that is,

$$d(f) := \inf_{z \in \mathbb{H}} d(z, f(z)).$$

Like the trace of a Möbius transformation, this number is invariant under conjugation by elements of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Is}^+(\mathbb{H})$. Indeed, if $g \in \mathrm{Is}^+(\mathbb{H})$ we have that $d(z, f(z)) = d(g(z), g \circ f(z))$, thus:

$$d(f) = \inf_{z \in \mathbb{H}} d(z, f(z)) = \inf_{g^{-1}(z), z \in \mathbb{H}} d(g^{-1}(z), f \circ g^{-1}(z)) = \inf_{z \in \mathbb{H}} d(g \circ g^{-1}(z), g \circ f \circ g^{-1}(z)) = d(g \circ f \circ g^{-1}).$$

Previously, we classified Möbius transformations according to their trace, now we do the same with the minimum displacement.

Proposition 8. The minimum displacement of elliptic transformations is 0 and it is attained, the minimum displacement of parabolic transformations is 0 but it is not attained and the minimum displacement of hyperbolic transformations is > 0.

Amongst real Möbius transformations, the hyperbolic ones along with their axis play the most important role in the theory to come. Let $f \in \mathrm{PSL}(2, \mathbb{R})$ be hyperbolic and $p_1 \neq p_2$ its fixed points in $\mathbb{R}$. There is a geodesic "joining" them, that is either a semicircle reaching $\mathbb{R}$ perpendicularly that would have p_1 and p_2 as endpoints on $\mathbb{R}$ or a straight vertical line joining $p_1 \in \mathbb{R}$ and $p_2 = \infty$. Since f sends geodesics to geodesics and fixes p_1 and p_2, it also fixes the geodesic "joining" them. This geodesic is called the axis of f and it is denoted by A_f. In other words $f(A_f) = A_f$.

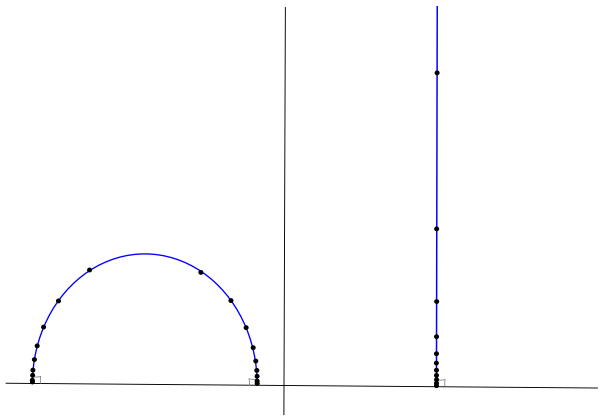

Figure A.1: The blue lines are axis of the hyperbolic transformations that have as fixed points the "intersections" of the axis with $\mathbb{R} \cup \{\infty\}$. The black dots are the orbit of a point, each is $d(f)$ apart.

Note that the axis itself is not quite a conjugacy invariant, we have that $A_{g \circ f \circ g^{-1}} = g(A_f)$. However, as a corollary to the previous proposition in [Bar21] we show that $d(f)$ is attained at every point of A_f for any hyperbolic transformation f. Lastly, we merge geometry and algebra by relating $d(f)$ and $\mathrm{tr}(f)^2$.

Lemma 9. Let $f \in \mathrm{PSL}(2, \mathbb{R})$ be a hyperbolic transformation. We have that

$$\mathrm{tr}(f)^2 = 4\cosh^2\left(\frac{d(f)}{2}\right).$$

Finally, that $\mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PSL}(2, \mathbb{C})$ acts 3-transitively on $\widehat{\mathbb{C}}$ does not imply that $\mathrm{Is}^+(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$ acts 3-transitively on $\mathbb{H}$. In fact, $\mathrm{Is}^+(\mathbb{H})$ could not even act 2-transitively on $\mathbb{H}$ since isometries can only take equidistant points to equidistant points. Taking this into account, $\mathrm{PSL}(2, \mathbb{R})$ actually acts 2-transitively on equidistant pairs of points.

Proposition 10. Let $P, P', Q, Q' \in \mathbb{H}$ such that $d = d_{\mathbb{H}}(P, P') = d_{\mathbb{H}}(Q, Q')$. Then there is a map $f \in \mathrm{Is}^+(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$ such that $f(P) = Q$ and $f(P') = Q'$.

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